Day 12

1. Calculating curvature

We saw last time that if $\mathbf{r}(s)$ is a parametrization of a curve C by arc length, then the curvature of C is defined to be

$$\kappa(s) = \|\mathbf{r}''(s)\|.$$

We used this to find the curvature of a circle of radius R. In practice, however, it's often not easy to work directly from the definition of curvature. For one thing, it requires you to find an arc length parametrization of the curve. While such a parametrization will always exist, it's not always possible to express it analytically. Fortunately, there are other formulas for curvature that do not involve arc length. Here are a few:

Formulas for the Curvature of a Space Curve

Let C be the graph of a twice-differentiable vector function $\mathbf{r}(t)$ defined on an interval I with unit tangent vector $\mathbf{T}(t)$. Then the curvature κ of C at a point on the curve is given by

(a) $\kappa = \dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$.

(b) $\kappa = \dfrac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$.

Note that given a vector function $\mathbf{r}(t)$, its unit tangent vector function $\mathbf{T}(t)$ is defined by $\mathbf{T}(t) = \frac{1}{\|\mathbf{r}'(t)\|}\mathbf{r}'(t)$.

Formulas for Curvature in the Plane

(a) Let $y = f(x)$ be a twice-differentiable function. Then the curvature of the graph of f is given by

$$\kappa = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}.$$

(b) Let C be the graph of a vector function $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ in the xy-plane, where x and y are twice-differentiable functions of t such that $x'(t)$ and $y'(t)$ are not simultaneously zero. Then the curvature κ of C at a point on the curve is given by

$$\kappa = \frac{|x'(t)y''(t) - x''(t)y'(t)|}{((x'(t))^2 + (y'(t))^2)^{3/2}}.$$

Let's return to the example of the parabola given by $y = x^2$. Since it's the graph of $f(x) = x^2$, we can calculate its curvature as

$$\kappa(x) = \frac{2}{(1 + 4x^2)^{3/2}}.$$

This agrees with our intuition that the curvature of the parabola is highest at the origin $(x = 0)$.

2. Osculating circle

As an application of curvature, let's look into how can find the circle that best approximates a curve at a given point. This is known as the *osculating circle*. Here's the definition:

Radius of Curvature and Osculating Circle

Let C be the graph of a vector function $\mathbf{r}$, and let $\mathbf{r}(t_0)$ be the position vector for a point on C at which the curvature $\kappa > 0$.

(a) The ***radius of curvature*** ρ of C at $\mathbf{r}(t_0)$ is given by

$$\rho = \frac{1}{\kappa}.$$

(b) The ***osculating circle*** to the curve C at $\mathbf{r}(t_0)$ is the circle in the osculating plane, with radius ρ, and whose center is the terminal point of the position vector $\mathbf{r}(t_0) + \rho\mathbf{N}(t_0)$.

Note that $\mathbf{N}(t)$ is defined as $\mathbf{N}(t) = \frac{1}{\|\mathbf{T}'(t)\|}\mathbf{T}'(t)$, where $\mathbf{T}(t)$ is the unit tangent vector function defined above. The normal vector $\mathbf{N}(t)$ will point toward the center of the osculating circle.

As an example, let's find the equation for the osculating circle at the point $(\frac{5\pi}{4}, -\frac{1}{\sqrt{2}})$ belonging to the graph of $y = \sin(x)$. The curvature of the graph at $x = \frac{5\pi}{4}$ is given by

$$\kappa = \frac{|-\sin(x)|}{(1+\cos(x)^2)^{3/2}}\bigg|_{x=\frac{5\pi}{4}} = \frac{1/\sqrt{2}}{(3/2)^{3/2}} = \frac{2}{3\sqrt{3}},$$

and thus the radius of the osculating circle is

$$\rho = \frac{3\sqrt{3}}{2}.$$

To find the center, we need to first find the appropriate normal direction. Using the parametrization $\mathbf{r}(t) = \langle t, \sin(t)\rangle$, we have

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\langle 1, \cos(t)\rangle}{\sqrt{1+\cos(t)^2}}.$$

