2MA918 Laboration I

Samuel Berg (sb224sc)

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1 Exercise 1: Warm-up LP-problem

This exercise examines a television production optimization problem for a company producing two types of TVs (type A and type B) with limited resources across three production stages.

i) Define the model

The linear programming model is formulated as follows:

Decision variables:

- x_1: Number of type A TVs to produce
- x_2: Number of type B TVs to produce

Objective function:

$$\text{maximize } z = 700x_1 + 1000x_2 \tag{1}$$

Constraints:

$$3x_1 + 5x_2 \leq 3900 \quad \text{(Stage I)} \tag{2}$$
$$x_1 + 3x_2 \leq 2100 \quad \text{(Stage II)} \tag{3}$$
$$2x_1 + 2x_2 \leq 2200 \quad \text{(Stage III)} \tag{4}$$
$$x_1, x_2 \geq 0 \tag{5}$$

In matrix form: maximize $c^T x$ subject to $Ax \leq b$, $x \geq 0$, where:

$$c = \begin{bmatrix} 700 \\ 1000 \end{bmatrix}, \quad A = \begin{bmatrix} 3 & 5 \\ 1 & 3 \\ 2 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 3900 \\ 2100 \\ 2200 \end{bmatrix} \tag{6}$$

ii) Restricted convex set region

The feasible region is bounded by the constraint lines and the non-negativity conditions. Figure 1 shows the three constraint lines along with the x_1 and x_2 axes.

iii) Study the figure of ii)

From the figure, we can identify five vertices that enclose the feasible region:

- $v_1 = (0, 0)$ — Origin
- $v_2 = (1100, 0)$ — Intersection of $x_2 = 0$ and Stage III constraint
- $v_3 = (800, 300)$ — Intersection of Stage I and Stage III constraints
- $v_4 = (300, 600)$ — Intersection of Stage I and Stage II constraints
- $v_5 = (0, 700)$ — Intersection of $x_1 = 0$ and Stage II constraint

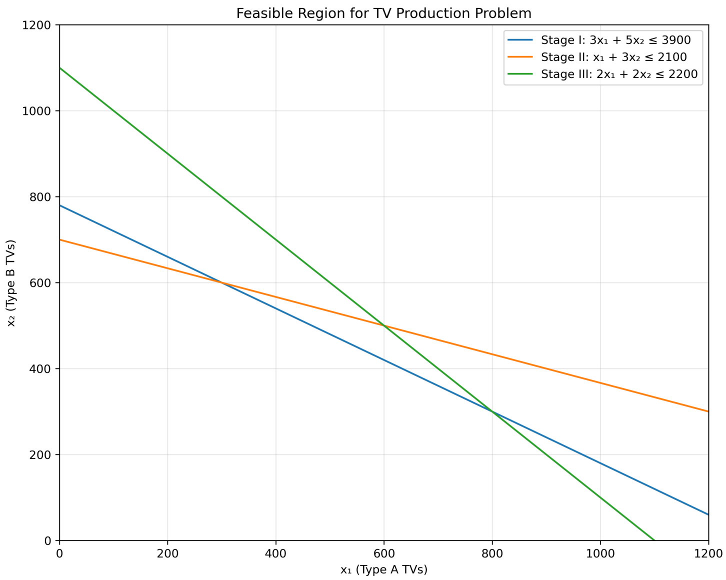

Figure 1: Feasible region for the TV production problem

iv) Adding level curves to convex hull plot

Figure 2 shows the feasible region with level curves of the objective function superimposed. The level curves are lines of constant profit $z = 700x_1 + 1000x_2 = k$ for increasing values of k. The curves have increasing red intensity for higher profit values. The maximum profit is achieved at the vertex where the highest level curve touches the feasible region, which is $v_3 = (800, 300)$.

v) Verify in all extreme points

Evaluating the objective function at each vertex:

$$\begin{aligned}
z(v_1) &= 700(0) + 1000(0) = 0 \\
z(v_2) &= 700(1100) + 1000(0) = 770{,}000 \\
z(v_3) &= 700(800) + 1000(300) = 860{,}000 \\
z(v_4) &= 700(300) + 1000(600) = 810{,}000 \\
z(v_5) &= 700(0) + 1000(700) = 700{,}000
\end{aligned}$$

The maximum value is $z^* = 860{,}000$ at vertex $v_3 = (800, 300)$.

vi) Find the maximum

Using `scipy.optimize.linprog` with the HiGHS method to solve the problem:

- Optimal solution: $x^* = (800, 300)$

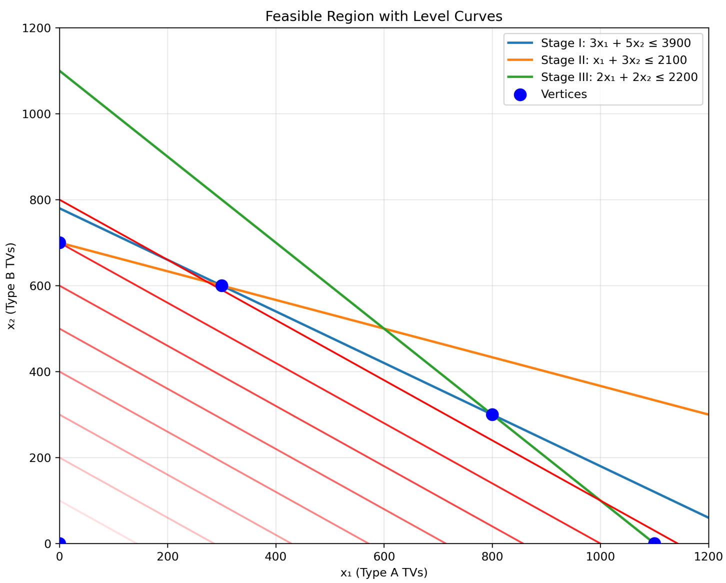

Figure 2: Feasible region with level curves of the objective function

- Maximum profit: $z^* = 860{,}000$

This confirms the result from the graphical analysis. The company should produce 800 type A TVs and 300 type B TVs to maximize profit at $860,000.

vii) Problem in *standard form*

Converting to standard form by introducing slack variables $s_1, s_2, s_3 \geq 0$:

$$
\begin{aligned}
3x_1 + 5x_2 + s_1 &= 3900 \qquad (7)\\
x_1 + 3x_2 + s_2 &= 2100 \qquad (8)\\
2x_1 + 2x_2 + s_3 &= 2200 \qquad (9)\\
x_1, x_2, s_1, s_2, s_3 &\geq 0 \qquad (10)
\end{aligned}
$$

Solving in standard form yields:

- Optimal solution: $x^* = (800, 300)$
- Slack variables: $s^* = (0, 400, 0)$
- Maximum profit: $z^* = 860{,}000$

The slack variable $s_2 = 400$ indicates that Stage II has 400 hours of unused capacity. Stage I and Stage III are fully utilized (slack = 0), making them binding constraints.

2 Exercise 2: Large LP-problems

This exercise investigates the computational performance of different optimization methods on large-scale linear programming problems.

i) Matrix and vectors solution, why?

Matrix and vector notation provides several critical advantages for large LP problems:

1. **Compact representation:** Instead of writing thousands of individual equations, we represent them as $Ax \leq b$.
2. **Computational efficiency:** Matrix operations are highly optimized in numerical libraries (e.g., BLAS, LAPACK), enabling fast computation.
3. **Scalability:** The same algorithm handles problems of any size without modification.
4. **Mathematical clarity:** The structure of the problem is clear, facilitating theoretical analysis.
5. **Implementation simplicity:** Reduces code complexity and potential errors when dealing with large systems.

ii) Solving example with simplex

For a small test problem with $m = n = 10$, the simplex method solves the problem in approximately 5–6 milliseconds on the test machine. This demonstrates that for small problems, the simplex method is quite efficient.

iii) Get simplex average time exceeds 1 second

By systematically increasing the problem size and averaging over 5 trials, we find that the simplex method exceeds 1 second average time at:

$$\boxed{m = n = 150} \tag{11}$$

The computational cost grows approximately cubically with problem size, as expected for the simplex method.

iv) Replace simplex with more sophisticated method

Using the HiGHS method (a state-of-the-art interior point algorithm), we find that the average time exceeds 1 second at:

$$\boxed{m = n = 1400} \tag{12}$$

This represents approximately a **9.3× performance improvement** over the simplex method. Figure 3 shows the comparison between both methods.

