Prediction Intervals in High-Dimensional Regression

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Abstract

We construct prediction intervals for future time-aggregates of an univariate response time series. This may depend on (potentially infinitely) many predictors. We chose to use LASSO but our results can be easily extended to other estimators. Allowing for general stationary error processes including long-memory, heavy tailed and non-linear, we provide consistency results for prediction intervals using a sharp tail probability inequality. Finally, we construct prediction intervals for hourly electricity prices over horizons spanning 17 weeks and compare them to selected Bayes and model-implied alternatives.

Keywords: consistency, LAD, LASSO, electricity prices, bootstrap, time-aggregation

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1. Introduction

Prediction intervals (PI's) help forecaster to access the uncertainty about future values of time series. Although, there are situations, where point-forecasts are preferred, here we consider PI's as our final goal. PI's provide more information than simple point-forecasts and it is theoretically and practically challenging to show their validity (Chatfield, 1993; Clements and Taylor, 2003), since not only the coverage probability but also their width matters. Suppose that univariate target time series $y_i, i = 1, \ldots, n$ follows a regression model

$$y_i = x_i^T \beta + e_i, \quad \beta \in \mathbb{R}^p. \tag{1.1}$$

For finite $p < n$, Zhou et al. (2010) showed that empirical quantiles obtained from rolling sums of past residuals provide theoretically valid asymptotic PI's for $S_m = y_{n+1} + \ldots + y_{n+m}$, when both $n, m \to \infty$. Zhou et al. (2010) utilize LAD[1] estimator for the β. However, if $p > n$ this and other conventional estimators, such as OLS, fail. Even if $p < n$ but large and β has many zero-elements, there are more efficient estimators, each leading to different PI $[l, U]$ such that for given α

$$P\left([L, U]_{\hat{\beta}} \ni S_m\right) = 1 - \alpha.$$

In the current paper, we therefore extend the theoretical properties of the PI's proposed by Zhou et al. (2010) to case of (potentially infinitely) many predictors. Although we present the results specifically for the LASSO estimator, they are applicable to other similar estimators as well. As a second contribution, we extend the theory of Zhou et al. (2010) to non-linear error process (e_i). In order to do so, we follow the functional dependence framework as proposed in a seminal paper by Wu (2005). Our error processes form much larger class including some popular examples such as TAR.

Our main motivation comes from fields such as macroeconomics, finance or energy, where the target generally depends on a large number of predictors. In addition, many of the time series are subject to structural breaks and other sources of complications for the forecaster (Cheng et al., 2016). It is generally accepted that inclusion of many (disaggregated) predictors provides some additional forecasting power over conventional univariate resp. low-dimensional approaches[2] (see Elliott et al., 2013; Kim and Swanson, 2014; Ludwig et al., 2015). But most empirical studies utilizing

[1] Least absolute deviation

[2] Yet there are empirical studies which do not corroborate these beliefs (Stock and Watson, 2012).

economic big data provide evidence based on short horizons and point-forecasts. This is partially because many series in finance (realized volatility of returns) or macroeconomics (GNP, inflation, interest rates, population, productivity and unemployment) exhibit joint long-run behaviour. These characteristics can overthrow any transitory behaviour and cannot be ignored. But they also depend on nuisance parameters (Elliott et al., 2015). Dealing with the latter is beyond our scope and remains as future challenge for any sensible frequentist approach. Instead, we found interesting applications of our regression framework in electricity price forecasting (EPF). The electricity prices generally depend on exogenous variables including weather conditions, local economy and environmental policy[3] (Knittel and Roberts, 2005; Huurman et al., 2012). Additionally, EPF is challenging also due to complex seasonality (daily, weekly, yearly), heteroscedasticity, heavy-tails and sudden price spikes. There is a substantial amount of literature about EPF (see Weron, 2014, for recent review). We focus on long- and medium-horizon PI's, which are essential for power portfolio risk management, derivatives pricing, medium-and long-term contracts evaluation and maintenance scheduling. Recently, Ludwig et al. (2015) found that inclusion of local (disaggregated) wind speed and temperature measured at 151 weather stations across Germany leads to forecasting improvements for EPEX SPOT electricity market. Since, they are focused on short-term point-forecasts, we try to verify if their findings hold for PI's over longer horizons spanning up to 17 weeks. For this, we adopt their data set. Additionally, we include deterministic seasonal predictors and day of week indicators. The model and implementation details are described in the empirical part. For visual out-of-sample comparison of our PI's we include alternative PI's such as Bayes PI's of Müller and Watson (2016) and bootstrap PI's obtained from methods such as exponential smoothing, neural networks and regression with auto-correlated errors, which can be easily computed with automatic forecasting R-package "forecast" (see Hyndman and Khandakar, 2008).

The rest of the article is organized as follows: In section 2, we construct the PI's of Zhou et al. (2010) under different scenarios for number of regressors and the error process (e_i). Section 3 and Section 4 summarize the asymptotic results for the cases without and with covariates. Section 5 shows simulation results and our real data analysis. Section 6 concludes.

[3]In 2005, Germany launched a program aiming at reducing emissions by increasing the share of renewable energy. The share was 25% during 2013-2014.

2. Construction of prediction intervals

In this paper, we cover forecasting in a regression-set-up for a large number of scenarios. We are able to capture both linear and non-linear errors, short-range or long-range dependence, light-tailed or heavy-tailed behavior of the noise process, linear or robust regression in case of finitely many regressors and LASSO or others in case of infinitely many regressors under proper sparsity condition. Thus organization and presentation of these results in a concise manner is essential. The two methods we use in this paper for forecasting are from Zhou et al. (2010). However we provide some data-driven adjustments to arrive at better forecasting performance. The first of these is based on a quenched central limit theorem for short-range dependence and light-tailed error processes. The second one is more generally applicable since it is based on empirical quantiles. We first discuss, as a primer, how to forecast if there is no covariates present. Then we introduce finitely many covariates and finally conclude with infinitely many covariates.

2.1. Primary model: Without covariates

For this set-up, we have $y_i = e_i$ where e_i are zero-mean noise processes. Depending on the nature of dependence and tail behavior we can have the following two methods to estimate the quantiles. Note that, these methods are similar to those reported in Zhou et al. (2010) and Chudy et al. (2017).

2.1.1. Quenched CLT inspired method

For predicting m-step ahead aggregated response i.e $e_{n+1} + \ldots + e_{n+m}$, one can estimate the long-run variance σ^2 of the e_i process

$$\hat{\sigma}^2 = \sum_{|i|\leq k_n} \hat{\gamma}_k = \sum_{|i|\leq k_n} \frac{1}{n} \sum_{j=1}^{n-|i|} (e_j - \bar{e})(e_{j+|i|} - \bar{e}),$$

and use the following version of the $100(1-\alpha)\%$ PI

$$[L, U] = \pm \hat{\sigma} t_{df,\alpha/2} \sqrt{m},$$

as desired prediction interval. The justification behind such an interval will be discussed in details as part of our asymptotic results where we show that under mild conditions the mean zero process $e_{n+1} + \ldots + e_{n+m}$ converge to an asymptotic normal distribution.

2.1.2. Empirical method based on quantiles

This is a substantially more general method that can account for long-range dependence or heavy-tailed behavior of the error process. The prediction intervals in this case will be

$$[L, U] = Q_{\alpha/2}, Q_{(1-\alpha)/2},$$

where Q_u is the u-th empirical quantiles of $\sum_{j=i-m+1}^{i} e_i; i = m, \ldots, n$. This simple prediction interval enjoys the advantage of interpretability, general applicability and still provides reasonable coverage for approprirate rate of growth of m compared to the size of observed sample n.

2.2. Finitely many covariates

We discuss two posible types of regression.

- *Least square regression* Assume the following model

$$y_i = x_i^T \beta + e_i, \quad i = 1, \ldots, n,$$

where β is a p-dimensional parameter vector. We wish to construct prediction interval for $y_{n+1} + \ldots + y_{n+m}$ after observing $(y_i, x_i); i = 1, \ldots, n$. If the error process show light tailed behavior and short-range dependence the popular least square regression estimates of β can lead to a good prediction interval. We estimate β by $\hat{\beta} = \arg\min \sum_i (y_i - x_i^T \beta)^2$ and then construct PI as

$$\sum_{i=n+1}^{n+m} x_i^T \hat{\beta} + \text{ PI for } \sum_{i=n+1}^{n+m} \hat{e}_i. \tag{2.1}$$

where $\hat{e}_i = y_i - x_i^T \hat{\beta}$ are the residuals from the estimation. Thus it suffices to discuss the construction of the PI for $\sum_{i=n+1}^{n+m} \hat{e}_i$ after observing $(y_i, x_i)_{i=1,\ldots,n}$. Since one of the major part of this paper also focuses on the scenario without the covariates we start discussing only how to construct the CI for $\sum_{i=n+1}^{n+m} e_i$ where $e_1, \ldots, e_n$ are observed. One can then replicate the same ideas by replacing e_i by $\hat{e}_i$. Theorem 4.5 shows the consistency properties for the case with the covariates.

