

## Chapter 1: First-Order Equations

### 1.3 Separable Equations & Applications

#### Steps to Solve Separable Equations:

1. **Separate variables:** Rewrite the differential equation in the form  $\frac{dy}{dx} = g(x)h(y)$ , then separate the variables as  $\frac{1}{h(y)}dy = g(x)dx$ .
2. **Integrate both sides:** Integrate both sides separately. The result will typically involve a constant of integration  $C$ .
3. **Solve for  $y$ :** If possible, solve the resulting equation for  $y$ .
4. **Apply initial condition (if provided):** Use the initial condition to solve for the constant  $C$ .

**Example Problem:** Solve  $\frac{dy}{dx} = 2xy$  with initial condition  $y(0) = 1$ .

**Solution:**

1. Rewrite the equation:  $\frac{1}{y}dy = 2xdx$ .
2. Integrate:  $\int \frac{1}{y}dy = \int 2xdx$ , so  $\ln|y| = x^2 + C$ .
3. Exponentiate both sides:  $y = e^{x^2+C} = Ce^{x^2}$ .
4. Use the initial condition  $y(0) = 1$ :  $1 = Ce^0$ , so  $C = 1$ .
5. Final solution:  $y = e^{x^2}$ .

### 1.2 Equilibrium and Stability

**Equilibrium Points:** Equilibrium points are the values where  $\frac{dy}{dx} = 0$ . These points are classified based on the behavior of nearby solutions:

- **Source:** If solutions move away from the equilibrium point as time increases, the point is called a *source*.
- **Sink:** If solutions move toward the equilibrium point, it is a *sink*.
- **Node:** Solutions either approach or move away in a non-oscillatory manner, depending on whether the node is stable (sink) or unstable (source).

#### Stability Classification:

- **Stable:** Nearby solutions approach the equilibrium point.
- **Unstable:** Nearby solutions move away from the equilibrium.
- **Semi-stable:** Solutions may approach from one side and move away from the other.

#### Steps to Determine Stability:

1. **Find equilibrium points:** Set  $\frac{dy}{dx} = 0$  and solve for  $y$ .
2. **Analyze the sign of  $\frac{dy}{dx}$ :** Determine how the solutions behave near each equilibrium point by analyzing the sign of  $\frac{dy}{dx}$  on either side of the equilibrium.

**Example Problem:** Find and classify the equilibrium points for  $\frac{dy}{dx} = y(2 - y)$ .

**Solution:**

1. Set  $y(2 - y) = 0$ , giving equilibrium points at  $y = 0$  and  $y = 2$ .
2. Analyze  $\frac{dy}{dx}$ :
  - For  $y < 0$ ,  $\frac{dy}{dx} > 0$ , so solutions move toward  $y = 0$ .
  - For  $0 < y < 2$ ,  $\frac{dy}{dx} > 0$ , so solutions move toward  $y = 2$ .
  - For  $y > 2$ ,  $\frac{dy}{dx} < 0$ , so solutions move toward  $y = 2$ .
3. Conclusion:  $y = 0$  is unstable, and  $y = 2$  is stable (sink).

## 1.4 Linear Equations & Applications

### Steps to Solve First-Order Linear Equations:

1. **Standard form:** Write the equation as  $y' + p(x)y = q(x)$ .
2. **Find the integrating factor:** Calculate the integrating factor  $I(x) = e^{\int p(x)dx}$ .
3. **Multiply through by the integrating factor:** Multiply both sides of the equation by  $I(x)$ .
4. **Integrate:** The left-hand side becomes  $(I(x)y)'$ , allowing you to integrate both sides.
5. **Solve for  $y$ :** Solve for  $y$ , and apply initial condition if given.

**Example Problem:** Solve  $y' + 3y = xe^{3x}$ .

**Solution:**

1. Already in standard form:  $y' + 3y = xe^{3x}$ .
2. Find the integrating factor:  $I(x) = e^{\int 3dx} = e^{3x}$ .
3. Multiply by  $I(x)$ :  $e^{3x}y' + 3e^{3x}y = xe^{6x}$ .
4. The left side becomes  $(e^{3x}y)'$ , so integrate both sides:

$$\int (e^{3x}y)' dx = \int xe^{6x} dx.$$

5. Solve the integral by parts and write the general solution.

## 1.5 Bernoulli and Exact Equations

### Steps to Solve Bernoulli Equations:

1. **Standard form:** Write the equation as  $y' + p(x)y = q(x)y^n$ .
2. **Substitution:** Let  $v = y^{1-n}$ , and differentiate to get  $v'$ .
3. **Transform into a linear equation:** Substitute  $v$  and  $v'$  into the original equation, turning it into a first-order linear equation in  $v$ .
4. **Solve for  $v$ :** Use the integrating factor method.
5. **Back-substitute for  $y$ :** Solve for  $y$  in terms of  $v$ .

