Семінар 13. Ейлерові і Гамільтонові цикли

13 червня 2023

• Definition

Let G be a graph, and let v and w be vertices in G.

A **walk from v to w** is a finite alternating sequence of adjacent vertices and edges of G. Thus a walk has the form

$$v_0 e_1 v_1 e_2 \cdots v_{n-1} e_n v_n,$$

where the v's represent vertices, the e's represent edges, $v_0 = v$, $v_n = w$, and for all $i = 1, 2, \ldots n$, v_{i-1} and v_i are the endpoints of e_i. The **trivial walk from v to v** consists of the single vertex v.

A **path from v to w** is a walk from v to w that does not contain a repeated edge. Thus a path from v to w is a walk of the form

$$v = v_0 e_1 v_1 e_2 \cdots v_{n-1} e_n v_n = w,$$

where all the e_i are distinct (that is, $e_i \neq e_k$ for any $i \neq k$).

A **simple path from v to w** is a path that does not contain a repeated vertex. Thus a simple path is a walk of the form

$$v = v_0 e_1 v_1 e_2 \cdots v_{n-1} e_n v_n = w,$$

where all the e_i are distinct and all the v_j are also distinct (that is, $v_j \neq v_m$ for any $j \neq m$).

A closed walk is a walk that starts and ends at the same vertex.

A circuit is a closed walk that does not contain a repeated edge. Thus a circuit is a walk of the form

$$v_0 e_1 v_1 e_2 \cdots v_{n-1} e_n v_n,$$

where $v_0 = v_n$ and all the e_i are distinct.

A **simple circuit** is a circuit that does not have any other repeated vertex except the first and last. Thus a simple circuit is a walk of the form

$$v_0 e_1 v_1 e_2 \cdots v_{n-1} e_n v_n,$$

where all the e_i are distinct and all the v_j are distinct except that $v_0 = v_n$.

• Definition

Let G be a graph. An **Euler circuit** for G is a circuit that contains every vertex and every edge of G. That is, an Euler circuit for G is a sequence of adjacent vertices and edges in G that starts and ends at the same vertex, uses every vertex of G at least once, and uses every edge of G exactly once.

Theorem 11.2.2

If a graph has an Euler circuit, then every vertex of the graph has even degree.

Contrapositive Version of Theorem 11.2.2

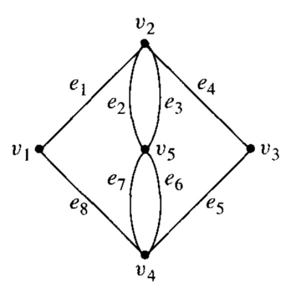
If some vertex of a graph has odd degree, then the graph does not have an Euler circuit.

Theorem 11.2.3

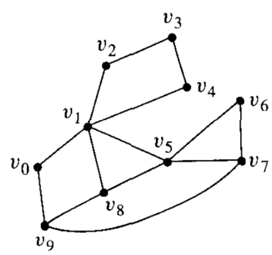
If every vertex of a nonempty graph has even degree and if the graph is connected, then the graph has an Euler circuit.

Determine which of the graphs in 12–17 have Euler circuits. If the graph does not have an Euler circuit, explain why not. If it does have an Euler circuit, describe one.

12.



13.


