Lesson 1

Part I: Systems of Linear Equations

Definition: A **linear equation** is in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $x_1, x_2, \ldots, x_n$ are variables, $a_1, a_2, \ldots, a_n$ and b are scalars, and n is a positive integer.

For example, (1) $3x_1 + 4x_2 = -2x_1$

(2) $(2 + x_1)x_2 = 6$

(3) $\sqrt{5}\, x_1 + 4x_2 = 0$

(4) $5\sqrt{x_1} + 4x_2 = 0$

Definition: A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations.

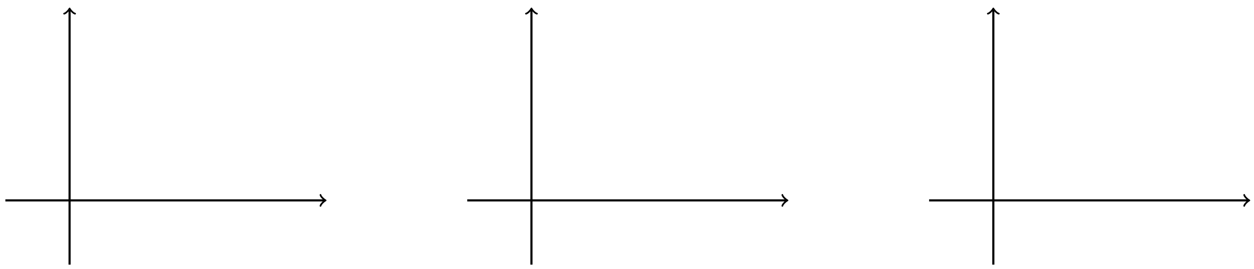
For example, $\begin{array}{rcl} 2x_1 + x_2 + x_3 & = & 1 \\ x_1 - 2x_2 - x_3 & = & 2 \end{array}$ is a linear system.

Definition: A **solution** of a system is a list of numbers $(s_1, s_2, \ldots, s_n)$ written as a column vector $\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$ in $\mathbb{R}^n$ such that every equation in the system is satisfied when each x_i is replaced by s_i.

For example, $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is a solution of the system $\begin{array}{rcl} 2x_1 + x_2 + x_3 & = & 1 \\ x_1 - 2x_2 - x_3 & = & 2 \end{array}$.

Geometrical interpretation of the solutions of a system of two linear equations in two variables:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$



Definition: A linear system is

- **consistent** if it has either one solution or infinitely many solutions.
- **inconsistent** if it has no solution.

Matrix Notation – In linear algebra, we often use matrix notation to represent a linear system. For example, consider a linear system:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

- $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$: **coefficient matrix** of the system
- $[\ A \mid \mathbf{b}\ ] = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$: **augmented matrix** of the system

The **size** of a matrix is the number of rows and number of columns. An $m \times n$ matrix is a rectangular array of numbers with m rows and n columns.

Example 1: Solve the system.

$$\begin{aligned} x_1 - x_2 &= -4 \\ x_1 + 2x_2 &= 2 \end{aligned}$$

Note: 1. Two linear systems are **equivalent** if they have the same solution set.

2. Solving a linear system is a step-by-step process (row operations) of replacing it with an equivalent system until it can be easily solved.

Example 2: Solve the system.

$$\begin{aligned} x_2 + 5x_3 &= -4 \\ x_1 + 4x_2 + 3x_3 &= -2 \\ 2x_1 + 7x_2 + 2x_3 &= -2 \end{aligned}$$

Part II: Row Operations & Echelon Forms

Elementary Row Operations

1. **Interchange** – Interchange two rows.
2. **Scaling** – Multiply every entry in a row by a nonzero constant.
3. **Replacement/Row addition** – Add a multiple of one row to another row.

Definition: Two matrices are called **row equivalent** if there is a sequence of elementary row operations that transforms one matrix into the other.

Statement: If the augmented matrices of two linear systems are row equivalent, then the systems have the same solution set.

Echelon Forms

Note: We call a row of a matrix a **zero row** if all its entries are 0 and a **nonzero row** otherwise. The **leading entry** of a nonzero row is the leftmost nonzero entry.

Definition:

• A matrix is in **row echelon form** (REF) if it satisfies the following three conditions:

1. Each nonzero row lies above every zero row.
2. The leading entry of a nonzero row lies in a column to the right of the column containing the leading entry of any preceding row.
3. All entries in a column below a leading entry are zero.

• A matrix in **reduced row echelon form** (RREF) satisfies two additional conditions:

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

Example 3: Determine which of the matrices is in reduced echelon form, which is only in echelon form, and which is not in echelon form.

REF satisfies 1--3 and RREF satisfies 1--5

1. row of zeros (if any) at the bottom
2. pivots go down and to the right
3. zeros below pivots
4. pivots = 1
5. zeros above and below pivots

(1) $\begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

(2) $\begin{bmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 2 \end{bmatrix}$

(3) $\begin{bmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 1 & 3 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$

(4) $\begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Theorem: (Uniqueness of the Reduced Echelon Form)
Each matrix is row equivalent to one and only one reduced echelon matrix.

Questions about Existence and Uniqueness of Solution

1. Is the system consistent; does at least one solution *exist*?
2. If a solution exists, is the solution *unique*?

Example 4: The augmented matrices of linear systems were reduced by row operations to the forms shown. Describe the solution sets of the original systems.

(1) $\begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

(2) $\begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

(3) $\begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Lesson 2

Part I: Row Reduction

Definition: Let A be a matrix.
- If A is in REF (or RREF), the first nonzero entry in each row is a **pivot**.
- A **pivot position** in A is a location that corresponds to a pivot.
- A **pivot column** is a column of A that contains a pivot position.
- The number of pivot positions in a matrix A is called the **rank** of A.

Note: $\text{rank}(A) = \text{number of pivots in } A_{\text{ref}}$
$= \text{number of pivots in } A_{\text{rref}}$

For example, if $A = \begin{bmatrix} 0 & 0 & 2 & -4 & -5 \\ 0 & 1 & -1 & 1 & 3 \\ 0 & 6 & 0 & -6 & 5 \end{bmatrix}$, then

$$A_{\text{ref}} = \begin{bmatrix} 0 & 1 & -1 & 1 & 3 \\ 0 & 0 & 2 & -4 & -5 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad A_{\text{rref}} = \begin{bmatrix} 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Circle the pivot positions.

pivot columns:

$\text{rank}(A) =$

Question: How do we reduce a matrix A in REF or RREF?

