Day 16

1. Limit along a path

At the end of last class, we recalled that the limit of a single-variable function exists if and only if the left-hand and right-hand limits exist and are equal. We saw that the analogous statement in the multivariable setting will be more complicated: There exist infinitely many paths along which $\mathbf{x}$ could approach $\mathbf{a}$, rather than only one path from the right and one from the left. Here's the formal definition of limit along a path for a multivariable function:

The Limit of a Function of Two or Three Variables Along a Path

(a) Let f be a function of two variables defined on an open set $S \subseteq \mathbb{R}^2$. If the graph of the continuous vector function $\langle x(t), y(t)\rangle$ is a curve $C \subset S$ such that $\lim_{t\to t_0} \langle x(t), y(t)\rangle = \langle a, b\rangle$, then we define the ***limit of f as (x, y) approaches (a, b) along*** C, denoted by $\lim_{\substack{(x,y)\to(a,b)\\ C}} f(x, y)$, to be $\lim_{t\to t_0} f(x(t), y(t))$.

(b) Let f be a function of three variables defined on an open set $S \subseteq \mathbb{R}^3$. If the graph of the continuous vector function $\langle x(t), y(t), z(t)\rangle$ is a curve $C \subset S$ such that $\lim_{t\to t_0} \langle x(t), y(t), z(t)\rangle = \langle a, b, c\rangle$, then we define the ***limit of f as (x, y, z) approaches (a, b, c) along*** C, denoted by $\lim_{\substack{(x,y,z)\to(a,b,c)\\ C}} f(x, y, z)$, to be $\lim_{t\to t_0} f(x(t), y(t), z(t))$.

Let's see an example: Let $f(x, y) = x^2 - y$, and consider the limit of $f(x, y)$ as $(x, y) \to (0, 1)$ along the unit circle C parametrized by $\langle x(t), y(t)\rangle = \langle \cos(t), \sin(t)\rangle$. We know that $\langle x(t), y(t)\rangle \to (0, 1)$ as $t \to \pi/2$. So

$$\lim_{(x,y)\xrightarrow[C]{}(0,1)} f(x, y) = \lim_{t\to\pi/2} f(x(t), y(t)) = \lim_{t\to\pi/2} [\cos(t)^2 - \sin(t)] = -1.$$

THEOREM 12.18 **The Limit of a Function of Two or More Variables Along Distinct Paths**

Let f be a function of two or more variables defined on an open set S. Then the limit $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ exists if and only if $\lim_{\substack{\mathbf{x}\to\mathbf{a}\\ C}} f(\mathbf{x}) = L$ for every path C containing the point (a, b).

Here's a classic example of a multivariable function that has different limits along different paths: Let $f(x, y) = \frac{xy}{x^2+y^2}$. We'll consider the behavior of $f(x, y)$ as $(x, y) \to (0, 0)$ along the x-axis and along the line $y = x$. Along the x-axis, we can use the parametrization $\langle x(t), y(t)\rangle = \langle t, 0\rangle$ with $t \to 0$. Thus

$$\lim_{(x,y)\xrightarrow[x\text{-axis}]{}(0,0)} \frac{xy}{x^2 + y^2} = \lim_{t\to 0} \frac{t \cdot 0}{t^2 + 0^2} = \lim_{t\to 0} 0 = 0.$$

Along the line $y = x$, we can use the parametrization $\langle x(t), y(t)\rangle = \langle t, t\rangle$ with $t \to 0$. This results in

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{xy}{x^2+y^2} = \lim_{t\to 0}\frac{t\cdot t}{t^2+t^2} = \lim_{t\to 0}\frac{1}{2} = \frac{1}{2}.$$

Because the limits along these paths are distinct, the function f does not have a limit at $(0,0)$.

In-class exercise[1]**:** Show that the function $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ does not a have a limit at $(0,0)$.

2. Continuity

Recall that a single-variable function f is continuous at a if $\lim_{x\to a} f(x) = f(a)$. The definition of continuity for a multivariable function is identical:

The Continuity of Functions of Two or Three Variables

Let f be a function of two or three variables defined on an open set S, and let $\mathbf{a}$ be a point in S. Then f is ***continuous*** at $\mathbf{a}$ if $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$.

