Inequalities / Law of Large Numbers

Sometimes, in probability theory, we may look to make statements, without having some specific knowledge. Often, we look to inequalities for assistance.

Note: For proofs, remember to consider inequalities for support.

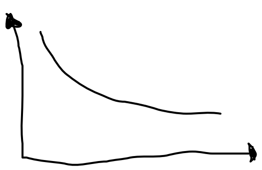
Jensen's I/E

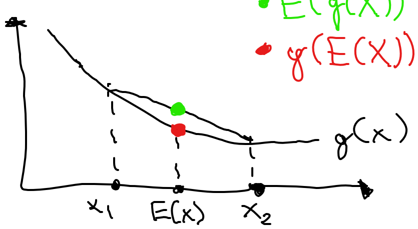
For any RV X, and convex function g
$$E(g(X)) \geq g(E(X))$$

(for concave g: $g(E(X)) \leq E(g(X))$)

Intuition

Convex





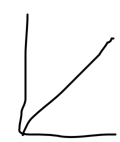
• $E(g(X))$ average value of $g(x)$ in interval

• $g(E(X))$ $g(x_0)$, where x_0 is the mean of interval

$f(x)$ is convex when $\frac{\partial^2 f}{\partial x^2} < 0$ for twice differentiable function.

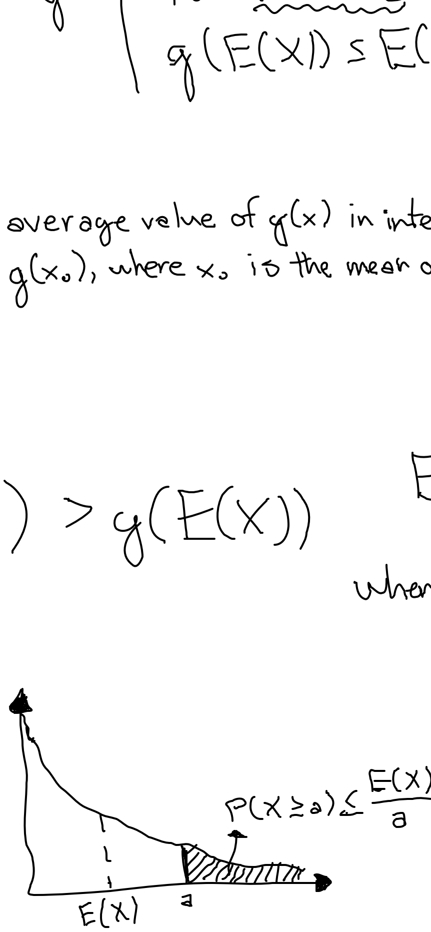
$$E(g(X)) > g(E(X))$$

$E(g(X)) = g(E(X))$ when $\frac{\partial^2 g}{\partial x^2} = 0$



Markov's I/E

For RV $X > 0$, $P(X \geq a) \leq \frac{E(X)}{a}$



Example

Consider 6-sided fair die roll as X

$$p_X(x) = \begin{cases} 1/6 : x \in \{1, \dots, 6\} \\ 0 : \text{o/w} \end{cases}, \quad E(X) = 3.5$$

$P(X \geq 1) = 1 \leq 3.5 = \frac{E(X)}{1}$

$P(X \leq 2) = \frac{5}{6} \leq \frac{3.5}{2} = \frac{E(X)}{2}$

$P(X \geq 3) = \frac{2}{3} \leq \frac{3.5}{3} = \frac{E(X)}{3}$

$P(X \leq 4) = \frac{1}{2} \leq \frac{3.5}{4} = \frac{E(X)}{4}$

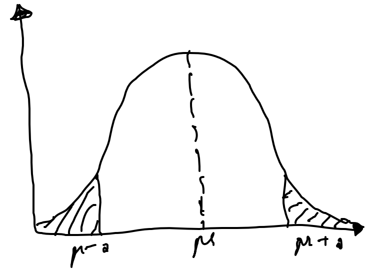
$P(X \geq 5) = \frac{1}{3} \leq \frac{3.5}{5} = \frac{E(X)}{5}$

$P(X \leq 6) = \frac{1}{6} \leq \frac{3.5}{6} = \frac{E(X)}{6}$

$P(X \geq 7) = 0 \leq \frac{3.5}{7} = \frac{E(X)}{7}$

Chebyshev's Inequality

For any RV X, $P(|X-\mu| \geq a) \leq \frac{V(X)}{a^2}$



Example

Suppose $X \in \{1, 2, \ldots, 20\}$

Where $\mu_x = 5.5$, $\sigma_x^2 = 1.6$

This is very broad.

