A Voyage to Chaos

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Contents

Dynamical Systems

In our route to chaos, we shall first embark on our journey at the port of dynamical systems. Dynamics means the motion, or the change with time. This directly leads to the definition - **dynamical systems** refers to a system which changes or evolves with time. If you've seen a video, you'll know that they're composed of a string of images played together. Every one of these images is called a frame. These frames play in a fixed rectangular window, with every colored dot in this grid having some information to be shown. Each of these frames can be called a **state**, a tuple of n numbers (every colored pixel in our video case) representing the n dynamic variables. Every state can be put into an abstract n dimensional space, $\mathcal{M}$, with every state encoded into a **representation point**. This representation point traces out the trajectory of the system in phase space and shows us the past and the future of the system's evolution.

1.1 Evolution Rule

An evolution rule represents the mathematical formalism to describe what happens to a system as it changes. It is given by a series of differential equations, so for n states, we have:

$$\dot{x_1} = f_1(x_1, x_2, ..., x_n)$$
$$\dot{x_2} = f_2(x_1, x_2, ..., x_n)$$
$$...$$
$$\dot{x_n} = f_n(x_1, x_2, ..., x_n)$$

These can be compressed into a vectorized form by taking:

$$\dot{\overrightarrow{x}} = \overrightarrow{f}(\overrightarrow{x})$$

How time affects evolution

When the evolution rule has no explicit time dependence, it is called an autonomous system. When there is time dependence, the system becomes a non autonomous system. This can be converted into an autonomous system by a mathematical trick. We can assume t to be a variable x_{n+1} which implies $\dot{x_{n+1}} = 1$. Now we can write our dynamical system of $n+1$ variables as:

$$\dot{x_1} = f_1(x_1, x_2, ..., x_n, x_{n+1})$$
$$\dot{x_2} = f_2(x_1, x_2, ..., x_n, x_{n+1})$$

$$\dots$$

$$\dot{x_n} = f_n(x_1, x_2, ..., x_n, x_{n+1})$$

$$\dot{x_{n+1}} = 1$$

If time is a continuous parameter for evolution, our system represents a flow. A flow map would yield us the state at a time t, if it starts from a intial state x_0.

$$\overrightarrow{x_0} \to \overrightarrow{x}(\overrightarrow{x_0}, t)$$

However, if time is a discretely sampled parameter, our system represents a discrete map, or simply map. For discrete systems, maps are defined recursively, given the n^{th} state, we have the following.

$$x_{n+1} = f(x_n)$$

The set of all points which we can reach from a point $\overrightarrow{x_0}$ under a dynamical flow is called the **orbit** of $\overrightarrow{x_0}$. An orbit is a dynamically invariant set. It would change only if the evolution rule is modified. The union of all possible orbits given any initial point gives us the phase space. If we have a well behaved function for our evolution rule, the trajectory of a deterministic system can be uniquely determined by solving the differential equations.

1.2 One Dimensional Systems

Getting down and dirty, we take a look at our first example of a dynamical system. Let us choose one dimension because obviously, it is the simplest example. A dynamical system can be given by the equation:

$$\dot{x} = 1$$

This system represents a velocity of 1 along the positive direction. Being an extremely simple flow, we can integrate this to get $x = t$. This means that if we take a look at the position of our system at any point t, we find that it would have moved to a numerically equal position. However there is nothing interesting about this system. To add some spice to our analysis, let's take a slightly more complex example.

$$\dot{x} = \sin x$$

Integrating this is pretty complex, and the result is straight up obnoxious if we want to calculate the position at any given time. Nevertheless, we can still gain a lot of information about this system. We start by plotting $\dot{x}$ with x.

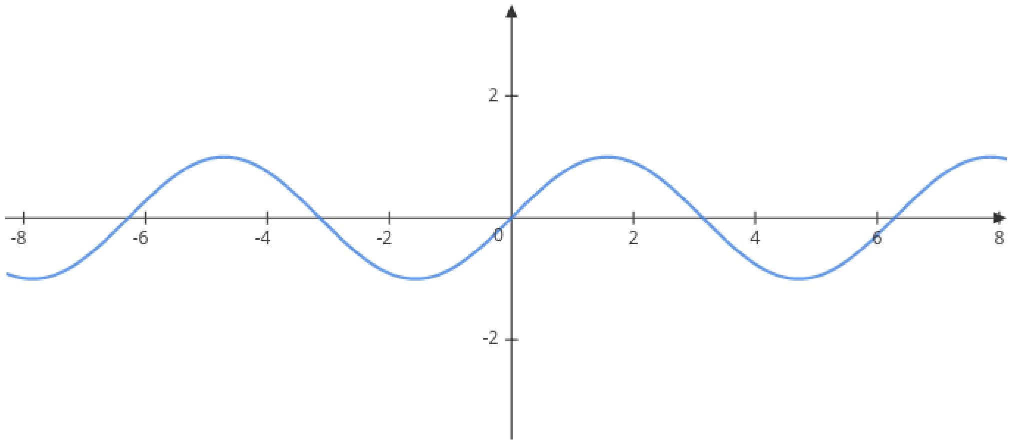

Figure 1.1

At certain positions, the system does not move ($\dot{x} = f(x^*) = 0$). These points where the system is stationary is called an equilibrium point or a **fixed point**. By virtue of its inertia, a system might move beyond a fixed point. However, it feels some fixed points call to it, and it returns to these fixed points. These are stable locations. Other fixed points might repel the system and these are unstable fixed points. For a 1-D system, the stability of fixed points alternate (why?).