By the quotient rule,

$$\mathbf{T}'(t) = \frac{\sqrt{1+\cos(t)^2}\langle 0, -\sin(t)\rangle - \langle 1, \cos(t)\rangle\frac{1}{2}(1+\cos(t)^2)^{-1/2}2\cos(t)(-\sin(t))}{1+\cos(t)^2}.$$

Plugging in $t = \frac{5\pi}{4}$ and simplifying, we find that

$$\mathbf{T}'\Big(\frac{5\pi}{4}\Big) = \Big\langle \frac{\sqrt{2}}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\Big\rangle$$

and thus

$$\mathbf{N}\Big(\frac{5\pi}{4}\Big) = \frac{\mathbf{T}'(\frac{5\pi}{4})}{\|\mathbf{T}'(\frac{5\pi}{4})\|} = \Big\langle \frac{1}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}\Big\rangle.$$

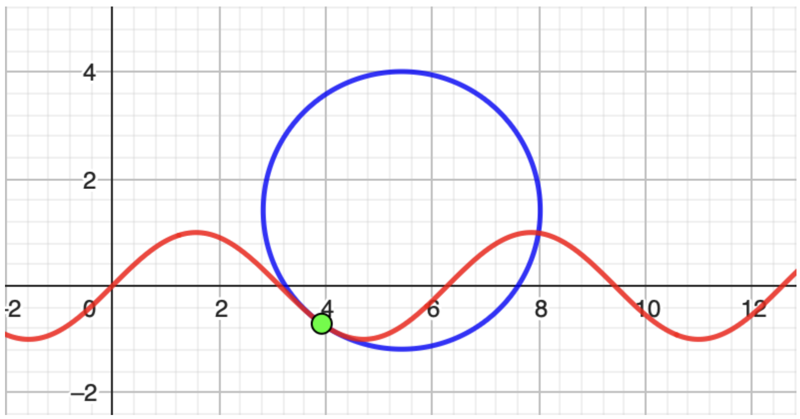
So the center of the osculating circle is the terminal point of the position vector

$$\mathbf{r}\Big(\frac{5\pi}{4}\Big) + \rho\mathbf{N}\Big(\frac{5\pi}{4}\Big) = \Big\langle \frac{5\pi}{4}, -\frac{1}{\sqrt{2}} \Big\rangle + \frac{3\sqrt{3}}{2}\Big\langle \frac{1}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}} \Big\rangle = \Big\langle \frac{5\pi+6}{4}, \sqrt{2} \Big\rangle.$$

Now that we know the radius and center, we can write down the equation:

$$\Big(x - \frac{5\pi+6}{4}\Big)^2 + (y-\sqrt{2})^2 = \frac{27}{4}.$$

Here's a plot from GeoGebra:

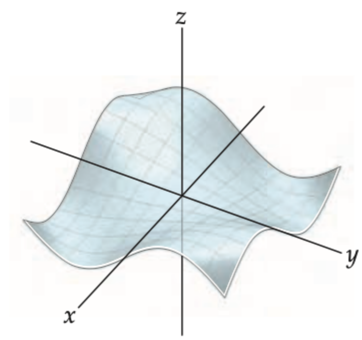


Day 13

1. Multivariable functions

Today we transition to the study of multivariable functions, the next unit of this course. Here "multivariable" refers to multiple independent variables (inputs), such as $f(x, y) = x^2 + y^2$ or $f(x, y, z) = 1 - \cos(xy + z)^3$. We've already seen vector functions, which have multiple dependent variables.

