Chapter 4

ELECTROSTATIC FIELDS

Take risks: if you win, you will be happy; if you lose you will be wise.

—PETER KREEFT

4.1 INTRODUCTION

Having mastered some essential mathematical tools needed for this course, we are now prepared to study the basic concepts of EM. We shall begin with those fundamental concepts that are applicable to static (or time-invariant) electric fields in free space (or vacuum). An electrostatic field is produced by a static charge distribution. A typical example of such a field is found in a cathode-ray tube.

Before we commence our study of electrostatics, it might be helpful to examine briefly the importance of such a study. Electrostatics is a fascinating subject that has grown up in diverse areas of application. Electric power transmission, X-ray machines, and lightning protection are associated with strong electric fields and will require a knowledge of electrostatics to understand and design suitable equipment. The devices used in solid-state electronics are based on electrostatics. These include resistors, capacitors, and active devices such as bipolar and field effect transistors, which are based on control of electron motion by electrostatic fields. Almost all computer peripheral devices, with the exception of magnetic memory, are based on electrostatic fields. Touch pads, capacitance keyboards, cathode-ray tubes, liquid crystal displays, and electrostatic printers are typical examples. In medical work, diagnosis is often carried out with the aid of electrostatics, as incorporated in electrocardiograms, electroencephalograms, and other recordings of organs with electrical activity including eyes, ears, and stomachs. In industry, electrostatics is applied in a variety of forms such as paint spraying, electrodeposition, electrochemical machining, and separation of fine particles. Electrostatics is used in agriculture to sort seeds, direct sprays to plants, measure the moisture content of crops, spin cotton, and speed baking of bread and smoking of meat.[1,2]

[1]For various applications of electrostatics, see J. M. Crowley, *Fundamentals of Applied Electrostatics.* New York: John Wiley & Sons, 1986; A. D. Moore, ed., *Electrostatics and Its Applications.* New York: John Wiley & Sons, 1973; and C. E. Jowett, *Electrostatics in the Electronics Environment.* New York: John Wiley & Sons, 1976.

[2]An interesting story on the magic of electrostatics is found in B. Bolton, *Electromagnetism and Its Applications.* London: Van Nostrand, 1980, p. 2.

We begin our study of electrostatics by investigating the two fundamental laws governing electrostatic fields: (1) Coulomb's law, and (2) Gauss's law. Both of these laws are based on experimental studies and they are interdependent. Although Coulomb's law is applicable in finding the electric field due to any charge configuration, it is easier to use Gauss's law when charge distribution is symmetrical. Based on Coulomb's law, the concept of electric field intensity will be introduced and applied to cases involving point, line, surface, and volume charges. Special problems that can be solved with much effort using Coulomb's law will be solved with ease by applying Gauss's law. Throughout our discussion in this chapter, we will assume that the electric field is in a vacuum or free space. Electric field in material space will be covered in the next chapter.

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4.2 COULOMB'S LAW AND FIELD INTENSITY

Coulomb's law is an experimental law formulated in 1785 by the French colonel, Charles Augustin de Coulomb. It deals with the force a point charge exerts on another point charge. By a *point charge* we mean a charge that is located on a body whose dimensions are much smaller than other relevant dimensions. For example, a collection of electric charges on a pinhead may be regarded as a point charge. Charges are generally measured in coulombs (C). One coulomb is approximately equivalent to 6×10^{18} electrons; it is a very large unit of charge because one electron charge $e = -1.6019 \times 10^{-19}$ C.

Coulomb's law states that the force F between two point charges Q_1 and Q_2 is:

1. Along the line joining them
2. Directly proportional to the product $Q_1 Q_2$ of the charges
3. Inversely proportional to the square of the distance R between them.[3]

Expressed mathematically,

$$F = \frac{k\,Q_1 Q_2}{R^2} \tag{4.1}$$

where k is the proportionality constant. In SI units, charges Q_1 and Q_2 are in coulombs (C), the distance R is in meters (m), and the force F is in newtons (N) so that $k = 1/4\pi\varepsilon_o$. The constant ε_o is known as the *permittivity of free space* (in farads per meter) and has the value

$$\boxed{\begin{aligned} \varepsilon_o &= 8.854 \times 10^{-12} \simeq \frac{10^{-9}}{36\pi}\ \text{F/m} \\ \text{or } k &= \frac{1}{4\pi\varepsilon_o} \simeq 9 \times 10^9\ \text{m/F} \end{aligned}} \tag{4.2}$$

[3]Further details of experimental verification of Coulomb's law can be found in W. F. Magie, *A Source Book in Physics*. Cambridge: Harvard Univ. Press, 1963, pp. 408–420.

Thus eq. (4.1) becomes

$$F = \frac{Q_1 Q_2}{4\pi\varepsilon_o R^2} \tag{4.3}$$

If point charges Q_1 and Q_2 are located at points having position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, then the force $\mathbf{F}_{12}$ on Q_2 due to Q_1, shown in Figure 4.1, is given by

$$\boxed{\mathbf{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_o R^2}\mathbf{a}_{R_{12}}} \tag{4.4}$$

where

$$\mathbf{R}_{12} = \mathbf{r}_2 - \mathbf{r}_1 \tag{4.5a}$$

$$R = |\mathbf{R}_{12}| \tag{4.5b}$$

$$\mathbf{a}_{R_{12}} = \frac{\mathbf{R}_{12}}{R} \tag{4.5c}$$

By substituting eq. (4.5) into eq. (4.4), we may write eq. (4.4) as

$$\mathbf{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_o R^3}\mathbf{R}_{12} \tag{4.6a}$$

or

$$\mathbf{F}_{12} = \frac{Q_1 Q_2 (\mathbf{r}_2 - \mathbf{r}_1)}{4\pi\varepsilon_o |\mathbf{r}_2 - \mathbf{r}_1|^3} \tag{4.6b}$$

It is worthwhile to note that

1. As shown in Figure 4.1, the force $\mathbf{F}_{21}$ on Q_1 due to Q_2 is given by

$$\mathbf{F}_{21} = |\mathbf{F}_{12}|\mathbf{a}_{R_{21}} = |\mathbf{F}_{12}|(-\mathbf{a}_{R_{12}})$$

or

$$\mathbf{F}_{21} = -\mathbf{F}_{12} \tag{4.7}$$

since

$$\mathbf{a}_{R_{21}} = -\mathbf{a}_{R_{12}}$$

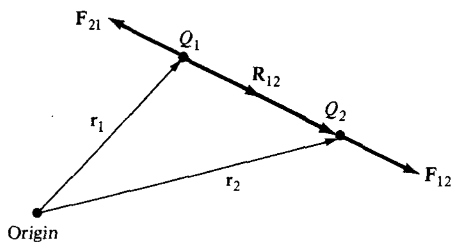

Figure 4.1 Coulomb vector force on point changes Q_1 and Q_2.

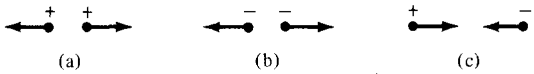

Figure 4.2 **(a)**, **(b)** Like charges repel; **(c)** unlike charges attract.

2. Like charges (charges of the same sign) repel each other while unlike charges attract. This is illustrated in Figure 4.2.
3. The distance R between the charged bodies Q_1 and Q_2 must be large compared with the linear dimensions of the bodies; that is, Q_1 and Q_2 must be point charges.
4. Q_1 and Q_2 must be static (at rest).
5. The signs of Q_1 and Q_2 must be taken into account in eq. (4.4).

If we have more than two point charges, we can use the *principle of superposition* to determine the force on a particular charge. The principle states that if there are N charges $Q_1, Q_2, \ldots, Q_N$ located, respectively, at points with position vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N$, the resultant force $\mathbf{F}$ on a charge Q located at point $\mathbf{r}$ is the vector sum of the forces exerted on Q by each of the charges $Q_1, Q_2, \ldots, Q_N$. Hence:

$$\mathbf{F} = \frac{QQ_1(\mathbf{r} - \mathbf{r}_1)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_1|^3} + \frac{QQ_2(\mathbf{r} - \mathbf{r}_2)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_2|^3} + \cdots + \frac{QQ_N(\mathbf{r} - \mathbf{r}_n)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_N|^3}$$

or

$$\boxed{\mathbf{F} = \frac{Q}{4\pi\varepsilon_o}\sum_{k=1}^{N}\frac{Q_k(\mathbf{r} - \mathbf{r}_k)}{|\mathbf{r} - \mathbf{r}_k|^3}} \tag{4.8}$$

We can now introduce the concept of *electric field intensity.*

1.4: Electric Field Intensity: -

The electric field intensity (or **electric field strength**) **E** is the force per unit charge when placed in the electric field.

Thus

$$\mathbf{E} = \lim_{Q \to 0} \frac{\mathbf{F}}{Q} \tag{4.9}$$

or simply

$$\boxed{\mathbf{E} = \frac{\mathbf{F}}{Q}} \tag{4.10}$$

The electric field intensity $\mathbf{E}$ is obviously in the direction of the force $\mathbf{F}$ and is measured in newtons/coulomb or volts/meter. The electric field intensity at point $\mathbf{r}$ due to a point charge located at $\mathbf{r}'$ is readily obtained from eqs. (4.6) and (4.10) as

$$\boxed{\mathbf{E} = \frac{Q}{4\pi\varepsilon_o R^2}\mathbf{a}_R = \frac{Q(\mathbf{r} - \mathbf{r}')}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}'|^3}} \tag{4.11}$$

For N point charges $Q_1, Q_2, \ldots, Q_N$ located at $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N$, the electric field intensity at point $\mathbf{r}$ is obtained from eqs. (4.8) and (4.10) as

$$\mathbf{E} = \frac{Q_1(\mathbf{r} - \mathbf{r}_1)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_1|^3} + \frac{Q_2(\mathbf{r} - \mathbf{r})_2}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_2|^3} + \cdots + \frac{Q_N(\mathbf{r} - \mathbf{r}_N)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_N|^3}$$

or

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_o}\sum_{k=1}^{N}\frac{Q_k(\mathbf{r} - \mathbf{r}_k)}{|\mathbf{r} - \mathbf{r}_k|^3} \tag{4.12}$$

EXAMPLE 4.1

Point charges 1 mC and −2 mC are located at (3, 2, −1) and (−1, −1, 4), respectively. Calculate the electric force on a 10-nC charge located at (0, 3, 1) and the electric field intensity at that point.

Solution:

$$\begin{aligned}
\mathbf{F} &= \sum_{k=1,2}\frac{QQ_k}{4\pi\varepsilon_o R^2}\mathbf{a}_R = \sum_{k=1,2}\frac{QQ_k(\mathbf{r} - \mathbf{r}_k)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_k|^3} \\
&= \frac{Q}{4\pi\varepsilon_o}\left\{\frac{10^{-3}[(0,3,1) - (3,2,-1)]}{|(0,3,1) - (3,2,-1)|^3} - \frac{2.10^{-3}[(0,3,1) - (-1,-1,4)]}{|(0,3,1) - (-1,-1,4)|^3}\right\} \\
&= \frac{10^{-3}\cdot 10\cdot 10^{-9}}{4\pi\cdot\dfrac{10^{-9}}{36\pi}}\left[\frac{(-3,1,2)}{(9+1+4)^{3/2}} - \frac{2(1,4,-3)}{(1+16+9)^{3/2}}\right] \\
&= 9\cdot 10^{-2}\left[\frac{(-3,1,2)}{14\sqrt{14}} + \frac{(-2,-8,6)}{26\sqrt{26}}\right] \\
\mathbf{F} &= -6.507\mathbf{a}_x - 3.817\mathbf{a}_y + 7.506\mathbf{a}_z \text{ mN}
\end{aligned}$$

At that point,

$$\begin{aligned}
\mathbf{E} &= \frac{\mathbf{F}}{Q} \\
&= (-6.507, -3.817, 7.506)\cdot\frac{10^{-3}}{10\cdot 10^{-9}} \\
\mathbf{E} &= -650.7\mathbf{a}_x - 381.7\mathbf{a}_y + 750.6\mathbf{a}_z \text{ kV/m}
\end{aligned}$$

PRACTICE EXERCISE 4.1

Point charges 5 nC and −2 nC are located at (2, 0, 4) and (−3, 0, 5), respectively.

(a) Determine the force on a 1-nC point charge located at (1, −3, 7).

(b) Find the electric field $\mathbf{E}$ at (1, −3, 7).

Answer: (a) $-1.004\mathbf{a}_x - 1.284\mathbf{a}_y + 1.4\mathbf{a}_z$ nN,
(b) $-1.004\mathbf{a}_x - 1.284\mathbf{a}_y + 1.4\mathbf{a}_z$ V/m.

EXAMPLE 4.2 Two point charges of equal mass m, charge Q are suspended at a common point by two threads of negligible mass and length ℓ. Show that at equilibrium the inclination angle α of each thread to the vertical is given by

$$Q^2 = 16\pi\,\varepsilon_o mg\ell^2 \sin^2\alpha \tan\alpha$$

If α is very small, show that

$$\alpha = \sqrt[3]{\frac{Q^2}{16\pi\varepsilon_o mg\ell^2}}$$

Solution:

Consider the system of charges as shown in Figure 4.3 where F_e is the electric or coulomb force, T is the tension in each thread, and mg is the weight of each charge. At A or B

$$T\sin\alpha = F_e$$

$$T\cos\alpha = mg$$

Hence,

$$\frac{\sin\alpha}{\cos\alpha} = \frac{F_e}{mg} = \frac{1}{mg}\cdot\frac{Q^2}{4\pi\varepsilon_o r^2}$$

But

$$r = 2\ell\sin\alpha$$

Hence,

$$Q^2\cos\alpha = 16\pi\varepsilon_o mg\ell^2\sin^3\alpha$$

or

$$Q^2 = 16\pi\varepsilon_o mg\ell^2\sin^2\alpha\tan\alpha$$

as required. When α is very small

$$\tan\alpha \simeq \alpha \simeq \sin\alpha$$

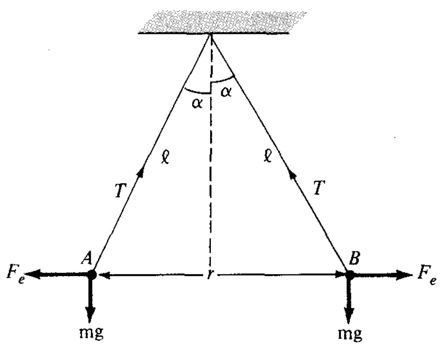

Figure 4.3 Suspended charged particles; for Example 4.2.

and so

$$Q^2 = 16\pi\varepsilon_o mg\ell^2\alpha^3$$

or

$$\alpha = \sqrt[3]{\frac{Q^2}{16\pi\varepsilon_o mg\ell^2}}$$

PRACTICE EXERCISE 4.2

Three identical small spheres of mass m are suspended by threads of negligible masses and equal length ℓ from a common point. A charge Q is divided equally between the spheres and they come to equilibrium at the corners of a horizontal equilateral triangle whose sides are d. Show that

$$Q^2 = 12\pi\varepsilon_o mgd^3\left[\ell^2 - \frac{d^2}{3}\right]^{-1/2}$$

where g = acceleration due to gravity.

Answer: Proof.

EXAMPLE 4.3

A practical application of electrostatics is in electrostatic separation of solids. For example, Florida phosphate ore, consisting of small particles of quartz and phosphate rock, can be separated into its components by applying a uniform electric field as in Figure 4.4. Assuming zero initial velocity and displacement, determine the separation between the particles after falling 80 cm. Take E = 500 kV/m and Q/m = 9 μC/kg for both positively and negatively charged particles.

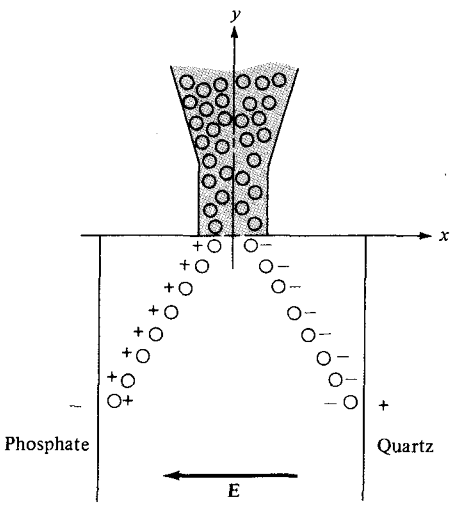

Figure 4.4 Electrostatic separation of solids; for Example 4.3.

Solution:

Ignoring the coulombic force between particles, the electrostatic force is acting horizontally while the gravitational force (weight) is acting vertically on the particles. Thus,

$$Q\mathbf{E} = m\frac{d^2x}{dt^2}\mathbf{a}_x$$

or

$$\frac{d^2x}{dt^2} = \frac{Q}{m}E$$

Integrating twice gives

$$x = \frac{Q}{2m}Et^2 + c_1t + c_2$$

where c_1 and c_2 are integration constants. Similarly,

$$-mg = m\frac{d^2y}{dt^2}$$

or

$$\frac{d^2y}{dt^2} = -g$$

Integrating twice, we get

$$y = -1/2gt^2 + c_3t + c_4$$

Since the initial displacement is zero,

$$x(t = 0) = 0 \rightarrow c_2 = 0$$

$$y(t = 0) = 0 \rightarrow c_4 = 0$$

Also, due to zero initial velocity,

$$\left.\frac{dx}{dt}\right|_{t=0} = 0 \rightarrow c_1 = 0$$

$$\left.\frac{dy}{dt}\right|_{t=0} = 0 \rightarrow c_3 = 0$$

Thus

$$x = \frac{QE}{2m}t^2 \qquad y = -\frac{1}{2}gt^2$$

When $y = -80 \text{ cm} = -0.8 \text{ m}$

$$t^2 = \frac{0.8 \times 2}{9.8} = 0.1633$$

and

$$x = 1/2 \times 9 \times 10^{-6} \times 5 \times 10^5 \times 0.1633 = 0.3673 \text{ m}$$

The separation between the particles is $2x = 73.47$ cm.

