ACM 157 PS5

(1) $H_0 : \theta = \frac{1}{2} \quad H_1 : \theta > \frac{1}{2} \Rightarrow R = \{ X : X_{(n)} > c \}$

(a) $\mathbb{P}(\text{Type I error}) = \mathbb{P}(X_{(n)} > c \mid \theta \in \Theta_0) = \mathbb{P}(X_{(n)} > c \mid \theta = \frac{1}{2})$

$= \begin{cases} 1-(2c)^n & c \le \frac{1}{2} \\ 0 & c > \frac{1}{2} \end{cases}$

$\mathbb{P}(\text{Type II error}) = 1 - \mathbb{P}(X_{(n)} > c \mid \theta \in \Theta_1)$

$\hookrightarrow = 1 - \mathbb{P}(X_{(n)} > c \mid \theta > \frac{1}{2})$

$\Rightarrow$ In general $\beta(\theta) = \mathbb{P}(X \in R \mid \theta) = \mathbb{P}(X_{(n)} > c \mid \theta)$

given $X \sim U[0, \theta] \Rightarrow \mathbb{P}(X_{(n)} > c \mid \theta) = 1 - \left(\frac{c}{\theta}\right)^n$

$$\Rightarrow \boxed{\beta(\theta) = \begin{cases} 1 - \left(\frac{c}{\theta}\right)^n & \theta > 0 \\ 1 & c < 0 \\ 0 & c > \theta \end{cases}}$$

(b) $\alpha = \sup_{\theta \in \Theta_0} \beta(\theta) = \sup_{\theta = \frac{1}{2}} \left(1 - \left(\frac{c}{\theta}\right)^n\right) = 1 - (2c)^n = \alpha$

$$\Rightarrow \boxed{c = \frac{(1-\alpha)^{1/n}}{2}}$$

(c) $n = 20 \quad X_{(n)} = .48$

$p(x) = \inf_{\alpha \in (0,1)} \{ \alpha : X \in R_\alpha \} = \inf_{\alpha \in (0,1)} \{ \alpha : X_{(n)} > c \}$

$\Rightarrow \inf_{\alpha \in (0,1)} \left\{ \alpha : X_{(n)} = .48 > \frac{(1-\alpha)^{1/n}}{2} \right\} \Rightarrow$ limit at $\alpha = 1 - (2 \cdot .48)^{20}$

$\hookrightarrow \approx .5579$

$\boxed{\text{p-value } (\alpha^*) \approx .5579}$

(2)

$X_1 \dots X_n \sim N(\mu, \sigma^2)$, $\sigma^2 = 1$, $H_0 : \mu = 0$ $H_1 : \mu = 1$

$\cancel{\beta(\mu)}$ $R = \{ x : \bar{X}_n > c \} \Rightarrow \beta(\mu) = \mathbb{P}(x \in R \mid \mu)$

$\beta(\mu) = \mathbb{P}(\bar{X}_n > c \mid \mu)$

$\Rightarrow$ given $X \sim N(\mu, \sigma^2) = N(\mu, 1) \Rightarrow \bar{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

thus $\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \sim N(0,1)$ the standard normal distribution

given $\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma}$ applies only linear transformations

$\Rightarrow \beta(\mu) = \mathbb{P}(\bar{X}_n > c \mid \mu) = \mathbb{P}\left(\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} > \frac{\sqrt{n}(c - \mu)}{\sigma} \mid \mu \right)$

$= 1 - \Phi\left(\frac{\sqrt{n}(c-\mu)}{\sigma} \right)$ where Φ is the standard normal CDF

$\alpha = \sup_{\mu \in H_0 \to \mu = 0} \beta(\mu) \Rightarrow = \beta(0) = 1 - \Phi\left(\frac{\sqrt{n}(c)}{\sigma} \right) = \alpha \Rightarrow c = \frac{\sigma \cdot \Phi^{-1}(1-\alpha)}{\sqrt{n}}$, $\sigma = 1$

$\Rightarrow$ $\boxed{\text{test of size } \alpha \text{ has } c = \frac{\Phi^{-1}(1-\alpha)}{\sqrt{n}}}$

(b) Power under $H_1 \Rightarrow \mu = 1 = \beta(1) = 1 - \Phi\left(\frac{\sqrt{n}(c-1)}{\cancel{\sigma}} \right) = 1 - \Phi\left(\sqrt{n}\left(\frac{\Phi^{-1}(1-\alpha)}{\sqrt{n}} - 1 \right) \right)$

$\Rightarrow$ Power under $H_1 = 1 - \Phi\left(\Phi^{-1}(1-\alpha) - \sqrt{n} \right)$

(c)