Also, f is ***continuous on*** S if f is continuous at every point $\mathbf{a} \in S$.

If the domain of f is $\mathbb{R}^2$ or $\mathbb{R}^3$ and f is continuous at every point in its domain, then f is ***continuous everywhere***.

Most of the functions you use regularly, such as polynomials, trigonometric functions, logarithms, etc. are continuous on their domains. Multivariable versions/combinations of these functions are also continuous. For example, the function

$$f(x,y) = \frac{x-\cos(xy^2)}{\ln(x^2+y^2)}$$

is continuous on its domain, because it's built from cosine, natural logarithm, and polynomial functions. In general, if you have a collection of continuous functions, then their sums, differences, products, quotients, and compositions will be continuous.

[1]The limit along the x-axis is 1, while the limit along the y-axis is -1.

Day 17

1. Practice with limits

Yesterday we saw that the limit of a multivariable function exists at $\mathbf{a}$ if and only if $f(\mathbf{x})$ converges to the same value along every curve that passes through $\mathbf{a}$. This gave us a method for proving that a limit doesn't exist: Find two paths along which $f(\mathbf{x})$ converges to different values.

To show that a limit does exist, we have two options: Verify directly from the definition of limit, or show that the function is continuous and then use continuity to evaluate the limit. Note that you can take for granted that all common functions are continuous on their domains.

In-class exercise[1]: Evaluate the limit, or show that it does not exist:

(a) $\displaystyle\lim_{(x,y)\to(3,4)} \frac{\sqrt{x^2+y^2}}{\cos((x-y)\pi)}$

(b) $\displaystyle\lim_{(x,y)\to(1,2)} y^2|x-5y|$

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$

(d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ (Hint: Consider the limit along the parabola $y=x^2$.)

2. Differentiation

Recall from single-variable calculus that a function f is differentiable at x_0 if there exists a number a such that

$$\lim_{x\to x_0} \frac{f(x)-[a(x-x_0)+f(x_0)]}{x-x_0} = 0.$$

This may look slightly different than the definition that you worked with, but it's equivalent. It's straightforward to show that $a = f'(x_0)$. The interpretation of the above limit is that near $(x_0, f(x_0))$, the graph of f looks like the line $y = a(x-x_0)+f(x_0)$, which is the tangent line at x_0.

We say that a two-variable function is *differentiable* at (x_0,y_0) if there exist numbers a and b such that

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)-[a(x-x_0)+b(y-y_0)+f(x_0,y_0)]}{\|\langle x,y\rangle - \langle x_0,y_0\rangle\|} = 0.$$

We can interpret this limit as saying that near the point $(x_0, y_0, f(x_0,y_0))$, the graph of f looks like the plane $z = a(x-x_0)+b(y-y_0)+f(x_0,y_0)$. The numbers a and b are called the *partial derivatives* of f at (x_0,y_0). Specifically, a is the partial derivative with respect to x and b is the partial derivative with respect to y. These values represent the instantaneous

[1] (a) -5 (by continuity)
(b) 36 (by continuity)
(c) Does not exist. (Converges to 1 along x-axis and 0 along y-axis.)
(d) Does not exist. (Converges to 0 along x-axis and $1/2$ along parabola $y=x^2$.)

rates of change of f in the x and y directions at the point (x_0, y_0). There are a couple of notations for partial derivatives:

$$a = \frac{\partial}{\partial x} f(x_0, y_0) = f_x(x_0, y_0) \qquad \text{and} \qquad b = \frac{\partial}{\partial y} f(x_0, y_0) = f_y(x_0, y_0).$$

3. Calculating partial derivatives

The partial derivatives of f at (x_0, y_0) can be calculated like ordinary derivatives by "freezing" one variable. For example, to calculate $f_x(x_0, y_0)$, we freeze y at the value $y = y_0$ and take the derivative of the function $f(x, y_0)$ at $x = x_0$. Similarly, to calculate $f_y(x_0, y_0)$ we would freeze x at $x = x_0$ and take the derivative of $f(x_0, y)$ at $y = y_0$.