The Row Reduction Algorithm

1. Working from the left, find the first column (pivot column) of the matrix that contains an nonzero entry.
2. Interchange rows (if necessary) so that the top entry (pivot) in the column is nonzero.

   Note: If 1 or -1 is in the column, choose either as a pivot.
3. Use row replacement operations to make all entries beneath the pivot into zeros.
4. Repeat steps 1 – 3 on the submatrix consisting of all rows below the most recently obtained pivot until we reach the bottom row or there are no remaining nonzero rows.
5. Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

Note: The combination of steps 1 – 4 is called the forward phase of the row reduction algorithm; step 5, which produces the unique reduced echelon form, is called the backward phase.

Example 1: Row reduce the matrix to RREF.

$$A = \begin{bmatrix} 0 & 0 & 2 & -4 & -5 \\ 0 & 1 & -1 & 1 & 3 \\ 0 & 6 & 0 & -6 & 5 \end{bmatrix}$$

Example 2: Solve the linear system whose augmented matrix has been reduced to the RREF.

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note: 1. The variables that correspond to pivot columns in the matrix are called **pivot variables** (or **basic variables**); the other variables are called **free variables**.

2. We can use free variables as parameters to describe the solution set.

3. If the system is inconsistent, the solution set is empty, even when the system has free variables.

Part II: Existence and Uniqueness of the Solutions

Theorem: (Existence and Uniqueness Theorem)
A linear system is **consistent** if and only if the rightmost column of the augmented matrix is not a pivot column – that is, if and only if an echelon form of the augmented matrix has no row of the form

$$[\,0 \;\; \cdots \;\; 0 \;\; b\,],\; b \neq 0$$

If a linear system is consistent, then
(i) it has a unique solution when there is no free variable.
(ii) it has infinitely many solutions when there is at least one free variable.

For example, if the augmented matrix of the system $A\mathbf{x} = \mathbf{b}$ has been reduced to a REF:

(1) $\begin{bmatrix} 3 & 2 & -2 & 2 & 0 \\ 0 & 2 & 4 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(2) $\begin{bmatrix} 3 & 2 & -2 & 2 & 0 \\ 0 & 2 & 4 & 1 & 3 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

An equivalent form of the Existence and Uniqueness Theorem:

Theorem: (Rouché–Capelli)
A linear system $A\mathbf{x} = \mathbf{b}$ is **consistent** if and only if

$$\operatorname{rank}(A) = \operatorname{rank}([\, A \;\; \mathbf{b}\,]),$$

where A is $m \times n$.

If a linear system is consistent, then
(i) it has a unique solution when $\operatorname{rank}(A) = n$.
(ii) it has infinitely many solutions when $\operatorname{rank}(A) < n$.

Recheck the previous examples using Rouché–Capelli Theorem:

(1) $\begin{bmatrix} 3 & 2 & -2 & 2 & 0 \\ 0 & 2 & 4 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(2) $\begin{bmatrix} 3 & 2 & -2 & 2 & 0 \\ 0 & 2 & 4 & 1 & 3 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

Use Row Reduction to Solve a Linear System

Example 3: Solve:

$$\begin{aligned} 2x_1 - x_2 + 3x_3 - 4x_4 &= 1 \\ x_1 + 2x_2 - x_3 - 2x_4 &= -2 \\ 2x_1 + 3x_2 + 2x_3 - 4x_4 &= 3 \end{aligned}$$

Lesson 3

Part I: Vectors

<u>Definition</u>: An ordered list of numbers is called a **vector**.

In linear algebra, if a matrix has only one row or only one column it is called a vector. For example, let u_1 and u_2 be real numbers.

- $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$: a column vector in $\mathbb{R}^2$
- $\mathbf{u} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$: a row vector

Operations on Vectors

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be two vectors in $\mathbb{R}^2$.

1. $\mathbf{u} = \mathbf{v} \iff$
2. $\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} =$
3. $c\mathbf{u} = c \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} =$

For example, if $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, then

$$3\mathbf{u} - \mathbf{v} =$$

Vectors in $\mathbb{R}^n$ can be written as $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$.

The **zero vector** (denoted by $\mathbf{0}$) in $\mathbb{R}^n$ is the vector whose n entries are all zero.

Properties of Vector Operations

For vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$ and for scalars c and d:

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$, where $-\mathbf{u} = (-1)\mathbf{u}$
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
7. $c(d\mathbf{u}) = (cd)\mathbf{u}$
8. $1\mathbf{u} = \mathbf{u}$

Geometric Description of vectors in $\mathbb{R}^2$

1. The column vector $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ can be graphed as a directed line segment from the origin $(0, 0)$ to the point (u_1, u_2). It is called the posotion vector.

2. The sum of two vectors $\mathbf{u} + \mathbf{v}$: use the Parallelogram Rule

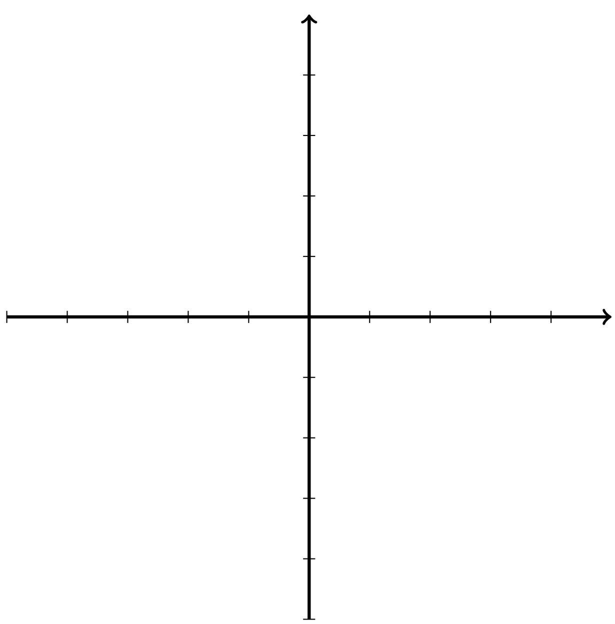
3. Scalar multiplication of a vector $c\mathbf{u}$:

(i) $c > 0$:

(ii) $c < 0$:

Note: The set of all scalar multiples of a nonzero vector $\mathbf{u}$ is a line through the origin and $\mathbf{u}$.