PRACTICE EXERCISE 4.3

An ion rocket emits positive cesium ions from a wedge-shape electrode into the region described by $x > |y|$. The electric field is $\mathbf{E} = -400\mathbf{a}_x + 200\mathbf{a}_y$ kV/m. The ions have single electronic charges $e = -1.6019 \times 10^{-19}$ C and mass $m = 2.22 \times 10^{-25}$ kg and travel in a vacuum with zero initial velocity. If the emission is confined to $-40 \text{ cm} < y < 40$ cm, find the largest value of x which can be reached.

Answer: 0.8 m.

4.3 ELECTRIC FIELDS DUE TO CONTINUOUS CHARGE DISTRIBUTIONS

1.2

So far we have only considered forces and electric fields due to point charges, which are essentially charges occupying very small physical space. It is also possible to have continuous charge distribution along a line, on a surface, or in a volume as illustrated in Figure 4.5.

It is customary to denote the line charge density, surface charge density, and volume charge density by ρ_L (in C/m), ρ_S (in C/m^2), and ρ_v (in C/m^3), respectively. These must not be confused with ρ (without subscript) used for radial distance in cylindrical coordinates.

The charge element dQ and the total charge Q due to these charge distributions are obtained from Figure 4.5 as

$$dQ = \rho_L \, dl \rightarrow Q = \int_L \rho_L \, dl \qquad \text{(line charge)} \qquad (4.13a)$$

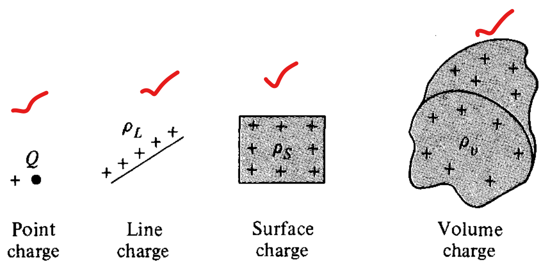

Figure 4.5 Various charge distributions and charge elements.

$$dQ = \rho_S \, dS \rightarrow Q = \int_S \rho_S \, dS \qquad \text{(surface charge)} \tag{4.13b}$$

$$dQ = \rho_v \, dv \rightarrow Q = \int_v \rho_v \, dv \qquad \text{(volume charge)} \tag{4.13c}$$

The electric field intensity due to each of the charge distributions ρ_L, ρ_S, and ρ_v may be regarded as the summation of the field contributed by the numerous point charges making up the charge distribution. Thus by replacing Q in eq. (4.11) with charge element $dQ =$ ρL dl, $\rho_S \, dS$, or $\rho_v \, dv$ and integrating, we get

$$\mathbf{E} = \int \frac{\rho_L \, dl}{4\pi\varepsilon_o R^2} \mathbf{a}_R \qquad \text{(line charge)} \tag{4.14}$$

$$\mathbf{E} = \int \frac{\rho_S \, dS}{4\pi\varepsilon_o R^2} \mathbf{a}_R \qquad \text{(surface charge)} \tag{4.15}$$

$$\mathbf{E} = \int \frac{\rho_v \, dv}{4\pi\varepsilon_o R^2} \mathbf{a}_R \qquad \text{(volume charge)} \tag{4.16}$$

It should be noted that R^2 and $\mathbf{a}_R$ vary as the integrals in eqs. (4.13) to (4.16) are evaluated. We shall now apply these formulas to some specific charge distributions.

A. A Line Charge

Consider a line charge with uniform charge density ρ_L extending from A to B along the z-axis as shown in Figure 4.6. The charge element dQ associated with element $dl = dz$ of the line is

$$dQ = \rho_L \, dl = \rho_L \, dz$$

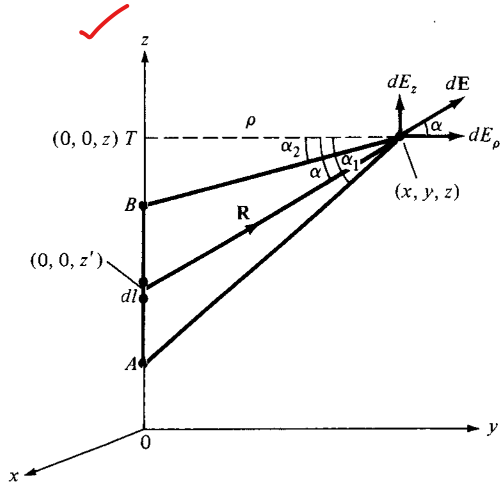

Figure 4.6 Evaluation of the **E** field due to a line charge.

and hence the total charge Q is

$$Q = \int_{z_A}^{z_B} \rho_L \, dz \tag{4.17}$$

The electric field intensity $\mathbf{E}$ at an arbitrary point $P(x, y, z)$ can be found using eq. (4.14). It is important that we learn to derive and substitute each term in eqs. (4.14) to (4.15) for a given charge distribution. It is customary to denote the field point[4] by (x, y, z) and the source point by (x', y', z'). Thus from Figure 4.6,

$$dl = dz'$$

$$\mathbf{R} = (x, y, z) - (0, 0, z') = x\mathbf{a}_x + y\mathbf{a}_y + (z - z')\mathbf{a}_z$$

or

$$\mathbf{R} = \rho\mathbf{a}_\rho + (z - z')\,\mathbf{a}_z$$

$$R^2 = |\mathbf{R}|^2 = x^2 + y^2 + (z - z')^2 = \rho^2 + (z - z')^2$$

$$\frac{\mathbf{a}_R}{R^2} = \frac{\mathbf{R}}{|\mathbf{R}|^3} = \frac{\rho\mathbf{a}_\rho + (z - z')\mathbf{a}_z}{[\rho^2 + (z - z')^2]^{3/2}}$$

Substituting all this into eq. (4.14), we get

$$\mathbf{E} = \frac{\rho_L}{4\pi\varepsilon_o} \int \frac{\rho\mathbf{a}_\rho + (z - z')\,\mathbf{a}_z}{[\rho^2 + (z - z')^2]^{3/2}}\, dz' \tag{4.18}$$

To evaluate this, it is convenient that we define α, α_1, and α_2 as in Figure 4.6.

$$R = [\rho^2 + (z - z')^2]^{1/2} = \rho \sec \alpha$$

$$z' = OT - \rho \tan \alpha, \qquad dz' = -\rho \sec^2 \alpha \, d\alpha$$

Hence, eq. (4.18) becomes

$$\begin{aligned} \mathbf{E} &= \frac{-\rho_L}{4\pi\varepsilon_o} \int_{\alpha_1}^{\alpha_2} \frac{\rho \sec^2 \alpha \,[\cos \alpha \,\mathbf{a}_\rho + \sin \alpha \,\mathbf{a}_z]\, d\alpha}{\rho^2 \sec^2 \alpha} \\ &= -\frac{\rho_L}{4\pi\varepsilon_o\rho} \int_{\alpha_1}^{\alpha_2} [\cos \alpha \,\mathbf{a}_\rho + \sin \alpha \,\mathbf{a}_z]\, d\alpha \end{aligned} \tag{4.19}$$

Thus for a *finite line charge,*

$$\mathbf{E} = \frac{\rho_L}{4\pi\varepsilon_o\rho}[-(\sin \alpha_2 - \sin \alpha_1)\mathbf{a}_\rho + (\cos \alpha_2 - \cos \alpha_1)\mathbf{a}_z] \tag{4.20}$$

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[4]The field point is the point at which the field is to be evaluated.

As a special case, for an *infinite line charge,* point B is at $(0, 0, \infty)$ and A at $(0, 0, -\infty)$ so that $\alpha_1 = \pi/2$, $\alpha_2 = -\pi/2$; the z-component vanishes and eq. (4.20) becomes

$$\boxed{\mathbf{E} = \frac{\rho_L}{2\pi\varepsilon_o\rho}\,\mathbf{a}_\rho} \tag{4.21}$$

Bear in mind that eq. (4.21) is obtained for an infite line charge along the z-axis so that ρ and $\mathbf{a}_\rho$ have their usual meaning. If the line is not along the z-axis, ρ is the perpendicular distance from the line to the point of interest and $\mathbf{a}_\rho$ is a unit vector along that distance directed from the line charge to the field point.

B. A Surface Charge

Consider an infinite sheet of charge in the xy-plane with uniform charge density ρ_S. The charge associated with an elemental area dS is

$$dQ = \rho_S\, dS$$

and hence the total charge is

$$Q = \int \rho_S\, dS \tag{4.22}$$

From eq. (4.15), the contribution to the **E** field at point $P(0, 0, h)$ by the elemental surface 1 shown in Figure 4.7 is

$$d\mathbf{E} = \frac{dQ}{4\pi\varepsilon_o R^2}\,\mathbf{a}_R \tag{4.23}$$

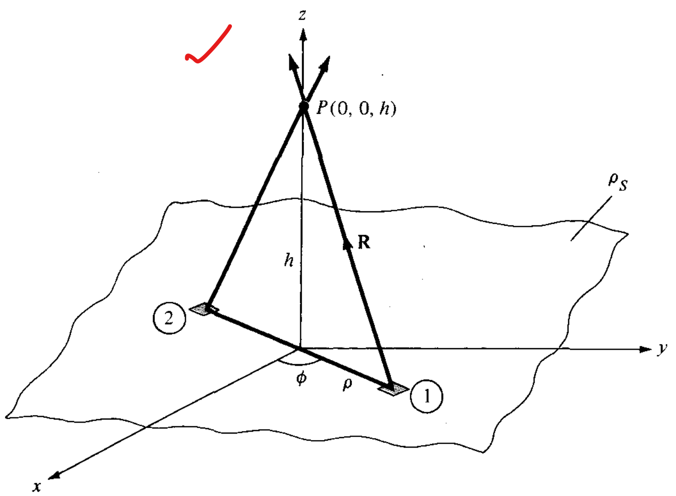

Figure 4.7 Evaluation of the **E** field due to an infinite sheet of charge.

From Figure 4.7,

$$\mathbf{R} = \rho(-\mathbf{a}_\rho) + h\mathbf{a}_z, \qquad R = |\mathbf{R}| = [\rho^2 + h^2]^{1/2}$$

$$\mathbf{a}_R = \frac{\mathbf{R}}{R}, \qquad dQ = \rho_S\, dS = \rho_S\, \rho\, d\phi\, d\rho$$

Substitution of these terms into eq. (4.23) gives

$$d\mathbf{E} = \frac{\rho_S\, \rho\, d\phi\, d\rho\,[-\rho\mathbf{a}_\rho + h\mathbf{a}_z]}{4\pi\varepsilon_o[\rho^2 + h^2]^{3/2}} \tag{4.24}$$

Due to the symmetry of the charge distribution, for every element 1, there is a corresponding element 2 whose contribution along $\mathbf{a}_\rho$ cancels that of element 1, as illustrated in Figure 4.7. Thus the contributions to E_ρ add up to zero so that $\mathbf{E}$ has only z-component. This can also be shown mathematically by replacing $\mathbf{a}_\rho$ with $\cos\phi\, \mathbf{a}_x + \sin\phi\, \mathbf{a}_y$. Integration of $\cos\phi$ or $\sin\phi$ over $0 < \phi < 2\pi$ gives zero. Therefore,

$$\begin{aligned}
\mathbf{E} &= \int dE_z = \frac{\rho_S}{4\pi\varepsilon_o}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{\infty}\frac{h\rho\, d\rho\, d\phi}{[\rho^2 + h^2]^{3/2}}\,\mathbf{a}_z \\
&= \frac{\rho_S h}{4\pi\varepsilon_o}\, 2\pi \int_0^{\infty}[\rho^2 + h^2]^{-3/2}\,\frac{1}{2}\, d(\rho^2)\,\mathbf{a}_z \\
&= \frac{\rho_S h}{2\varepsilon_o}\left\{-[\rho^2 + h^2]^{-1/2}\right\}_0^{\infty}\mathbf{a}_z
\end{aligned}$$

$$\mathbf{E} = \frac{\rho_S}{2\varepsilon_o}\,\mathbf{a}_z \tag{4.25}$$

that is, $\mathbf{E}$ has only z-component if the charge is in the xy-plane. In general, for an *infinite sheet* of charge

$$\boxed{\mathbf{E} = \frac{\rho_S}{2\varepsilon_o}\,\mathbf{a}_n} \tag{4.26}$$

where $\mathbf{a}_n$ is a unit vector normal to the sheet. From eq. (4.25) or (4.26), we notice that the electric field is normal to the sheet and it is surprisingly independent of the distance between the sheet and the point of observation P. In a parallel plate capacitor, the electric field existing between the two plates having equal and opposite charges is given by

$$\mathbf{E} = \frac{\rho_S}{2\varepsilon_o}\,\mathbf{a}_n + \frac{-\rho_S}{2\varepsilon_o}(-\mathbf{a}_n) = \frac{\rho_S}{\varepsilon_o}\,\mathbf{a}_n \tag{4.27}$$

C. A Volume Charge

Let the volume charge distribution with uniform charge density ρ_v be as shown in Figure 4.8. The charge dQ associated with the elemental volume dv is

$$dQ = \rho_v\, dv$$

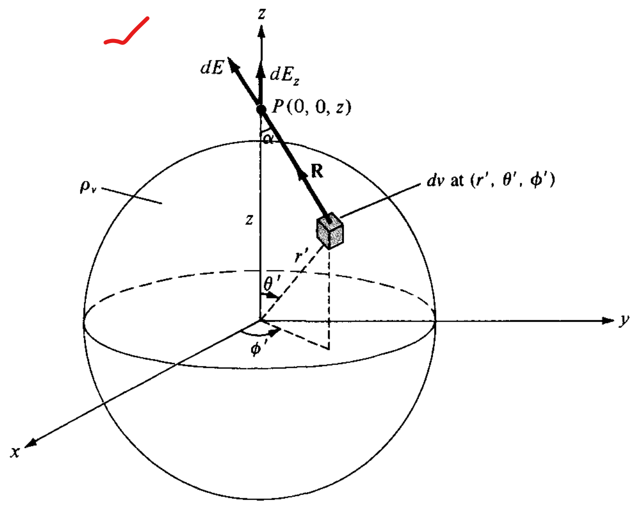

Figure 4.8 Evaluation of the **E** field due to a volume charge distribution.

and hence the total charge in a sphere of radius a is

$$Q = \int \rho_v \, dv = \rho_v \int dv$$
$$= \rho_v \frac{4\pi a^3}{3} \tag{4.28}$$

The electric field $d\mathbf{E}$ at $P(0, 0, z)$ due to the elementary volume charge is

$$d\mathbf{E} = \frac{\rho_v \, dv}{4\pi\varepsilon_o R^2} \mathbf{a}_R$$

where $\mathbf{a}_R = \cos\alpha \, \mathbf{a}_z + \sin\alpha \, \mathbf{a}_\rho$. Due to the symmetry of the charge distribution, the contributions to E_x or E_y add up to zero. We are left with only E_z, given by

$$E_z = \mathbf{E} \cdot \mathbf{a}_z = \int dE \cos\alpha = \frac{\rho_v}{4\pi\varepsilon_o} \int \frac{dv \cos\alpha}{R^2} \tag{4.29}$$

Again, we need to derive expressions for dv, R^2, and $\cos\alpha$.

$$dv = r'^2 \sin\theta' \, dr' \, d\theta' \, d\phi' \tag{4.30}$$

Applying the cosine rule to Figure 4.8, we have

$$R^2 = z^2 + r'^2 - 2zr' \cos\theta'$$
$$r'^2 = z^2 + R^2 - 2zR\cos\alpha$$

It is convenient to evaluate the integral in eq. (4.29) in terms of R and r'. Hence we express $\cos\theta'$, $\cos\alpha$, and $\sin\theta'\,d\theta'$ in terms of R and r', that is,

$$\cos\alpha = \frac{z^2 + R^2 - r'^2}{2zR} \tag{4.31a}$$

$$\cos\theta' = \frac{z^2 + r'^2 - R^2}{2zr'} \tag{4.31b}$$

Differentiating eq. (4.31b) with respect to θ' keeping z and r' fixed, we obtain

$$\sin\theta'\,d\theta' = \frac{R\,dR}{z\,r'} \tag{4.32}$$

Substituting eqs. (4.30) to (4.32) into eq. (4.29) yields

$$\begin{aligned}
E_z &= \frac{\rho_v}{4\pi\varepsilon_o}\int_{\phi'=0}^{2\pi} d\phi' \int_{r'=0}^{a}\int_{R=z-r'}^{z+r'} r'^2\,\frac{R\,dR}{zr'}\,dr'\,\frac{z^2+R^2-r'^2}{2zR}\,\frac{1}{R^2}\\
&= \frac{\rho_v 2\pi}{8\pi\varepsilon_o z^2}\int_{r'=0}^{a}\int_{R=z-r'}^{z+r'} r'\left[1+\frac{z^2-r'^2}{R^2}\right] dR\,dr'\\
&= \frac{\rho_v\pi}{4\pi\varepsilon_o z^2}\int_0^a r'\left[R-\frac{(z^2-r'^2)}{R}\right]_{z-r'}^{z+r'} dr'\\
&= \frac{\rho_v\pi}{4\pi\varepsilon_o z^2}\int_0^a 4r'^2\,dr' = \frac{1}{4\pi\varepsilon_o}\frac{1}{z^2}\left(\frac{4}{3}\pi a^3\rho_v\right)
\end{aligned}$$

or

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o z^2}\,\mathbf{a}_z \tag{4.33}$$

This result is obtained for $\mathbf{E}$ at $P(0, 0, z)$. Due to the symmetry of the charge distribution, the electric field at $P(r, \theta, \phi)$ is readily obtained from eq. (4.33) as

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o r^2}\,\mathbf{a}_r \tag{4.34}$$

which is identical to the electric field at the same point due to a point charge Q located at the origin or the center of the spherical charge distribution. The reason for this will become obvious as we cover Gauss's law in Section 4.5.

