MATH 231-01: Homework Assignment 6

20 October 2025

Due: 27 October 2025 by 10:00pm Eastern time, submitted on Moodle as a single PDF.

Instructions: Write your solutions on the following pages. If you need more space, you may add pages, but make sure they are in order and label the problem number(s) clearly. You should attempt each problem on scrap paper first, before writing your solution here. Excessively messy or illegible work will not be graded. You must show your work/reasoning to receive credit. You do not need to include every minute detail; however the process by which you reached your answer should be evident. You may work with other students, but please write your solutions in your own words.

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Score:

1. Find the local maxima, local minima, and saddle points of the function $f(x, y) = 2x^2 - 4xy + y^4 + 2$.

(1) Find the partial derivatives of the function:

$$f_x = 4x - 4y$$
$$f_y = -4x + 4y^3$$

(2) Set the partial derivatives equal to zero to find critical points:

$$4x - 4y = 0 \implies x = y$$
$$-4x + 4y^3 = 0 \implies x = y^3$$

(3) Substitute $x = y$ into $x = y^3$:

$$y = y^3 \implies y^3 - y = 0 \implies y(y^2 - 1) = 0$$
$$y = 0, y = 1, y = -1$$

(4) Find corresponding x values:

$$y = 0 \implies x = 0$$
$$y = 1 \implies x = 1$$
$$y = -1 \implies x = -1$$

(5) Critical points are: $(0, 0), (1, 1), (-1, -1)$

(6) Compute the second partial derivatives:

$$f_{xx} = 4$$
$$f_{yy} = 12y^2$$
$$f_{xy} = -4$$

(7) Calculate the discriminant D:

$$D = f_{xx} f_{yy} - (f_{xy})^2 = 4(12y^2) - (-4)^2 = 48y^2 - 16$$

(8) Evaluate D at each critical point:

- At $(0, 0)$:

$$D = 48(0)^2 - 16 = -16 < 0 \implies \text{saddle point}$$

- At $(1, 1)$:

$$D = 48(1)^2 - 16 = 32 > 0, f_{xx} = 4 > 0 \implies \text{local minimum}$$

- At $(-1, -1)$:

$$D = 48(-1)^2 - 16 = 32 > 0, f_{xx} = 4 > 0 \implies \text{local minimum}$$

(9) Summary of critical points:

- $(0, 0)$: Saddle point
- $(1, 1)$: Local minimum
- $(-1, -1)$: Local minimum

2. Find the local maxima, local minima, and saddle points of the function $f(x, y) = x^2 + y^2 + \frac{2}{xy}$.

(1) Find the partial derivatives of the function:

$$f_x = 2x - \frac{2}{x^2 y}$$

$$f_y = 2y - \frac{2}{xy^2}$$

(2) Set the partial derivatives equal to zero to find critical points:

$$2x - \frac{2}{x^2 y} = 0 \implies 2x^3 y - 2 = 0 \implies x^3 y = 1$$

$$2y - \frac{2}{xy^2} = 0 \implies 2y^3 x - 2 = 0 \implies y^3 x = 1$$

(3) From the equations $x^3 y = 1$ and $y^3 x = 1$, we can set them equal to each other:

$$x^3 y = y^3 x \implies x^2 = y^2 \implies y = x \text{ or } y = -x$$

(4) Substitute $y = x$ into $x^3 y = 1$:

$$x^4 = 1 \implies x = 1 \text{ or } x = -1$$

$$y = 1 \text{ or } y = -1$$

(5) Critical points are: $(1, 1), (-1, -1)$

(6) Compute the second partial derivatives:

$$f_{xx} = 2 + \frac{4}{x^3 y}$$

$$f_{yy} = 2 + \frac{4}{xy^3}$$

$$f_{xy} = \frac{2}{x^2 y^2}$$

(7) Calculate the discriminant D:

$$D = f_{xx} f_{yy} - (f_{xy})^2$$

(8) Evaluate D at each critical point:

- At $(1, 1)$:

$$f_{xx} = 6, f_{yy} = 6, f_{xy} = 2$$

$$D = 6 \cdot 6 - 2^2 = 36 - 4 = 32 > 0, f_{xx} = 6 > 0 \implies \text{local minimum}$$

- At $(-1, -1)$:

$$f_{xx} = 6, f_{yy} = 6, f_{xy} = 2$$

$$D = 6 \cdot 6 - 2^2 = 36 - 4 = 32 > 0, f_{xx} = 6 > 0 \implies \text{local minimum}$$

(9) Summary of critical points:

- $(1, 1)$: Local minimum
- $(-1, -1)$: Local minimum

3. Find the local maxima, local minima, and saddle points of the function $f(x, y) = 2xy^2 - x^2y + 4xy$.

(1) Find the partial derivatives of the function:

$$f_x = 2y^2 - 2xy + 4y$$
$$f_y = 4xy - x^2 + 4x$$

(2) Set the partial derivatives equal to zero to find critical points:

$$2y^2 - 2xy + 4y = 0$$
$$4xy - x^2 + 4x = 0$$

(3) Solve the system of equations to find critical points: From the first equation, factor out $2y$:

$$2y(y - x + 2) = 0 \implies y = 0 \text{ or } y = x - 2$$

Substitute $y = 0$ into the second equation:

$$4x(0) - x^2 + 4x = 0 \implies -x^2 + 4x = 0 \implies x(x - 4) = 0$$
$$x = 0 \text{ or } x = 4$$

Critical points from $y = 0$: $(0, 0), (4, 0)$ Now substitute $y = x - 2$ into the second equation:

$$4x(x - 2) - x^2 + 4x = 0$$
$$4x^2 - 8x - x^2 + 4x = 0$$
$$3x^2 - 4x = 0 \implies x(3x - 4) = 0$$
$$x = 0 \text{ or } x = \frac{4}{3}$$

Corresponding y values: For $x = 0$: $y = -2$ For $x = \frac{4}{3}$: $y = \frac{4}{3} - 2 = -\frac{2}{3}$
Critical points from $y = x - 2$: $(0, -2), (\frac{4}{3}, -\frac{2}{3})$

(4) Critical points are: $(0, 0), (4, 0), (0, -2), (\frac{4}{3}, -\frac{2}{3})$

(5) Compute the second partial derivatives:

$$f_{xx} = -2y$$
$$f_{yy} = 4x$$
$$f_{xy} = 4y + 4 - 2x$$

(6) Calculate the discriminant D:

$$D = f_{xx}f_{yy} - (f_{xy})^2$$

(7) Evaluate D at each critical point:

- At $(0, 0)$:

$$f_{xx} = 0, f_{yy} = 0, f_{xy} = 4$$
$$D = 0 \cdot 0 - 4^2 = -16 < 0 \implies \text{saddle point}$$

- At $(4, 0)$:

$$f_{xx} = 0, f_{yy} = 16, f_{xy} = -4$$
$$D = 0 \cdot 16 - (-4)^2 = -16 < 0 \implies \text{saddle point}$$

– At $(0, -2)$:

$$f_{xx} = 4, f_{yy} = 0, f_{xy} = -4$$
$$D = 4 \cdot 0 - (-4)^2 = -16 < 0 \implies \text{saddle point}$$

– At $(\frac{4}{3}, -\frac{2}{3})$:

$$f_{xx} = \frac{4}{3}, f_{yy} = \frac{16}{3}, f_{xy} = -\frac{4}{3}$$
$$D = \frac{4}{3} \cdot \frac{16}{3} - \left(-\frac{4}{3}\right)^2 = \frac{64}{9} - \frac{16}{9} = \frac{48}{9} > 0, f_{xx} = \frac{4}{3} > 0 \implies \text{local minimum}$$

(8) Summary of critical points:
- $(0, 0)$: Saddle point
- $(4, 0)$: Saddle point
- $(0, -2)$: Saddle point
- $(\frac{4}{3}, -\frac{2}{3})$: Local minimum

4. Complete Problem 12 in Section 12.6 of the textbook (p. 975).

Theorem 12.45 is inconclusive when the discriminant, $\det(H_f)$, is zero at a stationary point. In Exercises 10–12 we ask you to illustrate this fact by analyzing three functions of two variables with stationary points at the origin.

12. Show that the function $h(x, y) = x^3 + y^3$ has a stationary point at the origin. Show that the discriminant $\det(H_h(0, 0)) = 0$. Show that there are points arbitrarily close to the origin such that $h(x, y) > 0$. Show that there are points arbitrarily close to the origin such that $h(x, y) < 0$. Explain why all this shows that h has a saddle at the origin.

To analyze the function $h(x, y) = x^3 + y^3$, we first find its partial derivatives:

$$h_x = 3x^2, \quad h_y = 3y^2.$$

Setting these partial derivatives equal to zero gives us the stationary points:

$$3x^2 = 0 \implies x = 0, \quad 3y^2 = 0 \implies y = 0.$$

Thus, the only stationary point is at the origin $(0, 0)$.
Next, we compute the second partial derivatives:

$$h_{xx} = 6x, \quad h_{yy} = 6y, \quad h_{xy} = 0.$$

Evaluating these at the origin gives:

$$h_{xx}(0, 0) = 0, \quad h_{yy}(0, 0) = 0, \quad h_{xy}(0, 0) = 0.$$

The Hessian matrix $H_h(0, 0)$ is:

$$H_h(0, 0) = \begin{pmatrix} h_{xx}(0, 0) & h_{xy}(0, 0) \\ h_{xy}(0, 0) & h_{yy}(0, 0) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The discriminant is given by:

$$\det(H_h(0,0)) = h_{xx}(0,0)h_{yy}(0,0) - (h_{xy}(0,0))^2 = 0 \cdot 0 - 0^2 = 0.$$