Example 1: Given two vectors $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, find the vectors $\mathbf{u} + \mathbf{v}, 2\mathbf{v}, -\mathbf{v}$ and display them on a graph.



Part II: Vector Equations

Linear Combinations

Definition: Given vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p \in \mathbb{R}^n$ and scalars $c_1, c_2, \cdots, c_p$, the vector $\mathbf{b}$ defined by

$$\mathbf{b} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

is called a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$ with weights $c_1, c_2, \cdots, c_p$.

For Example, let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p \in \mathbb{R}^n$.
(1) A linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$ with weights $c_1 = c_2 = \cdots = c_p = 0$ is

(2) A linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$ with weights $c_1 = 1, c_2 = \cdots = c_p = 0$ is

(3) A linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$ with $c_i = 1$ and other weights $c_j = 0$ (if $j \neq i$) is

Statement: Let $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$, and $\mathbf{b}$ be vectors in $\mathbb{R}^m$. $\mathbf{b}$ can be written as a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ if and only if there exist weights $x_1, x_2, \cdots, x_n$ such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \tag{1}$$

that is, the **vector equation** (1) has a solution.

Statement: A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system $A\mathbf{x} = \mathbf{b}$.

Moreover, $\mathbf{b}$ can be generated by a linear combination of columns of A if and only if there exists a solution to the linear system $A\mathbf{x} = \mathbf{b}$.

Definition: If $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p \in \mathbb{R}^n$, then

$$\text{Span}\,\{\mathbf{v}_1, \cdots, \mathbf{v}_p\} = \{c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p \mid c_1, \cdots, c_p \text{ are scalars.}\}$$

is the subset of $\mathbb{R}^n$ **spanned** by $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$.

For example, let $S = \text{Span}\,\{\mathbf{v}_1, \cdots, \mathbf{v}_p\} = \{c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p \,|\, c_i \in \mathbb{R}\}$.

Then (1) $\mathbf{0} \in S$

(2) $\mathbf{v}_1 \in S$

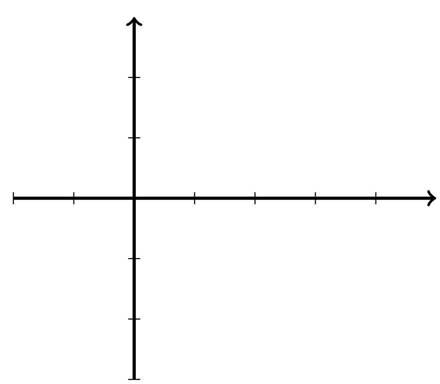
(3) $c\mathbf{v}_1 \in S$

Conclusion: S contains every scalar multiple of a vector $\mathbf{v}_i$ $(i = 1 : p)$.

Statement: The following statements are logically equivalent:

1. A vector $\mathbf{b}$ is in Span $\{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$.
2. A vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$ has a solution.
3. A linear system $A\mathbf{x} = \mathbf{b}$ with augmented matrix $[\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n\ \mathbf{b}]$ is consistent.

Example 2: Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$. Determine whether $\mathbf{b}$ can be written as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$. If yes, find weights and display the vectors on a graph.



Geometric Description of a Spanned Set in $\mathbb{R}^3$

1. Span $\{\mathbf{u}\}$

   (i) If $\mathbf{u} = \mathbf{0}$, then Span $\{\mathbf{u}\} =$

   (ii) If $\mathbf{u} \neq \mathbf{0}$, then Span $\{\mathbf{u}\} =$

2. Span $\{\mathbf{u}, \mathbf{v}\}$

   Assume that $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$.

   (i) If $\mathbf{v} = c\mathbf{u}$, then Span $\{\mathbf{u}, \mathbf{v}\} =$

   (ii) If $\mathbf{v} \neq c\mathbf{u}$, then Span $\{\mathbf{u}, \mathbf{v}\} =$

3. Span $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} =$

Example 3: Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$.

(1) What is Span $\{\mathbf{a}_1, \mathbf{a}_2\}$?

(2) Is $\mathbf{b}$ in Span $\{\mathbf{a}_1, \mathbf{a}_2\}$?

Lesson 4

Part I: The Matrix-Vector Product

Definition: Let $A = [\,\mathbf{a}_1\ \ \mathbf{a}_2\ \ \cdots\ \ \mathbf{a}_n\,]$ be an $m \times n$ matrix and $\mathbf{x} \in \mathbb{R}^n$. The **product of A and $\mathbf{x}$** is the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights.

$$A\mathbf{x} = [\,\mathbf{a}_1\ \ \mathbf{a}_2\ \ \cdots\ \ \mathbf{a}_n\,] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \stackrel{\text{def}}{=} x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

Note: The product $A\mathbf{x}$ is defined only when the number of columns in A is equal to the number of entries in $\mathbf{x}$.

Definition: An $n \times n$ **identity matrix** is defined as

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = [\,\mathbf{e}_1\ \ \mathbf{e}_2\ \ \cdots\ \ \mathbf{e}_n\,].$$

For example, consider a product of a 3×3 identity matrix I and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$:

Note: If I is an $n \times n$ identity matrix, $I\mathbf{x} = \mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^n$.

Properties of the Matrix-Vector Product

Theorem: If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then

1. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$
2. $A(c\mathbf{u}) = c(A\mathbf{u})$

How to Calculate the Matrix-Vector Product?

1. Use the definition:

Example 1: If $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 4 \\ 2 & -3 & -5 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, calculate

$$A\mathbf{x} \stackrel{\text{def}}{=}$$

Note: If $\mathbf{x} = [\, x_i \,]$ and $\mathbf{y} = [\, y_i \,]$ are vectors in $\mathbb{R}^n$, the **dot product** of $\mathbf{x}$ and $\mathbf{y}$ is

$$\mathbf{x} \cdot \mathbf{y} \stackrel{\text{def}}{=} x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

2. **Row-Vector Rule** for $A\mathbf{x}$:

When the product $A\mathbf{x}$ is defined, the ith entry in $A\mathbf{x}$ is the <u>dot product</u> of the the ith row of A and the vector $\mathbf{x}$.