EXAMPLE 4.4

A circular ring of radius a carries a uniform charge ρ_L C/m and is placed on the xy-plane with axis the same as the z-axis.

(a) Show that

$$\mathbf{E}(0, 0, h) = \frac{\rho_L a h}{2\varepsilon_o[h^2 + a^2]^{3/2}}\,\mathbf{a}_z$$

(b) What values of h gives the maximum value of $\mathbf{E}$?

(c) If the total charge on the ring is Q, find $\mathbf{E}$ as $a \to 0$.

Solution:

(a) Consider the system as shown in Figure 4.9. Again the trick in finding $\mathbf{E}$ using eq. (4.14) is deriving each term in the equation. In this case,

$$dl = a\,d\phi, \qquad \mathbf{R} = a(-\mathbf{a}_\rho) + h\mathbf{a}_z$$

$$R = |\mathbf{R}| = \left[a^2 + h^2\right]^{1/2}, \qquad \mathbf{a}_R = \frac{\mathbf{R}}{R}$$

or

$$\frac{\mathbf{a}_R}{R^2} = \frac{\mathbf{R}}{|\mathbf{R}|^3} = \frac{-a\mathbf{a}_\rho + h\mathbf{a}_z}{\left[a^2 + h^2\right]^{3/2}}$$

Hence

$$\mathbf{E} = \frac{\rho_L}{4\pi\varepsilon_o} \int_{\phi=0}^{2\pi} \frac{(-a\mathbf{a}_\rho + h\mathbf{a}_z)}{\left[a^2 + h^2\right]^{3/2}}\, a\,d\phi$$

By symmetry, the contributions along $\mathbf{a}_\rho$ add up to zero. This is evident from the fact that for every element dl there is a corresponding element diametrically opposite it that gives an equal but opposite dE_ρ so that the two contributions cancel each other. Thus we are left with the z-component. That is,

$$\mathbf{E} = \frac{\rho_L a h \mathbf{a}_z}{4\pi\varepsilon_o\left[h^2 + a^2\right]^{3/2}} \int_0^{2\pi} d\phi = \frac{\rho_L a h \mathbf{a}_z}{2\varepsilon_o\left[h^2 + a^2\right]^{3/2}}$$

as required.

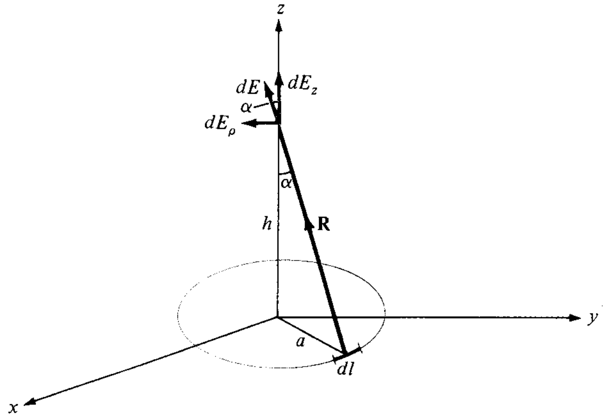

Figure 4.9 Charged ring; for Example 4.4.

(b)

$$\frac{d|\mathbf{E}|}{dh} = \frac{\rho_L a}{2\varepsilon_o}\left\{\frac{[h^2 + a^2]^{3/2}(1) - \frac{3}{2}(h)2h[h^2 + a^2]^{1/2}}{[h^2 + a^2]^3}\right\}$$

For maximum $\mathbf{E}$, $\dfrac{d|\mathbf{E}|}{dh} = 0$, which implies that

$$[h^2 + a^2]^{1/2}[h^2 + a^2 - 3h^2] = 0$$

$$a^2 - 2h^2 = 0 \qquad \text{or} \qquad h = \pm\frac{a}{\sqrt{2}}$$

(c) Since the charge is uniformly distributed, the line charge density is

$$\rho_L = \frac{Q}{2\pi a}$$

so that

$$\mathbf{E} = \frac{Qh}{4\pi\varepsilon_o[h^2 + a^2]^{3/2}}\mathbf{a}_z$$

As $a \to 0$

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o h^2}\mathbf{a}_z$$

or in general

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o r^2}\mathbf{a}_R$$

which is the same as that of a point charge as one would expect.

PRACTICE EXERCISE 4.4

A circular disk of radius a is uniformly charged with ρ_S C/m^2. If the disk lies on the $z = 0$ plane with its axis along the z-axis,

(a) Show that at point $(0, 0, h)$

$$\mathbf{E} = \frac{\rho_S}{2\varepsilon_o}\left\{1 - \frac{h}{[h^2 + a^2]^{1/2}}\right\}\mathbf{a}_z$$

(b) From this, derive the $\mathbf{E}$ field due to an infinite sheet of charge on the $z = 0$ plane.

(c) If $a \ll h$, show that $\mathbf{E}$ is similar to the field due to a point charge.

Answer: (a) Proof, (b) $\dfrac{\rho_S}{2\varepsilon_o}\mathbf{a}_z$, (c) Proof

EXAMPLE 4.5 The finite sheet $0 \leq x \leq 1$, $0 \leq y \leq 1$ on the $z = 0$ plane has a charge density $\rho_S = xy(x^2 + y^2 + 25)^{3/2}$ nC/m^2. Find

(a) The total charge on the sheet
(b) The electric field at (0, 0, 5)
(c) The force experienced by a -1 mC charge located at (0, 0, 5)

Solution:

(a) $Q = \int \rho_S \, dS = \int_0^1 \int_0^1 xy(x^2 + y^2 + 25)^{3/2} \, dx \, dy$ nC

Since $x\,dx = 1/2\, d(x^2)$, we now integrate with respect to x^2 (or change variables: $x^2 = u$ so that $x\,dx = du/2$).

$$\begin{aligned}
Q &= \frac{1}{2}\int_0^1 y \int_0^1 (x^2 + y^2 + 25)^{3/2} \, d(x^2) \, dy \text{ nC} \\
&= \frac{1}{2}\int_0^1 y \frac{2}{5}(x^2 + y^2 + 25)^{5/2} \Big|_0^1 dy \\
&= \frac{1}{5}\int_0^1 \frac{1}{2}\left[(y^2 + 26)^{5/2} - (y^2 + 25)^{5/2}\right] d(y^2) \\
&= \frac{1}{10} \cdot \frac{2}{7}\left[(y^2 + 26)^{7/2} - (y^2 + 25)^{7/2}\right] \Big|_0^1 \\
&= \frac{1}{35}\left[(27)^{7/2} + (25)^{7/2} - 2(26)^{7/2}\right] \\
Q &= 33.15 \text{ nC}
\end{aligned}$$

(b) $\mathbf{E} = \int \dfrac{\rho_S \, dS \, \mathbf{a}_R}{4\pi\varepsilon_o r^2} = \int \dfrac{\rho_S \, dS \, (\mathbf{r} - \mathbf{r}')}{4\pi\varepsilon_o |\mathbf{r} - \mathbf{r}'|^3}$

where $\mathbf{r} - \mathbf{r}' = (0, 0, 5) - (x, y, 0) = (-x, -y, 5)$. Hence,

$$\begin{aligned}
\mathbf{E} &= \int_0^1 \int_0^1 \frac{10^{-9} xy(x^2 + y^2 + 25)^{3/2}(-x\mathbf{a}_x - y\mathbf{a}_y + 5\mathbf{a}_z) dx \, dy}{4\pi \cdot \dfrac{10^{-9}}{36\pi}(x^2 + y^2 + 25)^{3/2}} \\
&= 9\left[-\int_0^1 x^2 \, dx \int_0^1 y \, dy \, \mathbf{a}_x - \int_0^1 x \, dx \int_0^1 y^2 dy \, \mathbf{a}_y + 5\int_0^1 x \, dx \int_0^1 y \, dy \, \mathbf{a}_z\right] \\
&= 9\left(\frac{-1}{6}, \frac{-1}{6}, \frac{5}{4}\right) \\
&= (-1.5, -1.5, 11.25) \text{ V/m}
\end{aligned}$$

(c) $\mathbf{F} = q\mathbf{E} = (1.5, 1.5, -11.25)$ mN

PRACTICE EXERCISE 4.5

A square plate described by $-2 \leq x \leq 2$, $-2 \leq y \leq 2$, $z = 0$ carries a charge $12\,|y|$ mC/m^2. Find the total charge on the plate and the electric field intensity at (0, 0, 10).

Answer: 192 mC, 16.46 $\mathbf{a}_z$ MV/m.

EXAMPLE 4.6

Planes $x = 2$ and $y = -3$, respectively, carry charges 10 nC/m^2 and 15 nC/m^2. If the line $x = 0$, $z = 2$ carries charge 10π nC/m, calculate $\mathbf{E}$ at $(1, 1, -1)$ due to the three charge distributions.

Solution:

Let

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3$$

where $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ are, respectively, the contributions to $\mathbf{E}$ at point $(1, 1, -1)$ due to the infinite sheet 1, infinite sheet 2, and infinite line 3 as shown in Figure 4.10(a). Applying eqs. (4.26) and (4.21) gives

$$\mathbf{E}_1 = \frac{\rho_{S_1}}{2\varepsilon_o}(-\mathbf{a}_x) = -\frac{10 \cdot 10^{-9}}{2 \cdot \dfrac{10^{-9}}{36\pi}}\mathbf{a}_x = -180\pi\mathbf{a}_x$$

$$\mathbf{E}_2 = \frac{\rho_{S_2}}{2\varepsilon_o}\mathbf{a}_y = \frac{15 \cdot 10^{-9}}{2 \cdot \dfrac{10^{-9}}{36\pi}}\mathbf{a}_y = 270\pi\,\mathbf{a}_y$$

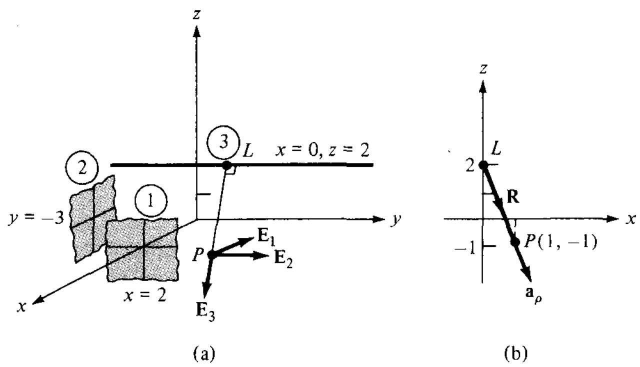

Figure 4.10 For Example 4.6: **(a)** three charge distributions; **(b)** finding ρ and $\mathbf{a}_\rho$ on plane $y = 1$.

and

$$\mathbf{E}_3 = \frac{\rho_L}{2\pi\varepsilon_o\rho}\,\mathbf{a}_\rho$$

where $\mathbf{a}_\rho$ (not regular $\mathbf{a}_\rho$ but with a similar meaning) is a unit vector along LP perpendicular to the line charge and ρ is the length LP to be determined from Figure 4.10(b). Figure 4.10(b) results from Figure 4.10(a) if we consider plane $y = 1$ on which $\mathbf{E}_3$ lies. From Figure 4.10(b), the distance vector from L to P is

$$\mathbf{R} = -3\mathbf{a}_z + \mathbf{a}_x$$

$$\rho = |\mathbf{R}| = \sqrt{10}, \qquad \mathbf{a}_\rho = \frac{\mathbf{R}}{|\mathbf{R}|} = \frac{1}{\sqrt{10}}\,\mathbf{a}_x - \frac{3}{\sqrt{10}}\,\mathbf{a}_z$$

Hence,

$$\begin{aligned}\mathbf{E}_3 &= \frac{10\pi \cdot 10^{-9}}{2\pi \cdot \dfrac{10^{-9}}{36\pi}} \cdot \frac{1}{10}(\mathbf{a}_x - 3\mathbf{a}_z)\\ &= 18\pi(\mathbf{a}_x - 3\mathbf{a}_z)\end{aligned}$$

Thus by adding $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$, we obtain the total field as

$$\mathbf{E} = -162\pi\mathbf{a}_x + 270\pi\mathbf{a}_y - 54\pi\mathbf{a}_z \text{ V/m}$$

Note that to obtain $\mathbf{a}_r$, $\mathbf{a}_\rho$, or $\mathbf{a}_n$, which we always need for finding $\mathbf{F}$ or $\mathbf{E}$, we must go from the charge (at position vector $\mathbf{r}'$) to the field point (at position vector $\mathbf{r}$); hence $\mathbf{a}_r$, $\mathbf{a}_\rho$, or $\mathbf{a}_n$ is a unit vector along $\mathbf{r} - \mathbf{r}'$. Observe this carefully in Figures 4.6 to 4.10.

PRACTICE EXERCISE 4.6

In Example 4.6 if the line $x = 0$, $z = 2$ is rotated through 90° about the point (0, 2, 2) so that it becomes $x = 0$, $y = 2$, find $\mathbf{E}$ at (1, 1, −1).

Answer: $-282.7\mathbf{a}_x + 564.5\mathbf{a}_y$ V/m.

1.5

4.4 ELECTRIC FLUX DENSITY

The flux due to the electric field $\mathbf{E}$ can be calculated using the general definition of flux in eq. (3.13). For practical reasons, however, this quantity is not usually considered as the most useful flux in electrostatics. Also, eqs. (4.11) to (4.16) show that the electric field intensity is dependent on the medium in which the charge is placed (free space in this chapter). Suppose a new vector field $\mathbf{D}$ independent of the medium is defined by

$$\boxed{\mathbf{D} = \varepsilon_o\mathbf{E}} \tag{4.35}$$

We define *electric flux* Ψ in terms of **D** using eq. (3.13), namely,

$$\Psi = \int \mathbf{D} \cdot d\mathbf{S} \tag{4.36}$$

In SI units, one line of electric flux emanates from +1 C and terminates on −1 C. Therefore, the electric flux is measured in coulombs. Hence, the vector field **D** is called the *electric flux density* and is measured in coulombs per square meter. For historical reasons, the electric flux density is also called *electric displacement.*

From eq. (4.35), it is apparent that all the formulas derived for **E** from Coulomb's law in Sections 4.2 and 4.3 can be used in calculating **D**, except that we have to multiply those formulas by ε_o. For example, for an infinite sheet of charge, eqs. (4.26) and (4.35) give

$$\mathbf{D} = \frac{\rho_S}{2}\,\mathbf{a}_n \tag{4.37}$$

and for a volume charge distribution, eqs. (4.16) and (4.35) give

$$\mathbf{D} = \int \frac{\rho_v\, dv}{4\pi R^2}\,\mathbf{a}_R \tag{4.38}$$

Note from eqs. (4.37) and (4.38) that **D** is a function of charge and position only; it is independent of the medium.

EXAMPLE 4.7

Determine **D** at (4, 0, 3) if there is a point charge -5π mC at (4, 0, 0) and a line charge 3π mC/m along the y-axis.

Solution:

Let $\mathbf{D} = \mathbf{D}_Q + \mathbf{D}_L$ where $\mathbf{D}_Q$ and $\mathbf{D}_L$ are flux densities due to the point charge and line charge, respectively, as shown in Figure 4.11:

$$\mathbf{D}_Q = \varepsilon_o \mathbf{E} = \frac{Q}{4\pi R^2}\,\mathbf{a}_R = \frac{Q\,(\mathbf{r} - \mathbf{r}')}{4\pi|\mathbf{r} - \mathbf{r}'|^3}$$

where $\mathbf{r} - \mathbf{r}' = (4, 0, 3) - (4, 0, 0) = (0, 0, 3)$. Hence,

$$\mathbf{D}_Q = \frac{-5\pi \cdot 10^{-3}(0, 0, 3)}{4\pi|(0, 0, 3)|^3} = -0.138\,\mathbf{a}_z \text{ mC/m}^2$$

Also

$$\mathbf{D}_L = \frac{\rho_L}{2\pi\rho}\,\mathbf{a}_\rho$$

In this case

$$\mathbf{a}_\rho = \frac{(4, 0, 3) - (0, 0, 0)}{|(4, 0, 3) - (0, 0, 0)|} = \frac{(4, 0, 3)}{5}$$

$$\rho = |(4, 0, 3) - (0, 0, 0)| = 5$$

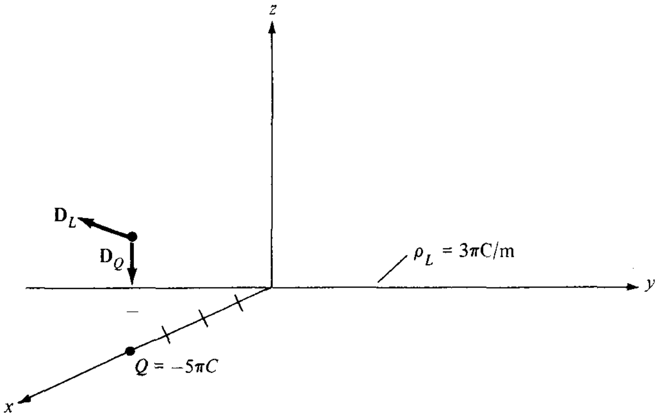

Figure 4.11 Flux density **D** due to a point charge and an infinite line charge.