For example, let $f(x, y) = x^2 e^{xy}$. Let's find $f_x(2, 3)$ and $f_y(2, 3)$. To find $f_x(2, 3)$, we freeze y at $y = 3$ and differentiate the function $f(x, 3)$ at $x = 2$:

$$\frac{d}{dx} f(x, 3) = \frac{d}{dx}[x^2 e^{3x}] = 2xe^{3x} + 3x^2 e^{3x} \qquad \Rightarrow \qquad f_x(2, 3) = 16e^6$$

To find $f_y(2, 3)$, we freeze x at $x = 2$ and differentiate the function $f(2, y)$ at $y = 3$:

$$\frac{d}{dy} f(2, y) = \frac{d}{dy}[4e^{2y}] = 8e^{2y} \qquad \Rightarrow \qquad f_y(2, 3) = 8e^6.$$

Here's the formal definition of the partial derivatives:

Partial Derivatives for a Function of Two Variables

Let f be a function of two variables defined on an open set S containing the point (x_0, y_0).

(a) The ***partial derivative with respect to x at*** (x_0, y_0), denoted by $f_x(x_0, y_0)$, is the limit

$$\lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

provided that this limit exists.

(b) Similarly, the ***partial derivative with respect to y at*** (x_0, y_0), denoted by $f_y(x_0, y_0)$, is the limit

$$\lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},$$

provided that this limit exists.

Notice that $f_x(x_0, y_0)$ is indeed the ordinary derivative of $f(x, y_0)$ at $x = x_0$ and $f_y(x_0, y_0)$ is the ordinary derivative of $f(x_0, y)$ at $y = y_0$.

We can also define the partial derivative functions $f_x(x, y)$ and $f_y(x, y)$ by treating (x, y) as an arbitrary point and calculating the partial derivatives as above. Let's again consider the function $f(x, y) = x^2 e^{xy}$. To find $f_x(x, y)$ we freeze y (i.e. treat y like a constant) and differentiate with respect to x:

$$f_x(x, y) = \frac{d}{dx}[x^2 e^{xy}] = 2xe^{xy} + x^2 e^{xy} y$$

Likewise, to find $f_y(x, y)$ we freeze x and differentiate with respect to y:

$$f_y(x, y) = \frac{d}{dy}[x^2 e^{xy}] = x^3 e^{xy}.$$

Notice that if we plug in $(x, y) = (2, 3)$, we get the same values as above: $f_x(2, 3) = 16e^6$ and $f_y(2, 3) = 8e^6$.

Partial derivatives of a three-variable function $f(x, y, z)$ can be defined in essentially the same way. Such a function will have three partials: f_x, f_y, and f_z. To find f_x, freeze y and z and differentiate with respect to x. Similarly, to find f_y freeze x and z and differentiate with respect to y, and to find f_z freeze x and y and differentiate with respect to z. The formal definition in terms of limits can be found in the textbook.

In-class exercise[2]**:** Find the partial derivatives of f:

(a) $f(x, y) = \sin(xy^2)$

(b) $f(x, y, z) = y\sqrt{x^2 - yz}$

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[2] (a) $f_x(x, y) = \cos(xy^2)y^2$,
$f_y(x, y) = \cos(x^2)2xy$

(b) $f_x(x, y, z) = xy(x^2 - yz)^{-1/2}$,
$f_y(x, y, z) = \sqrt{x^2 - yz} - \frac{1}{2}yz(x^2 - yz)^{-1/2}$,
$f_z(x, y, z) = -\frac{1}{2}y^2(x^2 - yz)^{-1/2}$

Day 18

1. Higher order partial derivatives

Yesterday we saw the definition of the partial derivatives of a multivariable function and how to find them. If f is a two variable function whose partial derivatives f_x and f_y are again differentiable functions, then we can take their partial derivatives:

$$f_{xx} = \frac{\partial^2}{\partial x^2} f, \qquad f_{xy} = \frac{\partial^2}{\partial y \partial x} f, \qquad f_{yx} = \frac{\partial^2}{\partial x \partial y} f, \qquad f_{yy} = \frac{\partial^2}{\partial y^2} f.$$

These are the *second partial derivatives* of f.