v_3
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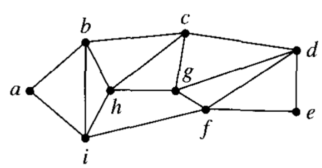
14.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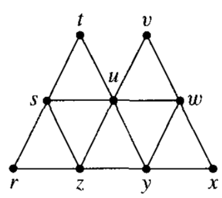
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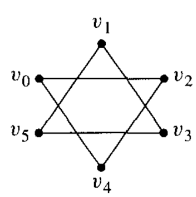
16.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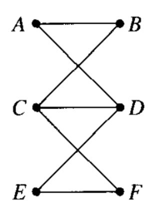
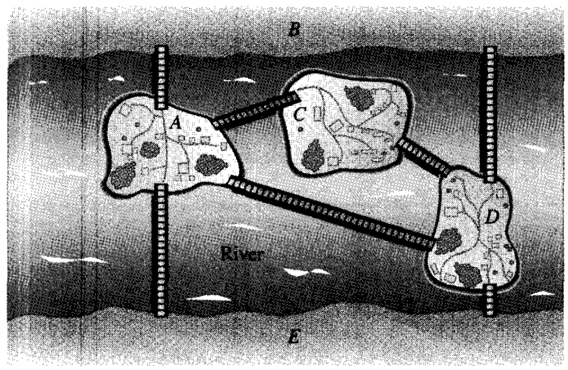
17.

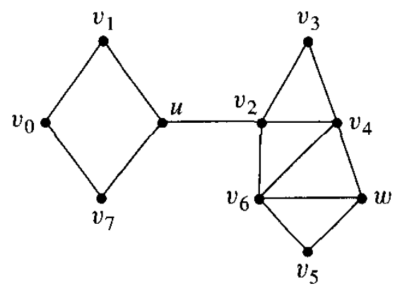

A
B
C
D
E
F

18. Is it possible to take a walk around the city whose map is shown below, starting and ending at the same point and crossing each bridge exactly once? If so, how can this be done?

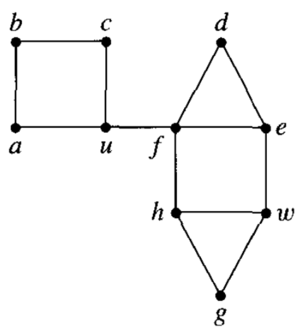


For each of the graphs in 19–21, determine whether there is an Euler path from u to w. If there is, find such a path.

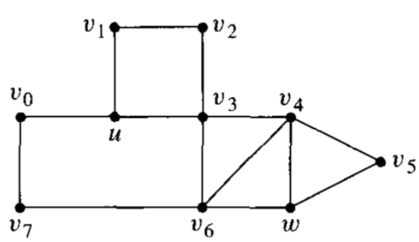
19.



20.



21.

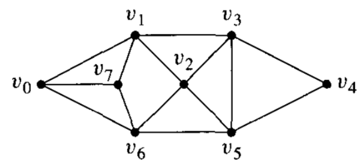


• Definition

Given a graph G, a **Hamiltonian circuit** for G is a simple circuit that includes every vertex of G. That is, a Hamiltonian circuit for G is a sequence of adjacent vertices and distinct edges in which every vertex of G appears exactly once, except for the first and the last, which are the same.

Find Hamiltonian circuits for each of the graphs in 23 and 24.

23.



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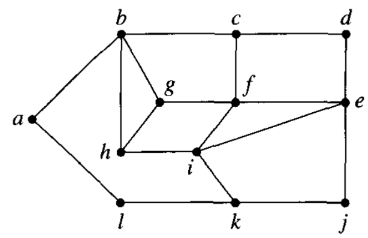
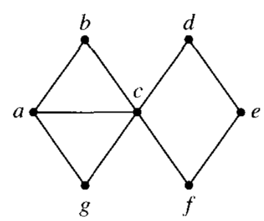
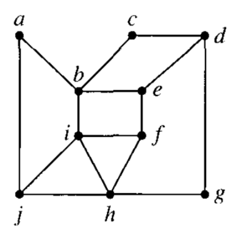


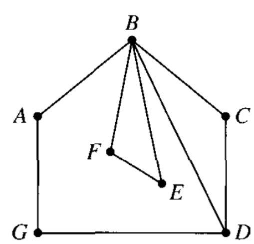
***H* 25.**



26.



27.



47. For what values of n does the complete graph K_n with n vertices have (a) an Euler circuit? (b) a Hamiltonian circuit?