# On the Perona-Malik approach to image processing

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# The aims

- eliminate disturbances
  i.e., filter out noise
- retain and enhance
  essential information

# Gaussian filtering

$\Omega \subset \mathbb{R}^n \quad (n = 2, 3)$ bounded domain

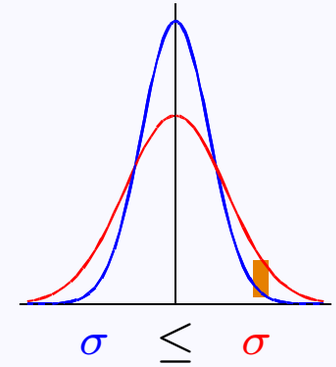
$u^0 : \Omega \mapsto [0, 1]$ grey level distribution of a noisy image

methods from signal transmission Marr & Hildreth (1980)

$$G_\sigma \star u^0(x) = \int_{\mathbb{R}^n} G_\sigma(x - y) u^0(y)\, dy, \quad x \in \mathbb{R}^n$$

$$G_\sigma(x) = (2\pi\sigma^2)^{-n/2} e^{-|x|^2/2\sigma^2}$$

$$u^0 \in L_1 \Longrightarrow G_\sigma \star u^0 \in C^\infty$$



# Frequency analysis

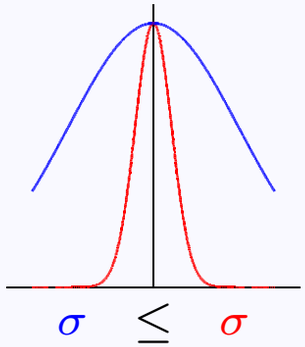
$$\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\, dx, \quad \xi \in \mathbb{R}^n$$

$$u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{u}(\xi)\, d\xi$$

$u(x)$ is a superposition of periodic oscillations of frequency $\sim |\xi|$

$$\widehat{G_\sigma \star u^0}(\xi) = \widehat{G_\sigma}(\xi)\widehat{u^0}(\xi)$$