Notice the order of the indices x and y in the two notations for second partial derivatives. We can think of f_{xy} as $(f_x)_y$; in other words, to find f_{xy} we first differentiate with respect to x, then y. We can think of $\frac{\partial^2}{\partial y \partial x} f$ as $\frac{\partial}{\partial y}(\frac{\partial}{\partial x} f)$, so again we first differentiate with respect to x, then y.

If f is a function of three variables, then it has 9 second partial derivatives: f_{xx}, f_{xy}, f_{xz}, f_{yx}, f_{yy}, f_{yz}, f_{zx}, f_{zy}, f_{zz}.

Once we have second partial derivatives, we can keep going: A partial derivative of a second partial derivative of f is a third partial derivative of f. If f is a function of two variables, then it has 8 third partial derivatives. If f is a function of 3 variables, then it has 27 third partial derivatives.

Let's revisit the function $f(x, y) = x^2 e^{xy}$. We found that

$$f_x(x, y) = (2x + x^2 y)e^{xy} \qquad \text{and} \qquad f_y(x, y) = x^3 e^{xy}.$$

Thus

$$\begin{aligned} f_{xx}(x, y) &= (2 + 2xy)e^{xy} + (2x + x^2 y)e^{xy} y, \\ f_{xy}(x, y) &= x^2 e^{xy} + (2x + x^2 y)e^{xy} x, \\ f_{yx}(x, y) &= 3x^2 e^{xy} + x^3 e^{xy} y, \\ f_{yy}(x, y) &= x^4 e^{xy}. \end{aligned}$$

Notice that f_{xy} and f_{yx} are actually the same. This is not a coincidence:

THEOREM 12.23 **Clairaut's Theorem: The Equality of the Mixed Second-Order Partial Derivatives**

Let $f(x, y)$ be a function defined on an open subset S of $\mathbb{R}^2$. If the second-order partial derivatives of f are continuous everywhere in S, then $f_{xy}(x, y) = f_{yx}(x, y)$ at every point in S.

This means that for almost any function $f(x, y)$ that you would commonly work with, the "mixed partials" f_{xy} and f_{yx} are equal. A similar theorem holds for functions of three or more variables.

In-class exercise[1]**:** Use Clairaut's theorem to show that there is no function $f(x, y)$ such that $f_x(x, y) = x^2 \sin(xy^2)$ and $f_y(x, y) = y^2 \sin(x^2 y)$.

2. Recovering a function from its partial derivatives

The above exercise shows that in general we cannot prescribe the partial derivatives of a function: If $f_x(x, y) = P(x, y)$ and $f_y(x, y) = Q(x, y)$, then we must have $P_y(x, y) = Q_x(x, y)$, and typically this criterion will not be satisfied. However, this also raises a question: If P and Q are two functions that do satisfy $P_y = Q_x$, can we always find a function f such that $f_x = P$ and $f_y = Q$? Yes! We won't prove this, but an example will illustrate the idea[2]:

Let $P(x, y) = x^2 + y$ and $Q(x, y) = x + \sin(y)$, and notice that $P_y = Q_x$. We want to find a function f such that $f_x = P$ and $f_y = Q$. Let's work backwards. If $f_x = P$, then we must have

$$f(x, y) = \int f_x(x, y)dx + C(y) = \int (x^2 + y)dx + C(y) = \frac{x^3}{3} + xy + C(y),$$

where $C(y)$ is a constant with respect to x (i.e. a function of y but not x). If $f_y = Q$, then we also know that

$$x + \sin(y) = f_y(x, y) = \frac{\partial}{\partial y}\left[\frac{x^3}{3} + xy + C(y)\right] = x + C'(y).$$

Thus

$$C'(y) = \sin(y),$$

which implies that

$$C(y) = -\cos(y) + A$$

for some constant A. We have shown that if $f_x = P$ and $f_y = Q$, then f must be of the form

$$f(x, y) = \frac{x^3}{3} + xy - \cos(y) + A.$$

It's easy to check that any such f actually works.

[1] If such a function f did exist, then by Clairaut's theorem it would satisfy $f_{xy} = f_{yx}$. However, we can check that $\frac{d}{dy}[x^2 \sin(xy^2)] \neq \frac{d}{dx}[y^2 \sin(x^2 y)]$.

[2] You can ask me about the proof or look it up if you're interested!

