X : domain

C : class of subsets of X

We know; any set S of cardinality n
↳ has 2^n possible subsets

C 'picks out' a subset, say s from $\{x_1, x_2, \cdots x_n\}$
if $\exists\ c \in C$ s.t s can be expressed as
: $c \cap \{x_1, x_2, \ldots, x_n\}$

- C 'shatters' $\{x_1, x_2, \ldots, x_n\}$
if all of its 2^n subsets can be 'picked out' by C

- Essentially, VC dim : largest cardinality of a set shattered by C.

($+\infty$) if arbitrarily large finite sets are shattered

define : $\Delta_n(C, x_1, x_2, \ldots, x_n)$

$$= \left| \{c \cap \{x_1, x_2, \ldots, x_n\} : c \in C\} \right|$$

→ denote VC dim of C as $V(C)$

$$V(C) = \sup\left\{n : \max_{(x_1, x_2, \ldots x_n)} \Delta_n(C, x_1, \ldots, x_n) = 2^n\right\}$$

if C is empty, $V(C) = -1$

- Old Notation: $S(C)$ = VC dim of C
and $V(C)$ = VC index of C
= VC dim of C + 1

$\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_m$: VC classes of subsets of $\mathcal{X}$

$V_1, V_2, \dots, V_m$: Corresponding Dimensions

define:

$$\bigsqcup_{j=1}^{m} \mathcal{C}_j \equiv \{ \cup_{j=1}^{m} c_j \ \text{ s.t } \ c_j \in \mathcal{C}_j \ \forall j = 1(1)m \}$$

$$\sqcap_{j=1}^{m} \mathcal{C}_j \equiv \{ \cap_{j=1}^{m} c_j \ \text{ s.t } \ c_j \in \mathcal{C}_j \ \forall j = 1(1)m \}$$

it is easy to see that if:

i) $\mathcal{C}_1 = \mathcal{C}_2 = \dots = \mathcal{C}_m = \mathcal{C}$

ii) $|\mathcal{C}_1| = |\mathcal{C}_2| = \dots = |\mathcal{C}_m| = m$, then:

$$V(\mathcal{C}_i) = V(\bigsqcup_{j=1}^{m} \mathcal{C}_j) = V(\sqcap_{j=1}^{m} \mathcal{C}_j)$$

define:

$\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_m$: VC classes of subsets of $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_m$

$$\boxtimes_{j=1}^{m} \mathcal{D}_j \equiv \{ d_1 \times d_2 \times \dots \times d_m \ \text{ s.t } \ d_j \in \mathcal{D} \ \forall j = 1(1)m \}$$

– VC class of subsets of $\mathcal{X}_1 \times \mathcal{X}_2 \times \dots \times \mathcal{X}_m$

• Consider $m = 2$

– define $V_1 = V(\mathcal{C}_1)$ & $V_2 = V(\mathcal{C}_2)$

Result: $\max V(\mathcal{C}_1 \sqcup \mathcal{C}_2)$

$$= \max V(\mathcal{C}_1 \sqcap \mathcal{C}_2)$$

$$= \max V(\mathcal{D}_1 \boxtimes \mathcal{D}_2)$$

$$= S(V_1, V_2)$$

Key Contribution —

define : $${}_{r}C_{\leq v} \equiv \sum_{j=0}^{v} \binom{r}{j}$$

· $T(V_1, V_2) \equiv \sup\{ r \in \mathbb{N} : {}_{r}C_{\leq V_1} \cdot {}_{r}C_{\leq V_2} \geq 2^{r} \}$

Result : $S(V_1, V_2) \leq T(V_1, V_2)$

Also : $S(1, k) \leq 2k+1 \quad \forall k \geq 1$

— equality holds for $k = 1, 2, 3$

> Extend bound to general case $\forall m \geq 2$

Main Contribution being the following theorem.

