Combinatorial Maps and Trace Moments of Wishart Matrices

by

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Chapter 1

Introduction

1.1 Background and Objectives

This paper focuses on calculating the large N-limit of the expected trace of powers of a Wishart matrix $S = XX^\dagger$, where X is an $N \times N$ matrix drawn from the complex Ginibre ensemble. Specifically, we aim to examine the moments of the normalized trace:

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E}[\mathrm{Tr}(S^k)].$$

Complex Ginibre matrices, composed of IID complex Gaussian entries, are fundamental in random matrix theory, with applications in fields like statistical physics and wireless communications. Understanding the trace behavior of these matrices in the large N-limit offers key insights into their limiting spectral properties and, therefore, provides valuable insight into the many applications where random matrices are ubiquitous.

In this paper, we consider two types of moments:

- **Single moments**: $m_k = \mathbb{E}[\mathrm{Tr}(S^k)]$
- **Multi-moments**: $m_{k_1,\ldots,k_n} = \mathbb{E}[\prod_{i=1}^n \mathrm{Tr}(S^{k_i})]$

The underlying combinatorics allow us to demonstrate that the cumulants of the random variables $\{\mathrm{Tr}(S^{k_i})\}$ exhibit a genus expansion. This expansion is critical for understanding the asymptotic behavior of the moments and cumulants as N becomes large, revealing connections to combinatorial maps and the topological properties of the underlying random matrix models.

Through this work, we aim to derive explicit expressions for these moments and cumulants in the large N-limit, providing a comprehensive understanding of the spectral properties of Wishart matrices constructed from complex Ginibre ensembles and related random matrix models.

1.2 Basics

Definition 1.2.1 (Complex Gaussian Random Variables)**.** A complex Gaussian random variable, $z = X + iY$, is formed by combining two real independent Gaussian random variables, X and Y.

where:

- X is the real part of z,
- Y is the imaginary part of z, and
- X and Y are independent and identically distributed (IID),

Variance of Real and Imaginary Parts

Let z have a mean of zero and a variance of $\frac{1}{N}$, i.e., $\mathbb{E}[z] = 0$ and $\text{Var}(z) = \frac{1}{N}$.

The variance of z can be expressed as:

$$\text{Var}(z) = \mathbb{E}[|z|^2] = \mathbb{E}[X^2] + \mathbb{E}[Y^2]$$

Since X and Y are independent and identically distributed, we have:

$$\text{Var}(z) = \text{Var}(X) + \text{Var}(Y)$$

For a complex Gaussian random variable with a variance of $\frac{1}{N}$, each of the real and imaginary parts has a variance of:

$$\text{Var}(X) = \text{Var}(Y) = \frac{1}{2N}$$

1.2.1 Complex Ginibre Matrices

We start by defining complex Ginibre matrices. A complex Ginibre matrix is a random matrix whose entries are independent and identically distributed (i.i.d.) complex Gaussian random variables. Specifically, the entries of X are given by:

$$X_{ij} \sim \mathcal{N}_{\mathbb{C}}(0, \sigma^2) \quad \text{with} \quad \sigma^2 = \frac{1}{N}$$

where $\mathcal{N}_{\mathbb{C}}(0, \sigma^2)$ denotes the complex normal distribution with mean zero and variance σ^2 as defined above.

This means that each element X_{ij} can be written as:

$$X_{ij} = \frac{1}{\sqrt{2N}}(a_{ij} + ib_{ij})$$

where a_{ij} and b_{ij} are i.i.d. real Gaussian random variables with mean zero and variance one. Additionally we define:

$$X^{\dagger}_{ab} := \overline{X}_{ba}$$

where $\overline{X}_{ba}$ denotes the complex conjugate of X_{ba}.

Concretely, the entries X_{ab} are distributed according to the probability distribution

$$\frac{N}{\pi} e^{-N|X_{ab}|^2} d\mathrm{Re}(X_{ab}) d\mathrm{Im}(X_{ab}),$$

where $d\mathrm{Re}(X_{ab})d\mathrm{Im}(X_{ab})$ is the product of two real measures, corresponding to the real and imaginary parts of X_{ab}.

We'll denote

$$dX^{\dagger}dX = \prod_{a,b} d\overline{X}_{ab} dX_{ab} = \prod_{a,b} 2i\, d\mathrm{Re}(X_{ab}) d\mathrm{Im}(X_{ab}),$$

so that X, a random matrix, has the distribution

$$d\mu(X) = \frac{N^{N^2}}{(2i\pi)^{N^2}} e^{-N\,\mathrm{Tr}(XX^{\dagger})} dX^{\dagger}dX.$$

1.2.2 The Wishart Ensemble and the Marchenko-Pastur distribution

Definition 1.2.2 (Wishart Ensemble)**.** A (complex) Wishart random matrix is the random variable defined as the product $S = XX^{\dagger}$, where X is a Ginibre matrix, i.e. X is an $N \times M$ matrix with independent, identically distributed complex Gaussian entries.

Wishart matrices are often considered in the context of studying the statistical properties of large sample covariance matrices of some M-dimensional variables observed N times. The empirical set consists of $N \times M$ data $\{x_{it}\}_{1 \leq i \leq N, 1 \leq t \leq M}$, where we have N observations, and each observation contains M variables. Examples abound: we could consider the daily returns of M stocks over a certain time period, or the number of spikes fired by M neurons during N consecutive time intervals of length Δt, etc. See [PB2020].

We have chosen to focus on the case $M = N$, so the number of observations is equal to the number of variables. However one can adjust the results in this paper for arbitrary M with little difficulty.

The k-th order moment m_k of a Wishart random matrix is defined as

$$m_k = \mathbb{E}\left[\mathrm{Tr}(S^k)\right].$$

Further, for any sequence of non-negative integers $k_1, \ldots, k_n$, we can define moments $m_{k_1,\ldots,k_n}$ of order $k_1, \ldots, k_n$. Similarly to the moments of order k, they are defined as the expectation of products of traces of powers of S:

$$m_{k_1,\ldots,k_n} = \mathbb{E}\left[\prod_{i=1}^{n} \mathrm{Tr}(S^{k_i})\right]$$

1.2.3 Theorem: Convergence of the Wishart Matrix density to the Marčenko-Pastur Distribution

Let W_N be a sequence of random Wishart matrices with parameters (N, M_N), where $c = \lim_{N\to\infty} \frac{M_N}{N}$.

Then for any integer $p \geq 1$, we have:

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\mathrm{Tr}\left(\frac{W_N}{N}\right)^p\right] = \int x^p d\mathrm{MP}_c(x),$$

where the Marčenko-Pastur distribution $d\mathrm{MP}_c(x)$ is given by:

$$d\mathrm{MP}_c = \max(1-c, 0)\delta_0 + \frac{\sqrt{(b-x)(x-a)}}{2\pi x}\mathbf{1}_{[a,b]}(x)\, dx,$$

and $a = (1-\sqrt{c})^2$ and $b = (1+\sqrt{c})^2$.[MS2018]

Special Case: $c = 1$

When $c = 1$, corresponding to the case $M = N$, the parameters a and b simplify to:

$$a = (1-1)^2 = 0, \quad b = (1+1)^2 = 4.$$

Therefore, the Marčenko-Pastur distribution for this case becomes:

$$d\mathrm{MP}_1(x) = \frac{\sqrt{4x - x^2}}{2\pi x}\mathbf{1}_{[0,4]}(x)\, dx.$$

This distribution describes the limiting spectral density of the normalized Wishart matrix $N^{-1}W_N$ as N becomes large, with the support of the distribution on the interval $[0, 4]$.

We'll prove this special case $c = 1$.

1.2.4 Organization of this Paper

Our goal is to compute the large N-limit for both single moments m_k and multi-moments $m_{k_1,...,k_n}$ of Wishart matrices. The paper is organized as follows:

1. **Preliminary Results:** We begin by collecting some essential preliminary results, particularly focusing on the tools for computing products of Gaussian random variables, such as Stein's lemma and Wick's theorem. These results will form the foundation for our later calculations.
2. **Combinatorial Maps and Permutation Representation:** We then define combinatorial maps and present their permutation representation. In this section, we state the Euler characteristic for the permutation model, establishing a crucial relation between the cycle decomposition in our combinatorial maps and the genus of the surface into which the map is embedded.
3. **Single Moment Expansion:** Building on the combinatorial framework, we derive the $1/N$ expansion of the single moment m_k, summed over the genus of the corresponding combinatorial maps.
4. **Multi-Moment Expansion:** We extend the analysis to multi-moments $m_{k_1,...,k_n}$, deriving their $1/N$ expansion, again summed over the genus of combinatorial maps.
5. **Cumulants and Moment-Cumulant Relation:** We then delve into classical cumulants and the moment-cumulant relation. Here, we define the trace cumulants over the random variable set $\{\mathrm{Tr}(S^{k_i})\}$.
6. **Cumulant Expansion via Rota-Möbius Inversion:** Using the multi-moment expansion, we apply the Rota-Möbius inversion over the lattice of pairings to obtain an expression for the trace cumulants summed over the genus of connected combinatorial maps.
7. **Schwinger-Dyson Loop Equations:** Finally, we bring together all the preceding results to derive the Schwinger-Dyson loop equations for the resolvent moments. This derivation leads to the key result:
8. **Wishart Density:** We can then use the resolvents we compute to find an expression for

the Wishart spectral density in the case $c = 1$

$$W_{0,1}(x) := \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left[\mathrm{Tr}\left((x-S)^{-1}\right)\right] = \sum_{p\geq 0} \frac{m_p^{[0]}}{x^{p+1}} = \frac{x - \sqrt{x^2-4x}}{2x}$$

Chapter 2

Preliminary Results

2.1 Stein's Lemma

Stein's Lemma, in the context of Gaussian random variables, provides a powerful tool for computing expectations involving products of random variables. It is particularly useful in the analysis of random matrices.

Theorem 2.1.1 (Stein's Lemma [LN2008]). For a set of Gaussian random variables $X_1, X_2, \ldots, X_n$ forming a zero-mean multivariate normal random vector, Stein's Lemma states that:

$$\mathbb{E}\left(X_1 f(X_1, X_2, \ldots, X_n)\right) = \sum_{i=1}^{n} \mathrm{Cov}(X_1, X_i)\mathbb{E}\left(\frac{\partial f(X_1, X_2, \ldots, X_n)}{\partial X_i}\right),$$

where:

- f is a differentiable function of the random variables $X_1, X_2, \ldots, X_n$.
- $\mathrm{Cov}(X_1, X_i)$ is the covariance between X_1 and X_i.

The significance of Stein's Lemma lies in its ability to express the expectation of the product of a random variable and a function of random variables in terms of the covariance and the partial derivatives of the function. This lemma is particularly useful for deriving recursive relations and moments in random matrix theory, as it simplifies complex expressions involving Gaussian random variables.

2.2 Wick's Formula (An application of Stein's Lemma)

Wick's formula, also known as Isserlis's theorem, is a direct application of Stein's Lemma and provides a method for computing the expectation of the product of an even number of zero-mean Gaussian random variables in terms of their covariances.

Theorem 2.2.1 (Wick's Formula)**.** For $2n$ zero-mean Gaussian random variables $X_1, X_2, \ldots, X_{2n}$, Wick's formula states:

$$\mathbb{E}\left[X_1 X_2 \cdots X_{2n}\right] = \sum_{\text{all pairings}} \prod_{(i,j)\in\text{pairing}} \mathrm{Cov}(X_i, X_j),$$

where the sum is taken over all possible pairings of the set $\{1, 2, \ldots, 2n\}$, and $\mathrm{Cov}(X_i, X_j)$ represents the covariance between X_i and X_j.

Proof. We proceed by induction on the number of variables.

Base Case: For $n = 1$, Wick's formula states that $\mathbb{E}[X_1X_2] = \mathrm{Cov}(X_1, X_2)$. This holds true by definition of the covariance for zero-mean Gaussian random variables.

Inductive Step: Assume Wick's formula holds for $2(n-1)$ variables. We now prove it for $2n$ variables.

Let $f(X_1, X_2, \ldots, X_{2n}) = X_2X_3 \ldots X_{2n}$. Then, applying Stein's Lemma:

$$\mathbb{E}[X_1X_2 \ldots X_{2n}] = \sum_{j=2}^{2n} \mathrm{Cov}(X_1, X_j)\mathbb{E}\left[X_2 \ldots \widehat{X_j} \ldots X_{2n}\right],$$

where $\widehat{X_j}$ indicates that X_j is omitted from the product.

By the inductive hypothesis, $\mathbb{E}[X_2 \ldots \widehat{X_j} \ldots X_{2n}]$ can be expanded as a sum over pairings of the remaining $2n-2$ variables. Thus:

$$\mathbb{E}[X_1X_2 \ldots X_{2n}] = \sum_{j=2}^{2n} \mathrm{Cov}(X_1, X_j) \sum_{\text{pairings}} \prod_{\text{pairs}} \mathrm{Cov}(X_k, X_l).$$

This result matches the form of Wick's formula, completing the proof by induction. □

2.3 Labeled Bi-coloured Combinatorial Maps

We'll now pause our discussion of trace moments to explore a seemingly unrelated topic of embedded maps on surfaces. Counting these objects will turn out to provide the key for calculating the large N-limit of our trace moments.

Definition: A **connected labeled bicolored combinatorial map**, or simply a **LBC map**, is a triplet $M = (E, \sigma^\circ, \sigma^\bullet)$ where:

- E is a set of edges labeled from 1 to p.
- σ° and $\sigma^\bullet$ are permutations of E.
- We say that M is connected when the group $\langle \sigma^\circ, \sigma^\bullet \rangle$ generated by σ° and $\sigma^\bullet$ acts transitively on E.

Definition: We call **white vertices** (resp. **black vertices**, resp. **faces**) the cycles in the unique decomposition of σ° (resp. $\sigma^\bullet$, resp. $\sigma^\bullet\sigma^\circ$) into disjoint cycles.

2.3.1 Graphical presentation of LBC maps

From each LBC map M, we can construct a graph $\Gamma(M)$ in the following way. We label the white vertices $W_1, \ldots, W_t$ and the black vertices $B_1, \ldots, B_r$.

For each edge $e \in E = \{1, \ldots, p\}$:

- Let W_* be the white vertex containing the integer e in its cycle.
- Let B_* be the black vertex containing the integer e in its cycle.
- Connect W_* and B_* with the edge e.

every graph constructed this way is unique up to relabeling of vertices.

Proposition 2.3.1. The LBC map M is connected (i.e., $\langle \sigma^\circ, \sigma^\bullet \rangle$ acts transitively on E), if and only if the graph $\Gamma(M)$ is connected.

2.3.2 Embedded graphs on Surfaces

Definition 2.3.2. An **embedded graph of genus** g is a connected graph Γ embedded into some connected orientable surface X of genus g such that each connected component of $X \setminus \Gamma$ is homeomorphic to a disk. We call the connected components of $X \setminus \Gamma$ the **faces**.

Definition 2.3.3. Two embedded graphs $\Gamma_1 \subset X_1$ and $\Gamma_2 \subset X_2$ are **isomorphic** if there exists an orientation-preserving homeomorphism $\varphi : X_1 \to X_2$ such that $\varphi|_{\Gamma_1}$ is a graph isomorphism between Γ_1 and Γ_2, and the restriction of φ to the marked edges is the identity.

Proposition 2.3.4. Every connected LBC map induces an embedded graph of genus g on some surface X where the edges do not cross.

Definition 2.3.5. Let M be an LBC map. Motivated by Proposition 2.3.4, we define $g(M)$ to be the minimal genus required to embed the graph $\Gamma(M)$ into a surface X of genus g without crossings. We also define $X(M)$ to be the surface of genus $g(M)$ into which M is embedded. The surface $X(M)$ is well defined up to isomorphism.

2.3.3 Example

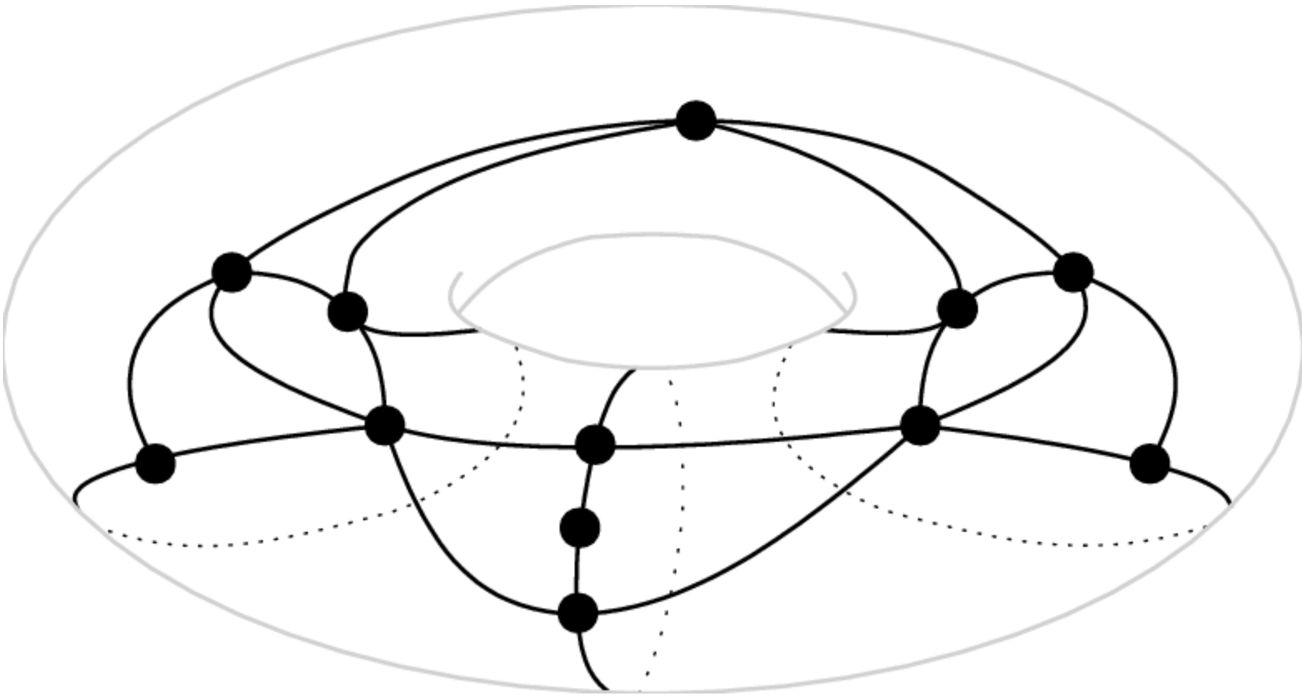


Figure 2.1: A graph embedded on the genus surface of a torus.[JS2020]

2.3.4 Embedding LBC maps into Surfaces of genus g

Given an LBC map M, it is not obvious what $g(M)$ could be, nor is it obvious how to go about constructing the surface $X(M)$ and the embedding of $\Gamma(M)$ into $X(M)$. We provide a procedure for generating the surface and the graph embedding as follows:

Given $M = (E = \{1, \ldots, p\}, \sigma^\circ, \sigma^\bullet)$, label the black and white vertices

for each face of M (i.e., for each disjoint cycle in the product $\sigma^\bullet\sigma^\circ$) in the following way:

$$L = (n_* \to \overline{\sigma^\circ(n_*)} \to \cdots \to \overline{\sigma^\circ(\ldots\sigma^\circ(n_*))})$$

where n_* is the first integer appearing in the k-cycle of that face.