Another elementary way to find out about the nature and locations of fixed points is to find the potential curve for the system. For the system $\dot{x} = f(x)$, the potential is given by V and can be evaluated as $f(x) = -dV/dx$. The valleys and pits represent locations of stability (stable fixed points) and the peaks represent instability (unstable fixed points).

Going back to our previous example, we find that whenever the our system has positive $\dot{x}$, it should move to the right and when it is negative, it moves to the left. This clearly separates out the stability of the fixed points of the system. Fixed points at $(2n+1)\pi$ (yellow points) are stable and fixed points at $2n\pi$ (red points) are unstable, where n is an integer.

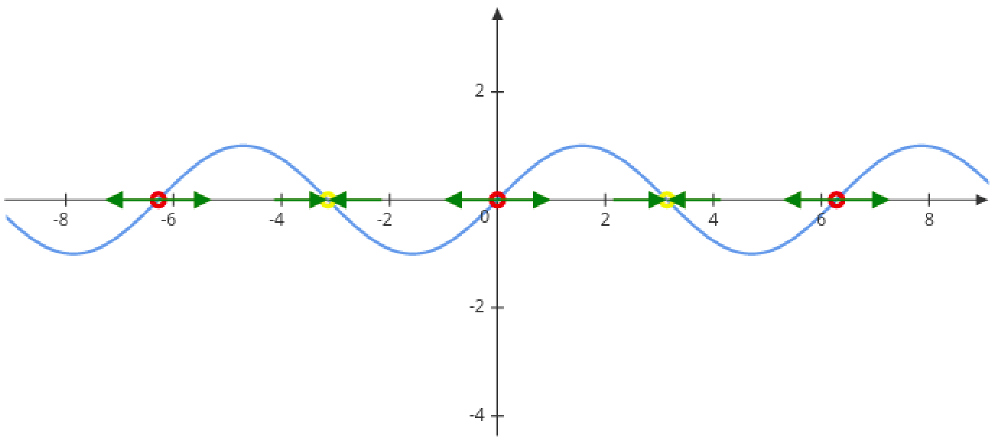

Figure 1.2

We also have a much more solid way of showing the stability of a fixed point quantitatively. Take a fixed point x^* and consider a small perturbation about it.

$$\eta(t) = x(t) - x^*$$

The perturbation changes as:

$$\dot{\eta} = \dot{x} = f(x) = f(x^* + \eta)$$

We can linearize this by:

$$f(x^* + \eta) = f(x^*) + \eta f'(x^*) + \mathcal{O}(\eta^2)$$

By definition, $f(x^*) = 0$ and higher terms are neglected, so we have:

$$\dot{\eta} = \eta f'(x^*)$$

When f' is positive, the perturbation grows exponentially. This means that the system is unstable at that point. However if f' is negative, the perturbation dies out and the system is stable at that point. Armed with this knowledge, we find that no matter where a system starts in a 1-D system, it has two fates - it goes to infinity or to a fixed point. A 1-D system can NOT oscillate, because it has to move between fixed points for this to happen and our analysis shows that this is not possible.

P.S. : When $f' = 0$, the neglected terms are not negligible. Linearization fails in this scenario.

1.3 Bifurcations

Sometimes the system in question would have some parameters which can vary the qualitative structure of a flow. This is called a bifurcation. For 1-D systems, there are three primary types of bifurcations:

1.3.1 Saddle-Node Bifurcation

In this case, fixed points of opposing stability collide and annihilate. Consider a dynamical system $\dot{x} = r + x^2$. We can vary the parameter r which would cause different dynamical systems to exist. When r is negative, we have two fixed points and a very normal scenario. As we increase r, we find an oddity at $r = 0$. The evolution rule pushes our system to the positive x direction no matter where the system starts except at one point. This is the origin, which is metastable (stable on the negative x side and unstable on the positive x side). Eventually, as we have positive r, there are no fixed points for the system.

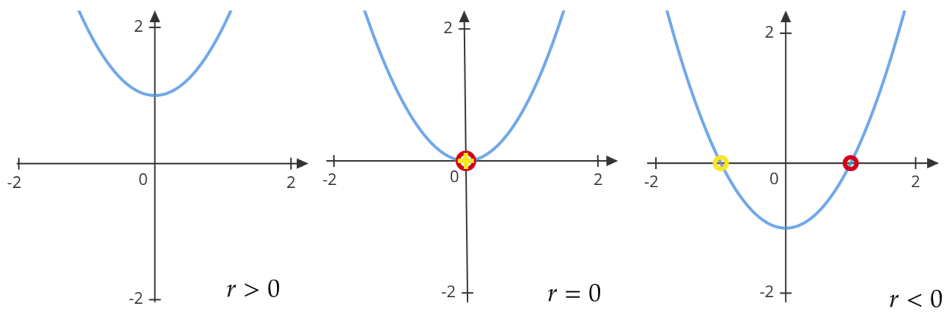

Figure 1.3

Since the system qualitatively changes around $r = 0$, a bifurcation occurs there. We can represent this by a bifurcation diagram. Given any r, we can find out the positions of fixed points from this diagram.