$\beta(\mu)$ has 2 inputs in input space $\mu = 0, 1$

$$\Rightarrow \lim_{n\to\infty} \beta(0) = 1 - \Phi\left(\sqrt{n}\left(\frac{\Phi^{-1}(1-\alpha)}{\sqrt{n}}\right)\right) = 1-(1-\alpha) = \alpha$$

$\Rightarrow \lim_{n\to\infty} \beta(1) = \lim_{n\to\infty} 1 - \Phi\left(\Phi^{-1}(1-\alpha) - \sqrt{n}\right)$, assuming $\Phi^{-1}(1-\alpha)$ is finite thus yields $\lim_{n\to\infty} \Phi(k - \sqrt{n})$, k is finite

$\hookrightarrow \alpha \neq 0$

$$\Rightarrow \lim_{n\to\infty} \Phi(-\sqrt{n}) = \Phi(-\infty) = 0 \Rightarrow$$

$$\lim_{n\to\infty} \beta(1) = \lim_{n\to\infty} 1 - \Phi\left(\Phi^{-1}(1-\alpha) - \sqrt{n}\right) = 1 - 0 = 1$$

$$\Rightarrow \boxed{\lim_{n\to\infty} \beta(0) = \alpha, \ \lim_{n\to\infty} \beta(1) = 1}$$

(3)

$$W = \left| \frac{\hat{\lambda} - \lambda_0}{\hat{se}_{\hat{\lambda}}} \right| > c \Rightarrow \alpha = \sup_{\lambda\ \theta \in \Theta_0} \beta(\theta) = \beta(\lambda_0) \quad \lambda = \lambda_0$$

$$\beta(\lambda_0) = \mathbb{P}(W > c \mid \lambda = \lambda_0) = \mathbb{P}\left(\left| \frac{\hat{\lambda} - \lambda_0}{\hat{se}} \right| > c \right)$$

$$\frac{\hat{\lambda} - \lambda_0}{\hat{se}} \sim N(0,1)$$

$$\Rightarrow \mathbb{P}\left(\left| \frac{\hat{\lambda} - \lambda_0}{\hat{se}} \right| > c \right) = 2\Phi(-c)$$

$$2\Phi(-c) = \alpha \Rightarrow c = -\Phi^{-1}\left(\frac{\alpha}{2}\right) = -z_{\frac{\alpha}{2}}$$

Wald test $\left| \frac{\hat{\lambda} - \lambda_0}{\hat{se}} \right| > -z_{\frac{\alpha}{2}} = z_{1-\frac{\alpha}{2}} \Rightarrow \left| \frac{\hat{\lambda} - \lambda_0}{\hat{se}} \right| > z_{1-\frac{\alpha}{2}}$

$$\hat{\lambda} = \operatorname*{argmax}_{\lambda} \left(L(x_1 \ldots x_n \mid \lambda) \right) = \operatorname*{argmax}_{\lambda} \left(\prod_{i=1}^{n} \frac{\lambda^{x_i}}{x_i!} e^{-\lambda} \right) = \operatorname*{argmax}_{\lambda} \ln\left(\prod_{i=1}^{n} \frac{\lambda^{x_i}}{x_i!} e^{-\lambda} \right)$$

$$\Rightarrow \hat{\lambda}_{MLE} = \operatorname*{argmax}_{\lambda} \left(\ln(L(x_1 \ldots x_n \mid \lambda)) \right) = \operatorname*{argmax}_{\lambda} \left(\sum_{i=1}^{n} \ln \frac{\lambda^{x_i}}{x_i!} e^{-\lambda} \right) =$$

$$= \operatorname*{argmax}_{\lambda} \left(\sum_{i=1}^{n} \ln\left(\frac{\lambda^{x_i}}{x_i!} e^{-\lambda} \right) \right) = \sum_{i=1}^{n} \underbrace{\ln(e^{-\lambda})}_{=-\lambda} + \ln(\lambda^{x_i}) - \ln(x_i!)$$

$$\frac{d}{d\lambda} \left(\sum_{i=1}^{n} -\lambda + \ln(\lambda^{x_i}) - \ln(x_i!) \right) = \sum_{i=1}^{n} -1 + \frac{x_i}{\lambda} = -n + \frac{1}{\lambda} \sum x_i = 0$$

second deriv. test satisfied

$$\Rightarrow \frac{d}{d\lambda}\left(\ln(L(x \mid \lambda)) \right) = 0 = -n + \frac{1}{\lambda} \sum x_i \Rightarrow \boxed{\lambda = \frac{\sum x_i}{n}} = \operatorname*{argmax}_{\lambda}, \quad \frac{d^2}{d\lambda^2} = -\frac{1}{\lambda^2} \sum x_i < 0$$

