# Day 1

## 1. What is multivariable calculus?

You probably already know that calculus is the mathematics of change. So far you've studied single-variable calculus, which describes how a single dependent variable $y = f(x)$ changes with respect to a single independent variable $x$. Multivariable calculus addresses the same kind of question, but for functions with multiple dependent variables or multiple independent variables or both. Naturally, the course will be organized into the following units:

- **Unit 0: Vectors and the geometry of $\mathbb{R}^3$.** You'll begin by developing the tools needed to analyze multivariable functions. A solid understanding of three-dimensional geometry will be essential. Vectors will allow you to make sense of geometric objects like lines and planes in a precise way.

- **Unit 1: Multiple dependent variables.** Next you'll study the calculus of functions with multiple dependent variables but only one independent variable. Visually, these functions look like *curves*, either in the plane or in space. This unit will feel a lot like the single-variable calculus you've already learned.

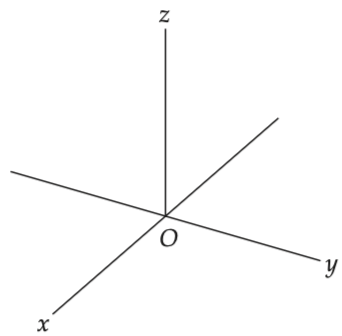
- **Unit 2: Multiple independent variables.** Here's where things get harder but also more interesting! In this unit, you'll learn the calculus of functions with multiple independent variables but only one dependent variable. Visually, these functions look like *surfaces* in space.

- **Unit 3: Multiple dependent variables and multiple independent variables.** Finally, you'll put everything together and study functions with multiple dependent variables and multiple independent variables. Visually, these functions assign a vector (or arrow) to each point in their domain and are known as *vector fields*.

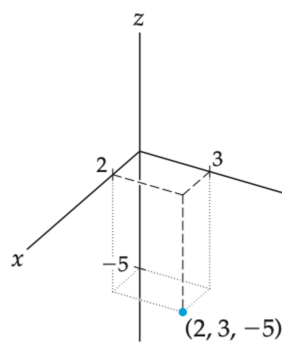
## 2. The Cartesian coordinate system in $\mathbb{R}^3$

In single-variable calculus, you studied functions of the form $y = f(x)$ and plotted their graphs in the $xy$-plane. If you consider a function with one additional variable (either dependent or independent), you'll need three dimensions in order to visualize it.

A point in three-dimensional space, or $\mathbb{R}^3$, can be defined using any of several different coordinate systems. The one that's used most often is the *Cartesian coordinate system*. It's also sometimes called the *rectangular coordinate system* because it's constructed from a set of three mutually perpendicular axes:




(Note: The diagram above is missing something important: an arrow on each axis to tell us in which direction it's increasing.) A point in $\mathbb{R}^3$ is specified by its coordinates along these axes. For example, the point $(2, 3, -5)$ can be located by moving 2 units along the $x$-axis in the positive direction, then 3 units along the $y$-axis in the positive direction, then 5 units along the $z$-axis in the negative direction:




To get a feel for Cartesian coordinates in $\mathbb{R}^3$, I recommend playing around with Geogebra's 3D Calculator: https://www.geogebra.org/3d

Each pair of axes determines a unique plane, namely the $xy$-plane (containing the $x$- and $y$-axes), the $xz$-plane (containing the $x$- and $z$-axes), and the $yz$-plane (containing the $y$- and $z$-axes). These are known as the *coordinate planes*. Each of the coordinate planes divides $\mathbb{R}^3$ into two "half-spaces"; the three planes together divide $\mathbb{R}^3$ into 8 *octants*. The octant in which $x$, $y$, and $z$ are all positive is called the *first octant*.

## 3. Distances and spheres

You probably remember how to find the distance between two points in the plane, $\mathbb{R}^2$: If $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, then the distance between $P$ and $Q$ is

$$\mathrm{dist}(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

(If you don't remember this, figure out why it's true. Hint: Pythagorean theorem!) The formula for distance in $\mathbb{R}^3$ is not much different: If $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$, then

$$\mathrm{dist}(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

In fact, it can be derived from the two-dimensional distance formula by considering distances in two different coordinate planes.