Take a $2p$-gon and label the edges according to L, labeling each vertex accordingly.

Now, glue the n_* edge with the $\overline{n_*}$ edge.

We state without proof that this procedure always produces a surface $X(M)$, and the graph $\Gamma(M)$ embedded onto the surface. For examples see section 3.1.7.

Proposition 2.3.6. Let M be an LBC map. The faces of M correspond exactly to the connected components of $X(M) \setminus \Gamma(M)$.

2.3.5 4g-gon gluing into genus g surface

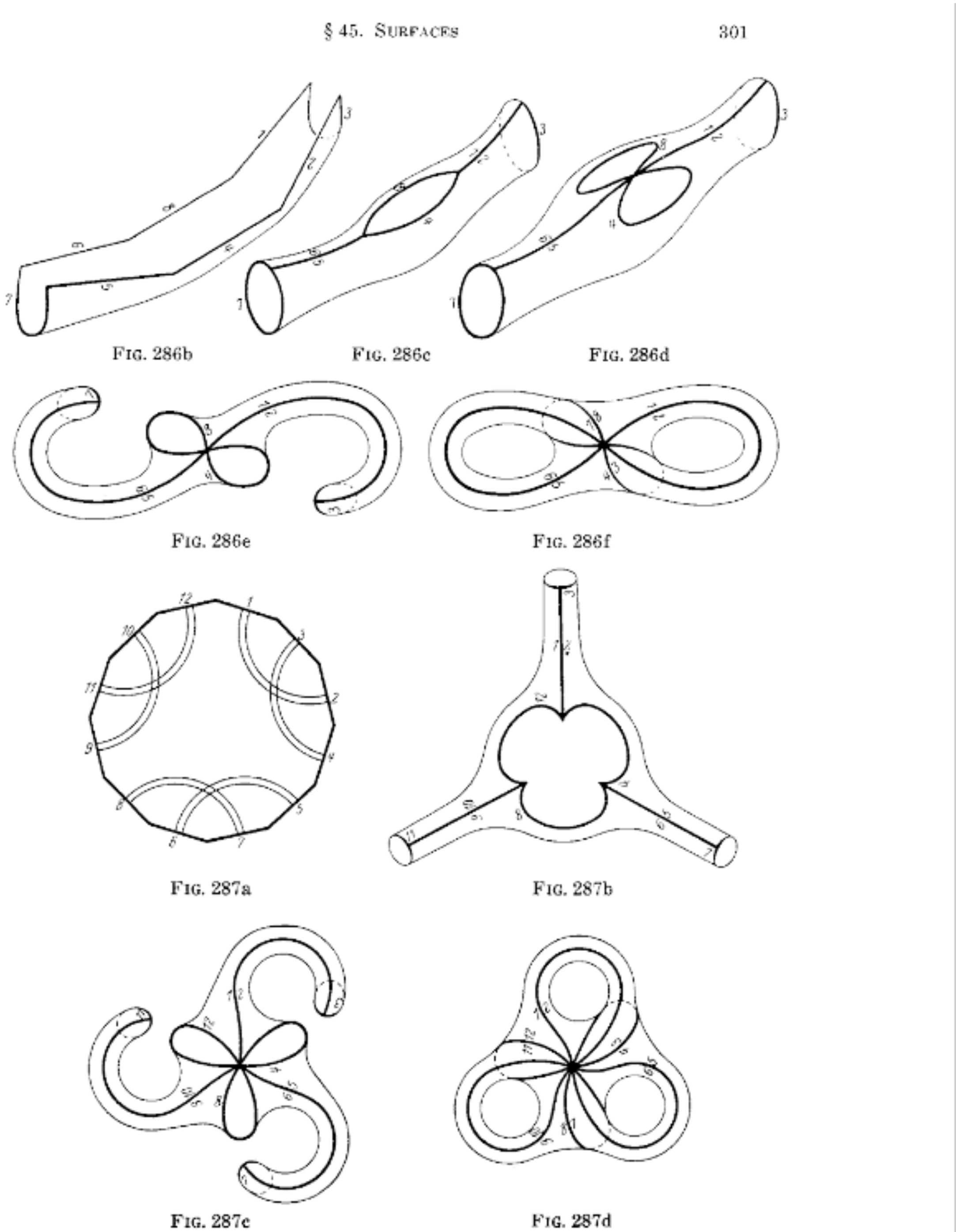


Figure 2.2: Obtaining a genus 2 and 3 surface from an octagon and dodecagon.[HC1999]

2.3.6 Euler Characteristic of Combinatorial Maps

Given a map $M = (E, \sigma^\circ, \sigma^\bullet)$, it would be useful to read off the genus without resorting to the elaborate construction detailed above. It turns out that we are able to determine the genus using only the three objects in the triple defining M. The key to this useful shortcut is the Euler Characteristic of the manifold $X(M)$.

Euler Characteristic of CW complex: Let X be a finite CW-complex with a_k number of k-cells. The **Euler characteristic** $\chi(X)$ is defined as:

$$\chi(X) = \sum_{k=0}^{\infty} (-1)^k a_k$$

This classical result in Topology tells us that for a surface X of genus g,

$$\chi(X) = 2 - 2g$$

[Hat2002].

Definition 2.3.7. The symbol $\#\alpha$ denotes the number of disjoint cycles of the permutation α in S_n. For example, $\#(1\ 2)(3\ 4) = 2$.

Theorem 2.3.8 (Euler Characteristic)**.** Given an LBC map $M = (E, \sigma^\circ, \sigma^\bullet)$, $g(M)$ satisfies:

$$\#\sigma^\circ + \#\sigma^\bullet + \#(\sigma^\bullet \sigma^\circ) - |E| = 2 - 2g(M),$$

where $g(M)$ is the minimal genus required to embed this map into the surface without crossing edges.

Proof. Proposition 2.3.4 tells us that $X(M)$ is a Riemann surface of genus g, which always has a CW complex decomposition. Proposition 2.3.6 connects the faces of M to the connected components of $X(M) \setminus \Gamma(M)$. Therefore, when we apply the classical Euler characteristic to the surface $X(M)$, we see that $g(M)$ must satisfy:

$$\#\sigma^\circ + \#\sigma^\bullet + \#(\sigma^\bullet \sigma^\circ) - |E| = 2 - 2g(M).$$

□

Armed with these results we are now ready to attack the original problem of calculating Trace moments.

Chapter 3

Back to Trace Moments

3.1 Computing Trace Moments and Cumulants using Combinatorial Maps

We first recall that the k-th moment is given by:

$$m_k = \mathbb{E}\left[\mathrm{Tr}(S^k)\right]$$

where $S = XX^\dagger$ and X is a Ginibre matrix.

We are interested in computing the normalized trace:

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left[\mathrm{Tr}(S^k)\right] = \lim_{N\to\infty} \frac{1}{N} m_k$$

3.1.1 Expanding m_k into Indices

First, we expand the trace into a sum over indices. The trace of S^k can be written as:

$$\mathrm{Tr}(S^k) = \sum_{i_1,i_2,\ldots,i_k} S_{i_1 i_2} S_{i_2 i_3} \ldots S_{i_k i_1}$$

This gives us:

$$m_k = \mathbb{E}\left[\sum_{i_1,i_2,\ldots,i_k} S_{i_1 i_2} S_{i_2 i_3} \ldots S_{i_k i_1}\right]$$

Here, the summation is taken over all possible k indices $i_1, i_2, \ldots, i_k$.

3.1.2 Substituting $S = XX^\dagger$

Next, we substitute $S_{ij} = (XX^\dagger)_{ij} = \sum_\alpha X_{i\alpha} X^\dagger_{\alpha j}$ into the expression for m_k:

$$m_k = \mathbb{E}\left[\sum_{i_1,i_2,\ldots,i_k} \sum_{\alpha_1,\alpha_2,\ldots,\alpha_k} X_{i_1\alpha_1} X^\dagger_{\alpha_1 i_2} X_{i_2\alpha_2} X^\dagger_{\alpha_2 i_3} \ldots X_{i_k\alpha_k} X^\dagger_{\alpha_k i_1}\right]$$

$$= \sum_{i_1,i_2,\ldots,i_k} \sum_{\alpha_1,\alpha_2,\ldots,\alpha_k} \mathbb{E}\left[X_{i_1\alpha_1} X^\dagger_{\alpha_1 i_2} X_{i_2\alpha_2} X^\dagger_{\alpha_2 i_3} \ldots X_{i_k\alpha_k} X^\dagger_{\alpha_k i_1}\right]$$

3.1.3 Applying Wick's Formula

At this point, we recall that $X^\dagger_{ab} = \overline{X}_{ba}$ and apply Wick's formula to compute the expectation of the product of Gaussian random variables:

$$m_k = \sum_{i_1,i_2,\ldots,i_k} \sum_{\alpha_1,\alpha_2,\ldots,\alpha_k} \sum_{\pi \in P_2(2k)} \prod_{(p,q)\in\pi} \mathrm{Cov}\left(X_{i_p\alpha_p}, \overline{X}_{i_q\alpha_q}\right)$$

Here, Wick's formula is used to express the expectation of the product as a sum over all possible pairings of the Gaussian random variables, with each pairing contributing a covariance factor.

Definition 3.1.1. Let $X_{i\alpha}$ and $\overline{X}_{i\alpha}$ be entries of a random matrix X and its conjugate transpose, respectively.

- **Homogeneous Pairs:** A pair of random variables $\mathrm{Cov}(X_{i_1\alpha_1}, X_{i_2\alpha_2})$ or $\mathrm{Cov}(\overline{X}_{i_1\alpha_1}, \overline{X}_{i_2\alpha_2})$ is called a *homogeneous pair* if both variables in the covariance come from the same "type" (i.e., both are X-type or both are $\overline{X}$-type).

  In particular, we define:

  $$\text{Homogeneous pair of } X \text{ terms: } \mathrm{Cov}(X_{i_1\alpha_1}, X_{i_2\alpha_2})$$

  $$\text{Homogeneous pair of } \overline{X} \text{ terms: } \mathrm{Cov}(\overline{X}_{i_1\alpha_1}, \overline{X}_{i_2\alpha_2})$$

- **Heterogeneous Pairs:** A pair of random variables $\mathrm{Cov}(X_{i_1\alpha_1}, \overline{X}_{i_2\alpha_2})$ is called a *heterogeneous pair* if the variables in the covariance come from different "types" (i.e., one is X-type and the other is $\overline{X}$-type).

In particular, we define:

$$\text{Heterogeneous pair: } \operatorname{Cov}(X_{i_1\alpha_1}, \overline{X}_{i_2\alpha_2})$$

Proposition 3.1.2 (Vanishing of Homogeneous Pairings)**.** Let $P_2^{\text{hom}}(2k)$ denote the set of pairings that contain at least one homogeneous pair. Then:

$$\lim_{N\to\infty} \frac{1}{N} \left(\sum_{i_1,i_2,\ldots,i_k} \sum_{\alpha_1,\alpha_2,\ldots,\alpha_k} \sum_{\eta\in P_2^{\text{hom}}(2k)} \prod_{(p,q)\in\eta} \operatorname{Cov}\left(X_{i_p\alpha_p}, X^{\dagger}_{\alpha_q i_q} \right) \right) = 0.$$

Proof: Suppose η is a homogeneous pairing. Then η contains at least one homogeneous pair, which is either: - $\operatorname{Cov}(X_{i_1\alpha_1}, X_{i_2\alpha_2})$, or - $\operatorname{Cov}(\overline{X}_{i_3\alpha_3}, \overline{X}_{i_4\alpha_4})$.

The IID structure of X ensures that the covariance of any such homogeneous pair is zero, i.e.,

$$\operatorname{Cov}(X_{i_1\alpha_1}, X_{i_2\alpha_2}) = 0 \quad \text{and} \quad \operatorname{Cov}(\overline{X}_{i_3\alpha_3}, \overline{X}_{i_4\alpha_4}) = 0.$$

This result arises because the covariance is defined as:

$$\operatorname{Cov}(X_{i_1\alpha_1}, X_{i_2\alpha_2}) = \mathbb{E}[X_{i_1\alpha_1}X_{i_2\alpha_2}] - \mathbb{E}[X_{i_1\alpha_1}]\mathbb{E}[X_{i_2\alpha_2}].$$

Since X_{**} is zero-mean ($\mathbb{E}[X_{i_1\alpha_1}] = 0$) and independent, $\mathbb{E}[X_{i_1\alpha_1}X_{i_2\alpha_2}] = 0$ unless $(i_1,\alpha_1) = (i_2,\alpha_2)$, which is not possible for homogeneous pairs.

Therefore, any homogeneous pair contributes zero to the overall sum.

□

3.1.4 Re-indexing the Sum with Symmetric Group

Motivated by Proposition 3.1.2, we can safely ignore those Wick pairings that contain homogeneous pairs and focus only on the heterogeneous pairs. Conveniently, we can parameterize all of the heterogeneous pairs by fixing the $X_{i_p\alpha_p}$ terms and permuting the sub indices of $\overline{X}_{i_q\alpha_q}$ terms using the Symmetric group in the following way:

$$m_k = \sum_{\substack{i_1,\ldots,i_k\\ \alpha_1,\ldots,\alpha_k}} \sum_{\sigma^\circ\in S_k} \Big(\operatorname{Cov}(X_{i_1\alpha_1}, \overline{X}_{i_{\sigma^\bullet(\sigma^\circ(1))}\alpha_{\sigma^\circ(1)}}) \cdot \operatorname{Cov}(X_{i_2\alpha_2}, \overline{X}_{i_{\sigma^\bullet(\sigma^\circ(2))}\alpha_{\sigma^\circ(2)}}) \ldots \operatorname{Cov}(X_{i_k\alpha_k}, \overline{X}_{i_{\sigma^\bullet(\sigma^\circ(k))}\alpha_{\sigma^\circ(k)}}) \Big)$$

$$= \sum_{i_1,i_2,\ldots,i_k} \sum_{\alpha_1,\alpha_2,\ldots,\alpha_k} \sum_{\sigma^\circ \in S_k} \prod_{l=1}^{k} \mathrm{Cov}\left(X_{i_l \alpha_l}, \overline{X}_{i_{\sigma^\bullet(\sigma^\circ(l))}\alpha_{\sigma^\circ(l)}}\right)$$

$$= \frac{1}{N^k} \sum_{i_1,i_2,\ldots,i_k} \sum_{\alpha_1,\alpha_2,\ldots,\alpha_k} \sum_{\sigma^\circ \in S_k} \prod_{l=1}^{k} \delta_{i_l, i_{\sigma_\bullet(\sigma^\circ(l))}} \delta_{\alpha_l, \alpha_{\sigma^\circ(l)}}$$

where:

- The sum over $\sigma^\circ \in S_k$ represents the sum over all possible pairings that permute the $\overline{X}_{i_q \alpha_q}$ terms in the covariance factors.
- The permutation $\sigma^\bullet$ ensures that the indices i_l align correctly according to the pairings defined by σ°, where $\sigma^\bullet = (1, 2, \ldots, k)$.
- The final expression comes from noting that $\mathrm{Cov}\left(X_{i_l \alpha_l}, \overline{X}_{i_{\sigma^\bullet(\sigma^\circ(l))}\alpha_{\sigma^\circ(l)}}\right) = \frac{1}{N}$ if and only if $i_l = i_{\sigma_\bullet(\sigma^\circ(l))}$ and $\alpha_l = \alpha_{\sigma^\circ(l)}$; otherwise, the covariance is zero.

Here, $\delta_{i_p, i_{\sigma^\circ(p)}}$ and $\delta_{\alpha_p, \alpha_{\sigma^\circ(p)}}$ are Kronecker deltas that ensure the indices are equal according to the permutation σ°. The form of each σ° permutation is crucial because it dictates how the indices are paired. In essence, σ° encodes the structure of valid pairings that survive the cancellations imposed by the covariance properties of the IID complex Gaussians. By summing over all possible σ°, we capture all configurations where the nonzero covariances align with the required index pairings, ensuring that the sum includes only those contributions where the covariance is nonzero.

This not only reduces the complexity of the expression but also directly links the combinatorial structure of the permutations to the algebraic properties of the Ginibre ensemble, illustrating how the permutation σ° naturally incorporates the cancellations that occur due to the independence and identically distributed nature of the Gaussian entries.

This final form shows that the sum is restricted to those terms where the indices are equal as dictated by the Wick pairings, effectively capturing the combinatorial structure of the problem.

3.1.5 Expressing m_k in terms of the cycle structure of σ° and $\sigma^\bullet$

After changing the order of summation we have the following expression:

$$m_k = \frac{1}{N^k} \sum_{\sigma^\circ \in S_k} \left(\sum_{i_1,i_2,\ldots,i_k} \sum_{\alpha_1,\alpha_2,\ldots,\alpha_k} \prod_{l=1}^{k} \delta_{i_l, i_{\sigma_\bullet(\sigma^\circ(l))}} \delta_{\alpha_l, \alpha_{\sigma^\circ(l)}} \right)$$

Expanding the Product:

The product $\prod_{l=1}^{k} \delta_{i_l, i_{\sigma_\bullet(\sigma^\circ(l))}} \delta_{\alpha_l, \alpha_{\sigma^\circ(l)}}$ contains $2k$ Kronecker delta factors, which pair the i_l and α_l indices according to the structure of σ° and $\sigma^\bullet$.