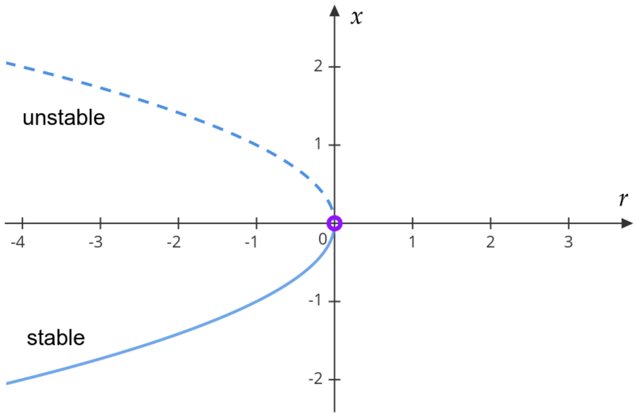

Figure 1.4

We have the blue sky bifurcation when the system has fixed points emerge out of nothing (emerge out of a clear blue sky). This can be seen in the case of the dynamical system $\dot{x} = r - x^2$. The two examples here aren't cherry picked. They happen to be the normal forms for saddle node bifurcations. Normal forms are representative of that class of bifurcation. If the local approximation of any system around a bifurcation matches this normal form (maybe a bit of scaling), it would have a saddle node bifurcation.

1.3.2 Transcritical Bifurcation

There is an exchange of stability of fixed points about a transcritical bifurcation. An initially unstable fixed point becomes stable and vice-versa after they collide at the bifurcation point. An example of this is the system $\dot{x} = rx - x^2$. This is also the normal form of the bifurcation. It has two fixed points, $x^* = 0, r$. The fixed point at r is unstable when $r < 0$ and becomes stable after $r > 0$.

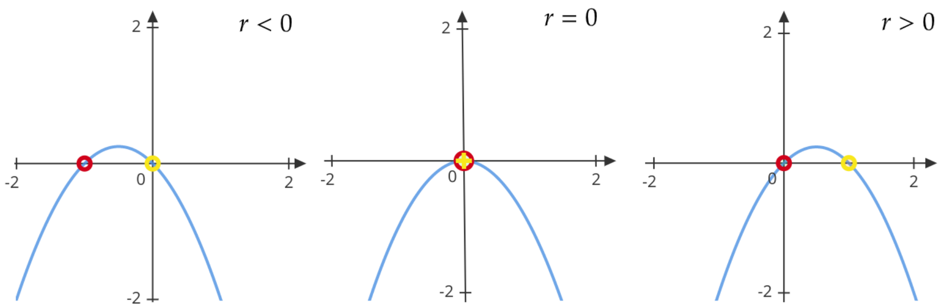

Figure 1.5

The bifurcation diagram for this system is given as:

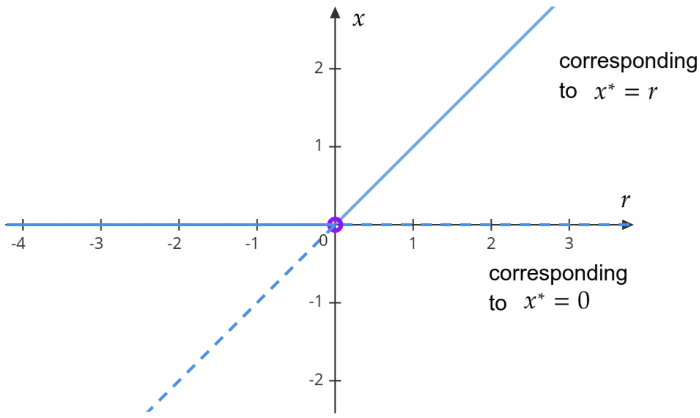

Figure 1.6

1.3.3 Pitchfork Bifurcation

Pitchfork bifurcation is common in systems with a left-right symmetry. The dynamic equation of these systems are invariant under the transformation $x \to -x$. There are two types of pitchfork bifurcation - supercritical and subcritical.

Supercritical Pitchfork Bifurcation

Supercritical pitchfork bifurcation has the normal form $\dot{x} = rx - x^3$. When $r < 0$, the origin is a stable fixed point. The cubic term is lost to linearization and the system is driven exponentially towards the fixed point. However, when r hits 0, the linearization is lost, and only the cubic term is significant. Here, the system critically slows down and reaches the fixed point much more slowly. Finally, as r reaches the positive realm, two more fixed points are spawned and are located symmetrically around the x axis at $x^* = \pm\sqrt{r}$. Plotting the bifurcation diagram shows the pitchfork structure of the bifurcation.

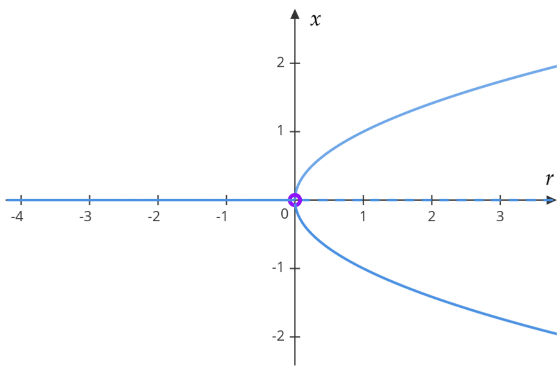

Figure 1.7

The cubic term here acts as a restoring force of sorts. Should the system move away, the cubic term forces the system back to the fixed point.