· Let $V \equiv \sum_{j=1}^{m} V_j$

$$\underline{V} \equiv (V_1, V_2, \ldots, V_m)$$

$$\lambda_1 = \frac{e}{(e-1)\log 2}$$

$$\lambda_2 = \frac{e}{\log 2} = (e-1)\lambda_1$$

$$\overline{V} \equiv \frac{1}{m} \sum_{j=1}^{m} V_j$$

$$\mathrm{Ent}(\underline{V}) = \frac{1}{m} \sum_{j=1}^{m} V_j \log V_j - \overline{V} \log \overline{V}$$

— entropy of V_j's

— discrete uniform distribution

- $$\left.\begin{array}{l} V(\sqcup_{j=1}^{m} C_j) \\ V(\sqcap_{j=1}^{m} C_j) \\ V(\boxtimes_{j=1}^{m} C_j) \end{array}\right\} \leq \lambda_1 V \log\left(\frac{\lambda_2 m}{\exp\left(\frac{\mathrm{Ent}(\underline{V})}{\overline{V}}\right)}\right)$$

$$\leq \lambda_1 V \log(\lambda_2 m)$$

→ Justification:

- Subsets picked out by $\sqcap_i C_i$ from $\{x_1, x_2, \dots, x_n\}$ in X are of the form : $c_1 \cap c_2 \cap \dots \cap c_m \cap \{x_1, \dots, x_n\}$

- Algorithm to generate :

→ Initially : form different sets of type :

$$c_1 \cap \{x_1, x_2, \dots, x_n\} \text{ for } c_1 \in C_1$$

→ Then, intersect each of the sets in the step above by sets in C_2 :

we have all sets of the form :

$$c_1 \cap c_2 \cap \{x_1, x_2, \dots, x_n\}$$

→ Proceed iteratively intersecting with sets in $C_3, C_4, \dots, C_m$ to generate all :

$$c_1 \cap c_2 \cap \dots \cap c_m \cap \{x_1, x_2, \dots, x_m\}$$

for $c_i \in C_i \;\; \forall\, i = 1(1)m$

Recall:

$$\Delta_n(C, y_1, \dots, y_n) = \left| \{ c \cap \{y_1, \dots, y_n\} \text{ s.t } c \in C \} \right|$$

Define:

$$\Delta_n(C) = \max_{(y_1, y_2, \dots, y_n)} \Delta_n(C, y_1, y_2, \dots, y_n)$$

$\therefore$ In the algorithmic scheme above:

- in step 1, we have atmost $\Delta_n(C_1)$ unique sets
- in step 2, we have atmost $\Delta_n(C_n)$ unique sets

$\vdots$

- in general, step i gives us $\Delta_n(C_i)$ unique sets

clearly, $\Delta_n(\sqcap_i C_i) \leq \prod_i \Delta_n(C_i)$

$\rightarrow$ Result: number of subsets (of a class of size n) of size $< s$ is bounded by $\left(\frac{en}{s}\right)^s$

Using this result: $\Delta_n(\sqcap_i C_i) \leq \prod_i \Delta_n(C_i)$

$$\leq \prod_i \left(\frac{en}{V_i}\right)^{V_i}$$

Now, if n is the VCdim of $\sqcap_i C_i$,
then $\Delta_n(\sqcap_i C_i) = 2^n$.

$$\therefore \quad 2^n \leq \prod_{i=1}^{m} \left(\frac{en}{V_i}\right)^{V_i}$$

Taking $\log(\cdot)$ on both sides:

$$n \log 2 \leq \sum_{i=1}^{m} V_i \log\left(\frac{en}{V_i}\right)$$

$$= \sum_{i=1}^{m} V_i \log\left(\frac{e}{V_i}\right) + \log(n) \sum_{i=1}^{m} V_i$$

define $r = \frac{en}{V}$ $\quad \therefore \quad r\frac{V}{e} = n$

$$\Rightarrow \frac{rV}{e}\log 2 \le \sum_i V_i \log\left(\frac{e}{V_i}\right) + \log\left(\frac{rV}{e}\right)\cdot V$$

$$= \log(e)\sum_i V_i - \sum_i V_i \log V_i$$

$$+ \log(r)V + \log(V).V - \log(e).V$$

$$\Rightarrow \frac{rV}{e}\log 2 = V\log r + V\log V - \sum_i V_i \log V_i$$

$$\Rightarrow \frac{r}{e}\log 2 \le \log r + \log V - \frac{\sum_i V_i \log V_i}{V}$$

$$= \log r + \log m + \log V - \log m - \frac{\sum_i V_i \log V_i}{V}$$

$$= \log r + \log m + \log\left(\frac{V}{m}\right) - \frac{\sum_i V_i \log V_i}{V}$$

