(1)

$H_0 : \mu \leq 0 \qquad H_1 : \mu > 0$

$$\lambda(x) = \frac{\sup_{\mu \in H_0 \cup H_1} L(\mu / x)}{\sup_{\mu \in H_0} L(\mu / x)}$$

$$= \frac{L(\hat{\mu}_{MLE} / x)}{L(\hat{\mu}_{0\text{-}MLE} / x)} \quad \text{notice } \hat{\mu}_{0}\text{-}MLE = \begin{cases} \bar{X}_n & \text{if } \bar{X}_n < 0 \\ 0 & \text{if } \bar{X}_n \geq 0 \end{cases}$$

$$L(\mu / x) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(x_i - \mu)^2}{2\sigma^2}\right) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\left(-\frac{\sum (x_i - \mu)^2}{2\sigma^2}\right)$$

$$\frac{L(\mu_1 / x)}{L(\mu_0 / x)} = \exp\left(\frac{\sum (x_i - \mu_0)^2 - \sum (x_i - \mu_1)^2}{2\sigma^2}\right)$$

$$\Rightarrow \frac{L(\hat{\mu}_{MLE} / x)}{L(\hat{\mu}_0\text{-}MLE / x)} \Rightarrow \begin{array}{l} \text{for } \hat{\mu}_0\text{-}MLE = \bar{x}_n \Rightarrow \exp\left(\frac{0}{2\sigma^2}\right) = 1 \\ \text{for } \hat{\mu}_0\text{-}MLE = 0 \Rightarrow \exp\left(\frac{\sum (x_i)^2 - \sum (x_i - \bar{x}_n)^2}{2\sigma^2}\right) \end{array}$$

$$= \exp\left(\frac{\sum x_i^2 - x_i^2 + 2x_i\bar{x}_n - \bar{x}_n^2}{2\sigma^2}\right) = \exp\left(\frac{n(\bar{x}_n)^2}{2\sigma^2}\right)$$

$\Rightarrow$ rejection region defined by: $\frac{L(\hat{\mu}_{MLE} / x)}{L(\hat{\mu}_{0\text{-}MLE} / x)} > c \Rightarrow \exp\left(\frac{n(\bar{X}_n)^2}{2\sigma^2}\right) > c$

$\Rightarrow \left|\frac{\sqrt{n}(\bar{X}_n)}{\sigma}\right| > \underbrace{\sqrt{2\ln(c)}}_{\text{denote } c'}$, where $\frac{\sqrt{n}(\bar{X}_n)}{\sigma} \sim N(0,1)$

size α test $\Rightarrow \mathbb{P}\left(\left|\frac{\sqrt{n}(\bar{X}_n)}{\sigma}\right| > c'\right) = \alpha \Rightarrow 1 - \mathbb{P}\left(\left|\frac{\sqrt{n}(\bar{X}_n)}{\sigma}\right| \leq c'\right) = \alpha = 1 - \Phi(c')$

this $\Rightarrow c' = \Phi^{-1}(1-\alpha) \Rightarrow \left|\frac{\sqrt{n}(\bar{X}_n)}{\sigma}\right| > c' = \Phi^{-1}(1-\alpha) \Rightarrow \bar{X}_n > \frac{\sigma(\Phi^{-1}(1-\alpha))}{\sqrt{n}}$

$\boxed{\text{the test is given by } \bar{X}_n > \frac{\sigma(\Phi^{-1}(1-\alpha))}{\sqrt{n}} \text{ which is the same as the ad hoc test of size } \alpha}$

(2)

$H_0: \mu = \mu_0 \quad H_1: \mu \neq \mu_0$

$$\lambda(x) = \frac{\sup_{\mu \in H_0 \cup H_1} L(\mu | x)}{\sup_{\mu \in H_0} L(\mu | x)} = \frac{L(\hat{\mu}_{MLE} | x)}{L(\mu_0 | x)}$$

$$L(\mu | x) = \left(\frac{1}{\sqrt{2\pi\sigma}}\right)^n \exp\left(\frac{-\sum (x_i - \mu)^2}{2\sigma^2}\right)$$

notice we must plug in $\hat{\sigma}$ for σ; ($\hat{\sigma}^2$ for σ^2):

$$\Rightarrow \hat{\sigma}_{MLE}^2 = \frac{\sum (x_i - \mu)^2 \to \hat{\mu}_{MLE}}{n}, \quad \sigma_0^2 = \frac{\sum (x_i - \mu_0)^2}{n}$$