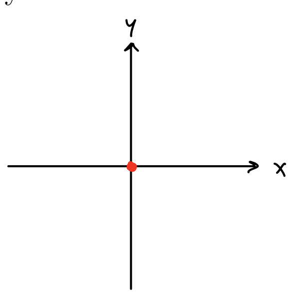
How can we visualize the graph of a multivariable function? As with ordinary functions, the graph of $f(x, y)$ is the set of pairs of inputs and corresponding outputs. Each input-output pair actually forms a triple of numbers, $(x, y, f(x, y))$. So the graph will be a surface in $\mathbb{R}^3$:

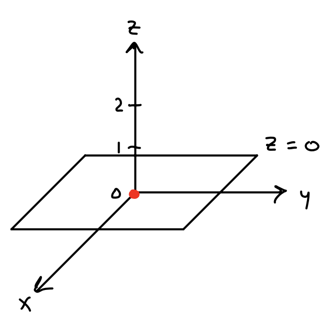


Let's see a familiar example: $f(x, y) = 2x+3y-5$. Why is this familiar? If we denote $f(x, y)$ by z, then we see that the graph of f is the set of all points (x, y, z) such that $z = 2x+3y-5$, or equivalently $2x+3y-z = 5$. This is an equation for a plane with normal vector $\langle 2, 3, -1\rangle$.

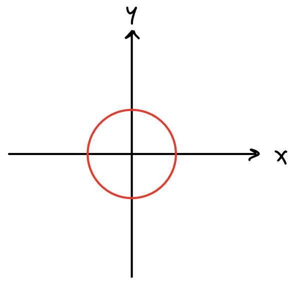
Let's consider another example: $f(x, y) = x^2+y^2$. Again, we can set $z = f(x, y)$ and consider the equation $z = x^2 + y^2$. What kind of shape do we get? The technique of *slicing* is often helpful for visualizing a surface. For various values of k, let's consider the intersection of the graph with the plane $z = k$.

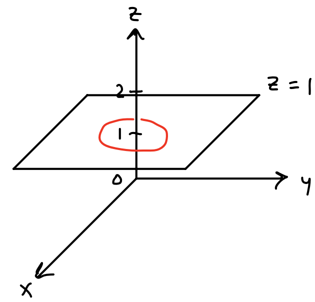
- When $k = 0$, the intersection is represented by the equation $0 = x^2 + y^2$. There's only one solution, $(x, y) = (0, 0)$. This means that the graph touches the plane $z = 0$ at the origin only:



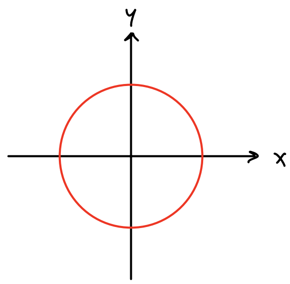


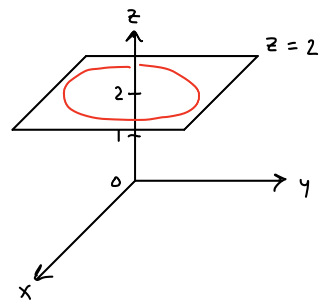
- When $k = 1$, the intersection is represented by $1 = x^2 + y^2$. The solutions form a circle of radius 1 centered at $(0, 0)$. So the intersection with the plane $z = 1$ looks like this:



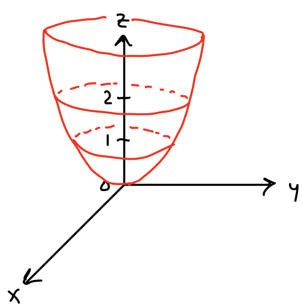


- When $k = 2$, the intersection is given by $2 = x^2 + y^2$, which describes a circle of radius $\sqrt{2}$ centered at $(0, 0)$. Therefore the intersection with the plane $z = 2$ looks like this:





You get the idea. We can stack these slices together to form a complete picture of the graph:



This surface is called a *paraboloid*. It can also be obtained by rotating an ordinary parabola about the z-axis.

In-class exercise[1]: Let $f(x, y) = \sqrt{x^2 + y^2}$. Sketch the graph of f. What kind of shape do you get?