Subcritical Pitchfork Bifurcation

If the cubic term were to push away our system from the fixed point, we'd have ourselves a subcritical pitchfork bifurcation. This is represented by the normal form $\dot{x} = rx + x^3$. It is a reverse form of the supercritical bifurcation. (Try drawing the three cases to see how the fixed points die)

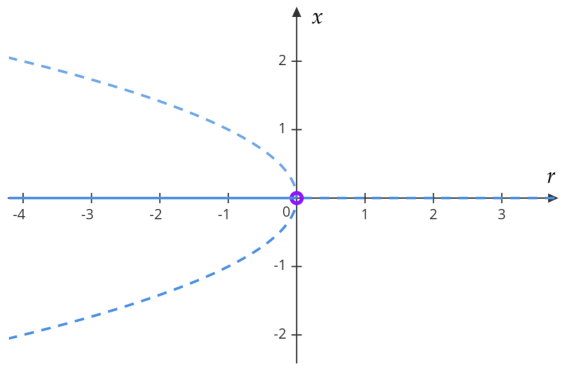

Figure 1.8

1.4 Two Dimensional Systems

Two dimensional systems offer much more freedom and diversity compared to one dimensional systems. Before stepping into the nonlinear generality, taking a look at linear systems helps in analysing the non linear systems. A linear 2D system is given by:

$$\dot{x} = ax + by$$

$$\dot{y} = cx + dy$$

This can be better represented with matrices and column vectors.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Taking the coefficient matrix to be A, we can find its eigenvalues λ_1 and λ_2, along with the corresponding eigenvectors $\vec{v_1}$ and $\vec{v_2}$. This would give us the solution to the system.

$$\vec{x}(t) = \alpha e^{\lambda_1 t}\vec{v_1} + \beta e^{\lambda_2 t}\vec{v_2}$$

While the location of the fixed point is given by solving the equation $\dot{\vec{x}} = 0$, as long as $\det(A) \neq 0$, we can be sure that $x^* = (0,0)$ is the fixed point for our linear system. There is a lot more diversity in the nature of fixed points for the 2D case and the eigenvalues tell us the whole story. They can be very easily extracted from the equation $\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0$. There are two cases at hand - λ can be complex or real. The complex case (shaded region) gives us centers, spirals and orbit-y stuff. The real case (unshaded region) gives us nodes and saddles. The specific cases can be determined in the plot below, where we can determine the nature from the trace and determinant directly. In general, we can talk about the stability by looking at the real parts of the eigenvalues. Positive real parts represent unstable sources - repellers, while negative real parts represent stable sinks - attractors.

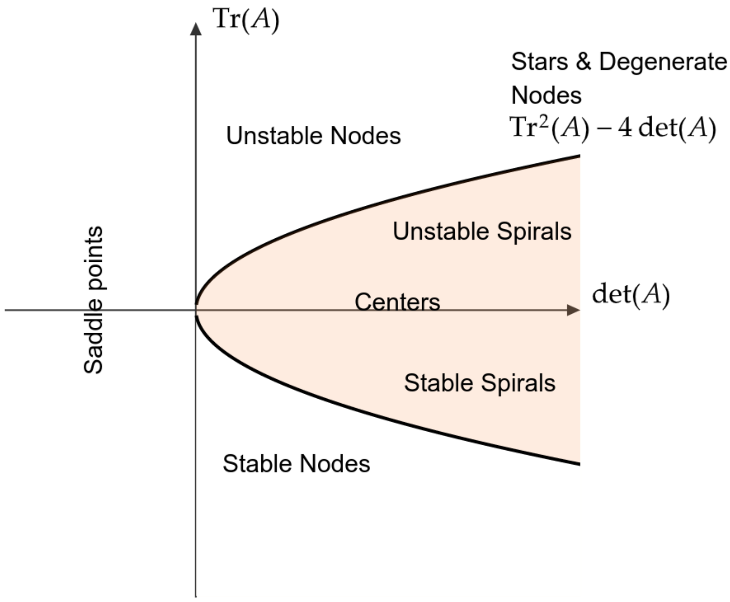

Figure 1.9

When the coefficient matrix is not a neat diagonal matrix, we would need to align the manifolds and direction of the flow around a fixed point accordingly. To do this, we would need to diagonalise the matrix and find the new coordinate axes before we make a plot of our own.

If we have a nonlinear system, we would need to - you guessed it - linearize it to simplify our analysis. We would need to find the fixed points for the system using the usual method. The local behaviour of the flow can be sufficiently approximated by linearization for most cases. This would mean we have a matrix for every fixed point in our system. To find the approximation at a fixed point $P(x^*, y^*)$, we would find the Jacobian at the point:

$$A = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}_P$$

Now we will proceed using the normal linear methods and find the nature of the fixed point and if necessary, align it along the new local coordinate space.

The stability of centers is highly sensitive to higher order non linear terms. If a fixed point does not have a center, it is called a hyperbolic fixed point. If we have a hyperbolic fixed point x_0, then the **Hartman-Grobman theorem** allows us to make the approximation of local flow by linearization. This implies that there exists a one-one continuous map H:

$$H\phi(x_0, t) = \phi_L(H(x_0), t) = e^{At}H(x_0)$$

Limit Cycles and Dealing with Chaos

Here we will pick up some tools for helping us quantify and classify points, systems, chaos and everything else.

2.1 Topological Index

The topological index (or winding number or Poincare index) of a closed curve C is a single integer which measures the winding of the vector field in C. Consider a starting point P. Over one circuit around C starting and ending at P, the vector field rotates by an angle ϕ, an integral multiple of 2π. This is the topological index.

$$I_C = \frac{1}{2\pi}\oint_C d\phi = \frac{1}{2\pi}[\phi]_C$$

- If a closed curve C can be continuously deformed into C' without passing through any fixed points, $I_C = I_{C'}$.
- If C doesn't enclose any fixed points, $I_C = 0$.
- I_C doesn't change if the direction of $\dot{\vec{x}}$ is flipped.
- $I_C + = 1$ for every node, center and spiral inside C and $I_C - = 1$ for every saddle.

It is useful to divide the region around a fixed point by straight lines in phase space. The local neighbourhood of a fixed point is topologically invariant, so we have m equiangular sectors around a fixed point. Each sector can be of three forms - hyperbolic (like saddles), parabolic (like spirals) or elliptic.

