Linnæus University
DFM
Peter Nyman

Examination in Optimization methods, 2MA104, 4.5 hp
Monday November 7, 2012, 8:00-13:00

For full point at the problem the solutions should be motivated and the calculations clear.
Allowed aids: Calculator and attached paper (Formula Collection 2MA104).

1. Consider the problem

$$\max z = 20y + x$$

$$10y + 50x \leq 200$$

$$y + 3x \leq 15$$

$$y \leq 10$$

$$x, y \geq 0$$

(a) Solve the problem graphically. (2p)

(b) Solve the Simplex method for this problem. (2p)

(c) Formulate the dual problem. (2p)

(d) Formulate phase-1 for the dual problem and perform one iteration by the Simplex method (phase-1). (2p)

2. (a) Show that the function

$$f(x, y) = x^2 + y^2 - xy$$

is convex. (1p)
(b) Consider the function

$$f(x, y) = x^a + y^b - cxy$$

where a, b, c are positive integers. Determine all values of a, b, c, such that this function is convex. (3p)

3. Consider the problem

$$\min f(x, y) = x^2 + e^{x+y} + xy^2 + y^3$$

$$\text{s.t.} \quad x + y = 1$$

(a) Find all KKT-points. (4p)

(b) Use the Hessian to determine if these points are/this point is local minimum points/point. (2p)

TURN PAGE PLEASE!

4. Consider the problem

$$\min f(x_1,\ x_2) = x_1^2 + 3x_2^2$$

$$\text{s.t.} \quad x_1 + x_2 = 4.$$

Introduce an appropriate penalty function and do one iteration. Use penalty parameter $\mu = 1$ (3p)

5. Consider the problem

$$\min f(x, y, z) = x^2 + y^2 - x + y + z^2$$

(a) Solve the problem by using the steepest descent method. Choose the starting point $(1/2, 1/2, 0)$. (3p)
(b) Solve the problem by using the Newton method. Choose the starting point $(1/2, 1/2, 0)$. (2p)

6. The company "Funny" will build an amusement park. Help the company "Funny" to formulate a mathematical model to maximize the profit every year. Assume that the cost to build the amusement park is independent of choice of entertainment attractions. The amusement park can have at most 6 entertainment attractions. The maximum budget is 10 million SEK/Year. They can hire at most 10 people for operating the amusement attractions. The total area of the amusement park is 10000 m^2. The park is open 10.00-18.00, four months during the summer. The details about the cost and so on are described in Table 1.

Attraction (No.)	Staff (No.)	Area (m^2)	Profit (SEK/Year)	Cost to run (SEK/Year)
1	3	2000	100000	90000
2	2	1500	120000	115000
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Table 1: Cost, profit and more

Observe that the amusement park can have several attractions of the same kind, but then the profit for this kind of attraction is reduced by 20 percent for each attraction of the same kind that is added. (4p)

GOOD LUCK!

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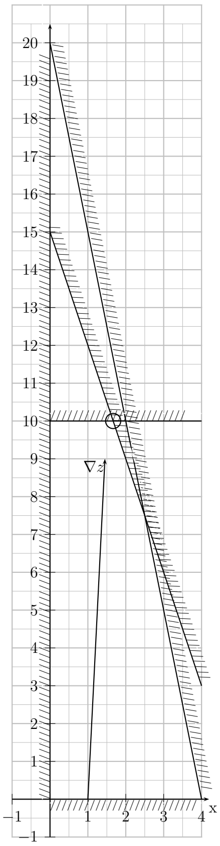
1. Consider the problem

$$\max z = 20y + x$$

$$\begin{aligned} 10y + 50x &\le 200 \\ y + 3x &\le 15 \\ y &\le 10 \\ x, y &\ge 0 \end{aligned}$$

Solution 1 (a):



Graphically solution.

The constrains $y + 3x \leq 15$ and $y \leq 10$ are active . Therefore is $10 = 15 - 3x \Leftrightarrow x = 5/3$ and the optimal point will be $(x^*, y^*) = (5/3, 10)$ with the optimal value $z^* = 20y^* + x^* = 605/3$.

