Modelling Systems for Control Studies - An Overview

S. Z. Sayed Hassen, *Student Member, IEEE* and F. Crusca, *Member, IEEE* and H. R. Abachi, *Member, IEEE*

***Abstract*—This paper describes a nonlinear control design problem. A model is manipulated in different mathematical forms and then interconnected using suitable weighting functions. The resulting models form the basis for different control design techniques that can be applied to synthesize a suitable controller.**

I. Introduction

OVER the past two decades there have been notable advances in the area of modeling and controller design, leading to the now popular robust control and gain-scheduled control formalisms [1], [2], [3], [4], [6], [7], [10], [12].

Firstly, there has been a significant advance in the area of modeling of plant uncertainty based on set-theoretic methods. While philosophically the idea that one can model uncertainty may seem paradoxical, one can certainly say that if the true model of the plant is contained within the allowed uncertain model set then the prescribed levels of system performance, robustness and stability can be guaranteed.

Secondly there has been a significant paradigm shift in the area of system optimization away from solving what are essentially quadratic objective functions and toward more general convex formulations. These advances have in turn been brought out by recent advances in semidefinite programming [9].

In this paper we focus on the advances in modeling and provide an overview of the paradigm shifts that have been witnessed in this field. We shall make the discussion more concrete by referring to a single specific example, namely, a two-link robotic manipulator operating in the horizontal plane.

II. Physical Model of Robot Manipulator

Robot manipulators are familiar examples of trajectory-controllable mechanical systems. However, their nonlinear dynamics present a challenging control problem, since traditional linear control approaches do not easily apply. With reference to the robotic manipulator example shown in Figure 1, the typical objective in control studies is to constrain the end-effector to track a particular trajectory, and this requires a model of the complete nonlinear dynamics of a 2-link manipulator.

S. Z. Sayed Hassen is with the Department of Electrical and Computer Systems Engineering, Monash University, Clayton 3168, AUS. Email: sayed.hassen@eng.monash.edu.au

F. Crusca is with the Department of Electrical and Computer Systems Engineering, Monash University, Clayton 3168, AUS. Email: f.crusca@eng.monash.edu.au

Hamid R. Abachi is with the Department of Electrical and Computer Systems Engineering, Monash University, Clayton 3168, AUS. Email: abachi@eng.monash.edu.au

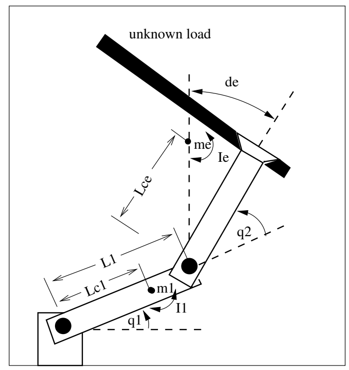

Fig. 1. Two-link manipulator

The physical modeling is detailed in full in literature [8], [11], and generally covers three types of dynamics torques that arise from the motion of the manipulator: (a) inertial torques (which are proportional to joint acceleration in accordance with Newton's second law); (b) centripetal torques (which arise from the centripetal forces which constrain a body to rotate about a point and are directed towards the center of the uniform circular motion, and are proportional to the square of the velocity); and (c) Coriolis torques (which arise from vertical forces derived from the interaction of two rotating links, and are proportional to the product of the joint velocities of those links).

The position of the planar, two-link, articulated manipulator can be described by a 2-vector $q = [q_1 \; q_2]^{\mathrm{T}}$ of joint angles, and the actuator inputs consist of a 2-vector $u = [u_1 \; u_2]^{\mathrm{T}}$ of torques applied at the manipulator joints. Allowing $\dot{q}$ to denote the joint velocities and $\ddot{q}$ the joint accelerations, and assuming that the manipulator is in the horizontal plane, then the dynamics of this simple manipulator is given by [11]:

$$H(q)\ddot{q} + C(q,\dot{q})\dot{q} = u \tag{1}$$

where $H(q) = \begin{bmatrix} h_{11}(q) & h_{12}(q) \\ h_{21}(q) & h_{22}(q) \end{bmatrix}$ is the manipulator inertia tensor matrix (which is symmetric positive definite), $C(q,\dot{q})\dot{q}$ is a 2-vector of Centripetal and Coriolis torques (with $C(q,\dot{q}) = \begin{bmatrix} -h(q)\dot{q}_2 & -h(q)(\dot{q}_1 + \dot{q}_2) \\ h(q)\dot{q}_1 & 0 \end{bmatrix}$ where

$$\begin{aligned} h_{11} &= a_1 + 2a_3 \cos q_2 + 2a_4 \sin q_2 \\ h_{12} &= h_{21} = a_2 + a_3 \cos q_2 + a_4 \sin q_2 \\ h_{22} &= a_2 \\ h &= a_3 \sin q_2 - a_4 \cos q_2 \end{aligned}$$

and the constants a_1, a_2, a_3 and a_4 are given as follows.

$$\begin{aligned} a_1 &= I_1 + m_1 l_{c1}^2 + I_e + m_e l_{ce}^2 + m_e l_1^2 \\ a_2 &= I_e + m_e l_{ce}^2 \\ a_3 &= m_e l_1 l_{ce} \cos\delta_e \\ a_4 &= m_e l_1 l_{ce} \sin\delta_e \end{aligned}$$

and

$$\begin{aligned} I_1 &= \text{inertia of arm 1} \\ I_e &= \text{inertia due to arm 2 and unknown load} \\ m_1 &= \text{mass of arm 1} \\ m_e &= \text{mass of arm 2 and unknown load} \\ l_1 &= \text{distance between joint 1 and joint 2} \\ l_{c1} &= \text{distance of joint 1 from centre of mass} \\ &= \text{of arm 1} \\ l_{ce} &= \text{distance of joint 2 from centre of mass} \\ & \quad \text{of arm 2 and unknown load} \\ \delta_e &= \text{angle between arm 2 and centre of mass} \\ & \quad \text{of unknown load} \end{aligned}$$

A typical control problem for such a system is to compute the required actuator inputs to perform desired tasks (*e.g*, move to a final desired position or follow a desired trajectory), given the measured systems state, namely the vector q of joint angles, and the vector $\dot{q}$ of joint velocities.

III. Models for Control

In this section we show how the model of equation (1) can be reformulated into a variety of different forms each suitable to different controller design techniques. Firstly we shall formulate the model in state-space form suitable for general nonlinear control studies. After that we shall reformulate the model as a constant but uncertain system of the type used in robust control studies such as Mixed sensitivity and k_m/μ-synthesis. Finally, we reformulated the model as a linear parameter varying system, suitable for use in gain scheduled controller designs.

A. Nonlinear Control

Let $x = \begin{bmatrix} q_1 & q_2 & \dot{q}_1 & \dot{q}_2 \end{bmatrix}$ then the dynamics of the two-link manipulator can be expressed as

$$\begin{bmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{H}(x) \end{bmatrix} \dot{x} = \begin{bmatrix} \mathbf{O} & \mathbf{I} \\ \mathbf{O} & -\mathbf{J}(x) \end{bmatrix} x + \begin{bmatrix} \mathbf{O} \\ \mathbf{I} \end{bmatrix} u$$

where

$$\mathbf{H}(x) = \begin{bmatrix} h_{11}(x) & h_{12}(x) \\ h_{21}(x) & h_{22}(x) \end{bmatrix}; \mathbf{I} = I_{2\times 2}$$

$$\mathbf{J}(x) = \begin{bmatrix} -2h(x)\,x_4 & -h(x)\,x_4 \\ h(x)\,x_3 & 0 \end{bmatrix}; \mathbf{O} = 0_{2\times 2}$$

In the case where $\Delta = h_{11}(x)h_{22}(x) - h_{12}^2(x) \neq 0$, the model can be written in the explicit form:

$$\dot{x} = A(x)\,x + B(x)\,u \qquad (2)$$

where

$$A(x) = \begin{bmatrix} \mathbf{O} & \mathbf{I} \\ \mathbf{O} & -\mathbf{H}(x)^{-1}\mathbf{J}(x) \end{bmatrix} \text{ and } B(x) = \begin{bmatrix} \mathbf{O} \\ \mathbf{H}(x)^{-1} \end{bmatrix}$$

This model is nonlinear, state-dependent and affine in the controls. It is used as the basis for various nonlinear control design methods.