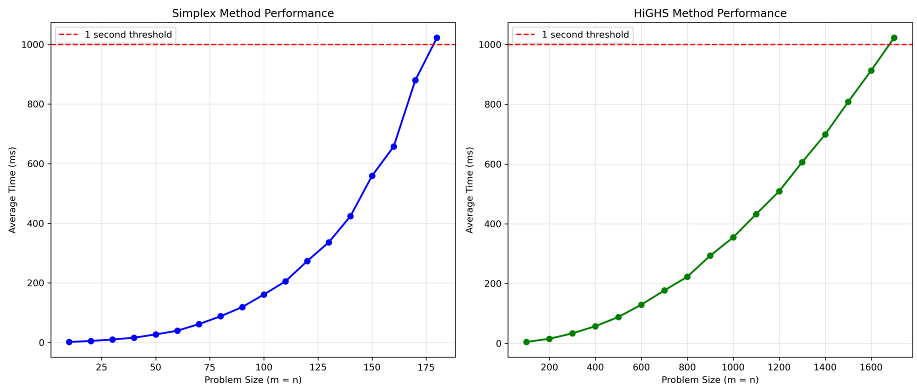

Figure 3: Performance comparison: Simplex vs HiGHS method

The HiGHS method demonstrates superior scalability for large problems, making it the preferred choice for industrial-scale optimization tasks.

3 Exercise 3: Sensitivity analysis

This exercise performs sensitivity analysis on the TV production problem from Exercise 1 to understand how changes in parameters affect the optimal solution.

i) Solve the dual problem and verify

The dual problem is formulated as:

$$\begin{aligned} \text{minimize} \quad & 3900y_1 + 2100y_2 + 2200y_3 && (13) \\ \text{subject to} \quad & 3y_1 + y_2 + 2y_3 \geq 700 && (14) \\ & 5y_1 + 3y_2 + 2y_3 \geq 1000 && (15) \\ & y_1, y_2, y_3 \geq 0 && (16) \end{aligned}$$

Solving the dual problem yields:

- Dual optimal solution: $y^* = (150, 0, 125)$
- Dual optimal value: $w^* = 860{,}000$

Verification: The primal optimal value equals the dual optimal value ($z^* = w^* = 860{,}000$), confirming strong duality.

ii) Shadow price of various constraints

The shadow prices (dual variables) are:

- Stage I (y_1): 150
- Stage II (y_2): 0
- Stage III (y_3): 125

Interpretation:

- Stage I has shadow price 150, meaning each additional hour increases profit by $150.
- Stage II has shadow price 0, indicating it has slack capacity (non-binding constraint).
- Stage III has shadow price 125, meaning each additional hour increases profit by $125.

iii) 100 extra working hours

Testing the addition of 100 hours to each stage:

Stage	Shadow Price	Predicted Increase	Actual Increase
Stage I	150	\$15,000	\$15,000
Stage II	0	\$0	\$0
Stage III	125	\$12,500	\$12,500

Recommendation:

- **INVEST** in Stage I (highest shadow price of 150)
- **DO NOT invest** in Stage II (shadow price of 0, has excess capacity)
- Stage III is also valuable (shadow price of 125)

iv) Price increase to change optimal solution

Testing price increases for type B TVs, we find that the optimal solution changes when the price increases by:

$$\boxed{\Delta p_B = \$200} \tag{17}$$

At price $p_B = \$1200$ (original \$1000 + \$200), the optimal solution shifts from $(800, 300)$ to $(300, 600)$, indicating a switch to producing more type B TVs.

v) New TV, should be produced or not?

Type C TV specifications:

- Profit: \$1,350
- Production times: $(7, 4, 2)$ hours for stages I, II, III

The reduced cost for type C is:

$$r_C = c_C - y^* \cdot a_C = 1350 - (150 \times 7 + 0 \times 4 + 125 \times 2) = 1350 - 1300 = 50 \tag{18}$$

Since the reduced cost is positive ($r_C = 50 > 0$), **type C should be PRODUCED**.

Solving the extended problem with type C included:

- Optimal production: $(950, 0, 150)$ units of types A, B, and C
- New optimal profit: \$867,500

This confirms that producing type C increases profit by \$7,500.

vi) Quality inspection working time increase

Quality inspection times:

- Type A: 0.5 hours
- Type B: 0.75 hours
- Type C: 0.1 hours

With optimal production of $(950, 0, 150)$ units:

$$\text{Total inspection hours} = 950 \times 0.5 + 0 \times 0.75 + 150 \times 0.1 = 475 + 0 + 15 = 490 \text{ hours} \tag{19}$$

Answer: The company must add $\boxed{\text{490 hours}}$ to the quality inspection line to maintain current production levels without interference.