Polytopic and LFT Approach to Gain-Scheduling: A Design Example

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***Abstract*— This paper is concerned with the design of gain-scheduling controllers for uncertain linear parameter-varying systems. Two alternative design techniques for constructing such controllers are presented. Both techniques are amenable to linear matrix inequality problems which are readily solvable. The validity of the results are demonstrated and compared on a simple missile design example.**

***Index Terms*— Gain-scheduling control, linear parameter-varying(LPV), robust control, linear matrix inequalities(LMI), uncertain systems.**

I. Introduction

THERE is a long history of gain-scheduling in applications, but there are few citations to the control literature before 1990. Gain-scheduling was considered an "applied" topic, and because of its superior performance capability, it was quickly adopted in military applications even though it came at a higher cost. The philosophy behind gain-scheduling is quite simple. Rather than seeking a single robust linear time invariant(LTI) controller for the entire operating range (which tends to be conservative if at all feasible) of a plant, gain scheduling consists of designing an LTI controller for each operating point and in switching controllers when the operating conditions change. The controller coefficients are continuously varied based on the current value of the "scheduling variable(s)", that may be either exogenous signals or endogenous signals with respect to the plant ([1], [2], [11]).

The traditional exogenously gain-scheduled controller, which is adjusted with reference to one or more externally measured variable(s), $\rho(t)$, has the form

$$\begin{aligned} \dot{x} &= A(\rho(t))x + B(\rho(t))y \\ u &= C(\rho(t))x + D(\rho(t))y \end{aligned} \tag{1}$$

The dynamic properties change with $\rho(t)$ and provided that the rate of change is not too rapid, then the dynamic properties of the time-varying controller (1) are similar to those of the linear controllers obtained by "freezing" the value of ρ; that is, the nonlinear controller inherits the dynamic properties of the family of linear controllers. There are no direct restrictions on the state, x, or the input, u and the only restriction is on the rate of change of the scheduling variable.

Rugh and Shamma [12] described the gain-scheduling controller design as a four-step procedure:

1) The first step is to compute a linear parameter-varying model for the plant and there are two approaches to this:
    a) The most common one is based on Jacobian linearization of the nonlinear plant about a family of equilibrium points, yielding a parametrized family of linearized plants.
    b) The other approach is what is known as "quasi-LPV" scheduling, in which the plant dynamics are rewritten to hide nonlinearities as time-varying parameters that are then used as scheduling variables.
2) Linear design controller techniques are then used for the LPV plant model that arises from either the linearization or quasi-LPV approach. This may result directly in a family of linear controllers corresponding to the linear parameter-dependent plant, or there may be an interpolation process to arrive at a family of linear controllers from a set of controller designs at isolated values of the scheduling variables.
3) The third step involves implementing the family of linear controllers such that the controller coefficients(gains) are varied(scheduled) according to the current value of the scheduling variable(s).
4) The final step is performance assessment. This may be relatively simple where analytical performance guarantees are part of the design process. More typically, the local stability and performance properties of the gain-scheduled controller might be subject to analytical investigation, while the nonlocal performance evaluation is based on extensive simulation studies.

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II. Polytopic Approach to Gain-Scheduling

The synthesis technique discussed below follows from [3] and is applicable to affine parameter-dependent plants with equations

$$\begin{aligned} \dot{x} &= A(p)\,x + B_1(p)\,w + B_2\,u \\ z &= C_1(p)\,x + D_{11}(p)\,w + D_{12}\,u \\ y &= C_2\,x + D_{21}\,w + D_{22}\,u \end{aligned} \tag{2}$$

where

$$p(t) = (p_1(t), \ldots, p_n(t)), \qquad \underline{p}_i \leq p_i(t) \leq \overline{p}_i$$

is a time-varying vector of physical parameters and $A(\cdot), B_1(\cdot), C_1(\cdot), D_{11}(\cdot)$ are affine functions of $p(t)$. The parameter vector, $p(t)$ may include part of the state vector x itself provided that the corresponding states are available for measurement.