Let's expand this out to see how the constraints are introduced. We can label the $2k$ Kronecker delta factors explicitly:

$$\delta_{i_1, i_{\sigma_\bullet(\sigma^\circ(1))}} \delta_{\alpha_1, \alpha_{\sigma^\circ(1)}} \cdot \delta_{i_2, i_{\sigma_\bullet(\sigma^\circ(2))}} \delta_{\alpha_2, \alpha_{\sigma^\circ(2)}} \cdot \cdots \cdot \delta_{i_k, i_{\sigma_\bullet(\sigma^\circ(k))}} \delta_{\alpha_k, \alpha_{\sigma^\circ(k)}}.$$

Kronecker Delta Constraints:

Each Kronecker delta introduces a constraint on the indices $i_1, \ldots, i_k$ and $\alpha_1, \ldots, \alpha_k$. Specifically:

- $\delta_{i_l, i_{\sigma_\bullet(\sigma^\circ(l))}}$ constrains the i_l indices. - $\delta_{\alpha_l, \alpha_{\sigma^\circ(l)}}$ constrains the α_l indices.

These constraints determine how the indices i_l and α_l are related based on the cycle structures of σ° and $\sigma^\bullet\sigma^\circ$.

Definition 3.1.3. A **Redundant Kronecker delta** is one that introduces a redundant constraint on the indices. For example, if we already have δ_{i_1,i_2} and δ_{i_2,i_3}, then δ_{i_1,i_3} is redundant because the values of i_1 and i_3 are already constrained.

Whenever a cycle closes in the decomposition of σ° and $\sigma^\bullet\sigma^\circ$, we get a redundant Kronecker delta, as the corresponding index has already been fixed by the earlier deltas in the cycle.

Counting the Cycles:

The number of independent Kronecker deltas that impose new constraints is exactly equal to the number of disjoint cycles in the permutation decompositions: - The number of disjoint cycles in σ° determines how the α_l indices are constrained. - The number of disjoint cycles in $\sigma^\bullet\sigma^\circ$ determines how the i_l indices are constrained.

Thus, the total contribution from each permutation $\sigma^\circ \in S_k$ is:

$$N^{\#(\sigma^\bullet\sigma^\circ)} \cdot N^{\#(\sigma^\circ)},$$

where $\#(\sigma^\circ)$ is the number of cycles in σ°, and $\#(\sigma^\bullet\sigma^\circ)$ is the number of cycles in the composition $\sigma^\bullet\sigma^\circ$.

Final Expression:

Therefore, summing over all possible permutations $\sigma^\circ \in S_k$, we get the final expression for m_k:

$$m_k = \frac{1}{N^k} \sum_{\sigma^\circ \in S_k} N^{\#(\sigma_\bullet \sigma^\circ)} \cdot N^{\#(\sigma^\circ)}.$$

This shows that each Kronecker delta constraint contributes factors of N based on the cycle structure of the permutations, and redundant constraints arise whenever a cycle closes.

$$m_k = \frac{1}{N^k} \sum_{\sigma^\circ \in S_k} N^{\#(\sigma_\bullet \sigma^\circ)} \cdot N^{\#(\sigma^\circ)}$$

where $\#(\sigma)$ denotes the number of disjoint cycles in the permutation σ.

where:

- $\#(\sigma^\bullet \sigma^\circ)$ is the number of faces, determined by the cycles in the product $\sigma^\bullet \sigma^\circ$.
- $\#(\sigma^\circ)$ is the number of white vertices, determined by the cycles in σ°.

Recognizing that each valid σ° corresponds to a unique combinatorial map M, the moment of order k can be expressed as a sum over these maps:

$$m_k = \sum_{M \in \mathcal{M}_k} N^{V_\circ(M) - k + F(M)},$$

where $\mathcal{M}_k$ denotes the set of combinatorial maps containing k edges.

where:

- $V(M) = \#(\sigma^\circ)$ denotes the number of vertices (white vertices in the map, corresponding to the cycles in σ°).
- $F(M) = \#(\sigma^\bullet \sigma^\circ)$ represents the number of faces in the map, corresponding to the cycles in the product $\sigma^\bullet \sigma^\circ$.

3.1.6 Final Expression and $1/N$ Expansion of a Single Moment m_k

Now that we have indexed our sum using the combinatorial maps M:

$$m_k = \sum_{M \in \mathcal{M}_k} N^{V_\circ(M) - k + F(M)}$$

we can apply the Euler characteristic:

$$\chi(M) = V_\circ(M) + V_\bullet - k + F(M) = 2 - 2g(M)$$

where:

- $V_\circ(M)$ denotes the number of white vertices in the map M.

- k is the number of edges.
- $F(M)$ represents the number of faces.

Using this, we obtain the expression for m_k:

$$m_k = \sum_{M \in \mathcal{M}_k} N^{1-2g(M)}$$

Grouping by Genus

We can group the maps by their genus g and sum over all maps with that genus:

$$m_k = \sum_{g \geq 0} N^{1-2g} \sum_{\substack{M \in \mathcal{M}_k \\ g(M)=g}} 1$$

We define:

$$m_k^{[g]} := \sum_{\substack{M \in \mathcal{M}_k \\ g(M)=g}} 1$$

leading to the final expression:

$$m_k = \sum_{g \geq 0} N^{1-2g} m_k^{[g]}$$

This final result shows that the moment m_k naturally has a genus expansion, where the leading order term corresponds to planar maps (genus $g = 0$), and higher-order terms correspond to maps with higher genus. The exponent $1 - 2g$ reflects the topological structure of the combinatorial maps contributing to each term in the expansion.

3.1.7 Worked out example for $m_3 = \mathbb{E}[\mathrm{Tr}(S^3)]$

We have:

$$E[\mathrm{Tr}(S^3)] = E\left[\sum_{i_1,i_2,i_3} S_{i_1 i_2} S_{i_2 i_3} S_{i_3 i_1}\right]$$

Since $S = XX^\dagger$, we substitute this expression for each S:

$$E[\mathrm{Tr}(S^3)] = E\left[\sum_{i_1,i_2,i_3} \sum_{\alpha_1,\alpha_2,\alpha_3} X_{i_1\alpha_1} \overline{X}_{i_2\alpha_1} X_{i_2\alpha_2} \overline{X}_{i_3\alpha_2} X_{i_3\alpha_3} \overline{X}_{i_1\alpha_3}\right]$$

numbering of terms:

$$= \sum_{i_1,i_2,i_3} \sum_{\alpha_1,\alpha_2,\alpha_3} E\left[\underbrace{X_{i_1\alpha_1}}_{3} \cdot \underbrace{\overline{X}_{i_2\alpha_1}}_{\overline{3}} \cdot \underbrace{X_{i_2\alpha_2}}_{2} \cdot \underbrace{\overline{X}_{i_3\alpha_2}}_{\overline{2}} \cdot \underbrace{X_{i_3\alpha_3}}_{1} \cdot \underbrace{\overline{X}_{i_1\alpha_3}}_{\overline{1}}\right]$$

applying Wick's formula:

$$= \sum_{\sigma^\circ \in S_3} \sum_{i_1,i_2,i_3} \sum_{\alpha_1,\alpha_2,\alpha_3} \prod_{l=1}^{3} \mathrm{Cov}\left(X_{i_l\alpha_l}, \overline{X}_{i_{\sigma^\bullet(\sigma^\circ(l))}\alpha_{\sigma^\circ(l)}}\right).$$

Equivalently:

$$= \frac{1}{N^3} \sum_{\sigma^\circ \in S_3} \sum_{i_1,i_2,i_3} \sum_{\alpha_1,\alpha_2,\alpha_3} \prod_{l=1}^{3} \delta_{i_l, i_{\sigma^\bullet(\sigma^\circ(l))}} \delta_{\alpha_l,\alpha_{\sigma^\circ(l)}}.$$

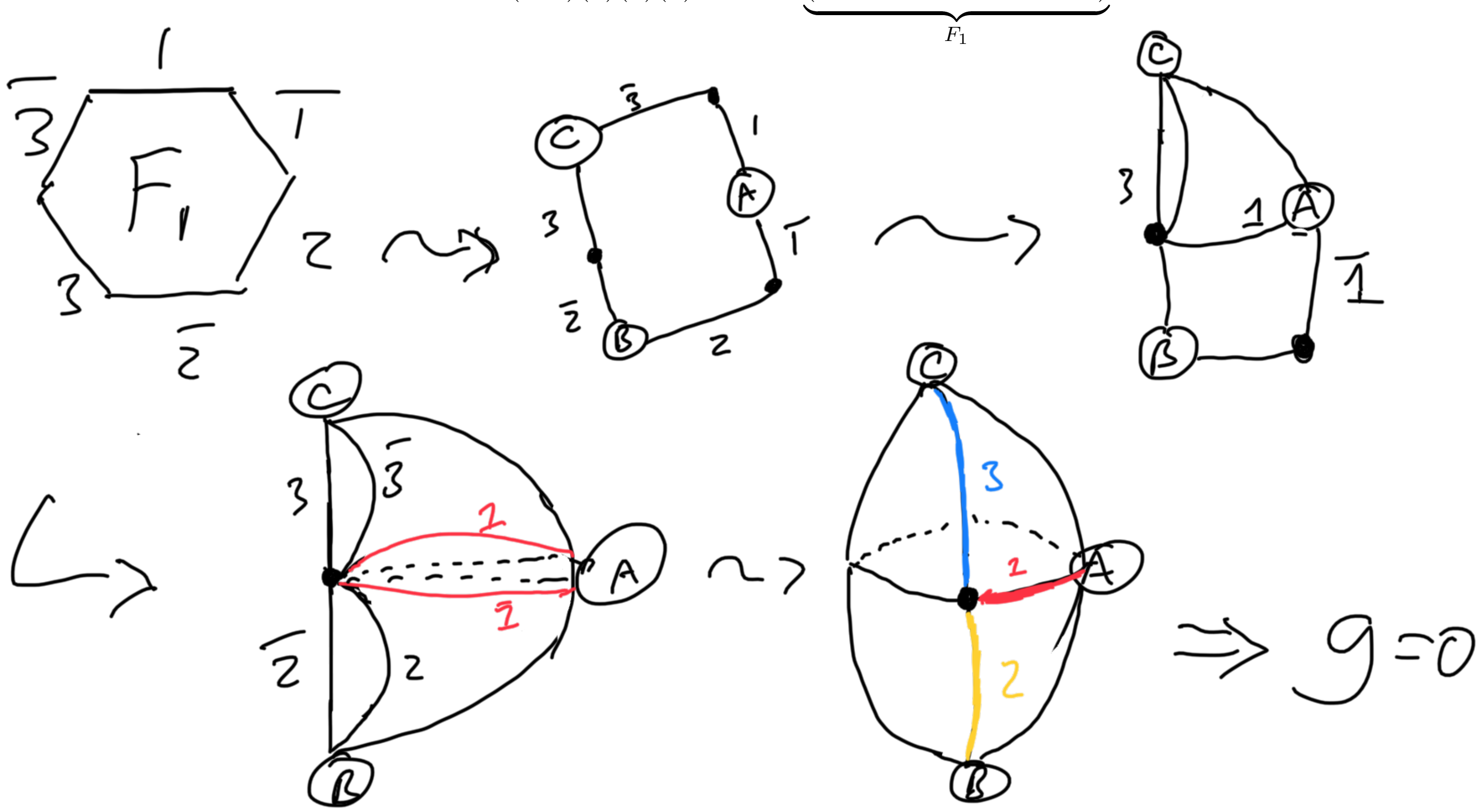
And finally we apply our genus expansion:

$$m_3 = \sum_{g \geq 0} N^{1-2g} m_3^{[g]}$$

There are 6 permutations in $\mathcal{S}_3$. In order to compute m_3, we will need to know the genus of the surface induced by each permutation, so we will study them accordingly.

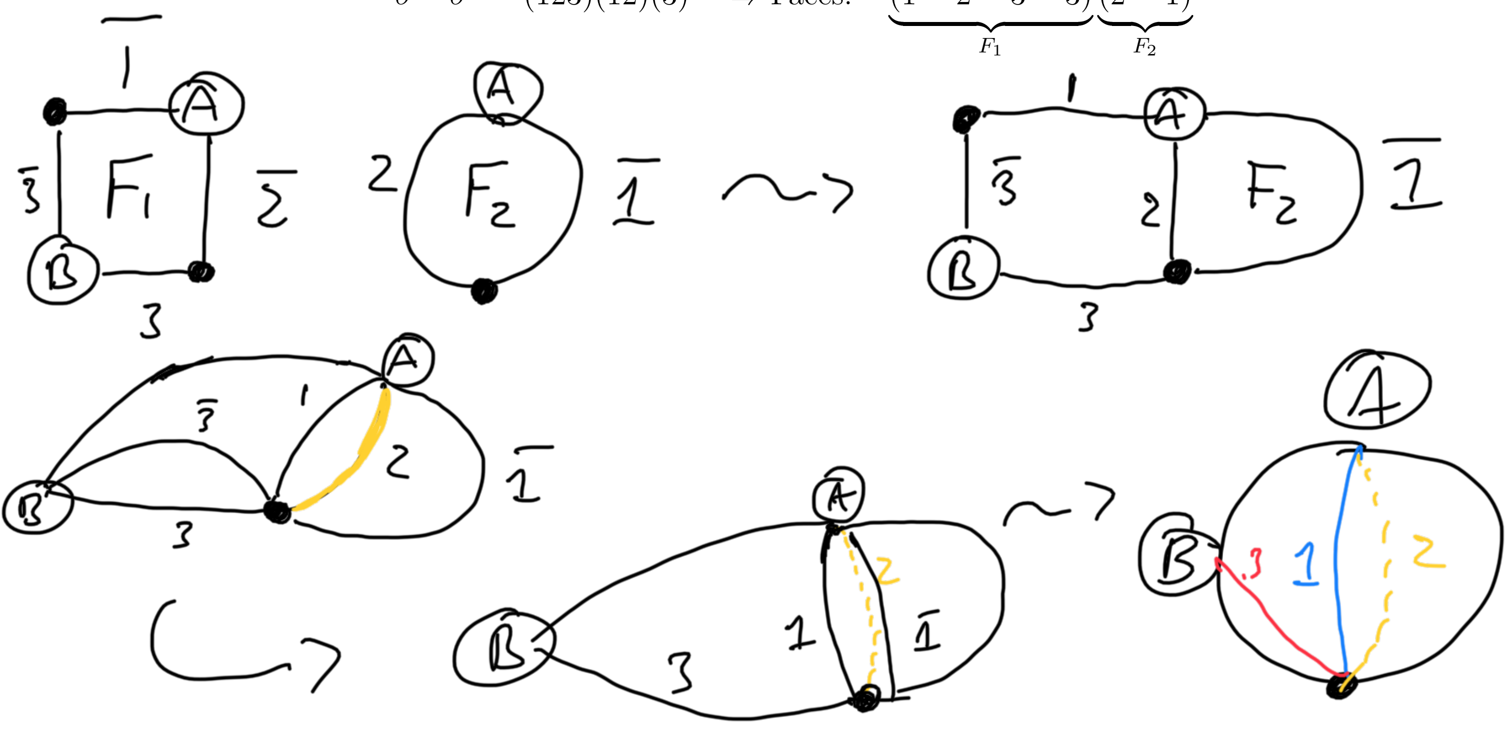
$\sigma^\circ = (1)(2)(3)$ White vertex labels: A, B, C $\underbrace{(1)}_{\text{A}}$ $\underbrace{(2)}_{\text{B}}$ $\underbrace{(3)}_{\text{C}}$

$$\sigma^\bullet \cdot \sigma^\circ = (123)(1)(2)(3) \quad \Rightarrow \quad \underbrace{(1 - \overline{1} - 2 - \overline{2} - 3 - \overline{3})}_{F_1}$$



$\sigma^\circ = (12)(3)$ White vertex labels: A, B $\underbrace{(12)}_{\text{A}}$ $\underbrace{(3)}_{\text{B}}$

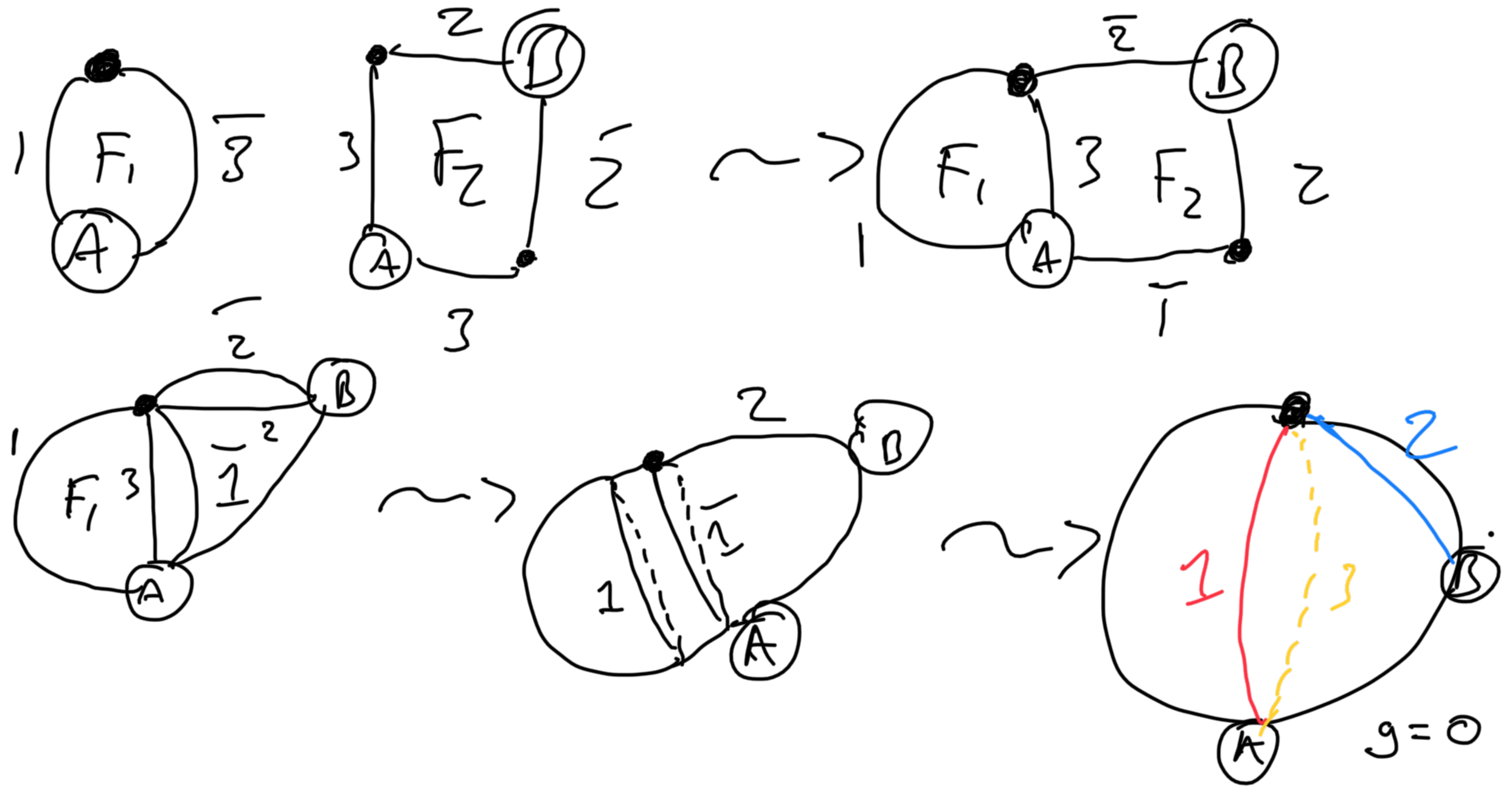
$$\sigma^\bullet \cdot \sigma^\circ = (123)(12)(3) \quad \Rightarrow \text{Faces:} \quad \underbrace{(1 - \overline{2} - 3 - \overline{3})}_{F_1}\underbrace{(2 - \overline{1})}_{F_2}$$