**Example Problem (Bernoulli):** Solve  $y' + y = y^2$ .

**Solution:**

1. Standard form:  $y' + y = y^2$ , where  $n = 2$ .
2. Substitution: Let  $v = y^{-1}$ , then  $v' = -y^{-2}y'$ .
3. Transform into the linear equation:  $v' - v = -1$ .
4. Solve for  $v$ , then substitute back  $y = \frac{1}{v}$ .

### Steps to Solve Exact Equations:

1. **Check for exactness:** For an equation of the form  $M(x, y)dx + N(x, y)dy = 0$ , the equation is exact if  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ .
2. **Find the potential function  $\Psi(x, y)$ :** Solve  $\frac{\partial \Psi}{\partial x} = M(x, y)$  and  $\frac{\partial \Psi}{\partial y} = N(x, y)$ .
3. **Solve for the solution:** The solution is  $\Psi(x, y) = C$ , where  $C$  is a constant.

**Example Problem (Exact Equation):** Solve  $(2xy + y^2)dx + (x^2 + 2xy)dy = 0$ .

**Solution:**

1. Check for exactness:  $M(x, y) = 2xy + y^2$  and  $N(x, y) = x^2 + 2xy$ . We have:

$$\frac{\partial M}{\partial y} = 2x + 2y, \quad \frac{\partial N}{\partial x} = 2x + 2y$$

Since  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ , the equation is exact.

2. Find the potential function  $\Psi(x, y)$ :

$$\frac{\partial \Psi}{\partial x} = M(x, y) = 2xy + y^2 \Rightarrow \Psi(x, y) = x^2y + xy^2 + h(y)$$

3. Differentiate with respect to  $y$ :

$$\frac{\partial \Psi}{\partial y} = x^2 + 2xy + h'(y)$$

Equate with  $N(x, y) = x^2 + 2xy$ , so  $h'(y) = 0$ , which implies  $h(y) = C$ .

4. Final solution:  $x^2y + xy^2 = C$ .

## Mixing Problems

### Steps to Solve Mixing Problems:

1. Let  $x(t)$  denote the amount of solute in the tank at time  $t$ .
2. The rate of change of the solute is given by the equation:

$$\frac{dx}{dt} = \text{Rate of solute in} - \text{Rate of solute out.}$$

3. Set up the equation for  $\frac{dx}{dt}$  based on the inflow and outflow rates and solve the resulting differential equation.

**Example Problem:** A tank initially contains 100 liters of water with 10 grams of salt. Brine containing 5 g/L of salt enters the tank at 4 L/min, and the well-mixed solution is drained at 2 L/min. Find the amount of salt after 10 minutes.

**Solution:**

1. Let  $x(t)$  be the amount of salt in the tank at time  $t$ .
2. Rate in:  $4 \times 5 = 20$  g/min. Rate out:  $\frac{x(t)}{V(t)} \times 2$ , where  $V(t) = 100 + 2t$ .
3. Set up the equation:

$$\frac{dx}{dt} = 20 - \frac{2x(t)}{100 + 2t}.$$

4. Solve the differential equation using an integrating factor.

## Newton's Law of Cooling

### Steps to Solve Newton's Law of Cooling:

1. The rate of change of temperature is proportional to the difference between the object's temperature and the ambient temperature:

$$\frac{dT}{dt} = -k(T - T_{\text{ambient}}).$$

2. Separate variables and integrate to solve for  $T(t)$ .
3. Apply initial conditions to solve for constants.

**Example Problem:** A cup of coffee at 90°C is left in a room at 20°C. After 10 minutes, the coffee is 70°C. When will it be 50°C?

**Solution:**

1. Set up the equation:

$$\frac{dT}{dt} = -k(T - 20).$$

2. Separate variables and integrate to get  $T(t) = 20 + Ce^{-kt}$ .
3. Use the initial condition  $T(0) = 90$  to find  $C$ , and use  $T(10) = 70$  to solve for  $k$ .
4. Solve for  $t$  when  $T(t) = 50$ .