<u>Example 2</u>: Calculate the product:

(1) $\begin{bmatrix} 2 & 0 & 6 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} =$

(2) $\begin{bmatrix} 1 & 2 & -2 \\ 0 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} =$

(3) $\begin{bmatrix} 3 & 1 \\ 2 & 4 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} =$

Part II: Matrix Equations

Definition: An equation of the form $A\mathbf{x} = \mathbf{b}$ is called a **matrix equation**.

We can write a linear system in three ways:

(1) as linear equations

$$\begin{array}{rcrcrcr} 2x_1 & + & 3x_2 & + & 4x_3 & = & -6 \\ x_1 & & & - & 2x_3 & = & 9 \\ & & 2x_2 & - & 3x_3 & = & 0 \end{array}$$

(2) as a vector equation

(3) as a matrix equation

Theorem: (Test for Consistency)

Let A be an $m \times n$ matrix. The following are logically equivalent:

1. The matrix equation $A\mathbf{x} = \mathbf{b}$ is consistent.
2. The vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$ has a solution.
3. $\mathbf{b}$ is in Span $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n\}$.
4. The linear system with augmented matrix $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$ is consistent.

Definition: Let each $\mathbf{v}_i \in \mathbb{R}^m$. We say that a set $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ **spans** $\mathbb{R}^m$ if every vector in $\mathbb{R}^m$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$, and we write

$$\mathbb{R}^m = \text{Span}\,\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}.$$

For example, if $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $\mathbb{R}^2 = \text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2\}$.

Theorem: Let A be an $m \times n$ matrix. The following statements are logically equivalent:

1. For each $\mathbf{b} \in \mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
2. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
3. The columns of A span $\mathbb{R}^m$.
4. A has a pivot position in every row.
5. $\text{rank}(A) = m$.

Example 3: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 4 \\ 2 & -6 & -7 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

(1) Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all possible b_1, b_2, b_3?

(2) Do the columns of A span $\mathbb{R}^3$?

Example 4: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 4 \\ 2 & -6 & -8 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

(1) Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all possible b_1, b_2, b_3?

(2) Do the columns of A span $\mathbb{R}^3$?

Lesson 5

Part I: Homogeneous Linear Systems

Definition: A linear system is called **homogeneous** if it can be written in the form $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix and $\mathbf{0} \in \mathbb{R}^m$ is the zero vector.

Note: A homogeneous linear system $A\mathbf{x} = \mathbf{0}$ always has at least one solution, $\mathbf{x} = \mathbf{0}$ (the zero vector in $\mathbb{R}^n$). It is called the **trivial solution**. Therefore, a homogeneous system is consistent.

Statement: A homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a **nontrivial solution** if and only if the equation has at least one free variable.

For example, consider a homogeneous equation $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}.$$

Summary: For a homogeneous equation $A\mathbf{x} = \mathbf{0}$:

1. If the system has no free variable, it has only the trivial solution and the solution set is Span $\{\mathbf{0}\}$.

2. If the system has one free variable, its solution can be expressed as a parametric vector equation

$$\mathbf{x} = t\mathbf{v}, \quad t \in \mathbb{R}$$

and the solution set is Span $\{\mathbf{v}\}$.

3. If the system has two free variables, its solution can be expressed as a parametric vector equation

$$\mathbf{x} = s\mathbf{u} + t\mathbf{v}, \quad s, t \in \mathbb{R}$$

and the solution set is Span $\{\mathbf{u}, \mathbf{v}\}$.

Example 1: The augmented matrix of $A\mathbf{x} = \mathbf{0}$ has been row reduced to the RREF below. Find the solution set in parametric vector form and describe it geometrically.

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 2: Describe all solutions of the homogeneous system:

$$x_1 - 3x_2 + 2x_3 = 0$$

Part II: Nonhomogeneous Linear Systems

Definition: A linear system is called **nonhomogeneous** if it can be written in the form $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} \neq \mathbf{0}$.

Compare solutions for homogeneous and nonhomogeneous systems:

Theorem: Let $\mathbf{v}_h$ be a general solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. Suppose $A\mathbf{x} = \mathbf{b}$ is consistent for a given vector $\mathbf{b}$ and let $\mathbf{p}$ be a solution to $A\mathbf{x} = \mathbf{b}$. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors in the form $\mathbf{x} = \mathbf{p} + \mathbf{v}_h$.

Summary: Geometrically, if $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{p}$,

1. and if $A\mathbf{x} = \mathbf{0}$ has no free variable, its solution $\mathbf{p}$ is unique and the solution set is a single point $\mathbf{p}$.

2. and if $A\mathbf{x} = \mathbf{0}$ has one free variable, its solution is

$$\mathbf{x} = \mathbf{p} + t\mathbf{v},\ t \in \mathbb{R}$$

and the solution set is a line through $\mathbf{p}$ parallel to the line spanned by $\mathbf{v}$.

3. and if $A\mathbf{x} = \mathbf{0}$ has two free variables, its solution is

$$\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v},\ s, t \in \mathbb{R}$$

and the solution set is a plane through $\mathbf{p}$ parallel to the plane spanned by $\mathbf{u}$ and $\mathbf{v}$.

Example 3: Find the solutions of the system in parametric vector form and describe the solution set geometrically.

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 5 \\ 2x_1 + x_2 - 3x_3 &= 13 \\ -x_1 + x_2 &= -8 \end{aligned}$$

Example 4: Find the solutions of the system in parametric vector form and describe the solution set geometrically.

$$2x_1 - 4x_2 + 6x_3 = 8$$

Lesson 6

Part I: Linear Independence

Definition:

• A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0} \qquad (1)$$

has only the trivial solution.

• If there is a nontrivial solution to the vector equation (1), then the set $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ is **linearly dependent**.

Note: Write $A = [\,\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_p\,]$.