To show that there are points arbitrarily close to the origin where $h(x, y) > 0$, consider the point (ϵ, ϵ) for a small positive number ϵ:

$$h(\epsilon, \epsilon) = (\epsilon)^3 + (\epsilon)^3 = 2\epsilon^3 > 0.$$

Similarly, to show that there are points arbitrarily close to the origin where $h(x, y) < 0$, consider the point $(-\epsilon, -\epsilon)$:

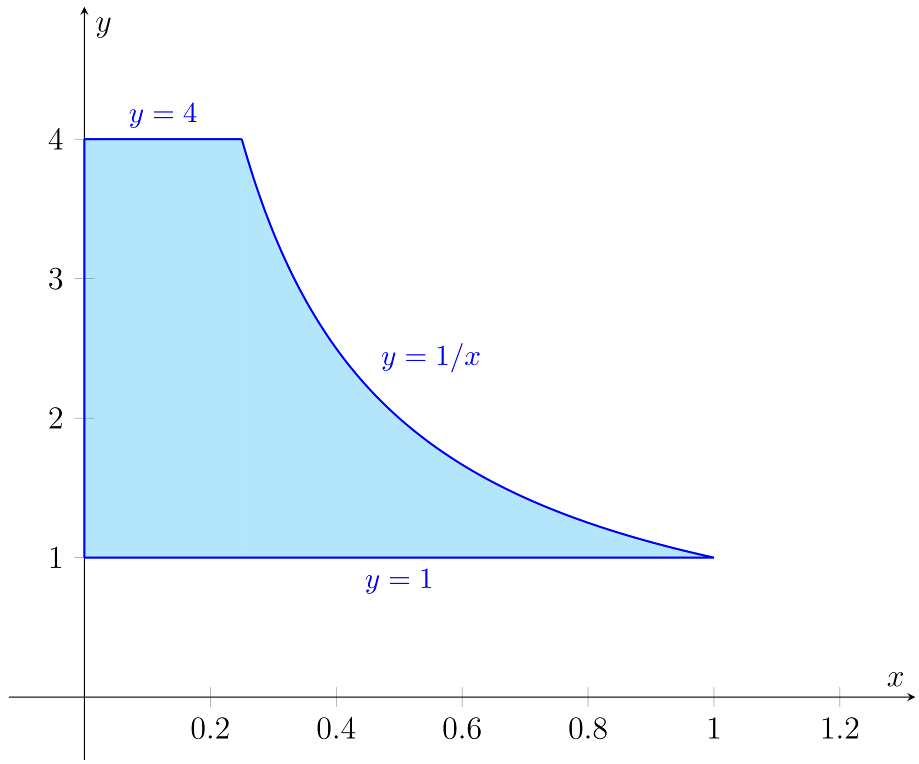
$$h(-\epsilon, -\epsilon) = (-\epsilon)^3 + (-\epsilon)^3 = -2\epsilon^3 < 0.$$

Since we can find points arbitrarily close to the origin where $h(x, y)$ is both positive and negative, this indicates that the function does not have a local maximum or minimum at the origin. Therefore, we conclude that h has a saddle point at the origin.

5. Consider the iterated integral

$$\int_1^4 \int_0^{1/y} \frac{x}{y} dx dy.$$

(a) Sketch the region determined by the bounds of the integral.



(b) Evaluate the integral.

To evaluate the integral

$$\int_1^4 \int_0^{1/y} \frac{x}{y} \, dx \, dy,$$

we first compute the inner integral with respect to x:

$$\int_0^{1/y} \frac{x}{y} \, dx = \frac{1}{y} \int_0^{1/y} x \, dx = \frac{1}{y} \left[\frac{x^2}{2} \right]_0^{1/y} = \frac{1}{y} \cdot \frac{(1/y)^2}{2} = \frac{1}{2y^3}.$$

Now we substitute this result into the outer integral:

$$\int_1^4 \frac{1}{2y^3} \, dy = \frac{1}{2} \int_1^4 y^{-3} \, dy.$$

We compute the integral:

$$\int y^{-3}\,dy = \int y^{-3}\,dy = -\frac{1}{2y^2} + C.$$

Evaluating this from 1 to 4 gives:

$$\left[-\frac{1}{2y^2}\right]_1^4 = -\frac{1}{2(4^2)} + \frac{1}{2(1^2)} = -\frac{1}{32} + \frac{1}{2} = \frac{16}{32} - \frac{1}{32} = \frac{15}{32}.$$

