Useful maths formulae

Trigonometric identities

Angle sum and difference

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Power reduction

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2} \qquad \cos^2\theta = \frac{1 + \cos(2\theta)}{2}$$

Series

Binomial series

$$(x+y)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^{n-k} y^k \qquad \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Taylor series

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Fourier series

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Integral transforms

Fourier transform

Integral substitutions

Tangent half-angle substitution

If $t = \tan\frac{x}{2}$ then

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad dx = \frac{2}{1+t^2} dt.$$

Euler's substitutions

This applies to integrals of the form:

$$\int R(x, \sqrt{ax^2 + bx + c})\, dx.$$

1. If $a > 0$, use
$$\sqrt{ax^2 + bx + c} = \pm x\sqrt{a} + t.$$

2. If $c > 0$, use
$$\sqrt{ax^2 + bx + c} = xt \pm \sqrt{c}.$$

3. If $ax^2 + bx + c$ has real roots α and β, use
$$\sqrt{ax^2 + bx + c} = \sqrt{a(x-\alpha)(x-\beta)} = (x-\alpha)t.$$

Integral tricks

Frullani integrals

Assuming $f(x)$ is continuous and both $\lim_{x\to 0} f(x)$ and $\lim_{x\to +\infty} f(x)$ are well-defined, then

$$\int_0^\infty \frac{f(ax) - f(bx)}{x}\, dx = (f(\infty) - f(0))\ln\frac{a}{b}.$$

Inverse functions

$$\int f^{-1}(x)\, dx = x f^{-1}(x) - (F \circ f^{-1})(x) + C$$

$$\int_a^b f(x)\, dx + \int_c^d f^{-1}(x)\, dx = bd - ac$$

Leibniz integral rule

$$\frac{d}{dx}\int_a^b f(x,t)\, dt = f(x,b)\cdot\frac{db}{dx} - f(x,a)\cdot\frac{da}{dx} + \int_a^b \frac{\partial}{\partial x} f(x,t)\, dt$$

assuming $-\infty < a(x)$ and $b(x) < \infty$.

Glasser's master theorem

$$\int_{-\infty}^\infty f(x)\, dx = \int_{-\infty}^\infty f\left(|\alpha| x - \sum_i \frac{\gamma_i}{x - \beta_i}\right) dx$$

for arbitrary $\alpha, \gamma_i, \beta_i$.

Gamma function

$$\int_0^\infty x^b e^{-ax}\, dx = \frac{\Gamma(b+1)}{a^{b+1}}$$

$$\int_0^\infty \frac{x^t}{e^x - 1}\frac{dx}{x} = \zeta(t)\gamma(t)$$

Coordinate systems

$$\mathbf{e}_u = \frac{\partial \mathbf{r}/\partial u}{\|\partial \mathbf{r}/\partial u\|} \qquad g_u = \|\partial \mathbf{r}/\partial u\|$$

Cylindrical coordinates

$$(R, \phi, z) \in [0, \infty) \times [0, 2\pi) \times (-\infty, \infty)$$

$$R = \sqrt{x^2 + y^2} \qquad x = R\cos\phi \qquad g_R = 1$$
$$\phi = \arctan\frac{y}{x} \qquad y = R\sin\phi \qquad g_\phi = R$$
$$z = z \qquad z = z \qquad g_z = 1$$

Spherical coordinates

$$(r, \theta, \phi) \in [0, \infty) \times [0, \pi] \times [0, 2\pi)$$

$$r = \sqrt{x^2 + y^2 + z^2} \qquad x = r\sin\theta\cos\phi \qquad g_r = 1$$
$$\theta = \arccos\frac{z}{r} = \arctan\frac{R}{z} \qquad y = r\sin\theta\sin\phi \qquad g_\theta = r$$
$$\phi = \arctan\frac{y}{x} \qquad z = r\cos\theta \qquad g_\phi = r\sin\theta$$

Vector calculus

$$\nabla \times (\nabla f) = 0 \qquad \nabla \cdot (\nabla \times \mathbf{F}) = 0$$

Divergence and curl in general orthogonal coordinates

$$\nabla = \mathbf{e}_u \frac{1}{g_u}\frac{\partial}{\partial u} + \mathbf{e}_v \frac{1}{g_v}\frac{\partial}{\partial v} + \mathbf{e}_u \frac{1}{g_w}\frac{\partial}{\partial w}$$

$$\nabla \cdot \mathbf{F} = \frac{1}{g_u g_v g_w}\left[\frac{\partial(g_v g_w F_u)}{\partial u} + \frac{\partial(g_w g_u F_v)}{\partial v} + \frac{\partial(g_u g_v F_w)}{\partial w}\right]$$

$$\mathbf{e}_u \cdot \nabla \times \mathbf{F} = \frac{1}{g_v g_w}\left[\frac{\partial g_w F_w}{\partial v} - \frac{\partial g_v F_v}{\partial w}\right]$$

Miscellaneous

2D matrix eigenvalues and eigenvectors

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0 \qquad \begin{pmatrix} b \\ \lambda - a \end{pmatrix}, \begin{pmatrix} \lambda - d \\ c \end{pmatrix}$$

Stirling's approxmiation

$$\ln n! \approx n\ln n - n + 1$$

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Descartes' rule of signs

If the non-zero terms of a single-variable polynomial with real coefficients are ordered by descending variable exponent, then the number of positive roots of the polynomial is either equal to the number of sign changes between consecutive coefficients, or is less than it by an even number. Example: $f(x) = x^3 + x^2 - x - 1$ has one sign change, so has exactly one positive root. $f(-x) = -x^3 + x^2 + x - 1$ has two sign changes, so has two or zero positive roots.