Classical Mechanics

MECHANICS OF A PARTICLE

* Let $\vec{r}$ be the radius vector of a particle from a given origin & v be its vector velocity

$$v = \frac{d\vec{r}}{dt}$$

* The linear momentum is given by

$$\vec{p} = m\vec{v}$$

* The mechanics of a particle is contained in Newton's Second Law of Motion

$$\vec{F} = \dot{\vec{p}} = \frac{d\vec{p}}{dt}$$

When the mass of the particle is constant

$$\vec{F} = \frac{d}{dt}(m\vec{v}) = m\frac{d}{dt}\vec{v} = m\vec{a}$$

* A reference frame where Newton's ~~Seq~~ Second Law of Motion holds is called an inertial frame.

* From this, we get the conservation theorem for linear momentum of a particle. When $\vec{F} = 0$, $\dot{\vec{p}} = 0$ and hence $\vec{p}$ is a constant.

* The angular momentum of a particle about a point O is defined as

$$\vec{L} = \vec{r} \times \vec{p}$$

* The moment of force, or torque is defined as

$$\vec{N} = \vec{r} \times \vec{F}$$

Analogous to Newton's Second Law,

$$\vec{N} = \vec{r} \times \frac{d}{dt}(m\vec{v})$$

It can also be written as

$$\frac{d}{dt}(\vec{r} \times m\vec{v}) = \underbrace{\vec{v} \times m\vec{v}}_{0} + \vec{r} \times \frac{d}{dt} m\vec{v} = \vec{N}$$

Hence,

$$\vec{N} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{L}}{dt} = \dot{\vec{L}}$$

* The torque equation also yields a conservation theorem - one for angular momentum. When N is zero, $\dot{\vec{L}}$ is zero, and hence $\vec{L}$ is constant.

* Consider work done by an external force F on a particle going from point 1 to point 2

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{s}$$

For constant mass

$$W_{12} = m \int_1^2 \frac{d\vec{v}}{dt} \cdot \vec{v}\, dt = \frac{m}{2} \int_1^2 \frac{dv^2}{dt}\, dt$$

$$W_{12} = \frac{m}{2}(v_2^2 - v_1^2)$$

The scalar quantity $mv^2/2$ is called the kinetic energy.

* A force is conservative if W_{12} is the same for any possible path between points 1 & 2. Alternately a force is conservative if the work done by it across a closed circuit is zero. i.e.

$$\oint \vec{F} \cdot d\vec{s} = 0$$

* A necessary & sufficient condition for the work to be independent of the physical path taken by the particle is the existence of a potential function V such that

$$\vec{F} = -\vec{\nabla} V$$

$$\vec{F} \cdot d\vec{s} = -dV \quad \Rightarrow \quad \vec{F_s} = -\frac{\partial V}{\partial s}$$

$\therefore$ For a conservative system;

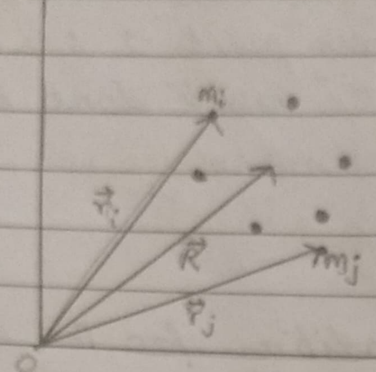
$$W_{12} = V_1 - V_2$$

* Energy Conservation Theorem for a particle: if the forces acting on a particle are conservative; total energy of the particle is conserved.

MECHANICS OF A SYSTEM OF PARTICLES

* Consider a system with i particles. They experience internal & external forces. For the i^{th} particle, we get

$$\sum_j \vec{F}_{ij} + \vec{F}_i^{(e)} = \dot{\vec{p}}_i$$



m_i - mass of i^{th} particle
$\vec{r}_i$ - position vector of i^{th} particle
$\vec{R}$ - position vector of CoM

∴ Summing over all particles

$$\frac{d^2}{dt^2} \sum_i m_i \vec{r}_i = \sum_i \vec{F}_i^{(e)} + \sum_{i,j} \vec{F}_{ij}$$

From Newton's 3rd Law,

$$\vec{F}_{ij} + \vec{F}_{ji} = 0$$

The center of mass is given by

$$\vec{R} = \frac{\sum m_i \vec{r}_i}{\sum m_i} \Rightarrow M\vec{R} = \sum m_i \vec{r}_i$$

$$\therefore \quad M \frac{d^2\vec{R}}{dt^2} = \sum_i \vec{F}_i^{(e)} = \vec{F}^{(e)}$$

→ The center of mass moves as if the total external force were acting entirely on it with the mass of the system concentrated on it.

→ The total linear momentum of the system

$$\vec{P} = \sum m_i \frac{d\vec{r}_i}{dt} = M \frac{d\vec{R}}{dt}$$

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* Conservation of linear momentum holds for a system of particles - if the total external force is zero, total linear momentum is conserved.

* The total angular momentum of the system can be determined by taking cross product $r \times p$ & summing over all the particles

$$\sum_i \vec{r}_i \times \dot{\vec{p}}_i = \sum_i \frac{d}{dt} (\vec{r}_i \times \vec{p}_i) = \dot{\vec{L}}$$

$$= \sum_i \vec{r}_i \times \vec{F}_i^{(e)} + \sum_{i,j} \vec{r}_i \times \vec{F}_{ji}$$

→ When forces obey the strong law of action & reaction sum of $\vec{r}_{ij} \times \vec{F}_{ji}$ pairs cancel

$$\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i^{(e)} = \vec{N}^{(e)}$$

* Conservation of angular momentum - $\vec{L}$ is constant in time if no external torque acts on the system.

* While the system behaves as if the total linear momentum is concentrated at the center of mass, it is different for angular momentum.

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i$$

$$= \sum_i \vec{r}_i \times m_i \vec{v}_i$$

* Let $\vec{R}$ be the position vector to the center of mass & $\vec{r}_i'$ the vector from the center of mass to the i^{th} particle

$$\therefore \vec{r}_i = \vec{R} + \vec{r}_i'$$

$$\Rightarrow \vec{v}_i = \vec{v} + \vec{v}_i'$$

The angular momentum takes the form

$$\vec{L} = \sum_i \vec{R} \times m_i \vec{v} + \sum_i \vec{r}_i' \times m_i \vec{v} + \sum_i \vec{R} \times m_i \vec{v}_i' + \sum_i \vec{r}_i' \times m_i \vec{v}_i'$$

$$\vec{L} = \vec{R} \times M\vec{v} + \sum_i \vec{r}_i' \times m_i \vec{v}_i'$$

$\sum_i m_i \vec{r}_i'$ is the null vector wrt center of mass

* The work done by all forces for an initial state 1 & final state 2 (conservative force)

$$W_{12} = \sum_i \int_1^2 \vec{F}_i \cdot d\vec{s}_i = \sum_i \int_1^2 \vec{F}_i^{(e)} \cdot d\vec{s}_i + \sum_{i,j} \int_1^2 \vec{F}_{ji} \cdot d\vec{s}_i$$

The work done can be written as the difference of the final & initial energies

$$K = \frac{1}{2} \sum_i m_i (\vec{v} + \vec{v}_i') \cdot (\vec{v} + \vec{v}_i')$$

$$= \frac{1}{2} \sum_i m_i v^2 + \frac{1}{2} \sum_i m_i |\vec{v}_i'|^2 + \vec{v} \cdot \frac{d}{dt} \left(\sum_i m_i \vec{r}_i'\right)$$

$$= \frac{1}{2} M v^2 + \frac{1}{2} \sum_i m_i |\vec{v}_i'|^2$$

* When the force can be represented as the gradient of potential,

$$\sum_i \int_1^2 \vec{F}_i^{(e)} \cdot d\vec{s}_i = -\sum_i \int_1^2 \vec{\nabla}_i V_i \cdot d\vec{s}_i = -\sum_i V_i \Big|_1^2$$

* When internal forces are conservative, forces between i^{th} particle & j^{th} particle can be obtained from V. To obey the strong form of the law of action & reaction

$$V_{ij} = V_{ij}(|\vec{r}_i - \vec{r}_j|)$$

Also, the forces lie along the line joining both particles

$$\vec{F}_{ji} = -\vec{\nabla}_i V_{ij} = +\vec{\nabla}_j V_{ij} = -\vec{F}_{ij}$$

$$\vec{\nabla} V_{ij}(|\vec{r}_i - \vec{r}_j|) = (\vec{r}_i - \vec{r}_j) f$$

where f is a scalar function

$$\therefore \int_1^2 \vec{F}_{ji} \cdot d\vec{s}_i = -\int_1^2 (\vec{\nabla}_i V_{ij} \cdot d\vec{s}_i + \vec{\nabla}_j V_{ij} \cdot d\vec{s}_j)$$

With $d\vec{s}_i - d\vec{s}_j = d\vec{r}_i - d\vec{r}_j = d\vec{r}_{ij}$

$$\vec{\nabla}_i V_{ij} = \vec{\nabla}_{ij} V_{ij} = -\vec{\nabla}_j V_{ij}$$

$$\therefore \int_1^2 \vec{F}_{ji} \cdot d\vec{s}_i = -\int_1^2 \vec{\nabla}_{ij} V_{ij} \cdot d\vec{r}_{ij}$$

