Discrete Random Variables

REVIEW

Random Variable: a function (often denoted as X) that maps the sample space to real line $[X: S \to \mathbb{R}]$

- we use RVs to assign quantities to events (numbers are consistent and easy to work with)

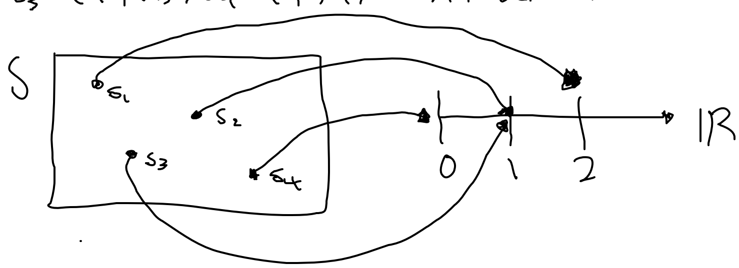
Example: Two Coin Flips.

Consider $S = \{s_1, s_2, s_3, s_4\}$

$s_1 = (H,H)$, $s_2 = (H,T)$

$s_3 = (T,H)$, $s_4 = (T,T)$

Let X denote the total number of heads in outcome $s \in S$



$$\forall s \in S,\ X(s) = \begin{cases} 0, & s = s_4 \\ 1, & s = s_2 \text{ or } s = s_3 \\ 2, & s = s_1 \end{cases}$$

Discrete RV

If we know that the output of X is among a finite set $(\{x_1, \dots, x_n\})$ or countably infinite set $(\{x_1, x_2, \dots\})$

*We will elaborate more on this when we cover Continuous RVs

Indicator RVs

- what it's name implies (indicates if event occurs)

For event A, the indicator RV, I_A outputs 1 when A occurs, and 0 when it doesn't.

$$I_A(s) = \begin{cases} 1, & s \in A \\ 0, & s \notin A \end{cases}$$

Distribution of RVs

The distribution of RV X is a collection of all the probabilities for any subset of $\mathbb{R}$

The distribution is how you know the probability for any event defined by X.

Probability Mass Function

For discrete RV X, where $X \in \{x_1, x_2, ..\}$
the probability mass function (PMF) is defined
$p_X(x_i) = P(X = x_i) = P(\{s \in \mathbb{R} : X(s) = x_i\})$

Note that $\sum_{\forall i} p_X(x_i) = 1$

Example: Two Coin Tosses
X: total # of heads (Assume 50% for either)
$s_1 = (H,H)$, $s_2 = (H,T)$
$s_3 = (T,H)$, $s_4 = (T,T)$

$$X(s) = \begin{cases} 0, & s = s_4 \\ 1, & s = s_2 \text{ or } s = s_3 \\ 2, & s = s_1 \end{cases}$$

$$p_X(x) = \begin{cases} 1/4, & x = 0 \\ 1/2, & x = 1 \\ 1/4, & x = 2 \end{cases} = P(X = x)$$

Cumulative Distribution Function

For an RV X, the cumulative distribution function (CDF) is defined
$F_X(x) = P(X \leq x) = P(\{s \in S : X(s) \leq x\}) \ \forall x \in \mathbb{R}$

* Since we know if $a < b$, $P(a < x \leq b) = P(x \leq b) - P(x \leq a)$
we can conclude $P(a < x \leq b) = F_X(b) - F_X(a)$

Properties: $F_X(-\infty) = P(X \leq -\infty) = 0$

$F_X(\infty) = P(X \leq \infty) = 1$

$\forall x_1 < x_2 : F_X(x_1) \leq F_X(x_2)$ (increasing) $\quad P(X \leq x_1) \leq P(X \leq x_2)$ $\quad X \leq 1 \Rightarrow X \leq 2$

$F_X(x) = P(X \leq x) = \sum_{x_i \leq x} p_X(x_i), \ \forall x \in \mathbb{R}$

Example: Two Coin Tosses
Recall $s_1 = (H,H)$ $s_2 = (H,T)$
$s_3 = (T,H)$ $s_4 = (T,T)$

$$X(s) = \begin{cases} 0, & s = s_4 \\ 1, & s = s_2 \text{ or } s = s_3 \\ 2, & s = s_1 \end{cases} \qquad p_X(x) = \begin{cases} 1/4, & x = 0 \\ 1/2, & x = 1 \\ 1/4, & x = 2 \end{cases}$$

Then $F_X(x) = P(X \leq x) = \begin{cases} 1/4, & x = 0 \\ 3/4, & x = 1 \\ 1, & x = 2 \end{cases}$

$F_X(x)$ represents the probability of getting at least x coins.

Bernoulli Distribution

- use for binary results (yes/no, success, fail)

If $X \sim Bernoulli(p)$, then
X is a Bernoulli RV.

$$X = \begin{cases} 1, & \text{success} \\ 0, & \text{fail} \end{cases}$$

$p_X(1) = P(X=1) = p$

$p_X(0) = P(X=0) = 1-p$

Example: Coin Toss
Consider fair coin. Let $X(s) = \begin{cases} 1, & s \text{ is heads} \\ 0, & s \text{ is tails} \end{cases}$

$$p_X(x) = \begin{cases} 0.5, & x=1 \\ 0.5, & x=0 \end{cases}$$ Thus $X \sim Bernoulli(0.5)$

Binomial Distribution

- Used when considering X successes from n binary trials.

If $X \sim Binomial(n, p)$, then
X is a binomial RV.

$$p_X(x) = P(X=x) = \binom{n}{x} (p)^x (1-p)^{n-x}$$

Example: Two die rolls. (fair die)
Let X represent the number of 6s obtained in 2 trials.

Then $X \sim Binomial(2, 1/6)$

$$p_X(0) = P(X=0) = \binom{2}{0} (1/6)^0 (5/6)^2$$

$$p_X(1) = P(X=1) = \binom{2}{1} (1/6)^1 (5/6)^1$$

$$p_X(2) = P(X=2) = \binom{2}{2} (1/6)^2 (5/6)^0$$

Binomial(1, p) similar to Bernoulli(p)

PS 4

Q6. X: most valuable prize

Only need to consider $\{5, 6, \dots, 100\}$

$$P(X=k) = \frac{\binom{k-1}{4}}{\binom{100}{5}}$$

Q3. X_1 label on chip of first draw

X_2 label on chip of second draw

$$W = X_1 + 10X_2$$

$$W \in \{00, 01, \dots, 98, 99\}$$

$$P(X_1 = x) = \frac{1}{10}$$

$$P(X_2 = x) = \frac{1}{10}$$

$$P(W=w) = P((X_1=a) \cap (X_2=b))$$

$w = ab$

$a \in X_1$

$b \in X_2$

$$= P(X_1=a)\,P(X_2=b)$$

$$= \frac{1}{10} \cdot \frac{1}{10}$$

$$= \frac{1}{100}$$