thus plugging in for $\lambda(x)$ we get $\Rightarrow L(\hat{\mu}_{MLE} | x)$

$$\lambda(x) = \frac{\left(\frac{1}{\sqrt{2\pi \hat{\sigma}_{MLE}}}\right)^n \exp\left(\frac{-\sum (x_i - \hat{\mu}_{MLE})^2}{2 \frac{\sum (x_i - \hat{\mu}_{MLE})}{n}}\right)}{L(\mu_0 | x) \Rightarrow \left(\frac{1}{\sqrt{2\pi \hat{\sigma}_0}}\right)^n \exp\left(\frac{-\sum (x_i - \hat{\mu}_0)^2}{2 \frac{\sum (x_i - \hat{\mu}_0)^2}{n}}\right)} = \left(\sqrt{\frac{\hat{\sigma}_0}{\hat{\sigma}_{MLE}}}\right)^n > c$$

$$\Rightarrow \left[\frac{\sum_{i=1}^{n} (x_i - \mu_0)^2}{\sum_{i=1}^{n} (x_i - \underset{\bar{x}_n}{\cancel{\mu}})^2}\right]^{\frac{n}{2}} > c \Rightarrow \frac{\sum_{i=1}^{n} (x_i - \mu_0)^2}{\sum_{i=1}^{n} (x_i - \underset{\bar{x}_n}{\cancel{\mu}})^2} > c^{\frac{2}{n}}$$

(2)

$$\Rightarrow \frac{\sum_{i=1}^{n}(x_i - \mu_0)^2}{\sum_{i=1}^{n}(x_i - \bar{x}_n)^2} > c^{\frac{2}{n}}$$

$$\sum_{i=1}^{n}(x_i - \mu_0)^2 = \sum_{i=1}^{n} x_i^2 - 2x_i\mu_0 + \mu_0^2$$

$$= \sum_{i=1}^{n} x_i^2 - 2\bar{x}_n\mu_0 + \mu_0^2$$

$$= \sum_{i=1}^{n} x_i^2 - \bar{x}_n^2 + \bar{x}_n^2 - 2\bar{x}_n\mu_0 + \mu_0^2$$

$$= \sum_{i=1}^{n} x_i^2 - \bar{x}_n^2 + (\bar{x}_n - \mu_0)^2$$

$$= \sum_{i=1}^{n} x_i^2 - 2x_i\bar{x}_n + \bar{x}_n^2 + (\bar{x}_n - \mu_0)^2$$

$$= \sum_{i=1}^{n}(x_i - \bar{x}_n)^2 + (\bar{x}_n - \mu_0)^2$$

$$= \sum_{i=1}^{n}(x_i - \bar{x}_n)^2 + n(\bar{x}_n - \mu_0)^2$$

$$= 1 + \frac{n(\bar{x}_n - \mu_0)^2}{\sum(x_i - \bar{x}_n)^2} > c^{\frac{2}{n}}$$

Very close to sample variance, might be possible to set up as a t-test

$$\Rightarrow \frac{n(\bar{x}_n - \mu_0)^2}{\sum(x_i - \bar{x}_n)^2} > c^{\frac{2}{n}} - 1 \Rightarrow \frac{n(\bar{x}_n - \mu_0)^2}{(n-1)S_n^2} > c^{\frac{2}{n}} - 1$$

, S_n is sample variance

$\Rightarrow$ simplifies to: $\left| \frac{\bar{x}_n - \mu_0}{S_n/\sqrt{n}} \right| > \sqrt{(c^{\frac{2}{n}} - 1)(n-1)}$; this is the same as a t-test

So we can just make an α-size test as:

$$\left| \frac{\bar{x}_n - \mu_0}{S_n/\sqrt{n}} \right| > t_{n-1, 1-\frac{\alpha}{2}}$$

(3) To find the least squares estimate of β

we want to minimize $\sum e_i^2 = \sum (y_i - \beta x_i)^2$

$\Rightarrow \min_{\beta} \left(\sum (y_i - \beta x_i)^2 \right)$, we can find this by

setting the first derivative wrt β of $\sum (y_i - \beta x_i)^2$ to 0

Consider $f = \sum (y_i - \beta x_i)^2 \Rightarrow \frac{df}{d\beta} = \sum -2x_i(y_i - \beta x_i) = 0$