$\Rightarrow g = 0$

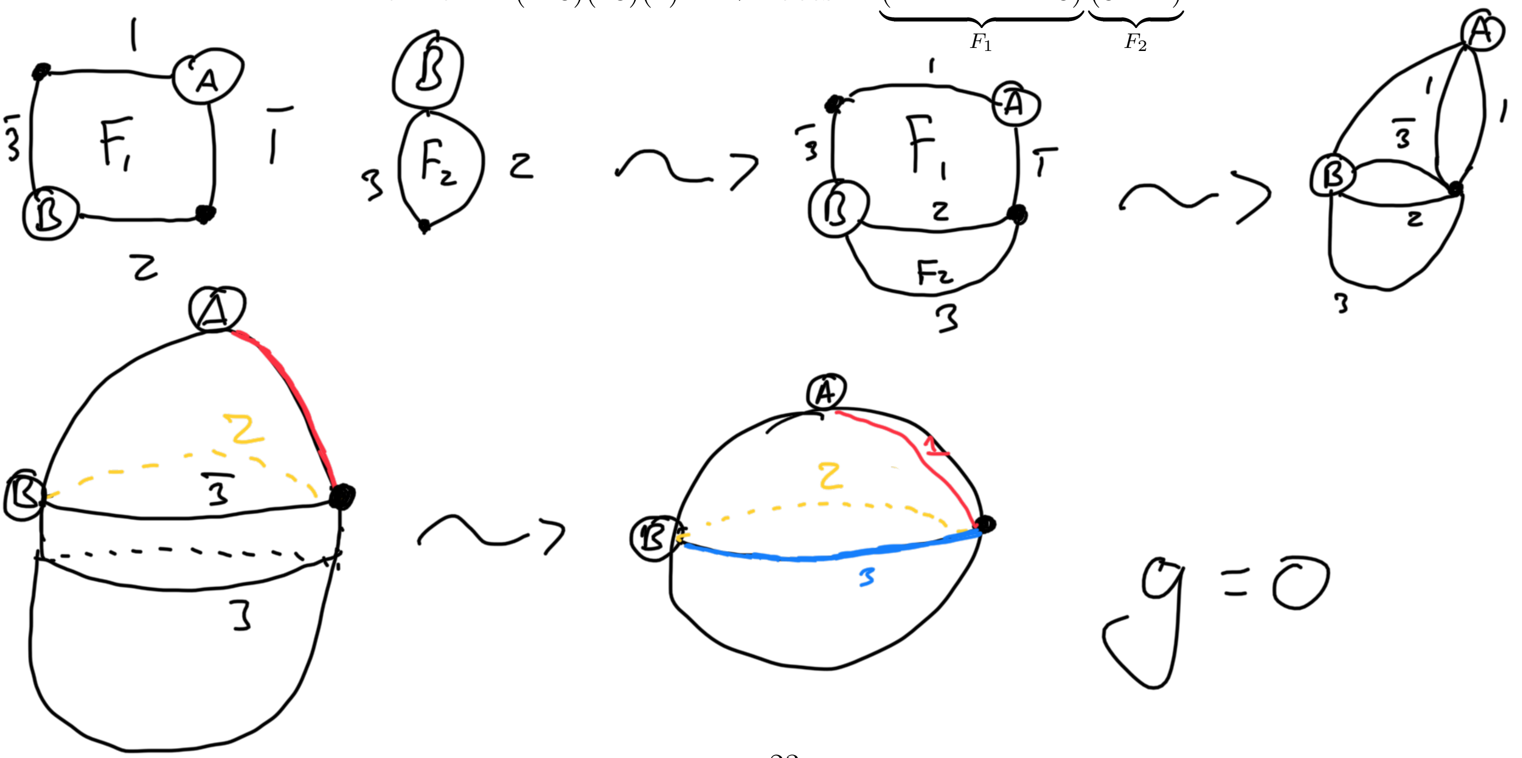
$\sigma^\circ = (13)(2)$ White vertex labels: A, B $\underbrace{(13)}_{\text{A}}$ $\underbrace{(2)}_{\text{B}}$

$\sigma^\bullet \cdot \sigma^\circ = (123)(13)(2) \quad \Rightarrow$ Faces: $\underbrace{(1-\bar{3})}_{F_1}\underbrace{(2-\bar{2}-3-\bar{1})}_{F_2}$



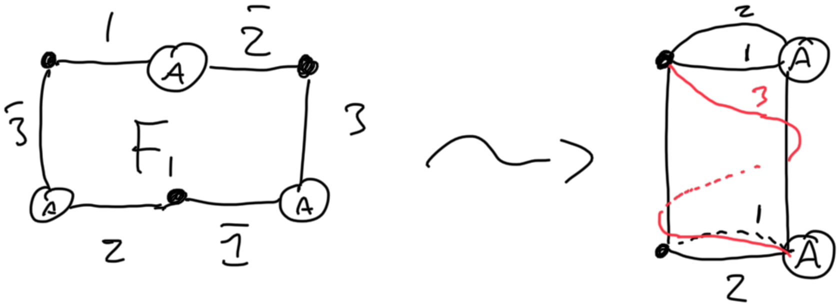
$\sigma^\circ = (23)(1)$ White vertex labels: A, B $\underbrace{(23)}_{\text{A}}$ $\underbrace{(1)}_{\text{B}}$

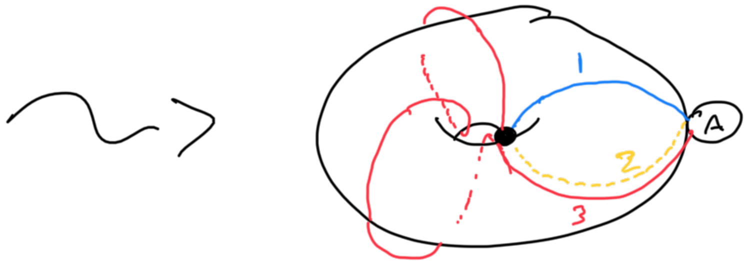
$\sigma^\bullet \cdot \sigma^\circ = (123)(23)(1) \quad \Rightarrow$ Faces: $\underbrace{(1-\bar{1}-2-\bar{3})}_{F_1}\underbrace{(3-\bar{2})}_{F_2}$



$\sigma^\circ = (123)$ White vertex label: A $\underbrace{(123)}_{\text{A}}$

$\sigma^\bullet \cdot \sigma^\circ = (123)(123) \quad \Rightarrow$ Faces: $\underbrace{(1 - \overline{2} - 3 - \overline{1} - 2 - \overline{3})}_{F_1}$

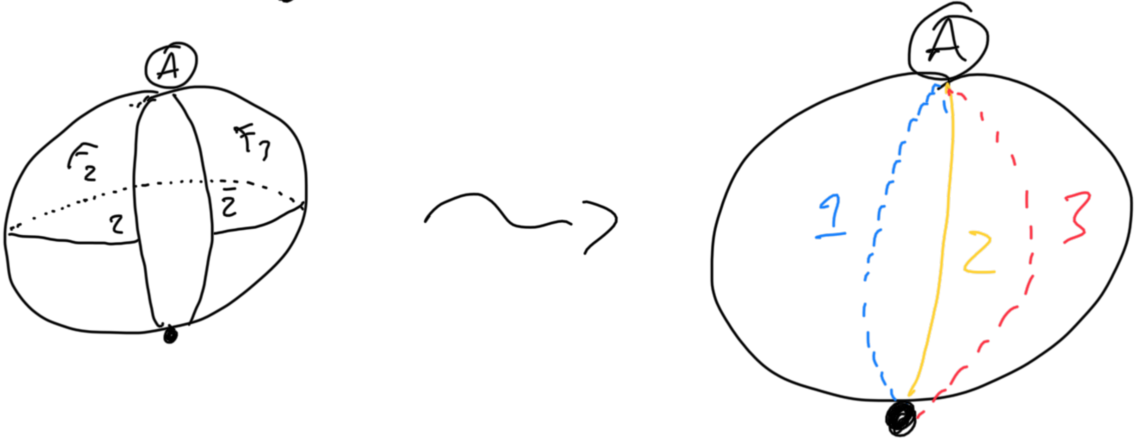




$g = 1$

$\sigma^\circ = (132)$ White vertex label: A $\underbrace{(132)}_{\text{A}}$

$\sigma^\bullet \cdot \sigma^\circ = (123)(132) \quad \Rightarrow$ Faces: $\underbrace{(1 - \overline{3})}_{F_1}\underbrace{(2 - \overline{1})}_{F_2}\underbrace{(3 - \overline{2})}_{F_3}$



$g = 0$

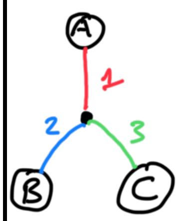
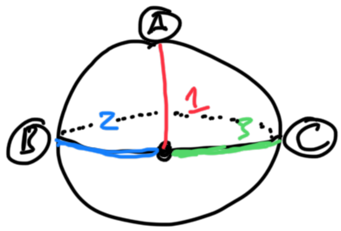
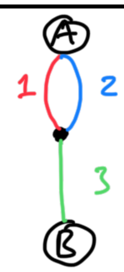
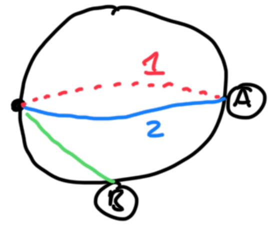
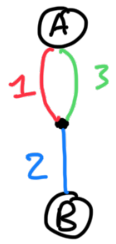
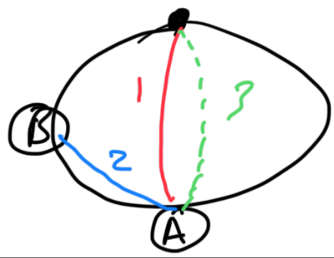
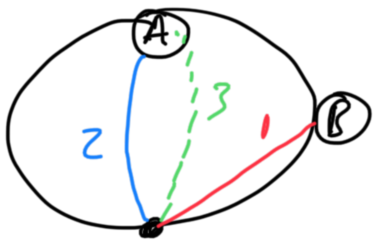
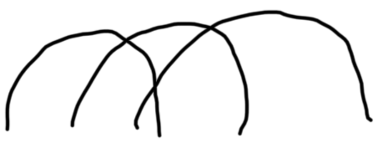
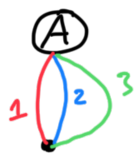
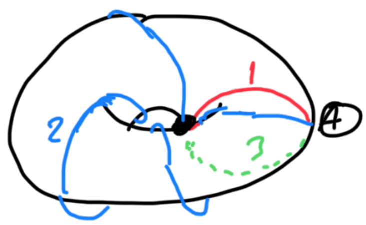
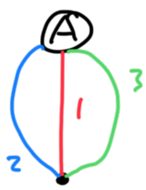
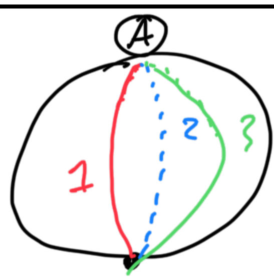
Pairing	**Heterogenous Wick-Pairing**	σ°	**Graph**	**Surface**
$\{1, \overline{1}\}$, $\{2, \overline{2}\}$, $\{3, \overline{3}\}$ 	$\mathrm{Cov}(X_{i_1\alpha_1}, \overline{X}_{\alpha_1 i_1})\cdot$ $\mathrm{Cov}(X_{i_2\alpha_2}, \overline{X}_{\alpha_2 i_2})\cdot$ $\mathrm{Cov}(X_{i_3\alpha_3}, \overline{X}_{\alpha_3 i_3})$	(1)(2)(3)		
$\{1, \overline{2}\}$, $\{2, \overline{1}\}$, $\{3, \overline{3}\}$ 	$\mathrm{Cov}(X_{i_1\alpha_1}, \overline{X}_{\alpha_2 i_1})\cdot$ $\mathrm{Cov}(X_{i_2\alpha_2}, \overline{X}_{\alpha_1 i_2})\cdot$ $\mathrm{Cov}(X_{i_3\alpha_3}, \overline{X}_{\alpha_3 i_3})$	(12)(3)		
$\{1, \overline{3}\}$, $\{2, \overline{2}\}$, $\{3, \overline{1}\}$ 	$\mathrm{Cov}(X_{i_1\alpha_1}, \overline{X}_{\alpha_3 i_1})\cdot$ $\mathrm{Cov}(X_{i_2\alpha_2}, \overline{X}_{\alpha_2 i_2})\cdot$ $\mathrm{Cov}(X_{i_3\alpha_3}, \overline{X}_{\alpha_1 i_3})$	(13)(2)		
$\{1, \overline{1}\}$, $\{2, \overline{3}\}$, $\{3, \overline{2}\}$ 	$\mathrm{Cov}(X_{i_1\alpha_1}, \overline{X}_{\alpha_1 i_1})\cdot$ $\mathrm{Cov}(X_{i_2\alpha_2}, \overline{X}_{\alpha_3 i_2})\cdot$ $\mathrm{Cov}(X_{i_3\alpha_3}, \overline{X}_{\alpha_2 i_3})$	(23)(1)		
$\{1, \overline{2}\}$, $\{2, \overline{3}\}$, $\{3, \overline{1}\}$ 	$\mathrm{Cov}(X_{i_1\alpha_1}, \overline{X}_{\alpha_2 i_1})\cdot$ $\mathrm{Cov}(X_{i_2\alpha_2}, \overline{X}_{\alpha_3 i_2})\cdot$ $\mathrm{Cov}(X_{i_3\alpha_3}, \overline{X}_{\alpha_1 i_3})$	(123)		
$\{1, \overline{3}\}$, $\{2, \overline{1}\}$, $\{3, \overline{2}\}$ 	$\mathrm{Cov}(X_{i_1\alpha_1}, \overline{X}_{\alpha_3 i_1})\cdot$ $\mathrm{Cov}(X_{i_2\alpha_2}, \overline{X}_{\alpha_1 i_2})\cdot$ $\mathrm{Cov}(X_{i_3\alpha_3}, \overline{X}_{\alpha_2 i_3})$	(132)		

Table 3.1: Wick Pairings and Corresponding Graphs/Surfaces for $E[\mathrm{Tr}(S^3)]$

3.2 Calculating Large N Limit of Wishart Normalized Trace Moments

3.2.1 Large N Limit of Moments

Recall we were interested in computing the normalized trace:

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left[\mathrm{Tr}(S^k)\right] = \lim_{N\to\infty} \frac{1}{N} m_k$$

Now, armed with the genus expansion of m_k:

$$m_k = \sum_{g\geq 0} N^{1-2g} m_k^{[g]}$$

We take the limit as $N \to \infty$:

$$\lim_{N\to\infty} \frac{1}{N} m_k = \lim_{N\to\infty} \frac{1}{N} \sum_{g\geq 0} N^{1-2g} m_k^{[g]} = m_k^{[0]}$$

In this limit, the contribution from maps with genus $g > 0$ vanishes, leaving only the planar (genus 0) maps, which is counted by $m_k^{[0]}$.

3.2.2 Non-Crossing Partitions and Catalan Numbers

Now that we have established that $m_k^{[0]}$ is the dominant contribution in the large N limit, it becomes relevant to count these planar maps with one vertex. We do so by showing that the number of planar, labeled, bicolored combinatorial maps with one black vertex and k edges is in one-to-one correspondence with non-crossing partitions.

Bijection Between Maps and Non-Crossing Partitions

Definition 3.2.1. The n-th Catalan number, denoted by C_n, is a sequence of natural numbers that appear in various counting problems, often involving recursive structures. The n-th Catalan number is given by the formula:

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}, \quad n \geq 0.$$

The Catalan numbers satisfy the recurrence relation:

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}, \quad C_0 = 1.$$

Definition 3.2.2. A *non-crossing partition* of a set of k elements is a partition in which no two blocks "cross" when the elements are arranged in a circle. For example, for the set $\{1, 2, 3, 4\}$, the partition $\{\{1, 2\}, \{3, 4\}\}$ is non-crossing, but $\{\{1, 3\}, \{2, 4\}\}$ is crossing.

Proposition 3.2.3. The n-th Catalan number C_n counts the number of non-crossing partitions of a set of n elements. In other words, the Catalan number C_n counts the number of ways to divide the set $\{1, 2, \ldots, n\}$ into non-crossing partitions.

Proposition 3.2.4.

$$m_k^{[0]} = C_k.$$

Proof. To establish this result, we will construct a bijection between the set of planar graphs with k vertices and the set of non-crossing partitions on k elements.