Finally, work done by internal forces reduces to

$$\sum_{i,j} \int_1^2 \vec{F}_{ji} \cdot d\vec{s}_i = -\frac{1}{2} \sum_{i,j} \int_1^2 \vec{\nabla}_{ij} V_{ij} \cdot d\vec{r}_{ij} = -\frac{1}{2} \sum_{ij} V_{ij} \Big|_1^2$$

CONSTRAINED FORCES

* In the system of particles where r_{ij} is fixed and does not vary with time
  - → Internal potential is always constant
  - → $d\vec{r}_{ij} \perp \vec{r}_{ij}$; $d\vec{r}_{ij} \perp \vec{F}_{ij}$ ⇒ $W = 0$
  - → such a system is called a rigid body

* Constraints are equations which limit the motion of the system. They are given by:

$$f(\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots, t) = 0 \quad \text{(holonomic constraints)}$$

  - → Eg: Rigid body constraint is $(\vec{r}_i - \vec{r}_j)^2 - c_{ij}^2 = 0$
  - → Any particle constrained to move along a curve or a surface follows a holonomic constraint
  - → Constraints not expressible by the above form are called non-holonomic

$$r^2 - a^2 \boxed{\geq} 0$$

* To work around the trouble of having r_i which are not independent; generalized coordinates are to be used. If k holonomic constraints exist, 3N − k independent coordinates are left, represented by $q_1, q_2 \ldots$ The transformation equations for jumping across coordinate systems is given by

$$\vec{r}_1 = r_1(q_1, q_2, \ldots, q_{3N-k}, t)$$
$$\vec{r}_2 = r_2(q_1, q_2, \ldots, q_{3N-k}, t)$$
$$\vdots$$
$$\vec{r}_N = r_N(q_1, q_2, \ldots, q_{3N-k}, t)$$

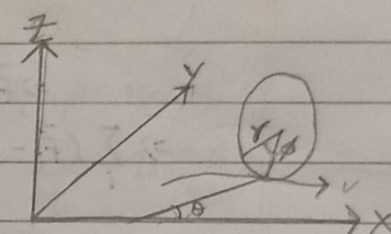
* The forces of constraint are unknown. To find them, the motion of the system must be solved. The mechanics must be formulated in such a way, forces of constraint vanish.

* Example for non holonomic constraint is rolling without slipping
  - → Cartesian coordinates specify location of the disk
  - → Angular coordinates specify orientation of the disk
  - → Rolling constraint connects them condn on velocities

$$v = r\dot{\phi}$$

$$\dot{x} = v\sin\theta \qquad \dot{y} = -v\cos\theta$$

$$dx = r\sin\theta\, d\phi \qquad dy = -r\cos\theta\, d\phi$$



However these equations can't be solved before solving the system

D'ALEMBERT'S PRINCIPLE

* A virtual displacement δx_i is defined as an infinitesimal instantaneous (dt=0) displacement of the coordinate x_i consistent with constraints on the system.

* From the above definition, we get virtual work $\vec{F_i}\cdot\delta\vec{r_i}$. Consider a system in equilibrium ($\vec{F_i}=0$)

$$\sum_i \vec{F_i}\cdot\delta\vec{r_i} = 0$$

On decomposing $\vec{F_i}$ into applied force & constraint forces

$$\vec{F_i} = \vec{F_i}^{(a)} + \vec{f_i}$$

$$\sum_i \vec{F_i}^{(a)}\cdot\delta\vec{r_i} + \sum_i \vec{f_i}\cdot\delta\vec{r_i} = 0$$

Rigid bodies

When the force of constraint is instantaneously perpendicular to the surface (and thus virtual displacement), $\vec{f_i}\cdot\delta\vec{r_i} = 0$

Force by actual disp doesn't vanish

$$\therefore \sum_i \vec{F}_i^{(a)} \cdot \delta \vec{r}_i = 0$$

which is called the principle of virtual work. $F_i^{(a)}$ need not equal zero.

→ The above equation gets rid of f_i but it is a static equation. To get a general condition, we use d'Alembert's principle.

* Consider the equation of motion

$$\vec{F}_i = \dot{\vec{p}}_i$$

$$\Rightarrow \vec{F}_i - \dot{\vec{p}}_i = 0$$

$$\Rightarrow \sum_i (\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta \vec{r}_i = 0$$

Splitting into applied & constraint forces

$$\sum_i (\vec{F}_i^{(a)} - \dot{\vec{p}}_i) \cdot \delta \vec{r}_i + \sum_i \vec{f}_i \cdot \delta \vec{r}_i = 0$$

Applying the same method to the constraint forces

$$\sum_i (\vec{F}_i^{(a)} - \dot{\vec{p}}_i) \cdot \delta \vec{r}_i = 0$$

which is called d'Alembert's principle

* Now d'Alembert's principle must be combined with generalized coordinates so that the coefficients of δq_i can be set to zero.

$$\vec{r}_i = r_i(q_1, q_2, \ldots, q_n, t)$$

$$\vec{v}_i = \frac{d\vec{r}_i}{dt} = \sum_k \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \vec{r}_i}{\partial t} = \sum_k \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k$$

Similarly for the virtual displacement

$$\delta \vec{r}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$$

$\therefore$ In terms of the generalized coordinates

$$\sum_i \vec{F}_i \cdot \delta \vec{r}_i = \sum_{i,j} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j = \sum_j Q_j \delta q_j$$

where Q_j's are the components of generalized force

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

Taking the other term of d'Alembert's principle

$$\sum_i \dot{\vec{p}}_i \cdot \delta \vec{r}_i = \sum_i m_i \ddot{\vec{r}}_i \cdot \delta \vec{r}_i = \sum_{i,j} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$$

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \left[\frac{d}{dt}\left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right) - m_i \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j} \right) \right]$$

$$\frac{d}{dt} \frac{\partial \vec{r}_i}{\partial q_j} = \frac{\partial \dot{\vec{r}}_i}{\partial q_j} = \sum_k \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_k} \dot{q}_k + \frac{\partial^2 \vec{r}_i}{\partial q_j \partial t} = \frac{\partial v_i}{\partial q_j}$$

Also, $\dfrac{\partial \vec{v}_i}{\partial \dot{q}_j} = \dfrac{\partial \vec{r}_i}{\partial q_j}$

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \left[\frac{d}{dt}\left(m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j} \right) - m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j} \right]$$

$$\therefore \sum_i \dot{\vec{p}}_i \cdot \delta \vec{r}_i = \sum_j \left[\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j} \sum_i \tfrac{1}{2} m_i v_i^2 \right] - \frac{\partial}{\partial q_j} \sum_i \tfrac{1}{2} m_i v_i^2 \right] \delta q_j$$

Finally d'Alembert's principle, $T = \sum_i \frac{1}{2} m_i v_i^2$

$$\sum_j \left[\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right) - Q_j \right] \delta q_j = 0$$

* When constraints are holonomic, it is possible to find independent coordinates q_j which contain the constraint conditions in the transformation equation.

$\therefore$ δq_j is independent of δq_k

$$\therefore \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$ (n such eqs)

The generalized forces can be written as

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i (-\vec{\nabla}_i V) \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

Also, the potential doesn't depend on the generalized velocities.

$$\Rightarrow \frac{\partial V}{\partial \dot{q}_j} = 0$$

$$\therefore \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial V}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = 0$$

$$\frac{d}{dt}\frac{\partial (T-V)}{\partial \dot{q}_j} - \frac{\partial (T-V)}{\partial q_j} = 0$$

Defining the Lagrangian as $L = T - V$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$ Lagrange's Equations

* Lagrange's equations can be used without a scalar potential function V, but with a generalized potential $U(q_j, \dot{q}_j)$

* When not all the forces acting on the system can be derived from a potential, Lagrange's equations are modified as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q_j$$

where Q_j represents forces not arising from a potential

* Frictional forces proportional to the velocity can be derived in terms of $\mathcal{F}$, Rayleigh's dissipation function,

$$\mathcal{F} = \frac{1}{2}\sum_i (k_x v_{ix}^2 + k_y v_{iy}^2 + k_z v_{iz}^2)$$

$$\vec{F}_{fx} = -k_x \vec{v}_x = -\frac{\partial \mathcal{F}}{\partial \dot{x}_i} \Rightarrow \vec{F}_f = -\nabla_v \mathcal{F}$$

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VARIATIONAL PRINCIPLES

* The instantaneous configuration of a system is described by the values of n generalized coordinates corresponding to the configuration space.

* As time goes on, the state of the system changes & the system point traces a curve in the configuration space described as the path of motion.

* Mechanical systems where the forces are derivable from a generalized scalar potential are called monogenic
  → If the potential is an explicit function of position coordinates, it is also conservative

* For monogenic systems, the Hamilton principle is stated as,
  "The motion of the system from time t_1 & time t_2 is such that the path integral (action integral) of the Lagrangian has a stationary value for the actual path of motion."

$$\delta I = \delta \int_{t_1}^{t_2} L(q_1, \dot{q}_1, \dots, q_n, \dot{q}_n, t)\, dt = 0$$

→ When the system constraints are holonomic, Hamilton's principle is both a necessary & sufficient condition for Lagrange's equations.