Hence,

$$\mathbf{D}_L = \frac{3\pi}{2\pi(25)}(4\mathbf{a}_x + 3\mathbf{a}_z) = 0.24\mathbf{a}_x + 0.18\mathbf{a}_z \text{ mC/m}^2$$

Thus

$$\begin{aligned}\mathbf{D} &= \mathbf{D}_Q + \mathbf{D}_L \\ &= 240\mathbf{a}_x + 42\mathbf{a}_z\ \mu\text{C/m}^2\end{aligned}$$

PRACTICE EXERCISE 4.7

A point charge of 30 nC is located at the origin while plane $y = 3$ carries charge 10 nC/m^2. Find **D** at (0, 4, 3).

Answer: $5.076\mathbf{a}_y + 0.0573\mathbf{a}_z$ nC/m^2.

1.6

4.5 GAUSS'S LAW—MAXWELL'S EQUATION

Gauss's[5] law constitutes one of the fundamental laws of electromagnetism.

Gauss's law states that the total electric flux Ψ through any *closed* surface is equal to the total charge enclosed by that surface.

[5]Karl Friedrich Gauss (1777–1855), a German mathematician, developed the divergence theorem of Section 3.6, popularly known by his name. He was the first physicist to measure electric and magnetic quantities in absolute units. For details on Gauss's measurements, see W. F. Magie, *A Source Book in Physics*. Cambridge: Harvard Univ. Press, 1963, pp. 519–524.

Thus

$$\Psi = Q_{enc} \tag{4.39}$$

that is,

$$\Psi = \oint d\Psi = \oint_S \mathbf{D} \cdot d\mathbf{S}$$
$$= \text{Total charge enclosed } Q = \int \rho_v \, dv \tag{4.40}$$

or

$$\boxed{Q = \oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v \, dv} \tag{4.41}$$

By applying divergence theorem to the middle term in eqs. (4.41)

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{D} \, dv \tag{4.42}$$

Comparing the two volume integrals in eqs. (4.41) and (4.42) results in

$$\boxed{\rho_v = \nabla \cdot \mathbf{D}} \tag{4.43}$$

which is the first of the four *Maxwell's equations* to be derived. Equation (4.43) states that the volume charge density is the same as the divergence of the electric flux density. This should not be surprising to us from the way we defined the divergence of a vector in eq. (3.32) and from the fact that ρ_v at a point is simply the charge per unit volume at that point.

Note that:

1. Equations (4.41) and (4.43) are basically stating Gauss's law in different ways; eq. (4.41) is the integral form, whereas eq. (4.43) is the differential or point form of Gauss's law.

2. Gauss's law is an alternative statement of Coulomb's law; proper application of the divergence theorem to Coulomb's law results in Gauss's law.

3. Gauss's law provides an easy means of finding **E** or **D** for symmetrical charge distributions such as a point charge, an infinite line charge, an infinite cylindrical surface charge, and a spherical distribution of charge. A continuous charge distribution has rectangular symmetry if it depends only on x (or y or z), cylindrical symmetry if it depends only on ρ, or spherical symmetry if it depends only on r (independent of θ and ϕ). It must be stressed that whether the charge distribution is symmetric or not, Gauss's law always holds. For example, consider the charge distribution in Figure 4.12 where v_1 and v_2 are closed surfaces (or volumes). The total flux leaving v_1 is $10 - 5 = 5$ nC because only 10 nC and -5 nC charges are enclosed by v_1. Although charges 20 nC and 15 nC outside v_1 do contribute to the flux crossing v_1, the net flux crossing v_1, according to Gauss's law, is irrespective of those charges outside v_1. Similarly, the total flux leaving v_2 is zero

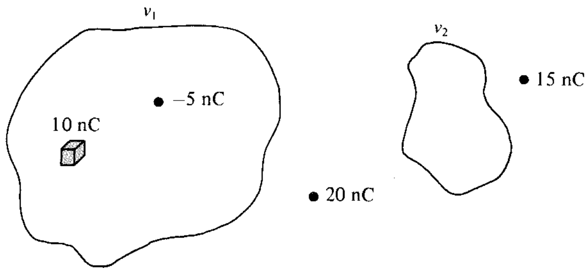

Figure 4.12 Illustration of Gauss's law; flux leaving v_1 is 5 nC and that leaving v_2 is 0 C.

because no charge is enclosed by v_2. Thus we see that Gauss's law, $\Psi = Q_{\text{enclosed}}$, is still obeyed even though the charge distribution is not symmetric. However, we cannot use the law to determine **E** or **D** when the charge distribution is not symmetric; we must resort to Coulomb's law to determine **E** or **D** in that case.

1. 6

4.6 APPLICATIONS OF GAUSS'S LAW

The procedure for applying Gauss's law to calculate the electric field involves first knowing whether symmetry exists. Once symmetric charge distribution exists, we construct a mathematical closed surface (known as a *Gaussian surface*). The surface is chosen such that **D** is normal or tangential to the Gaussian surface. When **D** is normal to the surface, $\mathbf{D} \cdot d\mathbf{S} = D\, dS$ because **D** is constant on the surface. When **D** is tangential to the surface, $\mathbf{D} \cdot d\mathbf{S} = 0$. Thus we must choose a surface that has some of the symmetry exhibited by the charge distribution. We shall now apply these basic ideas to the following cases.

A. Point Charge

Suppose a point charge Q is located at the origin. To determine **D** at a point P, it is easy to see that choosing a spherical surface containing P will satisfy symmetry conditions. Thus, a spherical surface centered at the origin is the Gaussian surface in this case and is shown in Figure 4.13.

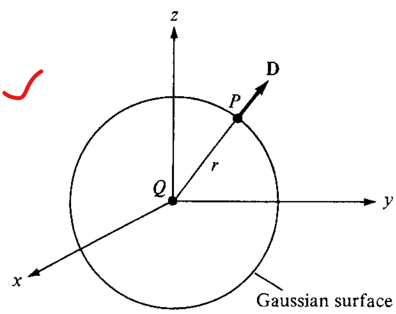

Figure 4.13 Gaussian surface about a point charge.

Since $\mathbf{D}$ is everywhere normal to the Gaussian surface, that is, $\mathbf{D} = D_r\mathbf{a}_r$, applying Gauss's law ($\Psi = Q_{\text{enclosed}}$) gives

$$Q = \oint \mathbf{D} \cdot d\mathbf{S} = D_r \oint dS = D_r\, 4\pi r^2 \tag{4.44}$$

where $\oint dS = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi = 4\pi r^2$ is the surface area of the Gaussian surface. Thus

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r \tag{4.45}$$

as expected from eqs. (4.11) and (4.35).

B. Infinite Line Charge

Suppose the infinite line of uniform charge ρ_L C/m lies along the z-axis. To determine $\mathbf{D}$ at a point P, we choose a cylindrical surface containing P to satisfy symmetry condition as shown in Figure 4.14. $\mathbf{D}$ is constant on and normal to the cylindrical Gaussian surface; that is, $\mathbf{D} = D_\rho \mathbf{a}_\rho$. If we apply Gauss's law to an arbitrary length ℓ of the line

$$\rho_L \ell = Q = \oint \mathbf{D} \cdot d\mathbf{S} = D_\rho \oint dS = D_\rho\, 2\pi\rho\ell \tag{4.46}$$

where $\oint dS = 2\pi\rho\ell$ is the surface area of the Gaussian surface. Note that $\int \mathbf{D} \cdot d\mathbf{S}$ evaluated on the top and bottom surfaces of the cylinder is zero since $\mathbf{D}$ has no z-component; that means that $\mathbf{D}$ is tangential to those surfaces. Thus

$$\mathbf{D} = \frac{\rho_L}{2\pi\rho}\mathbf{a}_\rho \tag{4.47}$$

as expected from eqs. (4.21) and (4.35).

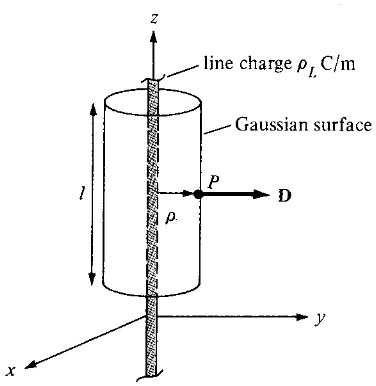

Figure 4.14 Gaussian surface about an infinite line charge.

C. Infinite Sheet of Charge

Consider the infinite sheet of uniform charge ρ_S C/m^2 lying on the $z = 0$ plane. To determine **D** at point P, we choose a rectangular box that is cut symmetrically by the sheet of charge and has two of its faces parallel to the sheet as shown in Figure 4.15. As **D** is normal to the sheet, $\mathbf{D} = D_z\mathbf{a}_z$, and applying Gauss's law gives

$$\rho_S \int dS = Q = \oint \mathbf{D} \cdot d\mathbf{S} = D_z\left[\int_{\text{top}} dS + \int_{\text{bottom}} dS\right] \tag{4.48}$$

Note that $\mathbf{D} \cdot d\mathbf{S}$ evaluated on the sides of the box is zero because **D** has no components along $\mathbf{a}_x$ and $\mathbf{a}_y$. If the top and bottom area of the box each has area A, eq. (4.48) becomes

$$\rho_S A = D_z (A + A) \tag{4.49}$$

and thus

$$\mathbf{D} = \frac{\rho_S}{2}\mathbf{a}_z$$

or

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon_o} = \frac{\rho_S}{2\varepsilon_o}\mathbf{a}_z \tag{4.50}$$

as expected from eq. (4.25).

D. Uniformly Charged Sphere

Consider a sphere of radius a with a uniform charge ρ_v C/m^3. To determine **D** everywhere, we construct Gaussian surfaces for cases $r \leq a$ and $r \geq a$ separately. Since the charge has spherical symmetry, it is obvious that a spherical surface is an appropriate Gaussian surface.

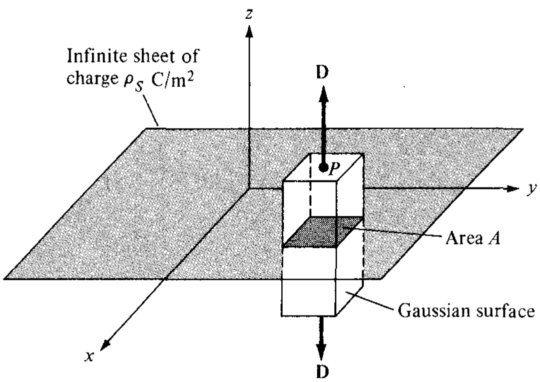

Figure 4.15 Gaussian surface about an infinite line sheet of charge.

For $r \leq a$, the total charge enclosed by the spherical surface of radius r, as shown in Figure 4.16 (a), is

$$\begin{aligned} Q_{\text{enc}} &= \int \rho_v \, dv = \rho_v \int dv = \rho_v \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{r} r^2 \sin\theta \, dr \, d\theta \, d\phi \\ &= \rho_v \frac{4}{3} \pi r^3 \end{aligned} \tag{4.51}$$

and

$$\begin{aligned} \Psi &= \oint \mathbf{D} \cdot d\mathbf{S} = D_r \oint dS = D_r \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi \\ &= D_r \, 4\pi r^2 \end{aligned} \tag{4.52}$$

Hence, $\Psi = Q_{\text{enc}}$ gives

$$D_r \, 4\pi r^2 = \frac{4\pi r^3}{3} \rho_v$$

or

$$\mathbf{D} = \frac{r}{3} \rho_v \, \mathbf{a}_r \qquad 0 < r \leq a \tag{4.53}$$

For $r \geq a$, the Gaussian surface is shown in Figure 4.16(b). The charge enclosed by the surface is the entire charge in this case, that is,

$$\begin{aligned} Q_{\text{enc}} &= \int \rho_v \, dv = \rho_v \int dv = \rho_v \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} r^2 \sin\theta \, dr \, d\theta \, d\phi \\ &= \rho_v \frac{4}{3} \pi a^3 \end{aligned} \tag{4.54}$$

while

$$\Psi = \oint \mathbf{D} \cdot d\mathbf{S} = D_r \, 4\pi r^2 \tag{4.55}$$

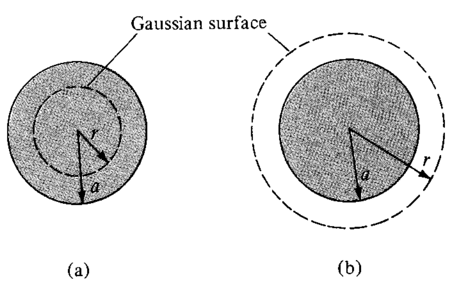

Figure 4.16 Gaussian surface for a uniformly charged sphere when: **(a)** $r \geq a$ and **(b)** $r \leq a$.

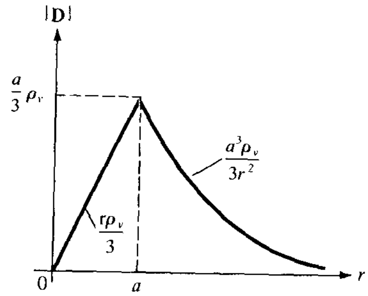

Figure 4.17 Sketch of $|\mathbf{D}|$ against r for a uniformly charged sphere.

just as in eq. (4.52). Hence:

$$D_r 4\pi r^2 = \frac{4}{3}\pi a^3 \rho_v$$

or

$$\mathbf{D} = \frac{a^3}{3r^2}\rho_v \mathbf{a}_r \qquad r \geqslant a \tag{4.56}$$

Thus from eqs. (4.53) and (4.56), **D** everywhere is given by

$$\mathbf{D} = \begin{cases} \dfrac{r}{3}\rho_v\,\mathbf{a}_r & 0 < r \leqslant a \\ \dfrac{a^3}{3r^2}\rho_v\,\mathbf{a}_r & r \geqslant a \end{cases} \tag{4.57}$$

and $|\mathbf{D}|$ is as sketched in Figure 4.17.

Notice from eqs. (4.44), (4.46), (4.48), and (4.52) that the ability to take **D** out of the integral sign is the key to finding **D** using Gauss's law. In other words, **D** must be constant on the Gaussian surface.

EXAMPLE 4.8

Given that $\mathbf{D} = z\rho\cos^2\phi\,\mathbf{a}_z$ C/m^2, calculate the charge density at $(1, \pi/4, 3)$ and the total charge enclosed by the cylinder of radius 1 m with $-2 \leq z \leq 2$ m.

Solution:

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{\partial D_z}{\partial z} = \rho\cos^2\phi$$

At $(1, \pi/4, 3)$, $\rho_v = 1 \cdot \cos^2(\pi/4) = 0.5$ C/m^3. The total charge enclosed by the cylinder can be found in two different ways.

Method 1: This method is based directly on the definition of the total volume charge.

$$
\begin{aligned}
Q &= \int_v \rho_v \, dv = \int_v \rho \cos^2 \phi \, \rho \, d\phi \, d\rho \, dz \\
&= \int_{z=-2}^{2} dz \int_{\phi=0}^{2\pi} \cos^2 \phi \, d\phi \int_{\rho=0}^{1} \rho^2 \, d\rho = 4(\pi)(1/3) \\
&= \frac{4\pi}{3} \text{ C}
\end{aligned}
$$

Method 2: Alternatively, we can use Gauss's law.

$$
\begin{aligned}
Q = \Psi &= \oint \mathbf{D} \cdot d\mathbf{S} = \left[\int_s + \int_t + \int_b \right] \mathbf{D} \cdot d\mathbf{S} \\
&= \Psi_s + \Psi_t + \Psi_b
\end{aligned}
$$

where Ψ_s, Ψ_t, and Ψ_b are the flux through the sides, the top surface, and the bottom surface of the cylinder, respectively (see Figure 3.17). Since $\mathbf{D}$ does not have component along $\mathbf{a}_\rho$, $\Psi_s = 0$, for Ψ_t, $d\mathbf{S} = \rho \, d\phi \, d\rho \, \mathbf{a}_z$ so

$$
\begin{aligned}
\Psi_t &= \int_{\rho=0}^{1} \int_{\phi=0}^{2\pi} z\rho \cos^2 \phi \, \rho \, d\phi \, d\rho \bigg|_{z=2} = 2 \int_0^1 \rho^2 d\rho \int_0^{2\pi} \cos^2 \phi \, d\phi \\
&= 2\left(\frac{1}{3}\right)\pi = \frac{2\pi}{3}
\end{aligned}
$$

and for Ψ_b, $d\mathbf{S} = -\rho \, d\phi \, d\rho \, \mathbf{a}_z$, so

$$
\begin{aligned}
\Psi_b &= -\int_{\rho=0}^{1} \int_{\phi=0}^{2\pi} z\rho \cos^2 \phi \, \rho \, d\phi \, d\rho \bigg|_{z=-2} = 2 \int_0^1 \rho^2 \, d\rho \int_0^{2\pi} \cos^2 \phi \, d\phi \\
&= \frac{2\pi}{3}
\end{aligned}
$$

Thus

$$
Q = \Psi = 0 + \frac{2\pi}{3} + \frac{2\pi}{3} = \frac{4\pi}{3} \text{ C}
$$

as obtained previously.