1. **From Non-Crossing Partitions to Planar Graphs**: Start with a non-crossing partition of $\{1, 2, \ldots, k\}$. Each block of the partition corresponds to a set of elements that can be connected without crossing edges. To construct a planar graph:
   (a) **Vertices**: Treat the elements $\{1, 2, \ldots, k\}$ as vertices.
   (b) **Edges**: For each block in the partition, connect the elements in that block to form a complete subgraph (clique). Since the partition is non-crossing, these connections form a planar graph.
2. **From Planar Graphs to Non-Crossing Partitions**: Conversely, given a planar graph on k vertices, we can form a non-crossing partition:
   (a) **Blocks**: Each connected component of the planar graph defines a block of the partition, grouping together all vertices in the same component.
   (b) **Non-Crossing Property**: Since the graph is planar, no edges cross. Thus, the partition formed by the connected components is non-crossing.

This construction demonstrates a one-to-one correspondence between planar graphs on k vertices and non-crossing partitions of k elements. Therefore, the number of planar graphs with k vertices is equal to the number of non-crossing partitions on k elements, both counted by the

Catalan number C_k. □

In Summary we have shown:

$$\lim_{N\to\infty}\frac{1}{N}m_k = \lim_{N\to\infty}\frac{1}{N}\sum_{g\geq 0}N^{1-2g}m_k^{[g]} = m_k^{[0]} = C_k$$

This analysis demonstrates the power of the combinatorial map perspective in computing the trace moments of Wishart matrices. We not only simplified the computation to counting planar maps but also uncovered some Topological structure underlying the large N behavior of these moments. That said, we still need to do more work to find the Wishart spectral density, and to do so we will expand our analysis to multi-moments.

3.3 Multi-Moment $m_{k_1,\ldots,k_n}$ $\frac{1}{N}$ Expansion

After computing the large N limit, of the normalized moment $\frac{1}{N}m_k$ we still need more in order to determine the N limit spectral density of the Wishart matrix. We'll start by studying the multi-moments.

3.3.1 Expanding the Product of Traces into S^{k_i}

We begin with the multi-moment expression:

$$m_{k_1,\ldots,k_n} = \mathbb{E}\left[\prod_{i=1}^{n}\mathrm{Tr}(S^{k_i})\right]$$

Expanding each trace $\mathrm{Tr}(S^{k_i})$ into its indexed form, we get:

$$\mathrm{Tr}(S^{k_i}) = \sum_{i_1,i_2,\ldots,i_{k_i}} S_{i_1 i_2}S_{i_2 i_3}\cdots S_{i_{k_i} i_1}$$

Here, S_{ij} represents the entries of the Wishart matrix S.

For a product of traces we have

$$\mathrm{Tr}(S^{k_1})\cdot\mathrm{Tr}(S^{k_2})\cdot\cdots\cdot\mathrm{Tr}(S^{k_n}) = \left(\sum_{i_1^{(1)},i_2^{(1)},\ldots,i_{k_1}^{(1)}} S_{i_1^{(1)}i_2^{(1)}}S_{i_2^{(1)}i_3^{(1)}}\cdots S_{i_{k_1}^{(1)}i_1^{(1)}}\right)$$

$$\cdot\left(\sum_{i_1^{(2)},i_2^{(2)},\ldots,i_{k_2}^{(2)}} S_{i_1^{(2)}i_2^{(2)}}S_{i_2^{(2)}i_3^{(2)}}\cdots S_{i_{k_2}^{(2)}i_1^{(2)}}\right)$$

$$\cdots \left(\sum_{i_1^{(n)}, i_2^{(n)}, \ldots, i_{k_n}^{(n)}} S_{i_1^{(n)} i_2^{(n)}} S_{i_2^{(n)} i_3^{(n)}} \cdots S_{i_{k_n}^{(n)} i_1^{(n)}} \right)$$

Which we can rewrite as

$$\prod_{j=1}^{n} \mathrm{Tr}(S^{k_j}) = \sum_{\substack{i_1^{(1)}, i_2^{(1)}, \ldots, i_{k_1}^{(1)} \\ i_1^{(2)}, i_2^{(2)}, \ldots, i_{k_2}^{(2)} \\ \vdots \\ i_1^{(n)}, i_2^{(n)}, \ldots, i_{k_n}^{(n)}}} \left(\prod_{j=1}^{n} S_{i_1^{(j)} i_2^{(j)}} S_{i_2^{(j)} i_3^{(j)}} \cdots S_{i_{k_j}^{(j)} i_1^{(j)}} \right)$$

$$= \sum_{\substack{i_1^{(1)}, i_2^{(1)}, \ldots, i_{k_1}^{(1)} \\ i_1^{(2)}, i_2^{(2)}, \ldots, i_{k_2}^{(2)} \\ \vdots \\ i_1^{(n)}, i_2^{(n)}, \ldots, i_{k_n}^{(n)}}} \sum_{\substack{\alpha_1^{(1)}, \alpha_2^{(1)}, \ldots, \alpha_{k_1}^{(1)} \\ \alpha_1^{(2)}, \alpha_2^{(2)}, \ldots, \alpha_{k_2}^{(2)} \\ \vdots \\ \alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_{k_n}^{(n)}}} \left(\prod_{j=1}^{n} X_{i_1^{(j)} \alpha_1^{(j)}} X^{\dagger}_{\alpha_1^{(j)} i_2^{(j)}} X_{i_2^{(j)} \alpha_2^{(j)}} X^{\dagger}_{\alpha_2^{(j)} i_3^{(j)}} \cdots X_{i_{k_j}^{(j)} \alpha_{k_j}^{(j)}} X^{\dagger}_{\alpha_{k_j}^{(j)} i_1^{(j)}} \right)$$

3.3.2 Now we take expectation

$$m_{k_1,\ldots,k_n} = \mathbb{E}\left[\sum_{\substack{i_1^{(1)}, i_2^{(1)}, \ldots, i_{k_1}^{(1)} \\ i_1^{(2)}, i_2^{(2)}, \ldots, i_{k_2}^{(2)} \\ \vdots \\ i_1^{(n)}, i_2^{(n)}, \ldots, i_{k_n}^{(n)}}} \sum_{\substack{\alpha_1^{(1)}, \alpha_2^{(1)}, \ldots, \alpha_{k_1}^{(1)} \\ \alpha_1^{(2)}, \alpha_2^{(2)}, \ldots, \alpha_{k_2}^{(2)} \\ \vdots \\ \alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_{k_n}^{(n)}}} \left(\prod_{j=1}^{n} X_{i_1^{(j)} \alpha_1^{(j)}} X^{\dagger}_{\alpha_1^{(j)} i_2^{(j)}} X_{i_2^{(j)} \alpha_2^{(j)}} X^{\dagger}_{\alpha_2^{(j)} i_3^{(j)}} \cdots X_{i_{k_j}^{(j)} \alpha_{k_j}^{(j)}} X^{\dagger}_{\alpha_{k_j}^{(j)} i_1^{(j)}} \right) \right]$$

$$=$$

$$m_{k_1,\ldots,k_n} = \sum_{\substack{i_1^{(1)}, \ldots, i_{k_1}^{(1)}, \\ \alpha_1^{(1)}, \ldots, \alpha_{k_1}^{(1)}, \\ \ldots, \\ i_1^{(2)}, \ldots, i_{k_2}^{(2)}, \\ \alpha_1^{(2)}, \ldots, \alpha_{k_2}^{(2)}, \\ \ldots, \\ i_1^{(n)}, \ldots, i_{k_n}^{(n)}, \\ \alpha_1^{(n)}, \ldots, \alpha_{k_n}^{(n)}}} \mathbb{E}\Big(X_{i_1^{(1)} \alpha_1^{(1)}} \overline{X}_{i_2^{(1)} \alpha_1^{(1)}} \ldots X_{i_{k_1}^{(1)} \alpha_{k_1}^{(1)}} \overline{X}_{i_1^{(1)} \alpha_{k_1}^{(1)}} X_{i_1^{(2)} \alpha_1^{(2)}} \overline{X}_{i_2^{(2)} \alpha_1^{(2)}} \ldots X_{i_{k_2}^{(2)} \alpha_{k_2}^{(2)}} \overline{X}_{i_1^{(2)} \alpha_{k_2}^{(2)}}$$

$$\ldots X_{i_1^{(n)} \alpha_1^{(n)}} \overline{X}_{i_2^{(n)} \alpha_1^{(n)}} \ldots X_{i_{k_n}^{(n)} \alpha_{k_n}^{(n)}} \overline{X}_{i_1^{(n)} \alpha_{k_n}^{(n)}} \Big)$$

Here, X represents the complex Ginibre matrix, and $\overline{X}$ denotes its conjugate transpose.

Notice each summand contributes $2p$ factors.

Now to simplify the indices, we relabel everything in terms of $i_1, \ldots, i_p$ and $\alpha_1, \ldots, \alpha_p$.

$$m_{k_1,\ldots,k_n} = \sum_{\substack{i_1,\ldots,i_p \\ \alpha_1,\ldots,\alpha_p}} \mathbb{E}\left(X_{i_1\alpha_1}\overline{X}_{i_2\alpha_1}X_{i_2\alpha_2}\overline{X}_{i_3\alpha_2}\ldots X_{i_{k_1}\alpha_{k_1}}\overline{X}_{i_1\alpha_{k_1}}X_{i_{k_1+1}\alpha_{k_1+1}}\overline{X}_{i_{k_1+2}\alpha_{k_1+1}}\ldots X_{i_p\alpha_p}\overline{X}_{i_1\alpha_p}\right)$$

This combines all indices $i_1, \ldots, i_p$ and $\alpha_1, \ldots, \alpha_p$ into one sum, with $p = k_1+k_2+\cdots+k_n$.

3.3.3 Applying Wick's Formula

Using Wick's formula, the expectation of the product of Gaussian variables is expressed as a sum over all possible pairings of these variables, with each term in the sum being a product of the covariances:

$$m_{k_1,\ldots,k_n} = \sum_{\substack{i_1,\ldots,i_p \\ \alpha_1,\ldots,\alpha_p}} \sum_{\text{all pairings}} \prod_{(p,q)\in\text{pairing}} \operatorname{Cov}\left(X_{i_p\alpha_p}, \overline{X}_{i_q\alpha_q}\right)$$

Given that the covariance $\operatorname{Cov}\left(X_{i_p\alpha_p}, \overline{X}_{\alpha_q i_q}\right)$ for independent identically distributed Gaussian entries of X is:

$$\operatorname{Cov}\left(X_{i_p\alpha_p}, \overline{X}_{\alpha_q i_q}\right) = \frac{1}{N}\delta_{i_p,i_q}\delta_{\alpha_p,\alpha_q}$$

This simplifies the expression to:

$$m_{k_1,\ldots,k_n} = \sum_{\substack{i_1^{(1)},\ldots,i_{k_1}^{(1)}, \\ \alpha_1^{(1)},\ldots,\alpha_{k_1}^{(1)}, \\ \ldots, \\ i_1^{(n)},\ldots,i_{k_n}^{(n)}, \\ \alpha_1^{(n)},\ldots,\alpha_{k_n}^{(n)}}} \sum_{\text{all pairings}} \prod_{(p,q)\in\text{pairing}} \frac{1}{N}\delta_{i_p,i_q}\delta_{\alpha_p,\alpha_q}$$

This equation represents the sum over all possible pairings of the indices, where each pairing enforces certain index equalities through the Kronecker deltas, and the contribution from each pairing is weighted by $\frac{1}{N}$.

3.3.4 Rewriting Using σ° and Summing over Symmetric Group

Once again we realize that we are only interested in summing over heterogeneous pairings of the form $XX^\dagger$, and therefore we can index our sum over the symmetric group permuting the $X^\dagger$ terms.

However, unlike the single moment case, we need to account for the redundancy in indices imposed by the multiple products of traces.

We can do so by adjusting our $\sigma^\bullet$ to take into account this redundancy:

$$m_{k_1,\ldots,k_n} = \sum_{\substack{i_1,\ldots,i_p\\ \alpha_1,\ldots,\alpha_p}} \sum_{\sigma^\circ \in S_p} \prod_{l=1}^{p} \mathrm{Cov}\left(X_{i_l\alpha_l}, \overline{X}_{i_{\sigma^\bullet(\sigma^\circ(l))}\alpha_{\sigma^\circ(l)}}\right)$$

where $\sigma_\bullet = (1,\ldots,k_1)(k_1+1,\ldots,k_1+k_2)\cdots(p-k_n+1,\ldots,p)$.

The Kronecker deltas enforce the condition that $i_p = i_q$ and $\alpha_p = \alpha_q$, which are captured by the cycles in the permutation σ°. Rewriting the sum over pairings as a sum over the permutation σ°, we have:

$$m_{k_1,\ldots,k_n} = \sum_{\substack{i_1,\ldots,i_p\\ \alpha_1,\ldots,\alpha_p}} \sum_{\sigma^\circ \in S_p} \prod_{l=1}^{p} \delta_{i_l, i_{\sigma_\bullet(\sigma^\circ(l))}} \delta_{\alpha_l, \alpha_{\sigma^\circ(l)}}$$

Changing order of summation and counting Kronecker deltas, as we did for the single moment case, gives us the final expression:

$$m_{k_1,\ldots,k_n} = \sum_{\sigma^\circ \in S_p} N^{V_\circ(M)-p+F(M)}$$

3.3.5 Final Multi-Moment Expansion

Summing over the relevant combinatorial maps M associated with each σ°, we conclude with:

$$m_{k_1,\ldots,k_n} = \sum_{M \in \mathcal{M}_{k_1,\ldots,k_n}} N^{V_\circ(M)-p+F(M)}$$

where:

- $V_\circ(M)$ is the number of white vertices (disjoint cycles in σ°).
- $p = k_1 + k_2 + \cdots + k_n$ is the total number of edges.
- $F(M)$ is the number of faces (cycles in $\sigma^\bullet\sigma^\circ$).
- $\mathcal{M}_{k_1,\ldots,k_n}$ denotes the set of combinatorial maps containing p edges and $\sigma_\bullet = (1,\ldots,k_1)(k_1+1,\ldots,k_1+k_2)\cdots(p-k_n+1,\ldots,p)$.

This result demonstrates the $\frac{1}{N}$ expansion of the multi-moments as a sum over combinatorial maps, weighted by the topological features of these maps. Notice that in this situation, we do not have a genus expansion for $m_{k_1,\ldots,k_n}$ like we had for single-moments. This is because the genus is only defined for combinatorial maps that have a connected CW complex associated with them. This happens precisely when the graph is connected i.e. $\sigma_\bullet$ and $\sigma_\circ$ act transitively on $\{1, \ldots, p\}$. This motivates us to try and re-index our sum over the connected maps instead, and we'll need cumulants to do so.

3.4 Moment-Cumulant Relation

3.4.1 Cumulants

In probability theory and statistics, moments and cumulants are two types of measures used to describe the shape of probability distributions.

The n-th moment of a random variable X is defined as:

$$\mu_n = \mathbb{E}[X^n]$$

The cumulants κ_n of a random variable X are defined through the cumulant-generating function $K(t)$, which is the natural logarithm of the moment-generating function $M_X(t)$:

$$K(t) = \log \mathbb{E}\left[e^{tX}\right].$$

This generating function can be expanded as a power series:

$$K(t) = \sum_{n=1}^{\infty} \kappa_n \frac{t^n}{n!}.$$

In this expansion, the coefficient κ_n is the n-th cumulant of the distribution of X.

First Cumulant κ_1:

The first cumulant is defined as the first derivative of $K(t)$ with respect to t at $t = 0$:

$$\kappa_1 = \left.\frac{dK(t)}{dt}\right|_{t=0}.$$

Using the chain rule:

$$\frac{dK(t)}{dt} = \frac{d}{dt} \log M(t) = \frac{1}{M(t)} \frac{dM(t)}{dt}.$$

Now, differentiate the moment-generating function $M(t)$:

$$\frac{dM(t)}{dt} = \frac{d}{dt}\mathbb{E}[e^{tX}] = \mathbb{E}\left[Xe^{tX}\right].$$

Substituting this into the expression for $K(t)$:

$$\frac{dK(t)}{dt} = \frac{1}{M(t)}\mathbb{E}[Xe^{tX}].$$

At $t = 0$, the moment-generating function $M(0) = \mathbb{E}[e^0] = 1$, and $\mathbb{E}[Xe^0] = \mathbb{E}[X]$. Therefore:

$$\kappa_1 = \left.\frac{dK(t)}{dt}\right|_{t=0} = \mathbb{E}[X].$$

Thus, the first cumulant κ_1 is indeed the mean μ of the distribution.

Second Cumulant κ_2:

$$\kappa_2 = \left.\frac{d^2K(t)}{dt^2}\right|_{t=0} = \mathbb{E}[X^2] - (\mathbb{E}[X])^2,$$

which is the variance σ^2 of the distribution.

Proof:

The second cumulant is defined as the second derivative of the cumulant-generating function $K(t)$:

$$\kappa_2 = \left.\frac{d^2K(t)}{dt^2}\right|_{t=0}.$$

First, differentiate $\frac{dK(t)}{dt}$ with respect to t again:

$$\frac{d^2K(t)}{dt^2} = \frac{d}{dt}\left(\frac{1}{M(t)}\mathbb{E}[Xe^{tX}]\right).$$

Using the product rule:

$$\frac{d^2K(t)}{dt^2} = \frac{-1}{M(t)^2}\left(\frac{dM(t)}{dt}\right)\cdot\mathbb{E}[Xe^{tX}] + \frac{1}{M(t)}\mathbb{E}\left[X^2e^{tX}\right].$$

At $t = 0$, we substitute $M(0) = 1$ and $\frac{dM(0)}{dt} = \mathbb{E}[X]$, so the second derivative simplifies to:

$$\left.\frac{d^2K(t)}{dt^2}\right|_{t=0} = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$

Therefore, the second cumulant κ_2 is the variance of the distribution:

$$\kappa_2 = \sigma^2 = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$

Third Cumulant κ_3:

$$\kappa_3 = \frac{d^3K(t)}{dt^3}\bigg|_{t=0} = \mathbb{E}[(X - \mathbb{E}[X])^3],$$

which is the third central moment, representing the skewness of the distribution.

Fourth Cumulant κ_4:

$$\kappa_4 = \frac{d^4K(t)}{dt^4}\bigg|_{t=0} = \mathbb{E}[(X - \mathbb{E}[X])^4] - 3(\mathbb{E}[(X - \mathbb{E}[X])^2])^2,$$

this is the first case in which cumulants are not simply moments or central moments.

