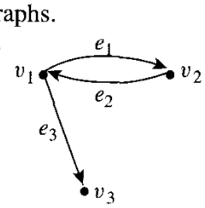
Семінар 14. Матричне представлення графів

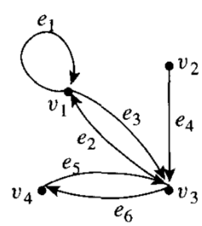
16 червня 2023

2. Find the adjacency matrices for the following directed graphs.

a.



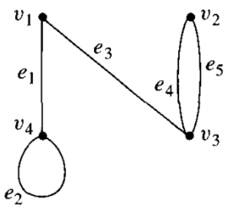
b.

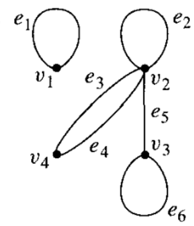


3. Find directed graphs that have the following adjacency matrices:

a. $\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ b. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

4. Find adjacency matrices for the following (undirected) graphs.

a. 

b. 

c. K_4, the complete graph on four vertices

d. $K_{2,3}$, the complete bipartite graph on (2, 3) vertices

5. Find graphs that have the following adjacency matrices.

a. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$ b. $\begin{bmatrix} 0 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Theorem 11.3.1

Let G be a graph with connected components $G_1, G_2, \ldots, G_k$. If there are n_i vertices in each connected component G_i and these vertices are numbered consecutively, then the adjacency matrix of G has the form

$$\begin{bmatrix} A_1 & O & O & \cdots & O & O \\ O & A_2 & O & \cdots & O & O \\ O & O & A_3 & \cdots & O & O \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ O & O & O & \cdots & O & A_k \end{bmatrix}$$

where each A_i is the $n_i \times n_i$ adjacency matrix of G_i, for all $i = 1, 2, \ldots, k$, and the O's represent matrices whose entries are all 0.

6. The following are adjacency matrices for graphs. In each case determine whether the graph is connected by analyzing the matrix without drawing the graph.

a. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ b. $\begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

Theorem 11.3.2

If G is a graph with vertices $v_1, v_2, \ldots, v_m$ and $\mathbf{A}$ is the adjacency matrix of G, then for each positive integer n,

$$\text{the } ij\text{th entry of } \mathbf{A}^n = \text{the number of walks of length } n \text{ from } v_i \text{ to } v_j \\ \text{for all integers } i, j = 1, 2, \ldots, m.$$

19. **a.** Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$. Find $\mathbf{A}^2$ and $\mathbf{A}^3$.

b. Let G be the graph with vertices v_1, v_2, and v_3 and with $\mathbf{A}$ as its adjacency matrix. Use the answers to part (a) to find the number of walks of length 2 from v_1 to v_3 and the number of walks of length 3 from v_1 to v_3. Do not draw G to solve this problem.

c. Examine the calculations you performed in answering part (a) to find five walks of length 2 from v_3 to v_3. Then draw G and find the walks by visual inspection.

20. The following is an adjacency matrix for a graph:

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Answer the following questions by examining the matrix and its powers only, not by drawing the graph:

a. How many walks of length 2 are there from v_2 to v_3?
b. How many walks of length 2 are there from v_3 to v_4?
c. How many walks of length 3 are there from v_1 to v_4?
d. How many walks of length 3 are there from v_2 to v_3?

22. a. Draw a graph that has

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 \\ 1 & 1 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \end{bmatrix}$$

as its adjacency matrix. Is this graph bipartite? (For a definition of bipartite, see exercise 37 in Section 11.1.)

Show that a graph with n vertices is bipartite if, and only if, for some labeling of its vertices, its adjacency matrix has the form

$$\begin{bmatrix} \mathrm{O} & \mathbf{A} \\ \mathbf{A}^t & \mathrm{O} \end{bmatrix}$$

where $\mathbf{A}$ is a $k \times (n-k)$ matrix for some integer k such that $0 < k < n$, the top left O represents a $k \times k$ matrix all of whose entries are 0, $\mathbf{A}^t$ is the transpose of $\mathbf{A}$, and the bottom right O represents an $(n-k) \times (n-k)$ matrix all of whose entries are 0.