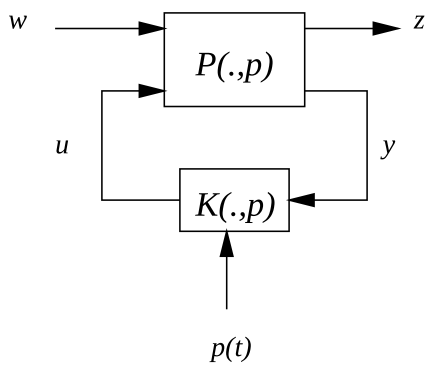

Fig. 1. Gain-Scheduled $\mathcal{H}_\infty$ problem

The plant system matrix for (2) is:

$$S(p) := \left(\begin{array}{c|cc} A(p) & B_1(p) & B_2 \\ C_1(p) & D_{11}(p) & D_{12} \\ \hline C_2 & D_{21} & D_{22} \end{array}\right)$$

If the parameter vector $p(t)$ takes values in a box of $\mathbb{R}^n$ with corners $\{\Pi_i\}_{i=1}^N (N = 2^n)$, then $S(p)$ ranges in a matrix polytope with vertices $S(\Pi_i)$. Specifically, given any convex decomposition

$$p(t) = \alpha_1 \Pi_1 + \ldots + \alpha_N \Pi_N, \quad \alpha_i \geq 0, \quad \sum_{i=1}^{N} \alpha_i = 1$$

of p over the corners of the parameter box, the system matrix $S(p)$ is given by

$$S(p) = \alpha_1 S(\Pi_1) + \ldots + \alpha_N S(\Pi_N).$$

With reference to the feedback system arrangement shown in Figure 1, we seek a parameter-dependent controller with equation

$$\begin{aligned} \dot{x}_K &= A_K(p)x_K + B_K(p)y \\ u &= C_K(p)x_K + D_K(p)y \end{aligned} \tag{3}$$

and having the following vertex property:

$$\begin{pmatrix} A_K(p) & B_K(p) \\ C_K(p) & D_K(p) \end{pmatrix} = \sum_{i=1}^{N} \alpha_i \begin{pmatrix} A_K(\Pi_i) & B_K(\Pi_i) \\ C_K(\Pi_i) & D_K(\Pi_i) \end{pmatrix}$$

That is, the controller state-space matrices at the operating point $p(t)$ are obtained by convex interpolation of the LTI vertex controllers

$$K_i = \begin{pmatrix} A_K(\Pi_i) & B_K(\Pi_i) \\ C_K(\Pi_i) & D_K(\Pi_i) \end{pmatrix}$$

yielding a smooth scheduling of the controller matrices by the parameter measurements $p(t)$.

For this class of controllers, let us consider the following $\mathcal{H}_\infty$-like synthesis problem relative to the interconnection of Figure 1. The aim is to design a gain-scheduled controller $K(., p)$ satisfying the vertex property such that

- the closed-loop system is stable for all admissible parameter trajectories $p(t)$
- the worst-case closed-loop $\mathcal{H}_\infty$ gain from w to z does not exceed some level $\gamma > 0$.

Using the notion of quadratic $\mathcal{H}_\infty$ performance to enforce the root mean square (RMS) gain constraint, Becker et al. [4] proposed that the synthesis be reduced to the following LMI problem of finding two symmetric matrices (R, S) such that (4)–(6) hold true where

$$\begin{pmatrix} A_i & B_{1i} \\ C_{1i} & D_{11i} \end{pmatrix} := \begin{pmatrix} A(\Pi_i) & B_1(\Pi_i) \\ C_1(\Pi_i) & D_{11}(\Pi_i) \end{pmatrix} \tag{7}$$

and $\mathcal{N}_{12}$ and $\mathcal{N}_{21}$ are bases of the null spaces of (B_2^T, D_{12}^T) and (C_2, D_{21}), respectively.

Having found R and S, the same approach as Gahinet and Apkarian [6] can be followed to solve for the controller at each vertex.

