Useful maths formulae

Trigonometric identities

Angle sum and difference

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Power reduction

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2} \qquad \cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

Series

Binomial series

$$(x+y)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^{n-k} y^k \qquad \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Taylor series

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Integral substitutions

Tangent half-angle substitution

If $t = \tan\frac{x}{2}$ then

$$\sin x = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-t^2}{1+t^2}, \qquad dx = \frac{2}{1+t^2}dt.$$

Euler's substitutions

This applies to integrals of the form:

$$\int R(x, \sqrt{ax^2+bx+c})\,dx.$$

1. If $a > 0$, use
$$\sqrt{ax^2+bx+c} = \pm x\sqrt{a} + t.$$
2. If $c > 0$, use
$$\sqrt{ax^2+bx+c} = xt \pm \sqrt{c}.$$
3. If ax^2+bx+c has real roots α and β, use
$$\sqrt{ax^2+bx+c} = \sqrt{a(x-\alpha)(x-\beta)} = (x-\alpha)t.$$

Integral tricks

Frullani integrals

Assuming $f(x)$ is continuous and both $\lim_{x\to 0} f(x)$ and $\lim_{x\to+\infty} f(x)$ are well-defined, then

$$\int_0^\infty \frac{f(ax)-f(bx)}{x}\,dx = (f(\infty)-f(0))\ln\frac{a}{b}.$$

Inverse functions

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - (F\circ f^{-1})(x) + C$$

$$\int_a^b f(x)\,dx + \int_c^d f^{-1}(x)\,dx = bd - ac$$

Leibniz integral rule

$$\frac{d}{dx}\int_a^b f(x,t)\,dt = f\big(x,b\big)\cdot\frac{db}{dx} - f\big(x,a\big)\cdot\frac{da}{dx} + \int_a^b \frac{\partial}{\partial x} f(x,t)\,dt$$

assuming $-\infty < a(x)$ and $b(x) < \infty$.

Glasser's master theorem

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f(|\alpha|x - \sum_i \frac{\gamma_i}{x-\beta_i})\,dx$$

for arbitrary α, γ_i, β_i.

Gamma function

$$\int_0^\infty x^b e^{-ax}dx = \frac{\Gamma(b+1)}{a^{b+1}}$$

$$\int_0^\infty \frac{x^t}{e^x-1}\frac{dx}{x} = \zeta(t)\gamma(t)$$

Coordinate systems

Cylindrical coordinates

$$(R,\phi,z) \in [0,\infty)\times[0,2\pi)\times(-\infty,\infty)$$

$$\begin{aligned} R &= \sqrt{x^2+y^2} & x &= R\cos\phi & g_R &= 1 \\ \phi &= \arctan\frac{y}{x} & y &= R\sin\phi & g_\phi &= R \\ z &= z & z &= z & g_z &= 1 \end{aligned}$$

Spherical coordinates

$$(r,\theta,\phi) \in [0,\infty)\times[0,\pi]\times[0,2\pi)$$

$$\begin{aligned} r &= \sqrt{x^2+y^2+z^2} & x &= r\sin\theta\cos\phi & g_r &= 1 \\ \theta &= \arccos\frac{z}{r} = \arctan\frac{R}{z} & y &= r\sin\theta\sin\phi & g_\theta &= r \\ \phi &= \arctan\frac{y}{x} & z &= r\cos\theta & g_\phi &= r\sin\theta \end{aligned}$$

Vector calculus

$$\nabla\times(\nabla f) = 0 \qquad \nabla\cdot(\nabla\times\mathbf{F}) = 0$$

$$\mathbf{e}_u = \frac{\partial\mathbf{r}/\partial u}{|\partial\mathbf{r}/\partial u|}$$

Miscellaneous

2D matrix eigenvalues and eigenvectors

$$\lambda^2 - (a+d)\lambda + (ad-bc) = 0 \qquad \begin{pmatrix} b \\ \lambda-a \end{pmatrix}, \begin{pmatrix} \lambda-d \\ c \end{pmatrix}$$

Stirling's approxmiation

$$\ln n! \approx n\ln n - n + 1$$
$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Descartes' rule of signs

If the non-zero terms of a single-variable polynomial with real coefficients are ordered by descending variable exponent, then the number of positive roots of the polynomial is either equal to the number of sign changes between consecutive coefficients, or is less than it by an even number.
Example: $f(x) = x^3 + x^2 - x - 1$ has one sign change, so has exactly one positive root. $f(-x) = -x^3 + x^2 + x - 1$ has two sign changes, so has two or zero positive roots.