1. The vector equation (1) can be written as $A\mathbf{x} = \mathbf{0}$, and therefore, the set $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ is linearly independent if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. ($\Longleftrightarrow$ $A\mathbf{x} = \mathbf{0}$ has)

2. The set $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ is linearly dependent $\Longleftrightarrow$ $A\mathbf{x} = \mathbf{0}$ has

3. When $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ is linearly dependent, the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0},\ c_1, \cdots c_p \text{ not all zero}$$

is a **linear dependence relation** among $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$.

Example 1: Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 3 \\ -6 \\ -1 \end{bmatrix}$.

(1) Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

(2) Find a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Statement: The columns of matrix A are linearly independent

$$\iff A\mathbf{x} = \mathbf{0} \text{ has only the trivial solution.}$$

Example 2: Determine if the columns of matrix A are linearly independent.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 4 & 5 \\ 3 & -2 & 1 \end{bmatrix}$$

Part II: Special Cases for Linear Dependence

1. A set of one vector: $\{\mathbf{v}_1\}$

   $\{\mathbf{v}_1\}$ is linearly dependent $\iff$ $\mathbf{v}_1 = \mathbf{0}$.

2. A set of two vectors: $\{\mathbf{v}_1, \mathbf{v}_2\}$

   $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent $\iff$ $\mathbf{v}_2 = c\mathbf{v}_1$.

3. A set of two or more vectors: $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$

Theorem: $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ is linearly dependent

$\iff$ at least one of the vectors in S is a linear combination of the others.

Special cases when linear dependence is automatic

Theorem:
If S contains more vectors than there are entries in each vector, then S is linearly dependent. That is, any set $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

Theorem:
If a set S in $\mathbb{R}^n$ contains the zero vector, then S is linearly dependent.

Geometric representation of dependence relations in $\mathbb{R}^3$

Example 3: Describe each set geometrically in $\mathbb{R}^3$.

(1) Span $\{\mathbf{u}, \mathbf{v}\}$, where $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -2 \\ -6 \\ -10 \end{bmatrix}$

(2) Span $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$

Example 4: Determine by inspection if the given set is linearly dependent.

(1) $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ -6 \\ -9 \end{bmatrix}$

(2) $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$

(3) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$

(4) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 8 \\ 3 \end{bmatrix}$

Lesson 7

Part I: Transformations

Definition: A **transformation** (**function** or **mapping**)

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

- $\mathbb{R}^n$ is the **domain** of T and $\mathbb{R}^m$ is the **codomain** of T.
- The vector $T(\mathbf{x})$ in $\mathbb{R}^m$ is the **image** of $\mathbf{x}$.
- The set of all images $\{\, T(\mathbf{x}) \mid \mathbf{x} \in \mathbb{R}^n \,\}$ is the **range** of T.

Definition: A transformation T is a **matrix transformation** if, for each $\mathbf{x} \in \mathbb{R}^n$, we can write $T(\mathbf{x}) = A\mathbf{x}$, where A is an $m \times n$ matrix.

Note: For a matrix transformation $T(\mathbf{x}) = A\mathbf{x}$, the range of T is the set of all linear combinations of the columns of A.

Example 1: Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the transformation defined by $T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \\ 1 & -4 \end{bmatrix}$. Let $\mathbf{b} = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$.

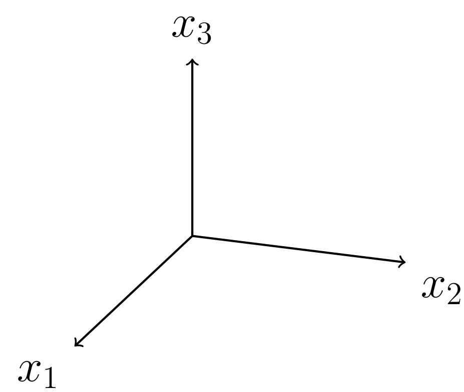
(1) Find the image of $\mathbf{u}$ under T if $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

(2) Find all $\mathbf{x} \in \mathbb{R}^2$, if any, whose image under T is $\mathbf{b}$.

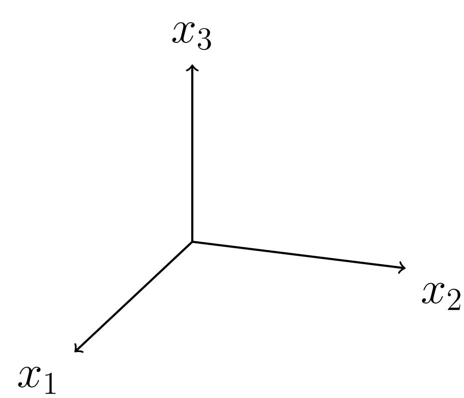
(3) Determine whether the vector $\mathbf{c}$ is in the range of T.

Example 2:

(1) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Show that the transformation $T : \mathbf{x} \longmapsto A\mathbf{x}$ **projects** each point in $\mathbb{R}^3$ onto the x_1x_2-plane. Find the domain, codomain, and the range of T.



(2) Let $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. If T is the transformation so that $T(\mathbf{x}) = A\mathbf{x}$, describe the transformation geometrically.



Part II: Linear Transformations

Definition: A transformation T is a **linear** if

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T, and
2. $T(c\mathbf{u}) = c\,T(\mathbf{u})$ for all scalar c and all $\mathbf{u}$ in the domain of T.

Note: If T is a linear transformation, then $T(\mathbf{0}) = \mathbf{0}$.

Statement: T is a linear transformation if and only if

$$T(c\mathbf{u} + d\mathbf{v}) = c\,T(\mathbf{u}) + d\,T(\mathbf{v})$$

for any scalars c, d, and for any $\mathbf{u}$, $\mathbf{v}$ in the domain of T.

For example, if $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation that maps $\mathbf{e}_1$ to the vector $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and maps $\mathbf{e}_2$ to the vector $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$, then

$$T\left(\begin{bmatrix} 2 \\ -3 \end{bmatrix}\right) =$$

Note: A matrix transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = A\mathbf{x}$, where A is an $m \times n$ matrix, is linear.

