Chapter 1

VECTOR ALGEBRA

One thing I have learned in a long life: that all our science, measured against reality, is primitive and childlike—and yet is the most precious thing we have.

—ALBERT EINSTEIN

1. Introduction

1.1 INTRODUCTION

1. Electromagnetics (EM) may be regarded as the study of the interactions between electric charges at rest and in motion. It entails the analysis, synthesis, physical interpretation, and application of electric and magnetic fields.

1 **Electromagnetics** (EM) is a branch of physics or electrical engineering in which electric and magnetic phenomena are studied.

1 EM principles find applications in various allied disciplines such as microwaves, antennas, electric machines, satellite communications, bioelectromagnetics, plasmas, nuclear research, fiber optics, electromagnetic interference and compatibility, electromechanical energy conversion, radar meteorology, and remote sensing.[1,2] In physical medicine, for example, EM power, either in the form of shortwaves or microwaves, is used to heat deep tissues and to stimulate certain physiological responses in order to relieve certain pathological conditions. EM fields are used in induction heaters for melting, forging, annealing, surface hardening, and soldering operations. Dielectric heating equipment uses shortwaves to join or seal thin sheets of plastic materials. EM energy offers many new and exciting possibilities in agriculture. It is used, for example, to change vegetable taste by reducing acidity.

1 EM devices include transformers, electric relays, radio/TV, telephone, electric motors, transmission lines, waveguides, antennas, optical fibers, radars, and lasers. The design of these devices requires thorough knowledge of the laws and principles of EM.

[1]For numerous applications of electrostatics, see J. M. Crowley, *Fundamentals of Applied Electrostatics*. New York: John Wiley & Sons, 1986.

[2]For other areas of applications of EM, see, for example, D. Teplitz, ed., *Electromagnetism: Paths to Research*. New York: Plenum Press, 1982.

†1.2 A PREVIEW OF THE BOOK

The subject of electromagnetic phenomena in this book can be summarized in Maxwell's equations:

$$\nabla \cdot \mathbf{D} = \rho_v \tag{1.1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.3}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{1.4}$$

where ∇ = the vector differential operator
$\mathbf{D}$ = the electric flux density
$\mathbf{B}$ = the magnetic flux density
$\mathbf{E}$ = the electric field intensity
$\mathbf{H}$ = the magnetic field intensity
ρ_v = the volume charge density
and $\mathbf{J}$ = the current density.

Maxwell based these equations on previously known results, both experimental and theoretical. A quick look at these equations shows that we shall be dealing with vector quantities. It is consequently logical that we spend some time in Part I examining the mathematical tools required for this course. The derivation of eqs. (1.1) to (1.4) for time-invariant conditions and the physical significance of the quantities **D**, **B**, **E**, **H**, **J** and ρ_v will be our aim in Parts II and III. In Part IV, we shall reexamine the equations for time-varying situations and apply them in our study of practical EM devices.

1.3 SCALARS AND VECTORS

Vector analysis is a mathematical tool with which electromagnetic (EM) concepts are most conveniently expressed and best comprehended. We must first learn its rules and techniques before we can confidently apply it. Since most students taking this course have little exposure to vector analysis, considerable attention is given to it in this and the next two chapters.[3] This chapter introduces the basic concepts of vector algebra in Cartesian coordinates only. The next chapter builds on this and extends to other coordinate systems.

A quantity can be either a scalar or a vector.

†Indicates sections that may be skipped, explained briefly, or assigned as homework if the text is covered in one semester.

[3]The reader who feels no need for review of vector algebra can skip to the next chapter.

A scalar is a quantity that has only magnitude.

Quantities such as time, mass, distance, temperature, entropy, electric potential, and population are scalars.

A vector is a quantity that has both magnitude and direction.

Vector quantities include velocity, force, displacement, and electric field intensity. Another class of physical quantities is called *tensors,* of which scalars and vectors are special cases. For most of the time, we shall be concerned with scalars and vectors.[4]

To distinguish between a scalar and a vector it is customary to represent a vector by a letter with an arrow on top of it, such as $\vec{A}$ and $\vec{B}$, or by a letter in boldface type such as **A** and **B.** A scalar is represented simply by a letter—e.g., A, B, U, and V.

EM theory is essentially a study of some particular fields.

A field is a function that specifies a particular quantity everywhere in a region.

If the quantity is scalar (or vector), the field is said to be a scalar (or vector) field. Examples of scalar fields are temperature distribution in a building, sound intensity in a theater, electric potential in a region, and refractive index of a stratified medium. The gravitational force on a body in space and the velocity of raindrops in the atmosphere are examples of vector fields.

1.4 UNIT VECTOR

A vector **A** has both magnitude and direction. The *magnitude* of **A** is a scalar written as A or $|\mathbf{A}|$. A *unit vector* $\mathbf{a}_A$ along **A** is defined as a vector whose magnitude is unity (i.e., 1) and its direction is along **A,** that is,

$$\boxed{\mathbf{a}_A = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{A}{A}} \tag{1.5}$$

Note that $|\mathbf{a}_A| = 1$. Thus we may write **A** as

$$\mathbf{A} = A\mathbf{a}_A \tag{1.6}$$

which completely specifies **A** in terms of its magnitude A and its direction $\mathbf{a}_A$.

A vector **A** in Cartesian (or rectangular) coordinates may be represented as

$$(A_x, A_y, A_z) \qquad \text{or} \qquad A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z \tag{1.7}$$

[4]For an elementary treatment of tensors, see, for example, A. I. Borisenko and I. E. Tarapor, *Vector and Tensor Analysis with Application.* Englewood Cliffs, NJ: Prentice-Hall, 1968.

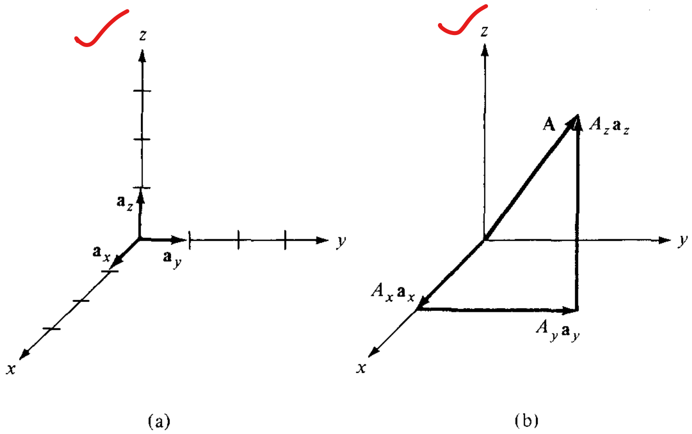

Figure 1.1 **(a)** Unit vectors $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$, **(b)** components of **A** along $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$.

where A_x, A_y, and A_z are called the *components of* **A** in the x, y, and z directions respectively; $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ are unit vectors in the x, y, and z directions, respectively. For example, $\mathbf{a}_x$ is a dimensionless vector of magnitude one in the direction of the increase of the x-axis. The unit vectors $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ are illustrated in Figure 1.1(a), and the components of **A** along the coordinate axes are shown in Figure 1.1(b). The magnitude of vector **A** is given by

$$\boxed{A = \sqrt{A_x^2 + A_y^2 + A_z^2}} \tag{1.8}$$

and the unit vector along **A** is given by

$$\mathbf{a}_A = \frac{A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}} \tag{1.9}$$

1.5 VECTOR ADDITION AND SUBTRACTION

Two vectors **A** and **B** can be added together to give another vector **C**; that is,

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \tag{1.10}$$

The vector addition is carried out component by component. Thus, if $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$.

$$\mathbf{C} = (A_x + B_x)\mathbf{a}_x + (A_y + B_y)\mathbf{a}_y + (A_z + B_z)\mathbf{a}_z \tag{1.11}$$

Vector subtraction is similarly carried out as

$$\begin{aligned}\mathbf{D} &= \mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}) \\ &= (A_x - B_x)\mathbf{a}_x + (A_y - B_y)\mathbf{a}_y + (A_z - B_z)\mathbf{a}_z\end{aligned} \tag{1.12}$$

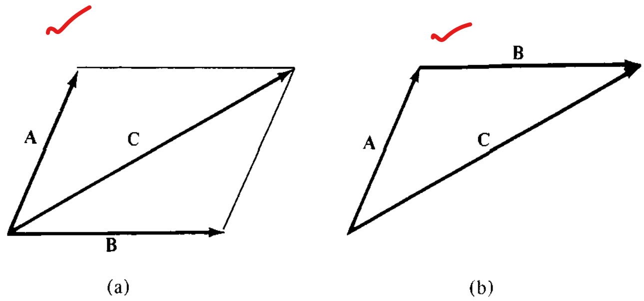

Figure 1.2 Vector addition $\mathbf{C} = \mathbf{A} + \mathbf{B}$: **(a)** parallelogram rule, **(b)** head-to-tail rule.

