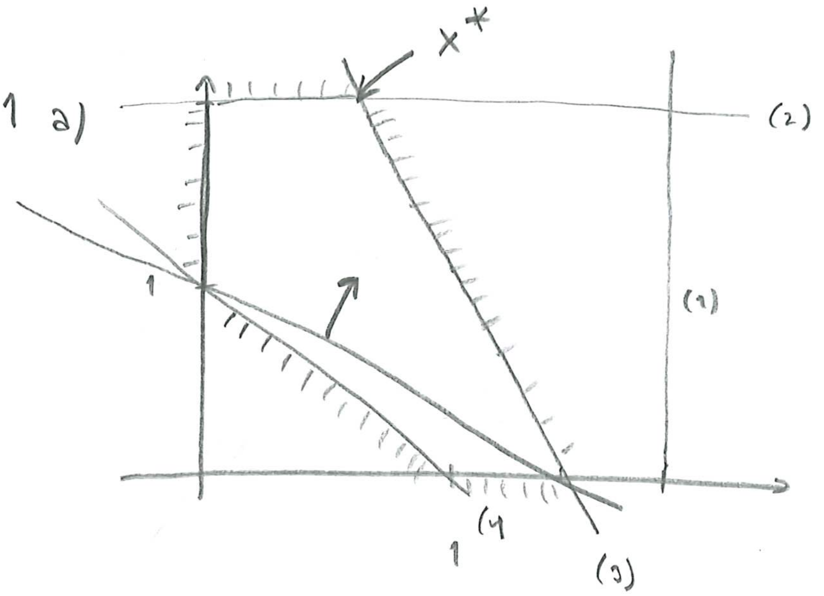
1 a)



we see graphically that max is obtained at the intersection of (2) and (3), so

$$x^* = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} 1/2 \\ 2 \end{pmatrix}$$

b)

$$\min \; w = 2y_1 + 2y_2 + 3y_3 + y_4$$

$$y_1 + \quad 2y_3 + y_4 \geq 2$$

$$y_2 + y_3 + y_4 \geq 3$$

$$y_1 \geq 0, \; y_2 \geq 0, \; y_3 \geq 0, \; y_4 \leq 0$$

c) For simplex we put $y_4' = -y_4$ and add s_1, s_2. We set

$$\min \; w = 2y_1 + 2y_2 + 3y_3 - y_4'$$

$$y_1 + 2y_3 - y_4' - s_1 = 2$$

$$y_2 + y_3 - y_4' - s_2 = 3$$

$$y; s \geq 0$$

Using $X_B = \begin{pmatrix} y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $X_N = \begin{pmatrix} y_1 \\ y_4' \\ s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

We see that

$$\begin{cases} y_1 + 2y_3 - y_4' - s_1 = 2 \\ y_2 + y_3 - y_4' - s_2 = 3 \end{cases} \sim \begin{cases} \frac{1}{2}y_1 + y_3 - \frac{1}{2}y_4' - \frac{1}{2}s_1 = 1 \\ -\frac{1}{2}y_1 + y_2 - \frac{1}{2}y_4' + \frac{1}{2}s_1 - s_2 = 2 \end{cases}$$

We may thus solve for y_2, y_3 and obtain

$$y_2 = 2 + \frac{1}{2}y_1 + \frac{1}{2}y_4' - \frac{1}{2}s_1 + s$$

$$y_3 = 1 - \frac{1}{2}y_1 + \frac{1}{2}y_4' + \frac{1}{2}s_1$$

Insertion into w yields

$$\begin{aligned} w &= 2y_1 + 2y_2 + 3y_3 - y_4' \\ &= 7 + \frac{3}{2}y_1 + \frac{3}{2}y_4' + \frac{1}{2}s_1 + \frac{7}{2}s_2. \end{aligned}$$

Conclusion, no need for simplex as the optimal solution was the given which is clear since the reduced costs are all non-negative.