- *Robust regression* For heavy-tailed or long-range dependent it is better (Huber and Ronchetti, 2009) to use robust regression to estimate the regression coefficient. In this case, the final prediction interval for the response y_i remains the same as (2.1) with $\hat{\beta}$ being estimated by a more general distance ρ

$$\hat{\beta} = \arg\min \sum_i \rho(y_i - x_i^T \beta).$$

Examples of such robust regression includes the $\mathbb{L}^q$ regression for $1 \leq q \leq 2$, quantile regressions $\rho(x) = qx^+ + (1-q)(-x)\cdot 0 < q < 1$, where $x^+ = max(x, 0)$ and Huber's estimate $(x^2 \mathbf{1}_{|x|\leq c})/2 + (c|x|c^2/2)\mathbf{1}_{|x|>c}, c > 0$ etc.

2.3. Infinitely many covariates

Consider the case where the number of covariates $p \gg n$. We use LASSO to find the estimates of β

$$\hat{\beta} = \arg\min_{\beta \in \mathcal{R}^p} \frac{1}{n}(y_i - x_i^T \beta)^2 + \lambda \sum_{j=1}^{p} |\beta_j|, \tag{2.2}$$

where λ is the penalty parameter. We have a prediction interval of the form (2.1) with $\hat{\beta}$ replaced by the LASSO estimator. It is important to note that, there are other regression estimates in the scenario $p \gg n$ that can work here as well. However, we keep the discussion concise by discussing just the LASSO.

3. Asymptotic results without covariates

3.1. Linear Process

Assume

$$e_i = \sum_{j=0}^{\infty} a_j \epsilon_{i-j}. \tag{3.1}$$

Assume $E(|e_i|^p) < \infty$ for some $p > 2$. Wu and Woodroofe (2004) proved that, if for some $q > 5/2$,

$$\|E(S_m|\mathcal{F}_0)\| = O\left(\frac{\sqrt{m}}{\log^q m}\right), \tag{3.2}$$

then we have the a.s. convergence

$$\Delta(\mathbb{P}(S_m/\sqrt{m} \leq \cdot|\mathcal{F}_0), N(0, \sigma^2)) = 0 \text{ a.s.},$$

where Δ denotes the Levy distance, $m \to \infty$ and $\sigma^2 = \lim_{m\to\infty} \|S_m\|^2/m$ is the long-run variance. Now, under linearity,

$$\|E(S_m|\mathcal{F}_0)\|^2 = \|(a_1 + \ldots + a_m)\epsilon_0 + (a_2 + \ldots + a_m)\epsilon_{-1} + \ldots\|^2 = \sum_{i=1}^{m} b_i^2, \tag{3.3}$$

where $b_i = a_i + \ldots + a_m$.

3.2. Non-linear Process

In this subsection, we propose extension of the results from Zhou et al. (2010) to non-linear processes. For the two methods mentioned above, we define a functional dependence measure of the non-linear process in two different ways. These assumptions are quite mild and easily verifiable compared to the more popularly used strong mixing conditions.

3.2.1. CLT for non-linear processes:Dependence structure

In this subsection, we relax the linearity assumption of e_i and assume a much more general set-up following Wu (2005)'s framework to formulate dependence through coupling. Assume e_i is a stationary process that admits the following representation

$$e_i = H(\mathcal{F}_i) = H(\epsilon_i, \epsilon_{i-1}, \ldots), \tag{3.4}$$

where H is such that e_i are well-defined random variable and $\epsilon_i, \epsilon_{i-1}, \ldots$ are independent innovations. One can see that it is a vast generalization from the linear structure of e_i assumed in (3.1). We need to define the dependence between (e_i) process. Define the following functional dependence measure

$$\delta_{j,p} = \sup_i \|e_i - e_{i,(i-j)}\|_p = \sup_i \|H_i(\mathcal{F}_i) - H_i(\mathcal{F}_{i,(i-j)}\|_p, \tag{3.5}$$

where $\mathcal{F}_{i,k}$ is the coupled version of $\mathcal{F}_i$ with ϵ_k in $\mathcal{F}_i$ replaced by an i. i. d copy ϵ'_k,

$$\mathcal{F}_{i,k} = (\epsilon_i, \epsilon_{i-1}, \ldots, \epsilon'_k, \epsilon_{k-1}, \ldots) \tag{3.6}$$

and $e_{i,\{i-j\}} = H(\mathcal{F}_{i,\{i-j\}})$. Clearly, $\mathcal{F}_{i,k} = \mathcal{F}_i$ is $k > i$. As Wu (2005) suggests, $\|H(\mathcal{F}_i) - H(\mathcal{F}_{i,(i-j)})\|_p$ measures the dependence of X_i on ϵ_{i-j}. This dependence measure can be thought as an input-output system. It facilitates easily verifiable and mild moment conditions on the dependence of the process and thus improves upon the usual strong mixing conditions which are often difficult to verify. Define the cumulative dependence measure

$$\Theta_{j,p} = \sum_{i=j}^{\infty} \delta_{i,p}, \tag{3.7}$$

which can be thought as cumulative dependence of $(X_j)_{j\geq k}$ on ϵ_k. For the quenched CLT in (3.2), we assume the following rate for $\Theta_{j,p}$.

$$\Theta_{j,p} = j^{-\chi}(\log j)^{-A} \text{ where } = \begin{cases} A > 0 \text{ for } 1 < \chi < 3/2, \\ A > 5/2 \text{ for } \chi \geq 3/2, \end{cases} \tag{3.8}$$

.

The m-dependence approximation is a key idea for the proof for the non-linear case,

$$\|E(\tilde{S}_m|\mathcal{F}_0) - E(S_m|\mathcal{F}_0)\| \leq \|S_m - \tilde{S}_m\| \leq m^{1/2}\Theta_{m,p} \ll m^{1/2}/(\log m)^{5/2},$$

where $\tilde{S}_m = \sum_{i=1}^m \tilde{X}_i = \sum_{i=1}^m E(X_i|\epsilon_i, \ldots \epsilon_{i-m})$. The proof of (3.2) follows along the line of (3.3) from the facts $\|P_j(\tilde{X}_i)\|_2 \leq \delta_{i-j,2}$,

$$E(\tilde{S}_m|\mathcal{F}_0) = \sum_{j=-m}^{0} P_j(\tilde{S}_m) = \sum_{j=-\infty}^{0} (E(\tilde{S}_m|\mathcal{F}_j) - E(\tilde{S}_m|\mathcal{F}_{j-1})).$$

However, one important limitation of the quenched CLT based method is that one cannot apply this if the tail behavior of the error process is heavy and the error process is long-range dependent. The quantile based method is more generally applicable.

3.2.2. Quantile estimation: Dependence structure

For the non-linear case however, one does not have such decomposition of the error process. Since the coefficients a_j in the decomposition measures how much e_i depend on ϵ_{i-j}, it will be beneficial to somehow control this dependence. With this motivation, we use the predictive density-based dependence measure. We assume e_i admits the following causal representation

$$e_i = H(\epsilon_i, \epsilon_{i-1}, \ldots), \tag{3.9}$$

where ϵ_i are i.i.d. Let $\mathcal{F}_k$ denote the σ-field generated by $(\epsilon_k, \epsilon_{k-1}, \ldots)$. Let (ϵ_i') be an i.i.d. copy of (ϵ_i) and

$$\mathcal{F}_k' = (\ldots, \epsilon_{-1}, \epsilon_0', \epsilon_1, \ldots, \epsilon_k),$$

be the coupled shift process. Let $F_1(u, t|\mathcal{F}_k) = P\{G(t; \mathcal{F}_{k+1}) \leq u|\mathcal{F}_k\}$ be the one-step ahead predictive or conditional distribution function and

$$f_1(u, t|\mathcal{F}_k) = \delta F_1(u, t|\mathcal{F}_k)/\delta u,$$

be the corresponding conditional density. We define the predictive dependence measure

$$\psi_{k,q} = \sup_{t\in[0,1]} \sup_{u\in\mathbb{R}} \|f_1(u,t|\mathcal{F}_k) - f_1(u,t|\mathcal{F}'_k)\|_q. \tag{3.10}$$

Quantity (3.10) measures the contribution of ϵ_0, the innovation at step 0, on the conditional or predictive distribution at step k. We shall make the following assumptions:

1. Smoothness (third order continuous differentiability): $f, m, \sigma \in C^3(\mathbb{R}[0,1])$;
2. For short-range dependence: $\Psi_{0,2} < \infty$ where $\Psi_{m,q} = \sum_{k=m}^{\infty} \psi_{m,q}$
   For long-range dependence: $\Psi_{0,2}$ can possibly be infinite.
3. (DEN) condition: There exists a constant $c_0 < \infty$ such that almost surely,

$$\sup_{t\in[0,1]} \sup_{u\in\mathbb{R}} \{f_1(u,t|\mathcal{F}_0) + |\delta f_1(u,t|\mathcal{F}_0)/\delta u|\} \le c_0.$$

The (DEN) Condition (3) implies that the marginal density $f(u,t) = Ef_1(u,t|\mathcal{F}_0) \le c_0$. Next define dependence adjusted norm

$$\|e.\|_{q,\alpha} = \sup_{t\ge 0}(t+1)^{\alpha} \sum_{i=t}^{\infty} \psi_{i,q}.$$