There is a nice relationship between spheres and distance: The *sphere with center $P$ and radius $r$* is the set of all points that lie at distance $r$ from $P$. If $P = (x_0, y_0, z_0)$, then the sphere can be described as the set of all points $(x, y, z)$ that satisfy the formula

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2.$$

**In-class exercise**[1]**:**

(a) Find the equation for the sphere of radius 5 centered around $(3, -1, 2)$.

(b) For each of the points $A = (7, 2, 4)$, $B = (6, 1, 4)$, and $C = (0, 3, 2)$, determine whether it lies inside, outside, or on the sphere from part (a).

(c) The equation $x^2 + y^2 - 2y + z^2 = 8$ describes a sphere. Find its center and radius.

[1] (a) $(x - 3)^2 + (y + 1)^2 + (z - 2)^2 = 25$
(b) $A$ outside, $B$ inside, $C$ on
(c) center $(0, 1, 0)$, radius 3

# Day 2

## 1. Cylinders

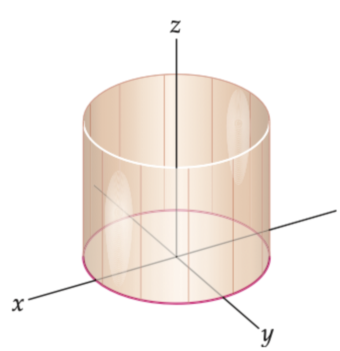
Yesterday we saw that the sphere in $\mathbb{R}^3$ with center $(x_0, y_0, z_0)$ and radius $r$ is described by the formula

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2.$$

This formula is similar to the formula for a circle in $\mathbb{R}^2$. For example, you'll remember that the unit circle centered at the origin is the set of all points $(x, y)$ such that

$$x^2 + y^2 = 1.$$

What happens if we interpret this as a formula for points in $\mathbb{R}^3$? In other words, what type of shape do we get by considering all points $(x, y, z)$ such that $x^2 + y^2 = 1$? To get a feel for this, take a point $(x_0, y_0, 0)$ on the unit circle in the $xy$-plane and consider the vertical line that passes through it. Every point $(x, y, z)$ on this line is of the form $x = x_0, y = y_0$ and therefore satisfies $x^2 + y^2 = 1$. So the equation $x^2 + y^2 = 1$ represents the *cylinder* of all points that lie directly above or directly below the unit circle in the $xy$-plane:

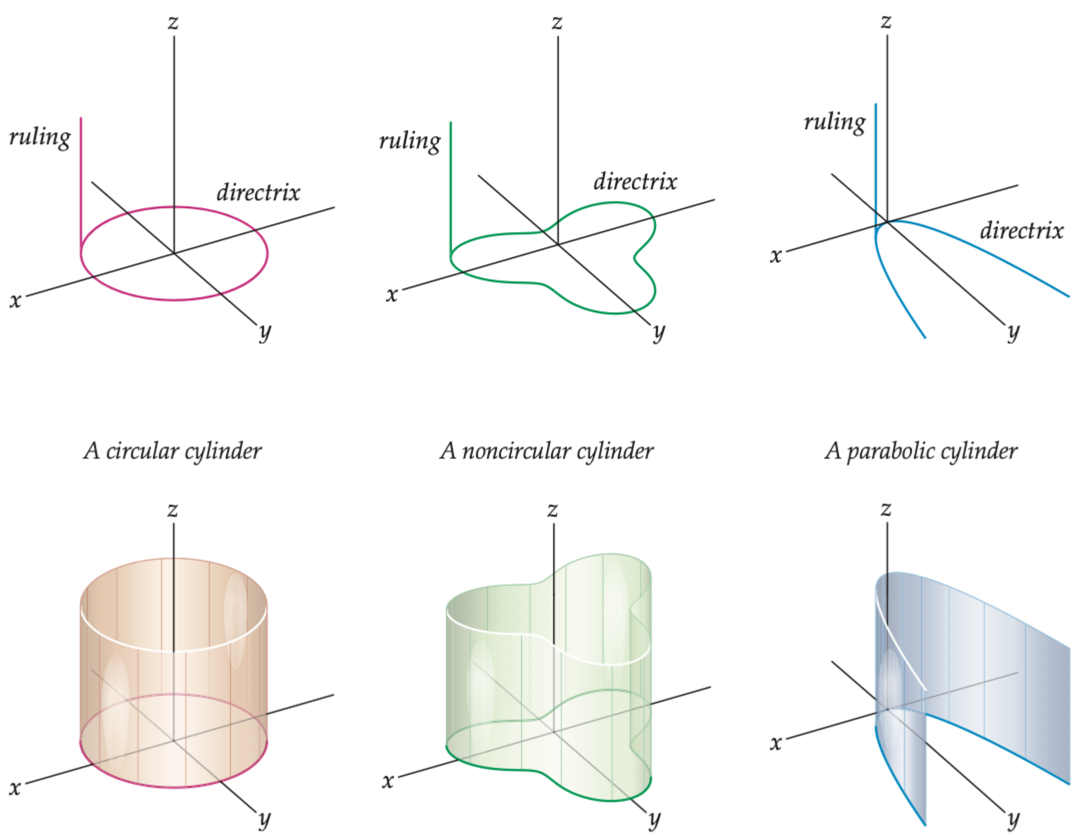



There are other kinds of cylinders. Here's the general definition and some examples from the textbook:

### Cylinders

Let $C$ be a curve in some plane $\mathcal{P}$, and let $l$ be a line that intersects $\mathcal{P}$, but does not lie in $\mathcal{P}$. A ***cylinder*** is the set of all points in $\mathbb{R}^3$ that are on lines parallel to $l$ that intersect $C$. The curve $C$ is called the ***directrix*** of the cylinder. The lines in the cylinder parallel to $l$ are called ***rulings*** of the cylinder.

In the following examples each directrix is a curve in the $xy$-plane and the rulings are parallel to the $z$-axis:




*A circular cylinder* *A noncircular cylinder* *A parabolic cylinder*

After a suitable rotation, the equation for a cylinder will always have a missing variable corresponding to the axis along which the cylinder is oriented. For example, each of the cylinders above is oriented along the $z$-axis, so its equation would involve $x$ and $y$ only.

**In-class exercise[1] (omitted, but I'm including it here):** Imagine you're standing on a perfectly straight coastline (the $y$-axis) looking directly out towards the water (in the direction of the negative $x$-axis). The surface of the water forms wavefronts that are parallel to the coastline. What's an equation that might plausibly model the shape of the water's surface?

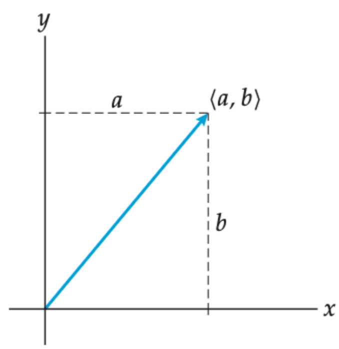
[1] This exercise is intentionally open-ended, but you could start with an equation like $z = \sin(x)$. It's a cylindrical sine wave, with wavefronts parallel to the $y$-axis.

## 2. Vectors in $\mathbb{R}^2$

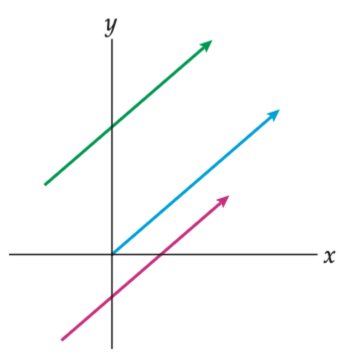
A *vector* in $\mathbb{R}^2$ is simply an ordered pair of numbers, such as $\langle 1, 2\rangle$, $\langle 0, 1/2\rangle$, or $\langle \pi, -4\rangle$. The numbers are called *components* of the vector. We use the angle brackets $\langle\rangle$ rather than parentheses () in order to distinguish vectors from points. (However, we'll see that these concepts are closely related.) We often denote a vector using either a bold letter or a letter with an arrow above it:

$$\mathbf{v} = \langle 1, 2\rangle \qquad \text{or} \qquad \vec{v} = \langle 1, 2\rangle$$

Vectors can be interpreted geometrically in a couple of ways. Consider the vector $\mathbf{v} = \langle a, b\rangle$. We can think of $\mathbf{v}$ as an arrow that starts at the origin and ends at the point $(a, b)$:



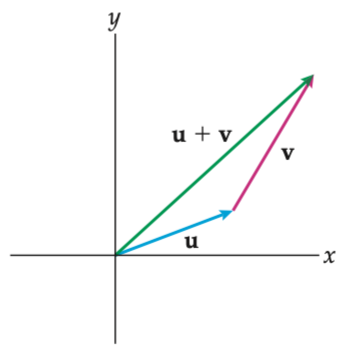

This arrow is the *position vector* for the point $(a, b)$. We can also think of $\mathbf{v}$ as any translation of the position vector, that is, any arrow that has the same length and points in the same direction. There will be many different arrows that represent the same vector:

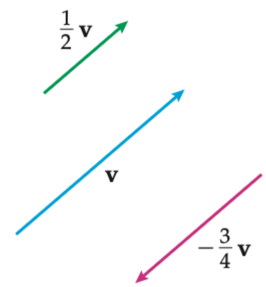



There are two basic operations that we can perform on vectors: addition and scalar multiplication:

- The *sum* of $\mathbf{v_1} = \langle x_1, y_1\rangle$ and $\mathbf{v_2} = \langle x_2, y_2\rangle$ is the vector
$$\mathbf{v_1} + \mathbf{v_2} = \langle x_1 + x + 2, y_1 + y_2\rangle.$$
- The *scalar multiple* of $\mathbf{v} = \langle x, y\rangle$ by a real number $c$ is the vector
$$c\mathbf{v} = \langle cx, xy\rangle.$$

Each operation can also be interpreted geometrically:




Note that vectors can also be subtracted. The expressions

$$\mathbf{v_1} - \mathbf{v_2}, \qquad \mathbf{v_1} + (-\mathbf{v_2}), \qquad \text{and} \qquad \mathbf{v_1} + (-1)\mathbf{v_2}$$

all mean the same thing.

**In-class exercise**[2]**:** Give two different arguments that vector addition is commutative, i.e. that $\mathbf{v_1} + \mathbf{v_2} = \mathbf{v_2} + \mathbf{v_1}$.

Here is a summary of the basic algebraic properties of vectors:

### Algebraic Properties of Vectors

**(a)** *Vector addition is commutative:*

For any two vectors **u** and **v** with the same number of components,

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

**(b)** *Vector addition is associative:*

For any three vectors **u**, **v**, and **w**, each with the same number of components,

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}).$$

**(c)** *Scalar multiplication distributes over vector addition:*

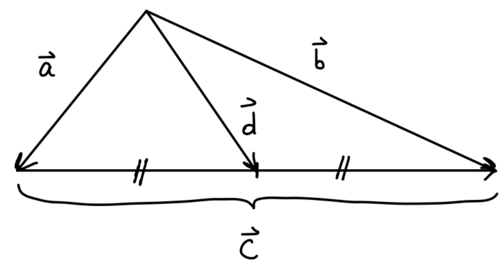
For any scalar $c$ and any two vectors **u** and **v** with the same number of components,

$$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}.$$

**In-class exercise**[3]**:** Using the diagram below, express **c** and **d** in terms of **a** and **b** only. (The slashes indicate that the segments they belong to have the same length.)

[2] Algebraic argument: $\langle x_1, y_1\rangle + \langle x_2, y_2\rangle = \langle x_1 + x_2, y_1 + y_2\rangle = \langle x_2 + x_1, y_2 + y_1\rangle = \langle x_2, y_2\rangle + \langle x_1, y_1\rangle$. Geometric argument: Draw the vector sums $\mathbf{v_1} + \mathbf{v_2}$ and $\mathbf{v_2} + \mathbf{v_1}$, with both sums starting from the origin. Together they form a parallelogram. Draw in the diagonal from the origin. This vector represents both sums, so they're equal.

[3] $\mathbf{c} = \mathbf{b} - \mathbf{a}$, $\mathbf{d} = \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a})$



$\vec{a}$
$\vec{b}$
$\vec{d}$
$\vec{c}$

# Day 3

## 1. Vectors in $\mathbb{R}^3$

Last time we saw vectors in $\mathbb{R}^2$. They were defined as ordered pairs of numbers but can also be thought of as arrows in the plane. Vectors in $\mathbb{R}^3$ are exactly what you would expect: ordered triples of numbers, like

$$\mathbf{v} = \langle 2, 3, 1 \rangle.$$

Equivalently, they can be thought of as arrows in $\mathbb{R}^3$. For example, the vector $\mathbf{v} = \langle 2, 3, 1 \rangle$ is (any translation of) the arrow extending from the origin to the point $(2, 3, 1)$. Vector addition and scalar multiplication also are defined in the same way as for vectors in $\mathbb{R}^2$. For example,

$$\langle 2, 3, 1 \rangle + \langle 1, 2, 3 \rangle = \langle 3, 5, 4 \rangle \qquad \text{and} \qquad 7\langle 2, 3, 1 \rangle = \langle 14, 21, 7 \rangle.$$

In a similar fashion, vectors can be defined in $\mathbb{R}^n$ for any integer $n \geq 1$. In this course, we'll mostly work in $\mathbb{R}^2$ and $\mathbb{R}^3$.

