Continuous Random Variables

RV X is called continuous if

- $P(X=x)=0, \forall x \in \mathbb{R}$
- $P(S)=1$

This means there are an uncountable number of values.

Intuition

Consider the sequence $\{a_n\} = \{1, 2, 3, 4, \dots, n, \dots\}$

Note that you would be able to track each element in the sequence.

This is a countably infinite sequence of numbers.

Now, consider interval $B = [0, 1]$

You would not be able to track each element in this interval.
(What is the smallest real number greater than 0?)

This is an uncountable interval of numbers.

Since $P(X=a)=0$ for continuous RV X, we are only interested in the probabilities of intervals of X.
(i.e/ $P(X \in (a, b]) = P(a < X \leq b), \forall a, b \in \mathbb{R}$)

Probability Density Function (PDF)

For continuous RV X, it's PDF is defined as $f_X(\cdot)$, where

$$P(a \leq X \leq b) = \int_a^b f_X(x)\,dx$$

Properties

- $f_X(x) \geq 0 \quad \forall x \in \mathbb{R}$ (but $f_X(x)$ can be greater than 1)
- $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$ (i.e/ $P(-\infty < X < \infty) = P(S) = 1$)
- $F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$ (Cumulative Distribution Function) $= P(X \leq x)$

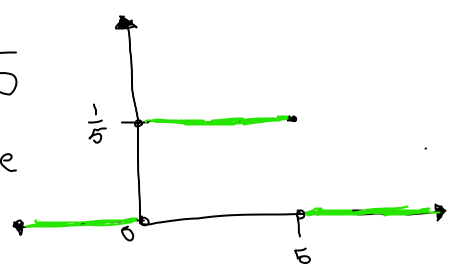
  $\hookrightarrow f_X(x) = \frac{d}{dx} F_X(x) = F_X'(x)$

Example

Let $X \sim \text{Uniform}(0,5)$ $\left(f_X(x) = \begin{cases} \frac{1}{5-0}, & 0 < X < 5 \\ 0, & \text{otherwise} \end{cases} \right)$

Then the PDF of X is

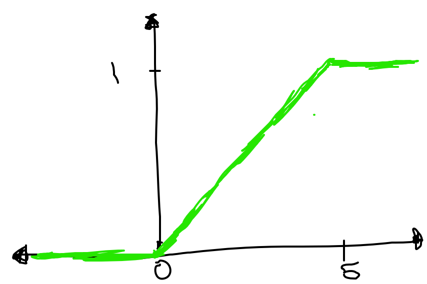
$$f_X(x) = \begin{cases} \frac{1}{5}, & 0 \leq x \leq 5 \\ 0, & \text{otherwise} \end{cases}$$



$$P(1 \leq X \leq 2) = \int_1^2 \frac{1}{5}\,dx = \left(\frac{1}{5}x\right)\Big|_1^2 = \frac{1}{5}$$

and the CDF is

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{5}, & 0 \leq x \leq 5 \\ 1, & x > 5 \end{cases}$$



$$P(1 \leq X \leq 2) = F_X(2) - F_X(1)$$
$$= \frac{2}{5} - \frac{1}{5}$$
$$= \frac{1}{5}$$

Change of Random Variable

For $B \subseteq \mathbb{R}$, let X be random variable, and $Y = h(X)$

Then $P(Y \in B) = P(X \in h^{-1}(B))$

where $h^{-1}(B) = \{x \in \mathbb{R} : h(x) \in B\}$

Example

$S = \{-2, -1, 0, 1, 2\}$

$X(s) = s$, $Y = X^3$

$(h(x) = x^3)$

$(h^{-1}(x) = \sqrt[3]{x})$

$$P(Y \in [8, \infty)) = P(X \in h^{-1}([8, \infty))) = P(X \in [2, \infty)) = \frac{1}{5}$$

Change of RVs using CDFs

For continuous, one-to-one, strictly increasing function h, where $Y = h(X)$

$$F_Y(y) = P(Y \leq y) = P(X \leq h^{-1}(y)) = F_X(h^{-1}(y))$$

NOTE: be sure to check for the function being one-to-one

Change of RVs Using PDFs

For continuous RV X, and continuous, one-to-one strictly increasing function h, let $Y = h(X)$

Then $f_Y(y) = \dfrac{f_X(h^{-1}(y))}{|h'(h^{-1}(y))|}$ (h' is the 1st derivative of h)

Example

Let $X \sim \text{Uniform}(0,3)$, $Y = 6X$ $\left(h(X) = 6X;\ h^{-1}(Y) = \frac{Y}{6}\right)$

Note that h is one-to-one, continuous, and strictly increasing.