* A one dimensional problem includes a function $f(y, \dot{y}, x)$ defined on a path $y(x)$. The objective is to find a stationary value for

$$J = \int_{x_1}^{x_2} f(y, \dot{y}, x)\, dx$$

For J to be stationary, the variation must be zero relative to a set of neighboring paths labelled by an infinitesimal parameter α. We choose $\eta(x)$ s.t $\eta(x_1) = \eta(x_2) = 0$

$$y(x, \alpha) = y(x, 0) + \alpha \eta(x)$$

y & η are assumed to be continuous non singular functions with continuous 1^{ST} & 2^{ND} order derivatives

$$\therefore J(\alpha) = \int_{x_1}^{x_2} f(y(x,\alpha), \dot{y}(x,\alpha), x)\, dx$$

$$\left.\frac{dJ}{d\alpha}\right|_{\alpha=0} = 0$$

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \underbrace{\frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha}} \right) dx$$

$$\left.\frac{\partial y}{\partial \alpha}\right|_{x_1} = \left.\frac{\partial y}{\partial \alpha}\right|_{x_2} = 0$$

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha}\, dx = \overbrace{\left.\frac{\partial f}{\partial \dot{y}}\frac{\partial y}{\partial \alpha}\right|_{x_1}^{x_2}}^{0} - \int_{x_1}^{x_2} \frac{\partial y}{\partial \alpha}\frac{d}{dx}\frac{\partial f}{\partial \dot{y}}\, dx$$

$$\therefore \frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \frac{\partial y}{\partial \alpha}\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}} \right] dx = 0$$

WHEN $\int_{x_1}^{x_2} M(x)\eta(x)dx = 0$ FOR ALL ARBITRARY FNS $\eta(x)$, $M(x)$ VANISHES IN THE INTERVAL (x_1, x_2).

$$\Rightarrow \quad \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}} = 0$$

The infinitesimal departure of the varied path from the correct path is designated by

$$\left(\frac{dJ}{d\alpha}\right)_0 d\alpha = \delta J$$

corresponds to virtual disp

$$\therefore \delta J = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}}\right) \delta y \, dx = 0$$

* Ex: Shortest distance b/w two points in a plane

In a plane,

$$dl = \sqrt{dx^2 + dy^2}$$

Length of path

$$I = \int_1^2 dl = \int_{x_1}^{x_2} \sqrt{1 + (dy/dx)^2}\, dx$$

We must minimise I, with $f = \sqrt{1+\dot{y}^2}$

$$\frac{\partial f}{\partial y} = 0 \qquad \frac{\partial f}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{1+\dot{y}^2}}$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}} = 0$$

$$\Rightarrow \frac{\dot{y}}{\sqrt{1+\dot{y}^2}} = c \Rightarrow \dot{y} = \frac{c}{\sqrt{1-c^2}} = a$$

$$y = ax + b$$

Substituting (x_1, y_1) & (x_2, y_2)

$$y - y_2 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_2)$$

* The Brachistochrone Problem: Find path $y = f(x)$ minimizing the time of travel from A to B under the influence of a gravitational field.

$$t_{AB} = \int_A^B \frac{ds}{v} = \int_A^B \frac{\sqrt{1+\dot{y}^2}}{\sqrt{2g(h-y)}}\,dx$$

$$\tfrac{1}{2}mv^2 + mgy = mgh \Rightarrow v = \sqrt{2g(h-y)}$$

Here we have $F(x, y, \dot{y}) = F(y, \dot{y}) \Rightarrow F$ is independent of x. We can use the Beltrami Identity

$$F - \dot{y}\frac{\partial F}{\partial y'} = c$$

$$\frac{\sqrt{1+\dot{y}^2}}{\sqrt{2g(h-y)}} - \frac{\dot{y}}{\sqrt{2g(h-y)}}\left[\frac{\dot{y}}{\sqrt{1+\dot{y}^2}}\right] = c$$

$$1 + \dot{y}^2 - \dot{y}^2 = c\sqrt{2g(h-y)(1+\dot{y}^2)}$$

$$\Rightarrow k = h - y + \dot{y}^2(h-y) \qquad k = \frac{1}{2gc^2}$$

$$\dot{y} = \sqrt{\frac{k-h+y}{h-y}}$$

$$\int dx = \int \sqrt{\frac{h-y}{k-h+y}}\,dy$$

$$y = h - k\sin^2(\theta/2) \Rightarrow dy = -k\cdot \sin\theta/2 \cos\theta/2\, d\theta$$

$$\int dx = \int \sqrt{\frac{k\sin^2\theta/2}{k - k\sin^2\theta/2}}\left(-k\cdot \sin\theta/2\cos\theta/2\, d\theta\right)$$

$$\int dx = \int -\frac{k}{2}(1-\cos\theta)\,d\theta$$

$$x = \frac{k_1}{2}(\theta - \sin\theta) + k_2 \quad \& \quad y = h + k_1\sin^2(\theta/2)$$

After using initial conditions & applying trigonometric identities,

$$x = \frac{k_1}{2}(\theta - \sin\theta)$$

$$y = h + \frac{k_1}{2}(1-\cos\theta)$$

* Consider a function f parameterized by n independent variables, their derivatives & an independent parameter. The variation of the integral J,

$$\delta J = \delta \int_1^2 f(y_1(x), \dot{y}_1(x), \dots, y_n(x), \dot{y}_n(x), x)\, dx$$

is obtained. Again, a parameter α for the n curves

$$y_1(x,\alpha) = y_1(x,0) + \alpha\eta_1(x)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$y_n(x,\alpha) = y_n(x,0) + \alpha\eta_n(x)$$

$$\delta J = \frac{\partial J}{\partial \alpha} d\alpha = \int_1^2 \sum_i \left(\frac{\partial f}{\partial y_i}\frac{\partial y_i}{\partial \alpha} d\alpha + \frac{\partial f}{\partial \dot{y}_i}\frac{\partial \dot{y}_i}{\partial \alpha} d\alpha \right) dx$$

Again after integration by parts of the second term

$$\delta J = \int_1^2 \sum_i \left(\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}_i} \right) \delta y_i \, dx$$

The y variables are independent & the so are the variations δy_i & $\eta_i(x)$. From an extension of the lemma $\int_1^2 M(x)\eta(x)dx = 0$, the coefficients of δy_i separately vanish

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}_i} = 0 \qquad i = 1, 2, \dots, n$$

* When Lagrange's equations are derived from d'Alembert's principle or Hamilton's principle, holonomic constraints appear when variations in q_i are considered independent.

* The virtual displacements may not be consistent with the constraints. The extra displacements can be eliminated by the method of Lagrange undetermined multipliers

$$I = \int_1^2 \left(L + \sum_{\alpha=1}^{m} \lambda_\alpha f_\alpha\right) dt$$

Variation of q_i gives

$$\delta I = \int_1^2 dt \left(\sum_{i=1}^{n} \left[\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} + \sum_{\alpha=1}^{m} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_i} \right] \delta q_i \right) = 0$$

→ For n variables & m constraints, $n-m$ equations are obtained independently, m equations are of the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} + \sum_{\alpha=1}^{m} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_k} = 0$$

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = -\sum_{\alpha=1}^{m} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_k} = Q_k$$

→ Semi non holonomic constraints can be written as

$$f_\alpha(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t) = 0$$

$$f_\alpha = \sum_{k=1}^{n} a_{\alpha k} \dot{q}_k + a_0 = 0$$

$$\delta \int_{t_1}^{t_2} \left(L + \sum_{\alpha=1}^{m} \mu_\alpha f_\alpha\right) dt = 0$$

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = Q_k = -\sum_{\alpha=1}^{m} \mu_\alpha \frac{\partial f_\alpha}{\partial \dot{q}_k}$$

* Consider a system of mass points under the influence of forces derived from potentials dependent on position only

$$\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} - \frac{\partial V}{\partial \dot{x}_i} = \frac{\partial}{\partial \dot{x}_i} \sum \tfrac{1}{2} m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)$$

$$= m_i \dot{x}_i = p_{ix}$$

$\therefore p_j = \dfrac{\partial L}{\partial \dot{q}_j}$ ⇒ generalized (or canonical or conjugate) momentum associated with cord q_j

* If the lagrangian of a system does not contain a given coordinate q_j, the coordinate is cyclic

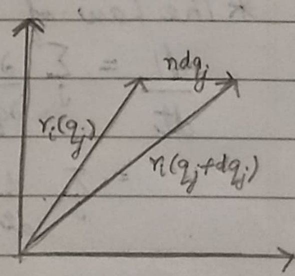
$$\Rightarrow \frac{\partial L}{\partial q_j} = 0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0$$

$$\Rightarrow p_j = k$$

∴ Generalized momentum conjugate to a cyclic coordinate is conserved.

