Day 8

1. Vector-valued functions

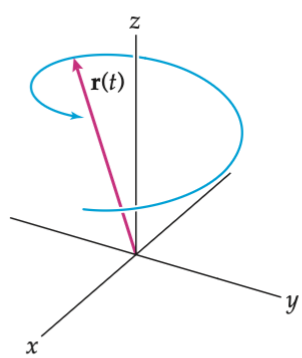
Recently we've been studying lines and planes in $\mathbb{R}^3$. We saw that the line containing the point $P = (x_0, y_0, z_0)$ and parallel to the vector $\mathbf{d} = \langle a, b, c\rangle$ is described by the function

$$\mathbf{r}(t) = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle.$$

(More specifically, a point $Q = (x, y, z)$ belongs to the line if and only if $\overrightarrow{OQ} = \mathbf{r}(t)$ for some value of t.) This function $\mathbf{r}$ is an example of a *vector-valued function*, a function whose output is a vector. Vector-valued functions are also called vector functions.

A function that outputs a number (as opposed to a vector) is sometimes called a scalar function. Vector functions can be thought of as vectors whose components are scalar functions. In the example above, the components of $\mathbf{r}$ are the scalar functions $x(t) = x_0 + at$, $y(t) = y_0 + bt$, and $z(t) = z_0 + ct$.

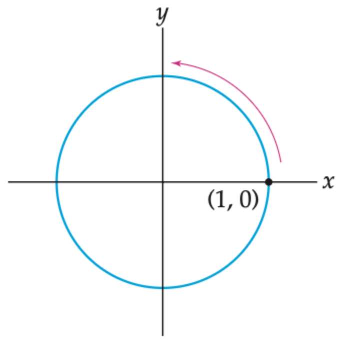
The image of a vector function $\mathbf{r}$ is a curve[1]:



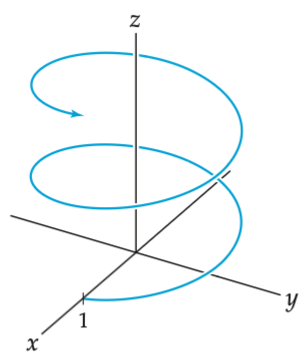
If $\mathbf{r}$ takes values in $\mathbb{R}^2$, then its image is a *plane curve*, and if $\mathbf{r}$ takes values in $\mathbb{R}^3$, then its image is a *space curve*. In the first example above, the image of $\mathbf{r}$ is a line in $\mathbb{R}^3$ (an example of a space curve).

Let's take a look at a different example: $\mathbf{r}(t) = \langle \cos(t), \sin(t), t\rangle$. What does the image of $\mathbf{r}$ look like? While we have a good grasp of each component of $\mathbf{r}$, it's difficult to understand how they interact to produce a curve in $\mathbb{R}^3$. It might help to first consider the projection of $\mathbf{r}$ onto one of the coordinate planes. If we ignore the z-component of $\mathbf{r}$, we're left with the standard parametrization of the unit circle: $x = \cos(t), y = \sin(t)$. Thus the projection of $\mathbf{r}$ onto the xy-plane looks like this:

[1] Actually, strictly speaking, the image is a set of vectors, but conventionally we identify the vector $\langle x, y, z\rangle$ with the point (x, y, z). The points corresponding to values of $\mathbf{r}(t)$ form a curve.



As t varies, the point $(\cos(t), \sin(t))$ moves around the unit circle. Now let's consider the z-component of $\mathbf{r}$, which was $z = t$. As t increases, the z component of $\mathbf{r}(t)$ increases. So the full output of $\mathbf{r}(t)$ is a point that is moving around the unit circle in the xy-plane while simultaneously moving upwards in the z direction:



This curve is called a *helix*.

Vector functions, scalar functions, and ordinary vectors and scalars can be combined in various ways. For example, if f is a scalar function and $\mathbf{r}$ is a vector function, then $f\mathbf{r}$ is the function defined by $f\mathbf{r}(t) = f(t)\mathbf{r}(t)$. Similarly, if $\mathbf{r_1}$ and $\mathbf{r_2}$ are two vector functions with outputs in $\mathbb{R}^3$, then $\mathbf{r_1} \times \mathbf{r_2}$ is the function defined by $\mathbf{r_1} \times \mathbf{r_2}(t) = \mathbf{r_1}(t) \times \mathbf{r_2}(t)$. The textbook gives a couple of other examples, but the point is that ordinary vector operations extend straightforwardly to vector functions.