III. LFT Approach to Gain Scheduling

LMIs can be used to fully characterize the existence of gain-scheduled controllers when the dependence on time-varying but measured parameters is *linear fractional* and usually, we can model or approximate the parameter dependence in an LPV system as an LFT. The problem is then less complex than for the general LPV case and the computational burden required is very similar to that for the standard linear time-invariant case.

LPV plants with a linear fractional dependence on θ can be represented by the upper LFT interconnection

$$\begin{pmatrix} q \\ y \end{pmatrix} = F_u(P(s), \Theta) \begin{pmatrix} w \\ u \end{pmatrix} \tag{8}$$

where $P(s)$ is a known LTI plant and Θ is some block diagonal time-varying operator specifying how θ enters the plant dynamics as depicted in Figure 2. Specifically

$$\Theta = \text{blockdiag}(\theta_1 I_{r_1}, \ldots, \theta_K I_{r_K})$$

where $r_i > 1$ whenever the parameter θ_i is repeated. The set of operators with structure (9) is denoted by

$$\Delta := \{\text{blockdiag}(\theta_1 I_{r_1}, \ldots, \theta_K I_{r_K}) : \theta_i(\tau) \in \mathbb{R}\}$$

The approach is based on the concept of parameter-dependent $\mathcal{H}_\infty$ controllers developed in ([1], [2]). The controllers depend on the varying parameters $\theta(t)$ through

$$\begin{aligned} \dot{x}_K(t) &= A_K(\theta(t))\, x_K(t) + B_K(\theta(t))\, y(t) \\ u(t) &= C_K(\theta(t))\, x_K(t) + D_K(\theta(t))\, y(t) \end{aligned} \tag{9}$$

where A_K, B_K, C_K, D_K are linear fractional functions of θ. This control structure is applicable whenever the value of $\theta(t)$ is measured at each time t. The resulting controller is time-varying and smoothly "scheduled" by the measurements of $\theta(t)$.

Consistently with (8), we seek LPV controllers are sought of the form

$$u = F_l(K(s), \Theta)y \tag{10}$$

which gives the rule for updating the controller state-space matrices based on the measurements of θ, as shown in Figure 2. The closed-loop operator from disturbance w to controlled output q is then given by

$$T(P, K, \Theta) = F_l(F_u(P, \Theta), F_l(K, \Theta)) \tag{11}$$

$$\begin{pmatrix} \mathcal{N}_{12} & 0 \\ 0 & I \end{pmatrix}^T \left(\begin{array}{cc|c} A_iR + RA_i{}^T & RC_{1i}^T & B_{1i} \\ C_{1i}R & -\gamma I & D_{11} \\ \hline B_1^T & D_{11i}^T & -\gamma I \end{array}\right) \begin{pmatrix} \mathcal{N}_{12} & 0 \\ 0 & I \end{pmatrix} < 0, \quad i = 1, \ldots, N \tag{4}$$

$$\begin{pmatrix} \mathcal{N}_{21} & 0 \\ 0 & I \end{pmatrix}^T \left(\begin{array}{cc|c} A_i{}^TS + SA_i & SB_{1i} & C_{1i}^T \\ B_{1i}^TS & -\gamma I & D_{11i}^T \\ \hline C_{1i} & D_{11i} & -\gamma I \end{array}\right) \begin{pmatrix} \mathcal{N}_{21} & 0 \\ 0 & I \end{pmatrix} < 0, \quad i = 1, \ldots, N \tag{5}$$

$$\begin{pmatrix} R & I \\ I & S \end{pmatrix} \geq 0 \tag{6}$$

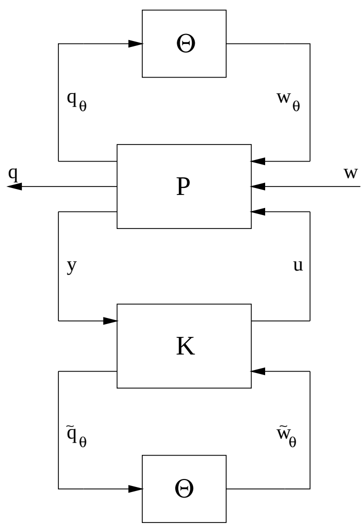

Fig. 2. LPV control structure

Given some LTI plant $P(s)$ mapping exogeneous inputs w and control inputs u to controlled outputs q and measured outputs y, the usual $\mathcal{H}_\infty$ control problem is concerned with finding an internally stabilizing LTI controller $K(s)$ such that

