Multivariate Continuous Distributions

$\boxed{\text{REVIEW}}$

Joint CDF

For arbitrary RVs X, Y, the joint CDF is defined

$$F_{X,Y}(x,y) = P(X \le x, Y \le y) = P(\{x : X \le x\} \cap \{y : Y \le y\})$$

and from the joint CDFs, you can derive the marginal CDFs.

$$F_X(x) = \lim_{y \to \infty} F_{X,Y}(x,y) = F_{X,Y}(x,\infty)$$

$$F_Y(y) = \lim_{x \to \infty} F_{X,Y}(x,y) = F_{X,Y}(\infty, y)$$

Joint PDFs

Often, you deal with X, Y where X and Y are dependent. Then, it may be more practical to use joint PDFs, defined as $f_{X,Y}$ where

$$P((X,Y) \in B) = \iint_B f_{X,Y}(x,y)\,dx\,dy$$

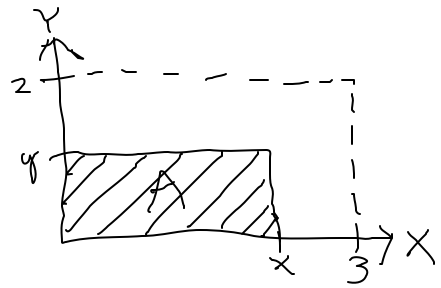
for any $B \in \mathbb{R}^2$.

Properties

- $f_{X,Y}(x,y) \ge 0 \quad \forall x,y \in \mathbb{R}$
- $\iint_{\mathbb{R}^2} f_{X,Y}(x,y)\,dx\,dy = 1$
- $F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(s,t)\,dt\,ds, \quad \forall x,y \in \mathbb{R}$
- $f_{X,Y}(x,y) = \dfrac{\partial^2 F_{X,Y}(x,y)}{\partial x\,\partial y} \quad \forall x,y$ (if derivative exists)
- $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$ (marginal PDFs)

Example

Consider X, Y as 2D Uniform over $[0,3]\times[0,2]$



Then for $A \subseteq [0,3]\times[0,2]$

$$P(A) = \frac{\text{Area}(A)}{\text{Area}([0,3]\times[0,2])} = \frac{\text{Area}(A)}{6}$$

Thus $$F_{X,Y}(x,y) = \begin{cases} 0 & : x<0 \text{ or } y<0 \\ \frac{xy}{6} & : 0\le x\le 3,\ 0\le y\le 2 \\ \frac{x}{6} & : 0<x\le 3,\ y>2 \\ \frac{y}{6} & : x>3,\ 0<y\le 2 \\ 1 & : x>3,\ y>2 \end{cases}$$

and $F_X(x) = \lim_{y\to\infty} F(x,y) = \frac{x}{6}$, $F_Y(y) = \lim_{x\to\infty} F_{X,Y}(x,y) = \frac{y}{6}$

Now, consider calculating $f_{X,Y}$

For $0\le x\le 3$, $0\le y\le 2$

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x\partial y}F_{X,Y}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{xy}{6}\right)\right) = \frac{\partial}{\partial x}\left(\frac{x}{6}\right) = \frac{1}{6}$$

Note that for other intervels (i.e. $(x,y)\notin[0,3]\times[0,2]$), $\frac{\partial^2}{\partial x\partial y}F_{X,Y}(x,y) = 0$ in this example. So, we heve

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{6} : & 0\le x\le 3,\ 0\le y\le 2 \\ 0 : & \text{otherwise} \end{cases}$$

and $$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^2 \frac{1}{6}\,dy = \frac{1}{6}y\Big|_0^2 = \frac{1}{3}, \quad 0\le x\le 3$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_0^3 \frac{1}{6}\,dx = \frac{1}{6}x\Big|_0^3 = \frac{1}{2}, \quad 0\le y\le 2$$

Conditional PDFs

For X, Y, the conditional PDF for X given $Y=y$ is defined

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$ (remember, it is a function of X)

Properties

- $f_{X|Y}(x|y) \geq 0$
- $\int_{\mathbb{R}} f_{X|Y}(x|y)\, dx = 1, \quad \forall y \in \mathbb{R}$
- $P(X \in A | Y = y) = \int_A f_{X|Y}(x|y)\, dx, \ \forall A \subseteq \mathbb{R}$

  $\hookrightarrow F_{X|Y}(x|y) = P(X \leq x | Y = y) = \int_{-\infty}^{x} f_{X|Y}(t|y)\, dt$
- $f_{X|Y}(x,y) = f_X(x) f_{Y|X}(y|X) \quad \forall x, y \in \mathbb{R}$