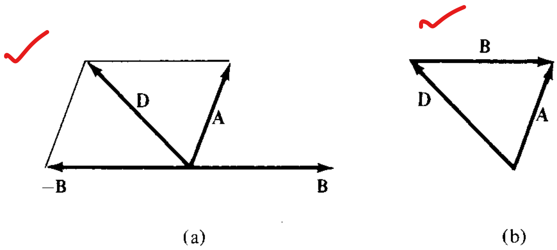

Figure 1.3 Vector subtraction $\mathbf{D} = \mathbf{A} - \mathbf{B}$: **(a)** parallelogram rule, **(b)** head-to-tail rule.

Graphically, vector addition and subtraction are obtained by either the parallelogram rule or the head-to-tail rule as portrayed in Figures 1.2 and 1.3, respectively.

The three basic laws of algebra obeyed by any giveny vectors **A**, **B**, and **C**, are summarized as follows:

Law	Addition	Multiplication
Commutative	$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$	$k\mathbf{A} = \mathbf{A}k$
Associative	$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$	$k(\ell\mathbf{A}) = (k\ell)\mathbf{A}$
Distributive	$k(\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B}$	

where k and ℓ are scalars. Multiplication of a vector with another vector will be discussed in Section 1.7.

1.6 POSITION AND DISTANCE VECTORS

1 A point P in Cartesian coordinates may be represented by (x, y, z).

The **position vector** $\mathbf{r}_P$ (or **radius vector**) of point P is as the directed sitance from the origin O to P; i.e.,

$$\mathbf{r}_P = OP = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z \tag{1.13}$$

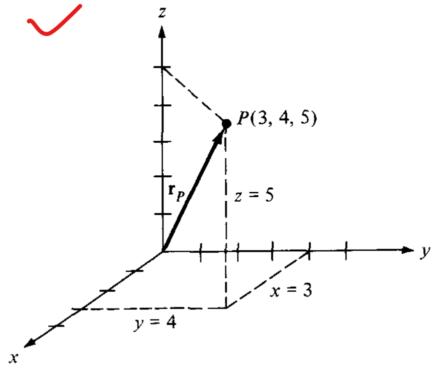

Figure 1.4 Illustration of position vector $\mathbf{r}_P = 3\mathbf{a}_x + 4\mathbf{a}_y + 5\mathbf{a}_z$.

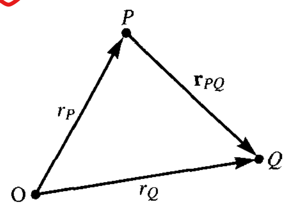

Figure 1.5 Distance vector $\mathbf{r}_{PQ}$.

The position vector of point P is useful in defining its position in space. Point (3, 4, 5), for example, and its position vector $3\mathbf{a}_x + 4\mathbf{a}_y + 5\mathbf{a}_z$ are shown in Figure 1.4.

The **distance vector** is the displacement from one point to another.

If two points P and Q are given by (x_P, y_P, z_P) and (x_Q, y_Q, z_Q), the *distance vector* (or *separation vector*) is the displacement from P to Q as shown in Figure 1.5; that is,

$$\begin{aligned}\mathbf{r}_{PQ} &= r_Q - r_P \\ &= (x_Q - x_P)\mathbf{a}_x + (y_Q - y_P)\mathbf{a}_y + (z_Q - z_P)\mathbf{a}_z\end{aligned} \tag{1.14}$$

The difference between a point P and a vector $\mathbf{A}$ should be noted. Though both P and $\mathbf{A}$ may be represented in the same manner as (x, y, z) and (A_x, A_y, A_z), respectively, the point P is not a vector; only its position vector $\mathbf{r}_P$ is a vector. Vector $\mathbf{A}$ may depend on point P, however. For example, if $\mathbf{A} = 2xy\mathbf{a}_x + y^2\mathbf{a}_y - xz^2\mathbf{a}_z$ and P is $(2, -1, 4)$, then $\mathbf{A}$ at P would be $-4\mathbf{a}_x + \mathbf{a}_y - 32\mathbf{a}_z$. A vector field is said to be *constant* or *uniform* if it does not depend on space variables x, y, and z. For example, vector $\mathbf{B} = 3\mathbf{a}_x - 2\mathbf{a}_y + 10\mathbf{a}_z$ is a uniform vector while vector $\mathbf{A} = 2xy\mathbf{a}_x + y^2\mathbf{a}_y - xz^2\mathbf{a}_z$ is not uniform because $\mathbf{B}$ is the same everywhere whereas $\mathbf{A}$ varies from point to point.

EXAMPLE 1.1

If $\mathbf{A} = 10\mathbf{a}_x - 4\mathbf{a}_y + 6\mathbf{a}_z$ and $\mathbf{B} = 2\mathbf{a}_x + \mathbf{a}_y$, find: (a) the component of $\mathbf{A}$ along $\mathbf{a}_y$, (b) the magnitude of $3\mathbf{A} - \mathbf{B}$, (c) a unit vector along $\mathbf{A} + 2\mathbf{B}$.

Solution:

(a) The component of **A** along $\mathbf{a}_y$ is $A_y = -4$.

(b) $$3\mathbf{A} - \mathbf{B} = 3(10, -4, 6) - (2, 1, 0) = (30, -12, 18) - (2, 1, 0) = (28, -13, 18)$$

Hence

$$|3\mathbf{A} - \mathbf{B}| = \sqrt{28^2 + (-13)^2 + (18)^2} = \sqrt{1277} = 35.74$$

(c) Let $\mathbf{C} = \mathbf{A} + 2\mathbf{B} = (10, -4, 6) + (4, 2, 0) = (14, -2, 6)$.

A unit vector along **C** is

$$\mathbf{a}_c = \frac{\mathbf{C}}{|\mathbf{C}|} = \frac{(14, -2, 6)}{\sqrt{14^2 + (-2)^2 + 6^2}}$$

or

$$\mathbf{a}_c = 0.9113\mathbf{a}_x - 0.1302\mathbf{a}_y + 0.3906\mathbf{a}_z$$

Note that $|\mathbf{a}_c| = 1$ as expected.

PRACTICE EXERCISE 1.1

Given vectors $\mathbf{A} = \mathbf{a}_x + 3\mathbf{a}_z$ and $\mathbf{B} = 5\mathbf{a}_x + 2\mathbf{a}_y - 6\mathbf{a}_z$, determine

(a) $|\mathbf{A} + \mathbf{B}|$

(b) $5\mathbf{A} - \mathbf{B}$

(c) The component of **A** along $\mathbf{a}_y$

(d) A unit vector parallel to $3\mathbf{A} + \mathbf{B}$

Answer: (a) 7, (b) (0, −2, 21), (c) 0, (d) ±(0.9117, 0.2279, 0.3419).