$$\|F_l(P, K)\|_\infty < \gamma$$

where γ is some prescribed performance level. The gain-scheduled version of this problem has a similar statement, except that both the plant and the controller are now LPV instead of LTI. Here the objective is to guarantee some closed loop performance $\gamma > 0$ from w to q for all admissible parameter trajectories $\theta_{(\tau)}$.

Apkrarian et al. [3] stated that a gain-scheduled controller can be designed for some minimal realization of an LTI plant $P(s)$ described by (12) if there exist pairs of symmetric matrices (R, S) in $\mathbb{R}^{n\times n}$ and (L_3, J_3) in $\mathbb{R}^{r\times r}$ such that (13)–(16) hold true.
The problem dimensions are given by

$$A \in \mathbb{R}^{n\times n}, D_{\theta\theta} \in \mathbb{R}^{r\times r}, D_{11} \in \mathbb{R}^{p_1\times p_1}, D_{22} \in \mathbb{R}^{p_2\times m_2}$$

and

$$\hat{B}_1 = (B_\theta, B_1),\ \hat{C}_1 = \begin{pmatrix} C_\theta \\ C_1 \end{pmatrix},\ \hat{D}_{11} = \begin{pmatrix} D_{\theta\theta} & D_{\theta 1} \\ D_{1\theta} & D_{11} \end{pmatrix}$$

Moreover, there exist γ-suboptimal controllers of order k if and only if the equations above hold for some quadruple (R, S, L_3, J_3) where (R, S) further satisfy the rank constraint

$$\text{rank } (I - RS) \leq k.$$

A. Computation of the Gain-Scheduled Controller

Provided the LMIs (13)–(16) are satisfied, the actual computation of the gain-scheduled controller can be addressed. Let the gain-scheduled controller $K(s)$ be defined as (17) and let

$$\Omega := \begin{pmatrix} A_K & B_{K1} & B_{K\theta} \\ C_{K1} & D_{K11} & D_{K1\theta} \\ C_{K\theta} & D_{K\theta 1} & D_{K\theta\theta} \end{pmatrix} \in \mathbb{R}^{(k+m_2+r)\times(k+p_2+r)} \tag{18}$$

Lemma 1: The singular value decomposition or SVD of a given matrix $M \in \mathbb{R}^{m\times n}$ having rank k is

$$M = U\Sigma V^T = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1{}^T \\ V_2{}^T \end{bmatrix}$$

where the matrices $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ are orthogonal (i.e., $U^TU = UU^T = I_m$ and $V^TV = VV^T = I_n$), $\Sigma_1 = \text{diag}(\sigma_1, \ldots, \sigma_k)$ and $\sigma_1 \geq \ldots \geq \sigma_k > 0$. The scalars $\sigma_1, \ldots, \sigma_{\min\{m,n\}}$ (where $\sigma_{k+1} = \ldots = \sigma_{\min\{m,n\}}$) are called the `singular values` of M.

Given any solution (R, S, L_3, J_3) of LMI system (13)-(16), the state-space data of Ω of some γ-suboptimal $K(s)$ is computed:

- From R, S the bounded real lemma matrix X_{cl} is derived using Lemma 1. Letting $MN^T = U\Sigma V^T = I - RS$, $\Rightarrow M = U\Sigma$ and $N = V$. Then X_{cl} is computed as the unique solution of the linear matrix equation
$$X_{cl}\begin{pmatrix} I & R \\ 0 & M^T \end{pmatrix} = \begin{pmatrix} S & I \\ N^T & 0 \end{pmatrix}$$
- Two matrices $L_1 \in L_\Delta$ and L_2 commuting with the structure Δ are then computed such that
$$L := \begin{pmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{pmatrix} > 0, \qquad L^{-1} = \begin{pmatrix} * & * \\ * & J_3 \end{pmatrix}$$
where * represents “don’t care”.
$$\begin{array}{rcl} J_3 & = & (L_3 - L_2^T L_1^{-1} L_2)^{-1} \\ L_3 - J_3^{-1} & = & L_2^T L_1^{-1} L_2 \quad \text{(From Lemma 1)} \\ \text{Let} \quad L_3 - J_3^{-1} & = & U\Sigma U^T \\ \text{and} \quad L_2^T L_1^{-1} L_2 & = & U\Sigma U^T \\ \therefore L_1 = \Sigma^{-1} & \text{and} & L_2 = U^T \end{array} \tag{19}$$

$$P(s) = \begin{pmatrix} D_{\theta\theta} & D_{\theta 1} & D_{\theta 2} \\ D_{1\theta} & D_{11} & D_{12} \\ D_{2\theta} & D_{21} & D_{22} \end{pmatrix} + \begin{pmatrix} C_\theta \\ C_1 \\ C_2 \end{pmatrix} (sI - A)^{-1} (B_\theta \quad B_1 \quad B_2) \tag{12}$$

$$\mathcal{N}_R^T \begin{pmatrix} AR + RA^T & R\hat{C}_1^T & \hat{B}_1 \begin{pmatrix} J_3 & 0 \\ 0 & I \end{pmatrix} \\ \hat{C}_1 R & -\gamma \begin{pmatrix} J_3 & 0 \\ 0 & I \end{pmatrix} & \hat{D}_{11} \begin{pmatrix} J_3 & 0 \\ 0 & I \end{pmatrix} \\ \begin{pmatrix} J_3 & 0 \\ 0 & I \end{pmatrix} \hat{B}_1^T & \begin{pmatrix} J_3 & 0 \\ 0 & I \end{pmatrix} \hat{D}_{11}^T & -\gamma \begin{pmatrix} J_3 & 0 \\ 0 & I \end{pmatrix} \end{pmatrix} \mathcal{N}_R < 0 \tag{13}$$

$$\mathcal{N}_S^T \begin{pmatrix} A^T S + SA & S\hat{B}_1 & \hat{C}_1^T \begin{pmatrix} L_3 & 0 \\ 0 & I \end{pmatrix} \\ \hat{B}_1^T S & -\gamma \begin{pmatrix} L_3 & 0 \\ 0 & I \end{pmatrix} & \hat{D}_{11}^T \begin{pmatrix} L_3 & 0 \\ 0 & I \end{pmatrix} \\ \begin{pmatrix} L_3 & 0 \\ 0 & I \end{pmatrix} \hat{C}_1 & \begin{pmatrix} L_3 & 0 \\ 0 & I \end{pmatrix} \hat{D}_{11} & -\gamma \begin{pmatrix} L_3 & 0 \\ 0 & I \end{pmatrix} \end{pmatrix} \mathcal{N}_S < 0 \tag{14}$$

$$\begin{pmatrix} R & I \\ I & S \end{pmatrix} \geq 0 \tag{15}$$

$$L_3 \in L_\Delta, \ J_3 \in L_\Delta, \ \begin{pmatrix} L_3 & I \\ I & J_3 \end{pmatrix} \geq 0. \tag{16}$$

where $L_\Delta = \{L > 0 : L\Theta = \Theta L, \forall \Theta \in \Delta\} \subset \mathbb{R}^{r \times r}$

$$K(s) \ = \ \begin{pmatrix} D_{K11} & D_{K1\theta} \\ D_{K\theta 1} & D_{K\theta\theta} \end{pmatrix} + \begin{pmatrix} C_{K1} \\ C_{K\theta} \end{pmatrix} (sI - A_K)^{-1} (B_{K1} \ B_{K\theta}) \tag{17}$$

$$A_K \in \mathbb{R}^{k \times k}, \ D_{K11} \in \mathbb{R}^{m_2 \times p_2}, \ D_{K\theta\theta} \in \mathbb{R}^{r \times r}$$

