2MA918 Laboration II

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1 Exercise 1: Minimum Cost Flow Problems

This exercise involves solving minimum cost network flow problems for a logistics company in the Kronoberg region.

1.1 Part 1: Initial Network Flow Problem

The company owns two production facilities in Älmhult (supply: 2500 units) and Markaryd (supply: 1000 units), with two storage terminals in Liatorp and Osby. They need to transport units to retail stores in Ljungby (demand: 1000), Alvesta (demand: 500), and Växjö (demand: 2000).

Formulation: This is formulated as a linear programming problem where we minimize the total transportation cost subject to:

- Flow conservation constraints at each node
- Capacity constraints on each edge
- Supply and demand constraints

Solution: Using scipy's linprog with the HiGHS method:

- **Minimum cost:** 250,000.00
- **Optimal flow:**
  - Älmhult → Liatorp: 1000 units (cost: 18)
  - Älmhult → Osby: 500 units (cost: 24)
  - Älmhult → Vaxjo: 1000 units (cost: 62)
  - Osby → Markaryd: 500 units (cost: 29)
  - Liatorp → Vaxjo: 1000 units (cost: 46)
  - Markaryd → Ljungby: 1500 units (cost: 51)
  - Ljungby → Alvesta: 500 units (cost: 42)

1.2 Part 2: New Production Facility Analysis

The company is considering adding a new production facility in either Värnamo or Vislanda.

1.2.1 Värnamo Option

With Värnamo producing 750 units and adjusted production at Älmhult (2000) and Markaryd (750):

- **Minimum cost:** 200,000.00

1.2.2 Vislanda Option

With Vislanda producing 380 units and adjusted production at Älmhult (2120) and Markaryd (1000):

- **Minimum cost:** 200,220.00

Recommendation: Värnamo should be chosen as it provides the lowest cost (200,000.00), saving 220.00 compared to Vislanda and reducing the total cost by 50,000.00 from the initial network.

2 Exercise 2: Unconstrained Optimization I

This exercise implements steepest descent and Newton's method to minimize $f(x,y) = (x+1)^2 - xy + 3(y-5)^2$.

2.1 Analytical Solution

The optimal point is found by solving $\nabla f = 0$:

$$\frac{\partial f}{\partial x} = 2(x+1) - y = 0$$
$$\frac{\partial f}{\partial y} = -x + 6(y-5) = 0$$

Solving this system yields:

- **Optimal point:** $x^* = (1.636364, 5.272727)$
- **Optimal value:** $f(x^*) = -1.454545$

The Hessian matrix is:

$$H_f = \begin{bmatrix} 2 & -1 \\ -1 & 6 \end{bmatrix}$$

with eigenvalues $\lambda_1 = 1.764$ and $\lambda_2 = 6.236$, confirming positive definiteness and that x^* is a minimum.

2.2 Steepest Descent Method

Starting from $x_0 = (1,1)$ with Armijo line search:

- **Number of iterations:** 18
- **Final point:** $(1.636346, 5.272606)$
- **Final function value:** -1.454545
- **Absolute error:** 4.24×10^{-8}
- **Final gradient norm:** 7.16×10^{-4}

2.3 Newton's Method

Starting from the same point $x_0 = (1,1)$:

- **Number of iterations:** 1
- **Final point:** $(1.636364, 5.272727)$
- **Final function value:** -1.454545
- **Absolute error:** 0.0
- **Final gradient norm:** 3.55×10^{-15}

2.4 Analysis

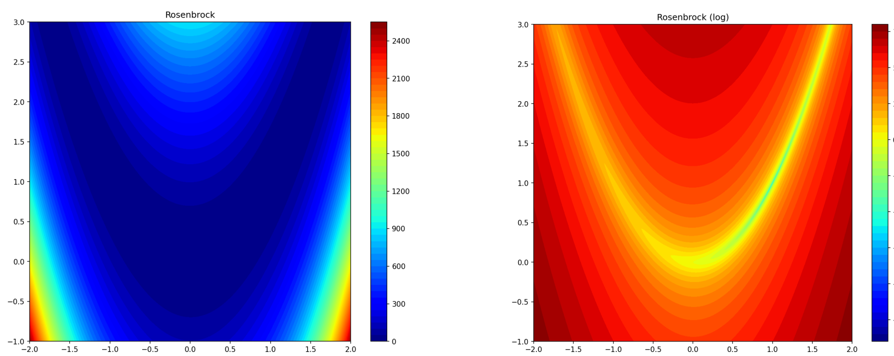
Newton's method is significantly more efficient for this quadratic problem, converging in a single iteration. This is because Newton's method uses second-order information (the Hessian) and achieves exact convergence for quadratic functions in one step when the Hessian is constant and positive definite. Steepest descent, using only first-order information, requires 18 iterations to reach similar accuracy.

3 Exercise 3: Unconstrained Optimization II

This exercise optimizes the Rosenbrock function $f(x, y) = (a - x)^2 + b(y - x^2)^2$ with $a = 1$ and $b = 100$.

3.1 Contour Plots

The Rosenbrock function is known as a challenging optimization problem due to its narrow, curved valley. Figure 1 shows both the standard and logarithmic contour plots.



(a) Standard contour plot

(b) Logarithmic contour plot

Figure 1: Contour plots of the Rosenbrock function

3.2 Steepest Descent Method

Starting from $x_0 = (-1, 1.5)$ with tolerance $\|\nabla f\| < 10^{-4}$:

- **Number of iterations:** 2664
- **Final point:** $(0.999891, 0.999781)$
- **Final function value:** 1.19×10^{-8}
- **Trajectory length:** 3.662594
- **Final gradient norm:** 9.79×10^{-5}

The trajectory is shown in Figure 2. The method struggles with the narrow valley, requiring many iterations to converge.

