1. Find the derivative of the following formula :-

$$f(z) = \log_e(1+z), \text{ where, } z = x^T x, \; x \in \mathbb{R}^d$$

Solution :-

If $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix}$ then $x^T = [x_1 \; x_2 \; \ldots \; x_d]$

$$x^T x = [x_1^2 + x_2^2 + x_3^2 + \ldots \; x_d^2]$$

By chain Rule,

$$\frac{df}{dx} = \frac{df}{dz} \cdot \frac{dz}{dx}$$

$$= \frac{d}{dz}\{\ln(1+z)\} \; \frac{d}{dx}(x_1^2 + x_2^2 + \ldots + x_d^2)$$

$$= \frac{1}{1+z}(2x_1 + 2x_2 + \ldots + 2x_d)$$

$$= \frac{2}{1+z} \sum_{i=1}^{d} x_i$$

Ans

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② Find the derivative of the following formula:-

$$f(z) = e^{-z/2}, \text{ where } z = g(y)$$

$$g(y) = y^T S^{-1} y, \quad y = h(x)$$

$$h(x) = x - \mu$$

Solution

Chain Rule,

$$\frac{df}{dx} = \frac{df}{dz} \cdot \frac{dz}{dy} \cdot \frac{dy}{dx}$$

here, $\frac{df}{dz} = \frac{d}{dz}\left(e^{-z/2}\right) = \frac{-1}{2} e^{-z/2}$

$$\frac{dz}{dy} = \frac{d}{dy}\left(y^T S^{-1} y\right)$$

$$= \lim_{h \to 0} \frac{g(y+h) - g(y)}{h}$$

$$= \lim_{h \to 0} \frac{(y^T + h) S^{-1} (y+h) - y^T S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{(y^T S^{-1} + h S^{-1})(y+h) - y^T S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{y^T S^{-1} y + y^T S^{-1} h + h S^{-1} y + h^2 S^{-1} - y^T S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{h\left(y^T S^{-1} + S^{-1} y + h S^{-1}\right)}{h}$$

$$= \lim_{h \to 0} \left(y^T s^{-1} + s^{-1} y + h s^{-1} \right)$$

$$= y^T s^{-1} + s^{-1} y$$

$$\frac{dy}{dx} = \frac{d}{dx}(x - \mu) = 1$$

$$\therefore \frac{df}{dx} = -\frac{e^{-7/2}}{2} \cdot \left(y^T s^{-1} + s^{-1} y \right) \cdot 1$$

$$= -\frac{e^{-7/2}}{2} \frac{1}{s} \left(y^T + y \right)$$

Ans