$$\widehat{G_\sigma}(\xi) = e^{-\sigma^2|\xi|^2/2}$$




fast oscillations are cut out

low pass filter

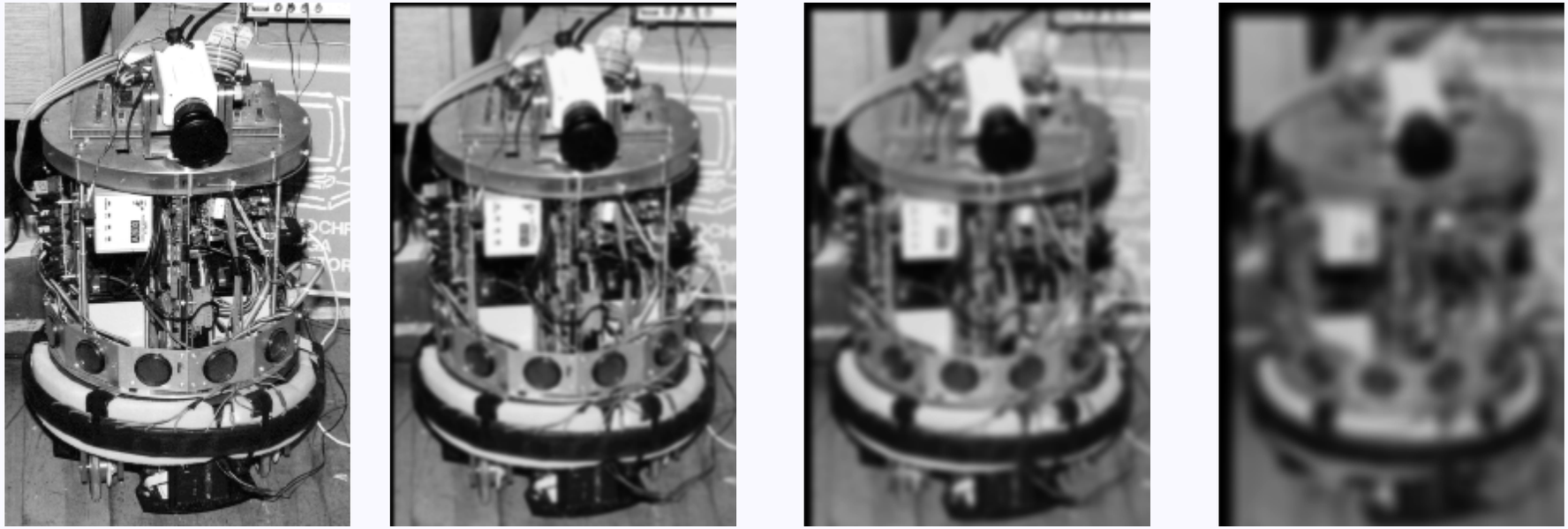
# Diffusion equation

$$
\begin{aligned}
u(x,t) &:= G_{\sqrt{2t}} \star u^0(x), \qquad && t \geq 0,\\
u(x,0) &= u^0(x)
\end{aligned}
$$

$$\Longleftrightarrow$$

$$
\begin{aligned}
\partial_t u &= \Delta u, \qquad && t \geq 0,\\
u(\cdot,0) &= u^0
\end{aligned}
$$

$G_\sigma \star u^0 = u(\cdot, \sigma^2/2)$ is embedded in a scale of smooth pictures



$\sigma = 1$ $\sigma = 2$ $\sigma = 4$

***HyperMedia Image Processing Reference***

# Anisotropic diffusion

$\partial_t u = \Delta u,$

$v(t) := u(at)$ $\quad a \gg 1$ strong smoothing,

$\quad a \ll 1$ weak smoothing,

$$\Longrightarrow \quad \partial_t v = a\Delta v = \nabla \cdot (a \nabla v)$$

anisotropic diffusion: $a = a(x)$

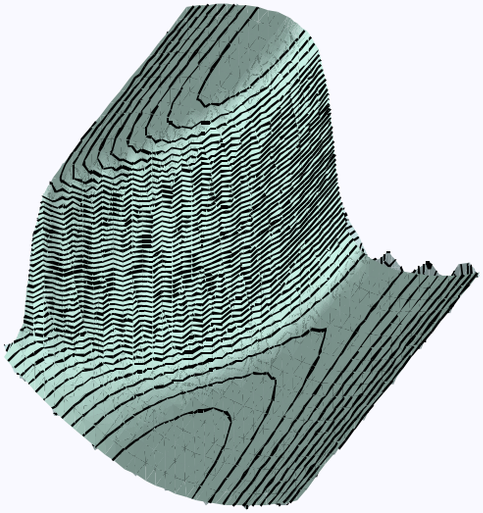
problem: where to put $a(x) \gg 1$ resp. $a(x) \ll 1$ ?

Perona & Malik, 1990 $\quad a = a(u) = g(|\nabla u|^2)$

$$a = g(|\nabla u|^2), \qquad g(s) > 0, \qquad g(s) \to 0 \quad (s \to \infty) \qquad g(s) = \frac{1}{1+s}$$

ideal

| | | | |
|---|---|---|---|
| $\lvert\nabla u(x)\rvert << 1$ | strong diffusion | $\Longrightarrow$ | smoothing |
| $\lvert\nabla u(x)\rvert >> 1$ | weak diffusion | $\Longrightarrow$ | preservation of edges |



# Perona-Malik equations

$$
\begin{aligned}
\partial_t u - \nabla \cdot \big(g(|\nabla u|^2)\nabla u\big) &= 0 && \text{on} \quad \Omega \times (0,\infty),\\
\partial_\nu u &= 0 && \text{on} \ \partial\Omega \times (0,\infty),\\
u(\cdot,0) &= u^0 && \text{on} \quad \Omega
\end{aligned}
$$

$$
g(s) = \frac{1}{1+s}
$$

## mathematical analysis

$$\nabla \cdot \big(g(|\nabla u|^2)\nabla u\big) = g(|\nabla u|^2)\Delta u + 2g'(|\nabla u|^2)D^2u\nabla u \cdot \nabla u$$

