Linnæus University
Department of mathematics
Jonas Nordqvist

Exam in Optimization methods, 2ma404, 7.5 hp
January 10, 2023, 8.00–13.00

Every answer needs to be carefully motivated! 50% is needed to pass the exam (grade E), 75% for C and 90% for A.
Allowed aids: One A4-sheet of handwritten notes may be brought to the exam. English dictionary. No calculator is allowed!

1. Consider the following linear programming-problem (6p)

$$\begin{array}{lrl} \max & z = & 2x_1 + 3x_2 \\ \text{s.t.} & & x_1 \leq 2 \\ & & x_2 \leq 2 \\ & & 2x_1 + x_2 \leq 3 \\ & & x_1 + x_2 \geq 1 \\ & & \mathbf{x} \geq 0 \end{array}$$

(a) Solve the problem graphically.

(b) Formulate the dual problem.

(c) Solve the dual problem using the simplex method and initial guess $(y_2, y_3)^T = (2, 1)^T$ as the basic variables.

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2. Solve the phase-1 problem of the following LP-problem, and conclude whether the feasible region is empty or not (you do not need to solve the problem afterwards) (4p)

$$\begin{array}{lrl} \max & z = & 2x_1 + 3x_2 + 4x_3 \\ \text{s.t.} & & 2x_1 + x_2 + x_3 \leq 1 \\ & & + x_2 - x_3 \geq 0 \\ & & x_1 + 2x_2 + x_3 = 2 \\ & & \mathbf{x} \geq 0 \end{array}$$

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3. Show, by using the definition of convex sets, that the set (3p)

$$\left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 2, x_1 \leq 1\right\}$$

is convex.

4. A company wants to optimize its production. The company produces four different details A, B, C and D in three different machines (cutting, punching, molding). The time that each machine is used for the different details is presented in Table 1. The company is confident that any produced amount will be sold, expect detail C which is slightly more expensive and should only be produced with at most 100 details. An important customer requires that at least 30 units of detail D will be produced, and anothter requires at least 10 details of A to be produced.

	Cutting	Punching	Molding
A	2	4	2
B	1	2	3
C	2	2	2
D	1	0	6

Table 1: Time in minutes needed for the different details in the different machines.

The staff is specific for each machine and cannot be replaced so there is no possibility change staff between machines. The total amount of staff hours on the different machines per day are 6, 6 and 8 for cutting, punching and molding respectively.

All details gives a profit of 100 SEK, except detail C which gives the profit 200 SEK. Formulate this as an linear programming problem, optimizing for largest possible profit (you don't have to solve the problem). (4p)

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5. Consider the unconstrained optimization problem (4p)

$$\min \quad f(x) = (x_1 - 2)^2 + (2x_2 + 1)^2 + 2x_1x_2 - 3x_1 - 3x_2^2.$$

(a) Is the problem convex?

(b) Compute the steepest descent direction, normalized to unit length.

(c) Consider the point $x = (1, 2)^T$ as a initial point. Perform, if possible, one iteration with Newton's method.

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6. Give a conceptual description of *penalty* and *barrier* methods in constrained non-linear optimization. Without solving the problem set up the following problem using either a penalty or barrier. (3p)

$$\begin{aligned} \min \quad f(x) = &\quad x_1^2 + x_2^2 + e^{x_1^2 + x_2^2} - x_1 \\ \text{s.t.} &\quad -x_1 - x_2^2 + 3 \leq 0 \end{aligned}$$

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7. Use Lagrange duality to solve the problem (4p)

$$\begin{aligned} \min \quad f(x) = &\quad x_1^2 + 4x_2^2 \\ \text{s.t.} &\quad x_1 + x_2 \geq 1 \end{aligned}$$

8. In each of the following subproblems determine if the statement is true or false. Motivate your answer. (7p)

(a) If $w_1 > 10$ and $w_2 > 12$ and $w_3 < 8$ then the cost of the minimal spanning tree in Figure 1 is $35 + w_3$.

(b) The point $x = \left(\frac{\sqrt{3}-1}{2}, \frac{\sqrt{3}}{2}\right)^T$ is a KKT-point for the minimization problem

$$\begin{array}{lll} \min & f(x_1, x_2) = -\ln(x_1+1) - x_2 & \\ \text{s.t.} & x_1^2 + x_2 & \leq 1 \\ & x_1 & \geq 0 \\ & x_2 & \geq 0 \end{array}$$

(c) Define a feasible region by the linear constraints

$$\begin{aligned} x_1 + x_2 &\geq 4 \\ x_1 + 2x_2 &\leq 6 \\ x_2 &\leq 2 \\ x_1, x_2 &\geq 0. \end{aligned}$$

Suppose that $x^* = (2,2)^T$ is optimal. Then the second constraint is *active* but *redudant.*

(d) Let $f(x_1, x_2) = (x_1 - 2)^2 + (2x_2 + 1)^2 + 2x_1x_2$, and $\hat{\mathbf{x}} = (0,0)^T$ then a minimal value of $f(\hat{\mathbf{x}} - t\nabla f(\hat{\mathbf{x}}))$ with respect to t is given by $t = 1$.

Good luck!

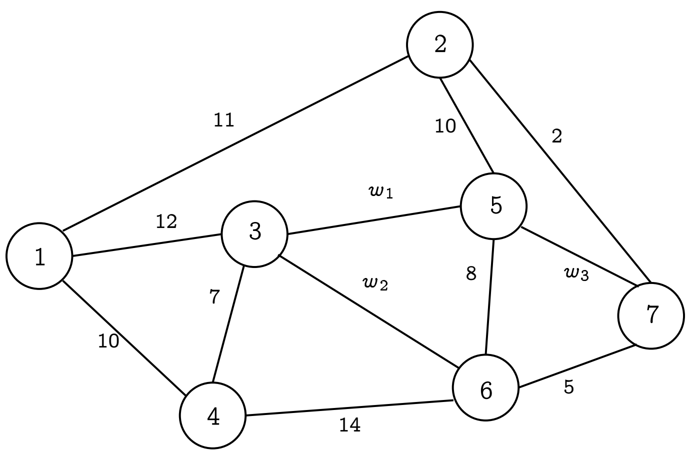

Figure 1: An undirected weighted graph