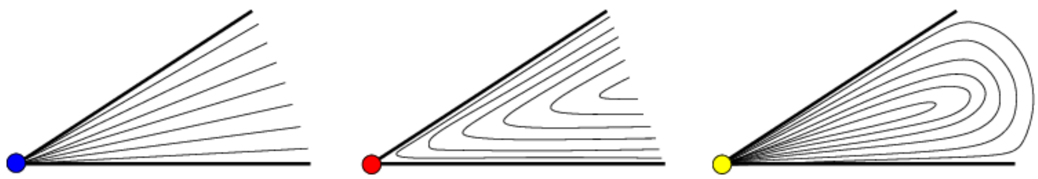

Figure 2.1: Parabolic, hyperbolic and elliptic sectors

There are a total of m sectors around our fixed point, giving each sector a $2\pi/m$ angle. If e elliptic sectors, h hyperbolic sectors and p parabolic sectors surround the fixed point, we can find the angle around them as:

$$\phi_e = \pi + \frac{2\pi}{m} \qquad \phi_p = \frac{2\pi}{m} \qquad \phi_h = -\pi + \frac{2\pi}{m}$$

Hence, we can find the total index number of a curve C around the fixed point as:

$$I_C = \frac{1}{2\pi}\left(p\frac{2\pi}{m} + e\left(\frac{2\pi}{2} + \frac{2\pi}{m}\right) + h\left(-\frac{2\pi}{2} + \frac{2\pi}{m}\right)\right)$$
$$I_C = \frac{p+e+h}{m} + \frac{e-h}{2}$$
$$I_C = 1 + \frac{e-h}{2}$$

Since the winding number must be an integer, we have the condition that $e-h$ must be an even number. If it happens to be odd, the phase diagram is topologically forbidden and hence, unphysical.

2.2 Conservative Systems

A conservative system put very simply is a system with a conserved quantity. We can define a conserved quantity as a real valued continuous function of the dynamic variables $E(\overrightarrow{x})$ that is constant on the trajectories, i.e, $\dot{E} = 0$. Additionally, E be non constant on every open set. A conservative system can not have any attracting or repelling fixed points as they would dissipate or boost the quantity. This means the only allowed geometries are saddles and centers. In a 2D system, if the fixed point is also a local minimum of E, then all trajectories near it are closed (orbits) implying a center at the fixed points. If $\overrightarrow{\nabla} \cdot \overrightarrow{f} = 0$ for a conservative field, it is volume preserving and the flow in it is incompressible.

An interesting oddity of conservative systems is the common occurence of homoclinic orbits. They are trajectories which start and end at the same point. It doesn't correspond to a periodic orbit. The trajectory reaches this point only as $t \to \pm\infty$.

2.3 Limit Cycles

A limit cycle is an isolated closed trajectory. If the neighbouring trajectories approach it, it is attracting or stable, and if the trajectories move away, it is unstable. Limit cycles are nonlinear phenomena, a linear system can't have isolated closed trajectories.

If we have the velocity vector of a non linear system given by the gradient of a single scalar function, then the system is a gradient system. A closed orbit is impossible in a gradient system.

To rule out closed orbits, we can make use of **Dulac's criterion**: Let $\dot{\overrightarrow{x}} = \overrightarrow{f}(\overrightarrow{x})$ be a continuously differentiable vector field defined on a simply connected subset R of the plane. If there exists a continuously differentiable, real-valued function $g(\overrightarrow{x})$ such that $\nabla \cdot (g\dot{\overrightarrow{x}})$ has only one sign throughout R, then there are no closed orbits lying entirely in R. In the special case that $g = 1$, we have Benedixson's criterion.

Instead of avoiding closed orbits, the Poincare-Benedixson's theorem tells us that closed orbits do exist in systems if in a closed, bounded subset of the plane, R (trapping region), which has no fixed points and a trajectory confined within it, then the trajectory is either a closed orbit or spirals towards one. This result depends on the 2-dimensionality of the phase plane. Plotting nullclines ($\dot{x} = 0$ or $\dot{y} = 0$) will help determine the flow and hence the trapping region.

2.4 Lyapunov Stability

For some systems, an energy like function can be constructed which decreases along the trajectories of the system. This is called a Lyapunov function, $V(\overrightarrow{x})$. It can describe the stability about a fixed point. This continuous, real-valued function satisfies the following:

$$\begin{aligned} V(x) &> 0 \quad \forall x \neq x^* \\ V(x^*) &= 0 \\ \dot{V}(x) &< 0 \quad \forall x \neq x^* \end{aligned}$$

Just like gradient systems, systems which have a Lyapunov function don't have closed orbits, the system simply flows into or away from the fixed point. This means that there are two classes of points on the phase space - points belonging to the stable manifold (S) and the unstable manifold (U).

$$S = \left\{ \overrightarrow{c} : \lim_{t\to\infty} \overrightarrow{x}(\overrightarrow{c}, t) = x^* \right\}$$
$$U = \left\{ \overrightarrow{c} : \lim_{t\to\infty} \overrightarrow{x}(\overrightarrow{c}, t) = \infty \right\}$$

The eigenvectors of the system are tangent to the manifolds at the fixed points. The eigenvectors corresponding to positive eigenvalues span the unstable manifold and the eigenvectors corresponding to negative eigenvalues span the stable manifold.