Day 19

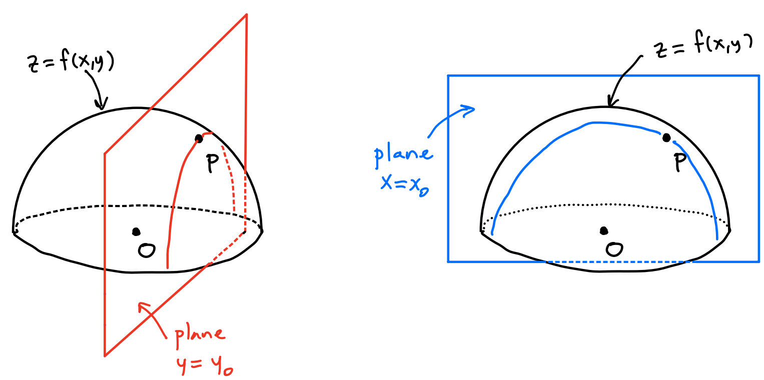
1. Geometric meaning of partial derivatives

Let $f(x, y)$ be a differentiable function. We know that f has two partial derivatives, f_x and f_y, and we've seen how to compute them. But what do the partial derivatives actually tell us?

The quantity $f_x(x_0, y_0)$ is the rate of change of $f(x, y)$ at the point $(x, y) = (x_0, y_0)$ as x varies and y stays fixed. In other words, $f_x(x_0, y_0)$ is the rate of change of f at (x_0, y_0) in the "x direction".

Similarly, $f_y(x_0, y_0)$ is the rate of change of $f(x, y)$ at the point $(x, y) = (x_0, y_0)$ as y varies and x stays fixed. It can be thought of as the change of f at (x_0, y_0) in the "y direction".

Let P be the point $P = (x_0, y_0, f(x_0, y_0))$, which belongs to the graph of f. Consider the intersection of the graph of f with the planes $y = y_0$ and $x = x_0$. It will look something like this:



In the above picture, $f_x(x_0, y_0)$ is the slope of the red curve at P. Note that along this curve, x varies but $y = y_0$ is fixed. Similarly, $f_y(x_0, y_0)$ is the slope of the blue curve at P. Along the blue curve, y varies and $x = x_0$ remains fixed.

2. Partial derivatives and differentiability

Earlier this week we saw a formal definition of differentiability for a function $f(x, y)$ at a point (x_0, y_0). We won't restate it here (look back to Day 17 or see Definition 12.27 in the textbook); the key takeaway was that differentiability corresponds to the existence of a tangent plane at the point $(x_0, y_0, f(x_0, y_0))$ on the graph of f.

A natural question arises: If the partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist, does it follow that f is differentiable at (x_0, y_0)? Perhaps surprisingly the answer is no. Consider the function

$$f(x, y) = \begin{cases} 0 & \text{if } xy = 0 \\ 1 & \text{otherwise.} \end{cases}$$

Since $f(x, 0) = 0$ for all x, we have $f_x(0, 0) = 0$. Similarly, since $f(0, y) = 0$ for all y, we have $f_y(0, 0)$. In particular, $f_x(0, 0)$ and $f_y(0, 0)$ both exist. But the function f is not continuous at $(0, 0)$, so it certainly isn't differentiable there, and visually, its graph doesn't have a tangent plane there.

It can be shown, however, that if f_x and f_y exist and are *continuous* at (x_0, y_0), then f is differentiable at (x_0, y_0):

THEOREM 12.28 **The Continuity of the Partial Derivatives Guarantees Differentiability**

If $f(x, y)$ is a function of two variables with partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ that are continuous on some open set containing the point (x_0, y_0), then f is differentiable at (x_0, y_0).