2. Limits and continuity of vector functions

We've seen that vector addition and scalar multiplication are performed componentwise. For example, $\langle 1, 2, 3\rangle + \langle 4, 5, 6\rangle = \langle 1 + 4, 2 + 5, 3 + 6\rangle$. Limits of vector functions are also defined componentwise:

The Limit of a Vector Function

Let $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ be a vector-valued function defined on a punctured interval around t_0. The ***limit*** of $\mathbf{r}(t)$ as t approaches t_0, denoted by $\lim\limits_{t\to t_0} \mathbf{r}(t)$, is defined by

$$\lim_{t\to t_0} \mathbf{r}(t) = \lim_{t\to t_0} \langle x(t), y(t), z(t)\rangle = \lim_{t\to t_0} x(t)\,\mathbf{i} + \lim_{t\to t_0} y(t)\,\mathbf{j} + \lim_{t\to t_0} z(t)\,\mathbf{k},$$

provided that each of the limits, $\lim\limits_{t\to t_0} x(t)$, $\lim\limits_{t\to t_0} y(t)$, and $\lim\limits_{t\to t_0} z(t)$, exists.

Similarly, the ***limit*** of the vector function $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ is

$$\lim_{t\to t_0} \mathbf{r}(t) = \lim_{t\to t_0} \langle x(t), y(t)\rangle = \lim_{t\to t_0} x(t)\mathbf{i} + \lim_{t\to t_0} y(t)\mathbf{j},$$

provided that each of the limits, $\lim\limits_{t\to t_0} x(t)$ and $\lim\limits_{t\to t_0} y(t)$, exists.

This means that if you understand limits of scalar functions, then you understand limits of vector functions. Anything that's defined in terms of limits is therefore also easy to extend from scalar functions to vector functions. For example:

The Continuity of a Vector Function

Let $\mathbf{r}(t)$ be a vector-valued function defined on an open interval $I \subseteq \mathbb{R}$, and let $t_0 \in I$. The function $\mathbf{r}(t)$ is said to be ***continuous*** at t_0 if

$$\lim_{t\to t_0} \mathbf{r}(t) = \mathbf{r}(t_0).$$

The Continuity of a Vector Function on an Interval

A vector function $\mathbf{r}(t)$ is ***continuous on an interval*** I if it is continuous at every interior point of I, right continuous at any closed left endpoint, and left continuous at any closed right endpoint.

Day 9

1. Limits and continuity of vector functions

Yesterday we saw that the limit of a vector function is defined by moving the limit inside. That is, if $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, then

$$\lim_{t\to t_0} \mathbf{r}(t) = \left\langle \lim_{t\to t_0} x(t), \lim_{t\to t_0} y(t), \lim_{t\to t_0} z(t)\right\rangle,$$

provided these limits exist. Once we've defined the notion of limit, we can define continuity of vector functions in the usual way: $\mathbf{r}$ is continuous at t_0 if $\lim_{t\to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$.

Let's take a look at a specific example. Let

$$\mathbf{r}(0) = \langle -8, 1, 0\rangle, \qquad \text{and} \qquad \mathbf{r}(t) = \left\langle \frac{3t-8}{t^2+1}, e^{4(\ln|t|)^3}, t^2 \sin\left(\frac{1}{t}\right)\right\rangle \quad \text{for } t \neq 0.$$

Is $\mathbf{r}$ continuous at 0? This is true if and only if the following three things hold: $\mathbf{r}(0)$ is defined, $\lim_{t\to 0} \mathbf{r}(t)$ exists, and $\lim_{t\to 0} \mathbf{r}(t)$ is equal to $\mathbf{r}(0)$. We know that $\mathbf{r}(0)$ is defined, so let's consider the limit. The first component of $\mathbf{r}$ is a rational function and its denominator is nonzero when $t = 0$. So its limit at $t = 0$ exists and is equal to its value at $t = 0$:

$$\lim_{t\to 0} \frac{3t-8}{t^2+1} = -8.$$

Looking at the second component of $\mathbf{r}$, we observe that $\ln|t| \to -\infty$ as $t \to 0$. Therefore we also have $4(\ln|t|)^3 \to -\infty$ as $t \to 0$. So

