Assignment 1

1) For any 2 distributions p, q on $\{1, 2, \dots, n\}$, s.t.

$$\max_{A \subseteq \{1,2,\dots,n\}} |p(A) - q(A)| = \sum_{i=1}^{n} |p(i) - q(i)|/2$$

Let $$Z = \sum_{i : p(i) \geq q(i)} |p(i) - q(i)| - \sum_{i : q(i) > p(i)} |p(i) - q(i)|$$

$$= \sum_{i : p(i) \geq q(i)} (p(i) - q(i)) - \sum_{i : q(i) > p(i)} (q(i) - p(i))$$

$$= \sum_{i : p(i) \geq q(i)} (p(i) - q(i)) + \sum_{i : q(i) > p(i)} (p(i) - q(i))$$

$$= \sum_{i=1}^{n} p(i) - \sum_{i=1}^{n} q(i)$$

$$= 1 - 1 = 0 \qquad \left[\text{As } \sum_i p(i) = \sum_i q(i) = 0\right]$$

$$\therefore Z = 0 \Rightarrow \sum_{i : p(i) \geq q(i)} |p(i) - q(i)|$$

$$= \sum_{i : q(i) > p(i)} |p(i) - q(i)|$$

Now, $$\sum_{i=1}^{n} |p(i) - q(i)| / 2$$

$$= \frac{\sum_{i : p(i) \geq q(i)} |p(i) - q(i)| + \sum_{i : q(i) > p(i)} |p(i) - q(i)|}{2}$$

$$= \frac{2 \sum_{i : p(i) \geq q(i)} |p(i) - q(i)|}{2}$$

$$= \sum_{i : p(i) \geq q(i)} |p(i) - q(i)|$$

$$= \sum_{i : q(i) > p(i)} |p(i) - q(i)|$$

Clearly, picking A s.t $A \subseteq \{1, 2, \dots, n\}$ & $p(i) \geq q(i) \ \forall i \in A$ OR $q(i) > p(i) \ \forall i \in A$

maximises $|p(A) - q(A)|$

$\therefore \max_{A \subseteq \{1,2,\dots,n\}} |p(A) - q(A)| = \sum_{i=1}^{n} |p(i) - q(i)| / 2$

example :

i	$p(i)$	$q(i)$
1	1/2	1/3
2	0	1/3
3	1/2	0
4	0	1/3

for $\max_{A \subseteq \{1,2,3,4\}} |p(A) - q(A)|$, $A = \{1,3\}$ or $A = \{2,4\}$,
either way: $|p(A) - q(A)| = 2/3$
& $\sum_{i=1}^{4} |p(i) - q(i)| / 2 = 2/3$.

2) P.T any DAG with a finite number of vertices has atleast one vertex with no incoming edges. Also, S.T there is atleast one vertex with no outgoing edges.

If possible, let there be no vertex of indegree 0 $\rightarrow$ every vertex is of indegree ≥ 1. Also, let the graph have 'n' vertices, $n \in \mathbb{N}$.

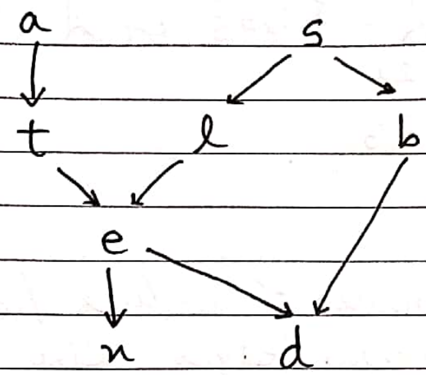
Start from vertex v_1 (say), picked randomly; go to any of its parents (a vertex v s.t $\exists$ an edge from v to v_1) and label it as v_2. Follow this sequence to generate the sequence $\{v_1, v_2, v_3, \dots, v_n, v_{n+1}\}$. By pigeonhole principle, $v_i = v_j$ for some $i \neq j$ & $i, j \in \{1, 2, \dots, n+1\}$ implying that $\exists$ a cycle. This contradicts the fact that the graph is acyclic.

$\therefore$ the graph cannot have all vertices of indegree ≥ 1
$\rightarrow$ the DAG with finite # vertices must have a vertex of indegree 0.

If possible, let all vertices have outdegree ≥ 1. Start from a random vertex v_1. outdegree $(v_1) \geq 1$ guarantees that $\exists$ an edge taking v_1 to some vertex v_2. Follow this procedure to generate the sequence of vertices $\{v_1, v_2, \dots, v_{n+1}\}$. By PHP, $v_i = v_j$ for some $i \neq j$; $i, j \in \{1, 2, \dots, n+1\}$ $\rightarrow \exists$ a cycle $v_i \rightarrow v_{i+1} \rightarrow \dots \rightarrow v_j = v_i$ which is a contradiction

$\therefore$ In a DAG, $\exists$ atleast 1 vertex of outdegree 0.