3.2.3. Quantile estimation consistency for non-linear process

Using the dependence measure on predictive densities, we will be able to extend the results from Zhou et al. (2010) to a more general non-linear set-up. Recall the sufficient conditions for the linear cases were based on the coefficients of the linear process. Here, however, the conditions will be based on the dependence measures. Recall the functional dependence measure defined at (3.10). Assume

$$\begin{aligned} &\text{(SRD)} \quad : \quad \sum_{j=0}^{\infty} |\psi_{j,q}| < \infty, \\ &\text{(LRD)} \quad : \quad \psi_{j,q} = j^{-\gamma} l(j), \gamma < 1, l(\cdot) \text{ is slowly varying function (s. v. f.) .} \end{aligned} \tag{3.11}$$

We propose the following result as our new contribution in this paper for the non-linear processes.
For a fixed $0 < q < 1$, let $\hat{Q}(q)$ and $\tilde{Q}(q)$ denote the q-th sample quantile and actual quantile of $\tilde{Y}_i$ $i = m, \ldots, n$ where

$$\tilde{Y}_i = \frac{\sum_{j=i-m+1}^{i} e_j}{H_m}, \quad i = m, m+1, \ldots \tag{3.12}$$

and

$$H_m = \begin{cases} \sqrt{m}, & \text{if (SRD) holds and } \mathbb{E}(\epsilon_j^2) < \infty, \\ \inf\{x : \mathbb{P}(|\epsilon_i| > x) \leq \frac{1}{m}\} & \text{if (SRD) holds and } \mathbb{E}(\epsilon_j^2) = \infty, \\ m^{3/2-\gamma} l(m) & \text{if (LRD) holds and } \mathbb{E}(\epsilon_j^2) < \infty, \\ \inf\{x : \mathbb{P}(|\epsilon_i| > x) m^{1-\gamma} l(m) & \text{if (LRD) holds and } \mathbb{E}(\epsilon_j^2) = \infty. \end{cases} \tag{3.13}$$

We have the following different rates of convergence of quantiles based on the nature of tail or dependence:

Theorem 3.1. *[Quantile consistency result:Non-linear error]*

- *Light tailed (SRD): Suppose (DEN) and (SRD) hold and* $\mathbb{E}(\epsilon_j^2) < \infty$. *If* $m^3/n \to 0$, *then for any fixed* $0 < q < 1$,

$$|\hat{Q}(q) - \tilde{Q}(q)| = O_{\mathbb{P}}(m/\sqrt{n}). \tag{3.14}$$

- *Light tailed (LRD): Suppose (LRD) and (DEN) hold with* γ *and* $l(\cdot)$ *in (3.11). If* $m^{5/2-\gamma} n^{1/2-\gamma} l^2(n) \to 0$, *then for any fixed* $0 < q < 1$,

$$|\hat{Q}(q) - \tilde{Q}(q)| = O_{\mathbb{P}}(mT^{1/2-\gamma}|l(n)|). \tag{3.15}$$

- *Heavy-tailed (SRD): Suppose (DEN) and (SRD) hold and* $\mathbb{E}(|\epsilon_j|^\alpha) < \infty$ *for some* $1 < \alpha < 2$. *If* $m = O(n^k)$ *for some* $k < (\alpha - 1)/(\alpha + 1)$, *then for any fixed* $0 < q < 1$,

$$|\hat{Q}(q) - \tilde{Q}(q)| = O_{\mathbb{P}}(mn^\nu) \text{ for all } \nu > 1/\alpha - 1. \tag{3.16}$$

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- *Heavy-tailed (LRD): Suppose (LRD) hold with* γ *and* $l(\cdot)$ *in (3.11). If* $m = O(n^k)$ *for some* $k < (\alpha\gamma - 1)/(2\alpha + 1 - \alpha\gamma)$, *then for any fixed* $0 < q < 1$,

$$|\hat{Q}(q) - \tilde{Q}(q)| = O_{\mathbb{P}}(mn^\nu) \text{ for all } \nu > 1/\alpha - \gamma. \tag{3.17}$$

4. Asymptotic results in presence of covariates

We divide our asymptotic results based on the estimation of regression coefficient β. The usual linear or robust regression is more straight-forward whereas one would need sparsity conditions imposed for the LASSO estimation for presence of infinitely many regressors.

4.1. Regression with finitely many regressors

Theorem 4.1 (Residual consistency for regression)**.**

$$\sum_i |\hat{e}_i - e_i| = o_P(\Pi(n)).$$

where the error bound $\Pi(n)$ *will be different for different behavior of the error process* e_i*.*

4.2. Regression with infinitely many regressors

4.2.1. Tail Probability inequality

We discuss a key tail probability inequality for the different settings as this can be of independent interest. Let $S_{n,b} = \sum b_i e_i$.

Theorem 4.2 (Nagaev inequality for linear processes)**.** *We have the following tail probability bounds of* $S_{n,b}$ *for the four different settings.*

- *Light-tailed SRD: If* $\sum_j |a_j| < \infty$ *and* $\epsilon_j \in \mathcal{L}^q$ *for some* $q > 2$*, then, for some constant* c_q*,*

$$P(|S_{n,b}| \geq x) \leq (1 + 2/q)^q \frac{|b|_q^q (\sum_j |a_j|)^q \|\epsilon_0\|_q^q}{x^q} + 2\exp\left(-\frac{c_q x^2}{n(\sum_j |a_j|)^2 \|\epsilon_0\|_2^2}\right) \quad (4.1)$$

- *Light-tailed LRD: If* $K = \sum_j |a_j|(1+j)^\beta < \infty$ *for* $0 < \beta < 1$ *and* $\epsilon_j \in \mathcal{L}^q$ *for some* $q > 2$*, then, for some constant* C_1, C_2 *depending on only* q *and* β*,*

$$P(|S_{n,b}| \geq x) \leq C_1 \frac{K^q |b|_q^q \|n^{q(1-\beta)}\epsilon_0\|_q^q}{x^q} + 2\exp\left(-\frac{C_2 x^2}{n^{3-2\beta}\|\epsilon_0\|_2^2 K^2}\right), \quad (4.2)$$

- *Heavy-tailed SRD: If* $\sum_j |a_j| < \infty$ *and* $\epsilon_j \in \mathcal{L}^q$ *for some* $1 < q \leq 2$*, then, for some constant* c_q

$$P(|S_{n,b}| \geq x) \leq (1+2/q)^q \frac{|b|_q^q (\sum_j |a_j|)^q \|\epsilon_0\|_q^q}{x^q} + 2\exp\left(-\frac{c_q x^2}{n(\sum_j |a_j|)^2 \|\epsilon_0\|_2^2}\right) \quad (4.3)$$

- *Heavy-tailed LRD: If* $K = \sum_j |a_j|(1+j)^\beta < \infty$ *for* $0 < \beta < 1$ *and* $\epsilon_j \in \mathcal{L}^q$ *for some* $q > 2$*, then, for some constants* C_1, C_2 *depending only on* q *and* β*,*

$$P(|S_{n,b}| \geq x) \leq C_1 \frac{K^q |b|_q^q \|n^{q(1-\beta)}\epsilon_0\|_q^q}{x^q} + 2\exp\left(-\frac{C_2 x^2}{n^{3-2\beta}\|\epsilon_0\|_2^2 K^2}\right), \quad (4.4)$$

For non-linear error process, however, it is difficult to discuss long-range dependence as one needs an appropriate model for that. For definiteness, we stick to short range dependence which means $\Theta_{0,q} < \infty$ where q is less or more than 2 depending on the tail-behavior of the error process

Theorem 4.3 (Nagaev inequality for non-linear processes)**.** *For short-range dependent processes, we have the following two versions of Nagaev inequality*

- *Light-tailed SRD:- Assume that* $\|e.\|_{q,\alpha} < \infty$ *where* $q > 2$ *and* $\alpha > 0$ *and* $\sum_{i=1}^n b_i^2 = n$*. Let* $r_n = 1$*(resp.* $(\log n)^{1+2q}$ *or* $n^{q/2-1-\alpha q}$ *) if* $\alpha > 1/2 - 1/q$ *(resp.* $\alpha = 0$ *or* $\alpha < 1/2 - 1/q$*). Then for all* $x > 0$*, for constants* C_1, C_2, C_3 *that depend on only* q *and* α*,*

$$P(|S_{n,b}| \geq x) \leq C_1 \frac{r_n}{(\sum_j |b_j|)^q \|e.\|_{q,\alpha}^q} x^q + C_2 \exp\left(-\frac{C_3 x^2}{n\|e.\|_{2,\alpha}^2}\right), \quad (4.5)$$

- *Heavy-tailed SRD:- Assume that* $\|e.\|_{q,\alpha} < \infty$ *where* $1 < q < 2$ *and* $\alpha > 0$ *and* $\sum_{i=1}^n b_i^2 = n$*. Let* $r_n = 1$*(resp.* $(\log n)^{1+2q}$ *or* $n^{q/2-1-\alpha q}$ *) if* $\alpha > 1/2 - 1/q$ *(resp.* $\alpha = 0$ *or* $\alpha < 1/2 - 1/q$*). Then for all* $x > 0$*, for constants* C_1, C_2, C_3 *that depend on only* q *and* α*,*

$$P(|S_{n,b}| \geq x) \leq C_1 \frac{r_n}{(\sum_j |b_j|)^q \|e.\|_{q,\alpha}^q} x^q + C_2 \exp\left(-\frac{C_3 x^2}{n\|e.\|_{2,\alpha}^2}\right), \quad (4.6)$$

Our main result for this section will be Theorem 4.5 and it will say that the error bounds obtained in Theorem 3.1 remain intact if we make a proper choice of the sparsity condition. Before that, we state a crucial lemma from Bickel et al. (2009).

