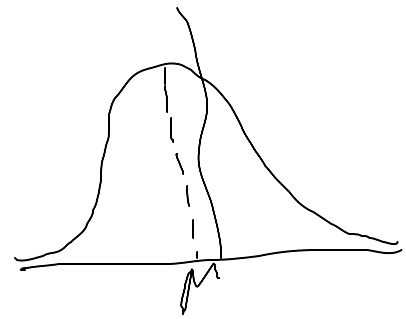
Normal Distribution

REVIEW

A continuous RV X follows Normal distribution w/ mean μ and variance σ^2 ($X \sim \text{Normal}(\mu, \sigma^2)$) if its PDF f_X is defined

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \quad \forall x \in \mathbb{R}$$



→ "the bell curve"

When $\mu = 0$, $\sigma^2 = 1$, called Standard Normal (often denoted $\phi(x)$)

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \quad \forall x \in \mathbb{R}$$

- cannot manually derive – must use computers/tables

Properties

Linear Functions: Let $X \sim N(\mu, \sigma^2)$, $Y = a + bX$

- then $Y \sim N(a + b\mu, b^2\sigma^2)$

↳ Standardization: can use Std. Norm. info for any Norm. RV
If $X \sim N(\mu, \sigma^2)$, Set $X = \mu + \sigma Z$, where $Z \sim N(0,1)$

Ex: $X \sim N(3, 9)$. Find $P(X \geq 0)$

Notation for CDF of std. norm.

$$P(X \geq 0) = 1 - P(X < 0) = 1 - P\left(\frac{X-3}{3} < \frac{0-3}{3}\right) = 1 - P(Z < -1) = 1 - \Phi(-1)$$

Bivariate Normal: X, Y are jointly normal if PDF $f_{X,Y}$ is

$$f_{X,Y}(x,y) = \frac{1}{\sqrt{(2\pi)^2 \begin{vmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{vmatrix}}} \exp\left(-\frac{1}{2}\begin{bmatrix} x-\mu_X \\ y-\mu_Y \end{bmatrix}^T \begin{bmatrix} \sigma_X^2 & \sigma_{XY}^2 \\ \sigma_{XY}^2 & \sigma_Y^2 \end{bmatrix}^{-1} \begin{bmatrix} x-\mu_X \\ y-\mu_Y \end{bmatrix}\right) \quad \forall x, y \in \mathbb{R}$$

OR equivalently

$$f_{X,Y}(x,y) = \frac{1}{\sqrt{(2\pi)^2 |\Sigma|}} \exp\left(-\frac{1}{2}(\vec{v} - \vec{\mu})^T \Sigma^{-1} (\vec{v} - \vec{\mu})\right) \quad \vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}, \vec{\mu} = \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}, \Sigma = \begin{bmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{bmatrix}$$

For joint Normal Distribution, marginals are also Normal

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\vec{\mu} = \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}, \Sigma = \begin{bmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{bmatrix}\right) \Rightarrow \begin{cases} X \sim N(\mu_X, \sigma_X^2) \\ Y \sim N(\mu_Y, \sigma_Y^2) \end{cases}$$

This means X, Y are Normal RVs

If X_1, X_2 are Normal RVs, and $Y = a_1 X_1 + a_2 X_2$
then $Y \sim \text{Normal}(\mu_Y = E(a_1 X_1 + a_2 X_2), \sigma^2 = Var(a_1 X_1 + a_2 X_2))$

Conditional Normal: Suppose for (X, Y), we have

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\vec{\mu} = \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}, \Sigma = \begin{bmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{bmatrix}\right)$$

Then for some $x \in \mathbb{R}$

$$Y \mid (X = x) \sim N\Bigg(\underbrace{\mu_Y + \frac{\sigma_{XY}}{\sigma_X^2}(x - \mu_X)}_{E(Y \mid X = x)}, \underbrace{\sigma_Y^2 - \frac{\sigma_{XY}^2}{\sigma_X^2}}_{V(Y \mid X = x)}\Bigg)$$

Poisson Distribution

If X is a discrete RV that follows the Poisson distribution w/ parameter λ ($X \sim \text{Poisson}(\lambda)$), then

$$P(X = x) = \frac{\lambda^x}{x!} e^{-\lambda}, \text{ for } x = 0, 1, 2, \ldots$$

Note that $\lambda > 0 \in \mathbb{R}$

→ when dealing w/ algebra using Poisson useful to know following equality

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

Important Approximations

For RV X,

- If $X \sim \text{Binomial}(n, \lambda/n)$, as $n \to \infty$, $X \sim \text{Poisson}(\lambda)$ [Discrete Approximation]
- If $X \sim \text{Binomial}(n, p)$, as $n \to \infty$, $X \sim N(np, np(1-p))$ [Continuous Approximation]
- If $X \sim \text{Poisson}(\lambda)$, as $\lambda \to \infty$, $X \sim N(\lambda, \lambda)$ [Continuous Approximation]