(2) The phase-1 problem may be set up as follows

$$\min \quad w = a_3$$

$$\text{s.t} \quad \begin{aligned} 2x_1 + x_2 + x_3 + s_1 &= 1 \\ -x_2 + x_3 + s_2 &= 0 \\ x_1 + 2x_2 + x_3 + a_3 &= 2 \end{aligned}$$

$$x, s, a \geq 0$$

Basic variables may be $x_B = \begin{pmatrix} s_1 \\ s_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

We set starting tableau

	w	x_1	x_2	x_3	s_1	s_2	a_3	b
$-w$	1	1	2	1	0	0	0	2
s_1	0	2	1	1	1	0	0	1
s_2	0	0	-1	1	0	1	0	0
a_3	0	1	2	1	0	0	1	2

1/2   -1/2   -1   1/2

$\max\{1, 2, 1\} = 2 \Rightarrow x_2$ entering

we can either s_1 or a_3 as leaving but as we want to have a_3 we take this

	w	x_1	x_2	x_3	s_1	s_2	a_3	b
W	1	0	0	0	0	0	-1	0
s_1	0	3/2	0	1/2	1	0	-1/2	0
s_2	0	1/2	0	3/2	0	1	1/2	1
x_2	0	1/2	1	1/2	0	0	1/2	1

We conclude that we've found optimum and $\begin{pmatrix} s_1 \\ s_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ can be used as basic variables in original problem.

(3.) suppose $x^{(1)}, x^{(2)} \in X = \{\text{the given set}\}$

we put $x' = \lambda x^{(1)} + (1-\lambda) x^{(2)}$

$$= \begin{pmatrix} \lambda x_1^{(1)} + (1-\lambda) x_1^{(2)} \\ \lambda x_2^{(1)} + (1-\lambda) x_2^{(2)} \end{pmatrix}$$

Clearly $x_1' \geq 0$, $x_2' \geq 0$ and $x_1' \leq 1$ as

$\underbrace{\lambda}_{\geq 0} \underbrace{x_1^{(1)}}_{\geq 0} + \underbrace{(1-\lambda)}_{\geq 0} \underbrace{x_1^{(2)}}_{\geq 0} \geq 0$ (same argument for $x_2' \geq 0$.

and

$\underbrace{\lambda}_{\in[0,1]} \underbrace{x_1^{(1)}}_{\leq 1} + \underbrace{(1-\lambda)}_{\in[0,1]} \underbrace{x_1^{(2)}}_{\leq 1}$

Lastly

$x_1' + x_2' = \lambda x_1^{(1)} + (1-\lambda) x_1^{(2)} + \lambda x_2^{(1)} + (1-\lambda) x_2^{(2)}$

$= \lambda (x_1^{(1)} + x_2^{(1)}) + (1-\lambda)(x_1^{(2)} + x_2^{(2)})$

$\leq 2\lambda + (1-\lambda) 2 = 2.$

4. Denote by x_A, x_B, x_C, x_D the number of details produced of each kind.

The model is the following for maximizing the profit

$$\max \quad z = 100x_A + 100x_B + 200x_C + 100x_D$$

s.t.

$$2x_A + x_B + 2x_C + x_D \le 6 \cdot 60 = 360$$

$$4x_A + 2x_B + 2x_C \le 6 \cdot 60 = 360$$

$$2x_A + 3x_B + 2x_C + 6x_D \le 8 \cdot 60 = 480$$

$$x_C \le 100$$

$$x_A \ge 10$$

$$x_D \ge 30$$

$$x_A, x_B, x_C, x_D \ge 0.$$

5.

(a) we study the Hessian

$$\nabla f = (2(x_1-2)+2x_2-3,\ 4(2x_2+1)+2x_1-6x_2)^T$$

$$H(f) = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \quad \det(H) = 0.$$

we find the eigenvalues of H by

$$\begin{vmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 4 = 4 - 4\lambda + \lambda^2 - 4 = \lambda(\lambda-4)$$

so the eigenvalues are 0 and 4, so the function is convex as the matrix is positive semidefinite.