Theorem: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \in \mathbb{R}^n,$$

and A is the $m \times n$ matrix in the form

$$A = [\, T(\mathbf{e}_1) \;\; T(\mathbf{e}_2) \;\; \cdots \;\; T(\mathbf{e}_n) \,],$$

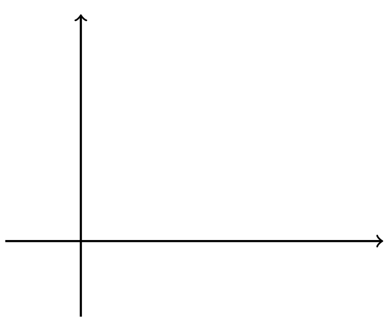
where $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$ are the columns of the $n \times n$ identity matrix I.

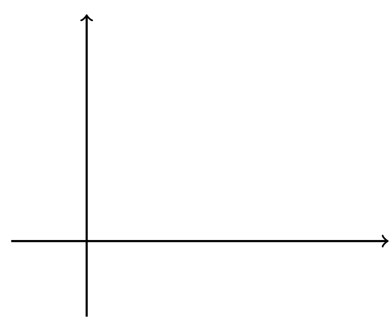
Definition: The matrix $A = [\, T(\mathbf{e}_1) \;\; T(\mathbf{e}_2) \;\; \cdots \;\; T(\mathbf{e}_n) \,]$ is called the **standard matrix** of the linear transformation T.

Note: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a transformation defined as

$$T(\mathbf{x}) = r\mathbf{x}, \ r \in \mathbb{R}.$$

1. T is linear.
2. T is called a contraction when $0 \leq r \leq 1$ and a dilation when $r > 1$.



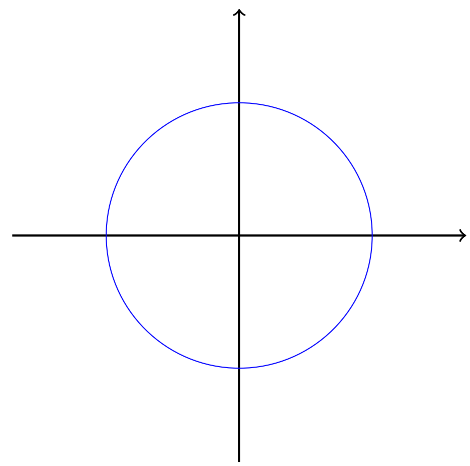


Example 3: Find the standard matrix A of the dilation transformation

$$T(\mathbf{x}) = 2\mathbf{x}, \ \mathbf{x} \in \mathbb{R}^3.$$

Example 4: Find the standard matrix A of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates each point of the plane through an angle θ.

Note: T is called a rotation transformation.



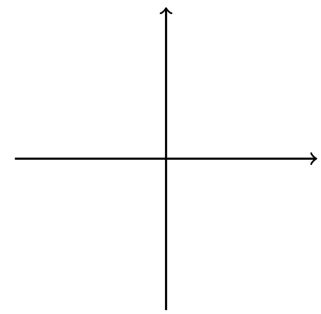
Lesson 8

Part I: Geometric Linear Transformations

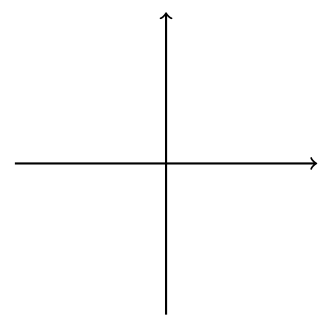
Recall: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, we can write $T(\mathbf{x}) = A\mathbf{x}$, where $A = [T(\mathbf{e}_1)\ T(\mathbf{e}_2)\ \cdots\ T(\mathbf{e}_n)]$ is the $m \times n$ standard matrix.

Example 1: Find the standard matrix A of each transformation T.

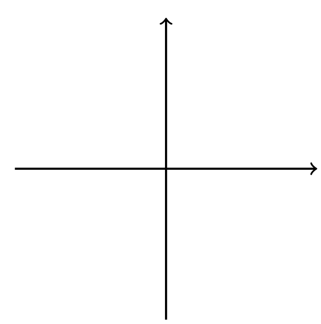
(1) reflection through the line $x_2 = x_1$:



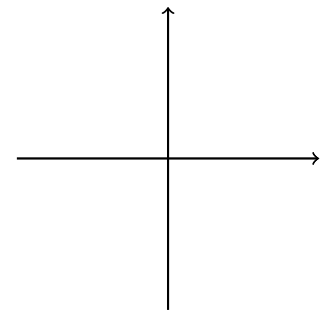
(2) reflection through the line $x_2 = -x_1$:



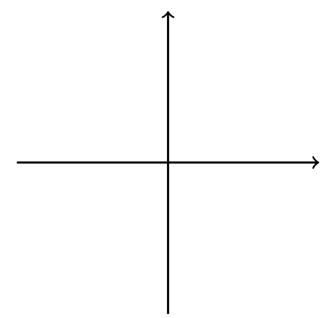
(3) reflection through the origin:



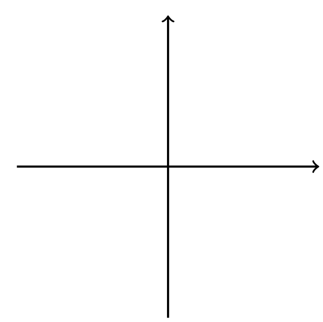
(4) reflection through the line x_2-axis:



(5) projection onto x_1-axis:



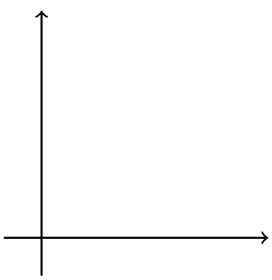
(6) vertical contraction and expansion by a factor of k ($k > 0$):

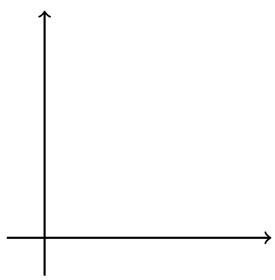


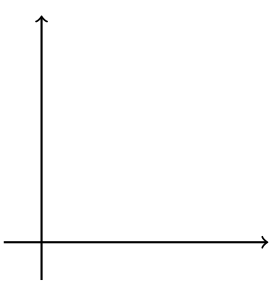
Example 2: The matrix transformations defined by the matrices

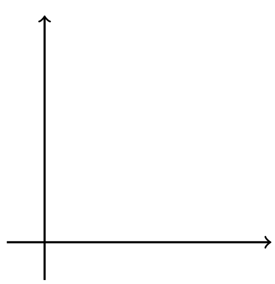
$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

are called horizontal shear and vertical shear transformations, respectively. Find the image of the square $[0, 1] \times [0, 1]$ under each of the transformations for $k > 0$.

