Laser Frequency Locking to an Optical Cavity using LQG Control

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Abstract. Systematic approaches to the design of quantum optical systems will become more and more important as their complexity grows. One such systematic technique is the Linear Quadratic Gaussian (LQG) methodology. Here we describe the design and implementation of a LQG feedback controller in a quantum optical experiment - frequency-locking an optical cavity to a laser. The successful implementation of the LQG design procedure to this particular problem lays the groundwork for the application of other modern control techniques to quantum optical systems. Our results are promising for future, more complex stabilisation problems in quantum optics, as the described approach is inherently multi-variable and naturally incorporates multiple sensors and actuators as well as nested loops.

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INTRODUCTION

For the development of quantum information technologies systematic approaches to design and synthesis will be critical. Systematic methods such as Linear Quadratic Gaussian (LQG) optimal control have been studied in control engineering since the 1960s. We describe the application of systematic methods of LQG optimal control to the ubiquitous problem of cavity locking in quantum optics. To some extent, the purpose of this paper is to describe the approach required to implement optimal LQG control in a real-life quantum optical experiment. We describe practical techniques which aid in the use of LQG optimal control in such experiments. Fig. 1 schematically illustrates the experiment.

LINEAR QUADRATIC GAUSSIAN OPTIMAL CONTROL

The LQG optimal controller design methodology begins with a linear state-space model comprising a vector of system variables x, a vector input to the system u, (potentially) a vector measured output y, and w is a Gaussian white noise disturbance acting on the system. Also, the system matrix is represented by A, the input matrix is represented by B, C is the output matrix, $D_1 w$ represents process noise and $D_2 w$ represents measurement noise.

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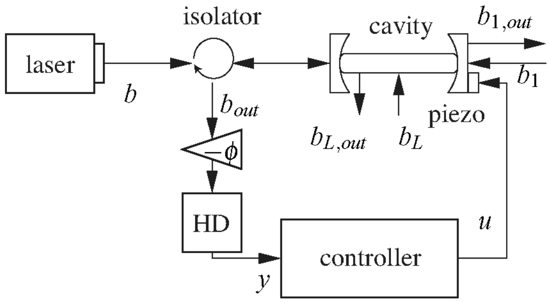

FIGURE 1. Cavity locking feedback control loop.

The goal of the LQG optimal control problem is to construct a feedback controller that minimises a quadratic cost functional. The cost functional codifies the design goal to minimise the system variables of interest (specifically, to keep the frequency detuning of the cavity small) and the requirement to limit the control energy.

The great advantage of the LQG optimal control approach to controller design is that it provides a tractable systematic way to construct output feedback controllers, even in the case of multi-input multi-output control systems. The solution to the LQG optimal control problem involves a Kalman filter which provides an optimal estimate $\hat{x}$ of the vector of system variables x based on the measured output y.

The standard LQG approach is not sufficient to generate a suitable controller as the system illustrated in Fig. 1 is subject to a large initial DC offset and slowly varying disturbances, which necessitates integral action. Here, an additional term is included in the cost function which involves the integral of a new quantity z. The new variable $\int z\,dt$ is also fed to the Kalman filter, which when combined with an optimal state-feedback control law leads to an integral LQG optimal controller; see [1].

Mathematically, the augmented system can be described in state-space form:

$$\dot{\tilde{x}} \;=\; \tilde{A}\tilde{x}+\tilde{B}w_1+\tilde{B}u; \quad \tilde{y}=\tilde{C}\tilde{x}+\begin{bmatrix} w_2 \\ w_3 \end{bmatrix}; \quad \tilde{x}=\begin{bmatrix} x \\ \int z\,d\tau \end{bmatrix} \text{ and } \tilde{y}=\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \quad (1)$$

Here the matrices $\tilde{A},\tilde{B},\tilde{C}$ are constructed from the matrices A,B,C as follows:

$$\tilde{A}=\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, \quad \tilde{B}=\begin{bmatrix} B \\ 0 \end{bmatrix}, \text{ and } \tilde{C}=\begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}.$$

The integral LQG performance criterion can be written as:

$$\mathscr{J}=\lim_{T\to\infty}\mathbf{E}\left[\frac{1}{T}\int_0^T [x^T Qx+L(z)^T\bar{Q}L(z)+u^T Ru]dt\right] \qquad (2)$$

where $L(z)=\int_0^t z(\tau)\,d\tau$ and we choose the matrices Q,R and $\bar{Q}$ such that $x^T Qx = |z|^2,\;\; u^T Ru = r|u|^2$, and $\bar{Q}=\bar{q}$.