Solution 1 (b):

Write the problem in standard form.

$$\begin{array}{lllllllllll}
\max & z & = & x & + & 20y & & & & & \\
\text{s.t} & & & 5x & + & y & + & s_1 & & = & 20 \\
& & & 3x & + & y & & + & s_2 & = & 15 \\
& & & & & y & & & + \; s_3 & = & 10
\end{array}$$

The simplex method:

bas	z	x	$y\downarrow$	s_1	s_2	s_3	b
z	1	-1	$\underline{-20}$				
s_1		5	1	1			20
s_2		3	1		1		15
$\leftarrow s_3$			1			1	10

bas	z	$x\downarrow$	y	s_1	s_2	s_3	b
z	1	1-				20	200
s_1		5		1		-1	10
$\leftarrow s_2$		3		1	-1		5
y			1			1	-0

bas	z	x	y	s_1	s_2	s_3	b
z	1				1/3	59/3	605/3
s_1				1	2/3		5/3
x		1			1/3	-1/3	5/3
y			1				10

The optimal point will be $(x^*, y^*) = (5/3, 10)$ with the optimal value $z^* = 605/3$

Solution 1 (c):

$$\min w = 20v_1 + 5v_2 + 4v_3$$

$$v_1 + v_2 + v_3 \geq 1$$

$$5v_1 + v_2 + v_3/3 \geq 3$$

$$v_1, v_2, v_3 \geq 0$$

Solution 1 (d):

The problem in standard form

$$\begin{array}{llllllllllllllll}
\min & w & = & 200v_1 & + & 15v_2 & + & 10v_3 & & & & & & & & \\
\text{s.t} & & & 10v_1 & + & v_2 & + & v_3 & + & s_1 & & & + \; a_1 & & = & 20 \\
& & & 50v_1 & + & 3v_2 & + & & & & + \; s_2 & & & + \; a_2 & = & 1.
\end{array}$$

The simplex method:

bas	w	v_1	v_2	v_3	s_1	s_2	a_1	a_2	b
$(w$	1						-1	$-1)$	
w	1	15	4	1	1		1		21
s_1		10	1	1	1		1		20
s_2		5	3			1		1	1

2. Show that the function

$$f(x, y) = x^2 + y^2 - xy$$

is convex. (1p)

Consider the function

$$f(x, y) = x^a + y^b - cxy$$

where a, b, c are positive constants. Determine all values of a, b, c, such that this function is convex. (3p)

Solution 2 (a):

The Hessian matrix will be

$$H = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

and this gives the eigenvalues $\lambda_1 = 1 > 0 \text{and} \lambda_2 = 3 > 0$. We have calculate $\det(H - I\lambda) = (2-\lambda)^2 - 1 = 0 \Leftrightarrow \begin{cases} \lambda_1 = 1, \\ \lambda_2 = 3 \end{cases}$ Therefore the Hessian matrix is positive definite and the problem is convex.

Solution 2 (b):

The Hessian matrix will be

$$H = \begin{pmatrix} a(a-1)x^{a-2} & -c \\ -c & b(b-1)y^{b-2} \end{pmatrix}$$

Then for $a = 1$ we have $\det h_1 = 0$ and $\det H = -c^2 < 0$ so not convex. For the same reason the function is not convex for $b = 1$. Assume that $a = 2$, then

$$H = \begin{pmatrix} 2 & -c \\ -c & b(b-1)y^{b-2} \end{pmatrix}$$

If $b > 2$, then $2b(b-1)y^{b-2} - c^2 < 0$ for sufficiently small values of y. If $b = 2$, then $4 - c^2 > 0$ for $-2 < c < 2$. So the function is convex for $a = b = 2, c = 1$. For $a, b > 2$ one may choose x, y sufficiently small such that $a(a-1)x^{a-2}b(b-1)y^{b-2} - c^2 < 0$. Therefore the function is only convex for $a = b = 2$ and $c = 1$.

3. Consider the problem

$$\min f(x, y) = x^2 + e^{x+y} + xy^2 + y^3$$

$$\text{s.t.} \quad x + y = 1$$

(a) Find all KKT-points. (4p)

(b) Use the Hessian to determine if these points are/this point is local minimum points/point. (2p)

Solution 3 (a):

The KKT points should satisfy the conditions

$$\begin{cases} 2x + e^{x+y} + y^2 = v_1 \\ e^{x+y} + 2xy + 3y^2 = v_1 \\ x + y = 1 \\ v_1(1 - x - y) = 0 \end{cases}$$

Then

$$2x + e^{x+y} + y^2 = e^{x+y} + 2xy + 3y^2$$

gives

$$x = xy + y^2$$

This together with $y = 1 - x$ gives us

$$x = x - x^2 + 1 - 2x + x^2$$

so $x = 1/2$ and $y = 1/2$.