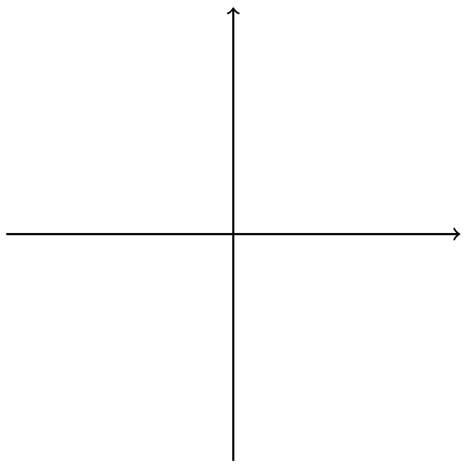


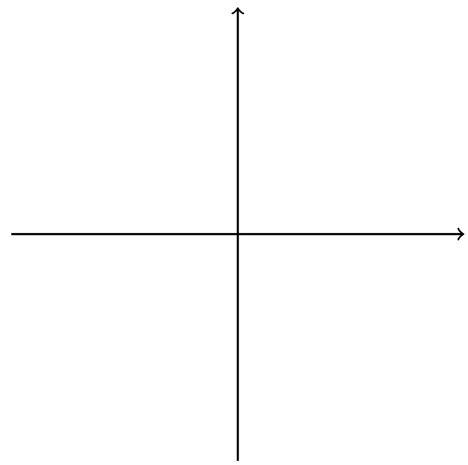






Example 3: Find the standard matrix A of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that first performs a horizontal shear that transforms $\mathbf{e}_2$ into $\mathbf{e}_2 + 2\mathbf{e}_1$ (leaving $\mathbf{e}_1$ unchanged) and then reflects the points through x_1-axis.





Part II: Onto and One-to-One Transformations

Definition: A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is a **onto** if the equation $T(\mathbf{x}) = \mathbf{b}$ has a solution for each $\mathbf{b} \in \mathbb{R}^m$.

Geometrically it means that each $\mathbf{b} \in \mathbb{R}^m$ is the image of at least one $\mathbf{x} \in \mathbb{R}^n$.

Definition: A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is a **one-to-one** if whenever $T(\mathbf{x}) = \mathbf{b}$ has a solution for $\mathbf{b} \in \mathbb{R}^m$, the solution is unique.

Geometrically it means that each $\mathbf{b}$ in the range of T is the image of exactly one $\mathbf{x} \in \mathbb{R}^n$.

Theorem: If $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, then

$$T \text{ is one-to-one} \iff T(\mathbf{x}) = \mathbf{0} \text{ has only the trivial solution.}$$

Example 4: Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation with the standard matrix $A = \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & -2 & 1 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$.

(1) Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$?

(2) Is T one-to-one?

Theorem: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix of T. Then

1. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ $\iff$ the columns of A span $\mathbb{R}^m$.

2. T is one-to-one $\iff$ the columns of A are linearly independent.

<u>Example 5</u>: Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation so that

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 3x_2 \\ 4x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}.$$

(1) Does T map $\mathbb{R}^2$ onto $\mathbb{R}^3$?

(2) Is T one-to-one?

Lesson 9

Part I: Matrices

Recall: We have used the following notation for an $m \times n$ matrix A:

$$A = [\, \mathbf{a}_1 \quad \cdots \quad \mathbf{a}_j \quad \cdots \quad \mathbf{a}_n \,]$$

where $\mathbf{a}_j = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{mj} \end{bmatrix} \in \mathbb{R}^m \ (j = 1 : n)$,

and we can rewrite matrix A in the form:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} = [\, a_{ij} \,] \ (i = 1 : m,\ j = 1 : n)$$

Equality of Two Matrices

Two matrices are equal if they have the same size and their corresponding entries are equal.

i.e., $A = B \iff a_{ij} = b_{ij},\ (i = 1 : m,\ j = 1 : n)$

Sums and Scalar Multiples of Matrices

Definition: Let $A = [\, a_{ij}\,]$ and $B = [\, b_{ij}\,]$ be two $m \times n$ matrices, and let r be a scalar.

- $A + B = [\, a_{ij} + b_{ij}\,]$
- $rA = [\, ra_{ij}\,]$

Note: $A + B$ is defined only when A and B have the same size.

Properties of Sums and Scalar Multiples of Matrices

Theorem: Let A, B, and C be $m \times n$ matrices, and let r and s be scalars. Then

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + 0 = A$, where 0 is the $m \times n$ zero matrix.
4. $r(A + B) = rA + rB$
5. $(r + s)A = rA + sA$
6. $r(sA) = (rs)A$

Example 1: Let $A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & 4 & -2 \end{bmatrix}$, and let 0 be the 2×3 zero matrix. Find:

$2(A - B) + 0 =$

Special Matrices

Definition: The **main diagonal** of a matrix $A = [\, a_{ij} \,]$ is the list of entries a_{ii}.

- A matrix $D_{n \times n}$ is a **diagonal matrix** if its non-diagonal entries are zero. (i.e., $d_{ij} = 0$ for all $i \neq j$)

Note: The $n \times n$ identity matrix I is a diagonal matrix.

- A matrix $L_{n \times n}$ is an **lower triangular matrix** if its non-zero entries are only in the lower triangle of the matrix. (i.e., $l_{ij} = 0$ for all $i < j$)

- A matrix $U_{n \times n}$ is an **upper triangular matrix** if its non-zero entries are only in the upper triangle of the matrix. (i.e., $u_{ij} = 0$ for all $i > j$)

Part II: Matrix Multiplication

Recall: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, we can write $T(\mathbf{x}) = A\mathbf{x}$, where A is an $m \times n$ matrix.