$S := S(t)$ a regular level set of $u(\cdot, t)$

near $S$:

$$g(|\nabla u|^2)\Delta u + 2g'(|\nabla u|^2)D^2u\nabla u \cdot \nabla u = g(|\nabla u|^2)\Delta_S u + h(|\nabla u|^2)\partial_\xi^2 u$$

$$h(s) := g(s) + 2sg'(s) = (1-s)/(1+s)^2$$

the Perona-Malik equations are ill-posed, in general

modifications

L. Wang and S. Zhou (2006), Zhaoyong Feng and Zhaoyang Yin (2009): $g \Longrightarrow \tilde{g}$, but $\tilde{h} \geq 0$

## space regularization

Catté, Lions, Morel, Coll (1992)

$u_\sigma := G_\sigma \star \widetilde{u}$

$G_\sigma$ the Gaussian with variance $\sigma$

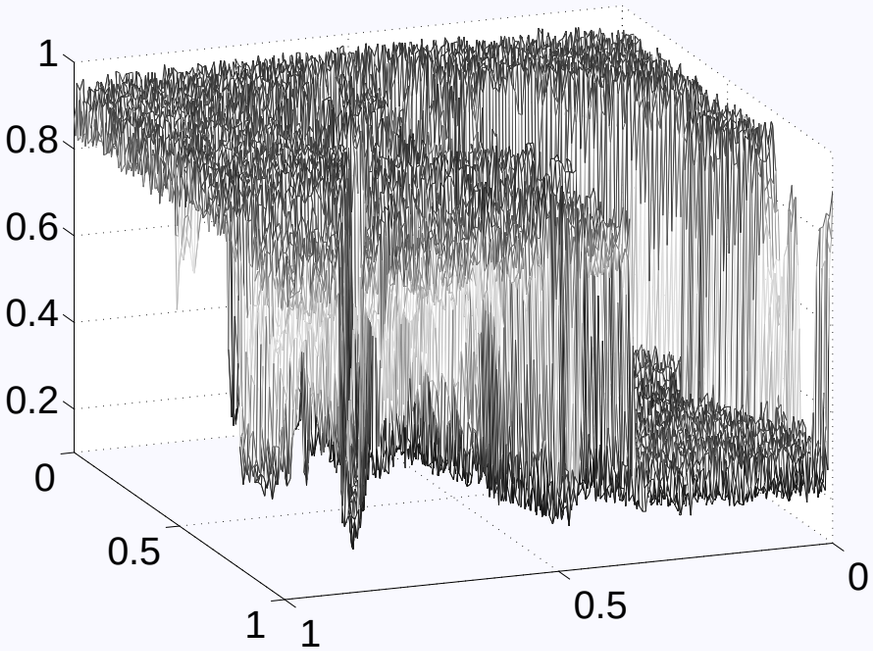
$\partial_t u - \nabla \cdot \big(g(|\nabla u_\sigma|^2)\nabla u\big) = 0$