(b) The general normalized steepest descent direction is

$$-\nabla f / \|\nabla f\| = -\begin{pmatrix} 2(x_1-2)+2x_2-3 \\ 4(2x_2+1)+2x_1-6x_2 \end{pmatrix} \cdot \frac{1}{\|\nabla f\|}$$

for the point $(1,2)$ we get

$$-\begin{pmatrix} 2(1-2)+4-3 \\ 4(2\cdot 2+1)+2-12 \end{pmatrix} / \|\nabla f(1,2)\|$$

$$= -\begin{pmatrix} -1 \\ 16 \end{pmatrix} / \|\nabla f(1,2)\| = \frac{1}{\sqrt{101}}\begin{pmatrix} 1 \\ -10 \end{pmatrix}$$

(c) Since H is singular there is no way to find a unique Newton direction as it depends on H^{-1}!

6. See text book for dascription

$$\min f(x) = x_1^2 + x_2^2 + e^{x_1^2 + x_2^2} - x_1$$

$$-x_1 - x_2^2 + 3 \le 0$$

For a penalty example we consider

$g(x) = -x_1 - x_2^2 + 3$, and we introduce

the penalty parameter $\mu > 0$.

Then

$$\min F_\mu(x) = f(x) + \mu \max(0, g(x))^2$$

is an unconstrained optimization task

and penalized version of the

original problem.

7. We introduce the Lagrangian

$$L(x,v) = x_1^2 + 4x_2^2 - v(x_1 + x_2 - 1)$$

The dual is the

$$h(v) = v + \min_{x_1}\{x_1^2 - v x_1\} + \min_{x_2}\{4x_2^2 - v x_2\}$$

which is explictly given by solving

$$\min_{x_1} x_1^2 - v x_1 = f_1(x_1),$$

we have

$$f_1'(x_1) = 2x_1 - v = 0 \iff x_1^* = \frac{v}{2}$$

and

$$\min_{x_2} 4x_2^2 - v x_2 = f_2(x_2)$$

$$f'(x_2) = 8x_2 - v = 0 \iff x_2^* = \frac{v}{8}.$$

Hence, we obtain the dual

$$h(v) = \frac{v^2}{4} + 4\frac{v^2}{64} - v\left(\frac{v}{2} + \frac{v}{8} - 1\right)$$

$$= \frac{v^2}{4} + \frac{v^2}{16} - \frac{v^2}{2} - \frac{v^2}{8} + v$$

$$= v - \frac{v^2}{4} - \frac{v^2}{16} = v - \frac{5v^2}{16}.$$

The dual problem is

$$\max_{v \geq 0} v - \frac{5v^2}{16},$$ which we solve by consitering

$$h'(v) = 1 - \frac{10v}{16} = 1 - \frac{5v}{8} = 0 \iff v = \frac{8}{5}.$$

Going back to $x_1^* = \frac{v}{2} = \frac{8/5}{2} = 4/5$, $x_2^* = \frac{8/5}{8} = \frac{1}{5}$

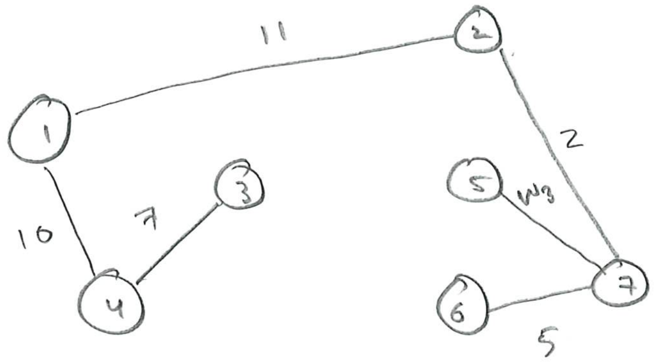
The sought solution is thus

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4/5 \\ 1/5 \end{pmatrix} \Rightarrow f(x) = \left(\frac{4}{5}\right)^2 + 4\left(\frac{1}{5}\right)^2$$