EXAMPLE 1.2

Points P and Q are located at (0, 2, 4) and (−3, 1, 5). Calculate

(a) The position vector P

(b) The distance vector from P to Q

(c) The distance between P and Q

(d) A vector parallel to PQ with magntude of 10

Solution:

(a) $\mathbf{r}_P = 0\mathbf{a}_x + 2\mathbf{a}_y + 4\mathbf{a}_z = 2\mathbf{a}_y + 4\mathbf{a}_z$

(b) $\mathbf{r}_{PQ} = \mathbf{r}_Q - \mathbf{r}_P = (-3, 1, 5) - (0, 2, 4) = (-3, -1, 1)$
or $\mathbf{r}_{PQ} = -3\mathbf{a}_x - \mathbf{a}_y + \mathbf{a}_z$

(c) Since $\mathbf{r}_{PQ}$ is the distance vector from P to Q, the distance between P and Q is the magnitude of this vector; that is,

$$d = |\mathbf{r}_{PQ}| = \sqrt{9 + 1 + 1} = 3.317$$

Alternatively:

$$\begin{aligned} d &= \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2 + (z_Q - z_P)^2} \\ &= \sqrt{9 + 1 + 1} = 3.317 \end{aligned}$$

(d) Let the required vector be $\mathbf{A}$, then

$$\mathbf{A} = A\mathbf{a}_A$$

where $A = 10$ is the magnitude of $\mathbf{A}$. Since $\mathbf{A}$ is parallel to PQ, it must have the same unit vector as $\mathbf{r}_{PQ}$ or $\mathbf{r}_{QP}$. Hence,

$$\mathbf{a}_A = \pm\frac{\mathbf{r}_{PQ}}{|\mathbf{r}_{PQ}|} = \pm\frac{(-3, -1, 1)}{3.317}$$

and

$$\mathbf{A} = \pm\frac{10(-3, -1, -1)}{3.317} = \pm(-9.045\mathbf{a}_x - 3.015\mathbf{a}_y + 3.015\mathbf{a}_z)$$

✓

PRACTICE EXERCISE 1.2

Given points $P(1, -3, 5)$, $Q(2, 4, 6)$, and $R(0, 3, 8)$, find: (a) the position vectors of P and R, (b) the distance vector $\mathbf{r}_{QR}$, (c) the distance between Q and R.

Answer: (a) $\mathbf{a}_x - 3\mathbf{a}_y + 5\mathbf{a}_z$, $3\mathbf{a}_x + 3\mathbf{a}_y$, (b) $-2\mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z$.

EXAMPLE 1.3

A river flows southeast at 10 km/hr and a boat flows upon it with its bow pointed in the direction of travel. A man walks upon the deck at 2 km/hr in a direction to the right and perpendicular to the direction of the boat's movement. Find the velocity of the man with respect to the earth.

Solution:

Consider Figure 1.6 as illustrating the problem. The velocity of the boat is

$$\begin{aligned} \mathbf{u}_b &= 10(\cos 45^\circ\, \mathbf{a}_x - \sin 45^\circ\, \mathbf{a}_y) \\ &= 7.071\mathbf{a}_x - 7.071\mathbf{a}_y \text{ km/hr} \end{aligned}$$

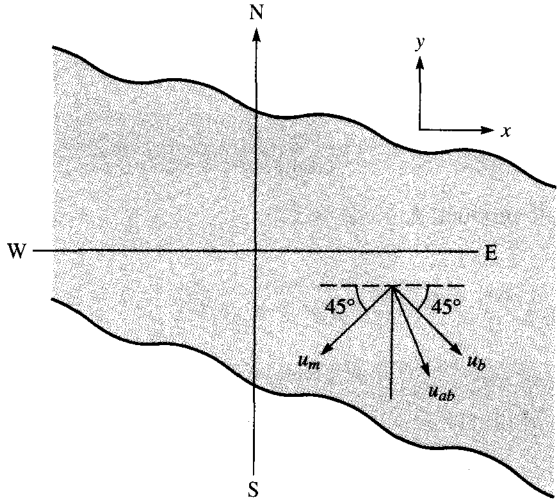

Figure 1.6 For Example 1.3.

The velocity of the man with respect to the boat (relative velocity) is

$$\begin{aligned} \mathbf{u}_m &= 2(-\cos 45^\circ\, \mathbf{a}_x - \sin 45^\circ\, \mathbf{a}_y) \\ &= -1.414\mathbf{a}_x - 1.414\mathbf{a}_y \text{ km/hr} \end{aligned}$$

Thus the absolute velocity of the man is

$$\begin{aligned} \mathbf{u}_{ab} &= \mathbf{u}_m + \mathbf{u}_b = 5.657\mathbf{a}_x - 8.485\mathbf{a}_y \\ |\mathbf{u}_{ab}| &= 10.2\underline{/-56.3^\circ} \end{aligned}$$

that is, 10.2 km/hr at 56.3° south of east.

PRACTICE EXERCISE 1.3

An airplane has a ground speed of 350 km/hr in the direction due west. If there is a wind blowing northwest at 40 km/hr, calculate the true air speed and heading of the airplane.

Answer: 379.3 km/hr, 4.275° north of west.

1.7 VECTOR MULTIPLICATION

When two vectors **A** and **B** are multiplied, the result is either a scalar or a vector depending on how they are multiplied. Thus there are two types of vector multiplication:

1. Scalar (or dot) product: $\mathbf{A} \cdot \mathbf{B}$
2. Vector (or cross) product: $\mathbf{A} \times \mathbf{B}$

Multiplication of three vectors **A**, **B**, and **C** can result in either:

3. Scalar triple product: $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$

or

4. Vector triple product: $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$

A. Dot Product

The **dot product** of two vectors **A** and **B**, written as $\mathbf{A} \cdot \mathbf{B}$, is defined geometrically as the product of the magnitudes of **A** and **B** and the cosine of the angle between them.

Thus:

$$\boxed{\mathbf{A} \cdot \mathbf{B} = AB \cos \theta_{AB}} \tag{1.15}$$

where θ_{AB} is the *smaller* angle between **A** and **B**. The result of $\mathbf{A} \cdot \mathbf{B}$ is called either the *scalar product* because it is scalar, or the *dot product* due to the dot sign. If $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$, then

$$\boxed{\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z} \tag{1.16}$$

which is obtained by multiplying **A** and **B** component by component. Two vectors **A** and **B** are said to be *orthogonal* (or perpendicular) with each other if $\mathbf{A} \cdot \mathbf{B} = 0$.

Note that dot product obeys the following:

(i) *Commutative law:*

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \tag{1.17}$$

(ii) *Distributive law:*

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \tag{1.18}$$

$$\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2 = A^2 \tag{1.19}$$

(iii)

Also note that

$$\mathbf{a}_x \cdot \mathbf{a}_y = \mathbf{a}_y \cdot \mathbf{a}_z = \mathbf{a}_z \cdot \mathbf{a}_x = 0 \tag{1.20a}$$

$$\mathbf{a}_x \cdot \mathbf{a}_x = \mathbf{a}_y \cdot \mathbf{a}_y = \mathbf{a}_z \cdot \mathbf{a}_z = 1 \tag{1.20b}$$

It is easy to prove the identities in eqs. (1.17) to (1.20) by applying eq. (1.15) or (1.16).

B. Cross Product

The **cross product** of two vectors **A** and **B**, written as **A** × **B**, is a vector quantity whose magnitude is the area of the parallelopiped formed by **A** and **B** (see Figure 1.7) and is in the direction of advance of a right-handed screw as **A** is turned into **B**.

Thus

$$\boxed{\mathbf{A} \times \mathbf{B} = AB \sin \theta_{AB} \mathbf{a}_n} \tag{1.21}$$

where $\mathbf{a}_n$ is a unit vector normal to the plane containing **A** and **B**. The direction of $\mathbf{a}_n$ is taken as the direction of the right thumb when the fingers of the right hand rotate from **A** to **B** as shown in Figure 1.8(a). Alternatively, the direction of $\mathbf{a}_n$ is taken as that of the advance of a right-handed screw as **A** is turned into **B** as shown in Figure 1.8(b).

The vector multiplication of eq. (1.21) is called *cross product* due to the cross sign; it is also called *vector product* because the result is a vector. If $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$ then

$$\boxed{\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}} \tag{1.22a}$$

$$= (A_yB_z - A_zB_y)\mathbf{a}_x + (A_zB_x - A_xB_z)\mathbf{a}_y + (A_xB_y - A_yB_x)\mathbf{a}_z \tag{1.22b}$$

which is obtained by "crossing" terms in cyclic permutation, hence the name cross product.