$$\lim_{t\to 0} e^{4(\ln|t|)^3} = 0.$$

This already implies that $\mathbf{r}$ is not continuous at 0, since the second component of $\mathbf{r}(0)$ was given to be 1. But let's consider the limit of the third component of $\mathbf{r}$ just for practice. As t approaches 0, the function $\sin(1/t)$ oscillates rapidly, but staying between -1 and 1. Thus $t^2 \sin(1/t)$ always stays between $-t^2$ and t^2. Since both $-t^2$ and t^2 approach 0 as $t \to 0$, it follows that

$$\lim_{t\to 0} t^2 \sin\left(\frac{1}{t}\right) = 0.$$

This can be formalized using the squeeze theorem. While we have shown that $\lim_{t\to 0} \mathbf{r}(t)$ exists, it is not equal to $\mathbf{r}(0)$, so $\mathbf{r}$ is not continuous at 0.

2. The derivative of a vector function

Just like with limits, the derivative of a vector function is defined by moving the derivative inside:

The Derivative of a Vector Function

Let

$$x = x(t), y = y(t), \text{ and } z = z(t)$$

be three real-valued functions, each of which is differentiable at every point in some interval $I \subseteq \mathbb{R}$. The ***derivative*** of the vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ is

$$\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}.$$

Similarly, the ***derivative*** of the vector function $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ is

$$\mathbf{r}'(t) = \langle x'(t), y'(t)\rangle = x'(t)\mathbf{i} + y'(t)\mathbf{j}.$$

In these cases we say that the function $\mathbf{r}$ is ***differentiable***.

Recall that the derivative of each of the scalar functions within $\mathbf{r}$ is defined as a limit of difference quotients. For example,

$$x'(t) = \lim_{h\to 0} \frac{x(t+h) - x(t)}{h}.$$

The derivative of $\mathbf{r}$ can also be defined in this way, and it's equivalent to the definition given above:

THEOREM 11.7 **The Derivative of a Vector-Valued Function**

Let $\mathbf{r}(t)$ be a differentiable vector function with either two or three components. The derivative of $\mathbf{r}(t)$ is given by

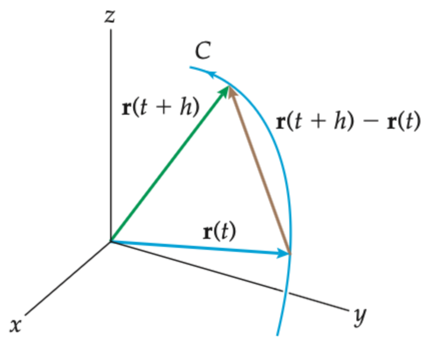
$$\mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}.$$

We can use either of these methods for calculating the derivative. But as a practical matter, it's often easier to use the componentwise method. For example, if $\mathbf{r}(t) = \langle \cos(t), \sin(t), t\rangle$ is the helix function that we looked at yesterday, then

$$\mathbf{r}'(t) = \left\langle \frac{d}{dt}\cos(t), \frac{d}{dt}\sin(t), \frac{d}{dt}t \right\rangle = \langle -\sin(t), \cos(t), 1\rangle.$$

3. Geometric meaning of the derivative

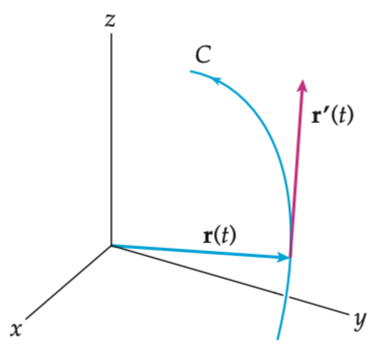
Let's use the difference quotient version (Theorem 11.7) to make sense of what the derivative means geometrically. We'll consider the difference $\mathbf{r}(t+h) - \mathbf{r}(t)$, where t is fixed and h is a small parameter. Let C denote the image of $\mathbf{r}$. If we position the tail of $\mathbf{r}(t+h) - \mathbf{r}(t)$ at the tip of $\mathbf{r}(t)$, then we get a picture like this:



You can see that $\mathbf{r}(t+h)-\mathbf{r}(t)$ joins nearby points on the curve C. As we let $h \to 0$, this vector will become tangent to the curve. Note that the vector will also become arbitrarily short; however the actual derivative $\mathbf{r}'(t)$ involves dividing this difference by h, which lengthens it:

$$\mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$$

So in the limit, we get a true tangent vector like this:



Given a vector function $\mathbf{r}$, we can use the derivative to find lines that are tangent to its image:

The Tangent Line to a Vector Curve

Let $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ be a differentiable vector function on some interval $I \subseteq \mathbb{R}$, and let t_0 be a point in I such that $\mathbf{r}'(t_0) \neq \mathbf{0}$. The ***tangent line*** to the vector curve defined by $\mathbf{r}(t)$ at $\mathbf{r}(t_0)$ is given by

$$\mathcal{L}(t) = \mathbf{r}(t_0) + t\,\mathbf{r}'(t_0).$$

Let's find the line tangent to the image of $\mathbf{r}(t) = \langle \cos(t), \sin(t), t\rangle$ at $t = \pi/3$. We saw above that $\mathbf{r}'(t) = \langle -\sin(t), \cos(t), 1\rangle$. Thus

$$\mathbf{r}'\Big(\frac{\pi}{3}\Big) = \Big\langle -\frac{\sqrt{3}}{2}, \frac{1}{2}, 1\Big\rangle.$$

We also have

$$\mathbf{r}\Big(\frac{\pi}{3}\Big) = \Big\langle \frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\pi}{3}\Big\rangle.$$

So the tangent line is given by the vector equation

$$\langle x, y, z\rangle = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\pi}{3} \right\rangle + t\left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2}, 1 \right\rangle.$$

4. Derivative rules

The derivative of a vector function exhibits the same types of properties that you're familiar with from ordinary calculus. In particular, it's linear:

$$\frac{d}{dt}(a\mathbf{r_1}(t) + b\mathbf{r_2}(t)) = a\mathbf{r_1}'(t) + b\mathbf{r_2}'(t) \qquad \text{for any scalars } a, b$$

It is also obeys a chain rule:

The Chain Rule for Vector-Valued Functions

Let $t = f(\tau)$ be a differentiable real-valued function of τ, and let $\mathbf{r}(t)$ be a differentiable vector function with either two or three components such that $f(\tau)$ is in the domain of $\mathbf{r}$ for every value of τ on some interval I. Then

$$\frac{d\mathbf{r}}{d\tau} = \frac{d\mathbf{r}}{dt}\frac{dt}{d\tau}.$$

Note that this is nothing more than the ordinary chain rule to each component of $\mathbf{r}$. For example, let $\mathbf{r}(t) = \langle e^t, \sin(t)\rangle$ where $t = \tau^2$. Using the chain rule as stated above, we see that

$$\frac{d\mathbf{r}}{d\tau} = \frac{d\mathbf{r}}{dt}\frac{dt}{d\tau} = \langle e^t, \cos(t)\rangle 2\tau = \langle e^{\tau^2}, \cos(\tau^2)\rangle 2\tau.$$

If instead we apply the ordinary chain rule to each component of $\mathbf{r}$, we get the same thing:

$$\frac{d\mathbf{r}}{d\tau} = \left\langle \frac{d}{d\tau}e^t, \frac{d}{d\tau}\sin(t) \right\rangle = \left\langle \frac{de^t}{dt}\frac{dt}{d\tau}, \frac{d\sin(t)}{dt}\frac{dt}{d\tau} \right\rangle = \left\langle e^t 2\tau, \cos(t)2\tau \right\rangle = \left\langle e^{\tau^2} 2\tau, \cos(\tau^2)2\tau \right\rangle.$$

Yesterday we saw that vector functions can be combined in various ways to form new functions. In particular, we can form product functions using, for example, the dot product and cross product. The derivatives of these products obey nice rules that resemble the product rule from ordinary calculus:

Derivatives of Products of Vector Functions

Let k be a scalar, $f(t)$ be a differentiable scalar function, and $\mathbf{r}(t)$ be a differentiable vector function. Then

(a) $\frac{d}{dt}(k\mathbf{r}(t)) = k\mathbf{r}'(t)$.

(b) $\frac{d}{dt}(f(t)\mathbf{r}(t)) = f'(t)\mathbf{r}(t) + f(t)\mathbf{r}'(t)$.

Furthermore, if $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ are differentiable vector functions with both having either two or three components, then

(c) $\frac{d}{dt}(\mathbf{r}_1(t) \cdot \mathbf{r}_2(t)) = \mathbf{r}_1'(t) \cdot \mathbf{r}_2(t) + \mathbf{r}_1(t) \cdot \mathbf{r}_2'(t)$.