And if f is differentiable, then its partial derivatives also give us an equation for the tangent plane to its graph:

THEOREM 12.29 **Using the Partial Derivatives to Find the Equation of the Tangent Plane**

Let $f(x, y)$ be a function of two variables that is differentiable at the point (x_0, y_0). Then the equation of the plane tangent to the surface defined by $f(x, y)$ at (x_0, y_0) is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = z - f(x_0, y_0).$$

This formula essentially follows from the definition of differentiability. But it's also helpful to understand it this way: Since $f_x(x_0, y_0)$ is the slope the the graph if the x direction, the vector $\mathbf{d}_x = \langle 1, 0, f_x(x_0, y_0)\rangle$ is tangent to the graph at $(x_0, y_0, f(x_0, y_0))$. Similarly, since $f_y(x_0, y_0)$ is the slope of the graph in the y direction, the vector $\mathbf{d}_y = \langle 0, 1, f_y(x_0, y_0)\rangle$ is also tangent to the graph at that point. Taking their cross product, we get a normal vector for the tangent plane, namely $\mathbf{n} = \mathbf{d}_x \times \mathbf{d}_y = -\langle f_x(x_0, y_0), f_y(x_0, y_0), -1\rangle$. After multiplying by -1, the components of $\mathbf{n}$ are the coefficients of x, y, z above.

3. Directional derivatives

We know how to take the derivative of a function $f(x, y)$ in the x direction or the y direction, namely we find the partial derivatives f_x and f_y. How do we take derivatives in other directions, and what would such a derivative represent? Here's the formal definition of directional derivative:

The Directional Derivative of a Function of Two Variables

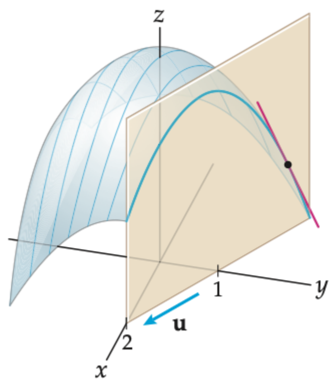
Let $f(x, y)$ be a function of two variables defined on an open set containing the point (x_0, y_0), and let $\mathbf{u} = \langle \alpha, \beta\rangle$ be a unit vector. The ***directional derivative*** of f at (x_0, y_0) in the direction of $\mathbf{u}$, denoted by $D_{\mathbf{u}} f(x_0, y_0)$, is given by

$$\lim_{h\to 0} \frac{f(x_0 + \alpha h, y_0 + \beta h) - f(x_0, y_0)}{h},$$

provided that this limit exists.

Notice that if $\mathbf{u} = \langle 1, 0\rangle$ then $D_{\mathbf{u}} f = f_x$, and if $\mathbf{u} = \langle 0, 1\rangle$ then $D_{\mathbf{u}} f = f_y$.

The geometric meaning of the directional derivative $D_{\mathbf{u}}f$ is similar to that of the partial derivatives f_x and f_y. As before, imagine intersecting the graph of f with a vertical plane that's parallel to $\mathbf{u}$:



Then $D_{\mathbf{u}}f(x_0, y_0)$ is the slope of the tangent line to intersection curve at the point $(x_0, y_0, f(x_0, y_0))$.

Let's see an example of finding a directional derivative using the definition. (Soon we'll learn a faster method that utilizes f_x and f_y.) Let $f(x, y) = xy^2$ and $\mathbf{u} = \langle \sqrt{3}/2, 1/2 \rangle$. Then

$$\begin{aligned}
D_{\mathbf{u}}f(x_0, y_0) &= \lim_{h\to 0} \frac{(x_0 + \frac{\sqrt{3}}{2}h)(y_0 + \frac{1}{2}h)^2 - x_0y_0^2}{h} \\
&= \lim_{h\to 0} \frac{(x_0 + \frac{\sqrt{3}}{2}h)(y_0^2 + y_0h + \frac{1}{4}h^2) - x_0y_0^2}{h} \\
&= \lim_{h\to 0} \frac{x_0y_0^2 + x_0y_0h + \frac{1}{4}x_0h^2 + \frac{\sqrt{3}}{2}y_0^2h + \frac{\sqrt{3}}{2}y_0h^2 + \frac{\sqrt{3}}{8}h^3 - x_0y_0^2}{h} \\
&= \lim_{h\to 0} \left(x_0y_0 + \frac{1}{4}x_0h + \frac{\sqrt{3}}{2}y_0^2 + \frac{\sqrt{3}}{2}y_0h + \frac{\sqrt{3}}{8}h^2 \right) \\
&= x_0y_0 + \frac{\sqrt{3}}{2}y_0^2.
\end{aligned}$$