Here are a few definitions you should know:

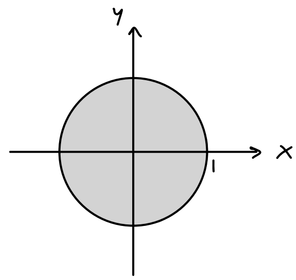
- The *domain* of a function f is the set of points at which f is defined. If f is a function of two variables, then the domain of f is a subset of $\mathbb{R}^2$. If f is a function of three variables, then the domain of f is a subset of $\mathbb{R}^3$.
- The *range* of a function f is the set of values that f outputs. If f outputs real numbers, then the range is a subset of $\mathbb{R}$.
- The *graph* of a function f of two variables is the set of all triples $(x, y, f(x, y))$, where (x, y) belongs to the domain of f. If f is a function of three variables, then its graph

[1] A cone, or more specifically the upper half of a cone

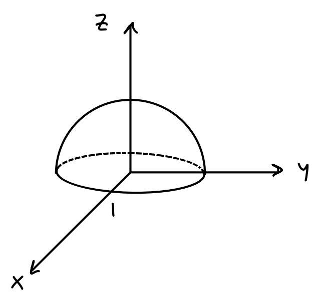
is the set of all quadruples $(x, y, z, f(x, y, z))$, where (x, y, z) belongs to the domain of f. (Note the graph of a three-variable function is a subset of $\mathbb{R}^4$!)

Let's consider the function $f(x, y) = x^2 + y^2$ once more. Its domain is the entirety of $\mathbb{R}^2$, since the expression $x^2 + y^2$ makes sense for any point (x, y). Its range is $[0, \infty)$, the set of nonnegative real numbers. We saw that its graph is a paraboloid.

What about the function $f(x, y) = \sqrt{1 - x^2 - y^2}$? Its domain is the set of all (x, y) such that $x^2 + y^2 \leq 1$. This is the unit disc centered at the origin, including the boundary:



Its range is the interval $[0, 1]$. To visualize the graph of f, set $z = f(x, y)$ and consider the equation $z = \sqrt{1 - x^2 - y^2}$. If we square both sides and rearrange, we get $x^2 + y^2 + z^2 = 1$, which you'll recognize as the equation for the unit sphere centered at the origin. But remember, z is nonnegative because it was defined as a square root. So the graph of f is the upper half of the unit sphere:



Day 14

1. Level curves and surfaces

You're probably aware that not every curve in $\mathbb{R}^2$ is a graph of a function $f(x)$. For example, there is no function $f(x)$ whose graph is the unit circle. (The unit circle does not pass the vertical line test.) The general form of an equation for a curve in $\mathbb{R}^2$ is $f(x, y) = k$, where k is a constant. For example, if $f(x, y) = x^2 + y^2$, then the unit circle centered at the origin is given by $f(x, y) = 1$. The set of points (x, y) that satisfy the equation $f(x, y) = k$ is called the *level curve at height* k for the function f.

We've already seen that the level curves of a function f can be used to help visualize its graph. Yesterday, when we were studying the function $f(x, y) = x^2 + y^2$, we looked at how its graph intersects the planes $z = k$. This led us to consider the equations $k = x^2 + y^2$ for $k = 0, 1, 2, \ldots$. The solution sets, which were concentric circles, were precisely the level sets of f. Stacking these level sets together according to their height k produced the graph of f.

Similarly, not every surface in $\mathbb{R}^3$ is a graph of a function $f(x, y)$. The general equation for a surface in $\mathbb{R}^3$ is $f(x, y, z) = k$, where k is a constant. The set of points (x, y, z) that satisfy $f(x, y, z) = k$ is called the *level surface at height* k for the function f. For example, the unit sphere give by $x^2 + y^2 + z^2 = 1$ is the level surface at height 1 for the function $f(x, y, z) = x^2 + y^2 + z^2$.

2. Quadric surfaces

The *quadric surfaces* are a well-known family of level surfaces in $\mathbb{R}^3$. They are solutions to the equation $Q(x, y, z) = 0$, where Q is a polynomial of degree 2, i.e.

