2.1 Let S be a finite set with n elements. Since S is finite, we can list out its elements $s_1, s_2, ..., s_n$. We can specify a function $f : S \to S$ by specifying one element of S for each element of S.

First we choose $f(s_1)$. There are n possible choices, these being any of s_1 through s_n. Then, in choosing the next element, there are $n-1$ choices, these being all the elements of S except the one chosen for $f(s_1)$. We may not choose the same element as $f(s_1)$, since then the function would no longer be injective. Similarly, we have $n-2$ choices for $f(s_3)$, $n-3$ choices for $f(s_4)$ and so on. Finally, at $f(s_n)$ there is only 1 choice. Ultimately, there are $n!$ injections from $S \to S$. These are also surjective, since we pick every element of S to output to. By the same logic, in reversing the arrows we count $n!$ surjections $S \to S$. So there are $n!$ bijections from $S \to S$.

2.2

($\Rightarrow$) Let $f : A \to B$ have a right inverse, with $A \neq \emptyset$. Want to show that f is surjective. Let $g : B \to A$ be the right inverse of f, so that $f \circ g = id_B$. Now let $b \in B$ be arbitrary. We know that $g(b) \in A$. And $f(g(b)) = (f \circ g)(b) = b$. So f is surjective.

($\Leftarrow$) Let $f : A \to B$ be surjective. Want to show that f has a right inverse. That f is surjective means that $(\forall b \in B)(\exists a \in A) f(a) = b$. Thus, since A is nonempty, we know that $f^{-1}(b)$ is nonempty for all $b \in B$. Now, consider the family of subsets of A indexed by B given by $\{f^{-1}(b) : b \in B\}$. This family is necessarily disjoint, since if not, then the function would not be well defined (e.g. one input would have more than one output). Now we apply the axiom of choice to define a function $g : B \to A$ choosing one element from each subset where the input to g is the index to the family. Now, for any member b of B, $b = f(g(b)) = (f \circ g)(b) = id_B(b)$. So g is a right inverse of f.

2.4

Let S be a set of sets. Let $\approx$ be a relation between members of S such that if $A, B \in S$, $A \approx B$ if they are isomorphic as sets. We will show that $\approx$ is an equivalence relation.

Reflexive. Let $A \in S$. Then id_A is a bijection between A and A.

Symmetric. Let $A, B \in S$. Then assume that $A \approx B$. So there is a bijection $f : A \to B$. Then $f^{-1} : B \to A$ is a bijection, so $B \approx A$.

Transitive. Let $A, B, C \in S$. Assume that $A \approx B$, $B \approx C$. Then there exist bijections $f : A \to B$, $g : B \to C$. Then $g \circ f : A \to C$ is a bijection, since it has a two sided inverse $f^{-1} \circ g^{-1} : C \to A$. So $A \approx C$.

2.5 (wrong)

Let $f : A \to B$ be an epimorphism if for all sets Z and for all functions $\beta : Z \to B$ there is a function $\alpha : Z \to A$ such that $\beta = f \circ \alpha$.

We will show that a function $f : A \to B$ is an epimorphism if and only if it is surjective.

($\Rightarrow$) We will say that f is an epimorphism. Let $b \in B$ be arbitrary. Now define a function $w : Z \to B$ so that w maps all elements of A to b. Now since f is an epimorphism, there must be $x : Z \to A$ so that $f \circ x = w$. Now apply this functional equation to an element z of Z. Then $f(x(z)) = w(z)$, so $f(x(z)) = b$. Now $x(z) \in A$, so f is surjective.

($\Leftarrow$) Say that f is surjective. Call its right inverse $g : B \to A$, so that $f \circ g = id_B$. Then let $\beta : Z \to B$ be arbitrary.

$$f \circ (g \circ \beta) = (f \circ g) \circ \beta = \beta.$$

So $g \circ \beta : Z \to A$ is a suitable α as above. Thus f is an epimorphism.

2.5 (for real this time)

Let $g : A \to B$ be an epimorphism if for all sets Z and all functions $\alpha, \alpha' : B \to Z$, $\alpha \circ g = \alpha' \circ g \implies \alpha = \alpha'$.

We will show that a function is surjective if and only if it is an epimorphism.

($\Rightarrow$) Assume that $g : A \to B$ is surjective, that $\alpha, \alpha' : B \to Z$ are arbitrary and $\alpha \circ g = \alpha' \circ g$. Then compose on the right by a right inverse h of g, which exists since g is surjective. Then $\alpha \circ g \circ h = \alpha' \circ g \circ h$, so $\alpha = \alpha'$.