3.4.2 Joint Cumulants for Products of Independent Random Variables and Relation to Mixed Moments

Definition 3.4.1. We first begin with the power series definition of joint cumulants following Peccati and Taqqu[PT2011]. The joint cumulant $\kappa(X_1, \ldots, X_n)$ of several random variables $X_1, \ldots, X_n$ is defined as the coefficient in the Maclaurin series expansion of the multivariate cumulant-generating function $G(t_1, \ldots, t_n)$, given by:

$$G(t_1, \ldots, t_n) = \log \mathbb{E}\left[e^{\sum_{j=1}^n t_j X_j}\right] = \sum_{l_1,\ldots,l_n} \kappa(X_1, \ldots, X_n) \frac{t_1^{l_1} \cdots t_n^{l_n}}{l_1! \cdots l_n!}.$$

The joint cumulants $\kappa(X_1, \ldots, X_n)$ are obtained by taking partial derivatives of the cumulant-generating function:

$$\kappa(X_1, \ldots, X_n) = \frac{\partial^n}{\partial t_1 \cdots \partial t_n} G(t_1, \ldots, t_n)\bigg|_{t_1=\cdots=t_n=0}.$$

Joint cumulants describe the dependence between multiple random variables and can be expressed as an alternate sum of products of their mixed moments. The formula for the joint cumulant of $X_1, X_2, \ldots, X_n$ is given by:

$$\kappa(X_1, \ldots, X_n) = \sum_{\pi} (|\pi| - 1)!(-1)^{|\pi|-1} \prod_{B\in\pi} \mathbb{E}\left(\prod_{i\in B} X_i\right),$$

where:

- π runs through all partitions of the set $\{1, \ldots, n\}$,
- B is one of the blocks in the partition π,

- $|\pi|$ is the number of parts in the partition π.

This expression is derived in detail in Mingo and Speicher's work[MS2018, Section 1.13].

Examples:

- The joint cumulant of a single random variable is its expected value:

$$\kappa(X) = \mathbb{E}(X).$$

- For two random variables X and Y, the joint cumulant is the covariance:

$$\kappa(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y).$$

- For three random variables X, Y, and Z, the joint cumulant is given by:

$$\kappa(X, Y, Z) = \mathbb{E}(XYZ) - \mathbb{E}(XY)\mathbb{E}(Z) - \mathbb{E}(XZ)\mathbb{E}(Y) - \mathbb{E}(YZ)\mathbb{E}(X) + 2\mathbb{E}(X)\mathbb{E}(Y)\mathbb{E}(Z).$$

For zero-mean random variables $X_1, \ldots, X_n$, any mixed moment of the form $\prod_{B\in\pi} \mathbb{E}\left(\prod_{i\in B} X_i\right)$ vanishes if any partition π contains a singleton block. As a result, the joint cumulant for such zero-mean variables simplifies. For example, if X, Y, Z are zero-mean random variables, we have:

$$\kappa(X, Y, Z) = \mathbb{E}(XYZ).$$

For four zero-mean random variables X, Y, Z, W, the joint cumulant becomes:

$$\kappa(X, Y, Z, W) = \mathbb{E}(XYZW) - \mathbb{E}(XY)\mathbb{E}(ZW) - \mathbb{E}(XZ)\mathbb{E}(YW) - \mathbb{E}(XW)\mathbb{E}(YZ).$$

Mixed Moments in Terms of Cumulants:

The mixed moments of random variables can be expressed in terms of joint cumulants. For example, for random variables $X_1, \ldots, X_n$, we have:

$$\mathbb{E}(X_1 \cdots X_n) = \sum_{\pi} \prod_{B\in\pi} \kappa(X_i : i \in B),$$

where π runs through all partitions of the set $\{1, \ldots, n\}$, and B is a block of the partition π.

This relationship is derived and discussed in Mingo and Speicher's work[MS2018, Section 1.13].

For instance, the mixed moment for three random variables X, Y, Z can be written as:

$$\mathbb{E}(XYZ) = \kappa(X, Y, Z) + \kappa(X, Y)\kappa(Z) + \kappa(X, Z)\kappa(Y) + \kappa(Y, Z)\kappa(X) + \kappa(X)\kappa(Y)\kappa(Z).$$

3.4.3 Moment Cumulation relation for Trace moments

Applying the above moment-cumulant relation on the random variables: $\{R_{k_i} := \mathrm{Tr}(S^{k_i})\}$ the relationship between trace moments $m_{k_1,\ldots,k_n}$, and trace cumulants $c_{k_1,\ldots,k_n}$ can be expressed as:

$$m_{k_1,\ldots,k_n} = \sum_{K \vdash \{k_1,\ldots,k_n\}} \prod_{\kappa_i \in K} c_{\kappa_i}$$

Here, the sum is taken over all partitions K of the set $\{k_1, \ldots, k_n\}$, and κ_i denotes the elements of the partition.

For example the moment $m_{4,2,6}$, we can list all possible partitions of the set $\{4, 2, 6\}$ as follows:

1. $\{\{4, 2, 6\}\}$
2. $\{\{4\}, \{2, 6\}\}$
3. $\{\{4, 2\}, \{6\}\}$
4. $\{\{4, 6\}, \{2\}\}$
5. $\{\{4\}, \{2\}, \{6\}\}$

The moment $m_{4,2,6}$ is then given by:

$$m_{4,2,6} = c_{4,2,6} + c_4 \cdot c_{2,6} + c_{4,2} \cdot c_6 + c_{4,6} \cdot c_2 + c_4 \cdot c_2 \cdot c_6$$

This expression sums over all partitions of the set $\{4, 2, 6\}$ and provides the relationship between the moment $m_{4,2,6}$ and the associated cumulants.

3.5 Cumulants and Connected Maps

Definition 3.5.1. $M^c_{k_1,\ldots,k_n} = (E, \sigma^\circ, \sigma^\bullet)$is the set of labeled bicolored combinatorial maps that are connected, i.e., the group $\langle \sigma^\circ, \sigma^\bullet \rangle$ generated by σ° and $\sigma^\bullet$ acts transitively on the set of edges E.

In this section we'll explore the correspondence between connected combinatorial maps and trace cumulants.

We'll prove these cumulants can be expressed as sums over connected combinatorial maps:

$$c_{k_1,\ldots,k_n} = \sum_{M \in M^c_{k_1,\ldots,k_n}} N^{V^\circ(M)-p+F(M)}.$$

To prove this we'll need to invoke Rota Möbius inversion.

3.6 Rota-Möbius Inversion and Moment-Cumulant Relations

Let f and g be two arithmetic functions defined on the positive integers, where the relation between them is given by:

$$g(n) = \sum_{d|n} f(d)$$

The Möbius inversion formula allows us to recover $f(n)$ from $g(n)$. Specifically,

$$f(n) = \sum_{d|n} \mu(d) g\left(\frac{n}{d}\right),$$

where μ is the Möbius function. This inversion technique is fundamentally tied to the lattice of integers partially ordered by divisibility and provides a means to transform between sums over divisors and sums over multiples.

3.6.1 Rota-Möbius Inversion on Posets

Rota extended the classical Möbius inversion to more general partially ordered sets (posets). In a poset $(P, \leq)$, let f and g be functions defined on the elements of P. If:

$$g(x) = \sum_{y \leq x} f(y),$$

then the Rota-Möbius inversion formula is given by:

$$f(x) = \sum_{y \leq x} \mu(x, y) g(y),$$

where $\mu(x, y)$ is the Möbius function for the poset $(P, \leq)$, which generalizes the classical Möbius function in this context.

Summing Moments Over Connected Components

The moment $m_{k_1,\ldots,k_n}$ counts both connected and disconnected combinatorial maps. To express this moment as a sum over connected components, we decompose it based on the possible ways a map can be disconnected into smaller connected maps.

Let:

- $\mathcal{M}^c_{k_1,\ldots,k_n}$ be the set of connected combinatorial maps.

The moment $m_{k_1,\ldots,k_n}$ can be written as a sum over partitions of the set of indices $\{1, 2, \ldots, n\}$, where each subset corresponds to a connected component.

For instance, a disconnected map can be represented as the disjoint union of smaller connected maps, where each connected component corresponds to a block in a partition of the vertex set. Thus, the total moment can be expressed as a sum over all possible ways to partition the vertices into connected subsets.

The Final Formula as a Sum Over Connected Components

Thus, the moment $m_{k_1,\ldots,k_n}$ can now be rewritten as:

$$m_{k_1,\ldots,k_n} = \sum_{P\in\mathcal{P}(\{1,\ldots,n\})} \prod_{B\in P} \sum_{M\in\mathcal{M}^c_{|B|}} N^{V_\circ(M)-|B|+F(M)}$$

where:

- The sum over $P \in \mathcal{P}(\{1, \ldots, n\})$ represents summing over all partitions of the vertices.
- $\mathcal{M}^c_{|B|}$ is the set of connected combinatorial maps corresponding to the block B.
- The term $N^{V_\circ(M)-|B|+F(M)}$ represents the contribution from a connected map M, where $V_\circ(M)$, $|B|$, and $F(M)$ are the number of white vertices, edges, and faces for that connected component.

3.6.2 Moment-Cumulant Relation

The moment-cumulant relation expresses a moment $m_{k_1,\ldots,k_n}$ in terms of cumulants c_{κ_i} as follows:

$$m_{k_1,\ldots,k_n} = \sum_{K\vdash\{k_1,\ldots,k_n\}} \prod_{\kappa_i\in K} c_{\kappa_i}$$

Here, K represents a partition of the set $\{k_1, \ldots, k_n\}$ and $\prod_{\kappa_i\in K} c_{\kappa_i}$ is the product of cumulants corresponding to each partition κ_i in K.

Moments and Cumulants Relation via Inclusion-Exclusion

The moment $m_{k_1,\ldots,k_n}$ is related to the cumulants (which count connected maps) via the following partition sum:

$$m_{k_1,\ldots,k_n} = \sum_{P \in \mathcal{P}(\{1,\ldots,n\})} \prod_{B \in P} c_{|B|}$$

where:

- $\mathcal{P}(\{1,\ldots,n\})$ is the set of all partitions of the set $\{1,2,\ldots,n\}$,
- P is a partition of the indices, with each block $B \in P$ corresponding to a connected component,
- $c_{|B|}$ is the cumulant corresponding to the connected map associated with the subset B,
- The product $\prod_{B \in P} c_{|B|}$ reflects the contribution from each partition by taking the product of cumulants over the connected components in that partition.

Lattice of Partitions

The lattice of partitions is a partially ordered set (POSET) where the order is determined by reverse refinement. Specifically:

- The largest partition is the single block containing all elements ($\{1,\ldots,n\}$).
- The smallest partition is the set of singletons (e.g., $\{\{1\},\{2\},\ldots,\{n\}\}$).
- For two partitions π_1 and π_2, we say $\pi_1 \leq \pi_2$ if every block of π_1 is contained in some block of π_2.

This ordering reflects the hierarchical structure of partitions and plays a key role in defining the Möbius function.

Möbius Inversion to Isolate Connected Components

By applying Möbius inversion, we can express the cumulants (which correspond to connected maps) in terms of moments. Specifically, we have:

$$c_{k_1,\ldots,k_n} = \sum_{P \leq \{k_1,\ldots,k_n\}} \mu(P,\{k_1,\ldots,k_n\}) m_{|P|}$$

where $\mu(P,\{k_1,\ldots,k_n\})$ is the Möbius function on the lattice of partitions, and $m_{|P|}$ is the moment corresponding to the partition P. This inversion removes the contributions from disconnected components and isolates the connected maps.

Interpretation

When we write $P \leq \{k_1, \ldots, k_n\}$, we're summing over all partitions P that are refinements of the "big block" partition $\{k_1, \ldots, k_n\}$. In this context, refinement means that every block in P is a subset of a block in $\{k_1, \ldots, k_n\}$. The blocks of P break down the larger block $\{k_1, \ldots, k_n\}$ into smaller components.

The Möbius function $\mu(P, \{k_1, \ldots, k_n\})$ alternates signs based on the structure of P, ensuring that disconnected contributions from the moments are systematically removed when refining down to the connected cumulants. The refinement order ensures that the Möbius inversion subtracts away the disconnected components.

By using $P \leq \{k_1, \ldots, k_n\}$, you're summing over all partitions that are derived from splitting $\{k_1, \ldots, k_n\}$. This is consistent with the hierarchical structure of the partition lattice. This refinement order plays a critical role in isolating connected components, ensuring that cumulants capture only the contributions of connected maps.

To invert this expression, we use the Möbius inversion formula in the context of partitions of sets. The goal is to express c_{κ_i} in terms of moments $m_{k_1,\ldots,k_n}$.

Defining the Partitions

Define the set of partitions $\mathcal{P}$ as:

$$\mathcal{P} = \{K \mid K \text{ is a partition of } \{k_1, \ldots, k_n\}\}$$

For each partition K, we denote the blocks as $b_1, \ldots, b_t \in K$.

The moment $m_{k_1,\ldots,k_n}$ can be written in terms of cumulants c_{κ_i} as:

$$m_{k_1,\ldots,k_n} = \sum_{K \in \mathcal{P}} \prod_{\kappa_i \in K} c_{\kappa_i}$$

Applying Möbius Inversion

To invert this sum, we use the Möbius inversion formula adapted for partitions. Specifically, the inverse relation is given by:

$$c_{\kappa_i} = \sum_{K \leq \kappa_i} \mu(K, \kappa_i) m_{k_1,\ldots,k_n}$$

where $\mu(K, \kappa_i)$ is the Möbius function defined on the poset of partitions. For partitions, μ

can be computed as:

$$\mu(K, \kappa_i) = \begin{cases} (-1)^{|K|-|\kappa_i|} & \text{if } \kappa_i \subseteq K \\ 0 & \text{otherwise} \end{cases}$$

Computing the Inverse

To compute c_{κ_i}, we use the formula:

$$c_{\kappa_i} = \sum_{K \leq \kappa_i} (-1)^{|K|-|\kappa_i|} \prod_{b_j \in K} m_{(b_j)}$$

where the sum is over all partitions K that refine κ_i. This allows us to express the cumulants c_{κ_i} in terms of moments $m_{k_1,\ldots,k_n}$.

Combinatorially, this inversion corresponds to performing inclusion-exclusion on the connected graphs, allowing us to recover an expression for cumulants over connected graphs. Specifically, we have:

$$c_{k_1,\ldots,k_n} = \sum_{M \in \mathcal{M}^c_{k_1,\ldots,k_n}} N^{V^\circ(M)-p+F(M)}$$

3.6.3 Genus of Combinatorial Maps and Cumulants

Thanks to the connectedness condition, we can meaningfully inquire about the genus of each of the combinatorial maps we are summing over. In particular we can apply the Euler Characteristic to the exponent in the expression above:

$$V_\bullet(M) + V^\circ(M) - p + F(M) = 2 - 2g(M)$$

leading us to the following expression for the cumulants:

$$c_{k_1,\ldots,k_n} = \sum_{M \in \mathcal{M}^c_{k_1,\ldots,k_n}} N^{2-2g(M)-n}$$

Finally, we can obtain a genus expansion for $c_{k_1,\ldots,k_n}$:

$$c_{k_1,\ldots,k_n} = \sum_{g \geq 0} N^{2-2g-n} \sum_{\substack{M \in \mathcal{M}^c_{k_1,\ldots,k_n} \\ g(M)=g}} 1 = \sum_{g \geq 0} N^{2-2g-n} c^{[g]}_{k_1,\ldots,k_n},$$

where

$$c^{[g]}_{k_1,\dots,k_n} := \sum_{\substack{M \in \mathcal{M}^c_{k_1,\dots,k_n} \\ g(M)=g}} 1.$$

Example: Moment-Cumulant Relation at the level of Surfaces

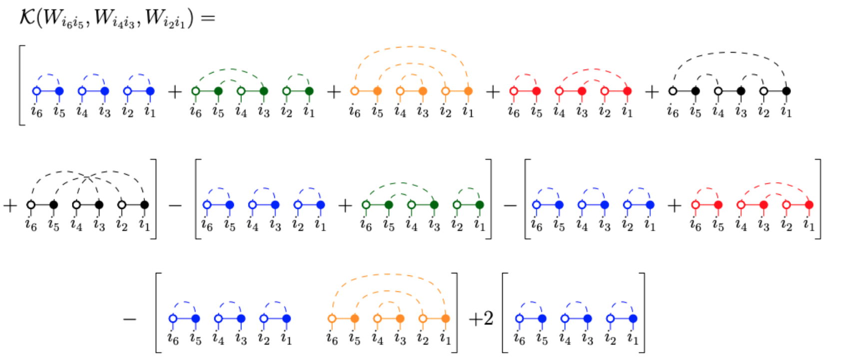


FIGURE 7. Graphical expansion of the cumulant $\mathcal{K}(W_{i_6 i_5}, W_{i_4 i_3}, W_{i_2 i_1})$ from its moments decomposition and the Wick theorem. Identical terms are shown in the same color.

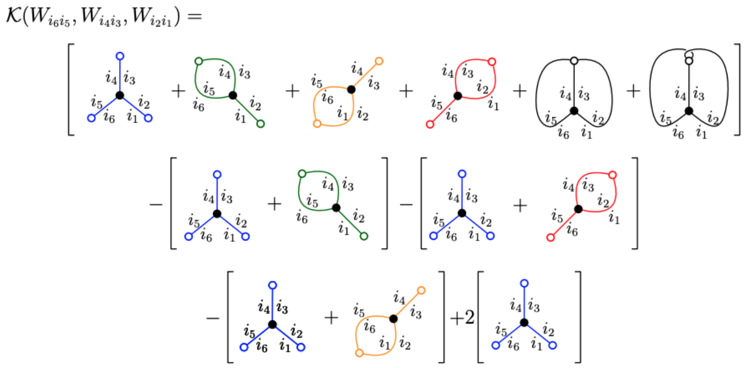


Figure 3.5: Moment-Cumulant Relation at the level of surfaces.[DLN2018]

3.7 Tutte Identity

In order to obtain a recursive relation between the moments and the double moments, we follow the approach of [DF2020] and prove the following identity:

Theorem 3.7.1 (Tutte Identity)**.**

$$\sum_{a,b=1}^{N} \int \frac{N^{N^2}}{(2i\pi)^{N^2}} dX^\dagger dX \, \partial_{X^\dagger_{ab}} \left((X^\dagger S^k)_{ab} e^{-N \operatorname{Tr}(XX^\dagger)} \right) = 0.$$

To prove this statement, we'll prove a slightly more general statement that implies each term in the sum in (1) is identically zero.