* Generalized Force,

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$



$$\frac{\partial \vec{r}_i}{\partial q_j} = \lim_{dq_j \to 0} \frac{\vec{r}_i(q_j + dq_j) - \vec{r}_i(q_j)}{dq_j} = \frac{\hat{n}\, dq_j}{dq_j}$$

$$= \hat{n}$$

$$Q_j = \hat{n} \cdot \vec{F}$$

$$T = \tfrac{1}{2} \sum m_i \dot{\vec{r}}_i^{\,2}$$

Conjugate momentum

$$p_j = \frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = \sum_i m_i \vec{v}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \hat{n} \cdot \sum m_i \vec{v}_i$$

* For an infinitesimal rotation of the vector $\vec{r}_i$

$$|d\vec{r}_i| = r_i \sin\theta \, dq_j \Rightarrow \left|\frac{\partial \vec{r}_i}{\partial q_j}\right| = r_i \sin\theta$$

$$\frac{\partial \vec{r}_i}{\partial q_j} = \hat{n} \times \vec{r}_i$$

$$\therefore Q_j = \sum_i \vec{F}_i \cdot \hat{n} \times \vec{r}_i = \sum_i \hat{n} \cdot \vec{r}_i \times \vec{F}_i = \hat{n} \cdot \sum_i \vec{N}_i$$

$$Q_j = \hat{n} \cdot \vec{N}$$

Assuming potential dependent only on position

$$p_j = \frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i \vec{v}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \hat{n} \cdot \vec{r}_i \times m_i \vec{v}_i = \hat{n} \cdot \sum_i \vec{L}_i = \hat{n} \cdot \vec{L}$$

$\Rightarrow$ Angular momentum is conserved

* If a generalized coordinate corresponding to a displacement is cyclic, the system is invariant under translation along the direction.

* The Law of Conservation of Energy can also be derived:

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j} \frac{dq_j}{dt} + \sum_j \frac{\partial L}{\partial \dot{q}_j} \frac{d\dot{q}_j}{dt} + \frac{\partial L}{\partial t}$$

$$= \sum_j \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j} \frac{d\dot{q}_j}{dt} + \frac{\partial L}{\partial t}$$

$$= \sum_j \frac{d}{dt}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{d}{dt}\left(\sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L\right) + \frac{\partial L}{\partial t} = 0$$

$\Rightarrow$ The energy function,

$$h(q_1, \dot{q}_1, \dots, q_n, \dot{q}_n, t) = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L$$

$$\therefore \frac{dL}{dt} = 0 = \frac{dh}{dt} + \frac{\partial L}{\partial t}$$

→ If the Lagrangian is not an explicit function of time, h is conserved.

→ Total KE can always be written as

$$T = T_0(q) + T_1(q,\dot{q}) + T_2(q,\dot{q})$$

$$\Rightarrow L(q,\dot{q},t) = L_0(q,t) + L_1(q,\dot{q},t) + L_2(q,\dot{q},t)$$

L_1 - first degree fn $\quad$ L_2 - second degree fn

$$\therefore h = 2L_2 + L_1 - L = L_2 - L_0 \quad \because \left(\sum_i x_i \frac{\partial f}{\partial x_i} = nf\right)$$

When potential doesn't depend on generalized velocities,

$$L_2 = T \quad \& \quad L_0 = -V$$

$$\therefore h = T + V = E$$

$\Rightarrow$ Energy conserved

* When the frictional forces are derivable from a dissipation function $\mathcal{F}$,

$$\frac{dh}{dt} + \frac{\partial L}{\partial t} = \sum_j \frac{\partial \mathcal{F}}{\partial \dot{q}_j} \dot{q}_j$$

$$\mathcal{F} = \frac{1}{2} \sum_i (k_x v_{ix}^2 + k_y v_{iy}^2 + k_z v_{iz}^2)$$

$$\Rightarrow \frac{dh}{dt} = -2\mathcal{F} - \frac{\partial L}{\partial t}$$

When L is not an explicit function of time

$$\frac{dE}{dt} = -2\mathcal{F}$$

* Mejik with the functional

→ To maximise (EXTREMIZE) $\int_{x_1}^{x_2} f(y, \dot{y})\, dx$, use the Beltrami Identity,

$$\dot{y}\frac{\partial f}{\partial \dot{y}} - f = k$$

ELE =

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial x} = 0$$

If $\partial F/\partial x = 0 \Rightarrow$ Beltrami ID

→ When the functional in the Hamiltonian is given by $\int_{x_1}^{x_2} f(y)\sqrt{1+\dot{y}^2}\, dx$

The $y(x)$ ~~maximizing~~ extremizing the integral satisfies

$$1+\dot{y}^2 = Bf(y)^2$$

→ To maximise $\int_{x_1}^{x_2} f(x, \dot{y})\, dx$,

$$\frac{\partial f}{\partial \dot{y}} = k$$

HAMILTONIAN EQUATIONS

* Legendre Transformation:

Consider a $f(x,y)$

$$df = u\,dx + v\,dy$$

$$\Rightarrow u = \frac{\partial f}{\partial x} \qquad v = \frac{\partial f}{\partial y}$$

Let g be a function defined by

$$g = f - ux$$

$$dg = df - u\,dx - x\,du$$

$$dg = v\,dy - x\,du$$

$$\Rightarrow x = -\frac{\partial g}{\partial u} \quad \& \quad v = \frac{\partial g}{\partial y}$$

* The canonical momentum is defined as

$$p_i = \frac{\partial L(q_i, \dot{q}_i, t)}{\partial \dot{q}_i}$$

The Lagrangian is defined as

$$L = \quad L(q_i, \dot{q}_i, t)$$

$$dL = \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial t} dt$$

$$= p_i\,d\dot{q}_i + \dot{p}_i\,dq_i + \frac{\partial L}{\partial t} dt$$

Taking the Legendre transform,

$$dH = \dot{q}_i\,dp_i - \dot{p}_i\,dq_i - \frac{\partial L}{\partial t} dt$$

$$\Rightarrow H(q, p, t) = \dot{q}_i p_i - L(q, \dot{q}, t)$$

The differential of the Hamiltonian is also given by

$$dH = \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt$$

$\therefore$ 2n+1 relations are obtained

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$-\dot{p}_i = \frac{\partial H}{\partial q_i} \quad \text{(CANONICAL EQUATIONS)}$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}$$

* In many cases, the Lagrangian can be expressed as (SYMPLECTIC APPROACH)

$$L = L_0(q_i, t) + L_1(q_i, t)\dot{q}_k + L_2(q_i, t)\dot{q}_k \dot{q}_m$$

L_0 - independent of generalized velocities

L_1 - coefficients of L homogeneous in $\dot{q}_i$ in 1st deg

L_2 - coefficients of L homogeneous in $\dot{q}_i$ in 2nd deg

→ If the equations defining the generalized coordinates don't depend on time,

$$L_2 \dot{q}_k \dot{q}_m = T$$

→ If the forces are derivable from a conservative potential,

$$L_0 = -V$$

→ When both these conditions hold,

$$H = T + V = E$$

→ If L_2 can be expressed as a quadratic function of generalized velocities & L_1 is a linear function of the same variable,

$$L(q_i, \dot{q}_i, t) = L_0(q,t) + \dot{q}_i a_i(q,t) + \dot{q}_i^2 T_i(q,t)$$

The q_i's can be formed into a column matrix

$$\therefore L = L_0(q,t) + \tilde{\dot{q}} a + \frac{1}{2} \tilde{\dot{q}} T \dot{q}$$

$\therefore H = \tilde{\dot{q}} p - L$ becomes

$$H = \tilde{\dot{q}}(p-a) - \frac{1}{2} \tilde{\dot{q}} T \dot{q} - L_0$$

→ Since T is symmetric,

$$p = T\dot{q} + a$$

$$\therefore \dot{q} = T^{-1}(p-a)$$

$$\tilde{\dot{q}} = (\tilde{p} - \tilde{a}) T^{-1}$$

Replacing $\dot{q}$ & $\tilde{\dot{q}}$ in the Hamiltonian,

$$H(q,p,t) = \frac{1}{2} (\tilde{p} - \tilde{a}) T^{-1} (p-a) - L_0(q,t) \qquad T^{-1} = \frac{\tilde{T_c}}{|T|}$$

* For a system with n degrees of freedom, we can describe of column matrix of size $2n$,

$$\eta_i = q_i \qquad \eta_{i+n} = p_i \qquad i \leq n$$

Hence,

$$\left(\frac{\partial H}{\partial \eta}\right)_i = \frac{\partial H}{\partial q_i} \qquad \left(\frac{\partial H}{\partial \eta}\right)_{i+n} = \frac{\partial H}{\partial p_i} \qquad i \leq n$$

Let J be the $2n \times 2n$ square matrix according to the scheme

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$\therefore$ Hamilton's equations of motion,

$$\dot{\eta} = J \frac{\partial H}{\partial \eta}$$

* A (ignorable) cyclic coordinate q_j is a coordinate which does not explicitly appear in the Lagrangian; hence by virtue of the Euler-Lagrange Equation, its conjugate momentum p_j is a constant.

$$\dot{p}_j = \frac{\partial L}{\partial q_j} = -\frac{\partial H}{\partial q_j}$$

→ A cyclic coordinate is absent from the Hamiltonian as well. While the Lagrangian remains a function of all n generalized velocities; the Hamiltonian is a function of $2n-2$ variables.