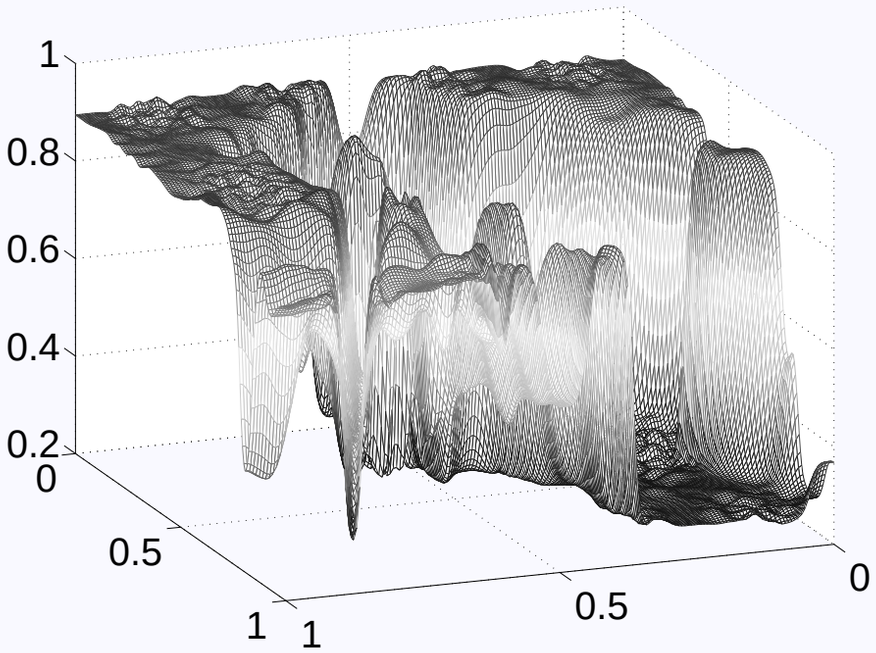
$\nabla u_\sigma = G_\sigma \star \nabla \widetilde{u}$ a weighted average of $\nabla u$

space regularization produces necessarily smearing of sharp edges

P. Guidotti (2007): regularizing by pseudo-differential operators

# numerical experiments






$$\delta = 10^{-2}, \quad k = 5, \quad u(\cdot\,, 0.05), \quad \sigma = 0.361, \quad t = 0.05 \sim \sigma^2/2$$

## why does PM – though mathematically ill-posed – produce extremely convincing results?

quasilinear evolution equation $\dot{u} + A(u)u = 0 \quad \text{in } (0,\infty), \qquad u(0) = u^0$

implicit (semi discrete) Euler scheme $\qquad u^k := u(k\delta)$

$$u^{k+1} - u^k + \delta A(u^{k+1})u^{k+1} = 0$$

$$u^{k+1} - u^k + \delta A(u^k)u^{k+1} = 0$$

induces a memory effect of the nonlinear term

$\theta := 1/2\delta$ on $(0, 2\delta)$, $\theta := 0$ otherwise

$\theta * |\nabla u|^2\,(t) = \frac{1}{2\delta}\int_{t-2\delta}^{t} |\nabla u(\tau)|^2\, d\tau \sim |\nabla u^k|^2$ midpoint rule

$$\partial_t u - \nabla \cdot \big(g(\theta * |\nabla u|^2)\nabla u\big) = 0$$

$$\sim \quad u^{k+1} - u^k + \delta\, \nabla \cdot \big(g(|\nabla u^k|^2)\nabla u^{k+1}\big) = 0$$

$$u^{k+1} - u^k + \delta A(u^k)u^{k+1} = 0$$

it cannot be decided whether the semidiscretization results from the local PM equation or from the time-delayed version.

## time regularization

$\theta \in L_\infty((0,S),\mathbb{R}^+)$

$$\partial_t u - \nabla \cdot \big(g(\theta * |\nabla u|^2)\nabla u\big) = 0$$

- nonlocal problem
- no approach via weak solutions possible, no variational structure,
- no maximum principle, viscosity solution approach fails

M. Nitzberg and T. Shiota (1992)

$$\partial_t u - \nabla \cdot \big(g(v)\nabla u\big) = 0$$
$$\partial_t v + \omega v = \omega |\nabla u|^2$$

well-posedness in Hölder spaces:

Belahmidi (2003), Belahmidi and Chambolle (2005)

variation of constants formula with $v(0) = 0$

$v(t) = \int_0^t \omega e^{-\omega(t-\tau)} |\nabla u|^2(\tau)\, d\tau \quad = \vartheta * (|\nabla u|^2), \qquad \vartheta := \omega e^{-\omega t}$

the Nitzberg-Shiota equations correspond to a time-delayed Perona-Malik equation where always the full history is taken into account

$$Q := \Omega \times (0,T), \qquad \mathcal{H}_p^{2,1}(Q) := L_p\big(J, H_p^2(\Omega)\big) \cap H_p^1\big(J, L_p(\Omega)\big)$$

$$u \in \mathcal{H}_p^{2,1}(Q) \Longleftrightarrow \int_0^T \int_\Omega \{|\partial_t u|^p + \sum_{i=0}^{2} |\nabla^i u|^p\}\, dxdt < \infty$$

$$v \in \mathcal{H}_p^{2,1}(Q) \hookrightarrow C^{1,0}(\overline{Q}), \qquad p > n+2$$

$$\partial_t u - \underbrace{g(\theta * |\nabla v|^2)}_{\in C(\overline{Q})} \Delta u = \underbrace{2g'(\theta * |\nabla v|^2)\, \theta * (D^2 v \nabla v) \cdot \nabla v}_{\in L_\infty(J, L_p(\Omega))}$$

$$\partial_t u - g(\theta * |\nabla v|^2)\Delta u = 2g'(\theta * |\nabla v|^2)\, \theta * (D^2 v \nabla v) \cdot \nabla v$$

$$\partial_\nu u = 0$$

$$u = u^0$$

has a unique solution $\qquad u = U(v) \in \mathcal{H}_p^{2,1}(Q)$

## well-posedness

Theorem

$\Omega$ smooth, $\quad p > n+2, \quad u^0 \in H^2_p(\Omega), \quad \partial_\nu u^0 = 0$

- There exists a unique local solution $u^* \in \mathcal{H}^{2,1}_p(Q)$ of the time-regularized PM equations
- It depends continuously on $u^0$, $\theta$, and $g$
- If $\mathrm{supp}(\theta)$ is in $[\sigma, S]$ for some $\sigma \in (0, S)$, then $u^*$ is global
- For $0 \le t < T_{\max}$, $\quad \min u^0 \le u^*(\cdot, t) \le \max u^0$

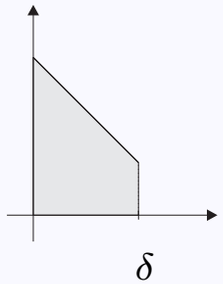
Proof based on a new general theorem on nonlocal evolution equations

## numerical experiments

$$\vartheta_\alpha(t) := 2(1+\alpha-\alpha t)/(2+\alpha)$$

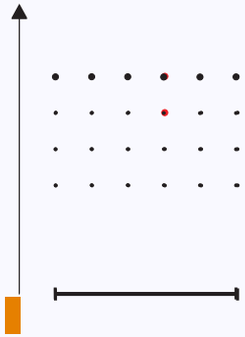
for $\quad 0 \le t \le 1, \qquad \alpha \ge 0$

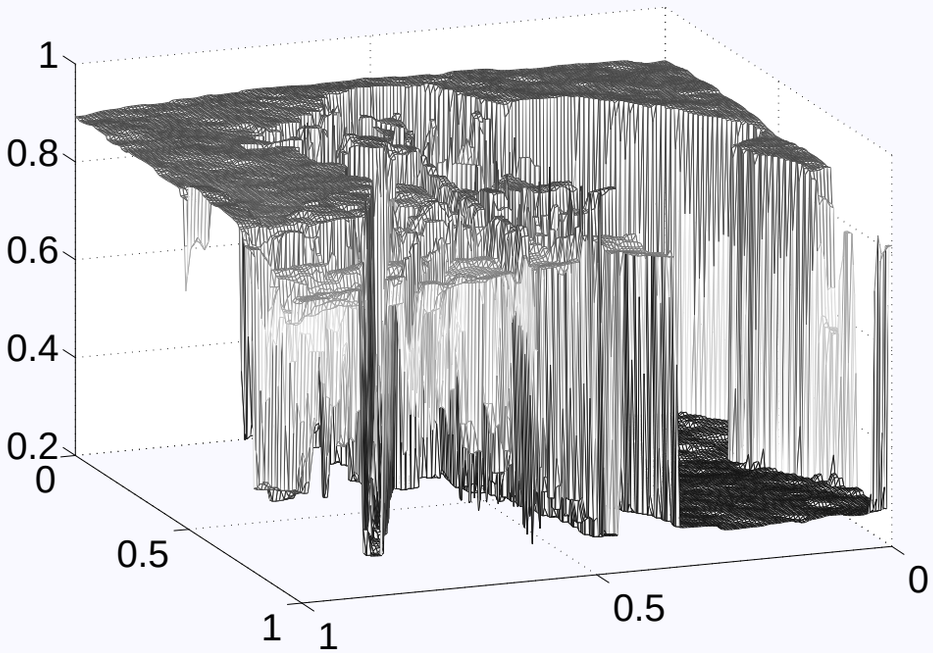
$$\theta(t) := \vartheta_\alpha(t/\delta)/\delta \qquad \text{for} \quad 0 \le t \le \delta$$