Proposition 3.7.2. For any differentiable function $f\colon \mathbb{C}^{N\times N}\to\mathbb{C}$, the following identity holds:

$$\int \partial_{X^\dagger_{ab}}\left(f(X)e^{-N\operatorname{Tr}(XX^\dagger)}\right)dX^\dagger dX=0,$$

Proof: Applying the product rule:

$$\partial_{X^\dagger_{ab}}\left(f(X)e^{-N\operatorname{Tr}(XX^\dagger)}\right)=\left(\partial_{X^\dagger_{ab}}f(X)\right)e^{-N\operatorname{Tr}(XX^\dagger)}+f(X)\partial_{X^\dagger_{ab}}\left(e^{-N\operatorname{Tr}(XX^\dagger)}\right).$$

The derivative of the exponential term is:

$$\partial_{X^\dagger_{ab}}e^{-N\operatorname{Tr}(XX^\dagger)}=-NX_{ba}e^{-N\operatorname{Tr}(XX^\dagger)}.$$

Thus, the expression becomes:

$$\int\left(\partial_{X^\dagger_{ab}}f(X)e^{-N\operatorname{Tr}(XX^\dagger)}-NX_{ba}f(X)e^{-N\operatorname{Tr}(XX^\dagger)}\right)dX^\dagger dX.$$

We rewrite this as an expectation:

$$\mathbb{E}\left[\partial_{X^\dagger_{ab}}f(X)\right]-N\mathbb{E}\left[X_{ba}f(X)\right].$$

Using the covariance version of **Stein's Lemma**, we know:

$$\mathbb{E}\left[X_{ba}f(X)\right]=\sum_{i,j}\operatorname{Cov}(X_{ba},X^\dagger_{ij})\mathbb{E}\left[\frac{\partial f(X)}{\partial X^\dagger_{ij}}\right].$$

Since X is i.i.d., the only nonzero covariance term is when $i=b$ and $j=a$, for which:

$$\operatorname{Cov}(X_{ba},X^\dagger_{ab})=\frac{1}{N}.$$

Thus, we have:

$$N\mathbb{E}\left[X_{ba}f(X)\right]=\mathbb{E}\left[\frac{\partial f(X)}{\partial X^\dagger_{ab}}\right].$$

Substituting this result into the original expression we obtain:

$$\mathbb{E}\left[\partial_{X^\dagger_{ab}}f(X)\right]-\mathbb{E}\left[\partial_{X^\dagger_{ab}}f(X)\right]=0.$$

Thus, the integral evaluates to zero, and the result holds for any differentiable function $f : \mathbb{C}^{N\times N} \to \mathbb{C}$.

$\square$

3.8 Tutte Moment Relation

Theorem 3.8.1 (Tutte Moment Relation)**.** Using the loop identity, we derive a recursive relationship between the moments of the Wishart random matrix.

Recall the moments are defined as:

$$m_k = \mathbb{E}\left[\mathrm{Tr}(S^k)\right],$$

and for any sequence of positive integers $k_1, \ldots, k_n$:

$$m_{k_1,\ldots,k_n} := \mathbb{E}\left[\prod_{i=1}^{n} \mathrm{Tr}(S^{k_i})\right].$$

We now prove the following recursive moment relation:

$$\sum_{p_1,p_2\geq 0,\, p_1+p_2=k} m_{p_1,p_2} - N m_{k+1} = 0.$$

Recall:

$$\sum_{a,b=1}^{N} \int \frac{N^{N^2}}{(2i\pi)^{N^2}} dX^\dagger dX\, \partial_{X^\dagger_{ab}} \left((X^\dagger S^k)_{ab} e^{-N\,\mathrm{Tr}(XX^\dagger)} \right) = 0.$$

Using the product rule, we have:

$$\sum_{a,b=1}^{N} \int \frac{N^{N^2}}{(2i\pi)^{N^2}} dX^\dagger dX\, \left[\left(\partial_{X^\dagger_{ab}} (X^\dagger S^k)_{ab} \right) e^{-N\,\mathrm{Tr}(XX^\dagger)} + (X^\dagger S^k)_{ab} \left(\partial_{X^\dagger_{ab}} e^{-N\,\mathrm{Tr}(XX^\dagger)} \right) \right] = 0.$$

$=$

$$\sum_{a,b=1}^{N} \int \frac{N^{N^2}}{(2i\pi)^{N^2}} dX^\dagger dX\, \left(\partial_{X^\dagger_{ab}} (X^\dagger S^k)_{ab} \right) e^{-N\,\mathrm{Tr}(XX^\dagger)} + \sum_{a,b=1}^{N} \int \frac{N^{N^2}}{(2i\pi)^{N^2}} dX^\dagger dX\, (X^\dagger S^k)_{ab} \left(\partial_{X^\dagger_{ab}} e^{-N\,\mathrm{Tr}(XX^\dagger)} \right) = 0. \quad (12)$$

We know the derivative of the second term in the sum:

$$\partial_{X^\dagger_{ab}} e^{-N\,\mathrm{Tr}(XX^\dagger)} = -N X_{ba} e^{-N\,\mathrm{Tr}(XX^\dagger)}$$

and so the second term

$$\sum_{a,b=1}^{N} \int \frac{N^{N^2}}{(2i\pi)^{N^2}} dX^\dagger dX\, (X^\dagger S^k)_{ab} \left(\partial_{X^\dagger_{ab}} e^{-N\,\mathrm{Tr}(XX^\dagger)}\right)$$
$$= -N \sum_{a,b=1}^{N} \int \frac{N^{N^2}}{(2i\pi)^{N^2}} dX^\dagger dX\, (X^\dagger S^k)_{ab} X_{ba} e^{-N\,\mathrm{Tr}(XX^\dagger)} \quad (3.1)$$

We start with the expression:

$$\sum_{a,b=1}^{N} X_{ba} (X^\dagger S^k)_{ab}$$

where $(X^\dagger S^k)_{ab}$ is the (a,b) element of the matrix $X^\dagger S^k$, and X_{ba} is the (b,a) element of the matrix X.

Expanding the product:

$$(X^\dagger S^k)_{ab} = \sum_{c=1}^{N} \overline{X}_{ca} (S^k)_{cb}$$

substituting this into the original expression gives:

$$\sum_{a,b=1}^{N} X_{ba} \left(\sum_{c=1}^{N} \overline{X}_{ca} (S^k)_{cb}\right)$$

interchanging the order of summation:

$$\sum_{c=1}^{N} \sum_{a,b=1}^{N} X_{ba} \overline{X}_{ca} (S^k)_{cb}$$

simplifying the expression using:

$$\sum_{a=1}^{N} X_{ba} \overline{X}_{ca} = (XX^\dagger)_{bc} = (S)_{bc}$$

we obtain:

$$\sum_{c=1}^{N} \sum_{b=1}^{N} (S)_{bc} (S^k)_{cb} = \sum_{c=1}^{N} (S^{k+1})_{cc} = \mathrm{Tr}(S^{k+1}).$$

Thus, we have shown that:

$$\sum_{a,b=1}^{N} X_{ba}(X^{\dagger}S^{k})_{ab} = \mathrm{Tr}(S^{k+1})$$

Substituting this back into expression (2) and taking expectations, we get that

$$-N\sum_{a,b=1}^{N}\int \frac{N^{N^2}}{(2i\pi)^{N^2}}dX^{\dagger}dX\,(X^{\dagger}S^{k})_{ab}X_{ba}e^{-N\,\mathrm{Tr}(XX^{\dagger})}$$

$$= -N\mathbb{E}\left[\mathrm{Tr}(S^{k+1})\right] = -N\cdot m_{k+1} \quad (3.2)$$

Now we return to (12) to deal with the first term. Recall that:

$$\frac{\partial}{\partial X^{\dagger}_{ab}} = \frac{\partial}{\partial \overline{X}_{ba}}$$

We need to differentiate $(X^{\dagger}S^{k})_{ab}$. Writing the matrix product in summation notation:

$$(X^{\dagger}S^{k})_{ab} = \sum_{c} X^{\dagger}_{ac}(S^{k})_{cb} = \sum_{c}\overline{X}_{ca}(S^{k})_{cb}$$

Expanding S^k in summation notation:

$$(S^{k})_{cb} = \sum_{c_1,c_2,\ldots,c_{k-1}}\sum_{m_1,m_2,\ldots,m_k} X_{cm_1}(X^{\dagger})_{m_1c_1}X_{c_1m_2}(X^{\dagger})_{m_2c_2}\cdots X_{c_{k-1}m_k}(X^{\dagger})_{m_kb}$$

$=$

$$(S^{k})_{cb} = \sum_{c_1,c_2,\ldots,c_{k-1}}\sum_{m_1,m_2,\ldots,m_k} X_{cm_1}\overline{X}_{c_1m_1}X_{c_1m_2}\overline{X}_{c_2m_2}\cdots X_{c_{k-1}m_k}\overline{X}_{bm_k}$$

Substituting this into $(X^{\dagger}S^{k})_{ab}$, we get:

$$(X^{\dagger}S^{k})_{ab} = \sum_{c}\overline{X}_{ca}\sum_{c_1,c_2,\ldots,c_{k-1}}\sum_{m_1,m_2,\ldots,m_k} X_{cm_1}\overline{X}_{c_1m_1}X_{c_1m_2}\overline{X}_{c_2m_2}\cdots X_{c_{k-1}m_k}\overline{X}_{bm_k}$$

$=$

$$(X^{\dagger}S^{k})_{ab} = \sum_{c_1,c_2,\ldots,c_k}\sum_{m_1,m_2,\ldots,m_k}\overline{X}_{c_1a}X_{c_1m_1}\overline{X}_{c_2m_1}\cdots X_{c_km_k}\overline{X}_{bm_k}$$

Differentiating $(X^\dagger S^k)_{ab}$ with respect to $\overline{X}_{ba}$:

$$\partial_{\overline{X}_{ba}}(X^\dagger S^k)_{ab} = \sum_{c_1,c_2,\ldots,c_k} \sum_{m_1,m_2,\ldots,m_k} \partial_{\overline{X}_{ba}} \overline{X}_{c_1 a} X_{c_1 m_1} \overline{X}_{c_2 m_1} \cdots X_{c_k m_k} \overline{X}_{b m_k}$$

Noticing that we have $k+1$ places where we'll see an $\overline{X}_{ba}$ term in this sum, we condition over $j = 1$ to $k+1$:

$$= \sum_{c_1,c_2,\ldots,c_k} \sum_{m_1,m_2,\ldots,m_k} \sum_{j=1}^{k+1} \overline{X}_{c_1 a} X_{c_1 m_1} \overline{X}_{c_2 m_1} \cdots X_{c_{j-1} m_{j-1}} [\partial_{\overline{X}_{ba}} \overline{X}_{c_j m_j}] X_{c_i m_i} \cdots X_{c_k m_k} \overline{X}_{b m_k}$$

$$= \sum_{j=1}^{k+1} \sum_{c_1,c_2,\ldots,c_{j-1},c_{j+1}\ldots c_k} \sum_{m_1,m_2,\ldots,m_{j-2}} (\overline{X}_{c_1 a} X_{c_1 m_1} \overline{X}_{c_2 m_1} \cdots X_{c_{j-1} a})(X_{b m_j} \cdots X_{c_k m_k} \overline{X}_{b m_k})$$

$$= \sum_{j=1}^{k+1} \left[\sum_{c_1,c_2,\ldots,c_{j-1}} \sum_{m_1,m_2,\ldots,m_{j-2}} (\overline{X}_{c_1 a} X_{c_1 m_1} \overline{X}_{c_2 m_1} \cdots X_{c_{j-1} a}) \right] \left[\sum_{c_{j+1},\ldots,c_k} \sum_{m_j,\ldots,m_k} (X_{b m_j} \cdots X_{c_k m_k} \overline{X}_{b m_k}) \right]$$

$$= \sum_{j=1}^{k+1} (X^\dagger X)^{j-1}_{aa} (X X^\dagger)^{k-(j-1)}_{bb}$$

$$= \sum_{j=1}^{k+1} \operatorname{Tr}\left((XX^\dagger)^{j-1}\right) \operatorname{Tr}\left((XX^\dagger)^{k-j+1}\right)$$

We have shown that

$$\partial_{\overline{X}_{ba}}(X^\dagger S^k)_{ab} = \sum_{j=1}^{k+1} \operatorname{Tr}\left((XX^\dagger)^{j-1}\right) \operatorname{Tr}\left((XX^\dagger)^{k-j+1}\right)$$

Substituting both terms into (12) and taking Expectations we get

$$\sum_{p_1,p_2 \geq 0,\, p_1+p_2=k} m_{p_1,p_2} - N m_{k+1} = 0$$

3.9 Matrix Resolvents

Definition 3.9.1. The resolvent of a square matrix S is defined as:

$$R(z) = (zI - S)^{-1},$$

where z is a complex parameter and I is the identity matrix.

The resolvent provides a tool to study the spectral properties of matrices. It is particularly useful for understanding the distribution of eigenvalues, and will be useful to us for studying the asymptotic behavior of eigenvalues.

Suppose X is a self-adjoint $n \times n$-matrix. Then the resolvent of X is $R(z) = (zI - X)^{-1}$. First, why is $R(z)$ well-defined, i.e., why is $(zI - X)$ invertible? By the Spectral Theorem, we know that the eigenvalues of X are real and $X = U\Lambda U^*$, where $U = (u_1, \ldots, u_n)$ is a unitary matrix and Λ is a diagonal matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ on the diagonal.

Therefore, $(zI - X) = U(zI - \Lambda)U^*$, where $(zI - \Lambda)$ is a diagonal matrix with entries $z - \lambda_i$. This matrix is invertible as long as $z \neq \lambda_i$ for all i. Thus, the resolvent $R(z) = (zI - X)^{-1}$ is well-defined when z is not an eigenvalue of X.

Using the rules for multiplication of block matrices, we have:

$$X = (u_1, \ldots, u_n)\Lambda \begin{pmatrix} u_1^* \\ \vdots \\ u_n^* \end{pmatrix} = \sum_{j=1}^{n} \lambda_j u_j u_j^*$$

We can derive the resolvent $R(z)$ as follows:

$$R(z) = (zI - X)^{-1} = (zI - U\Lambda U^*)^{-1} = U(zI - \Lambda)^{-1}U^*$$

Since Λ is diagonal, $(\Lambda - zI)$ is also diagonal with entries $(\lambda_j - z)$ on the diagonal. Thus,

$$(\Lambda - zI)^{-1} = \operatorname{diag}\left(\frac{1}{\lambda_1 - z}, \ldots, \frac{1}{\lambda_n - z}\right)$$

So,

$$R(z) = U(\Lambda - zI)^{-1}U^* = U \begin{pmatrix} \frac{1}{\lambda_1 - z} & & \\ & \ddots & \\ & & \frac{1}{\lambda_n - z} \end{pmatrix} U^* = \sum_{j=1}^{n} \frac{1}{\lambda_j - z} u_j u_j^*$$

Therefore, we have:

$$R(z) = \sum_{j=1}^{n} \frac{1}{\lambda_j - z} u_j u_j^*$$

This derivation shows that the resolvent $R(z)$ is well-defined when z is a complex number with a non-zero imaginary part, (i.e. not Real) ensuring that $(X - zI)$ is invertible.

3.10 Definitions of n-Point Resolvents

Returning to our Wishart matrix $S = XX^\dagger$, we define the n-point resolvents $\overline{W}_n(z_1, \ldots, z_n)$ and their connected counterpart $W_n(z_1, \ldots, z_n)$:

Definition 3.10.1. The **(Possibly) Disconnected n-Point Resolvent** is defined as:

$$\overline{W}_n(z_1, \ldots, z_n) := \mathbb{E}\left[\prod_{i=1}^{n} \mathrm{Tr}\left((z_i I - S)^{-1}\right)\right],$$

where $z_1, \ldots, z_n$ are complex parameters, and S is the matrix under consideration.

Proposition 3.10.2. The n-point resolvent $\overline{W}_n(x_1, \ldots, x_n)$ can be expressed as:

$$\overline{W}_n(x_1, \ldots, x_n) = \sum_{p_1, \ldots, p_n \geq 0} m_{p_1, \ldots, p_n} x_1^{-p_1-1} \cdots x_n^{-p_n-1},$$

where $m_{p_1, \ldots, p_n}$ are the moments.

Proof. We begin by expanding the resolvent $(x_i I - S)^{-1}$ as a power series:

$$(x_i I - S)^{-1} = \sum_{p_i=0}^{\infty} \frac{S^{p_i}}{x_i^{p_i+1}}.$$

Substituting this into the product of traces:

$$\prod_{i=1}^{n} \mathrm{Tr}\left((x_i I - S)^{-1}\right) = \prod_{i=1}^{n} \sum_{p_i=0}^{\infty} \frac{\mathrm{Tr}(S^{p_i})}{x_i^{p_i+1}}.$$

Taking the expectation, we expand:

$$\mathbb{E}\left[\prod_{i=1}^{n} \mathrm{Tr}\left((x_i I - S)^{-1}\right)\right] = \sum_{p_1, \ldots, p_n \geq 0} \mathbb{E}\left[\prod_{i=1}^{n} \mathrm{Tr}(S^{p_i})\right] \prod_{i=1}^{n} \frac{1}{x_i^{p_i+1}}.$$

Finally, using the definition of moments $m_{p_1, \ldots, p_n} = \mathbb{E}\left[\prod_{i=1}^{n} \mathrm{Tr}(S^{p_i})\right]$, we arrive at:

$$\overline{W}_n(x_1, \ldots, x_n) = \sum_{p_1, \ldots, p_n \geq 0} m_{p_1, \ldots, p_n} x_1^{-p_1-1} \cdots x_n^{-p_n-1}.$$

$\square$

Definition 3.10.3. The **Connected n-Point Resolvent** is defined as:

$$W_n(z_1, \ldots, z_n) := \sum_{p_1,\ldots,p_n \geq 0} c_{p_1,\ldots,p_n} z_1^{-p_1-1} \cdots z_n^{-p_n-1},$$

where $c_{p_1,\ldots,p_n}$ are the connected moments, and $z_1, \ldots, z_n$ are complex parameters.

