Different recursion schemas and their properties

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Morphisms

- Catamorphisms (destruction)
- Anamorphisms (construction)
- Hylomorphisms (combination of two)
- Paramorphisms (saves more information than hylomorphisms)

Catamorphism

- "generalized fold"
  $b :: B$

$$h :: \text{List } A \to B$$

$$h\ \mathit{Nil} = b$$
$$h\ (\mathit{Cons}\ a\ \mathit{as}) = f\ a\ (h\ \mathit{as})$$

- Notation: $h = (\!| b, f |\!)$
- Arises from algebra ($f\ a \to a$).

Anamorphism

- "generalized unfold"
  $p :: B \to Bool$
  $g :: B \to (A, B)$

$$\begin{aligned} & h :: B \to List\ A \\ & h\ b = Nil, && p\ b \\ & h\ b = Consa\ (h\ b'), && otherwise \end{aligned}$$

  where $(a, b') = g\ b$
- Notation $h = [\![(g, p)]\!]$
- Arises from coalgebra $(a \to f\ a)$.

Hylomorphism

- "call-tree looks like data structure"
  $c :: \ C$
  $f :: B \to C \to C$
  $g :: A \to (B, A)$
  $p : \ A \to Bool$

$$
\begin{aligned}
h &:: A \to C \\
h\ a &= c, && p\ a \\
&= f\ b (h\ a'), && otherwise
\end{aligned}
$$

  where $(b, a') = g\ a$
- Notation $h = [\![(c, f), (g, p)]\!]$

Hylomorphism-2

- Is a composition of anamorphism and catamorphism
  $[\![(c, f), (g, p)]\!] = (\!| c, f |\!) \circ [\!(g, p)\!]$
- *Look at the whiteboard for a fancy call-tree image*
- Example: factorial is $[\![(1, \times), (g, p)]\!]$, where:

$$p\ n = n == 0$$
$$g\ n = (n, n - 1)$$

Paramorphism

- Hylomorphism for *fac* is not inductively defined on *nat*.
- A nat paramorphism example:

$$
\begin{aligned}
h\,0 &= b \\
h\,(\mathrm{Succ}\ n) &= f\ n\ (h\ n)
\end{aligned}
$$

- A list example:

$$
\begin{aligned}
h\,\mathrm{Nil} &= b \\
h\,(\mathrm{Cons}\ x\ xs) &= f\ x\ xs\ (h\ xs)
\end{aligned}
$$

- Notation: $\langle\!\![b, f]\!\!\rangle$

We will provide some explanations about the algebra and coalgebra nature.

Category

$C = (obj, hom, \circ)$
Objects and morphisms between objects with their compositions.
Laws:

- Identity morphisms for each object
- Composition is associative
- Path equality

"Point" is synonymous to "Object"

Categories: *Set*

- $ob(Set)$ – all sets
- $hom(E, F)$ – functions between sets E and F
- $\circ$ – composition

Categories: *Set*

- $ob(Set)$ – all sets
- $hom(E, F)$ – functions between sets E and F
- $\circ$ – composition
- Note: $ob(Set)$ is not a set itself (is a class).

Categories

- *Mon*: (monoids, morphisms, composition)
- *Grp*: (groups, morphisms, composition)
- *Hask*: (haskell types, functions, `(.)`)
- ...

Functor

- Functor is a morphism between categories (preserves structures)

Functor in FP

- an (endo-)functor is an operation from types to types (from Hask to Hask)
- preserves identity and composition.
- functions can be 'mapped over' functors
- Basic ones: identity, product, sum (tagged), arrow ...

A usual recursive datatype

```
data Expr = Const Int
    | Add Expr Expr
    | Mul Expr Expr
```

is same as:

```
newtype Fix f = Fx (f (Fix f))

data ExprF a = Const Int | Add a a | Mul a a

type Expr = Fix ExprF
```

Test

```
testExpr = Fx $ (Fx $ (Fx $ Const 2 ) `Add` (Fx $ Const 3))
 `Mul` (Fx $ Const 4)
```

It is a functor!

```
instance Functor ExprF where
    fmap eval (Const i) = Const i
    fmap eval (Add x y) = Add (eval x) (eval y)
    fmap eval (Mul x y) = Mul (eval x) (eval y)
```

It is possible to construct an algebra on top of any functor.

```
type Algebra f a = f a → a
```

We expect to be able to 'evaluate' children of `Expr`. Example:

```
alg':: ExprF Int → Int
alg' (Const i) = i
alg' (Add x y) = x + y
alg' (Mul x y) = x ∗ y
```

What is an algebra?

(C, F, A, m)

- Category C (*Hask*, points are types of Haskell)
- Endofunctor F (maps Hask points into other Hask points)
- Point A (**carrier**) of category C (some type).
- Function m mapping $F\ A \to A$

Hence our definition:

```
type Algebra f a = f a → a
```

C is implied (*Hask*), F and A are type-level, the rest is m, the function itself (of type `Algebra f a`).
Look at the whiteboard for a fancy image!