Finally, if $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ are both differentiable three-component vector functions, then

(d) $\frac{d}{dt}(\mathbf{r}_1(t) \times \mathbf{r}_2(t)) = \mathbf{r}_1'(t) \times \mathbf{r}_2(t) + \mathbf{r}_1(t) \times \mathbf{r}_2'(t)$.

Part (a) is a special case of linearity of the derivative, which was already mentioned above. Each of parts (b), (c), and (d) can be proved using the product rule for scalar functions.

5. The integral of a vector function

Vector functions can also be integrated. By now you can probably predict how this works; we perform ordinary integration on each component:

The Integral of a Vector Function

Let $x = x(t)$, $y = y(t)$, and $z = z(t)$ be three real-valued functions with antiderivatives

$$\int x(t)\,dt, \quad \int y(t)\,dt, \quad \text{and} \quad \int z(t)\,dt,$$

respectively, on some interval $I \subseteq \mathbb{R}$. Then the vector function

$$\int \mathbf{r}(t)\,dt = \int (x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k})\,dt = \mathbf{i}\int x(t)\,dt + \mathbf{j}\int y(t)\,dt + \mathbf{k}\int z(t)\,dt$$

is an ***antiderivative*** of the vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$.

Similarly, if $x(t)$, $y(t)$, and $z(t)$ are all integrable on the interval $[a, b]$, then the ***definite integral*** of the vector function $\mathbf{r}(t)$ from a to b is the vector

$$\int_a^b \mathbf{r}(t)\,dt = \int_a^b (x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k})\,dt = \mathbf{i}\int_a^b x(t)\,dt + \mathbf{j}\int_a^b y(t)\,dt + \mathbf{k}\int_a^b z(t)\,dt.$$

Day 10

1. Velocity and acceleration

The output of a vector function $\mathbf{r}$ can be thought of as the position of a moving particle. Therefore, its derivative can be interpreted as the velocity of the particle, and its second derivative is the acceleration:

Velocity, Speed, and Acceleration along a Space Curve

Let $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ be a differentiable vector-valued function defined at every point in some time interval $I \subseteq \mathbb{R}$, and let C be the space curve defined by $\mathbf{r}(t)$.

(a) The ***velocity***, $\mathbf{v}$, of the particle as it moves along C is given by

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle.$$

(b) The ***speed*** of the particle as it moves along C is given by

$$\|\mathbf{v}(t)\| = \|\langle x'(t), y'(t), z'(t)\rangle\|.$$

(c) In addition, if $\mathbf{r}(t)$ is twice differentiable, the ***acceleration***, $\mathbf{a}$, of the particle as it moves along C is given by

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = \langle x''(t), y''(t), z''(t)\rangle.$$

If we know the velocity of the particle over time and we know its position at some particular time, then we can recover its position at all times. For example, suppose the velocity of the particle is $\mathbf{r}'(t) = \langle \sin(t), t^2, t^3\rangle$ and its initial position is $\mathbf{r}(0) = \langle 3, -2, 5\rangle$. Then

$$\mathbf{r}(t) = \int \mathbf{r}'(t)dt + \mathbf{c} = \Big\langle \int \sin(t)dt, \int t^2dt, \int t^3dt\Big\rangle + \mathbf{c} = \Big\langle -\cos(t), \frac{t^3}{3}, \frac{t^4}{4}\Big\rangle + \mathbf{c},$$

where $\mathbf{c}$ is an unknown vector (integration constant). Plugging in $t = 0$, we see that

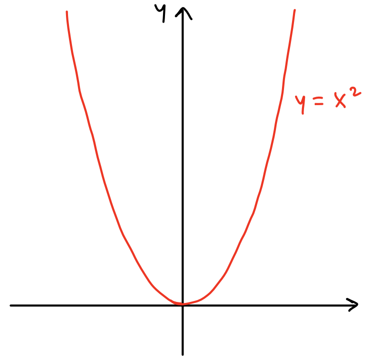
$$\langle 3, -2, 5\rangle = \mathbf{r}(0) = \langle -1, 0, 0\rangle + \mathbf{c},$$

and therefore $\mathbf{c} = \langle 4, -2, 5\rangle$. So altogether,

$$\mathbf{r}(t) = \Big\langle -\cos(t), \frac{t^3}{3}, \frac{t^4}{4}\Big\rangle + \langle 4, -2, 5\rangle.$$