However, suppose we want $P(\{X \leq 2\} \cup \{X \geq 9\})$

We may not be able to get our desired value, but we can provide a bound for it.

$$P(\{X \leq 2\} \cup \{X \geq 9\}) = P(|X - 5.5| \geq 3.5) \leq \frac{\sigma_x^2}{3.5^2} = \frac{1.6^2}{3.5^2} = 0.209$$

Weak Law of Large Numbers

For independent and identically distributed random variables $X_1, X_2, \ldots, X_n$

$$P(|\overline{X}_n - \mu| \geq \varepsilon) \to 0 \text{ as } n \to \infty, \forall \varepsilon > 0$$

Where $E(X_i) = \mu$ for $i = 1, 2, \ldots, n$. $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

Example

Let $X_1, X_2, \ldots, X_n$ represent the outcomes of n fair 6-sided die rolls.

X_i reproent the i-th outcome for $i = 1, \ldots, n$.

$E(X_i) = 3.5$

By WLLN, as $n \to \infty$, $\frac{1}{n}(X_1 + X_2 + \ldots + X_n) \to 3.5$

Q & A

PS9

Q7 n - sample size

$\hat{p}$ - prob. in favour of policy (sample)

p - prob. in favour of policy (pop.)

wts! $P(|\hat{p}-p| > c) \leq \frac{1}{4nc^2}$

$X \sim \text{Binomial}(n, p)$ (# people in favour in semple)

$$P(|\hat{p}-p| > c) \leq \frac{Var(\hat{p})}{c^2}$$

$$\hat{p} = \frac{X}{n}$$

$$= \frac{(p)(1-p)}{nc^2}$$

$$\leq \frac{1}{4nc^2}$$

$E(\hat{p}) = n \cdot \frac{\hat{p}}{n} = \hat{p}$

$Var(\hat{p}) = \frac{(p)(1-p)}{n}$

8. $X, Y > 0$; $X \neq kY, \forall k \in \mathbb{R}$

wts $E(X/Y)E(Y/X) > 1$

Let $W = Y/X$, $g(w) = 1/w$, convex for $w > 0$

$$E(X/Y) = E(1/W) > 1/E(W) = 1/E(Y/X)$$

Jensen's (note that strictly greater can be used when g is strictly convex)

$$E(X/Y) > \frac{1}{E(Y/X)} \Rightarrow E(X/Y)E(Y/X) > 1$$

9. X, where μ, σ^2 are both given

wts $E(X-\mu)^4 \geq \sigma^4$

$Y = (X-\mu)^2$, $g(x) = x^2$ (convex)

$$E((X-\mu)^4) = E(g(Y))$$
$$\geq g(E(Y))$$
$$= (E((X-\mu)^2))^2$$
$$= (\sigma^2)^2$$
$$= \sigma^4$$

10. $W_1, \ldots, W_n \overset{iid}{\sim} Exp(3)$

wts $P(W_1 + \ldots + W_n < \frac{n}{2}) > 0.999$

$M_n = \frac{W_1 + \ldots + W_n}{n}$ $E(W_i) = \frac{1}{3}$

By WLLN, $E(M_n) = \frac{1}{3}$

$Var(M_n) = Var\left(\frac{W_1 + \ldots + W_n}{n}\right) = \frac{1}{n^2} Var(W_1 + \ldots)$

$= \frac{n\left(\frac{1}{9}\right)}{n^2} = \frac{1}{9n}$

$P(W_1 + \ldots + W_n < \frac{n}{2}) > 0.999 \Leftrightarrow P(M_n < \frac{1}{2}) > 0.999$

$\Leftrightarrow P(M_n \geq \frac{1}{2}) \leq 0.001$

$P(|M_n - E(M_n)| \geq 1/6) \leq \frac{V(M_n)}{(1/6)^2}$

$\Rightarrow P(|M_n - 1/3| \geq 1/6) \leq \frac{6^2}{9n}$

$\Rightarrow P(\{M_n \leq 1/6\} \cup \{M_n \geq \frac{1}{2}\}) \leq \frac{4}{n}$

$\Rightarrow P(\{M_n \geq \frac{1}{2}\}) \leq \frac{4}{n}$

$n = 1000000$. $\exists n$ st. i/e holds.