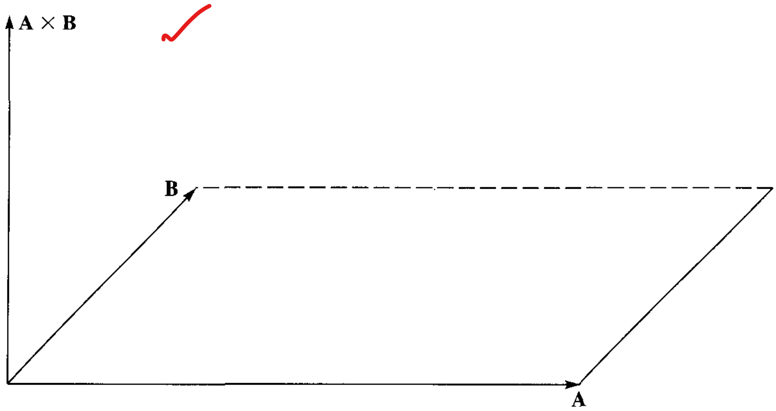

Figure 1.7 The cross product of **A** and **B** is a vector with magnitude equal to the area of the parallelogram and direction as indicated.

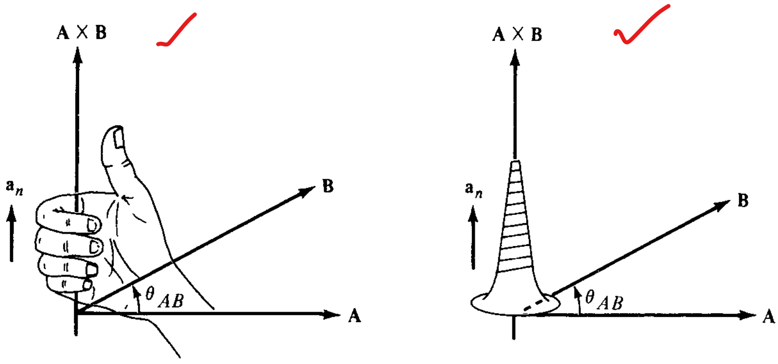

Figure 1.8 Direction of $\mathbf{A} \times \mathbf{B}$ and $\mathbf{a}_n$ using **(a)** right-hand rule, **(b)** right-handed screw rule.

Note that the cross product has the following basic properties:

(i) It is not commutative:

$$\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A} \tag{1.23a}$$

It is anticommutative:

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{1.23b}$$

(ii) It is not associative:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \tag{1.24}$$

(iii) It is distributive:

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \tag{1.25}$$

(iv)

$$\mathbf{A} \times \mathbf{A} = 0 \tag{1.26}$$

Also note that

$$\begin{aligned} \mathbf{a}_x \times \mathbf{a}_y &= \mathbf{a}_z \\ \mathbf{a}_y \times \mathbf{a}_z &= \mathbf{a}_x \\ \mathbf{a}_z \times \mathbf{a}_x &= \mathbf{a}_y \end{aligned} \tag{1.27}$$

which are obtained in cyclic permutation and illustrated in Figure 1.9. The identities in eqs. (1.25) to (1.27) are easily verified using eq. (1.21) or (1.22). It should be noted that in obtaining $\mathbf{a}_n$, we have used the right-hand or right-handed screw rule because we want to be consistent with our coordinate system illustrated in Figure 1.1, which is right-handed. A right-handed coordinate system is one in which the right-hand rule is satisfied: that is, $\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z$ is obeyed. In a left-handed system, we follow the left-hand or left-handed

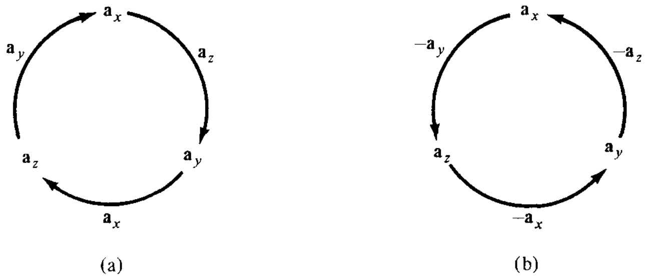

Figure 1.9 Cross product using cyclic permutation: **(a)** moving clockwise leads to positive results: **(b)** moving counterclockwise leads to negative results.

screw rule and $\mathbf{a}_x \times \mathbf{a}_y = -\mathbf{a}_z$ is satisfied. Throughout this book, we shall stick to right-handed coordinate systems.

Just as multiplication of two vectors gives a scalar or vector result, multiplication of three vectors **A**, **B**, and **C** gives a scalar or vector result depending on how the vectors are multiplied. Thus we have scalar or vector triple product.

C. Scalar Triple Product

Given three vectors **A**, **B**, and **C**, we define the scalar triple product as

$$\boxed{\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})} \tag{1.28}$$

obtained in cyclic permutation. If $\mathbf{A} = (A_x, A_y, A_z)$, $\mathbf{B} = (B_x, B_y, B_z)$, and $\mathbf{C} = (C_x, C_y, C_z)$, then $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ is the volume of a parallelepiped having **A**, **B**, and **C** as edges and is easily obtained by finding the determinant of the 3×3 matrix formed by **A**, **B**, and **C**; that is,

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \tag{1.29}$$

Since the result of this vector multiplication is scalar, eq. (1.28) or (1.29) is called the *scalar triple product.*

D. Vector Triple Product

For vectors **A**, **B**, and **C**, we define the vector tiple product as

$$\boxed{\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})} \tag{1.30}$$

obtained using the "bac-cab" rule. It should be noted that

$$(\mathbf{A} \cdot \mathbf{B})\mathbf{C} \neq \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) \tag{1.31}$$

but

$$(\mathbf{A} \cdot \mathbf{B})\mathbf{C} = \mathbf{C}(\mathbf{A} \cdot \mathbf{B}). \tag{1.32}$$

1.8 COMPONENTS OF A VECTOR

A direct application of vector product is its use in determining the projection (or component) of a vector in a given direction. The projection can be scalar or vector. Given a vector **A**, we define the *scalar component* A_B of **A** along vector **B** as [see Figure 1.10(a)]

$$A_B = A \cos \theta_{AB} = |\mathbf{A}|\,|\mathbf{a}_B| \cos \theta_{AB}$$

or

$$\boxed{A_B = \mathbf{A} \cdot \mathbf{a}_B} \tag{1.33}$$

The *vector component* $\mathbf{A}_B$ of **A** along **B** is simply the scalar component in eq. (1.33) multiplied by a unit vector along **B**; that is,

$$\boxed{\mathbf{A}_B = A_B \mathbf{a}_B = (\mathbf{A} \cdot \mathbf{a}_B)\mathbf{a}_B} \tag{1.34}$$

Both the scalar and vector components of **A** are illustrated in Figure 1.10. Notice from Figure 1.10(b) that the vector can be resolved into two orthogonal components: one component $\mathbf{A}_B$ parallel to **B**, another $(\mathbf{A} - \mathbf{A}_B)$ perpendicular to **B**. In fact, our Cartesian representation of a vector is essentially resolving the vector into three mutually orthogonal components as in Figure 1.1(b).

We have considered addition, subtraction, and multiplication of vectors. However, division of vectors **A/B** has not been considered because it is undefined except when **A** and **B** are parallel so that $\mathbf{A} = k\mathbf{B}$, where k is a constant. Differentiation and integration of vectors will be considered in Chapter 3.

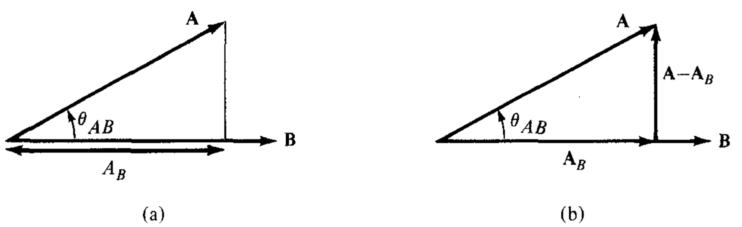

Figure 1.10 Components of **A** along **B**: **(a)** scalar component A_B, **(b)** vector component $\mathbf{A}_B$.