$$\hat{se} = \sqrt{\mathbb{V}(\hat{\lambda})} = \sqrt{\mathbb{V}\left(\frac{\sum x_i}{n} \right)} = \sqrt{\frac{1}{n^2} \mathbb{V}(\sum x_i)} = \sqrt{\frac{n}{n^2} \mathbb{V}(x_i)} = \sqrt{\frac{\hat{\lambda}}{n}} \approx \sqrt{\frac{\hat{\lambda}}{n}}$$

$$\hat{se} = \sqrt{\frac{\hat{\lambda}}{n}} \Rightarrow \boxed{W = \left| \frac{\frac{1}{n} \sum_{i=1}^{n} x_i - \lambda_0}{\sqrt{\frac{1}{n^2} \sum_{i=1}^{n} x_i}} \right| > z_{1-\frac{\alpha}{2}}}$$

```
clc; clear; close all;
```

Question 3

Part B

The results of the average wald_test below show a type 1 error rate of ~.054.

```
type1_error = 0;

lambda = 1;
n = 20;

alpha = .05;

for i = 1:10000
    sample = poissrnd(lambda, [n, 1]);
    wald_score = abs(((sum(sample)/ n) - 1) / sqrt(sum(sample) / (n^2)));
    if wald_score > norminv(1 - alpha / 2)
        type1_error = type1_error + 1;
    end
end

disp(type1_error / 10000)
```

```
0.0549
```

Question 4

Using permutation test since the number of samples are small and the normality assumption for the estimated parameter does not hold. The result of the permutation test yield a p-value on average of ~.03 < .05 which concludes that there is statistically significant evidence that the two distributions are different and the pH levels of the soil differ between the two locations.

```
error = 0;

sample1 = [7.58, 8.52, 8.01, 7.99, 7.93, 7.89, 7.85, 7.82, 7.80];
sample2 = [7.85, 7.73, 8.53, 7.40, 7.35, 7.30, 7.27, 7.27, 7.23];

initial_diff = abs(mean(sample1) - mean(sample2));

n = 10000;

Z = cat(2, sample1, sample2);

for i = 1:n
    Z_pi = Z(randperm(length(Z)));
    sample1_perm = Z_pi(1:length(sample1));
    sample2_perm = Z_pi(length(sample1) + 1:end);
    perm_diff = abs(mean(sample1_perm) - mean(sample2_perm));
```

```
    if perm_diff >  initial_diff
        error = error + 1;
    end
end

disp('p-value from permutation test:');
```

```
p-value from permutation test:
```

```
disp(error / n);
```

```
    0.0349
```

Question 5

Wald T-test

We notice that the p-value for the wald test is signficantly lower than the threshold for signficance of .05, which concludes that there is a statistically signficant difference between the distribution of Twain and Snodgrass, suggesting that Twain did not write the Snodgrass essays

```
sample_t = [0.225, 0.262, 0.217, 0.240, 0.230, 0.229, 0.235, 0.217];
sample_s = [0.209, 0.205, 0.196, 0.210, 0.202, 0.207, 0.224, 0.223, 0.220, 0.201];

standard_err = sqrt(var(sample_t) / length(sample_t) + var(sample_s) /
length(sample_s));

wald_score = abs((mean(sample_t) - mean(sample_s)) / standard_err);

p_value = 2 * normcdf(-1 * wald_score);

alpha = .05;

if wald_score > norminv(1 - alpha / 2)
    disp('H_0 rejected with p-value:');
    disp(p_value);
else
    disp('H_0 accepted with p-value:');
    disp(p_value);
end
```

```
H_0 rejected with p-value:
   2.1260e-04
```

Permutation test

We notice that the p-value for the wald test is signficantly lower than the threshold for signficance of .05, which concludes that there is a statistically signficant difference between the distribution of Twain and Snodgrass, suggesting that Twain did not write the Snodgrass essays. Additionally, we also notice that the permutation test yields a p-value different from the Wald T-test which also makes sense given that the normality assumption might not hold for the Wald T-test and thus the test produces results different from the permutation test, likely a result of the small quantity of data and the difference in method - both methods still conclude the same result though.

```
error = 0;

initial_diff = abs(mean(sample_t) - mean(sample_s));

n = 10000;

Z = cat(2, sample_t, sample_s);

for i = 1:n
    Z_pi = Z(randperm(length(Z)));
    sample1_perm = Z_pi(1:length(sample_t));
    sample2_perm = Z_pi(length(sample_t) + 1:end);
    perm_diff = abs(mean(sample1_perm) - mean(sample2_perm));
    if perm_diff >  initial_diff
        error = error + 1;
    end
end

disp('p-value from permutation test:');
```

```
p-value from permutation test:
```

```
disp(error / n);
```

```
    7.0000e-04
```