# Chapter 2: Second-Order Equations

## 2.1 Introduction

### Steps to Solve Second-Order Homogeneous Equations:

1. **Write the characteristic equation:** For  $ay'' + by' + cy = 0$ , solve the characteristic equation  $ar^2 + br + c = 0$ .
2. **Find the roots:**
  - Two distinct real roots:  $y = C_1e^{r_1x} + C_2e^{r_2x}$ .
  - Repeated root:  $y = (C_1 + C_2x)e^{rx}$ .
  - Complex roots:  $y = e^{\alpha x}(C_1 \cos(\beta x) + C_2 \sin(\beta x))$ , where  $r = \alpha \pm i\beta$ .

**Example Problem:** Solve  $y'' - 5y' + 6y = 0$ .

**Solution:**

1. Characteristic equation:  $r^2 - 5r + 6 = 0$ .
2. Solve for  $r$ : The roots are  $r = 2$  and  $r = 3$ .
3. General solution:  $y = C_1e^{2x} + C_2e^{3x}$ .

### Steps to Solve Non-Homogeneous Equations:

1. **Solve the homogeneous equation:** Solve the homogeneous equation  $ay'' + by' + cy = 0$  to find the complementary solution  $y_c(x)$ .
  - **Write the characteristic equation:** Solve the characteristic equation  $ar^2 + br + c = 0$ .
  - **Find the roots:**
    - Two distinct real roots:  $y_c(x) = C_1e^{r_1x} + C_2e^{r_2x}$ .
    - Repeated root:  $y_c(x) = (C_1 + C_2x)e^{rx}$ .
    - Complex roots:  $y_c(x) = e^{\alpha x}(C_1 \cos(\beta x) + C_2 \sin(\beta x))$ , where  $r = \alpha \pm i\beta$ .
2. **Find the particular solution:** Based on the form of the non-homogeneous term  $f(x)$ , guess the form of the particular solution  $y_p(x)$ .
  - If  $f(x)$  is a polynomial, assume  $y_p(x)$  is a polynomial of the same degree.
  - If  $f(x) = e^{ax}$ , assume  $y_p(x) = Ae^{ax}$ .
  - If  $f(x) = \sin(bx)$  or  $\cos(bx)$ , assume  $y_p(x) = A \cos(bx) + B \sin(bx)$ .
  - If  $f(x) = e^{ax} \sin(bx)$  or  $e^{ax} \cos(bx)$ , assume  $y_p(x) = e^{ax}(A \cos(bx) + B \sin(bx))$ .
3. **Substitute  $y_p(x)$  into the original equation:** Plug the particular solution into the non-homogeneous equation to solve for the unknown coefficients.
4. **Form the general solution:** The general solution is the sum of the complementary solution and the particular solution:
$$y(x) = y_c(x) + y_p(x)$$
5. **Solve for constants using initial conditions:** If initial conditions are provided, substitute them into the general solution to solve for the constants  $C_1$  and  $C_2$ .

**Example Problem:** Solve  $y'' + 3y' + 2y = 3x^2 + 5x + 7$ .

**Solution:**

1. **Solve the homogeneous equation:**
  - Characteristic equation:  $r^2 + 3r + 2 = 0$ .
  - Solve for  $r$ : The roots are  $r = -1$  and  $r = -2$ .
  - Complementary solution:  $y_c(x) = C_1e^{-x} + C_2e^{-2x}$ .
2. **Guess the particular solution:** Since the non-homogeneous term  $f(x) = 3x^2 + 5x + 7$  is a polynomial of degree 2, assume:

$$y_p(x) = Ax^2 + Bx + C$$

### 3. Substitute $y_p(x)$ into the original equation:

$$y_p'(x) = 2Ax + B, \quad y_p''(x) = 2A$$

Substitute into  $y'' + 3y' + 2y = 3x^2 + 5x + 7$ :

$$2A + 3(2Ax + B) + 2(Ax^2 + Bx + C) = 3x^2 + 5x + 7$$

Expand and collect like terms:

$$(2A) + (6A + 2B)x + (2A + 3B + 2C) = 3x^2 + 5x + 7$$

Solve for  $A$ ,  $B$ , and  $C$ :