Initial algebra

- There are infinitely many algebras based on same functor
- One is particular **Initial algebra**,
- Does not 'forget' anything, preserves all information about input.
- $\exists$ a homomorphism from initial algebra to any other algebra.
- Assume: carrier is `Fix f`, algebra is `Fx` (its ctor).

```
type Algebra f a = f a → a

type ExprInitAlgebra = Algebra ExprF (Fix ExprF)
ex_init_alg :: ExprF (Fix ExprF) → Fix ExprF
ex_init_alg = Fx
```

Functor *f* is fixed. Let *a* be the carrier object for a new algebra.

	Carrier	Evaluator
Initial algebra	Fix f	Fx
Some algebra	a	alg

Constructing any algebra-1

We want to get a homomorphism from initial algebra to some other algebra. Carrier mapper:

`g:: Fix f → a`

Remember:

```
newtype Fix f = Fx (f (Fix f))
```

Thanks to the fact that f is a functor, fmap is at our disposal to map :

`fmap g :: f (Fix f) → f a`

$$\begin{array}{ccc} f(\text{Fix } f) & \xrightarrow{\text{fmap } g} & f\ a \\ \downarrow Fx & & \downarrow alg \\ \text{Fix } f & \xrightarrow{g} & a \end{array}$$

Constructing any algebra-2

`Fx` is lossless, thus invertible.

```
unFix :: Fix f → f (Fix f)
unFix (Fx x) = x
```

$$
\begin{tikzcd}
f(\text{Fix } f) \arrow[r, "\text{fmap } g"] \arrow[d, "Fx"] & f\ a \arrow[d, "alg"] \\
\text{Fix } f \arrow[r, "g"] & a
\end{tikzcd}
$$

becomes

$$
\begin{tikzcd}
f(\text{Fix } f) \arrow[r, "\text{fmap } g"] & f\ a \arrow[d, "alg"] \\
\text{Fix } f \arrow[r, "g"] \arrow[u, "unFix"] & a
\end{tikzcd}
$$

g can be defined with unfix, fmap and evaluator:

```
g = alg . (fmap g) . unFix
```

Meet catamorphism

```
g = alg . (fmap g) . unFix
```

```
cata :: Functor f ⇒ (f a → a) → (Fix f → a)
cata alg = alg . fmap (cata alg) . unFix
```

Quick check:

```
alg :: ExprF String → String
alg (Const i) = [ chr (ord 'a' + i)]
alg (Add x y) = x ++ y
alg (Mul x y) = concat [[a,b] | a <− x, b <− y]
```

```
*Main> :t cata alg
cata alg :: Fix ExprF -> String
```

Coalgebra

Algebra: $f\ a \to a$
Coalgebra: $a \to f\ a$

$$
\begin{tikzcd}
f(\mathrm{Fix}\ f) \arrow[r, "\mathrm{fmap}\ g"] \arrow[d, "Fx"] & f\ a \arrow[d, "alg"] \\
\mathrm{Fix}\ f \arrow[r, "g"] & a
\end{tikzcd}
\quad \text{becomes} \quad
\begin{tikzcd}
f(\mathrm{Fix}\ f) & f\ a \arrow[l, "\mathrm{fmap}\ g"'] \\
\mathrm{Fix}\ f \arrow[u, "unFix"] & a \arrow[l, "g"'] \arrow[u, "coalg"]
\end{tikzcd}
$$

The same reasoning about initial coalgebra applies.

Coalgebra-2

μ combines Fx and unFix.

```
newtype Mu f = In {out :: f (Mu f)}

type CoAlgebra f a = a → f a

type Algebra f a = f a → a

catam alg = alg . fmap (catam alg) . out

anam :: Functor f ⇒ CoAlgebra f a → (a → Mu f)
anam coalg = In. fmap (anam coalg) . coalg
```

Coalgebra-3

As *out* is an initial algebra, *in* is a terminal coalgebra (there exists a unique homomorphism from any coalgebra to *in*).

$$\begin{array}{ccc} f(\mu\ f) & \xleftarrow{\text{fmap } g} & f\ a \\ \Big\downarrow{\scriptstyle in} & & \Big\uparrow{\scriptstyle coalg} \\ \mu\ f & \xleftarrow{g} & a \end{array}$$

Look at the whiteboard for a combined image.