2.5 Lyapunov Exponent

The Lyapunov exponent quantifies sensitivity to initial conditions. If the point is moving away very quickly from the initial point, it is a chaotic system and the exponent is positive. However, if stays around, then the exponent is negative. Consider the initial point x_0, which follows $x(t)$. The adjacent point, $x_0 + \epsilon$ follows $y(t)$. The exponent itself is given by the equation:

$$\lambda(x_0) = \lim_{t\to\infty} \lim_{\epsilon\to 0} \frac{1}{t} \ln \left| \frac{y(t) - x(t)}{\epsilon} \right|$$

An n-dimensional system will have n such exponents. If any λ is chaotic ($\lambda > 0$), then there is stretching in that direction. In the trivial case that $\lambda = 0$, the trajectories diverge exponentially in time (why?).

2.6 Lorenz Systems

The Lorentz system is a 3-D system, with 3 constants the Prandtl number (σ), the Rayleigh number (r, which represents the driving force), and b, all of which are positive. It would help us put together some of the tools we picked up here. The dynamic equation is given by:

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = rx - y - xz$$

$$\dot{z} = xy - bz$$

The divergence of the system's evolution rule is negative everywhere, indicating it is a dissipative system, which indicates there are no repellors. The system is also symmetric about $x = y$. $(0,0,0)$ is a fixed point for the system. To find other fixed points, we let $x = y$ and then solve the two equations and back-substitute to find that they are given by $c_{\pm} = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$.

The linearized matrix at the origin is found to be:

$$A = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}_{(0,0,0)}$$

The z direction has a e^{-bt} in the flow map (Hartman-Grobman Theorem), and hence is an attractor in this direction. In the 2×2 submatrix, the determinant is $\sigma(1-r)$ which is a stable node only when $r < 1$ (because $r, \sigma > 0$). Hence, the origin is an attractor when r is 1.

If we draw a bifurcation diagram for r, we find that till $r = 1$, we have the stable fixed point at the origin, which exists till $r = 1$ after which we have the birth of the two new fixed points, $C_{\pm}$ in a supercritical pitchfork bifurcation. They too exist, only until r_H, where we have a Hopf bifurcation.

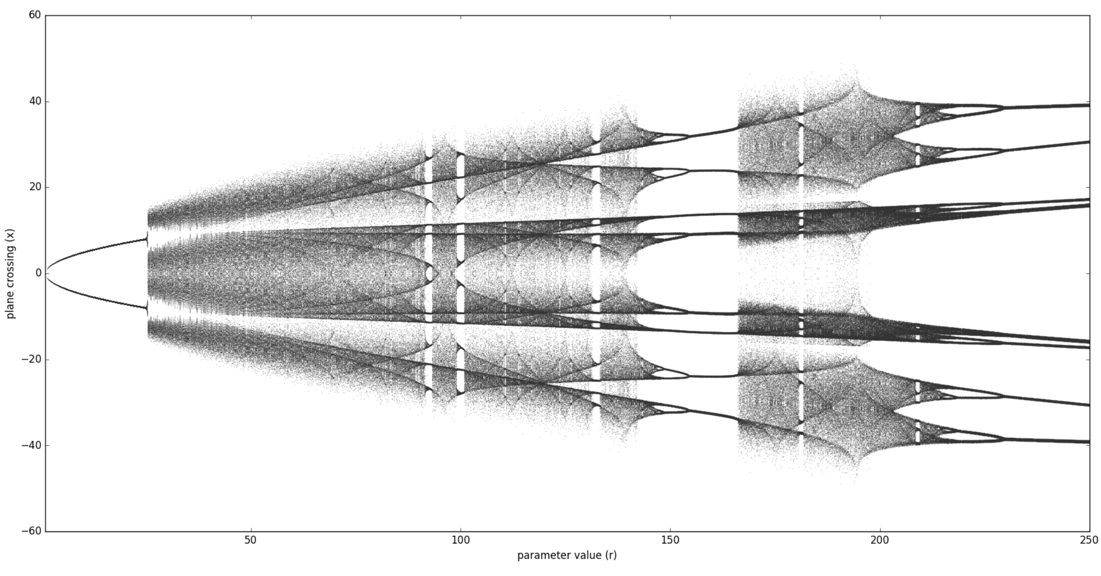

Figure 2.2: Lorenz Attractor, bifurcation diagram

The equation for r_H is given by:

$$r_H = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1}$$

Fixing our parameters at $\sigma = 10$ and $b = 8/3$, we vary r from 0. Until $r = 1$, we have explored above. Soon after, we have the rise of two new fixed points, and any perturbation from the unstable manifold about the origin drives the system to either of the two stable fixed points located symmetrically opposite to each other. We will have the rise of limit cycles around the fixed points at $r = 13.926$ where we have a homoclinic bifurcation. As we move further, we reach the preturbulent region with the onset of chaos. The trajectories wander intermittently and cross back and forth across both the fixed points before ending up in one of them. The trajectory no longer settles to a fixed point and crossed back and forth unpredictably. However, any point sufficiently close to a fixed point does sink into it. With the increase in the driving parameter, this basin reduces drastically and eventually vanishes. At $r = 24.06$, the time wandering around hits infinity and we have the strange attractor. At $r_H = 24.74$, the fixed points absorb an unstable limit cycle and undergo a subcritical Hopf bifurcation. We have explored the first chaotic system - the strange attractor.

There's more! At very large r (in the 100s), there are windows where all the initiall states tend to a periodic limit cycle. If we step in close to 230 and walk backwards, we find that the limit cycles bifurcate into new ones with twice the period which quickly pile up and become indistinguishable.