Lemma 4.4. *Let $\lambda = 2r$ in (2.2). Also assume,*

$$r = \max(A(n^{-1}\log p)^{1/2}\|e.\|_{2,\alpha}, B\|e.\|_{q,\alpha}|X|_q/n) \tag{4.7}$$

On the event

$$\mathcal{A} = \bigcup_{j=1}^{p}\{2|V_j| \le r\}, \text{ where } V_j = \frac{1}{n}\sum_{i=1}^{n} e_i x_{ij},$$

we have,

$$r|\hat{\beta} - \beta|_1 + |X(\hat{\beta} - \beta)|_2^2/n \le 4r|\hat{\beta}_J - \beta_J|_1 \le 4r\sqrt{s}|\hat{\beta}_J - \beta_J|_2.$$

Remark This allows us to use Nagaev inequality from Theorem 4.2 and 4.3 to V_j.

Let $\bar{Q}_n(q)$ be the qth empirical quantile of $(\tilde{\hat{Y}}_i)_m^n$.

Theorem 4.5. *[Quantile consistency for LASSO] Assume s, the number of non-zero coordinates in β satisfies the following CHECK MATH:*

$$s^2 \log p \ll n \tag{4.8}$$

$$s \ll \frac{n^{1-\max\{0,1/2-1/q-\alpha\}}}{|X|_q}, \tag{4.9}$$

$$s \ll \frac{n^{\frac{2}{\alpha}+\frac{\alpha-1}{\alpha+1}} L_1(n)}{|X|_q}, \tag{4.10}$$

Then the conclusions in Theorem 3.1, and Theorem 4.3 hold with $Q_n(q)$ replaced by $\bar{Q}_n(q)$.

5. Simulation and real data evaluation

5.1. Simulation results

The focus of our simulation study is on evaluation of PI's for the two cases from previous section, i.e. (i) $p < n$ and (ii) $p > n$. For the first case, we compare three estimators, i.e. OLS, LAD and LASSO and for the second, we have only LASSO, since the former estimators won't be identified. Similar to Zhou et al. (2010) we assume these two data generating processes with $\varphi = 0.6$, $\gamma = -0.8$ and $\sigma = 54.1$:

(i) $e_i = \varphi e_{i-1} + \sigma\epsilon_i$, for stable ϵ_i with heavy-tail index 1.5 and scale 1.

(ii) $e_i = \sigma \sum_{j=0}^{\infty}(j+1)^{\gamma}\epsilon_{t-j}$, with noise as in (i),

corresponding to (a) heavy-tail and short-memory (b) heavy-tail and long-memory (e_i). For each scenario, we generate sample of length $n+m$ and use first n for estimation and last m for evaluation. The exogenous covariates $x_i \in \mathbb{R}^p$ are same as in the following empirical study, namely, 151 weather predictors and 168 (for p¡n) resp. 336 (for p¿n) deterministic periodic functions. The elements of coefficient vector β are drawn randomly from uniform distribution $U[-1,1]$ and Cauchy distribution. The two distributions are supposed to represent two extreme situations, when (uniform) all values are equally probable, (Cauchy) most elements are close to 0 and few are large. In addition, we simulate different proportions of non-zero elements in the vector β, i.e. $||\beta||_0/p = 1\%, 10\%, 30\%, 50\%, 80\%$. The non-zero elements are selected on random bases. We think of this as robustness test for LASSO against violation of sparsity assumption. The sample is obtained as:

$$y_i = x_i^T\beta + e_i, i = 1, \ldots, m+n.$$

The experiment is repeated 1000 times for each of the scenarios (a,b), each distribution of *beta*-elements and each proportion of non-zero coefficients N_s. PI's nominal coverage is $(1-\alpha) = 0.9$. We compute the coverage probabilities

$$\widehat{(1-\alpha)} = \frac{1}{1000}\sum_{i=1}^{1000}\mathbb{I}\left([L,U]_i \ni \bar{y}_{i,+1:m}\right), \tag{5.1}$$

where $\mathbb{I}$ for the i-th trial is 1 when $\bar{y}_{i,+1:m}$ is covered by the $[L,U]_i$ and 0 otherwise. We start with the case (i), there $p < n$. This is supposed to resemble the set-up used in the empirical application, where we work with hourly data. Therefore we set $n = 8736$ (which is approximately 1 year of hourly data) and $m = 168, 336, 504, 672$ (i.e. 1,2,3,4 weeks respectively). The results are summarized in Table 1 . For the second case (p¿n), we used $n = 336$ and $m = 24, 72, 120, 168$ (i.e. 1,3,5,7 days). Both the sample size and the forecasting horizon become much shorter than in case (i), however the proportion m/n increases up to 1/2. This fact has a negative effect on the performance of the quntile-based PI's. Therefore, we employing two data-driven adjustments based on (boot) replication of the residual $\hat{e}_i = y_i - \hat{y}_i$ using stationary bootstrap (Politis and Romano, 1994) and (ker) kernel quantile estimator (Falk, 1984) instead of the sample version suggested by Zhou et al. (2010). As one can see in Table 2 this leads to significant improvements in terms of coverage probability.

For our forecasting of spot electricity prices, the horizon m will reach $m \approx n/3$ which is close to the current scenario. Therefore, we apply these adjustments also there (see Section 5 for details on implementation). Specifically

β		Uniform								Cauchy							
	DGP	short-heavy				long-heavy				short-heavy				long-heavy			
	m (days)	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
	ols	78.40	73.80	70.20	65.60	70.10	63.70	60.80	52.60	78.40	73.80	70.20	65.60	70.10	63.70	60.80	52.60
	lad	87.30	84.10	84.10	83.20	76.90	73.00	71.90	65.60	87.30	84.10	84.10	83.20	76.90	73.00	71.90	65.60
1%	lasso	89.00	88.20	87.70	84.70	87.10	86.40	83.30	81.40	88.30	87.70	85.30	83.70	86.20	85.90	82.50	80.80
	clt	85.70	81.00	76.80	72.70	71.90	59.90	53.00	47.20	78.00	72.70	66.50	65.40	71.70	59.90	53.00	47.40
10%	lasso	89.50	88.00	86.80	84.70	86.70	85.90	83.70	81.30	87.70	83.00	81.20	77.90	85.70	85.20	80.80	78.70
	clt	84.00	77.00	72.90	68.20	71.40	60.50	52.70	47.20	48.50	30.80	28.00	34.40	69.80	56.30	49.80	41.50
30%	lasso	88.60	86.40	86.20	82.80	86.60	85.60	83.40	81.10	86.00	78.10	77.00	79.00	83.80	82.20	79.70	74.20
	clt	82.50	71.90	67.30	64.90	71.40	59.90	51.80	47.80	26.40	16.90	12.30	17.10	65.50	52.90	48.40	33.50
50%	lasso	88.80	84.10	84.50	81.90	86.80	85.50	83.50	80.60	81.60	77.40	70.10	81.90	82.60	82.80	78.70	73.90
	clt	80.60	63.80	64.70	62.40	71.20	60.00	51.80	47.10	21.30	16.90	9.00	10.60	61.20	50.90	44.80	33.60
80%	lasso	88.70	80.90	84.20	82.80	86.40	85.70	83.60	81.50	72.70	74.60	52.60	80.60	78.50	83.30	77.30	74.80
	clt	83.40	54.40	65.00	65.40	72.70	59.60	53.20	45.80	11.90	11.50	5.30	7.90	53.90	49.40	40.00	35.80

Table 1: Comparison results of simulated forecasting experiment. Case $n > p$ for nominal coverage probability $100(1-\alpha) = 90\%$. We simulate following time series: short-memory & heavy tail and long-memory & heavy tail.The reported values are coverage probabilities.

β		Uniform								Cauchy							
	DGP	short-heavy				long-heavy				short-heavy				long-heavy			
	m (days)	1	3	5	7	1	3	5	7	1	3	5	7	1	3	5	7
1%	qtl	72.10	56.80	45.10	35.10	56.80	39.20	28.10	19.10	71.50	57.00	46.50	34.90	57.20	39.70	28.10	18.80
	clt	70.70	62.20	58.10	51.80	52.00	38.10	32.00	27.50	70.10	62.40	58.00	51.60	52.60	38.40	31.50	27.40
	adj	73.30	67.20	62.80	59.50	57.10	43.80	36.70	32.70	72.80	66.90	62.60	59.50	56.30	44.00	36.70	32.80
10%	qtl	71.50	56.50	45.40	34.90	56.90	38.90	28.20	19.30	69.50	57.60	43.00	33.80	56.20	39.10	28.90	17.90
	clt	71.20	61.70	58.20	52.00	52.20	38.10	32.20	27.10	68.50	62.80	55.80	53.40	51.40	38.30	32.50	24.90
	adj	73.10	66.60	62.40	60.10	57.10	43.80	36.40	32.20	71.30	66.80	60.20	58.90	54.60	44.40	36.10	30.90
30%	qtl	71.30	57.30	45.00	35.20	56.70	39.20	28.40	19.40	66.90	54.70	38.40	32.80	52.70	38.00	29.00	15.90
	clt	71.30	61.70	57.80	51.60	52.40	38.90	32.70	27.10	65.90	60.30	53.20	54.90	49.20	37.20	32.30	25.50
	adj	74.20	66.40	62.70	59.10	57.10	44.60	36.70	32.30	67.90	64.00	55.60	59.80	52.70	43.50	36.80	31.10
50%	qtl	71.80	57.40	45.60	35.10	57.20	39.10	28.70	19.40	66.10	51.80	35.10	31.30	50.50	38.10	29.40	16.30
	clt	71.10	62.50	58.30	51.70	51.80	38.40	33.00	27.20	64.50	56.80	48.90	53.20	46.20	37.50	32.10	24.70
	adj	73.70	66.60	62.40	59.50	56.50	44.50	36.40	32.70	67.50	60.70	52.30	57.90	49.70	43.60	36.90	30.50
80%	qtl	72.20	56.80	45.70	35.30	56.10	44.10	36.80	31.90	64.00	46.10	30.90	28.60	53.20	35.30	28.00	16.20
	clt	70.50	62.00	57.50	52.20	52.40	38.10	33.10	27.50	62.40	50.60	47.50	52.60	48.80	34.40	30.80	24.80
	adj	73.10	66.70	63.40	59.80	56.10	44.10	36.80	31.90	65.00	53.00	50.30	56.80	52.00	40.60	34.40	29.10