Using fixed point

We can define catamorphism and anamorphism in a non-recursive way.

```
mu f = f (mu f)

cata_fix :: Functor f ⇒ Algebra f a → (Mu f → a)
cata_fix = mu (\f alg→  alg . fmap (f alg) . out)

anam_fix :: Functor f ⇒ CoAlgebra f a → (a → Mu f)
anam_fix = mu (\f coalg→  In. fmap (f coalg) . coalg )
```

A note on notation

Authors use this rather unusual notation:
$(f \stackrel{F}{\leftarrow} g)h = f \circ h_F \circ g$
h_F is lifted inside the functor F.
Thus: $(f \stackrel{F}{\leftarrow} g) = \lambda h.f \circ h_F \circ g$
Also:
$(f \Delta g)(x, y) = (f\ x, g\ y)$

Notation summary

- Catamorphism:
  $(\!|\phi|\!)_F = \mu(\phi \overset{F}{\leftarrow} \mathit{out})$

  `cata_fix = mu (\f alg→ alg . fmap (f alg) . out)`

- Anamorphism:
  $[\!(\psi)\!]_F = \mu(\mathit{in} \overset{F}{\leftarrow} \psi)$

  `anam_fix = mu (\f coalg→ In. fmap (f coalg) . coalg )`

- Hylomorphism:
  $[\![\phi, \psi]\!]_F = \mu(\phi \overset{F}{\leftarrow} \psi)$
- Paramorphism:
  $\langle\![\mathcal{E}]\!\rangle = \mu(\lambda f.\mathcal{E} \circ (\mathit{id}\ \Delta\ f)_F \circ \mathit{out})$

Remember: $(f \overset{F}{\leftarrow} g) = \lambda h.f \circ h_F \circ g$

For each morphism type we give:

- Evaluation rule
- Uniqueness property
- Fusion law (class-preserving functions mapping over morphism)

Utility: fixed point fusion

$f(\perp) = \perp \wedge f \circ g = h \circ f \Rightarrow f(\mu\ g) = \mu\ h$ (without proof)

Catamorphism

- Evaluation rule:

$$(\!|\phi|\!) \circ in = \phi \circ (\!|\phi|\!)$$

  Apply recursively, then ϕ again. Compare with fixed point:

$$x = \mu f \Rightarrow x = f\ x$$

- **Uniqueness Property** (prove functions equality without explicit induction)
  $f = (\!|\phi|\!)_F \equiv f \circ \perp = (\!|\phi|\!)_F \circ \perp \wedge f \circ in = \phi \circ f_{\mu\ F}$
  Remember, `In` is also a type ctor, applying isomorphism between *a* and *Fa*

Catamorphisms - Fusion law

Blend $cata \circ f$ into a single catamorphism.

$$f \circ (\!|\phi|\!) = (\!|\psi|\!) \Leftarrow f \perp = \perp \wedge f \circ \phi = \psi \circ f_{\mu F}$$

Look at the whiteboard for a fancy diagram

Useful variation: f is strict, not “like $(\!|\psi|\!)$”:

$$f \circ (\!|\phi|\!) = (\!|\psi|\!) \Leftarrow f \circ \perp = (\!|\psi|\!) \circ \perp \wedge f \circ \phi = \psi \circ f_{\mu F}$$

Injective functions are catamorphisms

$f : A \to B$
$\phi : A_F \to A$
Then:

$$f \circ (\!|\phi|\!) = (\!|f \circ \phi \circ g_F|\!) \Leftarrow \begin{cases} f\perp = \perp \\ g \circ f = id \end{cases}$$

Take $(\!|\phi|\!) = in$:

$$f = (\!|f \circ in \circ g_F|\!) \Leftarrow \begin{cases} f\perp = \perp \\ g \circ f = id \end{cases}$$

Catamorphisms preserve strictness

A useful lemma for some proofs:

$$\mu F \circ \perp = \perp \Leftarrow \forall f : Ff \circ \perp = \perp$$

For cata:
$(\!|\phi|\!)_F \circ \perp = \perp \equiv \phi \circ \perp = \perp$

Example: fold-unfold

Whiteboard

Anamorphisms

Evaluation rule:

$$\mathit{out} \circ [\!(\psi)\!] = [\!(\psi)\!]_F \cdot \psi$$

Apply $[\!(\psi)\!]$, then apply $[\!(\psi)\!]_F$ to the result.

$$f = [\!(\psi)\!] \equiv \mathit{out} \circ f = f_F \cdot \psi$$

Fusion law

$$\llbracket\phi\rrbracket \circ f = \llbracket\psi\rrbracket \Leftarrow \phi \circ f = f_F \circ \psi$$

Proof by fixed point fusion theorem with $f := (\circ f)$, $g := in \leftarrow^F \phi$, $h := in \leftarrow^L \psi$

Surjective function is an anamorphism

Examples

$iterate f = [\!(firstOfSum \circ id \Delta f)\!]$
$takewhile\ p =$
$[\!(secondOfSum\ 'map'\ (VOID | id \circ (not\ p \circ second)?) \circ out]\!]$
$f' map' g \circ p?$ models if-then-else

“Actually, even in Haskell recursion is not completely first class because the compiler does a terrible job of optimizing recursive code. This is why F-algebras and F-coalgebras are pervasive in high-performance Haskell libraries like vector, because they transform recursive code to non-recursive code, and the compiler does an amazing job of optimizing non-recursive code.” (Gabriel Gonzales)

Further reading:

- Control.Functor.Algebra