The Lorenz Attractor in Python (try playing around with the values of the params)

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import odeint
from mpl_toolkits.mplot3d import Axes3D

r = 28.0
sigma = 10.0
b = 8.0 / 3.0

def f(state, t):
    x, y, z = state
    return sigma*(y-x), r*x-y-x*z, x*y-b*z

state0 = [1.0, 1.0, 1.0]
t = np.arange(0.0, 40.0, 0.01)

states = odeint(f, state0, t)

fig = plt.figure()
ax = fig.gca(projection="3d")
ax.plot(states[:, 0], states[:, 1], states[:, 2])
plt.draw()
plt.show()
```

Hamiltonian Systems

A dynamical system which satisfies Hamilton's equations is called a Hamiltonian system. It has $2n$ variables, n generalized coordinates and n generalized momenta. The Hamiltonian itself is a function of these $2n$ coordinates, satisfying:

$$\dot{q}_i = \frac{\partial H(\overrightarrow{q}, \overrightarrow{p})}{\partial p_i} \quad \dot{p}_i = -\frac{\partial H(\overrightarrow{q}, \overrightarrow{p})}{\partial q_i}$$

In symplectic matrix notation, the same can be rewritten as:

$$\dot{\overrightarrow{x}} = \mathbb{J}\overrightarrow{\nabla} H = \sum_j \mathbb{J}_{ij} \frac{\partial H}{\partial x_i} \qquad \mathbb{J} = \begin{bmatrix} 0_n & \mathbb{I}_n \\ -\mathbb{I}_n & 0_n \end{bmatrix}$$

3.1 Properties

Hamiltonian systems are conservative - they have no sources or sinks.

$$\begin{aligned}
\overrightarrow{f} &= (\dot{q}_1, \dot{q}_2, ..., \dot{p}_1, \dot{p}_2, ...) \\
&= (\frac{\partial H}{\partial p_1}, \frac{\partial H}{\partial p_2}, ..., -\frac{\partial H}{\partial q_1}, -\frac{\partial H}{\partial q_2}, ...) \\
\overrightarrow{\nabla} \cdot \overrightarrow{f} &= \frac{\partial^2 H}{\partial q_1 \partial p_1} + \frac{\partial^2 H}{\partial q_2 \partial p_2} + ... - \frac{\partial^2 H}{\partial p_1 \partial q_1} - \frac{\partial^2 H}{\partial p_2 \partial q_2} - ... \\
&= 0
\end{aligned}$$

Every Hamiltonian system has one conserved quantity - the Hamiltonian itself. If any other additional quantity $A(\overrightarrow{q}, \overrightarrow{p})$ has no time dependence, then the quantity is conserved - $dA/dt = \{A, H\} = 0$

Consider the Poisson bracket of two quantities $A(q,p)$ and $B(q,p)$. It is equal to their symplectic inner product:

$$\{A, B\} = \overrightarrow{\nabla} A^T \mathbb{J} \overrightarrow{\nabla} B$$

If the bracket goes to 0, A and B are in involution.

The accessible regions of phase space are compact and flow is volume preserving. After a sufficient duration of time, a point returns to the neighbourhood of its original location. This is called recurrence. If the dynamical system subject to the constraints of the phase space fills all the accessible locations, it is called ergodic. It also spends an equal time in equal volumes. If a set of points A_0 goes to $A(t)$ at a later time which has a non null intersection with a non null set B, the system is mixing. Mixing is a violent phenomenon. Ergodicity implies the system is mixing, but not vice versa.

3.2 Canonical Transforms

Consider the transform from $x_i(q,p)$ to $y_i(Q,P)$.

$$\begin{aligned}\dot{y}_i &= \sum_{j=1}^{2n} \frac{\partial y_i}{\partial x_j}\dot{x}_j \\ &= \sum_j \frac{\partial y_i}{\partial x_j} \sum_k \mathbb{J}_{jk} \frac{\partial H}{\partial x_k} \\ &= \sum_j \sum_k \sum_l \frac{\partial y_i}{\partial x_j} \mathbb{J}_{jk} \frac{\partial y_l}{\partial x_k} \frac{\partial H}{\partial y_l}\end{aligned}$$

From the other end, we have:

$$\dot{y}_i = \sum_i \mathbb{J}_{il} \frac{\partial H}{\partial y_l}$$

Merging the two results,

$$\mathbb{J}_{il} = \sum_j \sum_k S_{ij} \mathbb{J}_{jk} S^T_{kl} \qquad S_{ij} = \frac{\partial y_i}{\partial x_j}$$

In other words,

$$\mathbb{J} = S\mathbb{J}S^T = \begin{bmatrix} \{Q_i, Q_j\} & \{Q_i, P_j\} \\ \{P_i, Q_j\} & \{P_i, P_j\} \end{bmatrix}$$

If we look at how a region R in phase space morphs under a canonical transformation, we have the new region S.

$$\iint_R dqdp = \iint_S dQdP$$
$$\oint_C pdq = \oint_C PdQ$$

$$\oint_C (pdq - PdQ) = 0$$

This would mean we have an exact differential dF_1,

$$dF_1 = pdq - PdQ$$

The function F_1 is a generating function - it generates the transformation. There are three other basic generators:

$$\begin{aligned}dF_2(p,Q) &= -qdp - PdQ \\ dF_3(p,P) &= -qdp + QdP \\ dF_2(q,P) &= pdq + QdP\end{aligned}$$

3.3 Action Angle Variables

When the Hamiltonian for a system can be written as a function of only its momentum, we can transform it into action-angle variables by the means of a canonical transformation. This proves to be particularly useful in periodic systems where the frequencies can be determined without finding the trajectories. The "action" variable is found by isolating the momentum and transforming it.

$$H(x) = \frac{p^2}{2m} - V(x)$$

$$p = \sqrt{2m(E - V(x))}$$

$$I(E) = \frac{1}{2\pi}\int_{q_1}^{q_2} p\,dq$$

$\dot{I}$ is chosen to be 0 to freeze a certain Hamiltonian value for the system. This forces the time derivative of the angle variable to be a constant as well, which is the frequency of the system:

