Expectation & Variance

REVIEW

Expected Value

For discrete RV X, the expected value is defined as

$$E(X) = \sum_x x\, p_X(x)$$

for PMF $p_X(x)$

↳ If you repeat an experiment n times, and the observed samples are $(c_1, \ldots, c_n)$, the average of the n values is

$$\frac{1}{n}\sum_i c_i = \sum_x x \cdot \frac{(\#\text{ times } c_i = x)}{n} \approx E(X) \text{ as } n \to \infty$$

Example

Let $X \sim \text{Bernoulli}(p)$ $\left(p_X(x) = p^x(1-p)^{1-x}, x \in \{0,1\}\right)$

By defintion,

$$E(X) = \sum_x x p_X(x) = 0 \cdot p^0(1-p)^{1-0} + 1 \cdot p^1(1-p)^{1-1} = p$$

Suppose you roll a fair, 6-sided die nine times, and your outcomes are $(1,2,1,5,6,6,5,5,4)$

$$\frac{1}{n}\sum_i c_i = \frac{1}{n}\sum_x x \cdot (\#\text{ times } c_i = x) = \frac{1}{9}(2 \cdot 1 + 1 \cdot 2 + 0 \cdot 3 + 1 \cdot 4 + 3 \cdot 5 + 2 \cdot 6) = \frac{35}{9}$$

Properties of Expected Value

- For $Y = g(X)$, for some function $g(\cdot)$

  $E(Y) = E(g(X)) = \sum_x g(x) p_X(x)$ $\quad\left(E(X^2) = \sum_x x^2 p_X(x)\right)$

- For multivariate function $Z = g(X,Y)$ of discrete RVs X, Y

  $E(Z) = E(g(X,Y)) = \sum_x \sum_y g(x,y) p_{X,Y}(x,y)$ $\quad\left(E(X+Y) = \sum_x \sum_y (x+y) p_{X,Y}(x,y)\right)$

- For functions $g(\cdot), h(\cdot)$, and RVs X, Y, $a, b \in \mathbb{R}$
  $E(ag(X)+bh(Y)) = aE(g(X)) + bE(h(Y))$ $\quad \left(E(2X^2 - Y^3) = 2E(X^2) - E(Y^3) \right)$
- For functions $g(\cdot), h(\cdot)$, and RVs X, Y (where X, Y are independent)
  $E\left(g(X) \cdot h(Y)\right) = E(g(X))E(h(X))$ $\quad \left(E(X^2Y^2) = E(X^2)E(Y^2) \right)$

Variance

The variance (measure of spread) for RV X is

$$V(X) = E((X - \mu_x)^2) = E(X^2) - \mu_x^2$$

Used to measure how much a random variable deviates from its mean.

and is often denoted by σ^2.

↳ From the variance, we obtain the standard deviation

$SD(X) = \sqrt{\sigma^2}$

and is often denoted by σ.

Example

Let $X \sim \text{Bernoulli}(p)$ $\quad \left(p_x(x) = p^x (1-p)^{1-x}, x \in \{0,1\} \right)$

Recall $E(g(x)) = \sum_x g(x) p_x(x)$

First calculate $E(X^2)$ (We have $E(x) = p$)

$$E(X^2) = \sum_x x^2 p_x(x) = 0^2 p^0 (1-p)^{1-0} + 1^2 p^1 (1-p)^{1-1} = p$$

$$V(X) = E(X^2) - (E(X))^2 = p - p^2 = p(1-p)$$

Covariance

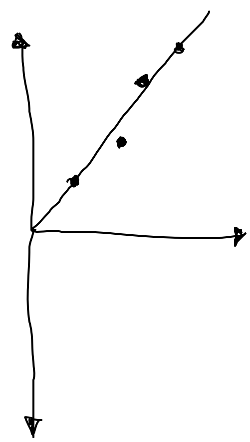
Sometimes, you may want to measure how random variables vary together. Consider the covariance.

$$Cov(X,Y) = E\big((X-\mu_X)(Y-\mu_Y)\big) = E(XY) - \mu_X\mu_Y$$

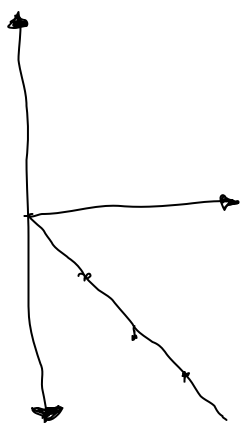
And the correlation.