3.3 Newton's Method

Starting from the same point $x_0 = (-1, 1.5)$:

- **Number of iterations:** 22

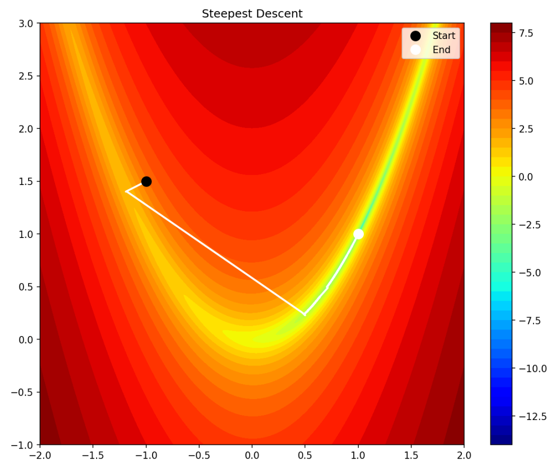


Figure 2: Steepest Descent trajectory on Rosenbrock function

- **Final point:** $(1.000000, 1.000000)$
- **Final function value:** 1.56×10^{-18}
- **Trajectory length:** 3.714547
- **Final gradient norm:** 3.63×10^{-8}

The trajectory is shown in Figure 3. Newton's method converges much faster (22 vs 2664 iterations), though the trajectory length is similar.

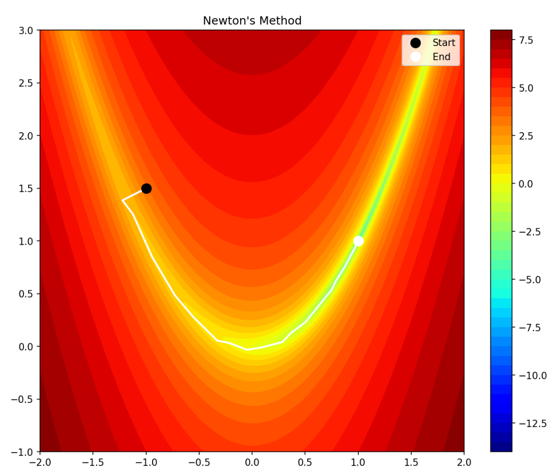


Figure 3: Newton's Method trajectory on Rosenbrock function

3.4 Scipy Optimization

Using scipy's minimize with BFGS method:

- **Number of iterations:** 35
- **Final point:** $(1.000000, 1.000000)$
- **Final function value:** 1.08×10^{-17}
- **Final gradient norm:** 6.75×10^{-8}

3.5 Comparison

The results clearly demonstrate the superiority of second-order methods for non-convex optimization:

- Steepest descent: 2664 iterations, very slow convergence
- Newton's method: 22 iterations, fast convergence with Hessian information
- BFGS (quasi-Newton): 35 iterations, good balance between efficiency and computational cost

For the Rosenbrock function, methods using curvature information (Newton, BFGS) are 100x more efficient than steepest descent.

4 Exercise 4: Constrained Optimization

This exercise solves the constrained optimization problem:

$$\begin{aligned} \min \quad & f(x,y) = (2-x-y)^2 + y^4 \\ \text{s.t.} \quad & x^2 + y^2 - 4 \leq 0 \\ & 4x + 5y - 25 \leq 0 \end{aligned}$$

using the penalty function method with $\mu_0 = 0.1$ and $\beta = 10$.

4.1 Initial Feasible Point

A feasible starting point must satisfy both constraints. The origin $x_0 = (0,0)$ is chosen:

- $g_1(x_0) = 0 + 0 - 4 = -4 \leq 0$ (satisfied)
- $g_2(x_0) = 0 + 0 - 25 = -25 \leq 0$ (satisfied)
- $f(x_0) = (2-0-0)^2 + 0^4 = 4$

4.2 Penalty Function Method

The penalty function is defined as:

$$\alpha(x) = \max(0, g_1(x))^2 + \max(0, g_2(x))^2$$

The augmented objective function is:

$$F(x,\mu) = f(x) + \mu \cdot \alpha(x)$$

We solve successive unconstrained problems, increasing μ by factor $\beta = 10$ until $\alpha(x) < 10^{-4}$.

4.3 Results

The method converged in **1 iteration**:

Iteration	μ	x	$f(x)$	$\alpha(x)$
0	0.10	$(1.9934, 0.0066)$	0.000000	0.00×10^0

4.4 Optimal Solution

- **Optimal point:** $x^* = (1.9934, 0.0066)$
- **Optimal function value:** $f(x^*) = 0.000000$
- **Constraint values:**

  - $g_1(x^*) = -0.0264 \leq 0$ (satisfied)
  - $g_2(x^*) = -16.9934 \leq 0$ (satisfied)

- **Gradient norm:** $\|\nabla f(x^*)\| = 7.97 \times 10^{-6}$
- **Penalty function:** $\alpha(x^*) = 0.0$

4.5 Analysis

The problem converged remarkably fast because the unconstrained minimum of $f(x, y) = (2 - x - y)^2 + y^4$ is very close to $(2, 0)$, which satisfies both constraints. The optimal solution $(1.9934, 0.0066)$ is near this point and remains well within the feasible region. The penalty function value is effectively zero, indicating a feasible solution with no constraint violations.