Now, $\log\left(\frac{V}{m}\right) - \frac{\sum_i V_i \log V_i}{V}$

$$= \log(\bar{V}) - \frac{\sum_i V_i \log V_i}{V} \qquad \text{As } \bar{V} = \frac{1}{m}\sum_{i=1}^{m} V_i = \frac{1}{m}.V$$

$$= \frac{1}{\bar{V}}\left[\bar{V}\log(\bar{V}) - \frac{\bar{V}}{V}\sum_i V_i \log V_i\right]$$

$$= \frac{1}{\bar{V}}\left[\bar{V}\log(\bar{V}) - \frac{1}{m}\sum_i V_i \log V_i\right]$$

$$= \frac{1}{\bar{V}}\left[-Ent(\underline{V})\right]$$

$$\therefore r\frac{\log 2}{e} \le \log r + \log m - \frac{Ent(\underline{V})}{\bar{V}} = \log\left(\frac{mr}{e^{Ent(\underline{V})/\bar{V}}}\right)$$

$$\Rightarrow \frac{r}{\log\left(\dfrac{mr}{\exp\left(\dfrac{\mathrm{Ent}(\underline{V})}{\bar{V}}\right)}\right)} \leq \frac{e}{\log 2}$$

Multiplying both sides with $m/\exp\left(\dfrac{\mathrm{Ent}(\underline{V})}{\bar{V}}\right)$

$$\rightarrow \frac{\dfrac{mr}{\exp(\mathrm{Ent}(\underline{V})/\bar{V})}}{\log\left(\dfrac{mr}{\exp(\mathrm{Ent}(\underline{V})/\bar{V})}\right)} \leq \frac{m}{\exp\left(\dfrac{\mathrm{Ent}(\underline{V})}{\bar{V}}\right)} \cdot \frac{e}{\log 2}$$

let $x \equiv \dfrac{mr}{\exp(\mathrm{Ent}(\underline{V})/\bar{V})}$, $y \equiv \dfrac{m}{\exp\left(\dfrac{\mathrm{Ent}(\underline{V})}{\bar{V}}\right)} \cdot \dfrac{e}{\log 2}$

$$\therefore \frac{x}{\log x} \leq y$$

if $g(x) = \dfrac{x}{\log x}$ & $g(x) \leq y$

- Need to look at $g(x)$.

$$g'(x) = \frac{\log x \cdot 1 - x \cdot 1/x}{x^2} = \frac{\log x - 1}{x^2}$$

$$\& \; g''(x) = \frac{x^2\left(\frac{1}{x}\right) - (\log x - 1) \cdot 2x}{x^4}$$

$$= \frac{x - 2x\log x + 2x}{x^4}$$

$$= \frac{3 - 2\log x}{x^3}$$

To find critical points :

$$g'(c) = 0 \Rightarrow \frac{\log c - 1}{c^2} = 0$$

$$\Rightarrow \log c = 1 \Rightarrow c = e$$

And $$g''(e) = \frac{3 - 2\log e}{e^3} = e^{-3} > 0$$

$\therefore$ $x = e$ is a minima for $g(x)$

And $g'(x) > 0 \ \forall \ x > e$

$\Rightarrow g(x)$ is increasing for $x > e$

Now, $$y \geq g(x) = \frac{x}{\log x}$$

$$\Rightarrow \log y \geq \log x - \log(\log x)$$

[As $\log(\cdot)$ is an increasing function]

$$= \log x \left(1 - \frac{\log(\log x)}{\log x}\right)$$

$$= \log x \left(1 - \frac{1}{g(\log x)}\right)$$

Now, $g(\log x)$ is minimum for $\log x = e$

$\rightarrow -\frac{1}{g(\log x)}$ is minimized for $\log x = e$

$$\therefore \log y \geq \log x \left(1 - \frac{1}{g(\log x)}\right) \geq \log x \left(1 - \frac{1}{e}\right)$$

[As $g(e) = e$] $\therefore \log x \leq \log y \left(1 - \frac{1}{e}\right)^{-1}$

$$\cdot \ \frac{x}{\log x} \leq y \Rightarrow x \leq y \log x \leq y\left(1 - \frac{1}{e}\right)^{-1} \log y$$

$$\therefore\ x \le \frac{e}{e-1}\, y \log y$$

$$\Rightarrow \frac{mr}{\exp\left(\frac{\mathrm{Ent}(\underline{V})}{\overline{V}}\right)} \le \frac{e}{e-1} \cdot \frac{me}{\exp\left(\frac{\mathrm{Ent}(\underline{V})}{\overline{V}}\right)} \cdot \frac{1}{\log 2} \cdot \log\left\{\frac{me}{\exp\left(\frac{\mathrm{Ent}(\underline{V})}{\overline{V}}\right) \cdot \log 2}\right\}$$