- Let

$$\mathcal{L} := \begin{pmatrix} L & 0 \\ 0 & I_{p_1} \end{pmatrix}, \qquad \mathcal{J} := \mathcal{L}^{-1} \tag{20}$$

- The following LMI is then solved for Ω

$$\Psi + \begin{pmatrix} X_{cl} & 0 \\ 0 & I \end{pmatrix} P^T \Omega Q + Q^T \Omega^T P \begin{pmatrix} X_{cl} & 0 \\ 0 & I \end{pmatrix} < 0 \tag{21}$$

where

$$\Psi = \begin{pmatrix} A_0^T X_{cl} + X_{cl} A_0 & X_{cl} B_0 & C_0^T \\ B_0^T X_{cl} & -\gamma \mathcal{L} & \mathcal{D}_{11}^T \\ C_0 & \mathcal{D}_{11} & -\gamma \mathcal{J} \end{pmatrix}$$

$$P := (\mathcal{B}^T, 0, \mathcal{D}_{12}^T), \qquad Q := (\mathcal{C}, \mathcal{D}_{21}, 0)$$

and

$$A_0 = \begin{pmatrix} A & 0 \\ 0 & 0_{k \times k} \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 & B_\theta & B_1 \\ 0_{k \times r} & 0 & 0 \end{pmatrix},$$

$$\mathcal{B} = \begin{pmatrix} 0 & B_2 & 0 \\ I_k & 0 & 0_{k \times r} \end{pmatrix}, \quad C_0 = \begin{pmatrix} 0 & 0_{r \times k} \\ C_\theta & 0 \\ C_1 & 0 \end{pmatrix},$$

$$\mathcal{D}_{11} = \begin{pmatrix} 0_{r \times r} & 0 & 0 \\ 0 & D_{\theta\theta} & D_{\theta 1} \\ 0 & D_{1\theta} & D_{11} \end{pmatrix}, \mathcal{D}_{12} = \begin{pmatrix} 0_{r \times k} & 0 & I_r \\ 0 & D_{\theta 2} & 0 \\ 0 & D_{12} & 0 \end{pmatrix},$$

$$\mathcal{C} = \begin{pmatrix} 0 & I_k \\ C_2 & 0 \\ 0 & 0_{r \times k} \end{pmatrix}, \quad \mathcal{D}_{21} = \begin{pmatrix} 0_{k \times r} & 0 & 0 \\ 0 & D_{2\theta} & D_{21} \\ I_r & 0 & 0 \end{pmatrix}$$

The controller parameters $(A_K, B_{K1}, \ldots, D_{K\theta\theta})$ can thus be obtained and linear fractionally transformed with the scheduling variables giving the gain-scheduled controller.

IV. Stability and Performance of Gain-Scheduled Controllers

Even though gain scheduled designs are based on linearizations, the overall system is still nonlinear [10]. This essentially renders difficult the possibility of nonconservative stability and performance analysis results. Stability and performance properties of a gain scheduled design were not easily guaranteed in the early approaches such as multi-model linearization ([5], [8], [9]) and they were typically inferred from extensive simulations. Such properties are not necessarily assured by the stability and performance properties of the fixed operating condition designs.

The polytopic and LFT approach presented however ensures stability as the synthesis method takes into account the parameter time variations during the design itself.