$$L = L(q_1, q_2, \dots, q_{n-1}; \dot{q}_1, \dot{q}_2, \dots, \dot{q}_n; t)$$

$$H = H(q_1, q_2, \dots, q_{n-1}; p_1, p_2, \dots, p_{n-1}; \alpha; t)$$

When q_n is ignorable, p_n is a constant, α

$$\dot{q}_n = \frac{\partial H}{\partial \alpha}$$

* Consider a system with $q_1, q_2, \dots, q_n$ where $q_{s+1}, \dots, q_n$ are cyclic. Define a quantity the Routhian,

$$R = \sum_{i=s+1}^{n} p_i \dot{q}_i - L$$

$$\therefore R = R(q_1, q_2, \dots, q_s; \dot{q}_1, \dot{q}_2, \dots, \dot{q}_s; p_{s+1}, p_{s+2}, \dots, p_n; t)$$

$$\frac{\partial R}{\partial q_i} = -\frac{\partial L}{\partial q_i} \qquad i = 1, 2, \dots, s$$

$$\frac{\partial R}{\partial \dot{q}_i} = -\frac{\partial L}{\partial \dot{q}_i} \qquad i = 1, 2, \dots, s$$

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$$\frac{\partial R}{\partial q_i} = -\dot{p}_i \qquad i = s+1, \dots, n$$

$$\frac{\partial R}{\partial p_i} = \dot{q}_i \qquad i = s+1, \dots, n$$

→ Routhian obeys Lagrange's equations for the first s coordinates.

⇒ The remaining coordinates & momenta obey the Hamiltonian equation with the Routhian as Hamiltonian

CANONICAL TRANSFORMATIONS

* Consider a Hamiltonian which is a constant of motion with all coordinates q_i as cyclic.

$$p_i = \alpha_i$$

$\therefore$ The Hamiltonian is

$$H = H(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$$

$$\dot{q}_i = \frac{\partial H}{\partial \alpha_i} = \omega_i$$

$$\Rightarrow \quad q_i = \omega_i t + \beta_i \quad \text{— init condns}$$

* Hamiltonian formulation also includes momenta as independent variables, hence transformation is given by the invertible equations

$$Q_i = Q_i(q, p, t)$$

$$P_i = P_i(q, p, t)$$

The new coordinates Q & P must be canonical, so the function K must play the role of a Hamiltonian.

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i}$$

$$\Rightarrow \quad \delta \int_{t_1}^{t_2} (P_i \dot{Q}_i - K(Q, P, t))\, dt = 0$$

With the old coordinates

$$\delta \int_{t_1}^{t_2} (p_i \dot{q}_i - H(q, p, t))\, dt = 0$$

$\therefore$ The integrands are connected by

ELE is INVARIANT for PT TRANSF. and for TOTAL DERIV

$$\lambda(p_i \dot{q}_i - H) = P_i \dot{Q}_i - K + \frac{dF}{dt}$$

λ - independent constant

F - Function of phase space with cont 2^{nd} derivatives.

* Coordinate transforms with
  $\lambda = 1$ scale transforms
  $\lambda \neq 1$ extended canonical transformation
  $dF = \lambda p_i dq_i - P_i dQ_i + (K - \lambda H) dt$

* The dF/dt term contributes to the variation of the action integral only at the end points & will vanish when F is a function of any mixture of phase space coordinates. (generating fn)

Consider

$$F = F_1(q, Q, t)$$

$$p_i \dot{q}_i - H = P_i \dot{Q}_i - K + \frac{dF_1}{dt}$$

$$p_i \dot{q}_i - H = P_i \dot{Q}_i - K + \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial q_i} \dot{q}_i + \frac{\partial F_1}{\partial Q_i} \dot{Q}_i$$

$$\cancel{p_i \dot{q}_i} - H = \cancel{P_i \dot{Q}_i} - K + \frac{\partial F_1}{\partial t} + \cancel{p_i \dot{q}_i} + \cancel{(-P_i) \dot{Q}_i}$$

$$K = H + \frac{\partial F_1}{\partial t}$$

If p_i can't be expressed as q, Q & t,

$$F = F_2(q, P, t) - Q_i P_i$$

$$p_i \dot{q}_i - H = -Q_i \dot{P}_i - K + \frac{d}{dt} F_2(q, P, t)$$

$$p_i \dot{q}_i - H = -Q_i \dot{P}_i - K + \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial q_i} \dot{q}_i + \frac{\partial F_2}{\partial P_i} \dot{P}_i$$

$$\Rightarrow K = H + \frac{\partial F_2}{\partial t}$$

* Four basic canonical transforms

→ $F = F_1(q, Q, t)$

$$p_i = \frac{\partial F_1}{\partial q_i} \qquad P_i = -\frac{\partial F_1}{\partial Q_i}$$

Trivial case:

$$F_1 = q_i Q_i \Rightarrow Q_i = p_i \ \& \ P_i = -q_i$$

→ $F = F_2(q, P, t) - Q_i P_i$

$$p_i = \frac{\partial F_2}{\partial q_i} \qquad Q_i = \frac{\partial F_2}{\partial P_i}$$

Trivial case:

$$F_2 = q_i P_i \Rightarrow Q_i = q_i \ \& \ P_i = p_i$$

→ $F = F_3(p, Q, t) + q_i p_i$

$$q_i = -\frac{\partial F_3}{\partial p_i} \qquad P_i = -\frac{\partial F_3}{\partial Q_i}$$

Trivial case:

$$F_3 = p_i Q_i \Rightarrow Q_i = -q_i \ \& \ P_i = -p_i$$

→ $F = F_4(p, P, t) + q_i p_i - Q_i P_i$

$$q_i = -\frac{\partial F_4}{\partial p_i} \qquad Q_i = \frac{\partial F_4}{\partial P_i}$$

Trivial case:

$$F_4 = p_i P_i \Rightarrow Q_i = p_i \ \& \ P_i = -q_i$$

* Consider the point transform

$$F = f_i(q_1, q_2, \dots, q_n; t) P_i + g(q_1, q_2, \dots, q_n; t)$$

$$p_j = \frac{\partial F}{\partial q_j} = \frac{\partial f_i}{\partial q_j} P_i + \frac{\partial g}{\partial q_j}$$

$$\begin{bmatrix} \vdots \\ p_j \\ \vdots \end{bmatrix} = \begin{bmatrix} \partial f_1/\partial q_1 & \cdots & \partial f_1/\partial q_n \\ \vdots & & \vdots \\ \partial f_n/\partial q_1 & \cdots & \partial f_n/\partial q_n \end{bmatrix} \begin{bmatrix} \vdots \\ P_i \\ \vdots \end{bmatrix} + \begin{bmatrix} \partial g/\partial q_1 \\ \vdots \\ \partial g/\partial q_n \end{bmatrix}$$

$$\therefore \vec{P} = \left[\partial f/\partial q\right]^{-1} \left[\vec{p} - \partial g/\partial \vec{q}\right]$$

* Consider a simple harmonic oscillator in one dimension.

$$H = \frac{p^2}{2m} + \frac{kx^2}{2}$$

$$= \frac{1}{2m}(p^2 + mkx^2)$$

Taking $k/m = \omega^2$,

$$H = \frac{1}{2m}(p^2 + m^2\omega^2 x^2)$$

→ As the Hamiltonian is a sum of squares, it may be possible to find a cyclic coordinate (Q)

$$p = f(P) \cos Q$$

$$x = \frac{f(P)}{m\omega} \sin Q$$

The Kamiltonian,

$$K = \frac{1}{2m} f^2(P) \cos^2 Q + \frac{1}{2m} f^2(P) \sin^2 Q$$

$$K = \frac{f^2(P)}{2m}$$

We find a generating function $F = F_1(q, Q)$ by

$$\frac{p}{x} = m\omega \cot Q$$

$$p = \frac{\partial F_1}{\partial x}$$

$$\Rightarrow F_1 = \int m\omega x \cot Q \, dx + f(Q)$$

$$F_1 = \frac{1}{2} m\omega x^2 \cot Q + f(Q)$$

Setting $= 0$

$$P = -\frac{\partial F_1}{\partial Q} = \frac{1}{2} m\omega x^2 \operatorname{cosec}^2 Q + \cancel{(-f'(Q))}$$

$$x^2 = \frac{2P}{m\omega} \sin^2 Q$$

$$x = \sqrt{\frac{2P}{m\omega}} \sin Q$$

$$\Rightarrow f(P) = \sqrt{2m\omega P}$$

$$\therefore K = P\omega$$

$$\therefore P = \frac{K}{\omega} = \frac{E}{\omega}$$

$$\dot{Q} = \frac{\partial K}{\partial P} = \omega \Rightarrow Q = \omega t + \phi$$

$$x = \sqrt{\frac{2E}{m\omega^2}} \sin(\omega t + \phi)$$

$$p = \sqrt{2mE} \cos(\omega t + \phi)$$

* Consider a restricted canonical transformation

$$Q_i = Q_i(q, p)$$
$$P_i = P_i(q, p)$$

The Hamiltonian does not change for such a transformation

$$\dot{Q}_i = \frac{\partial Q_i}{\partial q_j}\dot{q}_j + \frac{\partial Q_i}{\partial p_j}\dot{p}_j$$
$$= \frac{\partial Q_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial Q_i}{\partial p_j}\frac{\partial H}{\partial q_j}$$