EXAMPLE 1.4

Given vectors $\mathbf{A} = 3\mathbf{a}_x + 4\mathbf{a}_y + \mathbf{a}_z$ and $\mathbf{B} = 2\mathbf{a}_y - 5\mathbf{a}_z$, find the angle between **A** and **B**.

Solution:

The angle θ_{AB} can be found by using either dot product or cross product.

$$\begin{aligned}\mathbf{A} \cdot \mathbf{B} &= (3, 4, 1) \cdot (0, 2, -5)\\ &= 0 + 8 - 5 = 3\end{aligned}$$

$$|\mathbf{A}| = \sqrt{3^2 + 4^2 + 1^2} = \sqrt{26}$$

$$|\mathbf{B}| = \sqrt{0^2 + 2^2 + (-5)^2} = \sqrt{29}$$

$$\cos \theta_{AB} = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{3}{\sqrt{(26)(29)}} = 0.1092$$

$$\theta_{AB} = \cos^{-1} 0.1092 = 83.73°$$

Alternatively:

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 3 & 4 & 1 \\ 0 & 2 & -5 \end{vmatrix}\\ &= (-20 - 2)\mathbf{a}_x + (0 + 15)\mathbf{a}_y + (6 - 0)\mathbf{a}_z\\ &= (-22, 15, 6)\end{aligned}$$

$$|\mathbf{A} \times \mathbf{B}| = \sqrt{(-22)^2 + 15^2 + 6^2} = \sqrt{745}$$

$$\sin \theta_{AB} = \frac{|\mathbf{A} \times \mathbf{B}|}{|\mathbf{A}||\mathbf{B}|} = \frac{\sqrt{745}}{\sqrt{(26)(29)}} = 0.994$$

$$\theta_{AB} = \cos^{-1} 0.994 = 83.73°$$

PRACTICE EXERCISE 1.4

If $\mathbf{A} = \mathbf{a}_x + 3\mathbf{a}_z$ and $\mathbf{B} = 5\mathbf{a}_x + 2\mathbf{a}_y - 6\mathbf{a}_z$, find θ_{AB}.

Answer: 120.6°.

EXAMPLE 1.5

Three field quantities are given by

$$\mathbf{P} = 2\mathbf{a}_x - \mathbf{a}_z$$

$$\mathbf{Q} = 2\mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z$$

$$\mathbf{R} = 2\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z$$

Determine

(a) $(\mathbf{P} + \mathbf{Q}) \times (\mathbf{P} - \mathbf{Q})$

(b) $\mathbf{Q} \cdot \mathbf{R} \times \mathbf{P}$

(c) $\mathbf{P} \cdot \mathbf{Q} \times \mathbf{R}$

(d) $\sin \theta_{QR}$

(e) $\mathbf{P} \times (\mathbf{Q} \times \mathbf{R})$

(f) A unit vector perpendicular to both $\mathbf{Q}$ and $\mathbf{R}$

(g) The component of $\mathbf{P}$ along $\mathbf{Q}$

Solution:

(a)

$$\begin{aligned}
(\mathbf{P} + \mathbf{Q}) \times (\mathbf{P} - \mathbf{Q}) &= \mathbf{P} \times (\mathbf{P} - \mathbf{Q}) + \mathbf{Q} \times (\mathbf{P} - \mathbf{Q}) \\
&= \mathbf{P} \times \mathbf{P} - \mathbf{P} \times \mathbf{Q} + \mathbf{Q} \times \mathbf{P} - \mathbf{Q} \times \mathbf{Q} \\
&= 0 + \mathbf{Q} \times \mathbf{P} + \mathbf{Q} \times \mathbf{P} - 0 \\
&= 2\mathbf{Q} \times \mathbf{P} \\
&= 2 \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 2 & -1 & 2 \\ 2 & 0 & -1 \end{vmatrix} \\
&= 2(1 - 0)\, \mathbf{a}_x + 2(4 + 2)\, \mathbf{a}_y + 2(0 + 2)\, \mathbf{a}_z \\
&= 2\mathbf{a}_x + 12\mathbf{a}_y + 4\mathbf{a}_z
\end{aligned}$$

(b) The only way $\mathbf{Q} \cdot \mathbf{R} \times \mathbf{P}$ makes sense is

$$\begin{aligned}
\mathbf{Q} \cdot (\mathbf{R} \times \mathbf{P}) &= (2, -1, 2) \cdot \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 2 & -3 & 1 \\ 2 & 0 & -1 \end{vmatrix} \\
&= (2, -1, 2) \cdot (3, 4, 6) \\
&= 6 - 4 + 12 = 14.
\end{aligned}$$

Alternatively:

$$\mathbf{Q} \cdot (\mathbf{R} \times \mathbf{P}) = \begin{vmatrix} 2 & -1 & 2 \\ 2 & -3 & 1 \\ 2 & 0 & -1 \end{vmatrix}$$

To find the determinant of a 3×3 matrix, we repeat the first two rows and cross multiply; when the cross multiplication is from right to left, the result should be negated as shown below. This technique of finding a determinant applies only to a 3×3 matrix. Hence

$$\begin{aligned}
\mathbf{Q} \cdot (\mathbf{R} \times \mathbf{P}) &= \begin{array}{c} \\ \\ - \\ - \\ - \end{array} \begin{vmatrix} 2 & -1 & 2 \\ 2 & -3 & 1 \\ 2 & 0 & -1 \\ 2 & -1 & 2 \\ 2 & -3 & 1 \end{vmatrix} \begin{array}{c} \\ \\ + \\ + \\ + \end{array} \\
&= +6 + 0 - 2 + 12 - 0 - 2 \\
&= 14
\end{aligned}$$

as obtained before.

(c) From eq. (1.28)

$$\mathbf{P} \cdot (\mathbf{Q} \times \mathbf{R}) = \mathbf{Q} \cdot (\mathbf{R} \times \mathbf{P}) = 14$$

or

$$\begin{aligned} \mathbf{P} \cdot (\mathbf{Q} \times \mathbf{R}) &= (2, 0, -1) \cdot (5, 2, -4) \\ &= 10 + 0 + 4 \\ &= 14 \end{aligned}$$

(d)

$$\begin{aligned} \sin \theta_{QR} &= \frac{|\mathbf{Q} \times \mathbf{R}|}{|\mathbf{Q}||\mathbf{R}|} = \frac{|(5, 2, -4)|}{|(2, -1, 2)||(2, -3, 1)|} \\ &= \frac{\sqrt{45}}{3\sqrt{14}} = \frac{\sqrt{5}}{\sqrt{14}} = 0.5976 \end{aligned}$$

(e)

$$\begin{aligned} \mathbf{P} \times (\mathbf{Q} \times \mathbf{R}) &= (2, 0, -1) \times (5, 2, -4) \\ &= (2, 3, 4) \end{aligned}$$

Alternatively, using the bac-cab rule,

$$\begin{aligned} \mathbf{P} \times (\mathbf{Q} \times \mathbf{R}) &= \mathbf{Q}(\mathbf{P} \cdot \mathbf{R}) - \mathbf{R}(\mathbf{P} \cdot \mathbf{Q}) \\ &= (2, -1, 2)(4 + 0 - 1) - (2, -3, 1)(4 + 0 - 2) \\ &= (2, 3, 4) \end{aligned}$$

(f) A unit vector perpendicular to both **Q** and **R** is given by

$$\begin{aligned} \mathbf{a} &= \frac{\pm \mathbf{Q} \times \mathbf{R}}{|\mathbf{Q} \times \mathbf{R}|} = \frac{\pm(5, 2, -4)}{\sqrt{45}} \\ &= \pm(0.745, 0.298, -0.596) \end{aligned}$$

Note that $|\mathbf{a}| = 1$, $\mathbf{a} \cdot \mathbf{Q} = 0 = \mathbf{a} \cdot \mathbf{R}$. Any of these can be used to check **a**.

