Only answers do not give any points. Motivate your answers clearly. Allowed aids: Pocket calculator with empty memory and attached formulae sheet.

1. (20p) Solve the following problem by the simplex method starting at the initial point $(3, 0)$:

$$\begin{aligned} \min z = -5x_1 - 4x_2 \\ x_1 + 2x_2 \leq 8 \\ 3x_1 + 2x_2 \leq 9 \\ x_1, x_2 \geq 0. \end{aligned}$$

2. (20p) Consider the problem

$$\begin{aligned} \min z = 9x_1 - x_2 + 10x_3 \\ x_1 + x_2 - x_3 \leq 3 \\ x_1 + 3x_3 \geq -1 \\ x_1, x_2 \geq 0, x_3 \leq 0. \end{aligned}$$

   (a) Formulate the dual problem.
   (b) Solve the dual problem graphically.
   (c) Solve by help of complementarity the solution(s) to the dual problem.

3. (20p) Consider the problem

$$\begin{aligned} \min z = x_1^2 + x_2^2 - 2x_1 + 2x_2 + 2 \\ x_1, x_2 \in \mathbb{R}. \end{aligned}$$

   Show that the steepest descent method with exact line search converges to the optimal solution in just one iteration.

4. (20p) In finance on finding a portfolio of n stocks with smallest variance, the following problem is crucial:

$$\begin{aligned} \min z = \tfrac{1}{2} \textstyle\sum_{i,j=1}^{n} q_{ij} x_i x_j \\ x_1 + \ldots + x_n = 1. \end{aligned}$$

   where q_{ij} is the covariance of returns for stock i and j, and x_i is a fraction of the investment at stock i. Let $\mathbf{Q} = (q_{ij})$ be the $n \times n$ covariance matrix which is symmetric and positive semidefinite.

   (a) Show that the minimization problem can be reduced to

$$\begin{aligned} \mathbf{Q}\mathbf{x} &= v\mathbf{u} \\ \mathbf{u}^T\mathbf{x} &= 1 \end{aligned}$$

   where $\mathbf{u} = (1, \ldots, 1)^T \in \mathbb{R}^n$, $\mathbf{x} = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ and $v \in \mathbb{R}$.

(b) Assume furthermore that Q is positive definite. Show that

$$\mathbf{x} = \frac{\mathbf{Q}^{-1}\mathbf{u}}{\mathbf{u}^T\mathbf{Q}^{-1}\mathbf{u}}.$$

5. (20p) The company Woff produces two different dog fodders, Classic and Premium. The demand of Classic is 600 ton and the demand of Premium is 300 ton. The fodder are made of 'raw materials' chicken, wheat, rice och corn. The contents of nutrients in respectively raw material (in weight percent) and the cost (per kg) are given in the table below.

Raw material	Protein	Carbohydrate	Mineral X	Cost
Chicken	27	0	6	8
Wheat	15	65	0.018	1.5
Rice	3	25	0.042	2
Corn	4	20	0.003	2.5

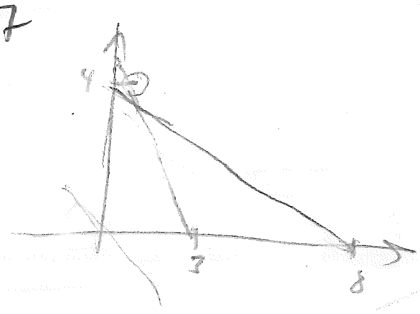
The content must fulfill the following minimum demand on nutrients (given in weight percentage):

Fodder	Protein	Carbohydrate	Mineral X
Classic	18	50	1
Premium	20	55	2

Formulate the optimization problem that minimizes the raw material costs and fulfill demand and the nutrient content.

① 2018-05-27

$$\begin{cases} x_1 + 2x_2 + x_3 = 8 \\ 3x_1 + 2x_2 + x_4 = 9 \end{cases}$$



$x_1 = 3,\ x_2 = 0$

$$\begin{cases} x_3 = 8 \\ 9 + x_4 = 9 \end{cases} \qquad \begin{cases} x_3 = 8 \\ x_4 = 0 \end{cases}$$

$x_1 = 3 - \frac{2}{3}x_2 - \frac{1}{3}x_4$ $\qquad x_3 = 8 - x_1 - 2x_2 = 8 - (3 - \frac{2}{3}x_2 - \frac{1}{3}x_4) - 2x_2$

$z = -5x_1 - 4x_2 = -5(3 - \frac{2}{3}x_2 - \frac{1}{3}x_4) - 4x_2$ $\qquad = -\frac{4}{3}x_2 + \frac{1}{3}x_4 + 5$

$= -\frac{2}{3}x_2 + \frac{5}{3}x_4 - 15$ $\qquad 3x_3 = -4x_2 + x_1 + 15$

	z	x_1	x_2 ↑	x_3	x_4				
z	1		$\frac{2}{3}$		$-\frac{5}{3}$	-15			6
x_1		3	2		1	9	(9/2)	(-2)	
→ x_3			4	3	-1	15	(15/4)	(1)	(-1)