Solution 3 (b):

The Hessian matrix will be

$$H = \begin{pmatrix} 2 + e^{x+y} & e^{x+y} + 2y \\ e^{x+y} + 2y & e^{x+y} + 2x + 6y \end{pmatrix}$$

For $x = y = 1/2$ the Hessian is

$$\begin{pmatrix} 2 + e & e + 1 \\ e + 1 & e + 4 \end{pmatrix}$$

which is positive definite since $2 + e > 0$ and $4e + 7 > 0$. So the KKT-point is a minimum.

4. Consider the problem

$$\min f(x_1,\ x_2) = x_1^2 + 3x_2^2$$

$$\text{s.t.} \quad x_1 + x_2 = 4.$$

Introduce an appropriate penalty function and do one iteration. Use penalty parameter $\mu = 1$ (3p)

Solution 4:

The penalty function is

$$F_s(x_1, x_2) = x_1^2 + 3x_2^2 + \mu(x_1 + x_2 - 4)^2 = x_1^2 + 3x_2^2 + (x_1 + x_2 - 4)^2.$$

Let the gradient be equal to zero, then we have $4x_1+2x_2-8 = 0$ and $8x_2+2x_1-8 = 0$. The system of equations has the solution $x_1 = 12/7$ and $x_2 = 4/7$ and $f(12/7, 4/7) = 192/49$.

5. Consider the problem

$$\min f(x, y, z) = x^2 + y^2 - x + y + z^2$$

(a) Solve the problem by using the steepest descent method. Choose the starting point $(1/2, 1/2, 0)$. (3p)
(b) Solve the problem by using the Newton method. Choose the starting point $(1/2, 1/2, 0)$. (2p)

Solution 5 (a):

The gradient of f is $(2x - 1, 2y + 1, 2z)$ and the Hessian matrix is

$$H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\nabla f(1/2, 1/2, 0) = (0, 2, 0)$$

then

$$d^{(0)} = -\nabla f(1/2, 1/2, 0) = -(0, 2, 0)$$

and

$$x^1 = (1/2, 1/2 - 2t, 0)$$

and

$$f(x^1) = 4t^2 - 4t - 1/2.$$

The zero of the derivative of f is given by $8t - 4 = 0$, so $t = 1/2$. Then $x^1 = (1/2, -1/2, 0)$ and $f(1/2, -1/2, 0) = -1/2$. This is the minimum since $d^{(1)} = (0, 0, 0)$.

Solution 5 (b):

then

$$d^{(0)} = -H^{-1}(1/2, 1/2, 0)\nabla f(1/2, 1/2, 0) = -(0, 1, 0)$$

and

$$x^1 = (1/2, 1/2 - 1, 0) = (1/2, -1/2, 0).$$

Then $d^{(1)} = (0, 0, 0)$, so minimum of f is $-1/2$.

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Table 2: Cost, profit and more

Observe that the amusement park can have several attractions of the same kind, but then the profit for this kind of attraction is reduced by 20 percent for each attraction of the same kind that is added. (4p)

Solution 6:

$$\max f(x_1, x_2, x_3, x_4, x_5, x_6) = (10x_1 \cdot 0.8^{x_1-1} + 12x_2 \cdot 0.8^{x_2-1} +$$
$$+9x_3 \cdot 0.8^{x_3-1} + 10x_4 \cdot 0.8^{x_4-1} + 11x_5 \cdot 0.8^{x_5-1} + 5x_6 \cdot 0.8^{x_6-1}) \cdot 10000$$

s.t.

$$\begin{aligned}
&20x_1 + 15x_2 + 14x_3 + 25x_4 + 13x_5 + 5x_6 \leq 100 \\
&3x_1 + 2x_2 + 2x_3 + 2x_4 + 3x_5 + x_6 \leq 10 \\
&x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 6 \\
&90x_1 + 115x_2 + 87x_3 + 96x_4 + 105x_5 + 40x_6 \leq 10000 \\
&x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{aligned}$$

$x_1, x_2, x_3, x_4, x_5, x_6$ are integers.