PRACTICE EXERCISE 4.8

If $\mathbf{D} = (2y^2 + z)\mathbf{a}_x + 4xy\mathbf{a}_y + x\mathbf{a}_z$ C/m^2, find

(a) The volume charge density at $(-1, 0, 3)$

(b) The flux through the cube defined by $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$

(c) The total charge enclosed by the cube

Answer: (a) -4 C/m^3, (b) 2 C, (c) 2 C.

EXAMPLE 4.9

A charge distribution with spherical symmetry has density

$$\rho_v = \begin{cases} \dfrac{\rho_o r}{R}, & 0 \le r \le R \\ 0, & r > R \end{cases}$$

Determine **E** everywhere.

Solution:

The charge distribution is similar to that in Figure 4.16. Since symmetry exists, we can apply Gauss's law to find **E**.

$$\varepsilon_o \oint \mathbf{E} \cdot d\mathbf{S} = Q_{\text{enc}} = \int \rho_v \, dv$$

(a) For $r < R$

$$\begin{aligned} \varepsilon_o E_r \, 4\pi r^2 = Q_{\text{enc}} &= \int_0^r \int_0^\pi \int_0^{2\pi} \rho_v \, r^2 \sin\theta \, d\phi \, d\theta \, dr \\ &= \int_0^r 4\pi r^2 \frac{\rho_o r}{R} \, dr = \frac{\rho_o \pi r^4}{R} \end{aligned}$$

or

$$\mathbf{E} = \frac{\rho_o r^2}{4\varepsilon_o R} \, \mathbf{a}_r$$

(b) For $r > R$,

$$\begin{aligned} \varepsilon_o E_r 4\pi r^2 = Q_{\text{enc}} &= \int_0^r \int_0^\pi \int_0^{2\pi} \rho_v r^2 \sin\theta \, d\phi \, d\theta \, dr \\ &= \int_0^R \frac{\rho_o r}{R} 4\pi r^2 \, dr + \int_R^r 0 \cdot 4\pi r^2 \, dr \\ &= \pi \rho_o R^3 \end{aligned}$$

or

$$\mathbf{E} = \frac{\rho_o R^3}{4\varepsilon_o r^2} \, \mathbf{a}_r$$

PRACTICE EXERCISE 4.9

A charge distribution in free space has $\rho_v = 2r$ nC/m^3 for $0 \le r \le 10$ m and zero otherwise. Determine **E** at $r = 2$ m and $r = 12$ m.

Answer: 226$\mathbf{a}_r$ V/m, 3.927$\mathbf{a}_r$ kV/m.

4.7 ELECTRIC POTENTIAL

From our discussions in the preceding sections, the electric field intensity **E** due to a charge distribution can be obtained from Coulomb's law in general or from Gauss's law when the charge distribution is symmetric. Another way of obtaining **E** is from the electric scalar potential V to be defined in this section. In a sense, this way of finding **E** is easier because it is easier to handle scalars than vectors.

Suppose we wish to move a point charge Q from point A to point B in an electric field **E** as shown in Figure 4.18. From Coulomb's law, the force on Q is $\mathbf{F} = Q\mathbf{E}$ so that the *work done* in displacing the charge by $d\mathbf{l}$ is

$$dW = -\mathbf{F} \cdot d\mathbf{l} = -Q\mathbf{E} \cdot d\mathbf{l} \tag{4.58}$$

The negative sign indicates that the work is being done by an external agent. Thus the total work done, or the potential energy required, in moving Q from A to B is

$$\boxed{W = -Q \int_A^B \mathbf{E} \cdot d\mathbf{l}} \tag{4.59}$$

Dividing W by Q in eq. (4.59) gives the potential energy per unit charge. This quantity, denoted by V_{AB}, is known as the *potential difference* between points A and B. Thus

$$\boxed{V_{AB} = \frac{W}{Q} = -\int_A^B \mathbf{E} \cdot d\mathbf{l}} \tag{4.60}$$

Note that

1. In determining V_{AB}, A is the initial point while B is the final point.
2. If V_{AB} is negative, there is a loss in potential energy in moving Q from A to B; this implies that the work is being done by the field. However, if V_{AB} is positive, there is a gain in potential energy in the movement; an external agent performs the work.
3. V_{AB} is independent of the path taken (to be shown a little later).
4. V_{AB} is measured in joules per coulomb, commonly referred to as volts (V).

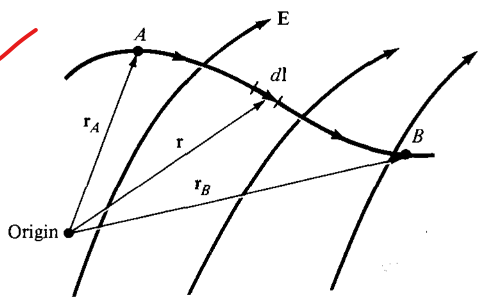

Figure 4.18 Displacement of point charge Q in an electrostatic field **E**.

As an example, if the **E** field in Figure 4.18 is due to a point charge Q located at the origin, then

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o r^2}\,\mathbf{a}_r \tag{4.61}$$

so eq. (4.60) becomes

$$V_{AB} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\varepsilon_o r^2}\,\mathbf{a}_r \cdot dr\,\mathbf{a}_r \tag{4.62a}$$

$$= \frac{Q}{4\pi\varepsilon_o}\left[\frac{1}{r_B} - \frac{1}{r_A}\right]$$

or

$$V_{AB} = V_B - V_A \tag{4.62b}$$

where V_B and V_A are the *potentials* (or *absolute potentials*) at B and A, respectively. Thus the potential difference V_{AB} may be regarded as the potential at B with reference to A. In problems involving point charges, it is customary to choose infinity as reference; that is, we assume the potential at infinity is zero. Thus if $V_A = 0$ as $r_A \to \infty$ in eq. (4.62), the potential at any point ($r_B \to r$) due to a point charge Q located at the origin is

$$\boxed{V = \frac{Q}{4\pi\varepsilon_o r}} \tag{4.63}$$

Note from eq. (4.62a) that because **E** points in the radial direction, any contribution from a displacement in the θ or ϕ direction is wiped out by the dot product $\mathbf{E} \cdot d\mathbf{l} = E\cos\theta\, dl = E\,dr$. Hence the potential difference V_{AB} is independent of the path as asserted earlier.

The **potential** at any point is the potential difference between that point and a chosen point at which the potential is zero.

In other words, by assuming zero potential at infinity, the potential at a distance r from the point charge is the work done per unit charge by an external agent in transferring a test charge from infinity to that point. Thus

$$V = -\int_{\infty}^{r} \mathbf{E} \cdot d\mathbf{l} \tag{4.64}$$

If the point charge Q in eq. (4.63) is not located at the origin but at a point whose position vector is $\mathbf{r}'$, the potential $V(x, y, z)$ or simply $V(\mathbf{r})$ at $\mathbf{r}$ becomes

$$V(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}'|} \tag{4.65}$$

We have considered the electric potential due to a point charge. The same basic ideas apply to other types of charge distribution because any charge distribution can be regarded as consisting of point charges. The superposition principle, which we applied to electric fields, applies to potentials. For n point charges $Q_1, Q_2, \ldots, Q_n$ located at points with position vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$, the potential at $\mathbf{r}$ is

$$V(\mathbf{r}) = \frac{Q_1}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_1|} + \frac{Q_2}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_2|} + \cdots + \frac{Q_n}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_n|}$$

or

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o} \sum_{k=1}^{n} \frac{Q_k}{|\mathbf{r} - \mathbf{r}_k|} \qquad \text{(point charges)} \tag{4.66}$$

For continuous charge distributions, we replace Q_k in eq. (4.66) with charge element $\rho_L\, dl$, $\rho_S\, dS$, or $\rho_v\, dv$ and the summation becomes an integration, so the potential at $\mathbf{r}$ becomes

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o} \int_L \frac{\rho_L(\mathbf{r}')dl'}{|\mathbf{r} - \mathbf{r}'|} \qquad \text{(line charge)} \tag{4.67}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o} \int_S \frac{\rho_S(\mathbf{r}')dS'}{|\mathbf{r} - \mathbf{r}'|} \qquad \text{(surface charge)} \tag{4.68}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o} \int_v \frac{\rho_v(\mathbf{r}')dv'}{|\mathbf{r} - \mathbf{r}'|} \qquad \text{(volume charge)} \tag{4.69}$$

where the primed coordinates are used customarily to denote source point location and the unprimed coordinates refer to field point (the point at which V is to be determined).

The following points should be noted:

1. We recall that in obtaining eqs. (4.63) to (4.69), the zero potential (reference) point has been chosen arbitrarily to be at infinity. If any other point is chosen as reference, eq. (4.65), for example, becomes

$$V = \frac{Q}{4\pi\varepsilon_o r} + C \tag{4.70}$$

where C is a constant that is determined at the chosen point of reference. The same idea applies to eqs. (4.63) to (4.69).

2. The potential at a point can be determined in two ways depending on whether the charge distribution or $\mathbf{E}$ is known. If the charge distribution is known, we use one of eqs. (4.65) to (4.70) depending on the charge distribution. If $\mathbf{E}$ is known, we simply use

$$V = -\int \mathbf{E} \cdot d\mathbf{l} + C \tag{4.71}$$

The potential difference V_{AB} can be found generally from

$$V_{AB} = V_B - V_A = -\int_A^B \mathbf{E} \cdot d\mathbf{l} = \frac{W}{Q} \tag{4.72}$$

EXAMPLE 4.10

Two point charges $-4\ \mu C$ and $5\ \mu C$ are located at $(2, -1, 3)$ and $(0, 4, -2)$, respectively. Find the potential at $(1, 0, 1)$ assuming zero potential at infinity.

Solution:

Let

$$Q_1 = -4\ \mu C, \qquad Q_2 = 5\ \mu C$$

$$V(\mathbf{r}) = \frac{Q_1}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_1|} + \frac{Q_2}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_2|} + C_o$$

If $V(\infty) = 0$, $C_o = 0$,

$$|\mathbf{r} - \mathbf{r}_1| = |(1, 0, 1) - (2, -1, 3)| = |(-1, 1, -2)| = \sqrt{6}$$
$$|\mathbf{r} - \mathbf{r}_2| = |(1, 0, 1) - (0, 4, -2)| = |(1, -4, 3)| = \sqrt{26}$$

Hence

$$\begin{aligned} V(1, 0, 1) &= \frac{10^{-6}}{4\pi \times \frac{10^{-9}}{36\pi}}\left[\frac{-4}{\sqrt{6}} + \frac{5}{\sqrt{26}}\right] \\ &= 9 \times 10^3\,(-1.633 + 0.9806) \\ &= -5.872\text{ kV} \end{aligned}$$

PRACTICE EXERCISE 4.10

If point charge $3\ \mu C$ is located at the origin in addition to the two charges of example 4.10, find the potential at $(-1, 5, 2)$ assuming $V(\infty) = 0$.

Answer: 10.23 kV.

EXAMPLE 4.11

A point charge 5 nC is located at $(-3, 4, 0)$ while line $y = 1, z = 1$ carries uniform charge 2 nC/m.

(a) If $V = 0$ V at $O(0, 0, 0)$, find V at $A(5, 0, 1)$.

(b) If $V = 100$ V at $B(1, 2, 1)$, find V at $C(-2, 5, 3)$.

(c) If $V = -5$ V at O, find V_{BC}.

Solution:

Let the potential at any point be

$$V = V_Q + V_L$$

where V_Q and V_L are the contributions to V at that point due to the point charge and the line charge, respectively. For the point charge,

$$V_Q = -\int \mathbf{E} \cdot d\mathbf{l} = -\int \frac{Q}{4\pi\varepsilon_o r^2} \mathbf{a}_r \cdot dr\, \mathbf{a}_r$$
$$= \frac{Q}{4\pi\varepsilon_o r} + C_1$$

For the infinite line charge,

$$V_L = -\int \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\rho_L}{2\pi\varepsilon_o \rho} \mathbf{a}_\rho \cdot d\rho\, \mathbf{a}_\rho$$
$$= -\frac{\rho_L}{2\pi\varepsilon_o} \ln \rho + C_2$$

Hence,

$$V = -\frac{\rho_L}{2\pi\varepsilon_o} \ln \rho + \frac{Q}{4\pi\varepsilon_o r} + C$$

where $C = C_1 + C_2 =$ constant, ρ is the perpendicular distance from the line $y = 1$, $z = 1$ to the field point, and r is the distance from the point charge to the field point.

(a) If $V = 0$ at $O(0, 0, 0)$, and V at $A(5, 0, 1)$ is to be determined, we must first determine the values of ρ and r at O and A. Finding r is easy; we use eq. (2.31). To find ρ for any point (x, y, z), we utilize the fact that ρ is the perpendicular distance from (x, y, z) to line $y = 1$, $z = 1$, which is parallel to the x-axis. Hence ρ is the distance between (x, y, z) and $(x, 1, 1)$ because the distance vector between the two points is perpendicular to $\mathbf{a}_x$. Thus

$$\rho = |(x, y, z) - (x, 1, 1)| = \sqrt{(y - 1)^2 + (z - 1)^2}$$

Applying this for ρ and eq. (2.31) for r at points O and A, we obtain

$$\rho_O = |(0, 0, 0) - (0, 1, 1)| = \sqrt{2}$$
$$r_O = |(0, 0, 0) - (-3, 4, 0)| = 5$$
$$\rho_A = |(5, 0, 1) - (5, 1, 1)| = 1$$
$$r_A = |(5, 0, 1) - (-3, 4, 0)| = 9$$

Hence,

$$V_O - V_A = -\frac{\rho_L}{2\pi\varepsilon_o} \ln \frac{\rho_O}{\rho_A} + \frac{Q}{4\pi\varepsilon_o}\left[\frac{1}{r_O} - \frac{1}{r_A}\right]$$
$$= \frac{-2 \cdot 10^{-9}}{2\pi \cdot \dfrac{10^{-9}}{36\pi}} \ln \frac{\sqrt{2}}{1} + \frac{5 \cdot 10^{-9}}{4\pi \cdot \dfrac{10^{-9}}{36\pi}}\left[\frac{1}{5} - \frac{1}{9}\right]$$
$$0 - V_A = -36 \ln \sqrt{2} + 45\left(\frac{1}{5} - \frac{1}{9}\right)$$

or

$$V_A = 36 \ln \sqrt{2} - 4 = 8.477 \text{ V}$$

Notice that we have avoided calculating the constant C by subtracting one potential from another and that it does not matter which one is subtracted from which.

(b) If $V = 100$ at $B(1, 2, 1)$ and V at $C(-2, 5, 3)$ is to be determined, we find

$$\begin{aligned}
\rho_B &= |(1, 2, 1) - (1, 1, 1)| = 1 \\
r_B &= |(1, 2, 1) - (-3, 4, 0)| = \sqrt{21} \\
\rho_C &= |(-2, 5, 3) - (-2, 1, 1)| = \sqrt{20} \\
r_C &= |(-2, 5, 3) - (-3, 4, 0)| = \sqrt{11}
\end{aligned}$$

$$V_C - V_B = -\frac{\rho_L}{2\pi\varepsilon_o} \ln \frac{\rho_O}{\rho_B} + \frac{Q}{4\pi\varepsilon_o}\left[\frac{1}{r_C} - \frac{1}{r_B}\right]$$

$$\begin{aligned}
V_C - 100 &= -36 \ln \frac{\sqrt{20}}{1} + 45 \cdot \left[\frac{1}{\sqrt{11}} - \frac{1}{\sqrt{21}}\right] \\
&= -50.175 \text{ V}
\end{aligned}$$

or

$$V_C = 49.825 \text{ V}$$

(c) To find the potential difference between two points, we do not need a potential reference if a common reference is assumed.

$$\begin{aligned}
V_{BC} &= V_C - V_B = 49.825 - 100 \\
&= -50.175 \text{ V}
\end{aligned}$$

as obtained in part (b).

PRACTICE EXERCISE 4.11

A point charge of 5 nC is located at the origin. If $V = 2$ V at $(0, 6, -8)$, find

(a) The potential at $A(-3, 2, 6)$

(b) The potential at $B(1, 5, 7)$

(c) The potential difference V_{AB}

Answer: (a) 3.929 V, (b) 2.696 V, (c) −1.233 V.

4.8 RELATIONSHIP BETWEEN E AND V—MAXWELL'S EQUATION

As shown in the previous section, the potential difference between points A and B is independent of the path taken. Hence,

$$V_{BA} = -V_{AB}$$

that is, $V_{BA} + V_{AB} = \oint \mathbf{E} \cdot d\mathbf{l} = 0$

or

$$\boxed{\oint \mathbf{E} \cdot d\mathbf{l} = 0} \tag{4.73}$$

This shows that the line integral of $\mathbf{E}$ along a closed path as shown in Figure 4.19 must be zero. Physically, this implies that no net work is done in moving a charge along a closed path in an electrostatic field. Applying Stokes's theorem to eq. (4.73) gives

$$\oint \mathbf{E} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = 0$$

or

$$\boxed{\nabla \times \mathbf{E} = 0} \tag{4.74}$$

Any vector field that satisfies eq. (4.73) or (4.74) is said to be conservative, or irrotational, as discussed in Section 3.8. Thus an electrostatic field is a conservative field. Equation (4.73) or (4.74) is referred to as *Maxwell's equation* (the second Maxwell's equation to be derived) for static electric fields. Equation (4.73) is the integral form, and eq. (4.74) is the differential form; they both depict the conservative nature of an electrostatic field.