Table 2: Comparison results of simulated forecasting experiment. Case $p > n$ for nominal coverage probability $100(1-\alpha) = 90\%$. We simulate following time series: short-memory & heavy tail and long-memory & heavy tail.The reported values are coverage probabilities.

5.2. Prediction intervals for European Power Exchange spot electricity prices

In this section, we forecasts the average of future m values $\bar{y}_{+1:m} = \sum_{t=1}^{m} y_{T+t}$. We conduct a graphical POOS comparison of following PI's:

(qtl) empirical PI's implied by LASSO-regression suggested in section 4,

(bayes) Bayes PI's of Müller and Watson (2016) with frequentist coverage,

(armx) PI's implied by ARMAX models,

(ets) PI's implied by exponential smoothing state space model (Hyndman et al., 2008),

(nnar) PI's implied by neural network auto-regression (Venables and Ripley, 2002; Hyndman and Athanasopoulos, 2013, section 9.3).

Data. We forecast hourly day-ahead spot electricity prices for Germany and Austria - the largest market at the European Power Exchange (EPEX SPOT). The prices arise from day-ahead hourly auctions where traders trade for specific hours of the next day. With market operating 24 hours a day, we have 11640 observations between 01/01/2013 00:00:00 UTC[4] and 04/30/2014 23:00:00 UTC. We split the data into a training period spanning from 01/01/2013 00:00:00 UTC till 12/31/2013 23:00:00 UTC and an evaluation period spanning from 01/01/2014 00:00:00 UTC till 04/30/2014 23:00:00 UTC (see Figure 1A). The forecasting horizon is $m = 1, 2, \ldots, 17$ weeks $(168, 336, \ldots, 2856$ hours).

Inspection of the periodogram for the prices in Figure 1C reveals peaks at periods 1 week, 1 day and 1/2 day. Such complex seasonality is difficult to model by SARIMA or ETS models which are suitable for monthly and quarterly data or by dummy variables. Instead, we use sums of sinusoids $g_t^k = R\sin(\omega_k t) + \phi) = \beta_k^{(s)}(R, \phi)\sin(\omega_k t) + \beta_k^{(c)}(R, \phi)\cos(\omega_k t)$ with seasonal Fourier frequencies $\omega_k = 2\pi k/168$, $k = 1, 2, \ldots, \frac{168}{2}$ corresponding to periods 1 week, 1/2 week,...,2 hours (see Bierbauer et al., 2007; Weron and Misiorek, 2008; Cartea and Figueroa, 2005; Hyndman and Athanasopoulos, 2013). The coefficients of linear combination $\beta_k^{(s)}, \beta_k^{(c)}$ can be estimated by least squares. In addition, we use 2 dummy variables as indicators for weekend.

As mentioned in Section 1, the local weather conditions are used as predictors too. The weather conditions implicitly capture seasonal patterns longer than a week, which is very important for long horizons. Local weather is represented by 151 hourly

[4]Coordinated Universal Time.

wind speed and temperature series observed over period of 5 years (2009-2013) i.e. including the training period but not the evaluation period (see above). In order to approximate some missing in-sample data and unobserved values for evaluation period, we take hourly-specific-averages[5] of each weather time series over these 5 years.

In total, we have 168 trigonometric predictors, 151 weather predictors and 2 dummies which gives a full set of 321 predictors. We denote these predictors

$$X_t = (d_{\text{sa}}, d_{\text{su}}, \sin(\omega_1 t), \cos(\omega_1 t), \ldots, , \sin(\omega_{84} t), \cos(\omega_{84} t), w_{1,t}, \ldots, w_{73,t}, \tau_{1,t}, \ldots, \tau_{78,t)}, \tag{5.2}$$

for $t = 1, \ldots, T$, with d as dummies for weekend, w_k, and τ_l as the wind speed and temperature measured at k-th, and l-th weather stations.

Methods. In Figure 1B, we see a drop of the price level during December 2013. Although the prices rise back in January 2014 the forecasts based on the whole training period would suffer from bias. On the contrary, using only the post-break December data leads to inefficiency. An optimal trade-off in such situations can be achieved by down-weighting older observations (see Pesaran et al., 2013) also called exponentially weighted regression (Taylor, 2010). In order to achieve better forecasting performance, we use the exponentially weighted regression with standardized exponential weights $v_{T-t+1} = \alpha^{t-1}((1-\alpha))/(1-\alpha^t)$, $t = 1, \ldots, T$, with $\alpha = 0.8$ for all regression models including the *qtl* and *nnar* methods. The *ets* and *armax* models provide exponential down-weighting implicitly. Finally, Müller and Watson (2016) showed that their methods are relatively robust to structural changes. The implementation of the competing methods follows:
empirical method (lasso):

1. Estimate linear regression model $y_t \sim X_t$, $t = 1 \ldots, T$ with LASSO (Friedman et al., 2010).

2. Replicate residuals $e_t = y_t - \hat{y}_t$, B times obtaining $e_t^b, t = 1, \ldots, T$, $b = 1, \ldots, B$.

3. Compute $(\bar{e}^b_{t(m)}) = m^{-1} \sum_{i=1}^{m} e^b_{t-i+1}$, $t = m, \ldots T$ from every replicated series.

4. Estimate the $\alpha/2$th and $(1-\alpha/2)$th quantile $\hat{Q}(\alpha/2)$ and $\hat{Q}(1-\alpha/2)$ using Gaussian kernel density estimator from $\bar{e}^b_{T(m)}$, $b = 1, \ldots, B$ (with $T = 260$).

[5] An alternative bootstrap approximation of unknown future observations was proposed by Hyndman and Fan (2010).

5. The PI for $\bar{y}_{+1:m}$ is $[L, U] = \bar{\hat{y}}_{T,1:m} + [\hat{Q}(\alpha/2), \hat{Q}(1-\alpha/2)]$, where $\bar{\hat{y}}_{T,1:m}$ is the average of h-step-ahead forecasts for $h = 1, \ldots, m$.

Bayes method (mn):

1. Set $q = 12$ and compute the cosine transformations $X = (X_1, , X_q)$ of series y_t. Standardize them as $Z = (Z_1, , Z_q) = X/\sqrt{X^{'}X}$.
2. For a grid of parameter values $\theta = (b, c, d)$ satisfying, $0.4 \leq d \leq 1$; $b, c \geq 0$, compute the matrix $\Sigma(\theta, q, m/T)$ using e.g. a numerical integration algorithm (for details see the supplementary Appendix of Müller and Watson (2016)).
3. Choose a prior for $\theta = (b, c, d)$ and compute the posterior distribution.
4. Decompose the covariance matrix as $\Sigma = \left(\begin{smallmatrix} \Sigma_{ZZ} & \Sigma_{Z\bar{e}} \\ \Sigma_{Z\bar{e}} & \Sigma_{\bar{e}\bar{e}} \end{smallmatrix}\right)$ and obtain covariance matrix of residuals $\Sigma_{UU} = \Sigma_{\bar{e}\bar{e}} - \Sigma'_{Z\bar{e}}(\Sigma^{-1}_{ZZ})\Sigma_{Z\bar{e}}$.
5. Compute weights for specific choice of q and m/T and the prior from step 3.
6. Numerically approximate s. c. least favorable distribution (LFD) of θ for specific choice of q and m/T (see Müller and Watson, 2016, supplementary appendix).
7. Compute the quantiles $Q_q^{\text{tmix}}(\alpha/2), Q_q^{\text{tmix}}(1-\alpha/2)$ of the conditional (mixture-t) distribution of $\bar{e}_{+1:m}$ using sequential bisection approximation.
8. Using the weights and the LFD solve the minimization problem (14) on page 1721 in Müller and Watson (2016) to get quantiles which give uniform coverage and minimize the expected PIs width.
9. The PI's are given by $[L, U] = \bar{y} + [Q_q^{\text{tmix}}(\alpha/2), Q_q^{\text{tmix}}(1-\alpha/2)]\sqrt{X^{'}X}$.