2. Toward defining curvature

As an application of vector functions, our next goal is to define the *curvature* of a curve. Curvature is a measure of how sharply a curve is turning at a given point. Let's consider the parabola given by $y = x^2$ as an example:



We can see that the parabola is most "curved" at the origin and flattens out the further we move away from the origin. But how do we quantify this? Curvature is supposed to measure how fast the direction of the curve is changing. We know that the direction of the curve is captured by a tangent vector, so maybe we should fix a parametrization of the parabola and look at how its tangent vector changes. The most natural parametrization of the parabola is $\mathbf{r}(t) = \langle t, t^2 \rangle$. As a first guess, we could define the curvature to be

$$\kappa(t) = \|\mathbf{r}''(t)\|, \qquad \text{(this is wrong!)}$$

the magnitude of the rate of change of the tangent vector $\mathbf{r}'(t)$. There are a few problems with this definition. First, observe that $\mathbf{r}''(t) = \langle 0, 2 \rangle$, so under this definition the curvature of the parabola would be constant. This does not agree with our intuition that the curvature of the parabola should be maximal at the origin. Secondly, and more significantly, if we were to use a different parametrization we would get different values for the curvature. For example, if we used $\mathbf{q}(t) = \langle t^3, t^6 \rangle$, then the curvature would be

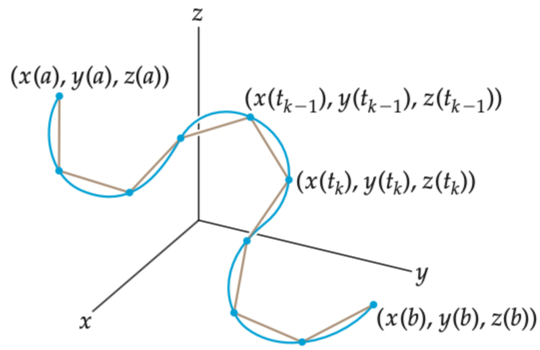
$$\kappa(t) = \|\mathbf{q}''(t)\| = \sqrt{36t^2 + 900t^8}, \qquad \text{(this is wrong!)}$$

which is very different from the curvature we found using $\mathbf{r}$. This is a problem, because curvature should be an intrinsic property of the curve itself; it should not depend on the specific parametrization we use to represent the curve.

The crux of the issue is that we're measuring rate of change with respect to the wrong parameter. If we imagine that the parametrization $\mathbf{r}(t)$ traces out the parabola over time, the quantity $\|\mathbf{r}''(t)\|$ represents the rate of change of the tangent vector with respect to time. This will depend heavily on how fast or slow the function $\mathbf{r}(t)$ is moving as it traces out the curve. The question we should instead ask is, how fast is the direction changing relative to the *arc length* traveled. Curvature corresponds not just to a change in direction, but a change in direction over a short distance (arc length).

3. Arc length

How do we calculate the length (or *arc length*) of a curve? To start, we know how to find the length of a line segment; it's the distance between its endpoints. But (essentially) every curve can be approximated by line segments. For example if we want to measure the length of a curve given by $\mathbf{r}(t)$ for $t \in [a, b]$, we could find n equally spaced points $t_1 < t_2 < \cdots < t_n$ in $[a, b]$ and approximate the curve with the line segments joining $\mathbf{r}(t_k)$ to $\mathbf{r}(t_{k+1})$:



We can add up their lengths, given by $\|\mathbf{r}(t_{k+1}) - \mathbf{r}(t_k)\|$, then take the limit as $n \to \infty$ to get the true arc length:

The Arc Length of a Vector Curve

Let C be a curve in $\mathbb{R}^3$ given by the vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ for $t \in [a, b]$, where x, y, and z are differentiable functions of t and such that the function is one-to-one from the interval $[a, b]$ to the curve C. The ***length of the curve*** C from a to b, denoted by $l(a, b)$, is

$$l(a, b) = \lim_{n \to \infty} \sum_{k=1}^{n} \|\mathbf{r}(t_k) - \mathbf{r}(t_{k-1})\|,$$

where $\Delta t = \frac{b-a}{n}$ and $t_k = a + k\Delta t$.