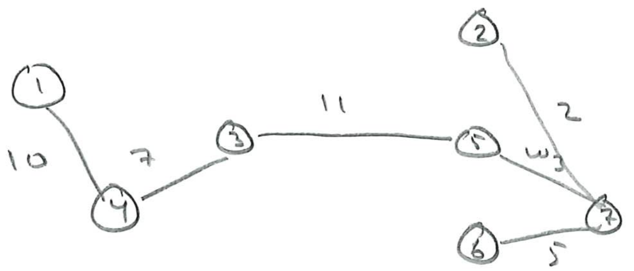
$$= \frac{16}{25} + \frac{4}{25} = \frac{20}{25} = \frac{4}{5}.$$

(8.) a) True, one way to see this is by
(7) considering the two cases $w_1 = 11$, and $w_1 > 11$.

In the latter case we get the minimal spanning tree



If $w_1 = 11$ there are two minimal spanning trees



and the previous.

However, the cost for these are the same

$10 + 7 + 11 + 2 + 5 + w_3 = 35 + w_3$.

8 (b) we rewrite $x_1 \geq 0$, $x_2 \geq 0$ to $-x_1 \leq 0$, $-x_2 \leq 0$

(7) x is a KKT-point if the following are satisfied

$$-\nabla f = y_1 \nabla g_1 + y_2 \nabla g_2 + y_3 \nabla g_3 \qquad (+)$$

$$y_1, y_2, y_3 \geq 0$$

$$x_1^2 + x_2 - 1 \leq 0 \qquad (*)$$

$$-x_1 \leq 0$$

$$-x_2 \leq 0$$

$$y_1(x_1^2 + x_2 - 1) = 0 \qquad (**)$$

$$-y_2 x_1 = 0$$

$$-y_3 x_2 = 0$$

Since $x_1 \neq 0$, $x_2 \neq 0$ we get $y_2 = y_3 = 0$ by complementarity regarding $(*)$ we get

$$\left(\frac{\sqrt{3}-1}{2}\right)^2 = \frac{1}{4}\left(3 - 2\sqrt{3} + 1\right) = 1 - \frac{\sqrt{3}}{2}$$

so

$x_1^2 + x_2 = 1 - \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = 1$, so $(*)$ and $(**)$ are satisfied.

Further investigation of $(+)$ yields

$$\nabla f = \left(-\frac{1}{1+x_1}, -1\right)^T, \quad \nabla g_1 = (2x_1, 1)^T, \quad \nabla g_2 = (-1, 0)^T, \quad \nabla g_3 = (0, 1)^T$$

↪

8(7) (b) cont.

Since $y_2 = y_3 = 0$ we get

$$-\nabla f = y_1 \nabla g_1$$

or equivalently

$$\frac{1}{1+x_1} = y_1 2 x_1 \qquad (1)$$

and

$$1 = y_1 1 \Rightarrow y_1 = 1$$

so left to prove is that $y_1 = 1$ satisfies (1)

$$\frac{1}{1+\left(\frac{\sqrt{3}-1}{2}\right)} = \frac{1}{\frac{1+\sqrt{3}}{2}} = \frac{2}{1+\sqrt{3}} \cdot \frac{(1-\sqrt{3})}{(1-\sqrt{3})}$$

$$= \frac{2(1-\sqrt{3})}{-2} = \sqrt{3}-1 = \text{LHS of } (1)$$

$$y_1 2 x_1 = 1 \cdot 2 \cdot \frac{\sqrt{3}-1}{2} = \sqrt{3}-1 = \text{RHS of } (1).$$