$$\Rightarrow r \le \frac{e^2}{(e-1)\log 2} \log\left\{\frac{m}{\exp\left(\mathrm{Ent}(\underline{V})/\overline{V}\right)} \cdot \frac{e}{\log 2}\right\}$$

$$\Rightarrow \frac{er}{V} \le \frac{e^2}{(e-1)\log 2} \log\left\{\frac{m}{\exp\left(\mathrm{Ent}(\underline{V})/\overline{V}\right)} \cdot \frac{e}{\log 2}\right\}$$

$$n \leq \frac{e}{(e-1)\log 2} \cdot V \cdot \log\left\{\frac{m}{\exp\left(\mathrm{Ent}(\underline{V})/\overline{V}\right)} \cdot \frac{e}{\log 2}\right\}$$

$$\Rightarrow n \le \lambda_1 V \log\left(\frac{\lambda_2 m}{\exp(\mathrm{Ent}(\underline{V})/\bar{V})}\right)$$

Also; $\mathrm{Ent}(\underline{V}) \ge 0 \Rightarrow \exp(\mathrm{Ent}(\underline{V})/\bar{V}) \ge 1$

$$\therefore n \le \lambda_1 V \log\left(\frac{\lambda_2 m}{\exp(\mathrm{Ent}(\underline{V})/\bar{V})}\right) \le \lambda_1 V \log(\lambda_2 m)$$

As $\frac{1}{\exp(\mathrm{Ent}(\underline{V})/\bar{V})} \le 1$ and $\log(\cdot)$ is an increasing fn

$$V\left(\prod_{j=1}^{m} C_j\right) \leq n \leq \lambda_1 V \log\left(\frac{\lambda_2 m}{\exp(\mathrm{Ent}(\underline{V})/\overline{V})}\right) \leq \lambda_1 V \log(\lambda_2 m)$$

For $V(\sqcup_{j=1}^{m} C_j) \longrightarrow$

We can write $\bigcup_i C_i = \left(\bigcap_i C_i'\right)'$ [Demorganization]

Since a class C of sets, and the class C' of their complements have the same VCdim.

$$\therefore V\left(\sqcup_{j=1}^{m} C_j\right) = V\left(\left(\sqcap_{j=1}^{m} C_j'\right)'\right) = V\left(\sqcap_{j=1}^{m} C_j\right)$$

$\rightarrow$ This justifies the corresponding statement for unions

For products :

$$\Delta_n\left(\boxtimes_{j=1}^{m} D_j\right) \leq \prod_{j=1}^{m} \Delta_n(D_j) \leq \prod_{j=1}^{m} \left(\frac{en}{V_j}\right)^{V_j}$$

- Rest of the steps are same as in the intersection case

Some Results based on the discussion:

Result 1:

$f(x) = \log x$ is concave.

- $f''(x) = -1/x^2$

let $P_j \equiv \dfrac{V_j}{\sum_{i=1}^{m} V_i}$

Now, $$\frac{\sum_{j=1}^{m} V_j \log V_j}{\sum_{j=1}^{m} V_j} = \sum_{j=1}^{m} \frac{V_j}{\sum_{i=1}^{m} V_i} \cdot \log V_j = \sum_{j=1}^{m} P_j \log V_j$$

$$\leq \log\left(\sum_{j=1}^{m} P_j V_j\right) \leq \log\left(\sum_{j=1}^{m} V_j\right)$$

Note: $\sum_{j=1}^{m} V_j \log V_j = m \, Ent(\underline{V}) + m \bar{V} \log \bar{V}$

- from the definition of $Ent(\underline{V})$.