From the way we defined potential, $V = -\int \mathbf{E} \cdot d\mathbf{l}$, it follows that

$$dV = -\mathbf{E} \cdot d\mathbf{l} = -E_x\, dx - E_y\, dy - E_z\, dz$$

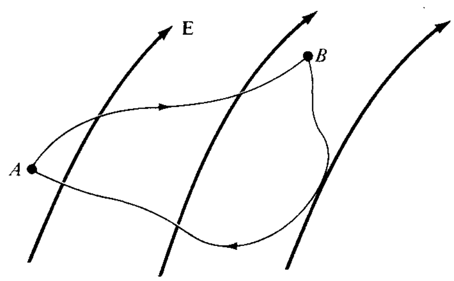

Figure 4.19 Conservative nature of an electrostatic field.

But

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

Comparing the two expressions for dV, we obtain

$$E_x = -\frac{\partial V}{\partial x}, \qquad E_y = -\frac{\partial V}{\partial y}, \qquad E_z = -\frac{\partial V}{\partial z} \tag{4.75}$$

Thus:

$$\boxed{\mathbf{E} = -\nabla V} \tag{4.76}$$

that is, the electric field intensity is the gradient of V. The negative sign shows that the direction of $\mathbf{E}$ is opposite to the direction in which V increases; $\mathbf{E}$ is directed from higher to lower levels of V. Since the curl of the gradient of a scalar function is always zero ($\nabla \times \nabla V = 0$), eq. (4.74) obviously implies that $\mathbf{E}$ must be a gradient of some scalar function. Thus eq. (4.76) could have been obtained from eq. (4.74).

Equation (4.76) shows another way to obtain the $\mathbf{E}$ field apart from using Coulomb's or Gauss's law. That is, if the potential field V is known, the $\mathbf{E}$ can be found using eq. (4.76). One may wonder how one function V can possibly contain all the information that the three components of $\mathbf{E}$ carry. The three components of $\mathbf{E}$ are not independent of one another: They are explicitly interrelated by the condition $\nabla \times \mathbf{E} = 0$. What the potential formulation does is to exploit this feature to maximum advantage, reducing a vector problem to a scalar one.

EXAMPLE 4.12

Given the potential $V = \dfrac{10}{r^2}\sin\theta\cos\phi$,

(a) Find the electric flux density $\mathbf{D}$ at $(2, \pi/2, 0)$.

(b) Calculate the work done in moving a 10-μC charge from point $A(1, 30°, 120°)$ to $B(4, 90°, 60°)$.

Solution:

(a) $\mathbf{D} = \varepsilon_o\mathbf{E}$

But

$$\begin{aligned}\mathbf{E} &= -\nabla V = -\left[\frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi\right] \\ &= \frac{20}{r^3}\sin\theta\cos\phi\,\mathbf{a}_r - \frac{10}{r^3}\cos\theta\cos\phi\,\mathbf{a}_\theta + \frac{10}{r^3}\sin\phi\,\mathbf{a}_\phi\end{aligned}$$

At $(2, \pi/2, 0)$,

$$\mathbf{D} = \varepsilon_o \mathbf{E}\,(r = 2, \theta = \pi/2, \phi = 0) = \varepsilon_o\left(\frac{20}{8}\mathbf{a}_r - 0\mathbf{a}_\theta + 0\mathbf{a}_\phi\right)$$
$$= 2.5\varepsilon_o \mathbf{a}_r \text{ C/m}^2 = 22.1\ \mathbf{a}_r \text{ pC/m}^2$$

(b) The work done can be found in two ways, using either **E** or V.

Method 1:

$$W = -Q\int \mathbf{E}\cdot d\mathbf{l} \qquad \text{or} \qquad -\frac{W}{Q} = \int \mathbf{E}\cdot d\mathbf{l}$$

and because the electrostatic field is conservative, the path of integration is immaterial. Hence the work done in moving Q from $A(1, 30°, 120°)$ to $B(4, 90°, 60°)$ is the same as that in moving Q from A to A', from A' to B', and from B' to B where

$$\begin{array}{ccc} A(1, 30°, 120°) & & B(4, 90°, 60°) \\ \downarrow\ d\mathbf{l} = dr\,\mathbf{a}_r & d\mathbf{l} = r\,d\theta\,\mathbf{a}_\theta & \uparrow\ d\mathbf{l} = r\sin\theta\,d\phi\,a_\phi \\ A'(4, 30°, 120°) & \rightarrow & B'(4, 90°, 120°). \end{array}$$

That is, instead of moving Q directly from A and B, it is moved from $A \rightarrow A'$, $A' \rightarrow B'$, $B' \rightarrow B$ so that only one variable is changed at a time. This makes the line integral a lot easier to evaluate. Thus

$$\begin{aligned}
\frac{-W}{Q} &= -\frac{1}{Q}(W_{AA'} + W_{A'B'} + W_{B'B}) \\
&= \left(\int_{AA'} + \int_{A'B'} + \int_{B'B}\right)\mathbf{E}\cdot d\mathbf{l} \\
&= \int_{r=1}^{4} \frac{20\sin\theta\cos\phi}{r^3}\,dr\,\Bigg|_{\theta=30°,\,\phi=120°} \\
&\quad + \int_{\theta=30°}^{90°} \frac{-10\cos\theta\cos\phi}{r^3}\,r\,d\theta\,\Bigg|_{r=4,\,\phi=120°} \\
&\quad + \int_{\phi=120°}^{60°} \frac{10\sin\phi}{r^3}\,r\sin\theta\,d\phi\,\Bigg|_{r=4,\,\theta=90°} \\
&= 20\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left[-\frac{1}{2r^2}\Bigg|_{r=1}^{4}\right] \\
&\quad - \frac{10}{16}\frac{(-1)}{2}\sin\theta\Bigg|_{30°}^{90°} + \frac{10}{16}(1)\left[-\cos\phi\Bigg|_{120°}^{60°}\right] \\
-\frac{W}{Q} &= \frac{-75}{32} + \frac{5}{32} - \frac{10}{16}
\end{aligned}$$

or

$$W = \frac{45}{16} Q = 28.125\ \mu\text{J}$$

Method 2:

Since V is known, this method is a lot easier.

$$\begin{aligned}
W &= -Q \int_A^B \mathbf{E} \cdot d\mathbf{l} = QV_{AB} \\
&= Q(V_B - V_A) \\
&= 10 \left(\frac{10}{16} \sin 90° \cos 60° - \frac{10}{1} \sin 30° \cos 120° \right) \cdot 10^{-6} \\
&= 10 \left(\frac{10}{32} - \frac{-5}{2} \right) \cdot 10^{-6} \\
&= 28.125\ \mu\text{J as obtained before}
\end{aligned}$$

PRACTICE EXERCISE 4.12

Given that $\mathbf{E} = (3x^2 + y)\,\mathbf{a}_x + x\mathbf{a}_y$ kV/m, find the work done in moving a $-2\ \mu$C charge from (0, 5, 0) to (2, −1, 0) by taking the path

(a) $(0, 5, 0) \rightarrow (2, 5, 0) \rightarrow (2, -1, 0)$

(b) $y = 5 - 3x$

Answer: (a) 12 mJ, (b) 12 mJ.

4.9 AN ELECTRIC DIPOLE AND FLUX LINES

An **electric dipole** is formed when two point charges of equal magnitude but opposite sign are separated by a small distance.

The importance of the field due to a dipole will be evident in the subsequent chapters.

Consider the dipole shown in Figure 4.20. The potential at point $P(r, \theta, \phi)$ is given by

$$V = \frac{Q}{4\pi\varepsilon_o}\left[\frac{1}{r_1} - \frac{1}{r_2}\right] = \frac{Q}{4\pi\varepsilon_o}\left[\frac{r_2 - r_1}{r_1 r_2}\right] \tag{4.77}$$

where r_1 and r_2 are the distances between P and $+Q$ and P and $-Q$, respectively. If $r \gg d$, $r_2 - r_1 \simeq d \cos\theta$, $r_2 r_1 \simeq r^2$, and eq. (4.77) becomes

$$V = \frac{Q}{4\pi\varepsilon_o} \frac{d\cos\theta}{r^2} \tag{4.78}$$

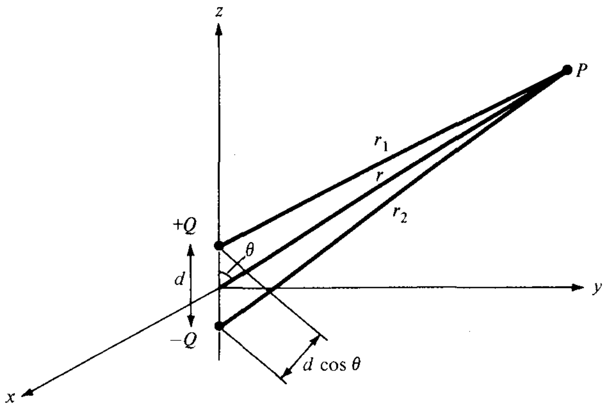

Figure 4.20 An electric dipole.

Since $d \cos \theta = \mathbf{d} \cdot \mathbf{a}_r$, where $\mathbf{d} = d\mathbf{a}_z$, if we define

$$\mathbf{p} = Q\mathbf{d} \tag{4.79}$$

as the *dipole moment*, eq. (4.78) may be written as

$$V = \frac{\mathbf{p} \cdot \mathbf{a}_r}{4\pi\varepsilon_o r^2} \tag{4.80}$$

Note that the dipole moment $\mathbf{p}$ is directed from $-Q$ to $+Q$. If the dipole center is not at the origin but at $\mathbf{r}'$, eq. (4.80) becomes

$$\boxed{V(\mathbf{r}) = \frac{\mathbf{p} \cdot (\mathbf{r} - \mathbf{r}')}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}'|^3}} \tag{4.81}$$

The electric field due to the dipole with center at the origin, shown in Figure 4.20, can be obtained readily from eqs. (4.76) and (4.78) as

$$\begin{aligned}\mathbf{E} = -\nabla V &= -\left[\frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta\right] \\ &= \frac{Qd\cos\theta}{2\pi\varepsilon_o r^3}\mathbf{a}_r + \frac{Qd\sin\theta}{4\pi\varepsilon_o r^3}\mathbf{a}_\theta\end{aligned}$$

or

$$\boxed{\mathbf{E} = \frac{p}{4\pi\varepsilon_o r^3}(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta)} \tag{4.82}$$

where $p = |\mathbf{p}| = Qd$.

Notice that a point charge is a *monopole* and its electric field varies inversely as r^2 while its potential field varies inversely as r [see eqs. (4.61) and (4.63)]. From eqs. (4.80) and (4.82), we notice that the electric field due to a dipole varies inversely as r^3 while its potential varies inversely as r^2. The electric fields due to successive higher-order multipoles (such as a *quadrupole* consisting of two dipoles or an *octupole* consisting of two quadrupoles) vary inversely as $r^4, r^5, r^6, \ldots$ while their corresponding potentials vary inversely as $r^3, r^4, r^5, \ldots$.

The idea of *electric flux* lines (or *electric lines of force* as they are sometimes called) was introduced by Michael Faraday (1791–1867) in his experimental investigation as a way of visualizing the electric field.

An **electric flux line** is an imaginary path or line drawn in such a way that its direction at any point is the direction of the electric field at that point.

In other words, they are the lines to which the electric field density **D** is tangential at every point.

Any surface on which the potential is the same throughout is known as an *equipotential surface*. The intersection of an equipotential surface and a plane results in a path or line known as an *equipotential line*. No work is done in moving a charge from one point to another along an equipotential line or surface ($V_A - V_B = 0$) and hence

$$\int \mathbf{E} \cdot d\mathbf{l} = 0 \tag{4.83}$$

on the line or surface. From eq. (4.83), we may conclude that the lines of force or flux lines (or the direction of **E**) are always normal to equipotential surfaces. Examples of equipotential surfaces for point charge and a dipole are shown in Figure 4.21. Note from these examples that the direction of **E** is everywhere normal to the equipotential

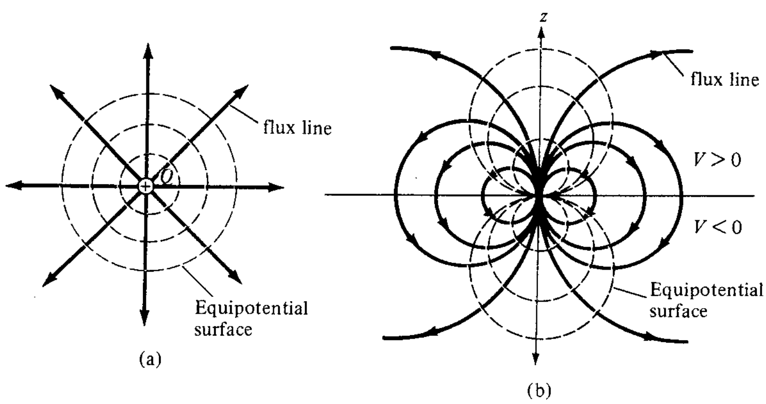

Figure 4.21 Equipotential surfaces for **(a)** a point charge and **(b)** an electric dipole.

lines. We shall see the importance of equipotential surfaces when we discuss conducting bodies in electric fields; it will suffice to say at this point that such bodies are equipotential volumes.

A typical application of field mapping (flux lines and equipotential surfaces) is found in the diagnosis of the human heart. The human heart beats in response to an electric field potential difference across it. The heart can be characterized as a dipole with the field map similar to that of Figure 4.21(b). Such a field map is useful in detecting abnormal heart position.[6] In Section 15.2, we will discuss a numerical technique for field mapping.

EXAMPLE 4.13

Two dipoles with dipole moments $-5\mathbf{a}_z$ nC/m and $9\mathbf{a}_z$ nC/m are located at points $(0, 0, -2)$ and $(0, 0, 3)$, respectively. Find the potential at the origin.

Solution:

$$
\begin{aligned}
V &= \sum_{k=1}^{2} \frac{\mathbf{p}_k \cdot \mathbf{r}_k}{4\pi\varepsilon_o r_k^3} \\
&= \frac{1}{4\pi\varepsilon_o}\left[\frac{\mathbf{p}_1 \cdot \mathbf{r}_1}{r_1^3} + \frac{\mathbf{p}_2 \cdot \mathbf{r}_2}{r_2^3}\right]
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{p}_1 &= -5\mathbf{a}_z, \qquad \mathbf{r}_1 = (0, 0, 0) - (0, 0, -2) = 2\mathbf{a}_z, \qquad r_1 = |\mathbf{r}_1| = 2 \\
\mathbf{p}_2 &= 9\mathbf{a}_z, \qquad \mathbf{r}_2 = (0, 0, 0) - (0, 0, 3) = -3\mathbf{a}_z, \qquad r_2 = |\mathbf{r}_2| = 3
\end{aligned}
$$

Hence,

$$
\begin{aligned}
V &= \frac{1}{4\pi \cdot \dfrac{10^{-9}}{36\pi}}\left[\frac{-10}{2^3} - \frac{27}{3^3}\right] \cdot 10^{-9} \\
&= -20.25 \text{ V}
\end{aligned}
$$

PRACTICE EXERCISE 4.13

An electric dipole of $100\,\mathbf{a}_z$ pC · m is located at the origin. Find V and $\mathbf{E}$ at points

(a) $(0, 0, 10)$
(b) $(1, \pi/3, \pi/2)$

Answer: (a) 9 mV, $1.8\mathbf{a}_r$ mV/m, (b) 0.45 V, $0.9\mathbf{a}_r + 0.7794\mathbf{a}_\theta$ V/m.

[6]For more information on this, see R. Plonsey, *Bioelectric Phenomena.* New York: McGraw-Hill, 1969.

4.10 ENERGY DENSITY IN ELECTROSTATIC FIELDS

To determine the energy present in an assembly of charges, we must first determine the amount of work necessary to assemble them. Suppose we wish to position three point charges Q_1, Q_2, and Q_3 in an initially empty space shown shaded in Figure 4.22. No work is required to transfer Q_1 from infinity to P_1 because the space is initially charge free and there is no electric field [from eq. (4.59), $W = 0$]. The work done in transferring Q_2 from infinity to P_2 is equal to the product of Q_2 and the potential V_{21} at P_2 due to Q_1. Similarly, the work done in positioning Q_3 at P_3 is equal to $Q_3(V_{32} + V_{31})$, where V_{32} and V_{31} are the potentials at P_3 due to Q_2 and Q_1, respectively. Hence the total work done in positioning the three charges is

$$\begin{aligned} W_E &= W_1 + W_2 + W_3 \\ &= 0 + Q_2V_{21} + Q_3(V_{31} + V_{32}) \end{aligned} \tag{4.84}$$

If the charges were positioned in reverse order,

$$\begin{aligned} W_E &= W_3 + W_2 + W_1 \\ &= 0 + Q_2V_{23} + Q_1(V_{12} + V_{13}) \end{aligned} \tag{4.85}$$

where V_{23} is the potential at P_2 due to Q_3, V_{12} and V_{13} are, respectively, the potentials at P_1 due to Q_2 and Q_3. Adding eqs. (4.84) and (4.85) gives

$$\begin{aligned} 2W_E &= Q_1(V_{12} + V_{13}) + Q_2(V_{21} + V_{23}) + Q_3(V_{31} + V_{32}) \\ &= Q_1V_1 + Q_2V_2 + Q_3V_3 \end{aligned}$$

or

$$W_E = \frac{1}{2}(Q_1V_1 + Q_2V_2 + Q_3V_3) \tag{4.86}$$

where V_1, V_2, and V_3 are total potentials at P_1, P_2, and P_3, respectively. In general, if there are n point charges, eq. (4.86) becomes

$$\boxed{W_E = \frac{1}{2}\sum_{k=1}^{n} Q_kV_k} \qquad \text{(in joules)} \tag{4.87}$$

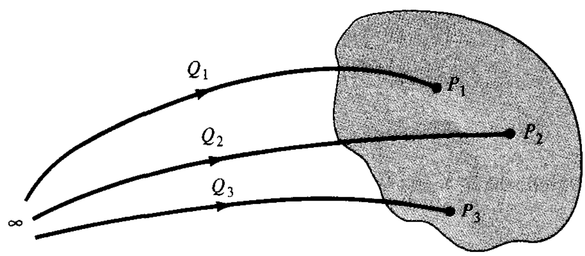

Figure 4.22 Assembling of charges.

