Linnæus University
Department of mathematics
Björn Lindenberg

Exam in Optimization Methods, 2MA404, 7.5 hp
January 08, 2024, 14.00–19.00

For full points on an exercise, remember to carefully motivate your answers. Scale: 50% for grade E and 90% for grade A.
Allowed aids: One A4 page of hand-written notes. Ruler. English dictionary.

1. Consider the following LP-problem (6p)

$$\begin{array}{ll} \max & z = 8x_1 + 5x_2 + 2x_3 \\ \text{s.t.} & 5x_1 + 9x_2 - 4x_3 \leq 5 \\ & -6x_1 + 6x_2 + 6x_3 \leq 2 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

(a) Solve it using the simplex method. State both the optimal point $\mathbf{x}^* = (x_1, x_2, x_3)$ and the optimal function value $z(\mathbf{x}^*)$.

(b) Using your derived data, state the optimal point $\mathbf{y}^*$ of the dual problem.

(c) What is the name and significance of the components in $\mathbf{y}^*$?

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2. The company *Urban Plant Vibes* needs your help with production planning to maximize profits. (6p)

The company does indoor vertical farming, where it has two different kinds of produce, namely *bok choy* and *lettuce*. However the one-cycle production is subject to a number of constraints:

- There are three *plants* of bok choy in every sold *item* of that type, but for lettuce there is only one plant.
- The combined total of cultivated *plants* needs to be at least 10 000.
- The number of bok choy *plants* subtracted by the number of lettuce *plants* cannot exceed 8 000.
- The number of lettuce *items* subtracted by the number of bok choy *items* cannot exceed 2 000.
- Each *item* requires 1 hour of cultivation and harvesting work for production, but the total amount of man-hours used cannot exceed 8 000.

The profit for an item of bok choy and lettuce is \$3 and \$2 respectively.

(a) Model the problem as an LP-problem.

(b) Solve it graphically and present the optimal production schedule and profit.

(c) What is the maximum positive change in profit for a lettuce item if we want the schedule to preserve optimality?

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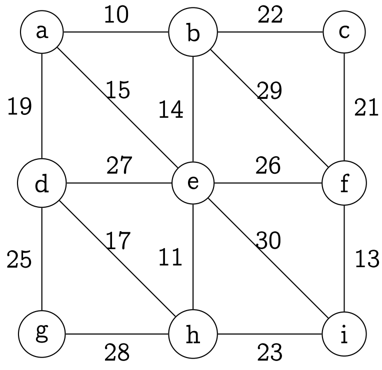
3. The following network shows a set of connected servers, where each arc has a reported latency. Find the minimal spanning tree. (3p)



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4. For each of the following functions determine if they are convex or not. If not then give a counter-example. (3p)

(a) $f(x, y, z) := e^{g(x,y,z)}$ on $\mathbb{R}^3$, where $g(x, y, z) = x^2 + 2y^2 + 3z^2 + 2xy + 2xz$.

(b) $f(x) := x^3 - x^2$ on $\mathbb{R}$.

(c) $f(x, y) := x^3 + y^3$ on $\mathbb{R}^2_+ := \{(x, y) \mid x \geq 0, y \geq 0\}$.

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5. Minimize (4p)

$$f(x, y) = x^2 + y^2 - xy - 3y + 3$$

using Newton's method and starting point $\mathrm{x}_0 = (1, 1)$. Perform a maximum of two iterations.

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6. Consider the problem (6p)

$$\begin{aligned} \min \quad & f(x, y) = x^2 + y^2 - 4x - 2y + 5, \\ \text{s.t.} \quad & 2x + 2y - x^2 - y^2 \geq 0, \\ & x^2 + y^2 - 2y \leq 1. \end{aligned}$$

(a) Formulate KKT-conditions for the problem.

(b) Show that $(x, y) = \left(\sqrt{2}, 1\right)$ is a local optimum.

(c) Is the point a global optimum? Motivate your answer.

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7. Consider the problem (6p)

$$\begin{aligned} \min \quad & f(x, y) = x^2 + y^2 - 4x - 2y + 5, \\ \text{s.t.} \quad & x + y \geq 4. \end{aligned}$$

(a) Formulate a Lagrangian function $L(\mathrm{x}, v)$ and its dual function $h(v)$.

(b) Evaluate $h(v)$ for all v by finding minimizers $x(v)$ and $y(v)$ of the Lagrangian subproblem. Find the optimal point v^*.

(c) Given the minimizer $\mathrm{x} := (x(v^*), y(v^*))$ of the v^*-subproblem, show that $h(v^*) = f(\mathrm{x})$. Deduce the optimal point x^* for this problem.

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8. (Only for 2MA404) Prove that if the problem (4p)

$$\begin{aligned} \min \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{x} \in X \end{aligned}$$

is convex, then every local minimum is also a global minimum.

Good luck!