## 2. Length of a vector

If we think of a vector as an arrow starting from the origin, then its length should be the distance between the origin and its terminal point. This is exactly the definition:

**The Magnitude, Norm, or Length of a Vector**

The ***magnitude***, ***norm***, or ***length*** of a vector $\mathbf{v}$ is denoted by $\|\mathbf{v}\|$.
In $\mathbb{R}^2$, when $\mathbf{v} = \langle a, b \rangle$,

$$\|\mathbf{v}\| = \sqrt{a^2 + b^2}.$$

In $\mathbb{R}^3$, when $\mathbf{v} = \langle a, b, c \rangle$,

$$\|\mathbf{v}\| = \sqrt{a^2 + b^2 + c^2}.$$

A nice fact (that you should check for yourself) is that for every vector $\mathbf{v}$ and every real number $c$, we have $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$. This means, for example, that if I multiply a vector by 7, then its length increases by factor of 7.

A vector whose length is 1 is called a *unit vector*. If $\mathbf{v}$ is a vector whose length is nonzero, then we can scale $\mathbf{v}$ to get a unit vector $\hat{\mathbf{v}}$ that points in the same direction as $\mathbf{v}$:

$$\hat{\mathbf{v}} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

The unit vectors that point in the positive directions of the coordinate axes are special; they're known as the *standard basis vectors*.

**Standard Basis Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$**

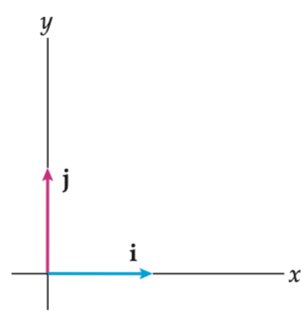
The ***standard basis vectors*** in $\mathbb{R}^2$ are

$$\mathbf{i} = \langle 1, 0\rangle \text{ and } \mathbf{j} = \langle 0, 1\rangle.$$

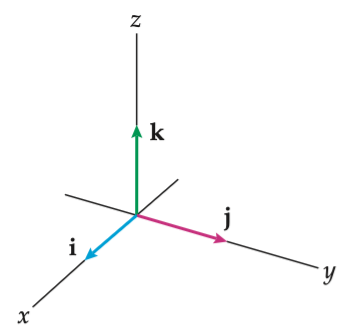
The ***standard basis vectors*** in $\mathbb{R}^3$ are

$$\mathbf{i} = \langle 1, 0, 0\rangle, \ \mathbf{j} = \langle 0, 1, 0\rangle, \text{ and } \mathbf{k} = \langle 0, 0, 1\rangle.$$

*The standard basis vectors in $\mathbb{R}^2$*




*The standard basis vectors in $\mathbb{R}^3$*




Every vector can be expressed (uniquely) as a combination of standard basis vectors. Specifically, if $\mathbf{v} = \langle a, b, c\rangle$, then $\mathbf{v} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$. A similar statement holds for vectors in $\mathbb{R}^2$.

One other special vector is the *zero vector*, the vector $\mathbf{0} = \langle 0, 0\rangle$ (in $\mathbb{R}^2$) or $\mathbf{0} = \langle 0, 0, 0\rangle$ (in $\mathbb{R}^3$). This is the only vector whose length is 0.