Statement: Let A be an $m \times n$ matrix, B is an $n \times p$ matrix, and $\mathbf{x} \in \mathbb{R}^p$. The **composition of linear transformations**

$$\begin{array}{rcccc} T: \mathbb{R}^p & \longrightarrow & \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \\ \mathbf{x} & \longmapsto & B\mathbf{x} & \longmapsto & A(B\mathbf{x}) \end{array}$$

is a linear transformation

$$\begin{array}{rcc} T: \mathbb{R}^p & \longrightarrow & \mathbb{R}^m \\ \mathbf{x} & \longmapsto & (AB)\mathbf{x} \end{array}$$

If $B = [\, \mathbf{b}_1 \quad \cdots \quad \mathbf{b}_j \quad \cdots \quad \mathbf{b}_p \,]$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}$, then

$B\mathbf{x} \stackrel{\text{def}}{=}$

$A(B\mathbf{x}) =$

How to Calculate the Matrix Multiplication?

1. Use the definition:

Definition: If A is an $m \times n$ matrix, and if $B = [\mathbf{b}_1 \quad \cdots \quad \mathbf{b}_j \quad \cdots \quad \mathbf{b}_p]$ is an $n \times p$ matrix, then the **product** AB is the $m \times p$ matrix with columns $A\mathbf{b}_j,\ j = 1 : p$.

$$\begin{aligned} AB &= A[\,\mathbf{b}_1 \quad \cdots \quad \mathbf{b}_j \quad \cdots \quad \mathbf{b}_p\,] \\ &\stackrel{\text{def}}{=} [\,A\mathbf{b}_1 \quad \cdots \quad A\mathbf{b}_j \quad \cdots \quad A\mathbf{b}_p\,] \end{aligned}$$

Note: For matrix multiplication, the number of columns in the first matrix must be equal to the number of rows in the second matrix.

Example 2: Compute AB if the product is defined.

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 \\ 1 & 4 \\ -1 & 2 \end{bmatrix}$$

2. **Row-Column Rule** for Computing AB

If A is an $m \times n$ matrix and B is an $n \times p$ matrix, the (i, j)-entry of AB is the dot product of the ith row of A and the jth column of B.

Example 3: Let $A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & -2 & 1 & 4 \\ 1 & 3 & -1 & 1 \end{bmatrix}$

(1) Find the second column of AB.

(2) $AB =$

(3) $BA =$

Properties of Matrix Multiplication

Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined. Then:

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(A + B)C = AC + BC$
4. $r(AB) = (rA)B = A(rB)$ for any scalar r
5. $I_m A = A = AI_n$, where I_m and I_n are the $m \times m$ and $n \times n$ identity matrices, respectively.

Warnings:

1. In general, $AB \neq BA$.

2. The **cancellation law** <u>does not hold</u> for matrix multiplication. That is, if $AB = AC$, then <u>it is not true</u> in general that $B = C$.

3. The **zero-product rule** <u>does not hold</u> for matrix multiplication. That is, if $AB = 0$ (the zero matrix), we <u>cannot</u> conclude in general that either $A = 0$ or $B = 0$.

Unit 1 Review

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation so that $T(\mathbf{x}) = A\mathbf{x}$, where $A_{m\times n} = [\,\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n\,]$.

- A has a pivot position in every column.

- A has a pivot position in every row.

1. Let $A = [\,\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\,] = \begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$.

(1) Determine if $\mathbf{b}$ is a linear combination of the columns of A.

(2) If yes, represent vector $\mathbf{b}$ as a linear combination of the columns of A by setting the parameter $x_3 = 0$. Is this the only representation possible?

(3) Do the columns of A span $\mathbb{R}^3$?

2. Let $A = \begin{bmatrix} 1 & -3 & 2 \\ -2 & 5 & -1 \\ 3 & -6 & -3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

(1) Determine if the equation $A\mathbf{x} = \mathbf{b}$ is consistent for all possible b_1, b_2, b_3.

(2)If not, give a description of the set of all $\mathbf{b}$ for which the equation is consistent.

(3) Describe the vectors $\mathbf{b}$ geometrically as a span of the least possible number of vectors.

3. Consider a homogeneous matrix equation $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

(1) Find the solutions in the parametric vector form, and describe the solutions geometrically.

(2) Does the nonhomogeneous system $A\mathbf{x} = \mathbf{b}$ has a solution for any $\mathbf{b} \in \mathbb{R}^3$?

(3) Find the general solution to $A\mathbf{x} = \mathbf{b}$ for $\mathbf{b} = \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}$, and describe the solution geometrically.

4. Let $A = [\,\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\,] = \begin{bmatrix} 1 & -3 & 2 \\ -2 & 7 & 1 \\ -4 & 6 & h \end{bmatrix}$, and let $\mathbf{b} = \begin{bmatrix} 1 \\ -4 \\ k \end{bmatrix}$.

(1) Find h so that the set of the columns of A is linearly dependent.

(2) Find h and k so that the system $A\mathbf{x} = \mathbf{b}$ has

a unique solution:

infinitely many solutions:

no solution:

5. Let $A = [\,\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \mathbf{a}_3\,] = \begin{bmatrix} 1 & -3 & 2 \\ 3 & -8 & 8 \\ 0 & 1 & 2 \\ 1 & 0 & 8 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 6 \\ 4 \\ 10 \end{bmatrix}$, and $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 6 \\ 3 \\ 10 \end{bmatrix}$.

(1) Find all vectors, if any, whose images under the transformation T are vector $\mathbf{b}_1$ and $\mathbf{b}_2$, respectively, where $T(\mathbf{x}) = A\mathbf{x}$.

(2) Whether the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ are in the range of T.

(3) Does range of $T = \mathbb{R}^4$?

(4) Is there only one vector $\mathbf{x} \in \mathbb{R}^3$, for every $\mathbf{b}$ in the range of T, that maps into $\mathbf{b}$?

6. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation so that

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 5x_2 + 4x_3 \\ x_2 - 6x_3 \end{bmatrix}.$$

(1) Find the standard matrix A of the transformation.

(2) Determine whether the equation $T(\mathbf{x}) = \mathbf{b}$ has a solution for any $\mathbf{b}$ in $\mathbb{R}^2$.

(3) Determine whether the equation $T(\mathbf{x}) = \mathbf{b}$ a unique solution for any $\mathbf{b}$ in the range of T.

7. True or False:

If a system $A\mathbf{x} = \mathbf{b}$ has no free variable, then it has a unique solution.