Consider inverse transforms

$$q_i = q_i(Q, P)$$
$$p_i = p_i(Q, P)$$

$$\therefore \frac{\partial H}{\partial P_i} = \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial P_i} + \frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial P_i}$$

$$\Rightarrow \dot{Q}_i = \frac{\partial H}{\partial P_i}$$

The transformation is canonical iff

$$\left(\frac{\partial Q_i}{\partial q_j}\right)_{q,p} = \left(\frac{\partial p_j}{\partial P_i}\right)_{Q,P} \; ; \; \left(\frac{\partial Q_i}{\partial p_j}\right)_{q,p} = -\left(\frac{\partial q_j}{\partial P_i}\right)_{Q,P} \qquad \dot{Q}_i = \frac{\partial H}{\partial P_i}$$

$$\left(\frac{\partial P_i}{\partial q_j}\right)_{q,p} = -\left(\frac{\partial p_j}{\partial Q_i}\right)_{Q,P} \; ; \; \left(\frac{\partial P_i}{\partial p_j}\right)_{q,p} = \left(\frac{\partial q_j}{\partial Q_i}\right)_{Q,P} \qquad -\dot{P}_i = \frac{\partial H}{\partial Q_i}$$

* Recalling matrix form of Hamiltonian,

$$\dot{\eta} = \mathbb{J}\frac{\partial H}{\partial \eta} \qquad \eta_i = q_i \; ; \; \eta_{i+n} = p_i$$

$$\mathbb{J} = \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix}$$

For the new coordinates, consider the column

$$\zeta = \zeta(\eta)$$

$$\dot{\zeta} = \frac{\partial \zeta}{\partial \eta_j}\dot{\eta}_j \qquad i,j = 1,2,\ldots,2n$$

$$\dot{\zeta} = M\dot{\eta} \quad \text{where } M \text{ is the Jacobian matrix}$$

$$M_{ij} = \frac{\partial \zeta_i}{\partial \eta_j}$$

$$\Rightarrow \dot{\zeta} = M\mathbb{J}\frac{\partial H}{\partial \eta}$$

$M\mathbb{J}\tilde{M}$ is antisymmetric

$$(M\mathbb{J}\tilde{M})^T = -(M\mathbb{J}\tilde{M})$$

$$\frac{\partial H}{\partial \eta_i} = \frac{\partial H}{\partial \zeta_j}\frac{\partial \zeta_j}{\partial \eta_i}$$

Extended Canonical Transform

$$M\mathbb{J}\tilde{M} = \lambda\mathbb{J}$$

In matrix notation,

$$\frac{\partial H}{\partial \eta} = \tilde{M}\frac{\partial H}{\partial \zeta}$$

$$\therefore \dot{\zeta} = M\mathbb{J}\tilde{M}\frac{\partial H}{\partial \zeta}$$

Hence the transformation is canonical if

$$M\mathbb{J}\tilde{M} = \mathbb{J}$$

$$\equiv$$

$$\tilde{M}\mathbb{J}M = \mathbb{J}$$

* Consider a transformation where the new variables differ infinitesimally.

$$Q_i = q_i + \delta q_i$$
$$P_i = p_i + \delta p_i$$
$$\therefore \zeta = \eta + \delta\eta$$

→ The generating function for an infinitesimal canonical transformation

$$F_2 = q_i P_i + \epsilon G(q, P, t)$$

$$p_j = \frac{\partial F_2}{\partial q_j} = P_j + \epsilon \frac{\partial G}{\partial q_j}$$

$$\delta p_j = P_j - p_j = -\epsilon \frac{\partial G}{\partial q_j}$$

$$Q_j = \frac{\partial F_2}{\partial P_j} = q_j + \underbrace{\epsilon \frac{\partial G}{\partial P_j}}_{\delta q_j}$$

$$\therefore \delta\eta = \epsilon J \frac{\partial G}{\partial \eta}$$

$$\zeta(t_0) \rightarrow \zeta(t_0 + dt)$$

$$M \equiv \frac{\partial \zeta}{\partial \eta} = 1 + \frac{\partial \delta\eta}{\partial \eta}$$

$$M = 1 + \epsilon J \frac{\partial^2 G}{\partial \eta \partial \eta}$$

$$\left(\frac{\partial^2 G}{\partial \eta \partial \eta}\right)_{i,j} = \frac{\partial^2 G}{\partial \eta_i \partial \eta_j}$$

$$\tilde{M} = 1 - \epsilon \frac{\partial^2 G}{\partial \eta \partial \eta} J$$

$$MJ\tilde{M} = \left(1 + \epsilon J \frac{\partial^2 G}{\partial \eta \partial \eta}\right) J \left(1 - \epsilon J \frac{\partial^2 G}{\partial \eta \partial \eta}\right) = J$$

* Poisson brackets of canonical variables are called fundamental Poisson brackets

$$[\eta, \eta]_\eta = J$$

$$[\zeta, \zeta]_\eta = J$$

→ Fundamental Poisson brackets are invariant under a canonical transformation.

* ALL Poisson brackets are invariant under a canonical transformation.

$$\frac{\partial v}{\partial \eta} = \tilde{M} \frac{\partial v}{\partial \zeta} \quad \text{and} \quad \widetilde{\frac{\partial u}{\partial \eta}} = \widetilde{\tilde{M}\frac{\partial u}{\partial \zeta}} = \widetilde{\frac{\partial u}{\partial \zeta}} M$$

$$\therefore [u, v]_\eta = \widetilde{\frac{\partial u}{\partial \eta}} J \frac{\partial v}{\partial \eta} = \widetilde{\frac{\partial u}{\partial \zeta}} M J \tilde{M} \frac{\partial u}{\partial \zeta} = [u, v]_\zeta$$

* Poisson brackets possess the following algebraic properties

$$[u, u] = 0$$

$$[u, v] = -[v, u]$$

$$[au + bv, w] = a[u, w] + b[v, w]$$

$$[uv, w] = [u, w]v + u[v, w]$$

→ Jacobi's identity

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$$

* Jacobi's identity says that the bracket product is not associative

→ Poisson's brackets also satisfy

$$[u_i, u_j] = \sum_k c_{ij}^k u_k$$

which form the non commutative Lie algebra

* Lagrange Bracket denoted by $\{u, v\}$ is another canonical invariant like the Poisson bracket. Suppose u & v are two functions out of a set of $2n$ independent functions of the canonical variables.

$$\{u, v\}_{q,p} = \frac{\partial q_i}{\partial u}\frac{\partial p_i}{\partial v} - \frac{\partial p_i}{\partial u}\frac{\partial q_i}{\partial v}$$

$$\{u, v\}_\eta = \frac{\partial \tilde{\eta}}{\partial u} J \frac{\partial \eta}{\partial v}$$

* If for u & v we take two members of the set of canonical variables, we have the fundamental Lagrange brackets

$$\{q_i, q_j\}_{qp} = \{p_i, p_j\}_{qp} = 0$$

$$\{q_i, p_j\}_{qp} = \delta_{ij}$$

In matrix notation,

$$\{\eta, \eta\} = J$$

* Lagrange brackets & Poisson brackets have a reciprocal nature; as

$$\{u, u\}[u, u] = -1$$

where u is a set of $2n$ independent functions of the canonical variables.

* Lagrange brackets DO NOT obey Jacobi's identity and don't qualify as a product operation in Lie algebra.

* The canonical transformation $\eta \to \zeta$ transforms the volume element

$$d\eta = dq_1\, dq_2 \cdots dq_n\, dp_1\, dp_2 \cdots dp_n$$

into the new element

$$d\zeta = dQ_1\, dQ_2 \cdots dQ_n\, dP_1\, dP_2 \cdots dP_n$$

via the Jacobian;

$$d\zeta = \|M\|\, d\eta$$

For example,

$$\Rightarrow dQ\, dP = \begin{vmatrix} \partial q/\partial Q & \partial q/\partial P \\ \partial P/\partial Q & \partial p/\partial P \end{vmatrix} dq\,dp = [q, p]_\zeta\, dq\, dp$$

Taking determinant of the symplectic condition,

$$|M|^2 \, |J| = |J|$$

$$\Rightarrow |M| = \pm 1$$

* Consider transformation equations for a 1D system

$Q = Q(q, p) \qquad P = P(q, p)$

We can consider it to be invertible s.t.