V. Missile Example

A. Dynamics of the Model

This model is drawn from [3]. The dynamics of the missile are strongly dependent on the angle of attack α, the air speed V and the altitude H and these three variables undergo large variations during operation. They completely define the

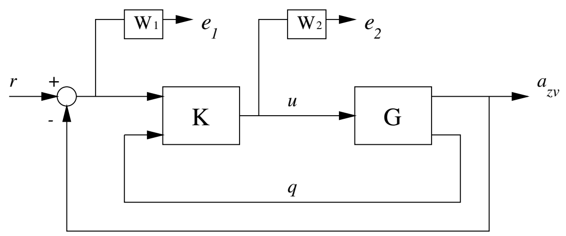

Fig. 3. Missile Control Autopilot

flight conditions(operating point) of the missile and they are assumed to be measured in real time. The LPV model is developed for this problem on the basis of linearization of the missile equations around its flight conditions. To simplify the design procedure while retaining the main difficulties of the problem, it is assumed that the pitch, yaw and roll axes are decoupled. A simple model of the linearized dynamics of the missile is the parameter-dependent model

$$\begin{aligned}\begin{bmatrix}\dot{\alpha}\\ \dot{q}\end{bmatrix} &= \begin{bmatrix}-Z_\alpha & 1\\ -M_\alpha & 0\end{bmatrix}\begin{bmatrix}\alpha\\ q\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}\delta_m\\ \begin{bmatrix}a_{zv}\\ q\end{bmatrix} &= \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\alpha\\ q\end{bmatrix}\end{aligned}$$

where a_{zv} is the normalized vertical acceleration, q is the pitch rate, δ_m the fin deflection, and Z_α, M_α are aerodynamical coefficients depending on α, V, and H. These two coefficients are functions of the flight conditions and are therefore available in real time. Though the parameter dependence of the original plant has been simplified, the control of the missile dynamics remains a hard task. The parameters Z_α and M_α abruptly change as functions of the flight conditions, and range over a large parameter domain where the stability properties of the missile vary greatly. As V, H and α vary in

$$V \in [0.5, 4]\mathrm{Mach}, \quad H \in [0, 18000]\mathrm{m}, \quad \alpha \in [0, 40]\deg$$

during operation, the coefficients Z_α and M_α range in

$$Z_\alpha \in [0.5, 4], \qquad M_\alpha \in [0, 106].$$

B. Gain-scheduled Control Design using LFT

Gahinet et al. [7] designed a gain-scheduled controller for this system using the polytopic approach, achieving a $\gamma = 0.205$. We will use the same stringent performance specifications as in [7] for our LFT design and then correlate the two results. Our goal is to control the vertical acceleration a_{zv} over this operating range. The control structure is as shown in Figure 3.

To enforce the performance and robustness requirements, we use the loop-shaping criterion

$$\left\| \begin{bmatrix} W_1 S \\ W_2 K S \end{bmatrix} \right\|_\infty < 1. \tag{22}$$

and the same shaping filters are chosen as in [7]

$$\begin{aligned} W_1(s) &= \frac{2.01}{s + 0.201}\\ W_2(s) &= \frac{9.678s^3 + 0.029s^2}{s^3 + 1.206e4s^2 + 1.136e7s + 1.066e10}\end{aligned}$$

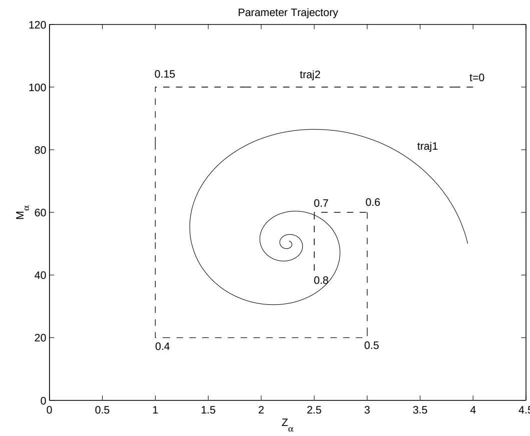

Fig. 4. Parameter trajectories 1 (solid), 2 (dashed)

The LMIs (13–16) are solved for $R, S, L_3, \mathrm{and}\ J_3$ and a 6^{th} order controller is computed yielding a performance level of $\gamma = 0.33$

C. Results

To assess the performance of the resulting LPV controller, the step response of the gain-scheduled system is simulated along the two parameter trajectories shown in Figure 4. Trajectory 1 is a smooth spiral trajectory which varies mathematically following:

$$\begin{aligned} Z_\alpha(t) &= 2.25 + 1.70\, e^{-4t}\, \cos(20t)\\ M_\alpha(t) &= 50 + 49\, e^{-4t}\, \sin(20t)\end{aligned}$$

In contrast, trajectory 2 is non-smooth and is intended to test the reaction of the LPV control law in the face of abrupt parameter changes.