Since LHS = RHS we've proven that

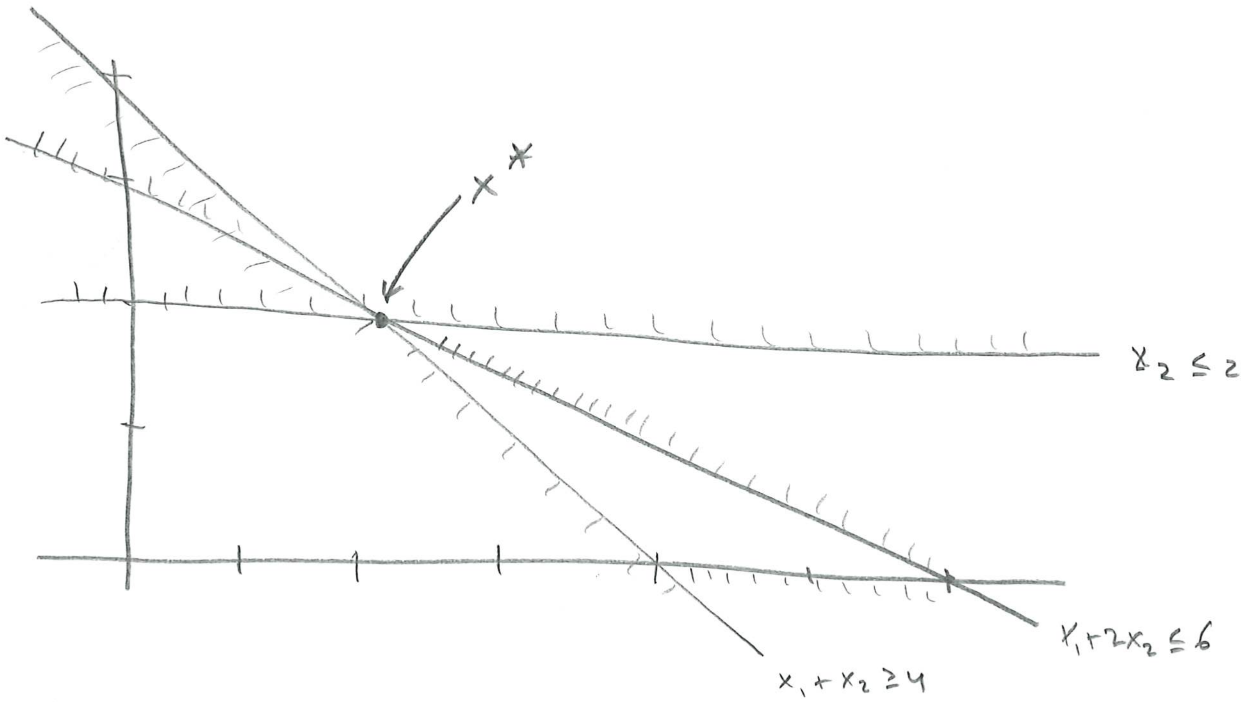
$x = \left(\frac{\sqrt{3}-1}{2}, \frac{\sqrt{3}}{2}\right)$ satisfies all KKT-conditions

and is thus a KKT-point.

8
(7)

(c) A constraint is redundant if it can be removed without changing the feasible set

A constraint is active if satisfied with equality for a given point.



The second constraint is certainly active as $2 + 2 \cdot 2 = 6$ but it is <u>not</u> redundant.

So, the statement is false, (but it is true for the third constraint).

(d) <u>FALSE!</u> As,

$$\nabla f = \left(2(x_1-2) + 2x_2,\ 4(2x_2+1) + 2x_1\right)^T$$

$$\nabla f(0,0) = (-4, 4)$$

$$-t\,\nabla f(0,0) = \begin{pmatrix} 4t \\ -4t \end{pmatrix}$$

$$\varphi(t) = f\begin{pmatrix} 4t \\ -4t \end{pmatrix} = (4t-2)^2 + (-8t+1)^2 - 32t^2$$

min $\varphi(t)$ is given by $\varphi'(t) = 0$ $= 8(4t-2) - 16(-8t+1) - 64t$

$= 32t - 16 + 128t - 16 - 64t$

$= 96t - 32$, so $t = \frac{32}{96} = \frac{1}{3} \neq 1$.