Simulation-from-model methods (armax, ets, nnar):

1. Adjust y_t for weekly periodicity using seasonal and trend decomposition by Cleveland et al. (1990).
2. Select model using AIC and fit it to seasonally adjusted y_t. For *armax* and *nnar* use aggregated weather data and dummies as exogenous predictors (see details in the supplementary appendix).
3. Simulate $b = 1, \ldots, B$ future paths $\hat{y}^b_{T,t}$, of length m from the estimated model.
4. Obtain bootstrap PI's as sample quantiles from set of averages $\bar{\hat{y}}^b_{T,T+1:T+m}$,$b = 1, \ldots, B$.

POOS results. Before we compare the *emp-LASSO* with competitors, we explore the benefits from incorporating disaggregated weather data. To do this, we compute the PI's using (i) no regressors in Figure 2IA, (ii) using only deterministic regressors in Figure 2IB, (iii) using both deterministic regressors and aggregated weather defined as $\bar{w}_t = \sum_{k=1}^{73} w_{k,t}$, $\bar{\tau}_t = \sum_{l=1}^{78} \tau_{l,t}$ and in Figure 2IC and finally, (iv) using all 321 predictors in Figure 2ID. We see only little difference between the first three plots as all three PI's seem to be upward-biased. Clearly, a striking improvement is achieved by including disaggregated weather series.

From the four other methods only *mw* and *ets* provide useful PI's. Figure 2IIA shows that *MN* works well over the whole 17-weeks-long evaluation period. However, when compared to *emp-LASSO*, we see a that the latter gives advantage in terms of precision. *ets* becomes too wide as the horizon grows. The *nnar* is clearly biased for large m. Surprisingly, the *armax*, which performs exponential smoothing by definition, perform worst of all. It might be that the down-weighting is simply too mild.

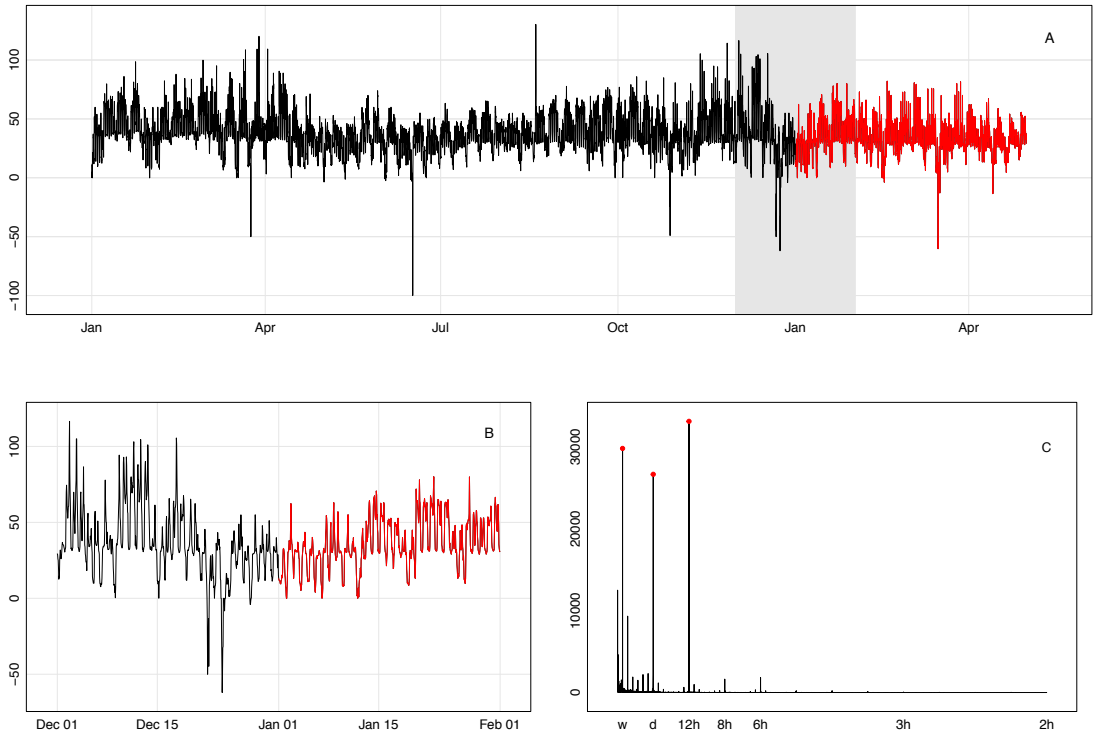
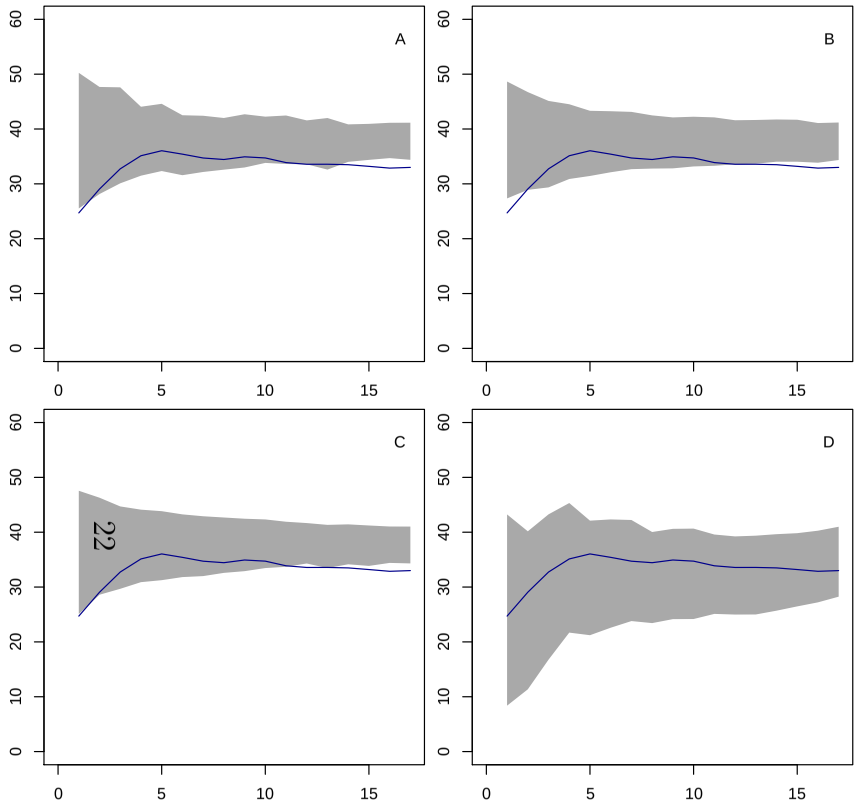
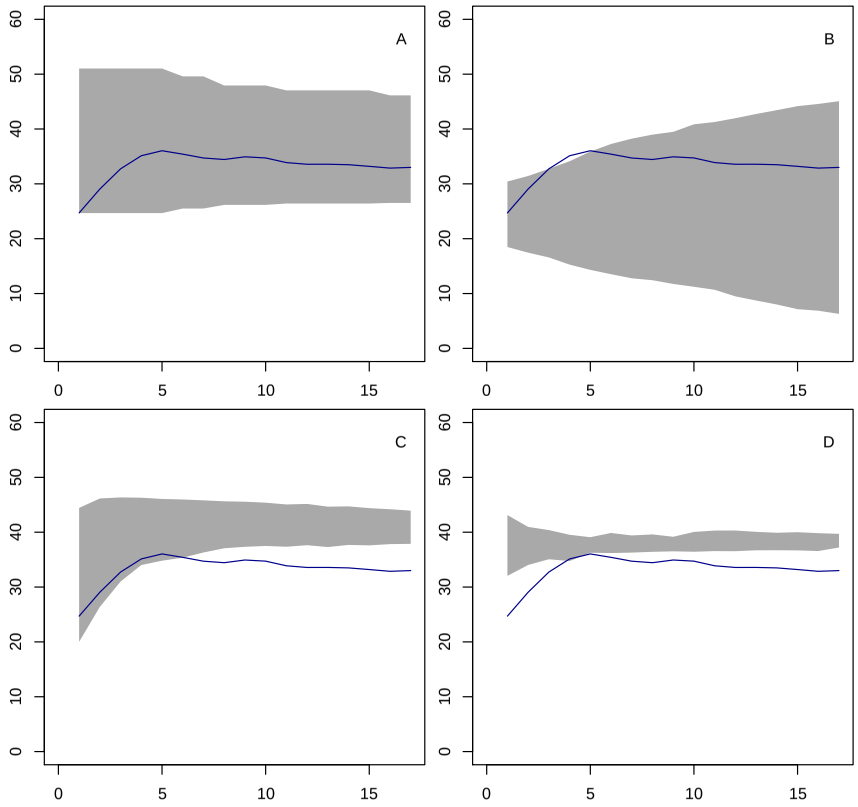


Figure 1: Electricity spot prices, A) Full sample, B) Drop in price level, C) Periodogram with peaks at periods 1 week, 1 day and 12 hours. According to European convention, the term spot refers to day-ahead rather than real-time unlike in the US, where term forward is common.



(I) A) emp (without predictors), B) emp-LASSO with 170 deterministic predictors, C) emp-LASSO with 170 deterministic predictors & 2 aggregated weather time series, D) emp-LASSO with 170 deterministic predictors & 151 disaggregated weather series.