$$\theta = \omega t + \theta_0$$

The action-angle transformation also requires the Hamiltonian system in consideration to be integrable. To test the integrability, we use the Liouville Arnold Theorem. It simply requires n independent constants of motion in the system with $2n$ dimensions to be in involution (their Poisson bracket evaluates to 0) with each other. These n constants of motion maybe written as the n conjugate momenta to be used for the transformation. The resulting Kamiltonian will depend on the action variables only. Integrable systems also happen to be non-ergodic. If we define a vector field ξ_i for each of the n constants such that $\xi_i = \mathbb{J}\overrightarrow{\nabla} F_i$. The n dimensional space where the phase point's motion is constrained to is tangential to all of these vector fields.

One Dimensional Maps

Dynamical systems with discrete time can be represented by difference equations, recurrence relations or maps. The rule:

$$x_{n+1} = f(x_n) \qquad f : \mathbb{R} \to \mathbb{R}$$

is an example of a 1D map and the sequence $x_0, x_1, ..., x_n$ is called the orbit. To determine the next point in the orbit, one would use a cobweb plot or a staircase plot.

If at any point of time, the phase point doesn't move, it is a fixed point and the orbit remains at that point for all future iterations.

$$f(x^*) = x^* = f(x_n) = x_{n+1} = x_{n+2} = ...$$

Again, the stability of the fixed point can be determined by linearization. Consider a nearby orbit, $x_n = x^* + \eta_n$. After linearizing, it gives us $\eta_{n+1} = \eta_n f'(x^*)$. This can be put in the form of an eigenvalue equation if we take $f'(x^*) = \alpha$. This gives us $\eta_n = \alpha^n \eta_0$. If the magnitude of this α is below 1, the system is stable and if it is above 1, the system is unstable. In the trivial case, one would have to use the higher terms to determine the stability.

1D maps can show periodic structures. If we have a cyclic relation like $f(a) = b$ and $f(b) = a$, we have a period 2 cycle. The stability of a period n cycle can be determined by evaluating $|f'(x_1)f'(x_2)...f'(x_n)|$.

To extend the definition of Lyapunov exponents to determine if the system will be chaotic or not, we can take a summation in place of our integral. Given our initial conditions, we have the Lyapunov exponent:

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln|f'(x_i)|$$

After multiple iterations, maps converge to a stationary probability density. If at $t = 0$, the distribution is $\rho_0(x)$, after an iteration it changes to $\rho_1(y) = \int \delta_D(y - f(x))\rho_0(x)dx$. Similarly after n iterations, $\rho_n(y) = \int \delta_D(y - f(x))\rho_{n-1}(x)dx = \int \delta_D(y - f^{(n)}(x))\rho_0(x)dx$. The invariant probability density is attained when $\rho_n^{eq} = \rho_{n-1}^{eq}$. At this juncture, we have $\rho^{eq}(y) = \int \delta_D(y - f(x))\rho^{eq}(x)dx$, the Frobenius-Perron equation.

4.1 Bernoulli Maps

The Bernoulli shift map is defined to be,

$$f(x_n) = 2x_n mod 1 \qquad x_n \in [0, 1]$$

which can be rewritten to give two linear equations,

$$f(x_n) = \begin{cases} 2x_n, & 0 \leq x_n < 0.5 \\ 2x_n - 1, & 0.5 \leq x_n < 1 \end{cases}$$

The action of a Bernoulli map is much like slicing the space at 0.5 and stacking them one on top of the other repeatedly. It is called a shift map because you can consider a binary number between 0 and 1 and shift the decimal one spot to the right while retaining the mantissa giving the rest of the operations. If this binary number has a finite mantissa or has a recurring end (in other words, is rational), then the map goes to either a fixed point or performs periodic oscillations. If this number is irrational, we have chaos. Any two points in the Bernoulli map can be mapped together and this property is called transitivity.

The Lyapunov exponent of the Bernoulli map comes out to be $\ln 2$. To find the solution to the Frobenius-Perron equation,

$$\begin{aligned} \rho^{eq}(y) &= \int_0^1 \delta_D(y - f(x))\rho^{eq}(x)dx \\ &= \int_0^{0.5} \delta_D(y - 2x)\rho^{eq}(x)dx + \int_{0.5}^1 \delta_D(y - 2x + 1)\rho^{eq}(x)dx \\ &= \frac{1}{2}\rho^{eq}(\frac{y}{2}) + \frac{1}{2}\rho^{eq}(\frac{y+1}{2}) \end{aligned}$$

If $\rho(x)$ is a solution to the equation, it must be unique, so we have

$$\int_0^1 c dx = 1 \Rightarrow c = 1$$

$$\rho^{eq}(x) = 1 \qquad x \in [0, 1]$$

4.2 Tent Maps

A tent map is defined on the domain $[0, 1] \to [0, \alpha]$ where $\alpha \leq 1$ and is given by

$$f(x_n) = \begin{cases} 2\alpha x_n, & 0 \leq x_n < 0.5 \\ 2\alpha(1 - x_n), & 0.5 \leq x_n < 1 \end{cases}$$

The tent map behaves like the Bernoulli map, however instead of stacking, we fold the space at 0.5. The map has a Lyapunov exponent of $\ln(2\alpha)$. This implies there is chaos only for $\alpha > 0.5$. The solution of the Frobenius-Perron equation again yields $\rho^{eq}(x) = 1 \forall x \in [0, 1]$.