## 3. Dot product

We've seen how vectors can be viewed both algebraically (as tuples of numbers) and geometrically (as arrows). Continuing this theme, we now introduce the dot product, which is an algebraic operation that provides geometric information about two vectors. Here's the definition: Let $\mathbf{v_1} = \langle x_1, y_1, z_1\rangle$ and $\mathbf{v_2} = \langle x_2, y_2, z_2\rangle$. The *dot product* of $\mathbf{v_1}$ and $\mathbf{v_2}$ is

$$\mathbf{v_1} \cdot \mathbf{v_2} = x_1 x_2 + y_1 y_2 + z_1 z_2.$$

For example, if $\mathbf{v_1} = \langle 2, 3, 1\rangle$ and $\mathbf{v_2} = \langle 1, 2, 3\rangle$, then their dot product is

$$\mathbf{v_1} \cdot \mathbf{v_2} = 2 \cdot 1 + 3 \cdot 2 + 1 \cdot 3 = 11.$$

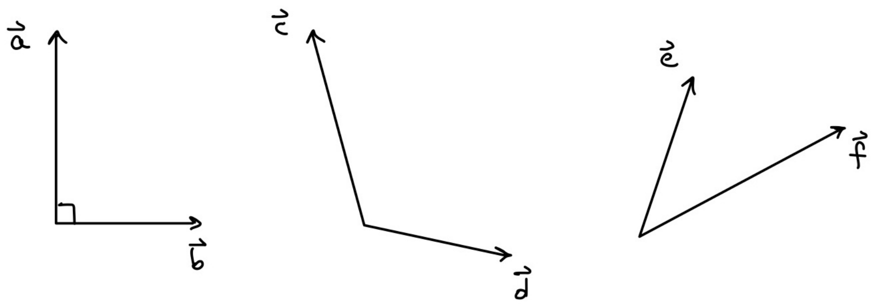
There is an analogous definition for vectors in $\mathbb{R}^2$ (or $\mathbb{R}^n$ more generally). Note that the dot product is only defined for vectors with the same number of components.

The key geometric property of the dot product is that it can tell us the angle between two vectors. More specifically, let $\mathbf{v_1}$ and $\mathbf{v_2}$ be vectors. If either $\mathbf{v_1}$ or $\mathbf{v_2}$ is the zero vector, then $\mathbf{v_1} \cdot \mathbf{v_2} = 0$. Otherwise,

$$\mathbf{v_1} \cdot \mathbf{v_2} = \|\mathbf{v_1}\|\|\mathbf{v_2}\| \cos\theta,$$

where $\theta$ is the (smallest) angle between $\mathbf{v_1}$ and $\mathbf{v_2}$. The proof is based on the Law of Cosines, which we'll see soon.

**In-class exercise**[1]**:** Using the figure below, arrange the dot products $\mathbf{a} \cdot \mathbf{b}$, $\mathbf{c} \cdot \mathbf{d}$, and $\mathbf{e} \cdot \mathbf{f}$ in increasing order.




Consequences of the cosine formula for the dot product:

- The angle between $\mathbf{v_1}$ and $\mathbf{v_2}$ is acute if $\mathbf{v_1} \cdot \mathbf{v_2} < 0$, a right angle if $\mathbf{v_1} \cdot \mathbf{v_2} = 0$, and obtuse if $\mathbf{v_1} \cdot \mathbf{v_2} > 0$.
- In particular, $\mathbf{v_1}$ and $\mathbf{v_2}$ are *orthogonal* (or *perpendicular*) if $\mathbf{v_1} \cdot \mathbf{v_2} = 0$. In fact, this is the definition of orthogonality.
- Cauchy–Schwarz inequality: $|\mathbf{v_1} \cdot \mathbf{v_2}| \leq \|\mathbf{v_1}\|\|\mathbf{v_2}\|$, or equivalently, $-\|\mathbf{v_1}\|\|\mathbf{v_2}\| \leq \mathbf{v_1} \cdot \mathbf{v_2} \leq \|\mathbf{v_1}\|\|\mathbf{v_2}\|$.
- The angle between $\mathbf{v_1}$ and $\mathbf{v_2}$ can be calculated as
$$\theta = \arccos\left(\frac{\mathbf{v_1} \cdot \mathbf{v_2}}{\|\mathbf{v_1}\|\|\mathbf{v_2}\|}\right).$$

The dot product has the algebraic properties you might expect (commutativity, distributivity across vector addition, etc.) Here's a list from the textbook:

**Algebraic Properties of the Dot Product**

For any vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, and any scalar $k$,

**(a)** $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

**(b)** $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

**(c)** $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$

**(d)** $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$

Property (d) may look less familiar, but it's very useful!

[1] $\mathbf{e} \cdot \mathbf{f} < \mathbf{a} \cdot \mathbf{b} < \mathbf{c} \cdot \mathbf{d}$