If, instead of point charges, the region has a continuous charge distribution, the summation in eq. (4.87) becomes integration; that is,

$$W_E = \frac{1}{2} \int \rho_L V \, dl \qquad \text{(line charge)} \tag{4.88}$$

$$W_E = \frac{1}{2} \int \rho_S V \, dS \qquad \text{(surface charge)} \tag{4.89}$$

$$W_E = \frac{1}{2} \int \rho_v V \, dv \qquad \text{(volume charge)} \tag{4.90}$$

Since $\rho_v = \nabla \cdot \mathbf{D}$, eq. (4.90) can be further developed to yield

$$W_E = \frac{1}{2} \int_v (\nabla \cdot \mathbf{D}) \, V \, dv \tag{4.91}$$

But for any vector $\mathbf{A}$ and scalar V, the identity

$$\nabla \cdot V\mathbf{A} = \mathbf{A} \cdot \nabla V + V(\nabla \cdot \mathbf{A})$$

or

$$(\nabla \cdot \mathbf{A})V = \nabla \cdot V\mathbf{A} - \mathbf{A} \cdot \nabla V \tag{4.92}$$

holds. Applying the identity in eqs. (4.92) to (4.91), we get

$$W_E = \frac{1}{2} \int_v (\nabla \cdot V\mathbf{D}) \, dv - \frac{1}{2} \int_v (\mathbf{D} \cdot \nabla V) \, dv \tag{4.93}$$

By applying divergence theorem to the first term on the right-hand side of this equation, we have

$$W_E = \frac{1}{2} \oint_S (V\mathbf{D}) \cdot d\mathbf{S} - \frac{1}{2} \int_v (\mathbf{D} \cdot \nabla V) \, dv \tag{4.94}$$

From Section 4.9, we recall that V varies as $1/r$ and $\mathbf{D}$ as $1/r^2$ for point charges; V varies as $1/r^2$ and $\mathbf{D}$ as $1/r^3$ for dipoles; and so on. Hence, $V\mathbf{D}$ in the first term on the right-hand side of eq. (4.94) must vary at least as $1/r^3$ while dS varies as r^2. Consequently, the first integral in eq. (4.94) must tend to zero as the surface S becomes large. Hence, eq. (4.94) reduces to

$$W_E = -\frac{1}{2} \int_v (\mathbf{D} \cdot \nabla V) \, dv = \frac{1}{2} \int_v (\mathbf{D} \cdot \mathbf{E}) \, dv \tag{4.95}$$

and since $\mathbf{E} = -\nabla V$ and $\mathbf{D} = \varepsilon_o \mathbf{E}$

$$\boxed{W_E = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} \, dv = \frac{1}{2} \int \varepsilon_o E^2 \, dv} \tag{4.96}$$

From this, we can define electrostatic energy density w_E (in J/m^3) as

$$w_E = \frac{dW_E}{dv} = \frac{1}{2}\mathbf{D}\cdot\mathbf{E} = \frac{1}{2}\varepsilon_o E^2 = \frac{D^2}{2\varepsilon_o} \tag{4.97}$$

so eq. (4.95) may be written as

$$W_E = \int w_E\, dv \tag{4.98}$$

EXAMPLE 4.14

Three point charges -1 nC, 4 nC, and 3 nC are located at (0, 0, 0), (0, 0, 1), and (1, 0, 0), respectively. Find the energy in the system.

Solution:

$$\begin{aligned}
W &= W_1 + W_2 + W_3 \\
&= 0 + Q_2V_{21} + Q_3(V_{31} + V_{32}) \\
&= Q_2 \cdot \frac{Q_1}{4\pi\varepsilon_o\,|(0,0,1) - (0,0,0)|} \\
&\quad + \frac{Q_3}{4\pi\varepsilon_o}\left[\frac{Q_1}{|(1,0,0) - (0,0,0)|} + \frac{Q_2}{|(1,0,0) - (0,0,1)|}\right] \\
&= \frac{1}{4\pi\varepsilon_o}\left(Q_1Q_2 + Q_1Q_3 + \frac{Q_2Q_3}{\sqrt{2}}\right) \\
&= \frac{1}{4\pi \cdot \dfrac{10^{-9}}{36\pi}}\left(-4 - 3 + \frac{12}{\sqrt{2}}\right)\cdot 10^{-18} \\
&= 9\left(\frac{12}{\sqrt{2}} - 7\right)\text{ nJ} = 13.37 \text{ nJ}
\end{aligned}$$

Alternatively,

$$\begin{aligned}
W &= \frac{1}{2}\sum_{k=1}^{3} Q_kV_k = \frac{1}{2}(Q_1V_1 + Q_2V_2 + Q_3V_3) \\
&= \frac{Q_1}{2}\left[\frac{Q_2}{4\pi\varepsilon_o(1)} + \frac{Q_3}{4\pi\varepsilon_o(1)}\right] + \frac{Q_2}{2}\left[\frac{Q_1}{4\pi\varepsilon_o(1)} + \frac{Q_3}{4\pi\varepsilon_o(\sqrt{2})}\right] \\
&\quad + \frac{Q_3}{2}\left[\frac{Q_1}{4\pi\varepsilon_o(1)} + \frac{Q_2}{4\pi\varepsilon_o(\sqrt{2})}\right] \\
&= \frac{1}{4\pi\varepsilon_o}\left(Q_1Q_2 + Q_1Q_3 + \frac{Q_2Q_3}{\sqrt{2}}\right) \\
&= 9\left(\frac{12}{\sqrt{2}} - 7\right)\text{ nJ} = 13.37 \text{ nJ}
\end{aligned}$$

as obtained previously.

PRACTICE EXERCISE 4.14

Point charges $Q_1 = 1$ nC, $Q_2 = -2$ nC, $Q_3 = 3$ nC, and $Q_4 = -4$ nC are positioned one at a time and in that order at (0, 0, 0), (1, 0, 0), (0, 0, −1), and (0, 0, 1), respectively. Calculate the energy in the system after each charge is positioned.

Answer: 0, −18 nJ, −29.18 nJ, −68.27 nJ.

EXAMPLE 4.15

A charge distribution with spherical symmetry has density

$$\rho_v = \begin{cases} \rho_o, & 0 \le r \le R \\ 0, & r > R \end{cases}$$

Determine V everywhere and the energy stored in region $r < R$.

Solution:

The **D** field has already been found in Section 4.6D using Gauss's law.

(a) For $r \geqslant R$, $\mathbf{E} = \dfrac{\rho_o R^3}{3\varepsilon_o r^2}\mathbf{a}_r$.

Once **E** is known, V is determined as

$$V = -\int \mathbf{E} \cdot d\mathbf{l} = -\frac{\rho_o R^3}{3\varepsilon_o}\int \frac{1}{r^2}\,dr$$

$$= \frac{\rho_o R^3}{3\varepsilon_o r} + C_1, \qquad r \geqslant R$$

Since $V(r = \infty) = 0$, $C_1 = 0$.

(b) For $r \leqslant R$, $\mathbf{E} = \dfrac{\rho_o r}{3\varepsilon_o}\mathbf{a}_r$.

Hence,

$$V = -\int \mathbf{E} \cdot d\mathbf{l} = -\frac{\rho_o}{3\varepsilon_o}\int r\,dr$$

$$= -\frac{\rho_o r^2}{6\varepsilon_o} + C_2$$

From part (a) $V(r = R) = \dfrac{\rho_o R^2}{3\varepsilon_o}$. Hence,

$$\frac{R^2\rho_o}{3\varepsilon_o} = -\frac{\rho_o R^2}{6\varepsilon_o} + C_2 \rightarrow C_2 = \frac{R^2\rho_o}{2\varepsilon_o}$$

and

$$V = \frac{\rho_o}{6\varepsilon_o}(3R^2 - r^2)$$

Thus from parts (a) and (b)

$$V = \begin{bmatrix} \dfrac{\rho_o R^3}{3\varepsilon_o r}, & r \geqslant R \\ \dfrac{\rho_o}{6\varepsilon_o}(3R^2 - r^2), & r \leqslant R \end{bmatrix}$$

(c) The energy stored is given by

$$W = \frac{1}{2}\int \mathbf{D} \cdot \mathbf{E}\, dv = \frac{1}{2}\varepsilon_o \int E^2\, dv$$

For $r \leqslant R$,

$$\mathbf{E} = \frac{\rho_o r}{3\varepsilon_o}\mathbf{a}_r$$

Hence,

$$W = \frac{1}{2}\varepsilon_o \frac{\rho_o^2}{9\varepsilon_o^2}\int_{r=0}^{R}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2 \cdot r^2 \sin\theta\, d\phi\, d\theta\, dr$$

$$= \frac{\rho_o^2}{18\varepsilon_o} 4\pi \cdot \frac{r^5}{5}\bigg|_0^R = \frac{2\pi\rho_o^2 R^5}{45\varepsilon_o}\ \text{J}$$

PRACTICE EXERCISE 4.15

If $V = x - y + xy + 2z$ V, find **E** at (1, 2, 3) and the electrostatic energy stored in a cube of side 2 m centered at the origin.

Answer: $-3\mathbf{a}_x - 2\mathbf{a}_z$ V/m, 0.2358 nJ.

SUMMARY

1. The two fundamental laws for electrostatic fields (Coulomb's and Gauss's) are presented in this chapter. Coulomb's law of force states that

$$\mathbf{F} = \frac{Q_1 Q_2}{4\pi\varepsilon_o R^2}\mathbf{a}_R$$

2. Based on Coulomb's law, we define the electric field intensity **E** as the force per unit charge; that is,

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o R^2}\mathbf{a}_R = \frac{Q\,\mathbf{R}}{4\pi\varepsilon R^3} \qquad \text{(point charge only)}$$

3. For a continuous charge distribution, the total charge is given by

$$Q = \int \rho_L \, dl \qquad \text{for line charge}$$

$$Q = \int \rho_S \, dS \qquad \text{for surface charge}$$

$$Q = \int \rho_v \, dv \qquad \text{for volume charge}$$

The **E** field due to a continuous charge distribution is obtained from the formula for point charge by replacing Q with $dQ = \rho_L \, dl$, $dQ = \rho_S \, dS$ or $dQ = \rho_v \, dv$ and integrating over the line, surface, or volume respectively.

4. For an infinite line charge,

$$\mathbf{E} = \frac{\rho_L}{2\pi\varepsilon_o\rho} \mathbf{a}_\rho$$

and for an infinite sheet of charge,

$$\mathbf{E} = \frac{\rho_S}{2\varepsilon_o} \mathbf{a}_n$$

5. The electric flux density **D** is related to the electric field intensity (in free space) as

$$\mathbf{D} = \varepsilon_o \mathbf{E}$$

The electric flux through a surface S is

$$\Psi = \int_S \mathbf{D} \cdot d\mathbf{S}$$

6. Gauss's law states that the net electric flux penetrating a closed surface is equal to the total charge enclosed, that is, $\Psi = Q_{\text{enc}}$. Hence,

$$\Psi = \oint \mathbf{D} \cdot d\mathbf{S} = Q_{\text{enc}} = \int \rho_v \, dv$$

or

$$\rho_v = \nabla \cdot \mathbf{D} \qquad \text{(first Maxwell's equation to be derived)}$$

When charge distribution is symmetric so that a Gaussian surface (where $\mathbf{D} = D_n \mathbf{a}_n$ is constant) can be found, Gauss's law is useful in determining **D**; that is,

$$D_n \oint dS = Q_{\text{enc}} \qquad \text{or} \qquad D_n = \frac{Q_{\text{enc}}}{S}$$

7. The total work done, or the electric potential energy, to move a point charge Q from point A to B in an electric field E is

$$W = -Q \int_A^B \mathbf{E} \cdot d\mathbf{l}$$

8. The potential at $\mathbf{r}$ due to a point charge Q at $\mathbf{r}'$ is

$$V(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}'|} + C$$

where C is evaluated at a given reference potential point; for example $C = 0$ if $V(\mathbf{r} \rightarrow \infty) = 0$. To determine the potential due to a continuous charge distribution, we replace Q in the formula for point charge by $dQ = \rho_L\, dl$, $dQ = \rho_S\, dS$, or $dQ = \rho_v\, dv$ and integrate over the line, surface, or volume, respectively.

9. If the charge distribution is not known, but the field intensity $\mathbf{E}$ is given, we find the potential using

$$V = -\int \mathbf{E} \cdot d\mathbf{l} + C$$

10. The potential difference V_{AB}, the potential at B with reference to A, is

$$V_{AB} = -\int_A^B \mathbf{E} \cdot d\mathbf{l} = \frac{W}{Q} = V_B - V_A$$

11. Since an electrostatic field is conservative (the net work done along a closed path in a static $\mathbf{E}$ field is zero),

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0$$

or

$$\nabla \times \mathbf{E} = 0 \qquad \text{(second Maxwell's equation to be derived)}$$

12. Given the potential field, the corresponding electric field is found using

$$\mathbf{E} = -\nabla V$$

13. For an electric dipole centered at $\mathbf{r}'$ with dipole moment $\mathbf{p}$, the potential at $\mathbf{r}$ is given by

$$V(\mathbf{r}) = \frac{\mathbf{p} \cdot (\mathbf{r} - \mathbf{r}')}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}'|^3}$$

14. $\mathbf{D}$ is tangential to the electric flux lines at every point. An equipotential surface (or line) is one on which $V =$ constant. At every point, the equipotential line is orthogonal to the electric flux line.

15. The electrostatic energy due to n point charges is

$$W_E = \frac{1}{2} \sum_{k=1}^{n} Q_k V_k$$

For a continuous volume charge distribution,

$$W_E = \frac{1}{2} \int_v \mathbf{D} \cdot \mathbf{E} \, dv = \frac{1}{2} \int \varepsilon_o |\mathbf{E}|^2 \, dv$$

REVIEW QUESTIONS

4.1 Point charges $Q_1 = 1$ nC and $Q_2 = 2$ nC are at a distance apart. Which of the following statements are incorrect?

(a) The force on Q_1 is repulsive.
(b) The force on Q_2 is the same in magnitude as that on Q_1.
(c) As the distance between them decreases, the force on Q_1 increases linearly.
(d) The force on Q_2 is along the line joining them.
(e) A point charge $Q_3 = -3$ nC located at the midpoint between Q_1 and Q_2 experiences no net force.

4.2 Plane $z = 10$ m carries charge 20 nC/m^2. The electric field intensity at the origin is

(a) $-10\,\mathbf{a}_z$ V/m
(b) $-18\pi\,\mathbf{a}_z$ V/m
(c) $-72\pi\,\mathbf{a}_z$ V/m
(d) $-360\pi\,\mathbf{a}_z$ V/m

4.3 Point charges 30 nC, -20 nC, and 10 nC are located at $(-1, 0, 2)$, $(0, 0, 0)$, and $(1, 5, -1)$, respectively. The total flux leaving a cube of side 6 m centered at the origin is:

(a) -20 nC
(b) 10 nC
(c) 20 nC
(d) 30 nC
(e) 60 nC

4.4 The electric flux density on a spherical surface $r = b$ is the same for a point charge Q located at the origin and for charge Q uniformly distributed on surface $r = a (a < b)$.

(a) Yes
(b) No
(c) Not necessarily

4.5 The work done by the force $\mathbf{F} = 4\mathbf{a}_x - 3\mathbf{a}_y + 2\mathbf{a}_z$ N in giving a 1 nC charge a displacement of $10\mathbf{a}_x + 2\mathbf{a}_y - 7\mathbf{a}_z$ m is

(a) 103 nJ
(b) 60 nJ
(c) 64 nJ
(d) 20 nJ

4.6 By saying that the electrostatic field is conservative, we do *not* mean that

(a) It is the gradient of a scalar potential.
(b) Its circulation is identically zero.
(c) Its curl is identically zero.
(d) The work done in a closed path inside the field is zero.
(e) The potential difference between any two points is zero.

4.7 Suppose a uniform electric field exists in the room in which you are working, such that the lines of force are horizontal and at right angles to one wall. As you walk toward the wall from which the lines of force emerge into the room, are you walking toward

(a) Points of higher potential?
(b) Points of lower potential?
(c) Points of the same potential (equipotential line)?