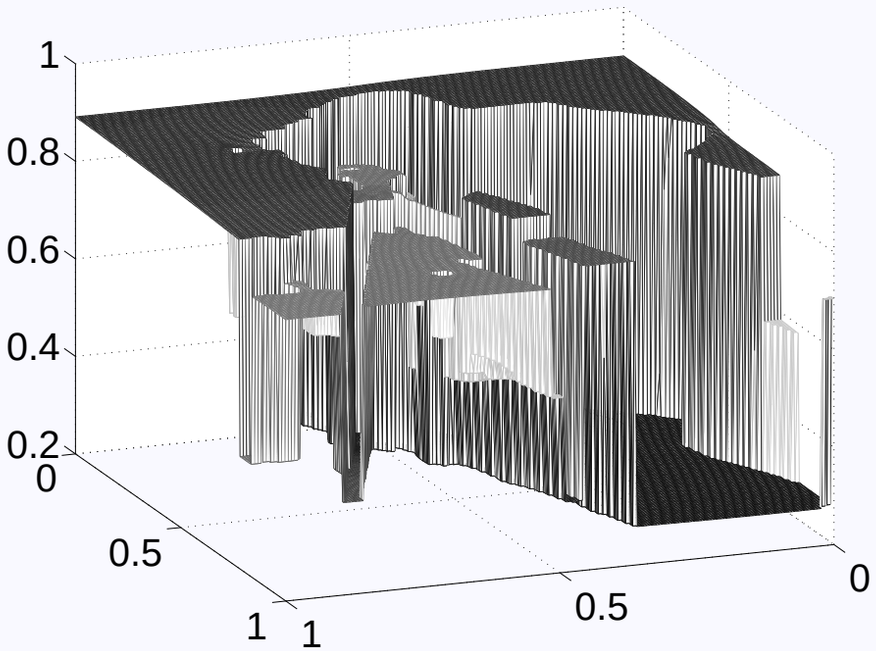

$$\theta * |\nabla u|^2(t) = \int_{t-\delta}^{t} \theta(t-\tau)\,|\nabla u(\tau)|^2\,d\tau$$

$$\sim \quad \frac{(1+\alpha)\,|\nabla u(t)|^2 + |\nabla u(t-\delta)|^2}{2+\alpha}$$