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} \quad \left(\text{note } Corr(X,Y) \in [-1,1]\right)$$

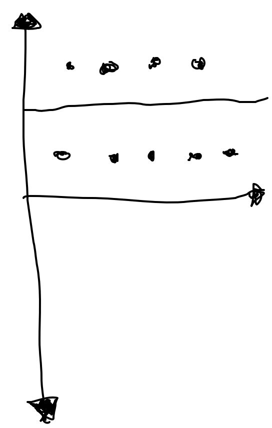
Intuition



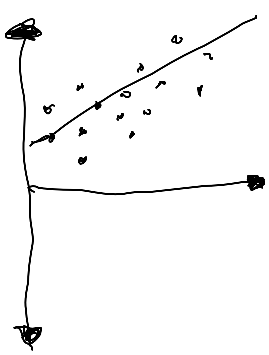
$Corr(X,Y) \approx 1$
$Cov(X,Y) > 0$



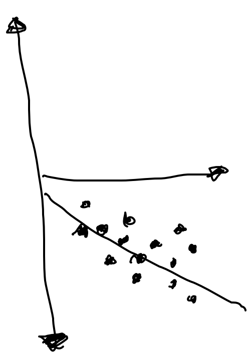
$Corr(X,Y) \approx -1$
$Cov(X,Y) < 0$



$Corr(X,Y) \approx 0$
$Cov(X,Y) \approx 0$



$Cov(X,Y) > 0$
$Corr(X,Y) > 0$



$Corr(X,Y) < 0$
$Cov(X,Y) < 0$

Covariance Properties

- For RV X

  $Cov(X,X) = V(X)$

- For RVs X, Y

  $$V(X+Y) = V(X) + V(Y) + 2Cov(X,Y)$$

- For RVs X, Y, where X and Y are independent,

  $Cov(X,Y) = 0$

  (However, you cannot assume that if $Cov(X,Y) = 0$, then X and Y are independent.)

Questions

2.a) X: money won from contest

$x = 16000$

$E(X) = 16000 \quad E(X^2) = E(16000^2) = 16000^2$

$V(X) = E(X^2) - (E(X))^2$

$= 16000^2 - 16000^2$

$= 0$

2.b) Lifeline applied on first question

$$X \in \{1000, 32000, 64000\}$$

$$p_X(x) = \begin{cases} \frac{1}{2} & , x=1000 \\ \frac{1}{2}\cdot\frac{3}{4} & , x=32000 \\ \frac{1}{2}\cdot\frac{1}{4} & , x=64000 \end{cases}$$

$$E(X) = 1000\cdot\frac{1}{2} + 32000\cdot\frac{1}{2}\cdot\frac{3}{4} + 64000\cdot\frac{1}{2}\cdot\frac{1}{4}$$
$$= 20500$$

$$E(X^2) = 1000^2\cdot\frac{1}{2} + 32000^2\cdot\frac{1}{2}\cdot\frac{3}{4} + 64000^2\cdot\frac{1}{2}\cdot\frac{1}{4}$$
$$= 896500000$$

$$V(X) = E(X^2) - (E(X))^2 = 896500000 - 20500^2$$

2.c)

$$X = \{1000, 32000, 64000\}$$

$$p_X(x) = \begin{cases} \frac{3}{4} & , x=1000 \\ \frac{1}{4}\cdot\frac{1}{2} & , x=32000 \\ \frac{1}{4}\cdot\frac{1}{2} & , x=64000 \end{cases}$$

$$E(X) = 1000 \cdot \frac{3}{4} + 32000 \cdot \frac{1}{8} + 64000 \cdot \frac{1}{8}$$

$$E(X^2) = 1000^2 \cdot \frac{3}{4} + 32000^2 \cdot \frac{1}{8} + 64000^2 \cdot \frac{1}{8}$$

$$V(X) = E(X^2) - (E(X))^2$$

3. Fair coin - keep flipping until head

Suppose you win 2^X, X is number of flips before head

a)

$$E(X) = \frac{1}{\frac{1}{2}} = 2$$

$$X \sim \text{Geometric}\left(\frac{1}{2}\right)$$

b)

$$\text{Expected Payoff} = \sum_{i=1}^{\infty} P(X=i)\, 2^i$$

$$= \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i 2^i$$

$$= \sum_{i=1}^{\infty} \left(\frac{2}{2}\right)^i = \sum_{i=1}^{\infty} 1 = \infty$$

c)

$$E\cancel{X}P = \sum_{i=1}^{M} P(X=i)2^i + \sum_{i=M+1}^{\infty} P(X=i)2^M$$

$$= \sum_{i=1}^{M} \left(\frac{1}{2}\right)^i 2^i + 2^M \sum_{i=M+1}^{\infty} \left(\frac{1}{2}\right)^i$$

$$= \sum_{i=1}^{M} 1 + \cancel{2^M} \frac{1}{\cancel{2^M}} \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i$$

$$= M + 1$$