$p = \phi(q, Q)$

$\Rightarrow P = \psi(q, Q)$

$p = \frac{\partial F_1(q, Q)}{\partial q} \qquad P = -\frac{\partial F_1(q, Q)}{\partial Q}$

In which case

$\frac{\partial \phi}{\partial Q} = -\frac{\partial \psi}{\partial q}$

We have the identity transformation

$\frac{\partial Q}{\partial Q} = 1$

$\Rightarrow 1 = \frac{\partial Q(q, \phi(q, Q))}{\partial Q}$

$1 = \frac{\partial Q}{\partial p} \frac{\partial \phi}{\partial Q}$

$\therefore$ The Poisson bracket,

$[Q, P] \equiv \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = 1$

The derivatives of P are derivatives of ψ

$[Q, P] = \frac{\partial Q}{\partial q}\frac{\partial \psi}{\partial Q}\frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial p}\left(\frac{\partial \psi}{\partial q} + \frac{\partial \psi}{\partial Q}\frac{\partial Q}{\partial q}\right)$

$$[Q, Q] = 0$$

$$[Q, P] = \frac{\partial \psi}{\partial Q}\left(\frac{\partial Q}{\partial q}\frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial Q}{\partial q}\right) - \frac{\partial Q}{\partial p}\frac{\partial \psi}{\partial q} = 1$$

$$\therefore \quad -\frac{\partial Q}{\partial p}\frac{\partial \psi}{\partial q} = 1$$

$\therefore$ We have

$$\frac{\partial Q}{\partial p}\frac{\partial \phi}{\partial Q} = -\frac{\partial Q}{\partial p}\frac{\partial \psi}{\partial q}$$

* The total time derivative of some function of the canonical variable, $u(q, p, t)$

$$\frac{du}{dt} = \frac{\partial u}{\partial q_i}\dot{q}_i + \frac{\partial u}{\partial p_i}\dot{p}_i + \frac{\partial u}{\partial t} = \frac{\partial u}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial u}{\partial t}$$

$$\Rightarrow \frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t}$$

or symplectically,

$$\frac{du}{dt} = \frac{\partial u}{\partial \eta}\dot{\eta} + \frac{\partial u}{\partial t} = \frac{\partial u}{\partial \eta} J \frac{\partial H}{\partial \eta} + \frac{\partial u}{\partial t}$$

$$\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t}$$

→ If u is a constant of motion,

$$[H, u] = \frac{\partial u}{\partial t}$$

* Poisson's theorem: If u & v are two constants of motion, then $[u, v]$ is also a constant of motion. (Proof by Jacobi's identity)

* Consider an infinite canonical transformation

$$\zeta = \eta + \delta\eta$$

with the change given in terms of the generator

$$\delta\eta = \epsilon \mathbb{J} \frac{\partial G(\eta)}{\partial \eta}$$

By the definition of the Poisson bracket,

$$[\eta, u] = \mathbb{J} \frac{\partial u}{\partial \eta}$$

$$\therefore \delta\eta = \epsilon [\eta, G]$$

Consider the continuous parameter of the ICT to be t, so that $\epsilon = dt$ & let the generating function be the Hamiltonian.

$$\delta\eta = dt [\eta, H] = \dot{\eta}\, dt = d\eta$$

* The Hamiltonian is the generator of system motion with time. System motion in finite time can be represented as successive infinitesimal contact transformations.

* A canonical transformation, in effect is the switching of phase space η to ζ where point $A(q,p)$ becomes $A'(Q,P)$. However we can consider a more active view by translating the point.

* Consider a function U. In the active canonical transform the system is translated from $A(q_A, p_A)$ to $B(q_B, p_B)$. If the transformation is infinitesimal,

$$\partial u = u(B) - u(A)$$

$$\Rightarrow \partial u = u(\eta + \delta\eta) - u(\eta)$$

$$\Rightarrow \partial u = \frac{\partial u}{\partial \eta}\,\delta\eta = \left(\frac{\tilde{\partial u}}{\partial \eta}\right)\left(\epsilon\, \mathbb{J}\, \frac{\partial G}{\partial \eta}\right)$$

$$\partial u = \epsilon\,[u, G]$$

Taking u for the phase space coordinates,

$$\partial \eta = \epsilon\,[\eta, G] = \delta\eta$$

→ While changing the phase space, the Hamiltonian also changes.

$$\therefore\ \partial H = H(B) - K(A')$$

Since $K = H + \frac{\partial F}{\partial t}$, we have for an I.C.T.,

$$K(A') = H(A') + \epsilon\frac{\partial G}{\partial t} = H(A) + \epsilon\frac{\partial G}{\partial t}$$

$$\therefore\ \partial H = H(B) - H(A) - \epsilon\frac{\partial G}{\partial t}$$

$$\Rightarrow \partial H = \epsilon\,[H, G] - \epsilon\frac{\partial G}{\partial t} = -\epsilon\frac{dG}{dt}$$

→ If G is a constant of the motion, it generates an ICT that doesn't change the Hamiltonian.

→ The constants of motion are the generating functions of the ICTs where the Hamiltonian is invariant.

* Consider a cyclic coordinate q_i, and a transformation generated by the generalized momentum conjugate to q_i.

$$G(q, p) = p_i$$

The resultant ICT is

$$\delta q_j = \epsilon \delta_{ij} \quad \& \quad \delta p_i = 0$$

→ If a coordinate is cyclic, its conjugate momentum is a constant of the motion.

→ If the generating function is defined as

$$G_l = (J\eta)_l = J_{lr}\eta_r$$

$\therefore$

$$\delta\eta_k = \epsilon J_{ks} \frac{\partial G_l}{\partial \eta_s} = \epsilon J_{ks} J_{lr} \delta_{rs} = \epsilon J_{ks} J_{ls}$$

$$\therefore \quad \delta\eta_k = \epsilon \delta_{kl} \quad (J \text{ is orthogonal})$$

* Consider the infinitesimal contact transformation of the dynamical variables that produce a rotation $d\theta$.

→ For an infinitesimal counter clockwise rotation oriented to the z axis,

$$\delta x_i = -y_i \, d\theta \qquad \delta y_i = x_i \, d\theta \qquad \delta z_i = 0$$

→ The effect on the conjugate momenta,

$$\delta p_{ix} = -p_{iy} \, d\theta \quad \delta p_{iy} = p_{ix} \, d\theta \qquad \delta p_{iz} = 0$$

→ The generating function is found as

$$\delta x_i = d\theta \frac{\partial G}{\partial p_{ix}} = -y_i \, d\theta \qquad \delta y_i = d\theta \frac{\partial G}{\partial p_{iy}} = x_i \, d\theta$$

$$\delta p_{ix} = -d\theta \frac{\partial G}{\partial x_i} = -p_{iy} \, d\theta \qquad \delta p_{iy} = -d\theta \frac{\partial G}{\partial y_i} = p_{ix} \, d\theta$$

$$\Rightarrow G = x_i p_{iy} - y_i p_{ix} = L_z = (r_i \times p_i)_z$$

→ The angular momentum is the generator of spatial rotations of the system.

* Consider each point of the trajectory in phase space to be encoded with α. Taking u as a function of the system configuration, u will be a continuous funct of α along the trajectory.

$$\partial u = d\alpha [u, G]$$

$$\frac{du}{d\alpha} = [u, G]$$

Taking Taylor series of $u(\alpha)$

$$u(\alpha) = u_0 + \alpha \frac{du}{d\alpha}\Big|_0 + \frac{\alpha^2}{2} \frac{d^2u}{d\alpha^2}\Big|_0 + \frac{\alpha^3}{3} \frac{d^3u}{d\alpha^3}\Big|_0 + \ldots$$

$$\frac{du}{d\alpha}\Big|_0 = [u, G]_0$$

$$\Rightarrow \frac{d^2u}{d\alpha^2} = [[u, G], G]$$

$$\therefore u(\alpha) = u_0 + \alpha [u, G]_0 + \frac{\alpha^2}{2} [[u, G], G]_0 + \frac{\alpha^3}{3} [[[u, G], G], G]_0 + \ldots$$

* For example, take the rotation about z axis

Taking $u = x_i$ & $G = L_z$

$$[x_i, L_z] = \frac{\partial x_i}{\partial x_i} \frac{\partial}{\partial p_{ix}} (x_i p_{iy} - y_i p_{ix}) - \cancel{\frac{\partial x_i}{\partial p_{ix}}} \frac{\partial}{\partial x_i} (x_i p_{iy} - y_i p_{ix})$$

$$= -y_i$$

$$[[x_i, L_z], L_z] = \frac{\partial(-y_i)}{\partial y_i} \frac{\partial}{\partial p_{iy}} (x_i p_{iy} - y p_{ix}) - \cancel{\frac{\partial(-y_i)}{\partial p_{iy}}}^{\,0} \frac{\partial}{\partial y} (x_i p_{iy} - y p_{ix})$$

$$= -x_i$$

$\vdots$

$$\therefore X_i = x_i + \theta(-y_i) + \frac{\theta^2}{2}(-x_i) + \frac{\theta^3}{3}(+y_i) + \frac{\theta^4}{4}(+x_i) - \dots$$

$$X_i = x_i \cos\theta - y_i \sin\theta$$

* For the angular momentum generator, if we consider a vector function of the system configuration, $\vec{F}$,

$$\partial \vec{F} = d\theta \, [\vec{F}, \vec{L} \cdot \vec{n}]$$

→ The generator induces spatial rotation of the system variables and not any external vector.