$\Rightarrow \sum x_i (y_i - \beta x_i) = 0 \Rightarrow \sum x_i y_i - \sum \beta x_i^2 = 0$

$\sum x_i y_i = \sum \beta x_i^2 = \beta \sum x_i^2 \Rightarrow \beta = \frac{\sum x_i y_i}{\sum x_i^2}$

thus we solve for β as being:

$$\boxed{\beta = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}}$$

this value can be verified as a minimum of $f = \sum_{i=1}^{n} (y_i - \beta x_i)^2$ via

the second derivative test: $\frac{df'}{d\beta} = \frac{d}{d\beta}\left(\sum -2x_i(y_i - \beta x_i) \right)$

$= \sum_{i=1}^{n} 2x_i^2 > 0$ for all values of x_i as long as $\forall x_i = 0$,

thus by second derivative test we have indeed

found the β least-squares estimate.

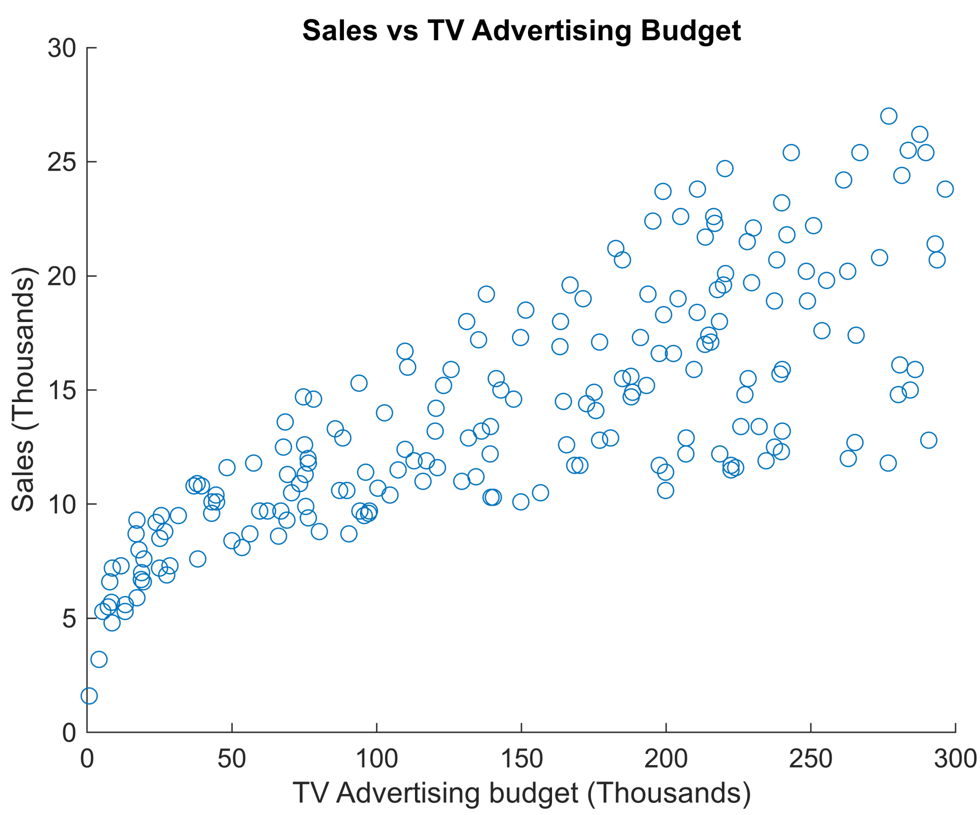
```
clc; clear; close all;
```

```
file = matfile('advertising.mat');

sales = file.sales;

tv = file.tv;
```

Problem 4

4a

```
figure;
scatter(tv, sales);
xlabel('TV Advertising budget (Thousands)')
ylabel('Sales (Thousands)')
title('Sales vs TV Advertising Budget')
```



4b

```
coefficient = dot((tv - mean(tv)), (sales - mean(sales))) / dot((tv - mean(tv)),
(tv - mean(tv)));
const = mean(sales) - coefficient * mean(tv);

disp('OLS coefficients (B1 then B0)')
```

```
OLS coefficients (B1 then B0)
```