($\Leftarrow$) Assume that $g : A \to B$ is not surjective. Then there is some element $b \in B$ such that $b \notin g(A)$. Let $\alpha, \alpha' : g(A) \cup \{b\} \to \{0, 1\}$, where $\alpha = \alpha' \equiv 0$ except that $\alpha(b) = 0$ and $\alpha'(b) = 1$. Now $\alpha \circ g = \alpha' \circ g$ but $\alpha \neq \alpha'$. So g is not an epimorphism.

2.9

Say that $A' \cong A''$ and $B' \cong B''$. Also say that $A' \cap B' = \emptyset$ and $A'' \cap B'' = \emptyset$. Want to show that $A' \cup B' \cong A'' \cup B''$. Let $\alpha : A' \to A''$ and $\beta : B' \to B''$ be bijections. Define a function $f : A' \cup B' \to A'' \cup B''$ such that

$$f(x) = \begin{cases} \alpha(x) & x \in A' \\ \beta(x) & x \in B' \end{cases}.$$

This function is well defined, since we will only ever have one of $x \in A'$ and $x \in B'$. Now, to show that f is a bijection, we will demonstrate an inverse. We know the inverses α^{-1}, β^{-1} exist since α, β are bijections. Then we will define $g : A'' \cup B'' \to A' \cup B'$ by

$$g(x) = \begin{cases} \alpha^{-1}(x) & x \in A'' \\ \beta^{-1}(x) & x \in B'' \end{cases}.$$

Again, this function is well defined since we know that A'' and B'' are disjoint. Now we will show that it is a left inverse. First, let $x \in A' \subset A' \cup B'$. Then

$g(f(x)) = g(\alpha(x)) = \alpha^{-1}(\alpha(x)) = x$. Otherwise, let $x \in B' \subset A' \cup B'$. Then $g(f(x)) = g(\beta(x)) = \beta^{-1}(\beta(x)) = x$. So g is a left inverse of f.

Now we will show that g is a right inverse. Let $x \in A'' \subset A'' \cup B''$. Then $f(g(x)) = f(\alpha^{-1}(x)) = \alpha(\alpha^{-1}(x)) = x$. Otherwise, let $x \in B'' \subset A'' \cup B''$. Then $f(g(x)) = f(\beta^{-1}(x)) = \beta(\beta^{-1}(x)) = x$. So g is a right inverse of f.

Since g is both a right and left inverse of f it is an inverse, and thus f is a bijection.

2.10

Let A, B be finite sets. A function $f : A \to B$ is determined by its graph, $\Gamma_f = \{(a,b) : a \in A, b \in B\}$, where there is exactly one occurrence of each $a \in A$ in the pairs in Γ_f. Since A is finite, we may enumerate its elements $a_1, a_2, ..., a_{|A|}$. Thus the form of any Γ_f must be a sequence $(a_1, b_?), (a_2, b_?), ..., (a_{|A|}, b_?)$.

Now, for each a_i, f must choose some element of B. There are $|B|$ such choices, so we exponentiate $|B|$ to the power $|A|$, for the number of inputs a_i. Thus we find that $|B^A| = |B|^{|A|}$.

2.11

I will construct a bijection between 2^A and $P(A)$. Let $f : 2^A \to P(A)$. The inputs to f are functions $A \to \{0,1\}$. The outputs are subsets of A. The map will take a function $x : A \to \{0,1\}$ to a set $\{a : a \in A, x(a) = 1\}$. To show that f is a bijection, I will propose an inverse, $g : P(A) \to 2^A$. The map g operates in the following way: Let $S \subset A$. Then the function $g(S) : A \to \{0,1\}$ is given by

$$g(S)(x) = \begin{cases} 0 & x \notin S \\ 1 & x \in S \end{cases}$$

Left inverse. Let $x \in 2^A$ so $x : A \to \{0,1\}$. Then $g(f(x)) = g(\{a \in A : x(a) = 1\})$

$$= y \mapsto \begin{cases} 0 & y \notin \{a \in A : x(a) = 1\} \\ 1 & y \in \{a \in A : x(a) = 1\} \end{cases} : A \to \{0,1\}$$

$$= y \mapsto x(y) = x$$

Right inverse. Let $S \in P(A)$ so $S \subset A$. Then

$$f(g(S)) = f\left(a \mapsto \begin{cases} 0 & a \notin S \\ 1 & a \in S \end{cases}\right)$$

$$= \left\{ b \in A : \left(a \mapsto \begin{cases} 0 & a \notin S \\ 1 & a \in S \end{cases}\right)(b) = 1 \right\} = \{b : b \in S\} = S$$

Since f has a right inverse and a left inverse, it is a bijection.