The optimal LQG controller is given by the following set of simultaneous equations (e.g., see [2]):

$$u = -r^{-1}\tilde{B}^T X\hat{\tilde{x}} = F\hat{\tilde{x}}, \tag{3}$$

$$0 = X\tilde{A} + \tilde{A}^T X + \tilde{Q} - r^{-1}X\tilde{B}^T\tilde{B}X, \quad \tilde{Q} = \tilde{C}^T \begin{bmatrix} 1 & 0 \\ 0 & \bar{q} \end{bmatrix} \tilde{C}, \tag{4}$$

$$\dot{\hat{\tilde{x}}} = \tilde{A}\hat{\tilde{x}} + \tilde{B}u + K[\tilde{y} - \tilde{C}\hat{\tilde{x}}], \quad K = P\tilde{C}^T V_2^{-1}, \tag{5}$$

$$0 = \tilde{A}P + P\tilde{A}^T + V_1 - P\tilde{C}^T V_2^{-1}\tilde{C}P. \tag{6}$$

LQG CONTROLLER FOR AN OPTICAL CAVITY

A complete state-space model of the system under control (or "plant"), the measurement and any noise sources is required in order to construct the optimal feedback controller. The cavity can be described in the Heisenberg picture by the standard cavity differential equations; e.g., see [3] and Section 9.2.4 of [4]. The linearised cavity dynamics and homodyne measurement can be expressed in state-space form in terms of the quadratures of the operators for the cavity mode and cavity inputs [5].

As it stands, that would not yet be a complete model of the system, as it does not include the dynamics for the detuning Δ. A reasonable hypothesis for these dynamics might be to treat them as arising from the dynamics of a cavity mirror, which is treated as a mass on a spring, subject to mechanical noise and laser frequency noise. We could then augment the optical state-space model with the equation $\Delta = Fx$, so that the matrix F relates the detuning to the vector of system variables, which includes those variables describing the mechanical dynamics of the cavity mirror as well as those for the optical mode.

Unfortunately, the true dynamics of the cavity mirror is much more complicated than a simple mass on a spring. Therefore, a more experimentally tractable approach must be adopted in order to arrive at the optimal integral LQG controller. In our modern control approach, the subspace identification method determines a state-space model from the input-output frequency response data and generates the system matrices A, B and C; see [6]. The frequency response data obtained is fitted to a 13^{th} order model (including an 8^{th}-order anti-aliasing filter with a corner frequency of 2.5 kHz) using subspace identification [7].

CONTROLLER AND RESULTS

The design values for the controller $\varepsilon_1 = 5 \times 10^{-2}$, $\varepsilon_2 = 500$ and $\varepsilon_3 = 3 \times 10^{-4}$ are the variances of the mechanical noise, detector noise and integrator noise. The control energy weighting in the controller cost function is $r = 1 \times 10^3$ and the integral output weighting is $\bar{q} = 1 \times 10^6$.

These parameter values lead to a 15^{th} order LQG controller which is reduced to a 6^{th} order controller using a frequency-weighted balanced controller reduction approach. The reduced controller is then discretised at a sampling rate of 50 kHz. This discretised controller provides good gain and phase margins of 16.2 dB and 63° respectively.

The discrete controller is implemented on a dSpace DS1103 Power PC DSP Board. This board is fully programmable from a Simulink block diagram and possesses 16-bit resolution. The controller successfully stabilises the frequency in the optical cavity, locking its resonance to that of the laser frequency, f_0; see [5]. This can be seen from the measured step response shown in Fig. 2, which is consistent with theory.

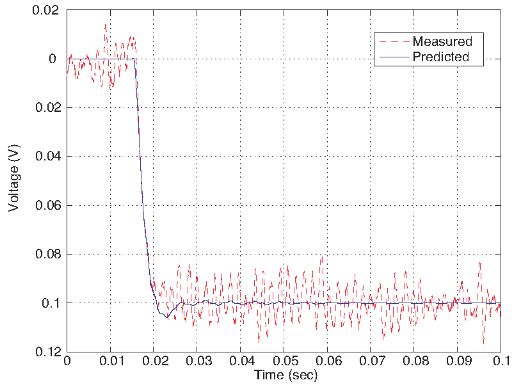

FIGURE 2. Step Response of the closed-loop system to an input of 0.1 V.

CONCLUSION

We have described the design and implementation of a LQG feedback controller in a quantum optical experiment - frequency-locking an optical cavity to a laser. Practical techniques are used to augment the basic LQG methodology, which facilitated the design and implementation of an optimal integral LQG controller. The controller was implemented digitally, and a step response measured. The performance of the closed-loop system is consistent with its state-space model.

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