Q&A

2.a) $X, Y \overset{\text{iid}}{\sim} N(0,1)$

$V = X+Y$, $W = X-Y$

Linear Comb. of norm variables is normal

$E(V) = E(X+Y) = E(X) + E(Y) = 0+0 = 0$

$E(W) = 0$ (similar)

$$\begin{aligned} Var(V) = Var(X+Y) &= Var(X) + Var(Y) + 2\,Cov(\underbrace{X,Y}_{\text{iid}}) \\ &= 1 + 1 + 2 \cdot 0 \\ &= 2 \end{aligned}$$

$Var(W) = 2$ (similar)

$V, W \sim N(0, 2)$

2.b) wts/ V, W are independent

$(f_{X,Y}(x,y) = f_X(x) f_Y(y))$

wts/ $\mathrm{Cov}(V,W) = 0$

$$\mathrm{Cov}(X+Y, X-Y) = \mathrm{Cov}(X,X) - \cancel{\mathrm{Cov}(X,Y)}$$
$$\cancel{+\mathrm{Cov}(Y,X)} - \mathrm{Cov}(Y,Y)$$
$$= \mathrm{Var}(X) - \mathrm{Var}(Y)$$
$$= 0$$

2.c) X, Y are now correlate w/ $\rho = 0.5$

$E(X+Y) = 0 \quad E(X-Y) = 0$

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y)$$
$$= 2 + 2 \cdot \frac{\rho}{\sigma_X \sigma_Y}$$
$$= 2 + 2 \cdot 0.5$$
$$= 3$$

$\mathrm{Var}(X-Y) = 3$ (similar)

$V, W \sim \mathrm{Normal}(0, 3)$

3.a) $\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}, \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{bmatrix} \right)$

Suppose $\sigma_{xy} = 0$

$$\overbrace{\begin{bmatrix} \frac{1}{\sigma_x^2} & 0 \\ 0 & \frac{1}{\sigma_y^2} \end{bmatrix}}$$

wts/ $X \perp Y$

$\left(f_{x,y}(x,y) = f_x(x)\, f_y(y) \right)$

$$f_{x,y}(x,y) = \frac{1}{\sqrt{(2\pi)^2 \begin{vmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{vmatrix}}} \exp\left(-\frac{1}{2} \begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix}^T \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{bmatrix}^{-1} \begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix} \right)$$

$$= \frac{1}{\sqrt{(2\pi)^2 \sigma_x^2 \sigma_y^2}} \exp\left(-\frac{1}{2} \begin{bmatrix} \frac{x-\mu_x}{\sigma_x^2} \\ \frac{y-\mu_y}{\sigma_y^2} \end{bmatrix}^T \begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix} \right)$$

$$= \frac{1}{\sqrt{(2\pi)^2 \sigma_x^2 \sigma_y^2}} \exp\left(-\frac{1}{2} \left(\left(\frac{(x-\mu_x)^2}{\sigma_x^2} \right) + \left(\frac{(y-\mu_y)^2}{\sigma_y^2} \right) \right) \right)$$

$$= \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left(-\frac{1}{2} \left(\frac{x-\mu_x}{\sigma_x} \right)^2 \right) \cdot \frac{1}{\sqrt{2\pi\sigma_y^2}} \exp\left(\left(\frac{y-\mu_y}{\sigma_y} \right)^2 \right)$$

$$= f_x(x) \cdot f_y(y)$$

7. Suppose $X \sim \text{Poisson}(\lambda)$

wts/ $E(X) = \lambda$ and $Var(X) = \lambda$

$$E(X) = \sum_{k=0}^{\infty} k \cdot p_x(k)$$

$$= \sum_{k=0}^{\infty} k \cdot \frac{\lambda^k}{k!} e^{-\lambda}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \lambda \cdot \frac{\lambda^{k-1}}{(k-1)!}$$

$$= e^{-\lambda} \cdot \lambda \sum_{j=0}^{\infty} \frac{\lambda^j}{j!}$$

$$(j = k-1)$$

$$= \cancel{e^{-\lambda}} \cdot \lambda \left(\cancel{e^{\lambda}}\right)$$

$$= \lambda$$

$$Var(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = \sum_{k=0}^{\infty} k^2 \frac{\lambda^k}{k!} e^{-\lambda}$$

$$= \lambda e^{-\lambda} \left(\sum_{k=0}^{\infty} k \frac{\lambda^{k-1}}{(k-1)!} \right)$$

$$= \lambda e^{-\lambda} \left(\sum_{k=0}^{\infty} (k-1) \frac{\lambda^{k-1}}{(k-1)!} + \sum_{k=0} 1 \cdot \frac{\lambda^{k-1}}{(k-1)!} \right)$$

$$= \lambda e^{-\lambda} \left(\lambda \sum_{k=0}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} + e^{\lambda} \right)$$

$$= \lambda e^{-\lambda} \left(\lambda \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} + e^{\lambda} \right)$$

$$= \lambda e^{-\lambda} \left(\lambda e^{\lambda} + e^{\lambda} \right)$$

$$= \lambda^2 + \lambda$$

$$Var(X) = E(X^2) - (E(X))^2$$

$$= \lambda^2 + \lambda - (\lambda)^2$$

$$= \lambda$$