(g) The component of **P** along **Q** is

$$\begin{aligned} \mathbf{P}_Q &= |\mathbf{P}| \cos \theta_{PQ} \mathbf{a}_Q \\ &= (\mathbf{P} \cdot \mathbf{a}_Q)\mathbf{a}_Q = \frac{(\mathbf{P} \cdot \mathbf{Q})\mathbf{Q}}{|\mathbf{Q}|^2} \\ &= \frac{(4 + 0 - 2)(2, -1, 2)}{(4 + 1 + 4)} = \frac{2}{9}(2, -1, 2) \\ &= 0.4444\mathbf{a}_x - 0.2222\mathbf{a}_y + 0.4444\mathbf{a}_z. \end{aligned}$$

PRACTICE EXERCISE 1.5

Let $\mathbf{E} = 3\mathbf{a}_y + 4\mathbf{a}_z$ and $\mathbf{F} = 4\mathbf{a}_x - 10\mathbf{a}_y + 5\mathbf{a}_z$.

(a) Find the component of **E** along **F**.

(b) Determine a unit vector perpendicular to both **E** and **F**.

Answer: (a) $(-0.2837, 0.7092, -0.3546)$, (b) $\pm(0.9398, 0.2734, -0.205)$.

EXAMPLE 1.6

Derive the cosine formula

$$a^2 = b^2 + c^2 - 2bc \cos A$$

and the sine formula

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

using dot product and cross product, respectively.

Solution:
Consider a triangle as shown in Figure 1.11. From the figure, we notice that

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$$

that is,

$$\mathbf{b} + \mathbf{c} = -\mathbf{a}$$

Hence,

$$\begin{aligned} a^2 &= \mathbf{a} \cdot \mathbf{a} = (\mathbf{b} + \mathbf{c}) \cdot (\mathbf{b} + \mathbf{c}) \\ &= \mathbf{b} \cdot \mathbf{b} + \mathbf{c} \cdot \mathbf{c} + 2\mathbf{b} \cdot \mathbf{c} \\ a^2 &= b^2 + c^2 - 2bc \cos A \end{aligned}$$

where A is the angle between $\mathbf{b}$ and $\mathbf{c}$.

The area of a triangle is half of the product of its height and base. Hence,

$$\left|\frac{1}{2}\mathbf{a} \times \mathbf{b}\right| = \left|\frac{1}{2}\mathbf{b} \times \mathbf{c}\right| = \left|\frac{1}{2}\mathbf{c} \times \mathbf{a}\right|$$

$$ab \sin C = bc \sin A = ca \sin B$$

Dividing through by abc gives

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

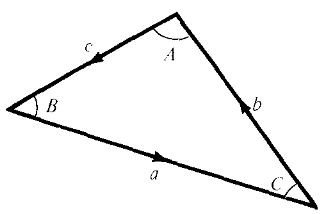

Figure 1.11 For Example 1.6.

PRACTICE EXERCISE 1.6

Show that vectors $\mathbf{a} = (4, 0, -1)$, $\mathbf{b} = (1, 3, 4)$, and $\mathbf{c} = (-5, -3, -3)$ form the sides of a triangle. Is this a right angle triangle? Calculate the area of the triangle.

Answer: Yes, 10.5.

EXAMPLE 1.7

Show that points $P_1(5, 2, -4)$, $P_2(1, 1, 2)$, and $P_3(-3, 0, 8)$ all lie on a straight line. Determine the shortest distance between the line and point $P_4(3, -1, 0)$.

Solution:

The distance vector $\mathbf{r}_{P_1P_2}$ is given by

$$\begin{aligned}\mathbf{r}_{P_1P_2} = \mathbf{r}_{P_2} - \mathbf{r}_{P_1} &= (1, 1, 2) - (5, 2, -4)\\ &= (-4, -1, 6)\end{aligned}$$

Similarly,

$$\begin{aligned}\mathbf{r}_{P_1P_3} = \mathbf{r}_{P_3} - \mathbf{r}_{P_1} &= (-3, 0, 8) - (5, 2, -4)\\ &= (-8, -2, 12)\end{aligned}$$

$$\begin{aligned}\mathbf{r}_{P_1P_4} = \mathbf{r}_{P_4} - \mathbf{r}_{P_1} &= (3, -1, 0) - (5, 2, -4)\\ &= (-2, -3, 4)\end{aligned}$$

$$\begin{aligned}\mathbf{r}_{P_1P_2} \times \mathbf{r}_{P_1P_3} &= \begin{vmatrix}\mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z\\ -4 & -1 & 6\\ -8 & -2 & 12\end{vmatrix}\\ &= (0, 0, 0)\end{aligned}$$

showing that the angle between $\mathbf{r}_{P_1P_2}$ and $\mathbf{r}_{P_1P_3}$ is zero ($\sin\theta = 0$). This implies that P_1, P_2, and P_3 lie on a straight line.

Alternatively, the vector equation of the straight line is easily determined from Figure 1.12(a). For any point P on the line joining P_1 and P_2

$$\mathbf{r}_{P_1P} = \lambda \mathbf{r}_{P_1P_2}$$

where λ is a constant. Hence the position vector $\mathbf{r}_P$ of the point P must satisfy

$$\mathbf{r}_P - \mathbf{r}_{P_1} = \lambda(\mathbf{r}_{P_2} - \mathbf{r}_{P_1})$$

that is,

$$\begin{aligned}\mathbf{r}_P &= \mathbf{r}_{P_1} + \lambda(\mathbf{r}_{P_2} - \mathbf{r}_{P_1})\\ &= (5, 2, -4) - \lambda(4, 1, -6)\end{aligned}$$

$$\mathbf{r}_P = (5 - 4\lambda, 2 - \lambda, -4 + 6\lambda)$$

This is the vector equation of the straight line joining P_1 and P_2. If P_3 is on this line, the position vector of P_3 must satisfy the equation; $\mathbf{r}_3$ does satisfy the equation when $\lambda = 2$.

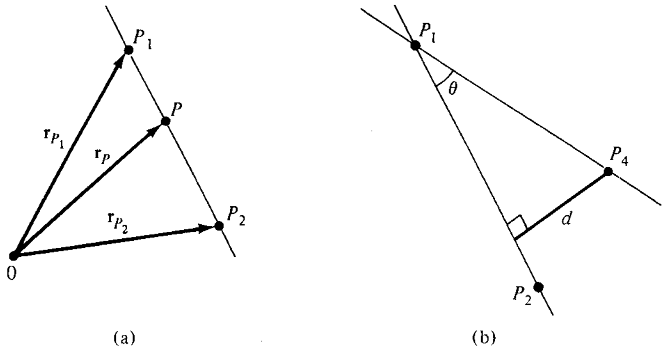

Figure 1.12 For Example 1.7.

The shortest distance between the line and point $P_4(3, -1, 0)$ is the perpendicular distance from the point to the line. From Figure 1.12(b), it is clear that

$$\begin{aligned} d &= r_{P_1P_4} \sin \theta = |\mathbf{r}_{P_1P_4} \times \mathbf{a}_{P_1P_2}| \\ &= \frac{|(-2, -3, 4) \times (-4, -1, 6)|}{|(-4, -1, 6)|} \\ &= \frac{\sqrt{312}}{\sqrt{53}} = 2.426 \end{aligned}$$

Any point on the line may be used as a reference point. Thus, instead of using P_1 as a reference point, we could use P_3 so that

$$d = |\mathbf{r}_{P_3P_4}| \sin \theta' = |\mathbf{r}_{P_3P_4} \times \mathbf{a}_{P_3P_1}|$$

PRACTICE EXERCISE 1.7

If P_1 is $(1, 2, -3)$ and P_2 is $(-4, 0, 5)$, find

(a) The distance P_1P_2

(b) The vector equation of the line P_1P_2

(c) The shortest distance between the line P_1P_2 and point $P_3(7, -1, 2)$

Answer: (a) 9.644, (b) $(1 - 5\lambda)\mathbf{a}_x + 2(1 - \lambda)\,\mathbf{a}_y + (8\lambda - 3)\,\mathbf{a}_z$, (c) 8.2.

SUMMARY

1. A field is a function that specifies a quantity in space. For example, $\mathbf{A}(x, y, z)$ is a vector field whereas $V(x, y, z)$ is a scalar field.
2. A vector $\mathbf{A}$ is uniquely specified by its magnitude and a unit vector along it, that is, $\mathbf{A} = A\mathbf{a}_A$.