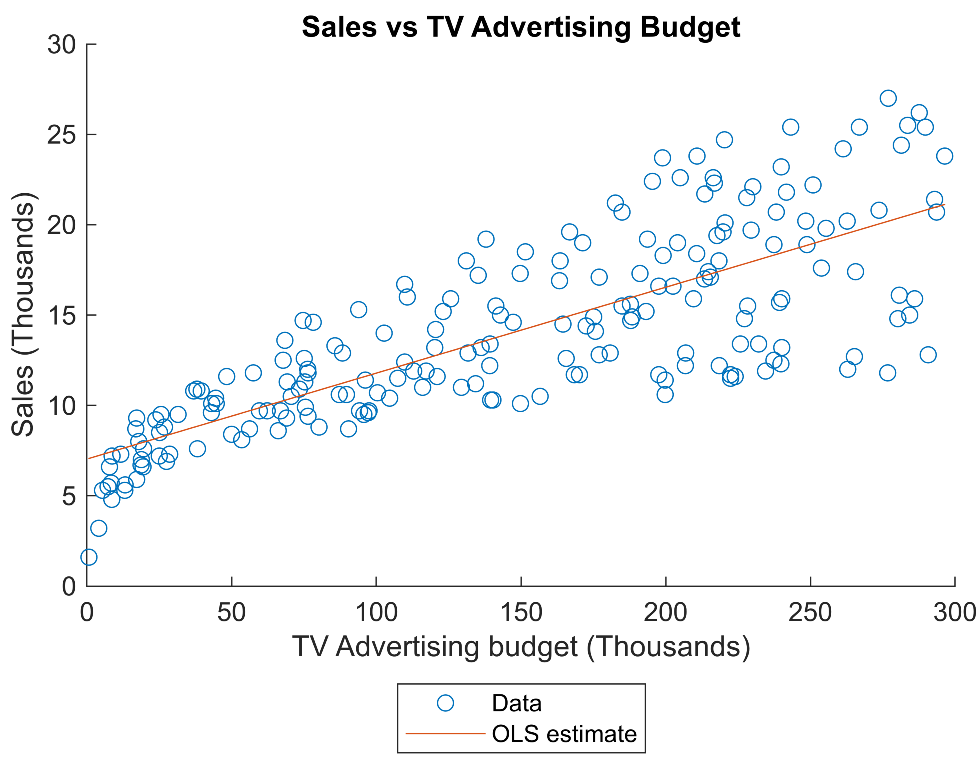
```
disp(coefficient);
```

```
    0.0475
```

```
disp(const);
```

```
    7.0326
```

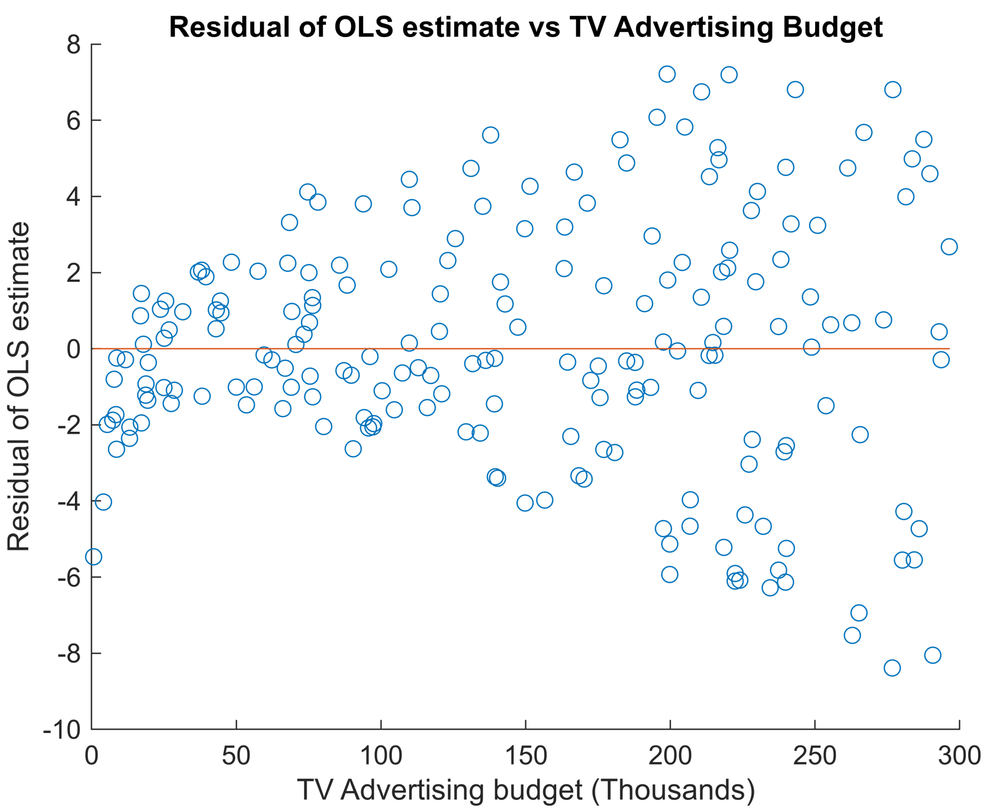
```
figure;
hold on;
scatter(tv, sales, 'DisplayName', 'Data')
plot(tv, coefficient * tv + const, 'DisplayName', 'OLS estimate')
xlabel('TV Advertising budget (Thousands)')
ylabel('Sales (Thousands)')
title('Sales vs TV Advertising Budget')
legend('Location', 'southoutside');
hold off;
```