(II) A) MN, B) ETS (A,N,N) with tuning parameter 0.0446, C) NNAR (38,22) with one hidden layer and with weekend dummies & aggregated weather as predictors, D) ARMA with weekend dummies & aggregated weather as exogenous predictors.

Figure 2: PI's (gray) for average spot electricity prices (blue) over forecasting horizon $m = 1, \ldots, 17$ weeks.

6. Discussion

We have considered problem of constructing empirically valid prediction intervals for high-dimensional regression.

From the theoretical perspective, we have extended the results of Zhou et al. (2010) into high-dimensional set-up by utilizing the LASSO estimator.

The quantile method was successfully applied to predict spot electricity prices for Germany and Austria using large set of local weather time series. The results proved superiority of conventional exponential smoothing and neural network approach as well as recently proposed low-frequency approach of Müller and Watson (2016). Regarding our application to electricity price forecasting, it would be interesting to consider even larger set of predictors, e.g., augmented by macroeconomic predictors like fuel prices, GDP.

Possible extensions to the current paper include multivariate target series and subsequent construction of simultaneous prediction intervals. Applications of such simultaneous intervals could include prediction of spot electricity prices for each hour simultaneously in the spirit of Raviv et al. (2015).

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Appendix A - Proofs
Proof of theorem 1
Lemma B.1 from 2009 lasso by bickel
Nagaev on V_j
Proof of lemma
Quantile consistency from Xiao, Xu and Wu
Proof for the non-linear case.
Define $\tilde{Z}_i$ as follows

$$\tilde{Z}_{i-1} = \frac{\sum_{j=1}^{\infty} \tilde{b}_j \epsilon_{i-j}}{H_m} \tag{6.1}$$

where $\tilde{b}_j = a_0 + a_1 + \ldots + a_j$ if $1 \leq j \leq m-1$ and $\tilde{b}_j = a_{j-m+1} + a_{j-m+2} + \ldots + a_j$ if $j \geq m$.

Define

$$\tilde{F}_n^*(x) = \frac{1}{n-m+1} \sum_{i=m}^{n} F_\epsilon(H_m(x - \tilde{Z}_{i-1})),$$

where $F_\epsilon(\dot{)}$ is the distribution function of ϵ. Let $tildeF(x) = P(\tilde{Y}_i \leq x)$. We write

$$\tilde{F}_n(x) - \tilde{F}(x) = \tilde{F}_n(x) - \tilde{F}_n^*(x) + \tilde{F}_n^*(x) - \tilde{F}(x) = M_n(x) + N_n(x)$$

Define $P_i(Y) = E(Y|\mathcal{F}_i) - E(Y|\mathcal{F}_{i-1})$. Using this, one can write $M_n(x)$ as follows

$$M_n(x) = \frac{1}{n-m+1} \sum_{i=m}^{n} P_i(I(Y_i \leq x)) \tag{6.2}$$

Lemma 6.1. *Under conditions of Theorem 4.2 and Theorem 4.3,*

$$\sup_{|u| \leq b_n} |M_n(x+u) - M_n(x)| = O_p\left(\sqrt{\frac{H_m b_n}{n}} \log^{1/2} n + n^{-3}\right), \tag{6.3}$$

where b_n is a positive bounded sequence with $\log n = o(H_m n b_n)$.

Proof. Let $c_0 = \sup_x |f_\epsilon(x)| < \infty$. Since $P(\tilde{Y}_i \leq x|\mathbb{F}_{i-1}) = F_\epsilon(H_m(x - \tilde{Z}_{i-1}))$, we have $P(x \leq \tilde{Y}_i \leq x+u|\mathbb{F}_{i-1}) \leq H_m c_0 u$ for all $u > 0$. Therefore for any $u \in [-b_n, b_n]$, we have

$$\sum_{i=m} n[E(V) - E^2(V)] \le c_0(n-m+1)H_m b_n \quad \text{where } V = I(x \le \tilde{Y}_i \le x+u|\mathbb{F}_{i-1}) \tag{6.4}$$

Applying Freedman's martingale inequality and a chaining argument, we have (6.3). Since the chaining argument is essentially similar to Lemma 5 in , Lemma 4 in and Lemma 6 in we skip the details □

Lemma 6.2. *Under conditions of SRD, DEN and light-tailed*

$$\| \sup_{|u| \le b_n} |N_n(x+u) - N_n(x)| \| = O\left(\frac{b_n m^{3/2}}{\sqrt{n}}\right) \tag{6.5}$$

Proof. Since $N_n(x) = \tilde{F}_n^*(x) - \tilde{F}(x)$, we have

$$N_n(x+u) - N_n(x) = \sqrt{m}\frac{\int_0^u R_n(x+t)dt}{n-m+1}$$

where

$$R_n(x) = \sum_{i=m}^{n}[f_\epsilon(H_m(x - \tilde{Z}_{i-1})) - E(f_\epsilon(H_m(x-\tilde{Z}_{i-1})))] \quad x \in \mathbb{R}.$$

. Since , we are left to prove

Hence,

$$\|R_n(x+u)\| \le Cmm\sqrt{n} \text{ for all } u \in [-b_n, b_n]$$

.

Let $(\epsilon_i')_{-\infty}^{\infty}$ be an i.i.d. copy of $(\epsilon_i)_{-\infty}^{\infty}$ and $\tilde{Z}_{i-1,k}^* = \tilde{Z}_{i-1} - \tilde{b}_k\epsilon_{i-k}/\sqrt{m} + \tilde{b}_k\epsilon_{i-k}'/\sqrt{m}$. Note that for $k \ge 1$,

$$\begin{aligned} \|\mathcal{P}_{i-k} f_\epsilon(\sqrt{m}(x+u-\tilde{Z}_{i-1}))\| &\le \|f_\epsilon(\sqrt{m}(x+u-\tilde{Z}_{i-1})) - f_\epsilon(\sqrt{m}(x+u-\tilde{Z}_{i-1,k}^*))\| \\ &\le \sup_{v\in\mathbb{R}}|f_\epsilon'(v)|\sqrt{m}(\tilde{Z}_{i-1} - \tilde{Z}_{i-1.k}^*)\|\| \le c_1\tilde{b}_k \end{aligned} \tag{6.6, 6.7}$$

where $c_1 = \sup_{vin\mathbb{R}} |\|\epsilon_0 - \epsilon_0'\| < \infty$. Further note that

$$R_n(x+u) = \sum_{k=1}^{\infty}\sum_{i=m}^{n}\mathcal{P}_{i-k}f_\epsilon(\sqrt{m}(x+u-\tilde{Z}_{i-1}))$$

and by the orthogonality of $\mathcal{P}_{i-k}, i = m, \ldots, n$

$$\|\sum_{i=m}^{n}\mathcal{P}_{i-k}f_\epsilon(\sqrt{m}(x+u-\tilde{Z}_{i-1}))\|^2=\sum_{i=m}^{n}\|\mathcal{P}_{i-k}f_\epsilon(\sqrt{m}(x+u-\tilde{Z}_{i-1}))\|^2\le c_1^2(n-m+1)\tilde{b}_k^2.$$

Therefore, for all $u\in[-b_n,b_n]$, by the short-range dependence condition as

$$\begin{aligned}\|R_n(x+u)\| &\le \sum_{k=1}^{\infty}\|\sum_{i=m}^{n}\mathcal{P}_{i-k}f_\epsilon(\sqrt{m}(x+u-\tilde{Z}_{i-1}))\|\\ &\le c_1\sqrt{n}\sum_{k=1}^{\infty}|\tilde{b}_k|\le c_1 m\sqrt{n}\sum_{j=0}^{\infty}|a_j|.\end{aligned}$$

Recall the functional dependence measure. The proofs for the linear cases will go through if we replace f_ϵ by f_1, a_j by $\delta_{j,2}$.

□

Lemma 6.3. *Under conditions of LRD, DEN and heavy-tailed, we have for any* $\rho\in(1/\gamma,\alpha)$

$$\|\sup_{|u|\le b_n}|N_n(x+u)-N_n(x)|\|_\rho=O\left(H_m b_n m n^{1/\rho-\gamma}|l(n)|\right)\tag{6.8}$$

Proof. Similar to the proof of Lemma 6.2, it suffices to prove, for some $0<C<\infty$,

$$\|R_n(x+u)\|_\rho\le Cmn^{1/\rho+1-\gamma}|l(n)|\text{ for all }u\in[-b_n,1-b_n]\tag{6.9}$$

Since $1<\rho<2$, by Burkholder's inequality of martingales, we have, with $C_\rho=[18\rho^{3/2}(\rho-1)^{-1/2}]^\rho$.

$$\begin{aligned}\|R_n(x+u)\|_\rho^\rho &= \|\sum_{k=-\infty}^{n-1}\mathcal{P}_k\sum_{i=m}^{n}f_\epsilon(H_m(x-\tilde{Z}_{i-1}))\|_\rho^\rho \qquad (6.10)\\ &\le C_\rho\sum_{k=-\infty}^{n-1}\|\mathcal{P}_k\sum_{i=m}^{n}f_\epsilon(H_m(x-\tilde{Z}_{i-1}))\|_\rho^\rho\\ &\le C_\rho\sum_{k=-\infty}^{n-1}(\sum_{i=m}^{n}\|\mathcal{P}_k f_\epsilon(H_m(x-\tilde{Z}_{i-1}))\|_\rho)^\rho\\ &\le C_\rho(\sum_{k=-\infty}^{-n}+\sum_{k=-n+1}^{0}+\sum_{k=1}^{n-1})(\sum_{i=m}^{n}\|\mathcal{P}_k f_\epsilon(H_m(x-\tilde{Z}_{i-1}))\|_\rho)^\rho\\ &\le C_\rho(I+II+III),\end{aligned}$$