4.8 A charge Q is uniformly distributed throughout a sphere of radius a. Taking the potential at infinity as zero, the potential at $r = b < a$ is

(a) $-\int_\infty^b \frac{Qr}{4\pi\varepsilon_o a^3}\,dr$

(b) $-\int_\infty^b \frac{Q}{4\pi\varepsilon_o r^2}\,dr$

(c) $-\int_\infty^a \frac{Q}{4\pi\varepsilon_o r^2}\,dr - \int_a^b \frac{Qr}{4\pi\varepsilon_o a^3}\,dr$

(d) $-\int_\infty^a \frac{Q}{4\pi\varepsilon_o r^3}\,dr$

4.9 A potential field is given by $V = 3x^2y - yz$. Which of the following is not true?

(a) At point $(1, 0, -1)$, V and $\mathbf{E}$ vanish.
(b) $x^2y = 1$ is an equipotential line on the xy-plane.
(c) The equipotential surface $V = -8$ passes through point $P(2, -1, 4)$.
(d) The electric field at P is $12\mathbf{a}_x - 8\mathbf{a}_y - \mathbf{a}_z$ V/m.
(e) A unit normal to the equipotential surface $V = -8$ at P is $-0.83\mathbf{a}_x + 0.55\mathbf{a}_y + 0.07\mathbf{a}_z$.

4.10 An electric potential field is produced by point charges 1 μC and 4 μC located at $(-2, 1, 5)$ and $(1, 3, -1)$, respectively. The energy stored in the field is

(a) 2.57 mJ
(b) 5.14 mJ
(c) 10.28 mJ
(d) None of the above

Answers: 4.1c,e, 4.2d, 4.3b, 4.4a, 4.5d, 4.6e, 4.7a, 4.8c, 4.9a, 4.10b.

PROBLEMS

4.1 Point charges $Q_1 = 5\ \mu$C and $Q_2 = -4\ \mu$C are placed at (3, 2, 1) and $(-4, 0, 6)$, respectively. Determine the force on Q_1.

4.2 Five identical 15-μC point charges are located at the center and corners of a square defined by $-1 < x, y < 1, z = 0$.

(a) Find the force on the 10-μC point charge at (0, 0, 2).
(b) Calculate the electric field intensity at (0, 0, 2).

4.3 Point charges Q_1 and Q_2 are, respectively, located at $(4, 0, -3)$ and (2, 0, 1). If $Q_2 = 4$ nC, find Q_1 such that

(a) The **E** at (5, 0, 6) has no z-component
(b) The force on a test charge at (5, 0, 6) has no x-component.

4.4 Charges $+Q$ and $+3Q$ are separated by a distance 2 m. A third charge is located such that the electrostatic system is in equilibrium. Find the location and the value of the third charge in terms of Q.

4.5 Determine the total charge

(a) On line $0 < x < 5$ m if $\rho_L = 12x^2$ mC/m
(b) On the cylinder $\rho = 3, 0 < z < 4$ m if $\rho_S = \rho z^2$ nC/m^2
(c) Within the sphere $r = 4$ m if $\rho_v = \dfrac{10}{r \sin\theta}$ C/m^3

4.6 Calculate the total charge due to the charge distributions labeled *A, B, C* in Fig. 4.23.

4.7 Find **E** at (5, 0, 0) due to charge distribution labeled *A* in Figure 4.23.

4.8 Due to the charge distribution labeled *B* in Figure 4.23,

(a) Find **E** at point (0, 0, 3) if $\rho_S = 5$ mC/m^2.
(b) Find **E** at point (0, 0, 3) if $\rho_S = 5 \sin\phi$ mC/m^2.

4.9 A circular disk of radius a carries charge $\rho_S = \dfrac{1}{\rho}$ C/m^2. Calculate the potential at $(0, 0, h)$.

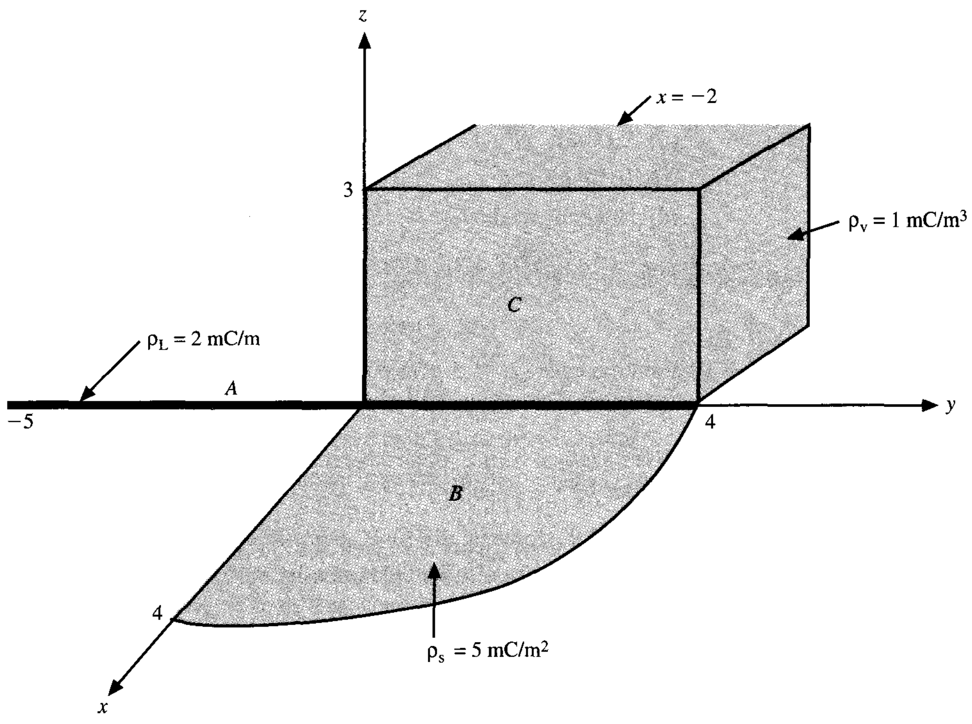

Figure 4.23 For Problem 4.6.

4.10 A ring placed along $y^2 + z^2 = 4$, $x = 0$ carries a uniform charge of 5 μC/m.

(a) Find **D** at $P(3, 0, 0)$.

(b) If two identical point charges Q are placed at $(0, -3, 0)$ and $(0, 3, 0)$ in addition to the ring, find the value of Q such that $\mathbf{D} = 0$ at P.

***4.11** (a) Show that the electric field at point $(0, 0, h)$ due to the rectangle described by $-a \leq x \leq a$, $-b \leq y \leq b$, $z = 0$ carrying uniform charge ρ_S C/m^2 is

$$\mathbf{E} = \frac{\rho_S}{\pi\varepsilon_o}\tan^{-1}\left[\frac{ab}{h(a^2 + b^2 + h^2)^{1/2}}\right]\mathbf{a}_z$$

(b) If $a = 2$, $b = 5$, $\rho_S = 10^{-5}$, find the total charge on the plate and the electric field intensity at $(0, 0, 10)$.

4.12 A point charge 100 pC is located at $(4, 1, -3)$ while the x-axis carries charge 2 nC/m. If the plane $z = 3$ also carries charge 5 nC/m^2, find **E** at $(1, 1, 1)$.

4.13 Line $x = 3$, $z = -1$ carries charge 20 nC/m while plane $x = -2$ carries charge 4 nC/m^2. Find the force on a point charge -5 mC located at the origin.

4.14 Point charges are placed at the corners of a square of size 4 m as shown in Figure 4.24. If $Q = 15\ \mu$C, find **D** at $(0, 0, 6)$.

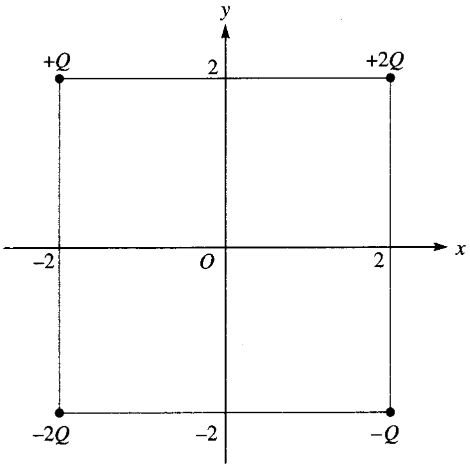

Figure 4.24 For Problem 4.14.

***4.15** State Gauss's law. Deduce Coulomb's law from Gauss's law thereby affirming that Gauss's law is an alternative statement of Coulomb's law and that Coulomb's law is implicit in Maxwell's equation $\nabla \cdot \mathbf{D} = \rho_v$.

4.16 Determine the charge density due to each of the following electric flux densities:

(a) $\mathbf{D} = 8xy\mathbf{a}_x + 4x^2\mathbf{a}_y$ C/m^2

(b) $\mathbf{D} = \rho \sin \phi \, \mathbf{a}_\rho + 2\rho \cos \phi \, \mathbf{a}_\phi + 2z^2\mathbf{a}_z$ C/m^2

(c) $\mathbf{D} = \dfrac{2 \cos \theta}{r^3} \mathbf{a}_r + \dfrac{\sin \theta}{r^3} \mathbf{a}_\theta$ C/m^2

4.17 Let $\mathbf{E} = xy\mathbf{a}_x + x^2\mathbf{a}_y$, find

(a) Electric flux density $\mathbf{D}$.

(b) The volume charge density ρ_v.

4.18 Plane $x + 2y = 5$ carries charge $\rho_S = 6$ nC/m^2. Determining $\mathbf{E}$ at $(-1, 0, 1)$.

4.19 In free space, $\mathbf{D} = 2y^2\mathbf{a}_x + 4xy - \mathbf{a}_z$ mC/m^2. Find the total charge stored in the region $1 < x < 2, 1 < y < 2, -1 < z < 4$.

4.20 In a certain region, the electric field is given by

$$\mathbf{D} = 2\rho(z + 1)\cos \phi \, \mathbf{a}_\rho - \rho(z + 1)\sin \phi \, \mathbf{a}_\phi + \rho^2 \cos \phi \, \mathbf{a}_z \ \mu\text{C/m}^2$$

(a) Find the charge density.

(b) Calculate the total charge enclosed by the volume $0 < \rho < 2$, $0 < \phi < \pi/2$, $0 < z < 4$.

(c) Confirm Gauss's law by finding the net flux through the surface of the volume in (b).

***4.21** The Thomson model of a hydrogen atom is a sphere of positive charge with an electron (a point charge) at its center. The total positive charge equals the electronic charge e. Prove that when the electron is at a distance r from the center of the sphere of positive charge, it is attracted with a force

$$F = \frac{e^2 r}{4\pi\varepsilon_o R^3}$$

where R is the radius of the sphere.

4.22 Three concentric spherical shells $r = 1$, $r = 2$, and $r = 3$ m, respectively, have charge distributions 2, -4, and 5 μC/m^2.

(a) Calculate the flux through $r = 1.5$ m and $r = 2.5$ m.

(b) Find **D** at $r = 0.5$, $r = 2.5$, and $r = 3.5$ m.

4.23 Given that

$$\rho_v = \begin{cases} 12\rho \text{ nC/m}^3, & 1 < \rho < 2 \\ 0, & \text{otherwise} \end{cases}$$

Determine **D** everywhere.

4.24 Let

$$\rho_v = \begin{cases} \dfrac{10}{r^2} \text{ mC/m}^3, & 1 < r < 4 \\ 0, & r > 0 \end{cases}$$

(a) Find the net flux crossing surface $r = 2$ m and $r = 6$ m.

(b) Determine **D** at $r = 1$ m and $r = 5$ m.

4.25 Find the work done in carrying a 5-C charge from $P(1, 2, -4)$ to $R(3, -5, 6)$ in an electric field

$$\mathbf{E} = \mathbf{a}_x + z^2\mathbf{a}_y + 2yz\mathbf{a}_z \text{ V/m}$$

4.26 Given that the electric field in a certain region is

$$\mathbf{E} = (z + 1)\sin\phi\,\mathbf{a}_\rho + (z + 1)\cos\,\mathbf{a}_\phi + \rho\sin\phi\,\mathbf{a}_z \text{ V/m}$$

determine the work done in moving a 4-nC charge from

(a) $A(1, 0, 0)$ to $B(4, 0, 0)$

(b) $B(4, 0, 0)$ to $C(4, 30°, 0)$

(c) $C(4, 30°, 0)$ to $D(4, 30°, -2)$

(d) A to D

4.27 In an electric field $\mathbf{E} = 20r\sin\theta\,\mathbf{a}_r + 10r\cos\theta\,\mathbf{a}_\theta$ V/m, calculate the energy expended in transferring a 10-nC charge

(a) From $A(5, 30°, 0°)$ to $B(5, 90°, 0°)$

(b) From A to $C(10, 30°, 0°)$

(c) From A to $D(5, 30°, 60°)$

(d) From A to $E(10, 90°, 60°)$

4.28 Let $V = xy^2z$, calculate the energy expended in transfering a 2-μC point charge from $(1, -1, 2)$ to $(2, 1, -3)$.

4.29 Determine the electric field due to the following potentials:

(a) $V = x^2 + 2y^2 + 4z^2$

(b) $V = \sin(x^2 + y^2 + z^2)^{1/2}$

(c) $V = \rho^2(z + 1)\sin \phi$

(d) $V = e^{-r} \sin \theta \cos 2\phi$

4.30 Three point charges $Q_1 = 1$ mC, $Q_2 = -2$ mC, and $Q_3 = 3$ mC are, respectively, located at $(0, 0, 4)$, $(-2, 5, 1)$, and $(3, -4, 6)$.

(a) Find the potential V_P at $P(-1, 1, 2)$.

(b) Calculate the potential difference V_{PQ} if Q is $(1, 2, 3)$.

4.31 In free space, $V = x^2y(z + 3)$ V. Find

(a) **E** at $(3, 4, -6)$

(b) the charge within the cube $0 < x,y,z < 1$.

4.32 A spherical charge distribution is given by

$$\rho_v = \begin{cases} \rho_o \dfrac{r}{a}, & r < a \\ 0, & r > a \end{cases}$$

Find V everywhere.

4.33 To verify that $\mathbf{E} = yz\mathbf{a}_x + xz\mathbf{a}_y + xy\mathbf{a}_z$ V/m is truely an electric field, show that

(a) $\nabla \times \mathbf{E} = 0$

(b) $\oint_L \mathbf{E} \cdot d\mathbf{l} = 0$, where L is the edge of the square defined by $0 < x,y < 2$, $z = 1$.

4.34 (a) A total charge $Q = 60\ \mu$C is split into two equal charges located at 180° intervals around a circular loop of radius 4 m. Find the potential at the center of the loop.

(b) If Q is split into three equal charges spaced at 120° intervals around the loop, find the potential at the center.

(c) If in the limit $\rho_L = \dfrac{Q}{8\pi}$, find the potential at the center.

4.35 For a spherical charge distribution

$$\rho_v = \begin{cases} \rho_o(a^2 - r^2), & r < a \\ 0, & r > a \end{cases}$$

(a) Find **E** and V for $r \geq a$

(b) Find **E** and V for $r \leq a$

(c) Find the total charge

(d) Show that **E** is maximum when $r = 0.145a$.

***4.36** (a) Prove that when a particle of constant mass and charge is accelerated from rest in an electric field, its final velocity is proportional to the square root of the potential difference through which it is accelerated.

(b) Find the magnitude of the proportionality constant if the particle is an electron.

(c) Through what voltage must an electron be accelerated, assuming no change in its mass, to require a velocity one-tenth that of light? (At such velocities, the mass of a body becomes appreciably larger than its "rest mass" and cannot be considered constant.)

***4.37** An electron is projected with an initial velocity $u_o = 10^7$ m/s into the uniform field between the parallel plates of Figure 4.25. It enters the field at the midway between the plates. If the electron just misses the upper plate as it emerges from the field.

(a) Find the electric field intensity.

(b) Calculate its velocity as it emerges from the field. Neglect edge effects.

4.38 An electric dipole with $\mathbf{p} = p\mathbf{a}_z$ C · m is placed at $(x, z) = (0, 0)$. If the potential at (0, 1) nm is 9 V, find the potential at (1, 1) nm.

4.39 Point charges Q and $-Q$ are located at $(0, d/2, 0)$ and $(0, -d/2, 0)$. Show that at point (r, θ, ϕ), where $r \gg d$,

$$V = \frac{Qd \sin\theta \sin\phi}{4\pi\varepsilon_o r^2}$$

Find the corresponding **E** field.

4.40 Determine the work necessary to transfer charges $Q_1 = 1$ mC and $Q_2 = -2$ mC from infinity to points $(-2, 6, 1)$ and $(3, -4, 0)$, respectively.

4.41 A point charge Q is placed at the origin. Calculate the energy stored in region $r > a$.

4.42 Find the energy stored in the hemispherical region $r \leq 2$ m, $0 < \theta < \pi$, where

$$\mathbf{E} = 2r \sin\theta \cos\phi\, \mathbf{a}_r + r\cos\theta\cos\phi\, \mathbf{a}_\theta - r\sin\phi\, \mathbf{a}_\phi \text{ V/m}$$

exists.

4.43 If $V = \rho^2 z \sin\phi$, calculate the energy within the region defined by $1 < \rho < 4$, $-2 < z < 2$, $0 < \phi < \pi/3$.

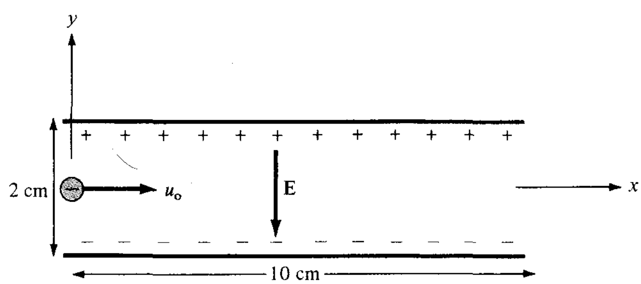

Figure 4.25 For Problem 4.37.