4c

We see from the plot below that the homoscedacity does not hold for this dataset as the variance in residuals of the OLS estimates clearly become larger over time, displaying heteroscedacity

```
res = zeros(length(tv), 1);

for i = 1: length(tv)
    res(i) = sales(i) - (coefficient * tv(i) + const);
end

figure;
hold on;
scatter(tv, res)
plot(tv, zeros(length(tv), 1))
xlabel('TV Advertising budget (Thousands)')
ylabel('Residual of OLS estimate')
title('Residual of OLS estimate vs TV Advertising Budget')
hold off;
```

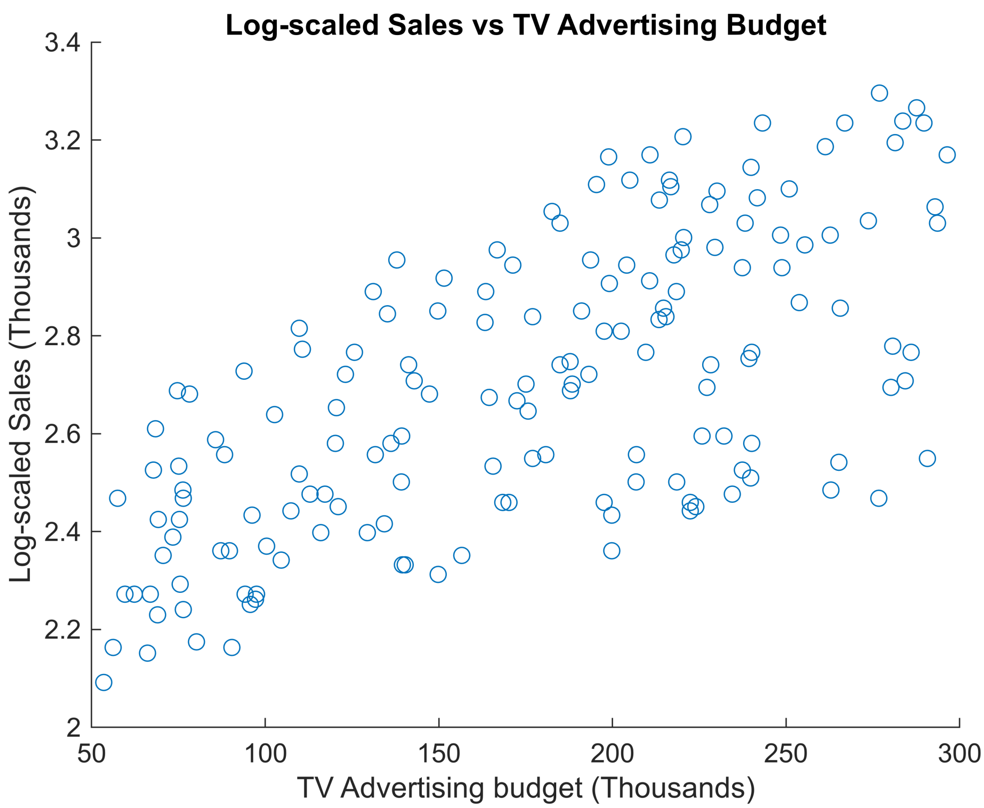


Problem 5

We see that the log-scaling of the sales data normalizes the variance of the data throughout the input space of TV advertisements spending. As a result, we can see the residual plot to display homoscedacity as the variance of estimates appears to be fixed over the input space and thus the homoscedacity assumption does hold for the log-scaled sales OLS estimate.

```
log_sales = sales;
tv_2 = tv;
```

```
for i = length(tv_2):-1:1
    if tv(i) <= 50
        log_sales(i) = [];
        tv_2(i) = [];
    else
        log_sales(i) = log(log_sales(i));
    end
end

figure;
scatter(tv_2, log_sales);