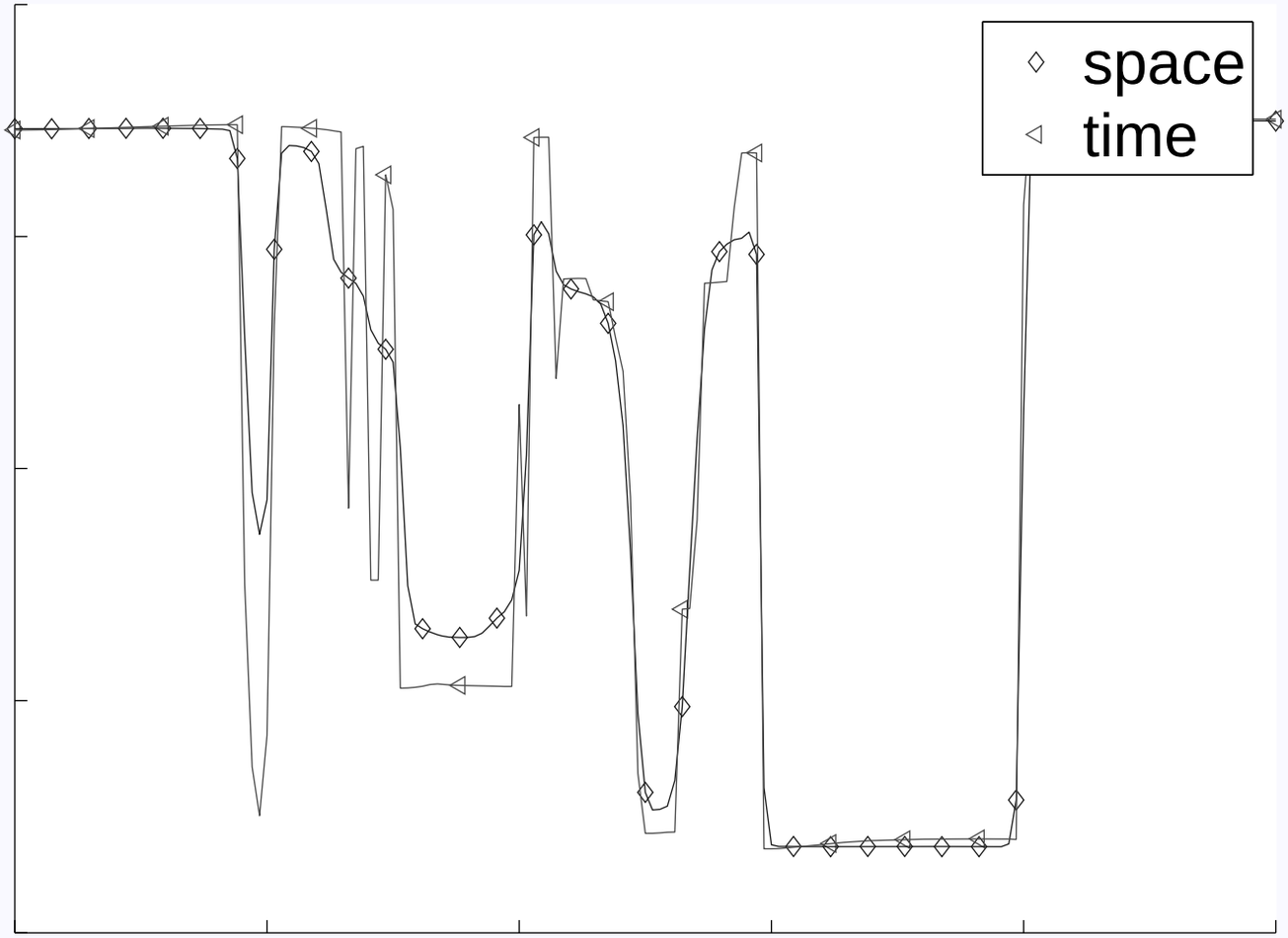





time regularized $\delta = 10^{-3}$ $k = 2$ $\alpha = 10$




time regularized $\delta = 10^{-3}$ $k = 50$ $u(\cdot\,, 0.05)$ $\alpha = 10$


space
time

## open problems

- gradient blow up

- domains with corners

  $\mathcal{H}_p^{2,1}(Q)$ has to be replaced by a space with weighted norms

- numerical approximation

  finite element, finite volume, or another method? error estimates?

# references

H. A. Time-delayed Perona Malik type problems
Acta Math. Univ. Comenianae, **76**, (2007), 15–38

H. A. Quasilinear parabolic problems via maximal regularity
Advances in Diff. Equ., **10** (2005), 1081–1110

www.math.unizh.ch/amann

implementation of the numerical schemes:

Jens Vogelgesang