Proposition 3.10.4. The relation between the connected and disconnected n-point resolvents is given by:

$$\overline{W}_n(x_1, \ldots, x_n) = \sum_{K \vdash \{1,\ldots,n\}} \prod_{K_i \in K} W_{|K_i|}(x_{K_i}),$$

where $K \vdash \{1, \ldots, n\}$ represents a partition of the set $\{1, \ldots, n\}$, and $x_{K_i} = \{x_j\}_{j \in K_i}$.

Proof. To prove this relation, we start by expanding the definition of the disconnected n-point resolvent $\overline{W}_n(x_1, \ldots, x_n)$ in terms of the moments $m_{p_1,\ldots,p_n}$:

$$\overline{W}_n(x_1, \ldots, x_n) = \sum_{p_1,\ldots,p_n \geq 0} m_{p_1,\ldots,p_n} x_1^{-p_1-1} \cdots x_n^{-p_n-1}.$$

Using the moment-cumulant relation, we express the moments $m_{p_1,\ldots,p_n}$ in terms of the cumulants $c_{p_1,\ldots,p_n}$:

$$m_{p_1,\ldots,p_n} = \sum_{K \vdash \{p_1,\ldots,p_n\}} \prod_{\kappa_i \in K} c_{\kappa_i}.$$

Substituting this into the expression for $\overline{W}_n(x_1, \ldots, x_n)$:

$$\overline{W}_n(x_1, \ldots, x_n) = \sum_{p_1,\ldots,p_n \geq 0} \left(\sum_{K \vdash \{p_1,\ldots,p_n\}} \prod_{\kappa_i \in K} c_{\kappa_i} \right) x_1^{-p_1-1} \cdots x_n^{-p_n-1}.$$

By grouping terms based on partitions K:

$$\overline{W}_n(x_1, \ldots, x_n) = \sum_{K \vdash \{1,\ldots,n\}} \prod_{K_i \in K} \left(\sum_{p_1,\ldots,p_{|K_i|} \geq 0} c_{p_1,\ldots,p_{|K_i|}} x_{K_i}^{-p_1-1} \cdots x_{K_i}^{-p_{|K_i|}-1} \right).$$

Recognizing that each term in the product corresponds to a connected n-point resolvent:

$$\overline{W}_n(x_1, \ldots, x_n) = \sum_{K \vdash \{1,\ldots,n\}} \prod_{K_i \in K} W_{|K_i|}(x_{K_i}),$$

which completes the proof of the relation between the connected and disconnected n-point resolvents. $\square$

3.11 Schwinger-Dyson Resolvent Loop equations

3.11.1 Moment Relation Leading to Connected Resolvent Equations

Note that $\overline{W}_1(x) = W_1(x)$.

Recall from the **Tutte Moment Relation** that we have the recursive relation:

$$\sum_{k\geq 0} \frac{1}{x^{k+1}} \left(\sum_{p_1,p_2\geq 0, p_1+p_2=k} m_{p_1,p_2} - N m_{k+1} \right) = 0. \qquad (*)$$

we can rewrite the above equation as:

$$\overline{W}_2(x,x) - N\overline{W}_1(x) + \frac{N^2}{x} = 0. \qquad (*)$$

Proof:

$$\overline{W}_2(x,x) = \sum_{p_1,p_2\geq 0, p_1+p_2=k} \left(\frac{m_{p_1,p_2}}{x^{p_1+1}x^{p_2+1}} \right) = \sum_{k\geq 0} \sum_{\substack{p_1,p_2\geq 0\\ p_1+p_2=k}} \left(\frac{m_{p_1,p_2}}{x^{p_1+p_2+2}} \right) = \sum_{k\geq 0} \frac{1}{x^{k+2}} \sum_{\substack{p_1,p_2\geq 0\\ p_1+p_2=k}} m_{p_1,p_2}$$

from (*) we have

$$= N \left(\sum_{k\geq 0} \frac{m_{k+1}}{x^{k+2}} \right) = N \left(\sum_{k\geq 1} \frac{m_k}{x^{k+1}} - \frac{m_0}{x} \right) = N W_1(x) - \frac{N^2}{x}.$$

Using the cumulant relation between resolvents in (*):

$$\overline{W}_2(x,x) = W_2(x,x) + W_1(x)W_1(x).$$

Allowing us to write (*) in terms of connected resolvents as

$$W_1(x)^2 + W_2(x,x) - N W_1(x) + \frac{N^2}{x} = 0.$$

$\square$

3.11.2 Recovering the Marchenko-Pastur Distribution

We start with the expression for the connected resolvents in terms of the cumulants:

$$W_n(x_1,\ldots,x_n) := \sum_{p_1,\ldots,p_n\geq 0} c_{p_1,\ldots,p_n} x_1^{-p_1-1}\cdots x_n^{-p_n-1} \tag{1}$$

then we use our genus expansion for the cumulants:

$$c_{k_1,\ldots,k_n} = \sum_{g\geq 0} N^{2-n-2g} c^{[g]}_{k_1,\ldots,k_n}$$

where each term $c^{[g]}_{k_1,\ldots,k_n}$ records the contributions from connected combinatorial maps of genus g.

Substituting into (1) we get:

$$W_n(x_1,\ldots,x_n) = \sum_{g\geq 0} N^{2-2g-n} W_{g,n}(x_1,x_2,\ldots,x_n)$$

where

$$W_{g,n}(x_1,\ldots,x_n) = \sum_{p_1,\ldots,p_n\geq 0} c^{[g]}_{p_1,\ldots,p_n} x_1^{-p_1-1} x_2^{-p_2-1}\cdots x_n^{-p_n-1}$$

3.11.3 computing (*)

Recall that

$$W_1(x)^2 + W_2(x,x) - NW_1(x) + \frac{N^2}{x} = 0.$$

To compute $W_1(x)$, we start with:

$$W_1(x) = \sum_{g\geq 0} N^{2-2g-1} W_{g,1}(x).$$

Expanding $W_{g,1}(x)$ using the definition:

$$W_{g,1}(x) = \sum_{p\geq 0} x^{-p-1} c^{[g]}_p$$

we get:

$$W_1(x) = \sum_{g\geq 0} N^{2-2g-1} \sum_{p\geq 0} x^{-p-1} c^{[g]}_p.$$

Thus, we have:

$$W_1(x) = \sum_{g\geq 0} N^{1-2g} W_{g,1}(x).$$

To compute $W_2(x,x)$, we start with:

$$W_2(x,x) = \sum_{g\geq 0} N^{2-2g-2} W_{g,2}(x,x).$$

Expanding $W_{g,2}(x,x)$ using the definition:

$$W_{g,2}(x,x) = \sum_{p_1,p_2\geq 0} x^{-p_1-1} x^{-p_2-1} c^{[g]}_{p_1,p_2}$$

we get:

$$W_2(x,x) = \sum_{g\geq 0} N^{2-2g-2} \sum_{p_1,p_2\geq 0} x^{-p_1-1} x^{-p_2-1} c^{[g]}_{p_1,p_2}$$

Thus, we have:

$$W_2(x,x) = \sum_{g\geq 0} N^{-2g} W_{g,2}(x,x).$$

Substitute the genus expressions for $W_1(x)$ and $W_2(x,x)$ into $W_1(x)^2 + W_2(x,x) - NW_1(x) + \frac{N^2}{x}$. We obtain:

$$W_1(x) = \sum_{g\geq 0} N^{1-2g} W_{g,1}(x)$$

$$W_2(x,x) = \sum_{g\geq 0} N^{-2g} W_{g,2}(x,x)$$

$$W_1(x)^2 = \left(\sum_{g\geq 0} N^{1-2g} W_{g,1}(x)\right)^2$$

$$-NW_1(x) = -\sum_{g\geq 0} N^{2-2g} W_{g,1}(x).$$

Combining all terms,

$$W_1(x)^2 + W_2(x,x) - NW_1(x) + \frac{N^2}{x} = 0$$

becomes:

$$\left(\sum_{g\geq 0} N^{1-2g} W_{g,1}(x)\right)^2 + \sum_{g\geq 0} N^{-2g} W_{g,2}(x,x) - \sum_{g\geq 0} N^{2-2g} W_{g,1}(x) + \frac{N^2}{x} = 0.$$

We start with the following equation:

$$\left(\sum_{g\geq 0} N^{1-2g}W_{g,1}(x)\right)^2 + \sum_{g\geq 0} N^{-2g}W_{g,2}(x,x) - \sum_{g\geq 0} N^{2-2g}W_{g,1}(x) + \frac{N^2}{x} = 0$$

Step 1: Breaking off the $W_{0,1}(x)$ term from the first sum

The first sum is:

$$\sum_{g\geq 0} N^{1-2g}W_{g,1}(x)$$

Breaking off the $g = 0$ term gives:

$$NW_{0,1}(x) + \sum_{g\geq 1} N^{1-2g}W_{g,1}(x)$$

Step 2: Squaring the first sum

Now, square the first sum:

$$\left(NW_{0,1}(x) + \sum_{g\geq 1} N^{1-2g}W_{g,1}(x)\right)^2$$

Expanding this:

$$(NW_{0,1}(x))^2 + 2NW_{0,1}(x)\sum_{g\geq 1} N^{1-2g}W_{g,1}(x) + \left(\sum_{g\geq 1} N^{1-2g}W_{g,1}(x)\right)^2$$

Thus, we have:

$$N^2W_{0,1}(x)^2 + 2NW_{0,1}(x)\sum_{g\geq 1} N^{1-2g}W_{g,1}(x) + \left(\sum_{g\geq 1} N^{1-2g}W_{g,1}(x)\right)^2$$

Step 3: Breaking off the $W_{0,2}(x,x)$ term from the second sum

The second sum is:

$$\sum_{g\geq 0} N^{-2g}W_{g,2}(x,x)$$

Breaking off the $g = 0$ term gives:

$$W_{0,2}(x,x) + \sum_{g\geq 1} N^{-2g}W_{g,2}(x,x)$$

Assuming $W_{0,2}(x,x) = 0$, this reduces to:

$$\sum_{g\geq 1} N^{-2g} W_{g,2}(x,x)$$

Step 4: Breaking off the $W_{0,1}(x)$ term from the third sum

The third sum is:

$$\sum_{g\geq 0} N^{2-2g} W_{g,1}(x)$$

Breaking off the $g = 0$ term gives:

$$N^2 W_{0,1}(x) + \sum_{g\geq 1} N^{2-2g} W_{g,1}(x)$$

Step 5: Substituting all expressions back into the equation

Now, substitute the expanded terms back into the original equation:

$$\begin{aligned} N^2 W_{0,1}(x)^2 + 2N W_{0,1}(x) \sum_{g\geq 1} N^{1-2g} W_{g,1}(x) + \left(\sum_{g\geq 1} N^{1-2g} W_{g,1}(x) \right)^2 \\ + \sum_{g\geq 1} N^{-2g} W_{g,2}(x,x) - N^2 W_{0,1}(x) - \sum_{g\geq 1} N^{2-2g} W_{g,1}(x) + \frac{N^2}{x} = 0 \end{aligned} \quad (3.3)$$

Step 6: Combine like terms

Now, we can combine the terms that involve $W_{0,1}(x)$:

$$\begin{aligned} N^2 W_{0,1}(x)^2 - N^2 W_{0,1}(x) + \frac{N^2}{x} \\ + 2N W_{0,1}(x) \sum_{g\geq 1} N^{1-2g} W_{g,1}(x) \\ + \left(\sum_{g\geq 1} N^{1-2g} W_{g,1}(x) \right)^2 + \sum_{g\geq 1} N^{-2g} W_{g,2}(x,x) - \sum_{g\geq 1} N^{2-2g} W_{g,1}(x) = 0 \end{aligned} \quad (3.4)$$

Now dividing by N^2, we get:

$$\begin{aligned} W_{0,1}(x)^2 - W_{0,1}(x) + \frac{1}{x} \\ + \frac{2}{N} W_{0,1}(x) \sum_{g\geq 1} N^{1-2g} W_{g,1}(x) \\ + \frac{1}{N^2} \left(\sum_{g\geq 1} N^{1-2g} W_{g,1}(x) \right)^2 + \frac{1}{N^2} \sum_{g\geq 1} N^{-2g} W_{g,2}(x,x) - \frac{1}{N^2} \sum_{g\geq 1} N^{2-2g} W_{g,1}(x) = 0 \end{aligned} \quad (3.5)$$

The above relation expresses equality between two power series. And since the RHS is the zero power series we can conclude the following statement:

$$W_{0,1}(x)^2 - W_{0,1}(x) + \frac{1}{x} = 0$$

Multiplying by x, we get:

$$xW_{0,1}(x)^2 - xW_{0,1}(x) + 1 = 0$$

Which we can solve for $W_{0,1}(x)$ using the quadratic formula. The equation:

$$xW_{0,1}(x)^2 - xW_{0,1}(x) + 1 = 0$$

is a quadratic in $W_{0,1}(x)$. Using the quadratic formula:

$$W_{0,1}(x) = \frac{-(-x) \pm \sqrt{(-x)^2 - 4(x)(1)}}{2x}.$$

Simplifying this:

$$W_{0,1}(x) = \frac{x \pm \sqrt{x^2 - 4x}}{2x}.$$

Further simplification gives:

$$W_{0,1}(x) = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4}{x}}\right).$$

We can determine the correct branch of the square root by examining the series expansion of $W_{0,1}(x)$. The series expansion for $W_{0,1}(x)$ begins with:

$$W_{0,1}(x) = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots,$$

which contains no constant term.

Now, if we expand the + branch of the expression:

$$W_{0,1}(x) = \frac{1}{2}\left(1 + \sqrt{1 - \frac{4}{x}}\right),$$

we obtain the following series:

$$W_{0,1}(x) = \frac{1}{2}\left(1 + \left(1 - \frac{2}{x} - \frac{2}{x^2} - \frac{2}{x^3} + O\left(\frac{1}{x^4}\right)\right)\right).$$

Simplifying this, we get:

$$W_{0,1}(x) = 1 - \frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} + O\left(\frac{1}{x^4}\right).$$

Notice that the series contains a constant term, 1, which is inconsistent with the correct expansion of $W_{0,1}(x)$, which starts with $\frac{1}{x}$ and has no constant term. This rules out the $+$ branch as a valid solution.

On the other hand, expanding the $-$ branch:

$$W_{0,1}(x) = \frac{1}{2}\left(1 - \sqrt{1 - \frac{4}{x}}\right),$$

we find that the constant term cancels out, and the series starts with $\frac{1}{x}$, matching the expected behavior.

Therefore, the correct branch is:

$$W_{0,1}(x) = \frac{1}{2}\left(1 - \sqrt{1 - \frac{4}{x}}\right),$$

3.12 Recovering the Wishart density using Resolvents

As previously discussed, the function $W_{0,1}(x)$ is defined as the large N limit of the resolvent:

$$W_{0,1}(x) = \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left[\mathrm{Tr}\left((xI - S)^{-1}\right)\right]$$

This can also be written as $W_{0,1}(x) = \frac{1}{N}R(x)$, where $R(x)$ is the resolvent of the random matrix S. In terms of the eigenvalue density $\rho(\lambda)$, the resolvent can be expressed as the Stieltjes transform:

$$W_{0,1}(x) = \frac{x - \sqrt{x^2 - 4x}}{2x} = \int \frac{\rho(\lambda)}{x - \lambda} d\lambda$$

Here, $\rho(\lambda)$ represents the eigenvalue density of the matrix S, and the integral provides the analytic continuation of the resolvent in the complex plane.

Using the Stieltjes transform inversion, we could recover the expression for $\rho(\lambda)$. In this specific case, obtaining $\rho(\lambda)$ is straightforward.

3.12.1 Stieltjes Inversion

To perform Stieltjes inversion and obtain $\rho(\lambda)$ from the given resolvent $W_{0,1}(x)$, we use the fact that the resolvent is related to the density of eigenvalues $\rho(\lambda)$ through:

$$W_{0,1}(x) = \int \frac{\rho(\lambda)}{x - \lambda} d\lambda.$$

The Stieltjes inversion formula is given by:

$$\rho(\lambda) = -\frac{1}{\pi} \lim_{\epsilon \to 0^+} \operatorname{Im} W_{0,1}(\lambda + i\epsilon).$$

The resolvent $W_{0,1}(x)$ is:

$$W_{0,1}(x) = \frac{x - \sqrt{x^2 - 4x}}{2x}.$$

To analyze the behavior of $W_{0,1}(x)$, focus on the square root term:

$$\sqrt{x^2 - 4x} = \sqrt{x(x-4)}.$$

For real x, the square root becomes imaginary for $x \in (0, 4)$, which is the region where the eigenvalue density $\rho(\lambda)$ is non-zero.

Find $\operatorname{Im} W_{0,1}(x)$

When $x = \lambda \in (0, 4)$, the square root $\sqrt{\lambda^2 - 4\lambda}$ has an imaginary part. Writing $x = \lambda + i\epsilon$, we approximate the square root for small ϵ and find:

$$\sqrt{(\lambda + i\epsilon)^2 - 4(\lambda + i\epsilon)} = i\sqrt{4\lambda - \lambda^2}.$$

Thus, for $\lambda \in (0, 4)$, the imaginary part of $W_{0,1}(\lambda + i\epsilon)$ is:

$$\operatorname{Im} W_{0,1}(\lambda + i\epsilon) = \frac{\sqrt{4\lambda - \lambda^2}}{2\lambda}.$$

Apply Stieltjes inversion

Using the Stieltjes inversion formula:

$$\rho(\lambda) = -\frac{1}{\pi} \lim_{\epsilon \to 0^+} \operatorname{Im} W_{0,1}(\lambda + i\epsilon),$$

we obtain:

$$\rho(\lambda) = \frac{1}{2\pi} \frac{\sqrt{4\lambda - \lambda^2}}{\lambda}.$$

This is the density of eigenvalues $\rho(\lambda)$ for $\lambda \in (0, 4)$.

Theorem 3.12.1 (Marchenko-Pastur Distribution)**.** The eigenvalue density $\rho(\lambda)$ of the Wishart matrix is given by:

$$\rho(\lambda) = \frac{1}{2\pi} \frac{\sqrt{4\lambda - \lambda^2}}{\lambda}, \quad \text{for } \lambda \in (0, 4).$$

This is the Marchenko-Pastur distribution with parameter $c = 1$. The proof that this is a probability measure on $\mathbb{C}$ is given in section 2.1 of Mingo and Speicher [MS2018].

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