xlabel('TV Advertising budget (Thousands)')
ylabel('Log-scaled Sales (Thousands)')
title('Log-scaled Sales vs TV Advertising Budget')
```




```
coef2 = dot((tv_2 - mean(tv_2)), (log_sales - mean(log_sales))) / dot((tv_2 -
mean(tv_2)), (tv_2 - mean(tv_2)));
const2 = mean(log_sales) - coef2 * mean(tv_2);

disp('Log-scaled dataset OLS coefficients (B1 then B0)')
```

```
Log-scaled dataset OLS coefficients (B1 then B0)
```

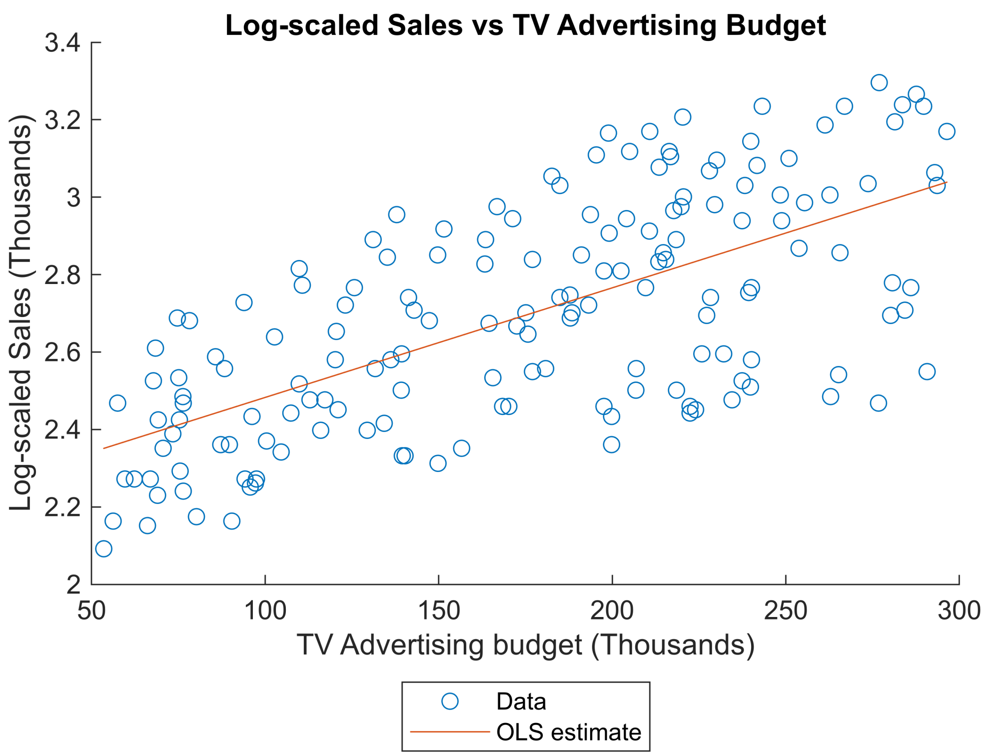
```
disp(coef2);
```

```
    0.0028
```

```
disp(const2);
```

```
    2.1997
```

```

figure;
hold on;
scatter(tv_2, log_sales, 'DisplayName', 'Data')
plot(tv_2, coef2 * tv_2 + const2, 'DisplayName', 'OLS estimate')
xlabel('TV Advertising budget (Thousands)')
ylabel('Log-scaled Sales (Thousands)')
title('Log-scaled Sales vs TV Advertising Budget')
legend('Location', 'southoutside');
hold off;
```




```
res = zeros(length(tv_2), 1);

for i = 1: length(tv_2)
    res(i) = log_sales(i) - (coef2 * tv_2(i) + const2);
end

figure;
hold on;
scatter(tv_2, res)
```

```
plot(tv_2, zeros(length(tv_2), 1))
xlabel('TV Advertising budget (Thousands)')
ylabel('Residual of OLS estimate')
title('Residual of OLS estimate vs TV Advertising Budget')
hold off;
```