→ The generator induces a rotation in the vector $\vec{F}$ only when it is a function of the system variables & does not involve any external quantities which aren't affected by the ICT. These are called system vectors.

$$d\vec{F} = \vec{n} d\theta \times \vec{F}$$

We also have

$$\partial \vec{F} = d\theta \, [\vec{F}, \vec{L} \cdot \vec{n}] = \vec{n} d\theta \times \vec{F}$$

$$\Rightarrow [\vec{F}, \vec{L} \cdot \vec{n}] = \vec{n} \times \vec{F}$$

$$\Rightarrow [F_i, L_j n_j] = \epsilon_{ijk} n_j F_k$$

$$\Rightarrow [F_i, L_j] = \epsilon_{ijk} F_k$$

The Levi-Civita goes to 1 when i, j, k are cyclic.

→ Consider the vectors $\vec{A}$ & $\vec{B}$

$$[\vec{A} \cdot \vec{B}, \vec{L} \cdot \vec{n}] = \vec{A} \cdot [\vec{B}, \vec{L} \cdot \vec{n}] + \vec{B} \cdot [\vec{A}, \vec{L} \cdot \vec{n}]$$

$$= \vec{A} \cdot \vec{n} \times \vec{B} + \vec{B} \cdot \vec{n} \times \vec{A}$$

$$= \vec{A} \cdot \vec{n} \times \vec{B} + \vec{A} \cdot \vec{B} \times \vec{n}$$

$$= 0$$

* Taking the system vector as L,

$$[\vec{L}, \vec{L}\cdot\vec{n}] = \vec{n} \times \vec{L}$$

$$[L_i, L_j] = \epsilon_{ijk} L_k$$

$$[L^2, \vec{L}\cdot\vec{n}] = 0$$

$$\therefore [p, \vec{L}\cdot\vec{n}] = \vec{n} \times \vec{p}$$

$$[p_i, L_j] = \epsilon_{ijk} p_k$$

→ If L_x & L_y are constants of the motion, by Poisson's theorem, L_z is also constant. Consider p_z is also constant

$$\therefore [p_z, L_x] = p_y \ \& \ [p_z, L_y] = -p_x$$

$\therefore \vec{p}$ & $\vec{L}$ are conserved.

KINEMATICS OF RIGID BODIES

* consider a system of N particles, $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N$ s.t $|\vec{r}_i - \vec{r}_j| = c_{ij}$ $\forall i, j$, the system constitutes a rigid body.
  - → 3N degrees of freedom
  - → $\frac{1}{2}(N-1)N$ constraints, not necessarily independent
  - → Rigid body needs 6 coordinates to be specified, 3 for location & 3 for orientation.

* Consider the orientation of the axes to be represented by (x', y', z') specified by direction cosines $\alpha_1, \alpha_2, \alpha_3$ wrt (x, y, z) axes.

→ The direction cosines are given by

$$\cos\theta_{11} = \hat{i}' \cdot \hat{i} \qquad \cos\theta_{12} = \hat{i}' \cdot \hat{j} \qquad \cos\theta_{13} = \hat{i}' \cdot \hat{k}$$
$$\cos\theta_{21} = \hat{j}' \cdot \hat{i} \qquad \cos\theta_{22} = \hat{j}' \cdot \hat{j} \qquad \cos\theta_{23} = \hat{j}' \cdot \hat{k}$$
$$\cos\theta_{31} = \hat{k}' \cdot \hat{i} \qquad \cos\theta_{23} = \hat{k}' \cdot \hat{j} \qquad \cos\theta_{33} = \hat{k} \cdot \hat{k}$$

→ The basis vectors of coordinates in both systems are orthogonal,

$$\sum_{l=1}^{3} \cos\theta_{lm'} \cos\theta_{lm} = \delta_{m'm}$$

* Let us consider the transformation of set (x_1, x_2, x_3) to (x_1', x_2', x_3').

$$\Rightarrow \quad x_1' = a_{11} x_1 + a_{12} x_2 + a_{13} x_3$$
$$x_2' = a_{21} x_1 + a_{22} x_2 + a_{23} x_3$$
$$x_3' = a_{31} x_1 + a_{32} x_2 + a_{33} x_3$$

→ As the vectors are invariant,

$$\sum_i x_i x_i = \sum_i x_i' x_i' = \sum a_{ij} x_j \sum a_{ik} x_k$$

$$\Rightarrow \sum_i a_{ij} a_{ik} = \delta_{jk} \qquad \text{(orthogonality condition)}$$

* The transformation matrix can be an operator which transforms the unprimed system to the primed.

$$\vec{r}' = A\vec{r} \qquad \text{where} \qquad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

* In a transformation where the vectors are invariant and only the components change, a passive transformation occurs.

* Consider the inversion of the coordinate axes.

inversion = (rotate about z)(reflect about xy)

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

* The determinant of an orthogonal real matrix is always ± 1. Orthogonal transforms with determinant $+1$ are called proper & transforms with determinant -1 are called improper.

* The Euler angles are defined as three successive rotations performed in a specific sequence.

→ Rotate xyz by ϕ counter clockwise about z

$$xyz \rightarrow R(z,\phi) \rightarrow \xi\eta\zeta \qquad \xi = Dx$$

Heading / Yaw

$$x = \tilde{D}\xi$$

→ Rotate $\xi\eta\zeta$ by θ counter clockwise about ξ

$$\xi\eta\zeta \rightarrow R(\xi,\theta) \rightarrow \xi'\eta'\zeta' \qquad \xi' = C\xi = C(Dx)$$

Pitch / Altitude

$$\xi = \tilde{C}\xi' \qquad \tilde{D}\tilde{C}\xi' = x$$

→ Rotate $\xi'\eta'\zeta'$ by ψ counter clockwise about ζ'

$$\xi'\eta'\zeta' \rightarrow R(\zeta',\psi) \rightarrow x'y'z' \qquad x' = B\xi' = B(CDx)$$

Roll / Bank

$$\xi' = \tilde{B}x' \qquad x = \tilde{D}\tilde{C}\tilde{B}\xi'$$

→ In a compact matrix notation,

$$x' = (BCD)x = Ax$$

$$D = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

$$B = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\psi\sin\phi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix}$$

As it is orthogonal, $A^{-1} = \tilde{A}$

* Euler's Theorem states that the general displacement of a rigid body with one fixed point is equivalent to a rotation about some axis.

→ If the fixed point is taken as the origin of the body axes, then the displacement is only a change of orientation.

→ During rotation one direction, the axis of rotation is unaffected by the operation.

→ The magnitude of vectors must be unaffected.

$$R' = AR = \lambda R \qquad (\lambda = 1)$$

→ All eigenvalues are 1. Trivial case. or

→ One eigenvalue is +1 & the other two are −1, inversion in two coordinate axes. or

→ One eigenvalue is +1 & the other two are complex conjugates, $e^{i\phi}$ & $e^{-i\phi}$

$$\mathrm{Tr}\, A = \Sigma \lambda_i \qquad |A| = \Pi \lambda_i$$

* Chasles' Theorem: The most general displacement of a rigid body is a translation and a rotation.

* Angular velocity with $\dot{\phi}, \dot{\theta}, \dot{\psi}$

$$\omega = \underset{\omega_\phi}{\hat{n}_z \dot{\phi}} + \underset{\omega_\theta}{\hat{n}_{\xi'} \dot{\theta}} + \underset{\omega_\psi}{\hat{n}_{z'} \dot{\psi}} = \begin{array}{cc} \begin{bmatrix} \dot{\phi} \sin\psi \sin\theta + \dot{\theta} \cos\psi \\ \dot{\phi} \cos\psi \sin\theta - \dot{\theta} \sin\psi \\ \dot{\phi} \cos\theta + \dot{\psi} \end{bmatrix} & \begin{array}{c} \omega_x' \\ \omega_y' \\ \omega_z' \end{array} \end{array}$$

WRT NORMAL STUFF

* For writing the Lagrangian of a multi particle system, the kinetic energy is

$$T = \frac{1}{2} M v^2 + \frac{1}{2} M_i v_i'^2$$

→ If the body axis is the center of mass,

$$T = \frac{1}{2} M (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + T'(\dot{\phi}, \dot{\theta}, \dot{\psi})$$

* The potential energy can also be separated

$$V = V_1(x, y, z) + V_2(\phi, \theta, \psi)$$

* Hence, the Lagrangian is

$$L = L_t(x, y, z, \dot{x}, \dot{y}, \dot{z}) + L_r(\phi, \theta, \psi, \dot{\phi}, \dot{\theta}, \dot{\psi})$$

* Proof of Euler's theorem:

$$(A - 1)\tilde{A} = A\tilde{A} - \tilde{A} = 1 - \tilde{A} \qquad (\tilde{A} = A^{-1})$$

$$|A - 1||\tilde{A}| = |1 - \tilde{A}| \qquad (|\tilde{A}| = 1)$$

$$|A - 1| = |1 - A|$$

For 3 dimensions,

$$|A - 1| = 0$$

From eigen value equation,

$$AR = \lambda R \Rightarrow (A - \lambda) R = 0$$

$$\lambda = 1$$

WRT BODY AXES

* Angular velocity again:

$$\omega = \hat{n}_x \omega_x + \hat{n}_y \omega_y + \hat{n}_z \omega_z = \begin{bmatrix} \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi \\ \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi \\ \dot{\psi}\cos\theta + \dot{\phi} \end{bmatrix}$$