Independence

X, Y are independent if $\forall A, B \subseteq \mathbb{R}$

$$P(X \in A, Y \in B) = P(X \in A) P(Y \in B)$$

For continuous RVs, it follows that

- $F_{X,Y}(x,y) = F_X(x) F_Y(y)$
- $f_{X,Y}(x,y) = f_X(x) f_Y(y)$

Sums of RVs

Suppose you have RVs X, Y, with the joint CDF and $Z = X+Y$.

$$F_Z(z) = P(Z \le z) = P(X+Y \le z) = P(Y \le z - X) = \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f_{X,Y}(x,y)\,dy\,dx$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x)\,dx = \int_{-\infty}^{\infty} f_{X,Y}(z-y, y)\,dy$$

Q & A

Q3. $$f_{X,Y}(x,y) = \begin{cases} c(x^2+y) & : -1 \le x \le 1,\ 0 < y < 1-x^2 \\ 0 & : o/w \end{cases}$$

want c

$P(0 \le X \le 1/2)$, $P(Y \ge X)$, $P(Y = X^2)$

$$\iint_{\mathbb{R}^2} f_{X,Y}(x,y) = \int_{-1}^{1}\int_{0}^{1-x^2} c(x^2+y)\,dy\,dx = 1$$

$$= \int_{-1}^{1} c\,\tfrac{1}{2}y^2 \Big|_0^{1-x^2}$$

$$= \frac{c}{2}\int_{-1}^{1}(1-x^2)^2$$

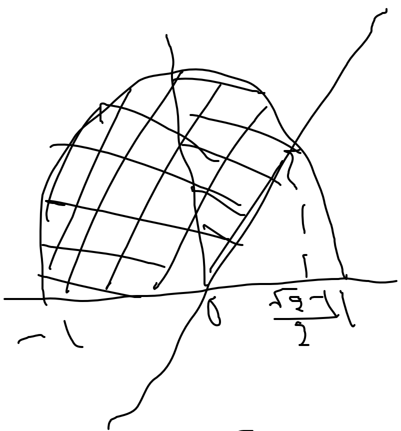
$$= \quad c = \frac{5}{4}$$

$$P\left(0 \le X \le \tfrac{1}{2}\right) = \int_0^{\frac{1}{2}} \int_0^{1-x^2} \frac{5}{4}(x^2+y)\,dy\,dx$$

$$= \frac{5}{2\times 4}\int_0^{\frac{1}{2}} (1-x^2)^2$$

$$= \frac{79}{256}$$

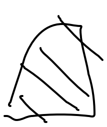
c)

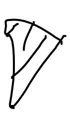


$Y > X$

$y = x$

$y = 1 - x^2$

$\int$  $= \int_{-1}^{0} \int_0^{1-x^2} \frac{5}{4}(x^2+y)\,dy\,dx$

$\int$  $= \int_0^{\frac{\sqrt{5}-1}{2}} \int_x^{1-x^2} \frac{5}{4}(x^2+y)\,dy\,dx$

14. Dealing w/ 1 hour

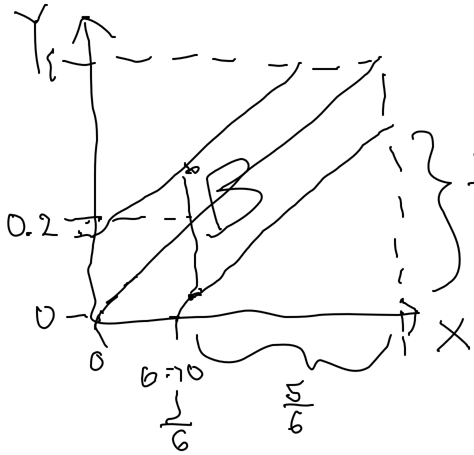
X - time of one friend

Y - time of other friend

X, Y are independent

$X, Y \sim \text{Uniform}(0,1)$

$\{(x,y) \in [0,1]^2 : |x-y| \le \frac{1}{6}\}$



$\frac{1-\left(\frac{5}{6}\right)^2 \cdot \cancel{2}}{\cancel{2}}$

$P(B) = 1-\left(\frac{5}{6}\right)^2$

10. stick w/ length >0

break point — $X \sim \text{Uniform}(0, l)$

after breaking from X - break again — Y

$Y|X = x \sim \text{Uniform}(0, x)$

$f_{X,Y}(x,y) = f_{Y|X}(y|x) f_X(x)$

$= \begin{cases} \frac{1}{x} & ; 0<y<x \\ 0 & ; o/w \end{cases} \times \begin{cases} \frac{1}{l} & ; 0<x<l \\ 0 & ; o/w \end{cases}$

$= \begin{cases} \frac{1}{xl} & ; 0<y<x<l \\ 0 & ; o/w \end{cases}$