Report: Advection-Diffusion

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1 Discretization of the $\underline{\mathbf{u}} \cdot \nabla c$ term.

Discretization allows for incremental iteration through c. The upwind portion of the advection-diffusion equation results in the $\underline{\mathbf{u}}$ element becoming (u_x, u_y) and ∇c becoming $(\frac{dc}{dx}, \frac{dc}{dy})$. Result is of the form:

$$u_x \frac{dc}{dx} + u_y \frac{dc}{dy}$$

Depending on the value of $\underline{\mathbf{u}}$, the terms will be:

$$\text{for } \underline{\mathbf{u}} > 0,\ \frac{dc}{dx} = \frac{c_{i,j}^{n+1} - c_{i-1,j}^{n+1}}{\Delta x},\ \frac{dc}{dy} = \frac{c_{i,j}^{n+1} - c_{i,j-1}^{n+1}}{\Delta y}$$

$$\text{for } \underline{\mathbf{u}} \leq 0,\ \frac{dc}{dx} = \frac{c_{i+1,j}^{n+1} - c_{i,j}^{n+1}}{\Delta x},\ \frac{dc}{dy} = \frac{c_{i,j+1}^{n+1} - c_{i,j}^{n+1}}{\Delta y}$$

2 Linear system satisfied by $\underline{\mathbf{c}}^{n+1}$

The linear system is of the form $A\underline{\mathbf{c}}^{n+1} = RHS$ where A is a matrix consisting of the bottom, left, centre, right, and top elements, $\underline{\mathbf{c}}_{i,j}^{n+1}$ is the advection vector and the right-hand side consists of the diffusion equation $\underline{\mathbf{c}}_{i,j}^{n} - \nabla c + S$

3 Scheme

The linear system can be described as:
Pseudo-code:

```
Declare variables;

A = sparse(m,m);

while t < t_final,
        for boundary pixels,
                set to exact solution
        end;
```

```
for interior nodes,
        calculate C;
        calculate L;
        calculate R;
        calculate T;
        calculate B;

        calculate RHS;

        cplus1 = A\backslash RHS;
        c = ctp1
```

4 Source/sink

$\mathrm{S}_{exact} = \frac{dc}{dt} + \underline{\mathbf{u}}\nabla c - D\nabla^2 c$ where

$c = (e^{-t} - 1)(sin(\pi x) + sin(\pi y))$

$\nabla c = (e^{-t} - 1)(-\pi cos(\pi x) - \pi cos(\pi y))$, and

$\nabla^2 c = (e^{-t} - 1)(-\pi^2 sin(\pi x) - \pi^2 sin(\pi y))$

5 Plot

6 Error

$error = max|c - c_{exact}|$

Figure 1: Exact plot