	z	x_1	x_2	x_3	x_4	
z	6			-3	-9	-105
x_1		-6		3	-3	-3
x_2			4	3	-1	15

$x_1^* = \frac{1}{2}$ $\qquad z^* = -35$

$x_2^* = \frac{15}{4} = 3{,}5$

① $\min z = 9x_1 - x_2 + 10x_3$

$x_1 + x_2 - x_3 \leq 3$

$x_1 \quad + 3x_3 \geq -1$

$x_1, x_2 \geq 0, \quad x_3 \leq 0$

a) $\max w = 3v_1 - v_2$

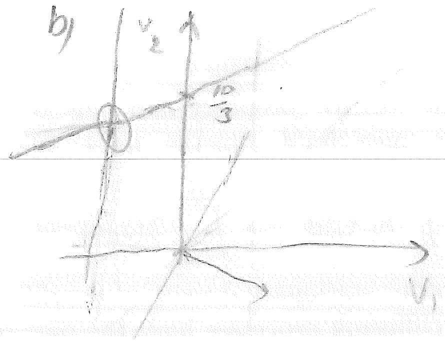
$v_1 + v_2 \leq 9$

$v_1 \quad \leq -1$

$-v_1 + 3v_2 \geq 10$

$v_1 \leq 0, \quad v_2 \geq 0$

b)



$v_1 = -1$

$v_2 = 3$

c) $x_1(v_1 + v_2 - 9) = 0$ $\qquad v_1(x_1 + x_2 - x_3 - 3) = 0$

$x_2(v_1 + 1) = 0$ $\qquad v_2(x_1 + 3x_3 + 1) = 0$

$x_3(-v_1 + 3v_2 - 10) = 0$

Candidate: $v_1 = -1,\ v_2 = 3,\ x_1 = 0,\ \begin{cases} x_2 - x_3 = 3 \\ 3x_3 = -1 \end{cases} \quad \begin{cases} x_2 = 3 - \frac{1}{3} = \frac{8}{3} \\ x_3 = -\frac{1}{3} \end{cases}$

Then $z = -\frac{8}{3} - \frac{10}{3} = -6, = -3-3 = w.$

By the Complementary Slackness Thm, $x^* = (0, \frac{8}{3}, -\frac{1}{3})$ is optimum and $z^* = -6$

$$f = x_1^2 + x_2^2 - 2x_1 + 2x_2 + 2 = (x_1-1)^2 + (x_2+1)^2 \quad \text{minimum at } (1,-1).$$

$$\nabla f = \begin{pmatrix} 2x_1 - 2 \\ 2x_2 + 2 \end{pmatrix} = 2\begin{pmatrix} x_1 - 1 \\ x_2 + 1 \end{pmatrix}.$$

$$x^{(0)} = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}, \quad d^{(1)} = -\nabla f(x^{(0)}) = -2\begin{pmatrix} x_1^0 - 1 \\ x_2^0 + 1 \end{pmatrix}$$

$$x^{(1)} = x^{(0)} + t d^{(1)} = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} - 2t\begin{pmatrix} x_1^0 - 1 \\ x_2^0 + 1 \end{pmatrix} = (1-2t)\begin{pmatrix} x_1^0 - 1 \\ x_2^0 + 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$f(x^{(1)}) = (1-2t)^2 x_1^0 + (1-2t)^2 x_2^0 = 0 \quad \text{for } t = 1/2$$

(4)

a) The problem is of the form $\min z = f(x)$, $g(x) = 1$

$f(x) = x^T Q x$

$g(x) = x^T u$

$\nabla f = Qx \qquad \nabla g = u$

KKT-points:

$$\begin{cases} \nabla f = v \nabla g \\ v(g(x) - 1) = 0 \\ v \text{ free} \end{cases} \qquad \begin{cases} Qx = vu \\ v(x^T u - 1) = 0 \end{cases}$$

$v = 0$: $Qx = 0$, $x^T Q x = 0 \Rightarrow x = 0$ but $x^T u = 1$

$v \neq 0$: $\begin{cases} Qx = vu \\ x^T u = 1 \end{cases}$

b) $\begin{cases} Qx = vu \\ x^T u = 1 \end{cases}$

$x = vQ^{-1}u$

$x^T u = v u^T Q^{-1} u$

$1 = v u^T Q^{-1} u$

$v = \dfrac{1}{u^T Q^{-1} u}$

$x = \dfrac{Q^{-1} u}{u^T Q^{-1} u}$

Chicken x_{11} Classic, x_{12} Premium (kg) Wheat x_{21} Classic, x_{22} Premium Rice x_{31} Class, x_{32} Prem Corn x_{41} Class, x_{42} Prem

$$\min\ z = 8(x_{11}+x_{12}) + 1{,}5(x_{21}+x_{22}) + 2(x_{31}+x_{32}) + 2{,}5(x_{41}+x_{42}) \quad \text{Cost}$$

$x_{11} + x_{21} + x_{31} + x_{41} = 600$ Classic demand

$x_{12} + x_{22} + x_{32} + x_{42} = 3600$ Premium demand

$0{,}27x_{11} + 0{,}15x_{21} + 0{,}03x_{31} + 0.04x_{41} \geq 0{,}18(x_{11} + x_{21} + x_{31} + x_{41})$

$0x_{11} + 0.65x_{21} + 0{,}25x_{31} + 0{,}20x_{41} \geq 0{,}50(x_{11} + x_{21} + x_{31} + x_{41})$

$6x_{11} + 0{,}018x_{31} \ldots + 0.005x_{41} \geq 0.01(x_{11} + \ldots + x_{41})$

$0.27x_{12} + \ldots + 0.04x_{42} \geq 0{,}20(x_{12} + \ldots + x_{42})$

$0x_{12} + \ldots + 0{,}20x_{42} \geq 0{,}55(x_{12} + \ldots + x_{42})$

$6x_{12} + \ldots + 0.005x_{42} \geq 0.02(x_{12} + \ldots + x_{42})$