# Day 24

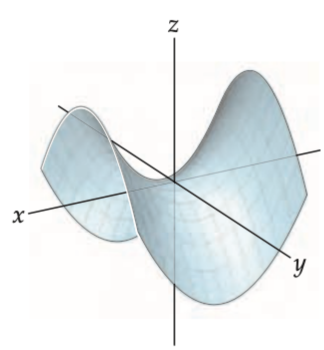
## 1. Classifying critical points

Last time we saw the definition of the local extrema (i.e. local minima and maxima) of a function. We saw that local extrema only occur at critical points. If we've found a critical point, how can we tell whether it's a local minimum or local maximum (or both or neither)? The *second derivative test* will provide a criterion that answers this question in most cases. We will only state the second derivative test for functions of two variables. (There is a second derivative test for functions with three or more variables, but it requires some knowledge of linear algebra.) First we need a few more definitions:

**Saddle Points of a Function of Two Variables**

A point $(x_0, y_0)$ in the domain of a function $f(x, y)$ is called a ***saddle point*** of $f$ if $(x_0, y_0)$ is a stationary point of $f$ at which there is neither a maximum nor a minimum.

The prototypical example of a function with a saddle point is the hyperbolic paraboloid $f(x, y) = x^2 - y^2$, whose graph looks like this:




This function has a saddle point at the origin $(0, 0)$. There the tangent plane is horizontal (i.e. the gradient is zero), but the function has neither a local maximum nor a local minimum.

Recall that a function of $x$ and $y$ has four second partial derivatives: $f_{xx}$, $f_{xy}$, $f_{yx}$, and $f_{yy}$. The second derivative test will involve all of these. It's convenient to organize these derivatives into a $2 \times 2$ matrix, know as the Hessian matrix.

### The Hessian and the Discriminant of a Function of Two Variables

Let $f(x, y)$ be a function with continuous second-order partial derivatives on some open set $S$.

**(a)** The ***Hessian*** of $f$ is the $2 \times 2$ matrix of second-order partial derivatives:

$$H_f = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial y \partial x} \\[2ex] \dfrac{\partial^2 f}{\partial x \partial y} & \dfrac{\partial^2 f}{\partial y^2} \end{bmatrix}.$$

**(b)** The ***discriminant*** of $f$ is the determinant of the Hessian. That is,

$$\det(H_f) = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2.$$

**In-class exercise**[1]**:** Let $f(x, y) = x^2 + xy - y^2$

(a) Find the critical point(s) of $f$.
(b) At each critical point, find the Hessian matrix $H_f$ and the discriminant $\det(H_f)$.

**THEOREM 12.45**

### The Second-Order Partial-Derivative Test for Classifying Stationary Points

Let $f(x, y)$ be a function with continuous second-order partial derivatives on some open disk containing the point at $(x_0, y_0)$. If $f$ has a stationary point at $(x_0, y_0)$, then

**(a)** $f$ has a relative maximum at $(x_0, y_0)$ if $\det(H_f(x_0, y_0)) > 0$ with $f_{xx}(x_0, y_0) < 0$ or $f_{yy}(x_0, y_0) < 0$.

**(b)** $f$ has a relative minimum at $(x_0, y_0)$ if $\det(H_f(x_0, y_0)) > 0$ with $f_{xx}(x_0, y_0) > 0$ or $f_{yy}(x_0, y_0) > 0$.

**(c)** $f$ has a saddle point at $(x_0, y_0)$ if $\det(H_f(x_0, y_0)) < 0$.

**(d)** If $\det(H_f(x_0, y_0)) = 0$, no conclusion may be drawn about the behavior of $f$ at $(x_0, y_0)$.

[1] (a) $(0, 0)$
(b) $-5$