First, we will find F_Y using F_X, $\left(\text{We have } F_X(x) = \begin{cases} 0 : x < 0 \\ x/3 : 0 \le x < 3 \\ 1 : x \ge 3 \end{cases}\right)$

$$F_Y(y) = P(Y \le y) = P(X \le h^{-1}(y)) = P\left(X \le \frac{Y}{6}\right) = \begin{cases} 0 : y/6 < 0 \\ y/18 : 0 \le y/6 < 3 \\ 1 : y/6 \ge 3 \end{cases} = \begin{cases} 0 : y < 0 \\ y/18 : 0 \le y < 18 \\ 1 : y \ge 18 \end{cases}$$

Now, we will find f_Y using f_X $\left(\text{We have } f_X(x) = \begin{cases} 1/3 : 0 \le x \le 3 \\ 0 : \text{otherwise} \end{cases}\right)$

$h'(x) = 6$

$$f_Y(y) = \frac{f_X(h^{-1}(y))}{|h'(h^{-1}(y))|} = \begin{cases} \frac{1}{6} \cdot \frac{1}{3} : 0 \le y/6 \le 3 \\ \frac{1}{6} : \text{otherwise} \end{cases} = \begin{cases} \frac{1}{18} : 0 \le y \le 18 \\ 0 : \text{otherwise} \end{cases}$$

Q&4

Q6 #1

X, Y, $Var(X) = 9$

$Var(Y) = 4$

$Corr(X, Y) = -0.5$

$$Z = X + tY$$

$$Var(Z) = Var(X) + t^2 Var(Y) + 2t\,Cov(X, Y)$$

$$\frac{d}{dt} Var(Z) = 2t\,Var(Y) + 2\,Cov(X, Y)$$

Zero the left hand side

$$2t\,Var(Y) = -2\,Cov(X, Y)$$

$$t = -\frac{Cov(X, Y)}{Var(Y)}$$

$$= -\frac{3}{4}$$

$$\frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} = Corr(X,Y)$$

$$Cov(X,Y) = Corr(X,Y)\,SD(X) \times SD(Y)$$

$$= -0.5 \times \sqrt{9} \times \sqrt{4}$$

$$= -3$$

2. $f_X(x) = e^{-x} \exp(-e^{-x})$.

- show $f_X(x) \geq 0$ (product of two positive fxns)
- $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{\infty} e^{-x} \exp(-e^{-x})\,dx$$
$$= \exp(-e^{-x}) \Big|_{-\infty}^{\infty}$$
$$= \exp(-e^{-\infty}) - \exp(-e^{\infty})$$
$$= \exp(0) - \exp(-\infty)$$
$$= 1 - 0 = 1$$

3. $F_X(x) = \begin{cases} 1 - e^{-x^2/2} & , x > 0 \\ 0 & , x \leq 0 \end{cases}$

- show $F_X(x)$ is a increasing function $F'(x) \geq 0$
- $\lim_{x \to \infty} F_X(x) = 1$
- $\lim_{x \to -\infty} F_X(x) = 0$

$$F'(x) = \frac{d}{dx}\left(1 - e^{-x^2/2}\right) = +e^{-x^2/2} \cdot \frac{+2x}{2} = xe^{-x^2/2} > 0 \quad \forall x$$

$$\lim_{x \to \infty} F_X(x) = 1 - \lim_{x \to \infty} e^{-x^2/2} = 1 - 0 = 1$$

$$\lim_{x \to -\infty} F_X(x) = 0$$

10. $X \sim \text{Exponential}(\lambda)$

$Y = X^{1/4}$ $\quad (h(X) = x^{1/4})$

$h(X)$ easy to see that its continuous, one-to-one, and strictly increasing

$$f_Y(y) = \frac{f_x(h^{-1}(y))}{|h'(h^{-1}(y))|} = \frac{\lambda e^{-\lambda y^4}}{\frac{1}{4}(y^4)^{-\frac{3}{4}}} = \frac{\lambda e^{-\lambda y^4}}{\frac{1}{4} y^{-3}} = 4\lambda y^3 e^{-\lambda y^4}$$

$h' = \frac{1}{4} x^{-\frac{3}{4}}$ $\quad h^{-1}(y) = y^4$ $\quad f_x(x) = \lambda e^{-\lambda x}, \ x > 0$

11.a) $f_X(x) = \begin{cases} x^3/4 & : 0 < x < 2 \\ 0 & : o/w \end{cases}$

$Y = X^2$ $\quad (f_Y(y))$

$h(x) = x^2$ (over $(0,2)$, $h(x) = x^2$ is strictly increasing also 1-1 and continuous)

$h'(x) = 2x$ $\quad h^{-1}(x) = \sqrt{x}$ $\quad Y \in (0,4)$

$$f_Y(y) = \frac{f_x(h^{-1}(y))}{|h'(h^{-1}(y))|} = \frac{(\sqrt{x})^3/4}{|2(\sqrt{x})|} = \frac{x^{3/2}/4}{2\sqrt{x}} = \frac{x}{8}$$

$$f_Y(y) = \begin{cases} y/8 & : 0 < y \leq 4 \\ 0 & : o/w \end{cases}$$