B. Local Control

In this section, the two-link manipulator is linearized about a certain equilibrium operating point (x_o, u_o). Linearization about the given point results in the linear system

$$E\Delta\dot{x} = A\Delta x + B\Delta u \qquad (3)$$

where

$$E = \begin{bmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{H}(x_0) \end{bmatrix};\ A = \begin{bmatrix} \mathbf{O} & \mathbf{I} \\ \mathbf{O} & \mathbf{O} \end{bmatrix};\ B = \begin{bmatrix} \mathbf{O} \\ \mathbf{I} \end{bmatrix}$$

Note that $\mathbf{H}(x_0)$ is now a constant matrix depending on the point of linearization.

Equation (3) is used as the basis for small signal analysis/synthesis; and any standard methods of linear control such as Root-Locus, pole-placement, optimal control, can be used. In k_m/μ-synthesis, one proceeds to treat the system as if it were a constant system with a number of uncertain but constant parameters, $(\delta_1, \delta_2, \ldots)$. This is achieved as follows. Define $\delta_1 = \cos q_2$, and $\delta_2 = \sin q_2$, and write:

$$E(\delta_1,\delta_2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a_1 & a_2 \\ 0 & 0 & a_2 & a_2 \end{bmatrix}$$
$$+\delta_1 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2a_3 & a_3 \\ 0 & 0 & a_3 & 0 \end{bmatrix} + \delta_2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2a_4 & a_4 \\ 0 & 0 & a_4 & 0 \end{bmatrix}$$

Note that we simply assume that (δ_1, δ_2) are constant but uncertain and in the range $[-1, 1]$. In addition it is typical to assume some non-parametric uncertainty, to represent high frequency unmodelled dynamics. For example, one can represent input multiplicative uncertainty in the plant model, $G(\delta_1, \delta_2)$ by introducing a constant but uncertain matrix, Δ, in the plant model as follows:

$$G(\delta_1, \delta_2, \Delta) = (I + W_u\Delta)G(\delta_1, \delta_2) \qquad (4)$$

where W_u may represent a suitable weighting function.

In essence this forms the basic model for use in such studies, which when augmented with appropriate frequency weightings, as shown in Figure 2 provides a reasonably systematic method for shaping the feedback quantities of interest and obtain good performance over all allowed uncertainty variations.

Then the problem becomes one of synthesizing a controller K that minimizes the so-called error signals e as shown in Figure 3 and meet the control objectives.

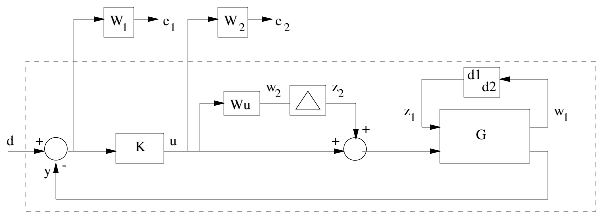

Fig. 2. Perturbed System with Multiplicative Uncertainty used for k_m/μ-Synthesis