3. Multiplying two vectors **A** and **B** results in either a scalar $\mathbf{A} \cdot \mathbf{B} = AB \cos \theta_{AB}$ or a vector $\mathbf{A} \times \mathbf{B} = AB \sin \theta_{AB} \mathbf{a}_n$. Multiplying three vectors **A**, **B**, and **C** yields a scalar $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ or a vector $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$.
4. The scalar projection (or component) of vector **A** onto **B** is $A_B = \mathbf{A} \cdot \mathbf{a}_B$ whereas vector projection of **A** onto **B** is $\mathbf{A}_B = A_B \mathbf{a}_B$.

REVIEW QUESTIONS

1.1 Identify which of the following quantities is not a vector: (a) force, (b) momentum, (c) acceleration, (d) work, (e) weight.

1.2 Which of the following is not a scalar field?

(a) Displacement of a mosquito in space
(b) Light intensity in a drawing room
(c) Temperature distribution in your classroom
(d) Atmospheric pressure in a given region
(e) Humidity of a city

1.3 The rectangular coordinate systems shown in Figure 1.13 are right-handed except:

1.4 Which of these is correct?

(a) $\mathbf{A} \times \mathbf{A} = |\mathbf{A}|^2$
(b) $\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{A} = 0$
(c) $\mathbf{A} \cdot \mathbf{B} \cdot \mathbf{C} = \mathbf{B} \cdot \mathbf{C} \cdot \mathbf{A}$
(d) $\mathbf{a}_x \cdot \mathbf{a}_y = \mathbf{a}_z$
(e) $\mathbf{a}_k = \mathbf{a}_x - \mathbf{a}_y$
where $\mathbf{a}_k$ is a unit vector.

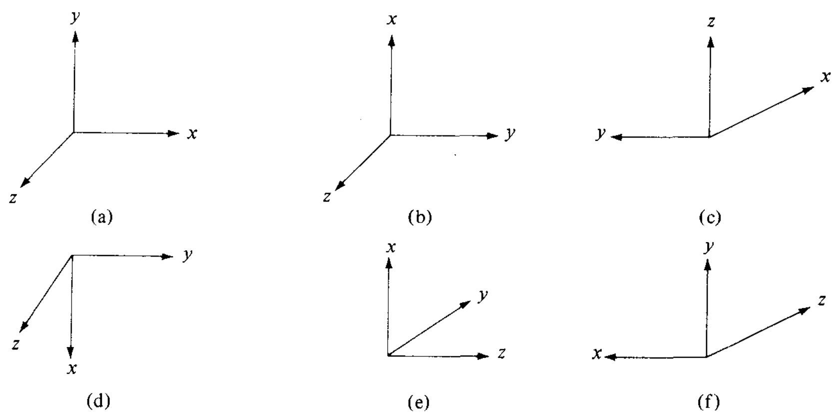

Figure 1.13 For Review Question 1.3.

1.5 Which of the following identities is not valid?

(a) $\mathbf{a}(\mathbf{b} + \mathbf{c}) = \mathbf{ab} + \mathbf{bc}$
(b) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
(c) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
(d) $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = -\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})$
(e) $\mathbf{a}_A \cdot \mathbf{a}_B = \cos \theta_{AB}$

1.6 Which of the following statements are meaningless?

(a) $\mathbf{A} \cdot \mathbf{B} + 2\mathbf{A} = 0$
(b) $\mathbf{A} \cdot \mathbf{B} + 5 = 2\mathbf{A}$
(c) $\mathbf{A}(\mathbf{A} + \mathbf{B}) + 2 = 0$
(d) $\mathbf{A} \cdot \mathbf{A} + \mathbf{B} \cdot \mathbf{B} = 0$

1.7 Let $\mathbf{F} = 2\mathbf{a}_x - 6\mathbf{a}_y + 10\mathbf{a}_z$ and $\mathbf{G} = \mathbf{a}_x + G_y\mathbf{a}_y + 5\mathbf{a}_z$. If **F** and **G** have the same unit vector, G_y is

(a) 6
(b) -3
(d) 0
(e) 6

1.8 Given that $\mathbf{A} = \mathbf{a}_x + \alpha\mathbf{a}_y + \mathbf{a}_z$ and $\mathbf{B} = \alpha\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z$, if **A** and **B** are normal to each other, α is

(a) -2
(b) $-1/2$
(c) 0
(d) 1
(e) 2

1.9 The component of $6\mathbf{a}_x + 2\mathbf{a}_y - 3\mathbf{a}_z$ along $3\mathbf{a}_x - 4\mathbf{a}_y$ is

(a) $-12\mathbf{a}_x - 9\mathbf{a}_y - 3\mathbf{a}_z$
(b) $30\mathbf{a}_x - 40\mathbf{a}_y$
(c) 10/7
(d) 2
(e) 10

1.10 Given $\mathbf{A} = -6\mathbf{a}_x + 3\mathbf{a}_y + 2\mathbf{a}_z$, the projection of **A** along $\mathbf{a}_y$ is

(a) -12
(b) -4
(c) 3
(d) 7
(e) 12

Answers: 1.1d, 1.2a, 1.3b,e, 1.4b, 1.5a, 1.6b,c, 1.7b, 1.8b, 1.9d, 1.10c.

PROBLEMS

1.1 Find the unit vector along the line joining point (2, 4, 4) to point (−3, 2, 2).

1.2 Let $\mathbf{A} = 2\mathbf{a}_x + 5\mathbf{a}_y - 3\mathbf{a}_z$, $\mathbf{B} = 3\mathbf{a}_x - 4\mathbf{a}_y$, and $\mathbf{C} = \mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z$. (a) Determine $\mathbf{A} + 2\mathbf{B}$. (b) Calculate $|\mathbf{A} - 5\mathbf{C}|$. (c) For what values of k is $|k\mathbf{B}| = 2$? (d) Find $(\mathbf{A} \times \mathbf{B})/(\mathbf{A} \cdot \mathbf{B})$.

1.3 If

$$\mathbf{A} = 2\mathbf{a}_x + \mathbf{a}_y - 3\mathbf{a}_z$$
$$\mathbf{B} = \mathbf{a}_y - \mathbf{a}_z$$
$$\mathbf{C} = 3\mathbf{a}_x + 5\mathbf{a}_y + 7\mathbf{a}_z$$

determine:

(a) $\mathbf{A} - 2\mathbf{B} + \mathbf{C}$

(b) $\mathbf{C} - 4(\mathbf{A} + \mathbf{B})$

(c) $\dfrac{2\mathbf{A} - 3\mathbf{B}}{|\mathbf{C}|}$

(d) $\mathbf{A} \cdot \mathbf{C} - |\mathbf{B}|^2$

(e) $\frac{1}{2}\mathbf{B} \times \left(\frac{1}{3}\mathbf{A} + \frac{1}{4}\mathbf{C}\right)$

1.4 If the position vectors of points T and S are $3\mathbf{a}_x - 2\mathbf{a}_y + \mathbf{a}_z$ and $4\mathbf{a}_x + 6\mathbf{a}_y + 2\mathbf{a}_x$, respectively, find: (a) the coordinates of T and S, (b) the distance vector from T to S, (c) the distance between T and S.

1.5 If

$$\mathbf{A} = 5\mathbf{a}_x + 3\mathbf{a}_y + 2\mathbf{a}_z$$
$$\mathbf{B} = -\mathbf{a}_x + 4\mathbf{a}_y + 6\mathbf{a}_z$$
$$\mathbf{C} = 8\mathbf{a}_x + 2\mathbf{a}_y$$

find the values of α and β such that $\alpha\mathbf{A} + \beta B + C$ is parallel to the y-axis.

1.6 Given vectors

$$\mathbf{A} = \alpha\mathbf{a}_x + \mathbf{a}_y + 4\mathbf{a}_z$$
$$\mathbf{B} = 3\mathbf{a}_x + \beta\mathbf{a}_y - 6\mathbf{a}_z$$
$$\mathbf{C} = 5\mathbf{a}_x - 2\mathbf{a}_y + \gamma\mathbf{a}_z$$

determine α, β, and γ such that the vectors are mutually orthogonal.