So $$\frac{\sum_{j=1}^{m} V_j \log V_j}{\sum_{j=1}^{m} V_j} \leq \log\left(\sum_{j=1}^{m} V_j\right)$$

$$\Rightarrow \sum_{j=1}^{m} V_j \log V_j \leq \log\left(\sum_{j=1}^{m} V_j\right) \sum_{j=1}^{m} V_j$$

Replacing $\sum_j v_j \log v_j$ with $m\,Ent(\underline{V}) + m\bar{V}\log\bar{V}$

$\longrightarrow$

$$m\,Ent(\underline{V}) + m\bar{V}\log\bar{V} \le \log\left(\sum_j v_j\right)\cdot\sum_j v_j$$

$$\Rightarrow m\,Ent(\underline{V}) \le \left(\sum_j v_j\right)\cdot\log\left(\sum_j v_j\right) - m\bar{V}\log\bar{V}$$

Dividing both sides with $\bar{V}m$, we have

$$\frac{Ent(\underline{V})}{\bar{V}} \le \frac{1}{m}\log\left(\sum_j v_j\right)\cdot\frac{\left(\sum_j v_j\right)}{\bar{V}} - \log\bar{V}$$

$$= \frac{1}{m}\log(\bar{V}m)\cdot\frac{\bar{V}m}{\bar{V}} - \log\bar{V}$$

$\left(\text{As } \sum_{j=1}^{m} v_j = m\bar{V}\right)$

$$= \log(\bar{V}m) - \log(\bar{V}) = \log m$$

$$\Rightarrow \frac{Ent(\underline{V})}{\bar{V}} \le \log m$$

$$\Rightarrow \exp\left(\frac{Ent(\underline{V})}{\bar{V}}\right) \le m \Rightarrow \frac{m}{\exp\left(\frac{Ent(\underline{V})}{\bar{V}}\right)} \ge 1$$

$$\Rightarrow 1 \le \frac{m}{\exp(Ent(\underline{V})/\bar{V})} \le m$$

$\left\{\text{as } \frac{Ent(\underline{V})}{\bar{V}} > 0\right\}$

Also, $0 \le Ent(\underline{V}) \le \bar{V}\log m$

- $\frac{m}{e^{Ent(\underline{V})/\bar{V}}}$ can be very close to 1
  - provided V_i's are sufficiently heterogeneous
  - even for large m

Result 2:

if r is a large integer
and $V_i = r^i \quad \forall\, i = 1(1)m$

$$\lim_{m\to\infty} \frac{m}{\exp(\mathrm{Ent}(\underline{V})/\bar{V})} = \frac{r}{r-1}\, r^{\frac{1}{r-1}}$$

$$= \frac{r}{r-1} \exp\left((r-1)^{-1} \log r\right)$$

as r increases, this limit gets arbitrarily closer to 1.

Result 3:

if $m = 2$, $V_1 = k$, $V_2 = rk$ s.t $r, k \in \mathbb{N}$

$$\frac{\mathrm{Ent}(\underline{V})}{\bar{V}} = \frac{1}{m} \frac{\sum_j V_j \log V_j}{\bar{V}} - \log \bar{V}$$

replacing $\bar{V}$ with $\frac{1}{m}\sum_j V_j \longrightarrow$

$$= \frac{1}{m} \frac{\sum_j V_j \log V_j}{1/m \sum_j V_j} - \log\left(\sum_j V_j\right) + \log m$$

$$= \frac{k \log k + rk \log rk}{k + rk} - \log(k + rk) + \log 2$$

$$= \frac{k(\log k + r\log r + r\log k)}{k(1+r)} - \log\{k(1+r)\} + \log 2$$

$$= \frac{\log k}{1+r} + \frac{r \log r}{1+r} + \frac{r\log k}{1+r} - \log k - \log(1+r) + \log 2$$

$$= \frac{\log k}{1+r} + \frac{r \log k}{1+r} - \log k + \frac{r \log r}{1+r} - \log(1+r) + \log 2$$

$$= \frac{1+r}{1+r} \log k - \log k + \log 2 + \frac{\log r^r}{1+r} - \frac{\log (1+r)^{(1+r)}}{1+r}$$

$$= \log 2 - \frac{1}{1+r} \log\left\{\frac{(1+r)^{1+r}}{r^r}\right\}$$

which is independent of k

$$= \log 2 - \frac{1}{1+r} \log\left\{(1+r)\frac{(1+r)^r}{r^r}\right\}$$

$$= \log 2 - \frac{1}{1+r} \log\left\{(1+r)\left(1+\frac{1}{r}\right)^r\right\}$$

As $r \to \infty$, $\frac{1}{1+r} \log\left\{(1+r)\left(1+\frac{1}{r}\right)^r\right\} \longrightarrow 0$

$$\therefore \lim_{r\to\infty} \frac{Ent(\underline{V})}{\bar{V}} = \log 2$$

or as $r \to \infty$: $\frac{2}{\exp\{Ent(\underline{V})/\bar{V}\}} \longrightarrow 1$