MGFs and CLT

Moments

For RV X, the k-th moment of X is defined as $E(X^k)$. (1^{st} moment of X is $E(X)=\mu$)

↳ The k-th central moment is defined as $E((X-\mu)^k)$, where $E(X)=\mu$
(2^{nd} central moment of X is $E((X-\mu)^2)=\sigma^2$

Moment Generating Functions

For RV X, we can use moment generating functions (MGFs) to calculate the k-th moment.

- the MGF of X is given as $m(t)=E(e^{tX})$
  ↳ to get the k^{th} moment of X, we use the following formula

$$E(X^k)=m^{(k)}(0) \quad [k^{th} \text{ derivative of } m, \text{ at } t=0]$$

Example

Suppose you have X, $P(X=x)=\begin{cases}\frac{1}{4}, k=2\\ \frac{3}{4}, k=4\end{cases}$. Calculate $E(X^2)$ using MGFs.

$$m_X(t)=\frac{1}{4}e^{2t}+\frac{3}{4}e^{4t}$$

$$m^{(1)}(t)=2\cdot\frac{1}{4}e^{2t}+4\cdot\frac{3}{4}e^{4t}$$
$$=\frac{1}{2}e^{2t}+3e^{4t}$$
$$m^{(2)}(t)=2\cdot\frac{1}{2}e^{2t}+4\cdot 3e^{4t}$$
$$=e^{2t}+12e^{4t}$$

$$E(X^2)=m^{(2)}(0)=1+12=13$$

$$E(X^2)=4^2\cdot\frac{3}{4}+2^2\cdot\frac{1}{4}=13$$

Properties About MGFs

- For X, Y, if you have $m_x(t), m_y(t)$

$$m_x(t) = m_y(t) \iff X, Y \text{ follow same distribution}$$

Very useful if you want to show equivalence of variables w/ limited info

- Let $X_1, X_2, \dots, X_n$ be independent w/ MGFs and $Y = a_1X_1 + \dots + a_nX_n$. Then

$$m_Y(t) = m_{X_1}(a_1t) \times \dots \times m_{X_n}(a_nX_n)$$

Central Limit Theorem

An incredibly important theorem in statistics

- Suppose you have $X_1, \dots, X_n$ iid RVs. w/ mean μ, variance σ^2

Then, for $\overline{X}_n = \frac{1}{n}(X_1 + \dots + X_n)$

w/ $Z_n = \sqrt{n}\left(\frac{\overline{X}_n - \mu}{\sigma}\right)$

then $Z_n \xrightarrow{D} N(0,1)$ [Z_n converges in distribution to $N(0,1)$]

and $\overline{X}_n \rightarrow N(\mu, \sigma^2/n)$

Convergence

Consider sequence of continuous RVs $X_1, X_2, \dots$ and RV Y

- X_n converges in probability to Y, as $n \to \infty$ if ($X_n \xrightarrow{P} Y$)

$$\lim_{n \to \infty} P(|X_n - Y| \geq \varepsilon) = 0, \ \forall \varepsilon > 0$$

- X_n converges in distribution to Y, as $n \to \infty$, if ($X_n \xrightarrow{D} Y$)

$$\lim_{n \to \infty} P(X_n \leq x) = P(Y \leq x) \ \forall x \in \mathbb{R}$$

↳ very useful for approximating probabilities

Q&A

7. b) $X \sim Exp(1)$, $Y = X^3$

$$E(Y^n) = E(X^{3n})$$
$$= \int_0^\infty x^{(3n+1)-1} e^{-x}\, dx$$
$$= \Gamma(3n+1) \int_0^\infty \underbrace{\frac{1}{\Gamma(3n+1)} x^{(3n+1)-1} e^{-x}}_{Gamma(3n+1,\,1)}\, dx$$
$$= \Gamma(3n+1) = (3n)!$$

$$E(Y) = 3! = 6$$
$$V(Y) = E(Y^2) - (E(Y))^2$$
$$= 6! - (3!)^2$$
$$= 720 - 36 = 684$$

8. Weekly Output $\sim U(20, 30)$

$X_1, X_2, \ldots, X_{52} \sim U(20, 30)$

$E(X_i) = 25$, $Var(X_i) = \frac{100}{12}$

$X = \sum_{i=1}^{52} X_i \qquad \overline{X_{52}} = \frac{\sum_{i=1}^{52} X_i}{52}$

want: $P\left(\sum_{i=1}^{52} X_i < 1280\right) = P\left(\overline{X_{52}} < \frac{1280}{52}\right)$

$$= P\left(\sqrt{52}\,\frac{\overline{X_{52}} - 25}{\sqrt{100/12}} < \sqrt{52}\,\frac{1280/52 - 25}{\sqrt{100/12}}\right)$$

$$= P(Z < -0.961) = \Phi(-0.961)$$

F19 Q5

a) $V(X_i) = E(X_i^2) - (E(X_i)^2)$

$E(X_i^2) = V(X_i) + \mu^2$

$\frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{P} \sigma^2 + \mu^2$

b) $\bar{X}_n \xrightarrow{P} \mu$, $f(x) = x^2$

$(\bar{X}_n)^2 \xrightarrow{P} \mu^2$

F17 Q5

X, Y indepent

c) $X \sim Exp(1)$ want m_Z

$Y \sim Exp(2)$ $Z = X + Y$

$m_Z(t) = m_{X+Y}(t) = m_X(t)\, m_Y(t) = \left(\frac{1}{1-t}\right)\left(\frac{2}{2-t}\right)$

h) want $P(Z > 4 \mid W = 2)$ $W = \sqrt{X}$

$P(Z > 4 \mid W = 2) = P(X + Y > 4 \mid \sqrt{X} = 2)$

$= P(Y > 4 - X \mid X = 4)$

$= P(Y > 0 \mid X = 4)$

X, Y i.i.d

$= P(Y > 0)$

$= 1$