Since $E(|\epsilon_i|^\rho) < \infty$, similarly as (6.6), we have for $k \leq i-1$ that

$$\|\mathcal{P}_k f_\epsilon(H_m(x - Z_{i-1}))\|_\rho \leq c_1 |\tilde{b}_{i-k}|, \tag{6.11}$$

where $c_1 = \sup_{v\in\mathbb{R}} |f'_\epsilon(v)| \|\epsilon_0 - \epsilon'_0\|_\rho < \infty$. Thus using Karamata's theorem for the term I, we have

$$\begin{aligned} I \leq c_1^\rho \sum_{k=-\infty}^{-n} (\sum_{i=m}^{n} |\tilde{b}_{i-k}|)^\rho &\leq c_1^\rho \sum_{k=n}^{\infty} (m \sum_{i=1}^{n} |a_{k+i}|)^\rho \\ &\leq c_1^\rho m^\rho n^{\rho-1} \sum_{k=n}^{\infty} \sum_{i=1}^{n} |a_{k+i}|^\rho \\ &= O[m^\rho n^{1+\rho(1-\gamma)} |l(n)|^\rho] \end{aligned} \tag{6.12}$$

Since $\rho > 1$ and $\rho\gamma > 1$, we use H"o"lder inequality to manipulate term III as follows:

$$\begin{aligned} III \leq c_1^\rho \sum_{k=1}^{n-1} (\sum_{i=\max(m,k+1)}^{n} |\tilde{b}_{i-k}|)^\rho &\leq c_1^\rho \sum_{k=1}^{n-1} (m \sum_{i=0}^{n-k} |a_i|)^\rho \\ &= m^\rho \sum_{k=1}^{n-1} O[(n-k)^{1-\gamma} |l(n-k)|]^\rho \\ &= O[m^\rho n^{1+\rho(1-\gamma)} |l(n)|^\rho]. \end{aligned} \tag{6.13}$$

Similarly for term II we have, $II = O[m^\rho n^{1+\rho(1-\gamma)} |l(n)|^\rho]$. Combining this with (6.12) and (6.13), we finish the proof of the lemma. □

of Theorem 4.2. By central limit theorem of Hannan (1979), we have $\tilde{Y}_i \xrightarrow{D} N(0, \sigma^2)$, where $\sigma = \|\sum_{i=0}^{\infty} \mathcal{P}_0 e_i\| < \infty$. Hence $\tilde{Q}_q$ is well-defined and it converges to qth quantile of a $N(0, \sigma^2)$ distribution as $m \to \infty$. Furthermore, note that e_i is a weighted sum of i.i.d. random variables and the density $f_\epsilon(\cdot)$ is bounded. Hence a standard characteristic function argument yields

$$\sup_x |f_m(x) - \phi(x/\sigma)/\sigma| \to 0, \tag{6.14}$$

where $f_m(\cdot)$ is the density of $\tilde{Y}_i$ and $\phi(x)$ is the density of a standard normal random variable. Let (c_n) be an arbitrary sequence of positive numbers that goes to infinity. Let $\bar{c}_n = \min(c_n, n^{1/4}/m^{3/4})$. Then $\bar{c}_n \to \infty$. Lemma 6.1 and 6.2 imply that

$$\begin{aligned} |\tilde{F}_n(\tilde{Q}_q + B_n) - \tilde{F}(\tilde{Q}_q + B_n) - [F_n(\tilde{Q}_q) - \tilde{F}(\tilde{Q}_q)]| &= O_P(\frac{B_n m^{3/2}}{\sqrt{n}} + m^{1/4}\sqrt{\frac{B_n}{n}}(\log n)^{1/2}) \\ &= o_p(B_n), \end{aligned} \tag{6.15}$$

where $B_n = \bar{c}_n m/\sqrt{n}$. Furthermore, similar arguments as those in Lemma 6.1 and 6.2 imply

$$|\tilde{F}_n(\tilde{Q}_q) - \tilde{F}(\tilde{Q}_q)| = O_p(\frac{m}{\sqrt{n}}) = o_P(B_n). \tag{6.16}$$

Using Taylor's expansion of $\tilde{F}(\cdot)$, we have

$$\tilde{F}(\tilde{Q}_q + B_n) - \tilde{F}(\tilde{Q}_q) = B_n f_m(\tilde{Q}_q) + O(B_n)^2. \tag{6.17}$$

By (6.14), $f_m(\tilde{Q}_q) > 0$ for sufficiently large n. Plugging in (6.16) and (6.17) into (6.15), we have $P(\tilde{F}_n(\tilde{Q}_q + B_n) > q) \to 1$. Hence $P(\hat{Q}_n(q) > \tilde{Q}_q + B_n) \to 0$ by the monotonicity of $\tilde{F}_n(\cdot)$. Similar arguments yield $P(\hat{Q}_n(q) < \tilde{Q}_q - B_n) \to 0$. Using the fact that c_n can approach infinity arbitrarily slowly, we finish the proof of Theorem 4.2.

□

Proof of Theorem 4.5. From Lemma 4.4, we have

$$\sup_{m \le i \le n} |\sum_{k=i-m+1}^{i} (\hat{e}_i - e_i)| = O_p(\pi(n)), \tag{6.18}$$

for a suitable π depending on the λ and sparsity s. Thus

$$\bar{Q}_n(q) - \hat{Q}_n(q) = O_p\left(\frac{\pi(n)}{H_m}\right). \tag{6.19}$$

□

Appendix B: Additional information for section 5

Additional notes on implementation of emp-LASSO

We use LASSO implementation in R-package glmnet with tuning parameter λ chosen by cross validation and with weights argument $(v_1 \ldots, v_T) = ((1-\alpha)\alpha^{(T-1)}/(1-\alpha^T), \ldots, 1)$ to account for the structural change in coefficients. $\alpha = 0.8$.

Additional notes on implementation of ets, nnar and armax with software output

The ETS(A,N,N) with tuning parameter = 0.0446, NNAR$(38, 22)$ with one hidden layer and ARMA$(2, 1)$ were selected by AIC and estimated by R-package forecast. NNAR and ARMAX allow for exogenous predictors, therefore we include aggregated weather series $\bar{w}_t = \sum_{k=1}^{73} w_{k,t}$, $\bar{\tau}_t = \sum_{l=1}^{78} \tau_{l,t}$, and weekend dummies as well. For NNAR, we can provide weights for the predictor observations. We use the same exponential down-weighting scheme as for the emp-LASSO, but with $\alpha = 0.98$, which gave better results.

First the price series y_t is seasonally adjusted using STL decomposition (R-core function). The seasonally adjusted prices z_t is used as input for the models implemented in R-package forecast. The models are specified as follows:

ETS The model is selected according to AIC criterion. We restrict the model in that we dont use trend component, because the prices do not show any trend pattern (see 1). However, probably due to breaks in price-level, the AIC would select a trend component. This results in too varying future paths. For optimization criterion, for, we use Average MSFE, over maximal possible horizon=30 hours. This results into model with tuning parameter 0.0446 selected by AIC. This is gives better forecasting results than minimizing in-sample MSE which would result in tuning parameter 0.99 and huge PI's.

```
ETS (A, N, N)
# means additive model, without trend and seasonal components.
Call:
ets(y = y, model = "ZNZ", opt.crit = "amse", nmse = 30)

Smoothing parameters:
 alpha = 0.0446

Initial states:
 l = 34.1139
```

```
 sigma: 8.4748

  AIC     AICc    BIC
116970.9 116970.9 116992.1
```

NNAR The model is selected according to AIC criterion. The model is restrited in that it allows only one hidden layer. The number of nodes in this layer is by default given as (#AR lags + #exogenous predictors)/2. In this case, we use aggregated wind speed and temperatures, and dummies for weekend so the number of exogenous predictors is 4. In order to get fair comparison with the emp-LASSO, we also use exponential downweighting on the exogenous predictors, this time with tuning parameter 0.95.

```
NNAR (38,22)
# means that order of AR component is 38 and there are 22 nodes in the hidden layer
Call:  nnetar(y = y, xreg = cbind(Weather_agg, dummy_12), weights = expWeights(alpha=

Average of 20 networks, each of which is
a 42-22-1 network with 969 weights
options were - linear output units

sigma^2 estimated as 15.34
```

ARMA The model is selected according to AIC criterion. we use aggregated wind speed and temperatures, and dummies for weekend.

```
Regression with ARIMA(2,0,1) errors

Coefficients:
         ar1     ar2     ma1  intercept    xreg1    xreg2   xreg3   xreg4
      0.5062  0.3284  0.4908    59.3783  -4.5514  -0.4894  0.4811  0.1333
s.e.  0.0601  0.0548  0.0568     1.6441   0.4037   0.0602  0.5382  0.5382

sigma^2 estimated as 24.35:  log likelihood=-26410.34
AIC=52838.68   AICc=52838.7   BIC=52902.38
```