Probabilistic Graphical Models

Assignment 3

Sayar Ghosh Roy 20171047

Section 6.8

1) H, H' are two hypotheses classes s.t $H' \subseteq H$.
Let $C = \{c_1, c_2, \dots, c_m\}$ be a set of samples i.e $C \subseteq X$.
Now, $H' \subseteq H$
$\Leftrightarrow$ if H' shatters C, then H shatters C as well.
[$\exists$ a $h' \in H'$ s.t h' shatters C and since $H' \subseteq H$, $h' \in H$ and H shatters C]
Now, C could be any arbitrary set s.t $C \subseteq X$, and $\forall$ such C, H' shattering C implies H shatters C. $\therefore$ VC dim$(H) \geq$ VC dim (H').

3) Domain X : $\{0,1\}^n$; $n \in \mathbb{N}$.
if $I \subseteq \{1,2,3,\dots,n\}$, we can define parity function

$$h_I(X) = \left(\sum_{i \in I} x_i\right) \bmod 2$$

where $X = (x_1, x_2, \dots, x_n) \in \{0,1\}^n$

$$\mathcal{H}_{n\text{-parity}} = \{h_I : I \subseteq \{1,2,\dots,n\}\}$$

Now, $\text{VC dim}(\mathcal{H}_{n\text{-parity}}) \leq \log_2(|\mathcal{H}_{n\text{-parity}}|)$
$= \log_2(2^n) = n$.

$\mathcal{H}_{n\text{-parity}}$ also shatters the standard basis in n-dimensi-ons i.e $\{e_j\}_{j=1}^n$ being the set of basis vectors.
if $(y_1, y_2, \dots, y_n) \in \{0,1\}^n$ is a vector of labels;
let $J = \{j \in [n] \text{ s.t } y_j = 1\}$
Now $\forall j \in [n]$, $h_J(e_j) = y_j$
$\therefore$ VC dim$(\mathcal{H}_{n\text{-parity}}) = n$.

5) $\mathcal{H}^d_{rec}$: class of axis-aligned rectangles in $\mathbb{R}^d$.
Given real constants : $a_1 \le b_1$, $a_2 \le b_2$, ..., $a_d \le b_d$.
Let classifier $h_{(a_1, b_1, \dots, a_d, b_d)}(x_1, x_2, \dots, x_d)$

$$= \prod_{i=1}^{d} \mathbb{1}[x_i \in [a_i, b_i]] \quad \begin{cases} \text{i.e } 1 \text{ if within rectangle} \\ \quad \& \ 0 \text{ if outside} \end{cases}$$

Now, $\mathcal{H}^d_{rec} = \{h_{(a_1, b_1, \dots, a_d, b_d)} \text{ s.t } a_i \le b_i \ \forall\, i \in [d]\}$.
Let $S = \{x_1, x_2, \dots, x_{2d}\}$ where $\begin{cases} x_i = e_i & \forall\, i \in [d] \\ x_i = -e_{i-d} & \forall\, i > d \end{cases}$

Clearly, $\mathcal{H}^d_{rec}$ shatters S.
if $(y_1, y_2, \dots, y_{2d}) \in \{0,1\}^{2d}$, construct a s.t and construct b s.t

$$a_i = \begin{cases} -2 & \text{if } y_{i+d} = 1 \\ 0 & \text{else} \end{cases}$$

$$b_i = \begin{cases} 2 & \text{if } y_i = 1 \\ 0 & \text{else} \end{cases}$$

Then, $h_{(a_1, b_1, \dots, a_d, b_d)}(x_i) = y_i \quad \forall\, i \in [2d]$

$\Rightarrow \text{VCdim}(\mathcal{H}^d_{rec}) \ge 2d$ —— (I)

Now, let C be of size atleast $2d+1$. Using PHP, $\exists\, x \in C$ s.t $\forall\, j \in [d]$; $\exists\, x' \in C$ with $x'_j \le x_j$ and $\exists\, x''_j \in C$ with $x''_j \ge x_j$. $\therefore$ We cannot have a labelling s.t x is 0 (outside) while all other elements in C are 1 (inside) [By definition of $h_{(a_1, \dots, b)}$ or rather the geometry of a rectangle].

$\therefore \text{VCdim}(\mathcal{H}^d_{rec}) < 2d+1$ —— (II)

by (I) & (II) $\rightarrow \text{VCdim}(\mathcal{H}^d_{rec}) = 2d$ [Proved]

9) $\mathcal{H} = \{ h_{a,b,s} : a \leq b,\ s \in \{-1, 1\} \}$

$$h_{a,b,s}(x) = \begin{cases} s & \text{if } x \in [a,b] \\ -s & \text{if } x \notin [a,b] \end{cases}$$

Let $C = \{x_1, x_2, x_3\}$. WLG let $x_1 = 1$, $x_2 = 2$, $x_3 = 3$.

Now; the following assignments clearly show that a set of 3 points can be shattered:

x_1	x_2	x_3	a	b	s
−	−	−	0.5	3.5	−1
−	−	+	2.5	3.5	1
−	+	−	1.5	2.5	1
−	+	+	1.5	3.5	1
+	−	−	0.5	1.5	1
+	−	+	1.5	2.5	−1
+	+	−	0.5	2.5	1
+	+	+	0.5	3.5	1

$\therefore$ VCdim$(\mathcal{H}) \geq 3$. —— (I)

if $C = \{x_1, x_2, x_3, x_4\}$; WLG let $x_1 < x_2 < x_3 < x_4$

& $y_1 = y_3 = +$ & $y_2 = y_4 = -$

— this labelling cannot be achieved by any $h \in \mathcal{H}$.

$\therefore$ VCdim$(\mathcal{H}) < 4$. —— (II)

From (I) & (II); VCdim$(\mathcal{H}) = 3$