This is essentially a limit of Riemann sums, and thus can also be expressed as an integral:

THEOREM 11.18 **The Arc Length of a Vector Curve**

Let C be a curve in $\mathbb{R}^3$ given by the vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ such that the function is one-to-one from the interval $[a, b]$ to the curve C. If x, y, and z are differentiable functions of t such that x', y', and z' are continuous on $[a, b]$, then the length of the curve C from $\mathbf{r}(a)$ to $\mathbf{r}(b)$ is given by

$$l(a, b) = \int_a^b \|\mathbf{r}'(t)\| dt.$$

In practice, it is usually easier to evaluate the integral rather than the limit. (However, in many cases, the integral will not be nice either!)

In-class exercise[1]**:** Find the arc length of the curve defined by $\mathbf{r}(t) = \langle 4\sin(t), t^{3/2}, -4\cos(t) \rangle$ for $t \in [0, 4]$.

[1] $\frac{488}{27}$ or about 18.1. (Use Theorem 11.18, and don't forget the Pythagorean identity $\sin(t)^2 + \cos(t)^2 = 1$!)

Day 11

1. Parametrization by arc length

Last time we were thinking about the concept of curvature. We tried to write down a sensible definition, but found that our first guess of $\kappa = \|\mathbf{r}''(t)\|$ didn't work. Curvature is an intrinsic property of a curve, and unfortunately this definition of κ changes with our choice of parametrization $\mathbf{r}$. To fix this, we need to standardize the parametrization we're using, so that it traces out the curve at a uniform unit speed.

We say that a vector function $\mathbf{r}(t)$ is an arc length parametrization if, over any T units of time, $\mathbf{r}(t)$ traverses an arc of length T. (In other words, $\mathbf{r}(t)$ is moving at unit speed.) Here's the formal definition:

Arc Length Parametrization

Let C be the graph of a differentiable vector function $\mathbf{r}(t)$ defined on an interval I. The function $\mathbf{r}(t)$ is said to be an ***arc length parametrization*** for C if

$$\int_c^d \|\mathbf{r}'(t)\|\, dt = d - c$$

for every c and $d \in I$. Also, C is said to be ***parametrized by arc length.***

(Recall from last time that the integral above is equal to the arc length of the curve between $t = c$ and $t = d$.)

Importantly, $\mathbf{r}$ is an arc length parametrization if and only if $\|\mathbf{r}'(t)\| = 1$ for every t.

In-class exercise[1]**:** For each curve and vector function below, determine whether the vector function is an arc length parametrization of the curve. If it's not, find one.

(a) The unit circle parametrized by $\mathbf{r}(t) = \langle \cos(2\pi t), \sin(2\pi t)\rangle$
(b) The helix parametrized by $\mathbf{r}(t) = \langle \cos(t), \sin(t), 1\rangle$

2. Curvature

As hinted above, we can use parametrization by arc length to properly define curvature: Let $\mathbf{r}(s)$ be an arc length parametrization of a curve C. Then the *curvature* of C is the scalar function

$$\kappa(s) = \|\mathbf{r}''(s)\|.$$

We often use the parameter s to indicate that the parametrization is by arc length. This definition of curvature differs very slightly from the one given in the textbook (in terms of the unit tangent vector); however the two definitions are equivalent.

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1 (a) Not an arc length parametrization: The arc length (circumference) of the unit circle is 2π, but it takes only 1 unit of time ($t = 0$ to $t = 1$) for $\mathbf{r}(t)$ to cover all of the circle. An arc length parametrization is given by $\mathbf{q}(s) = \langle \cos(s), \sin(s)\rangle$.
(b) Not an arc length parametrization: The length of the curve between $t = a$ and $t = b$ works out to be $\sqrt{2}(b - a)$, rather than $b - a$. An arc length parametrization is given by $\mathbf{q}(s) = \langle \cos(s/\sqrt{2}), \sin(s/\sqrt{2}), s/\sqrt{2}\rangle$.

In-class exercise[2]: Calculate the curvature of each curve below. (You'll need to convert to an arc length parametrization first.)

(a) The circle of radius R parametrized by $\mathbf{r}(t) = \langle R\cos(t), R\sin(t)\rangle$
(b) The helix of radius R and "climb" C given by $\mathbf{r}(t) = \langle R\cos(t), R\sin(t), Ct\rangle$

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2 (a) $\frac{1}{R}$ (arc length parametrization is $\mathbf{q}(s) = \langle R\cos(s/R), R\sin(s/R)\rangle$)
(b) $\frac{R}{R^2+C^2}$ (arc length parametrization is $\mathbf{q}(s) = \langle R\cos(s/\rho), R\sin(s/\rho), Cs/\rho\rangle$, where $\rho = R^2 + C^2$)