1.7 (a) Show that

$$(\mathbf{A} \cdot \mathbf{B})^2 + (\mathbf{A} \times \mathbf{B})^2 = (AB)^2$$

(b) Show that

$$\mathbf{a}_x = \frac{\mathbf{a}_y \times \mathbf{a}_z}{\mathbf{a}_x \cdot \mathbf{a}_y \times \mathbf{a}_z}, \qquad \mathbf{a}_y = \frac{\mathbf{a}_z \times \mathbf{a}_x}{\mathbf{a}_x \cdot \mathbf{a}_y \times \mathbf{a}_z}, \qquad \mathbf{a}_z = \frac{\mathbf{a}_x \times \mathbf{a}_y}{\mathbf{a}_x \cdot \mathbf{a}_y \times \mathbf{a}_z}$$

1.8 Given that

$$\mathbf{P} = 2\mathbf{a}_x - \mathbf{a}_y - 2\mathbf{a}_z$$
$$\mathbf{Q} = 4\mathbf{a}_x + 3\mathbf{a}_y + 2\mathbf{a}_z$$
$$\mathbf{C} = -\mathbf{a}_x + \mathbf{a}_y + 2\mathbf{a}_z$$

find: (a) $|\mathbf{P} + \mathbf{Q} - \mathbf{R}|$, (b) $\mathbf{P} \cdot \mathbf{Q} \times \mathbf{R}$, (c) $\mathbf{Q} \times P \cdot \mathbf{R}$, (d) $(\mathbf{P} \times \mathbf{Q}) \cdot (\mathbf{Q} \times \mathbf{R})$, (e) $(\mathbf{P} \times \mathbf{Q}) \times (\mathbf{Q} \times \mathbf{R})$, (f) $\cos \theta_{PR}$, (g) $\sin \theta_{PQ}$.

1.9 Given vectors $\mathbf{T} = 2\mathbf{a}_x - 6\mathbf{a}_y + 3\mathbf{a}_z$ and $\mathbf{S} = \mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z$, find: (a) the scalar projection of $\mathbf{T}$ on $\mathbf{S}$, (b) the vector projection of $\mathbf{S}$ on $\mathbf{T}$, (c) the smaller angle between $\mathbf{T}$ and $\mathbf{S}$.

1.10 If $\mathbf{A} = -\mathbf{a}_x + 6\mathbf{a}_y + 5\mathbf{a}_z$ and $\mathbf{B} = \mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_x$, find: (a) the scalar projections of $\mathbf{A}$ on $\mathbf{B}$, (b) the vector projection of $\mathbf{B}$ on $\mathbf{A}$, (c) the unit vector perpendicular to the plane containing $\mathbf{A}$ and $\mathbf{B}$.

1.11 Calculate the angles that vector $\mathbf{H} = 3\mathbf{a}_x + 5\mathbf{a}_y - 8\mathbf{a}_z$ makes with the x-, y-, and z-axes.

1.12 Find the triple scalar product of $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$ given that

$$\mathbf{P} = 2\mathbf{a}_x - \mathbf{a}_y + \mathbf{a}_z$$
$$\mathbf{Q} = \mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z$$

and

$$\mathbf{R} = 2\mathbf{a}_x + 3\mathbf{a}_z$$

1.13 Simplify the following expressions:

(a) $\mathbf{A} \times (\mathbf{A} \times \mathbf{B})$

(b) $\mathbf{A} \times [\mathbf{A} \times (\mathbf{A} \times \mathbf{B})]$

1.14 Show that the dot and cross in the triple scalar product may be interchanged, i.e., $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$.

1.15 Points $P_1(1, 2, 3)$, $P_2(-5, 2, 0)$, and $P_3(2, 7, -3)$ form a triangle in space. Calculate the area of the triangle.

1.16 The vertices of a triangle are located at $(4, 1, -3)$, $(-2, 5, 4)$, and $(0, 1, 6)$. Find the three angles of the triangle.

1.17 Points P, Q, and R are located at $(-1, 4, 8)$, $(2, -1, 3)$, and $(-1, 2, 3)$, respectively. Determine: (a) the distance between P and Q, (b) the distance vector from P to R, (c) the angle between QP and QR, (d) the area of triangle PQR, (e) the perimeter of triangle PQR.

***1.18** If $\mathbf{r}$ is the position vector of the point (x, y, z) and $\mathbf{A}$ is a constant vector, show that:

(a) $(\mathbf{r} - \mathbf{A}) \cdot \mathbf{A} = 0$ is the equation of a constant plane

(b) $(\mathbf{r} - \mathbf{A}) \cdot \mathbf{r} = 0$ is the equation of a sphere

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*Single asterisks indicate problems of intermediate difficulty.

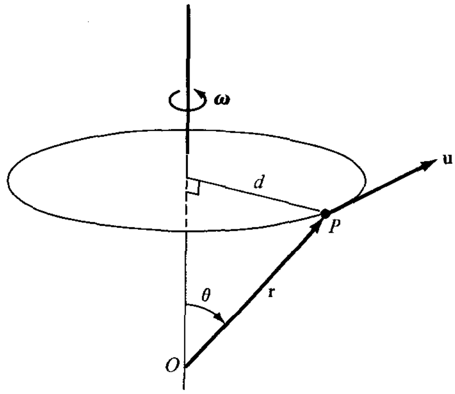

Figure 1.14 For Problem 1.20.

(c) Also show that the result of part (a) is of the form $Ax + By + Cz + D = 0$ where $D = -(A^2 + B^2 + C^2)$, and that of part (b) is of the form $x^2 + y^2 + z^2 = r^2$.

***1.19** (a) Prove that $\mathbf{P} = \cos\theta_1\mathbf{a}_x + \sin\theta_1\mathbf{a}_y$ and $\mathbf{Q} = \cos\theta_2\mathbf{a}_x + \sin\theta_2\mathbf{a}_y$ are unit vectors in the xy-plane respectively making angles θ_1 and θ_2 with the x-axis.

(b) By means of dot product, obtain the formula for $\cos(\theta_2 - \theta_1)$. By similarly formulating $\mathbf{P}$ and $\mathbf{Q}$, obtain the formula for $\cos(\theta_2 + \theta_1)$.

(c) If θ is the angle between $\mathbf{P}$ and $\mathbf{Q}$, find $\frac{1}{2}|\mathbf{P} - \mathbf{Q}|$ in terms of θ.

1.20 Consider a rigid body rotating with a constant angular velocity ω radians per second about a fixed axis through O as in Figure 1.14. Let $\mathbf{r}$ be the distance vector from O to P, the position of a particle in the body. The velocity $\mathbf{u}$ of the body at P is $|\mathbf{u}| = d\omega = |\mathbf{r}| \sin\theta\, |\boldsymbol{\omega}|$ or $\mathbf{u} = \boldsymbol{\omega} \times \mathbf{r}$. If the rigid body is rotating with 3 radians per second about an axis parallel to $\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z$ and passing through point $(2, -3, 1)$, determine the velocity of the body at $(1, 3, 4)$.

1.21 Given $\mathbf{A} = x^2y\mathbf{a}_x - yz\mathbf{a}_y + yz^2\mathbf{a}_z$, determine:

(a) The magnitude of $\mathbf{A}$ at point $T(2, -1, 3)$

(b) The distance vector from T to S if S is 5.6 units away from T and in the same direction as $\mathbf{A}$ at T

(c) The position vector of S

1.22 $\mathbf{E}$ and $\mathbf{F}$ are vector fields given by $\mathbf{E} = 2x\mathbf{a}_x + \mathbf{a}_y + yz\mathbf{a}_z$ and $\mathbf{F} = xy\mathbf{a}_x - y^2\mathbf{a}_y + xyz\mathbf{a}_z$. Determine:

(a) $|\mathbf{E}|$ at $(1, 2, 3)$

(b) The component of $\mathbf{E}$ along $\mathbf{F}$ at $(1, 2, 3)$

(c) A vector perpendicular to both $\mathbf{E}$ and $\mathbf{F}$ at $(0, 1, -3)$ whose magnitude is unity