- Coefficient of  $x^2$ :  $2A = 3 \Rightarrow A = \frac{3}{2}$
- Coefficient of  $x$ :  $6A + 2B = 5 \Rightarrow 6\left(\frac{3}{2}\right) + 2B = 5 \Rightarrow B = -2$
- Constant term:  $2A + 3B + 2C = 7 \Rightarrow 2\left(\frac{3}{2}\right) + 3(-2) + 2C = 7 \Rightarrow C = 5$

Therefore, the particular solution is:

$$y_p(x) = \frac{3}{2}x^2 - 2x + 5$$

### 4. General solution:

$$y(x) = C_1e^{-x} + C_2e^{-2x} + \frac{3}{2}x^2 - 2x + 5$$

## Wronskian and Linearity

**Wronskian Determinant:** The Wronskian is used to determine the linear independence of two solutions to a second-order differential equation. For solutions  $y_1(x)$  and  $y_2(x)$ , the Wronskian is given by:

$$W(y_1, y_2) = y_1y_2' - y_1'y_2.$$

- If  $W(y_1, y_2) \neq 0$  for all  $x$ , the solutions are linearly independent.
- If  $W(y_1, y_2) = 0$ , the solutions are linearly dependent.

### Steps to Check Linearity:

1. Find the general solutions  $y_1$  and  $y_2$ .
2. Compute the Wronskian determinant  $W(y_1, y_2)$ .
3. If  $W \neq 0$ , the functions are linearly independent, and the general solution is a linear combination of the two.

**Example:** Given solutions  $y_1 = e^x$  and  $y_2 = e^{-x}$  for the equation  $y'' - y = 0$ , find whether the solutions are linearly independent.

**Solution:**

$$W(y_1, y_2) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2.$$

Since  $W \neq 0$ , the solutions are linearly independent.

## 2.4 Free and Forced Mechanical Vibrations

**Free Vibrations:** The motion of a system is governed by:

$$mx'' + cx' + kx = 0$$

where  $m$  is the mass,  $c$  is the damping coefficient, and  $k$  is the spring constant.

- **Undamped motion** ( $c = 0$ ):

$$mx'' + kx = 0$$

Characteristic equation:  $r^2 + \frac{k}{m} = 0$ . Solution:  $x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)$ , where  $\omega = \sqrt{\frac{k}{m}}$ .

- **Damped motion** ( $c \neq 0$ ): The general solution depends on whether the system is underdamped, overdamped, or critically damped, based on the discriminant  $c^2 - 4mk$ .

**Forced Vibrations:** The motion is governed by:

$$mx'' + cx' + kx = F(t)$$

where  $F(t)$  is the external force. The solution is the sum of the complementary solution (solving the homogeneous equation) and a particular solution, often found using **undetermined coefficients** or **variation of parameters**.

## 2.5 Method of Undetermined Coefficients

If

$$ay'' + by' + cy = A(x)$$

where:

1.  $A(x)$  is a polynomial in  $x$ , then the particular solution  $y_p$  is:

$$y_p = x^k \times (\text{a general polynomial of the same degree})$$

where  $k$  is the number of times that 0 is a root of the characteristic equation.

2. If  $A(x) = e^{ax}$ , then:

$$y_p = x^k e^{ax} \times (\text{a general polynomial of the same degree})$$

where  $k$  is the number of times that  $a$  is a root of the characteristic equation.

3. If  $A(x) = e^{ax} \cos(bx)$  or  $A(x) = e^{ax} \sin(bx)$ , then:

$$y_p = x^k e^{ax} [(\text{polynomial of the same degree}) \cos(bx) + (\text{another polynomial of the same degree}) \sin(bx)]$$

where  $k$  is the number of times that  $a + bi$  are roots of the characteristic equation.

## 2.6 Variation of Parameters

**Steps for Variation of Parameters:**

1. **Solve the complementary equation** to get the general solution  $y_c = C_1 y_1(x) + C_2 y_2(x)$ .
2. **Assume a particular solution** of the form:

$$y_p = v_1(x) y_1(x) + v_2(x) y_2(x)$$

where  $v_1(x)$  and  $v_2(x)$  are functions to be determined.

3. **Solve for  $v_1(x)$  and  $v_2(x)$**  using Wronskian matrix:

$$v_1 = - \int \frac{y_2 f(x)}{W(y_1, y_2)} dx, \quad v_2 = \int \frac{y_1 f(x)}{W(y_1, y_2)} dx$$

4. **Integrate** to find  $v_1(x)$  and  $v_2(x)$ .
5. **Find the particular solution:** Use  $y_p = v_1(x) y_1(x) + v_2(x) y_2(x)$ .