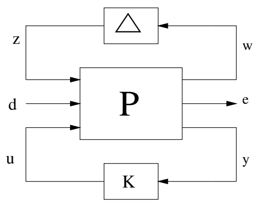

Fig. 3. General Configuration

C. Gain-Scheduled Control

We may reformulate our nonlinear model of equation (1) into a form suitable for use in gain scheduled control studies by representing the model as a constant known model with uncertain time-varying parameters using the following substitutions:

$$m_1(x_2) \;=\; a_3 \cos x_2 + a_4 \sin x_2 \qquad (5)$$

$$\text{and } m_2(x_2) \;=\; a_3 \sin x_2 - a_4 \cos x_2 \qquad (6)$$

Let us for convenience introduce three new parameters (p_1, p_2, p_3):

$$p_1 = m_1; \quad p_2 = m_2 x_3; \quad p_3 = m_2 x_4$$

Then the model of equation (1) can be expressed as a linear parameter-varying system:

$$E(p)\dot{x} = A(p)x + Bu$$

where

$$E(p) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a_1 & a_2 \\ 0 & 0 & a_2 & a_2 \end{bmatrix} + p_1 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A(p) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + p_2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$+ p_3 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly then, we ended up with a model having 3 scheduled variables p_1, p_2, p_3 about which a Linear Parameter Varying (LPV) Controller can be designed [6].

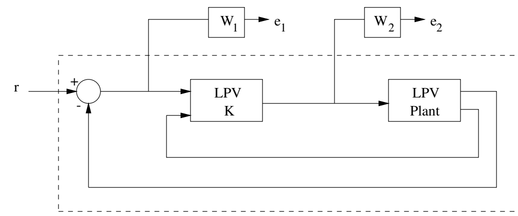

Fig. 4. Interconnected System that can be used for Gain-Scheduled Controller Synthesis

IV. Conclusions

We looked at two different methods of representing the two-link robot manipulator. Firstly, we linearized the model and added some non-parametric uncertainty to account for high frequency dynamic uncertainty and put the model in a suitable form for k_m/μ-synthesis. After that, the nonlinear model was set up in such a way that it became an affine model dependent on 3 parameters. These 3 parameters are then the scheduling variables about which a gain-scheduled controller can be designed using suitable controller design techniques [5].

References

[1] P Apkarian. On the Discretization of LMI-synthesized Linear Parameter-Varying Controllers. *Automatica*, 33(4):655–661, 1997.

[2] P Apkarian and R J Adams. Advanced Gain-Scheduling Techniques for Uncertain Systems. *IEEE Trans. Control Syst. Technology*, 6(1):21–32, Jan 1998.

[3] P Apkarian and P Gahinet. A Convex Characterization of Gain-Scheduled H_∞ Controllers. *IEEE Trans. Automatic Control*, 40(5):853–863, May 1995.

[4] P Apkarian, P Gahinet, and G Becker. Self-scheduled H_∞ Control of Linear Parameter-Varying systems : A Design Example . *Automatica*, 31(9):1251–1261, 1995.

[5] P Gahinet, A Nemirovski, A J Laub, and M Chilali. *LMI Control Toolbox*. The MATH WORKS Inc, 1995.

[6] H Kajiwara, P Apkarian, and P Gahinet. Wide-Range Stabilization of an Arm-Driven Inverted Pendulum Using Linear Parameter-Varying Techniques. In *American Institute of Aeronautics and Astronautics Inc.*, pages 1853–1865, 1998.

[7] D A Lawrence and W J Rugh. Gain Scheduling Dynamic Linear Controllers for a Nonlinear Plant. *Automatica*, 31(3):381–390, 1995.

[8] P J McKerrow. *Introduction to Robotics*. Addison-Wesley Publishing Company, Inc, 1991.

[9] Y Nesterov and A Nemirovskii. *Interior-Point Polynomial Algorithms in Convex Programming*. Society for Industrial and Applied Mathematics, 1993.

[10] J S Shamma and M Athans. Guaranteed Properties of Gain Scheduled Control for Linear Parameter-varying Plants. *Automatica*, 27(3):559–564, 1991.

[11] J E Slotine and W Li. *Applied Nonlinear Control*. Prentice Hall, 1991.

[12] F Wu, X H Yang, A Packard, and G Becker. Induced L_2-Norm Control for LPV Systems with Bounded Parameter Variation Rates. *Int J. Robust and Nonlinear Control*, 6:983–998, 1996.