The corresponding nonlinear simulations are presented for both the LFT and the polytopic approaches in Figure 5 (trajectory 1) and Figure 6 (trajectory 2) respectively. Clearly, both controllers provide acceptable performances in the face of quick parameter variation. The settling time specification ($t_s < 0.5$ s) is clearly met for trajectory 1. A slight degradation of performance is noticed for trajectory 2. This degradation is due to the corner points of the parameter trajectory, where there is an infinite derivative of the parameter. However, from a practical point of view, this response is still satisfactory, since infinite derivatives are not realistic. It should also be noted that the initial control signal generated by the polytopic controller is much larger than that of LFT controller(explaining its slightly better performance), even though it finally settles at a value which is about half that of the LFT.

VI. Conclusion

The polytopic and LFT approaches to gain-scheduling have been laid out and applied to a parameter varying plant. Both controllers designed behaved very well in the face of uncertainty as predicted by theory and managed to meet tight performance and robustness specifications with little apparent conservatism over a wide operating region. Moreover, the specifications were maintained even for rapidly changing parameters (as demonstrated with trajectory 2).

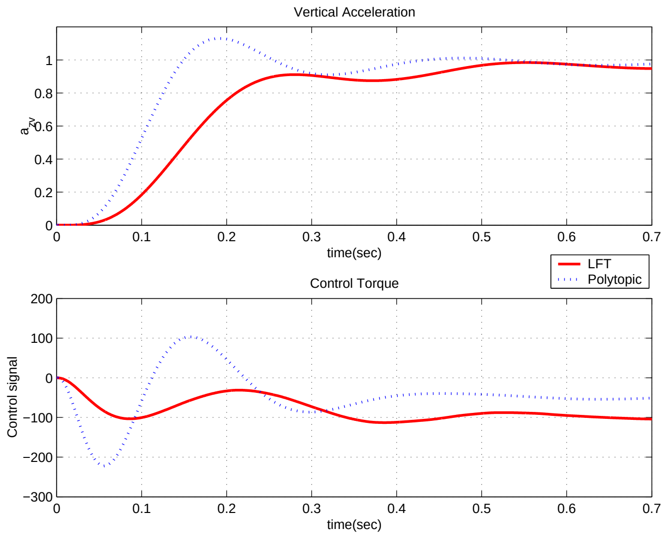

Fig. 5. Nonlinear simulation of missile over trajectory 1 for the polytopic and LFT controllers

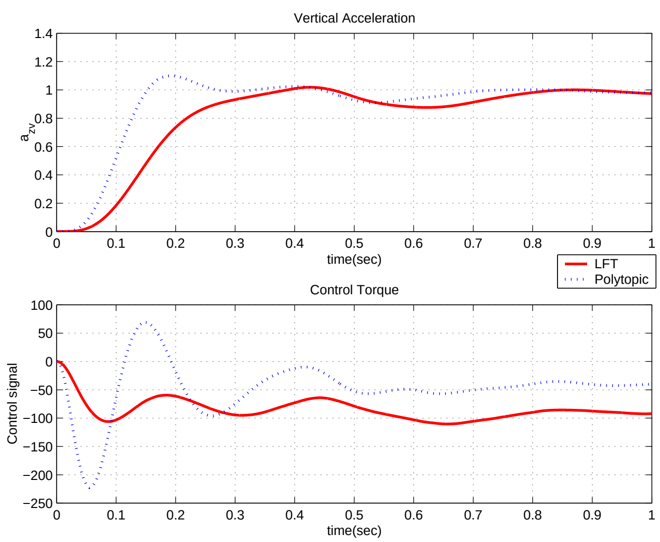

Fig. 6. Nonlinear simulation of missile over trajectory 2 for the polytopic and LFT controllers

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