MAE 5010 | DATA ASSIMILATION

HOMEWORK ASSIGNMENT 7

(Due: April 9th class time)

Feel free to use any computer language you like and you can use available packages (i.e., you do not have to write standard tools from scratch, you can use built-in packages).

Please report your findings clearly and concisely, and return via soft copy by email to osan@okstate.edu (you can embed code snaps into your report if you wish, or preferably you can provide GitHub links for your codes if it is easier for you).

Please study the materials provided in Module 8.3- Nonlinear Filtering

https://www.youtube.com/watch?v=HjYUu1JbtWo

Lecture notes can be found at Canvas.

There are 3 methods described in this module: zeroth-order filter (linearized Kalman filter), first-order filter (Extended Kalman Filter), and second-order filter (Gauss filter). Compare these 3 filters for the following example problem. Further details of the problem can be found in Julier, Uhlmann, Durrant-Whyte, IEEE Transactions on Automatic Control Volume 45, 2000. Digital copy of this paper can be also found at Canvas. (Note that our model problem is slightly modified, e.g., added gravity, down direction velocity is negative.)

Problem: Consider a nonlinear dynamics describing a falling body given by

$$\dot{x}_1 = x_2 + w_1$$

$$\dot{x}_2 = \frac{\rho_0}{2} \exp\left(\frac{-x_1}{k}\right) x_2^2 x_3 - g + w_2$$

$$\dot{x}_3 = w_3$$

where x_1 is the altitude, x_2 is the velocity and x_3 is the ballistic coefficient. $w_i \sim N(\mu, \sigma^2_i)$ is the process noise. ρ_0 density of air at sea level, g is the acceleration due to gravity and k is a constant that relates air density and altitude. Let

$$Z_k = [M^2 + (x_1(t_k) - a)^2]^{½} + v_k$$

Where a is the altitude of the range measuring instrument and M is the horizontal distance between the measuring device and the body and $v_i \sim N(0, \sigma^2_0)$

The constants are given by:

ρ_0 = 2 pounds – sec^2/ft^4, g = 32.2ft/sec^2,

k = 20,000, M = 100,000 ft, a = 100,000 ft, σ^2_0 = 10,000 ft^2

Let $\hat{x}_0$ = [300,000, -20,000, 0.001]^T

$$\hat{P}_0 = \begin{bmatrix} 10^6 & 0 & 0 \\ 0 & 4 \times 10^6 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

- Implement zeroth-, first- and second-order Kalman filters

(i) with $\sigma_i^2 \equiv 0$, i = 1,2,3

(ii) Pick arbitrary but small values for σ_i^2, $1 \leq i \leq 3$, e.g. $[0.1, 0.01, 0.001]^T$