3)



All variables are binary

a	P(a)
t	0.01
f	0.99

s	P(s)
t	0.5
f	0.5

t	a	P(t|a)
t	t	0.05
f	t	0.95
t	f	0.01
f	f	0.99

l	s	P(l|s)
t	t	0.1
f	t	0.9
t	f	0.01
f	f	0.99

b	s	P(b|s)
t	t	0.6
f	t	0.4
t	f	0.3
f	f	0.7

n	e	P(n|e)
t	t	0.98
f	t	0.02
t	f	0.05
f	f	0.95

e	t	l	P(e|t l)
t	f	f	0
f	f	f	1
t	t	f	1
f	t	f	0
t	f	t	1
f	f	t	0
t	t	t	1
f	t	t	0

d	e	b	P(d | e b)
t	t	t	0.9
f	t	t	0.1
t	t	f	0.7
f	t	f	0.3
t	f	t	0.8
f	f	t	0.2
t	f	f	0.1
f	f	f	0.9

$$P(d) = \sum_{e,b,l,t,s,a \,\in\, \{true,\, false\}} P(d=true, e, b, l, t, s, a)$$

$$= \sum_{e,b,l,t,s,a \,\in\, \{true,\, false\}} P(d=true \mid e\, b) \cdot P(e \mid t\, l) \cdot P(b \mid s) \cdot P(l \mid s) \cdot P(t \mid a) \cdot P(a) \cdot P(s)$$

$= 0.43597$

$$P(d \mid s=true) = \frac{\sum P(d=true, e, b, l, t, s=true, a)}{P(s=true)}$$

$$= \frac{\sum P(d=true \mid e b) \cdot P(e \mid t l) \cdot P(b \mid s=true) \cdot P(l \mid s=true) \cdot P(t \mid a) \cdot P(a) \cdot \cancel{P(s=true)}}{\cancel{0.5}}$$

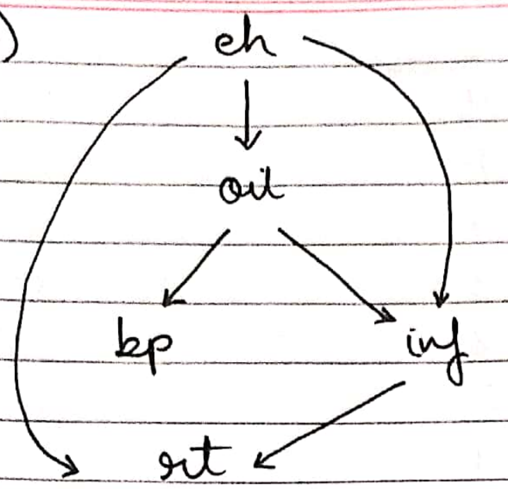
$= 0.55281$

$$P(d \mid s=false) = \frac{\sum P(d=true, e, b, l, t, s=false, a)}{P(s=false)}$$

$$= \frac{\sum (d=true \mid e b) \cdot P(e \mid t l) \cdot P(b \mid s=false) \cdot P(l \mid s=false) \cdot P(t \mid a) \cdot P(a) \cdot \cancel{P(s=false)}}{\cancel{0.5}}$$

$= 0.319133$

4)



eh	P(eh)
l	0.2
h	0.8

bp	oil	P(bp | oil)
l	h	0.1
n	h	0.4
h	h	0.5
l	l	0.9
n	l	0.1
h	l	0

oil	eh	P(oil | eh)
l	l	0.9
h	l	0.1
l	h	0.05
h	h	0.95

rt	inf	eh	P(rt | inf eh)
l	l	l	0.9
h	l	l	0.1
l	l	h	0.1
h	l	h	0.9
l	h	l	0.1
h	h	l	0.9
l	h	h	0.01
h	h	h	0.99

inf	oil	eh	P(inf | oil eh)
l	l	l	0.9
h	l	l	0.1
l	l	h	0.1
h	l	h	0.9
l	h	l	0.1
h	h	l	0.9
l	h	h	0.01
h	h	h	0.99

$$P(inf = h \mid bp = n, rt = h)$$

$$= \frac{P(inf = h, bp = n, rt = h)}{P(bp = n, rt = h)}$$

$$= \frac{\sum_{oil, eh \in \{l,h\}} P(inf = h, bp = n, rt = h, oil, eh)}{\sum_{inf, oil, eh \in \{l,h\}} P(inf, bp = n, rt = h, oil, eh)}$$

$$= \sum_{oil, eh \in \{l,h\}} P(rt = h \mid inf = h, eh) \cdot P(inf = h \mid oil, eh) \cdot P(bp = n \mid oil) \cdot P(oil \mid eh) \cdot P(eh)$$

$$\overline{\sum_{\substack{inf, oil, eh \\ \in \{l,h\}}} P(inf \mid oil, eh) \cdot P(bp = n \mid oil) \cdot P(rt = h \mid inf\ eh) \cdot P(oil \mid eh) \cdot P(eh)}$$

$$= \frac{0.309614}{0.31441} = 0.98474$$