Conditional Probability

For events A, B, the probability of A, given B has occurred (denoted as $P(A|B) = \frac{P(A \cap B)}{P(B)}$

* $P(B)$ must be > 0

Independence

If there are 2 events, A, B, they are (pairwise) independent when $P(A \cap B) = P(A)P(B)$

If there are $n \geq 3$ events $A_1, \dots, A_n$, they are mutually indepent if for any arbitrary subcollection of the A_i's $(A_{k_1}, \dots, A_{k_m})$

$$P(A_{k_1} \cap \cdots \cap A_{k_m}) = P(A_{k_1}) \cdots P(A_{k_m})$$

REMEMBER: Pairwise independence DOES NOT imply mutual independence

$$\left(\left.\begin{matrix} P(A \cap B) = P(A)P(B) \\ P(B \cap C) = P(B)P(C) \\ P(A \cap C) = P(A)P(C) \end{matrix}\right\} \not\Rightarrow \begin{matrix} P(A \cap B \cap C) = \\ P(A)P(B)P(C) \end{matrix} \right)$$

Bayes' Rule

For events, A, B,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

* One of the most useful rules for calculating an unknown, when you are given conditional probabilities.

Conditional Independence

Events A, B, are conditionally independent given C, when

$$P(A \cap B|C) = P(A|C)P(B|C)$$

* implies $P(A|B,C) = P(A|C)$
$P(B|A,C) = P(B|C)$

REMEMBER: Conditional independence DOES NOT imply independence (unless A, B, C are mutually independent)

$$\left(P(A \cap B|C) = P(A|C)P(B|C) \not\Rightarrow P(A \cap B) = P(A)P(B) \right)$$

Useful Things to Remember

Law of Total Probability

For partition of S: $A_1, A_2, \ldots, A_n$ and event B.

$$P(B) = P(B \cap A_1) + \ldots + P(B \cap A_n)$$

From this, we have

$$\boxed{P(B) = P(B|A_1)P(A_1) + \ldots + P(B|A_n)P(A_n)}$$

* Very useful when using Bayes' Rule

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$$= \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

Multiplication Rule

For events $A_1, A_2, \ldots, A_n$

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1) \times P(A_2|A_1) \times P(A_3|A_1, A_2) \times \ldots P(A_n|A_1, A_2, \ldots A_{n-1})$$

5.i) $P(A_1 \cap A_2) \neq P(A_1)P(A_2)$

We want to show this   B - the event of biased die

$$P(A_1) = P(A_1|B)P(B) + P(A_1|B^c)P(B^c)$$
$$= \frac{2}{6} \cdot \frac{1}{2} + \frac{1}{6} \cdot \frac{1}{2}$$
$$= \frac{1}{4}$$

Similarly, $P(A_2) = \frac{1}{4}$

$$P(A_1 \cap A_2) = P(A_1 \cap A_2|B)P(B) + P(A_1 \cap A_2|B^c)P(B^c)$$

Biased

$\{1,2,3,4,6,6\}$

$$= \frac{4}{36} \cdot \frac{1}{2} + \frac{1}{36} \cdot \frac{1}{2}$$
$$= \frac{5}{72}$$

$$P(A_1)P(A_2) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$$

$P(A_1 \cap A_2) \neq P(A_1)P(A_2)$

5.ii) a) $P(A_1 \cap A_2 | B) = P(A_1|B)P(A_2|B)$

$$P(A_1 \cap A_2 | B) = \frac{4}{36}$$

Die Outcomes: $\{1, 2, 3, 4, 6_a, 6_b\}$

$A_1 \cap A_2 | B = \{(6_a, 6_a), (6_a, 6_b), (6_b, 6_a), (6_b, 6_b)\}$

$$P(A_1|B) = \frac{2}{6}, \quad P(A_2|B) = \frac{2}{6}$$

b) $P(A_1 \cap A_2 | B^c) = P(A_1|B^c)P(A_2|B^c)$

$$P(A_1 \cap A_2 | B^c) = \frac{1}{36}$$

$$P(A_1|B^c) = \frac{1}{6}, \quad P(A_2|B^c) = \frac{1}{6}$$

Show

11. ii) $P(A|B,C) < P(A|C)$

Given

- $P(A \cap B) > 0$
- $P(A \cup B) < 1$
- $C = A \cup B$
- A, B are independent given C

Clarify: $P(A|B,C) = P(A|B \cap C)$

$$P(A|B,C) = P(A|B \cap C)$$
$$= \frac{P(A \cap B \cap C)}{P(B \cap C)}$$
$$= \frac{P(A \cap B \cap (A \cup B))}{P(B \cap (A \cup B))}$$
$$= \frac{P(A \cap B)}{P(B)}$$
$$= \frac{P(A)\,P(B)}{P(B)} = P(A)$$

$$P(A|C) = \frac{P(A \cap C)}{P(C)}$$
$$= \frac{P(A)}{P(C)}$$

$P(A|C) < P(A|B,C)$ since $0 < P(C) < 1$