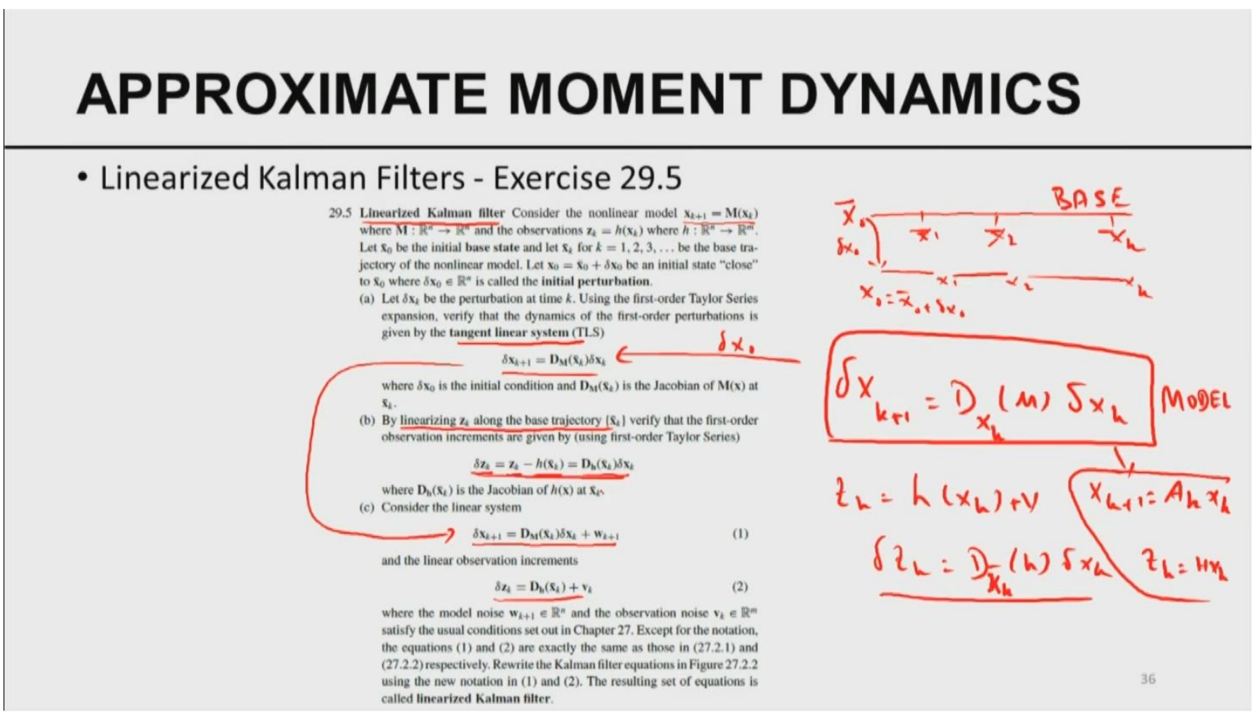
(iii) Compare the results

Note: You can use a second order Runge-Kutta scheme for time stepping. Maximum time can be set to Tmax = 60 sec. You can use $\Delta t = 1/64$ sec, and record your observations Z_k at every 64 steps (one observation per second). In your twin experiments, you can generate observations by running model starting from

$[x_1, x_2, x_3]^T = [300{,}000, -20{,}000, 0.00003]^T$

Although the initial estimates of altitude and velocity are correct, since x_3 is wrong in our initial analysis we should be able to correct our trajectories using variants of Kalman filters applied at every 64 iterations (when the observation is available).

Details of the algorithms are provided below:



APPROXIMATE MOMENT DYNAMICS

- First-order filter:
  - Set all second-moments to zero
  - Extended Kalman Filters

Model $\mathbf{x}_{k+1} = \mathbf{M}(\mathbf{x}_k) + \mathbf{w}_{k+1}$

Observation $\mathbf{z}_k = h(\mathbf{x}_k) + \mathbf{v}_k$

Forecast Step $\mathbf{x}^{\mathrm{f}}_{k+1} = \mathbf{M}(\widehat{\mathbf{x}}_k)$

$$\mathbf{P}^{\mathrm{f}}_{k+1} = \mathbf{D}_{\mathbf{M}}\widehat{\mathbf{P}}_k\mathbf{D}^{\mathrm{T}}_{\mathbf{M}} + \mathbf{Q}_{k+1}$$

Data Assimilation Step

$$\widehat{\mathbf{x}}_{k+1} = \mathbf{x}^{\mathrm{f}}_{k+1} + \mathbf{K}[\mathbf{z}_{k+1} - h(\mathbf{x}^{\mathrm{f}}_{k+1})]$$

$$\mathbf{K} = \mathbf{P}^{\mathrm{f}}_{k+1}\mathbf{D}^{\mathrm{T}}_{\mathbf{h}}[\mathbf{D}_{\mathbf{h}}\mathbf{P}^{\mathrm{f}}_{k+1}\mathbf{D}^{\mathrm{T}}_{\mathbf{h}} + \mathbf{R}_{k+1}]^{-1}$$

$$\widehat{\mathbf{P}}_{k+1} = (\mathbf{I} - \mathbf{K}\mathbf{D}_{\mathbf{h}})\mathbf{P}^{\mathrm{f}}_{k+1}$$

APPROXIMATE MOMENT DYNAMICS

- Second-order filter is.

Model $\mathbf{x}_{k+1} = \mathbf{M}(\mathbf{x}_k) + \mathbf{w}_{k+1}$

$X_0 \sim N(m_0, P_0)$

Observation $\mathbf{z}_k = h(\mathbf{x}_k) + \mathbf{v}_k$

$\boxed{\begin{array}{l} M(x) := M(x) \\ D_x(m) := M \\ D^2 M_i := 0 \end{array}}$

II - order ⇓ K.F

Forecast Step

$$\mathbf{x}^{\mathrm{f}}_{k+1} = \mathbf{M}(\widehat{\mathbf{x}}_k) + \overbrace{\tfrac{1}{2}\partial^2(\mathbf{M}, \widehat{\mathbf{P}}_k)}^{\mathrm{II}}$$

$$\mathbf{P}^{\mathrm{f}}_{k+1} = \mathbf{D}_{\mathbf{M}}\widehat{\mathbf{P}}_k\mathbf{D}^{\mathrm{T}}_{\mathbf{M}} + \mathbf{Q}_{k+1}$$

Data Assimilation Step

$$\widehat{\mathbf{x}}_{k+1} = \mathbf{x}^{\mathrm{f}}_{k+1} + \mathbf{K}[\mathbf{z}_{k+1} - h(\mathbf{x}^{\mathrm{f}}_{k+1}) - \overbrace{\tfrac{1}{2}\partial^2(h, \mathbf{P}^{\mathrm{f}}_{k+1})}^{\mathrm{II}}]$$

$$\mathbf{K} = \mathbf{P}^{\mathrm{f}}_{k+1}\mathbf{D}^{\mathrm{T}}_{\mathbf{h}}[\mathbf{D}_{\mathbf{h}}\mathbf{P}^{\mathrm{f}}_{k+1}\mathbf{D}^{\mathrm{T}}_{\mathbf{h}} + \mathbf{R}_{k+1}]^{-1}$$

$$\widehat{\mathbf{P}}_{k+1} = (\mathbf{I} - \mathbf{K}\mathbf{D}_{\mathbf{h}})\mathbf{P}^{\mathrm{f}}_{k+1}$$