Therefore, the value of the integral is:

$$\frac{1}{2} \cdot \frac{15}{32} = \frac{15}{64}.$$

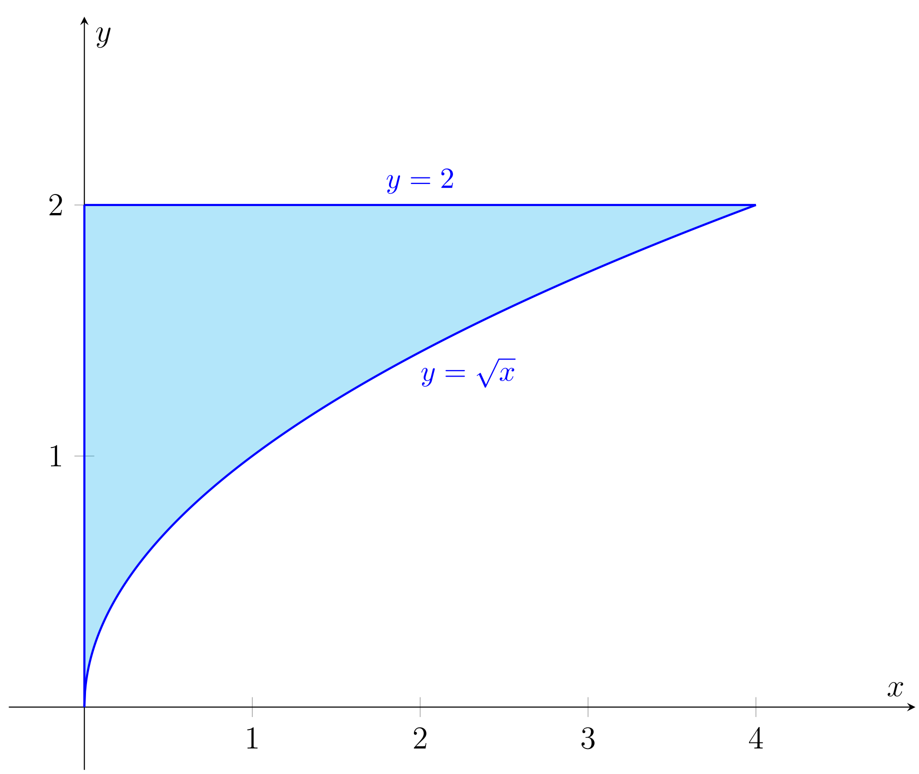
Thus, the final answer is:

$$\frac{15}{64}.$$

6. Consider the iterated integral

$$\int_0^4 \int_{\sqrt{x}}^2 \sin(y^3) dydx.$$

(a) Sketch the region determined by the bounds of the integral.



(b) Evaluate the integral by reversing the order of integration.

To reverse the order of integration for the integral

$$\int_0^4 \int_{\sqrt{x}}^2 \sin(y^3)\, dy\, dx,$$

we first identify the region of integration. The bounds indicate that x ranges from 0 to 4, and for each fixed x, y ranges from $\sqrt{x}$ to 2.

Rewriting the bounds in terms of y: - The lower bound for y is $\sqrt{x}$, which implies $x = y^2$. - The upper bound for y is 2, which implies x can go up to 4.

Thus, for a fixed y, x ranges from 0 to y^2. The new bounds for y are from 0 to 2. Therefore, the reversed integral is:

$$\int_0^2 \int_0^{y^2} \sin(y^3)\, dx\, dy.$$

Now we evaluate the inner integral with respect to x:

$$\int_0^{y^2} \sin(y^3)\, dx = \sin(y^3) \cdot (y^2 - 0) = y^2 \sin(y^3).$$

Substituting this into the outer integral gives:

$$\int_0^2 y^2 \sin(y^3)\, dy.$$

To evaluate this integral, we use the substitution $u = y^3$, which gives $du = 3y^2 dy$ or $y^2 dy = \frac{du}{3}$. When $y = 0$, $u = 0$, and when $y = 2$, $u = 8$. Thus, the integral becomes:

$$\int_0^8 \sin(u) \cdot \frac{1}{3}\, du = \frac{1}{3} \int_0^8 \sin(u)\, du.$$

Evaluating this integral:

$$\frac{1}{3} \left[-\cos(u)\right]_0^8 = \frac{1}{3} \left(-\cos(8) + \cos(0)\right) = \frac{1}{3} \left(1 - \cos(8)\right).$$

Therefore, the value of the integral is:

$$\frac{1 - \cos(8)}{3}.$$