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$

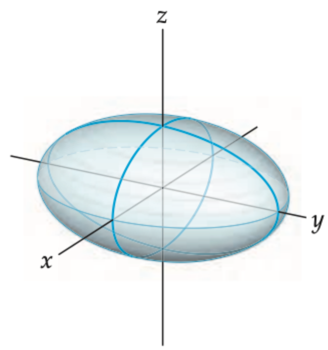
There are 17 different "types" of solution sets, not all of which are actually surfaces. However, here are few examples that may be familiar:

- $x^2 + y^2 + z^2 - 1 = 0$ (sphere)
- $x^2 + y^2 - 1 = 0$ (cylinder)
- $x^2 + y^2 - z^2 = 0$ (cone)

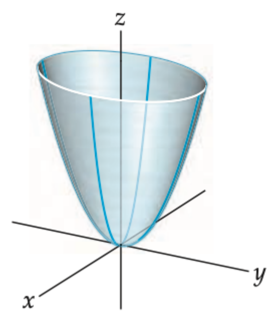
The quadric surfaces are a kind of $\mathbb{R}^3$ counterpart to the conic sections on $\mathbb{R}^2$ (parabolas, hyperbolas, ellipses, etc.). If you intersect a quadric surface with a plane, you will aways get a conic section.

While the general equation for a quadric surface is quite complicated, any such surface can be translated and rotated so that it has a simpler equation. After such a transformation, the surface is said to be *standard form*. The standard forms of several common quadric surfaces are given below. Assume throughout that A, B, and C are positive real numbers.

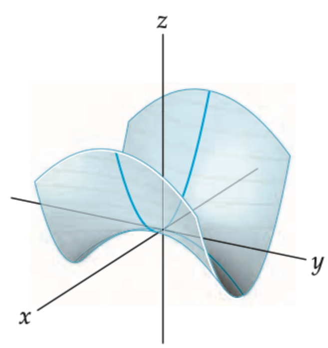
- Ellipsoid: $Ax^2 + By^2 + Cz^2 - 1 = 0$



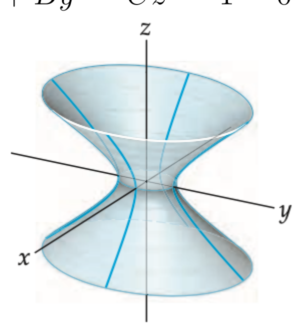
- Elliptic paraboloid: $Ax^2 + By^2 - z = 0$



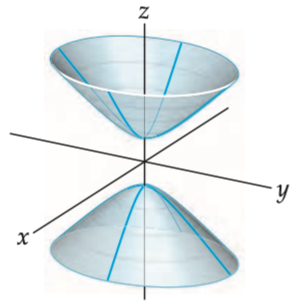
- Hyperbolic paraboloid: $Ax^2 - By^2 - z = 0$



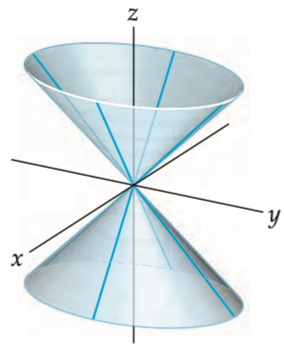
- Hyperboloid of one sheet: $Ax^2 + By^2 - Cz^2 - 1 = 0$



- Hyperboloid of two sheets: $Ax^2 + By^2 - Cz^2 + 1 = 0$



- Cone: $Ax^2 + By^2 - z^2 = 0$



3. Open sets and closed sets

Recall that many of the main theorems from calculus require a function to be defined on a certain kind of set, such as an open interval or a closed interval. For example, the mean value theorem begins:

Let f be a continuous function defined on the closed interval $[a, b]$ and differentiable on the open interval (a, b), where $a < b$. Then...

To extend the results of single-variable calculus to the multivariable setting, we'll need to understand the multidimensional counterparts for these kinds of sets. The $\mathbb{R}^2$ analogue of an open interval is an *open disc*, and the $\mathbb{R}^3$ analogue is an *open ball*.

Open Disks and Open Balls

Let $\epsilon > 0$.

(a) Let $(x_0, y_0) \in \mathbb{R}^2$. A subset of $\mathbb{R}^2$ of the form

$$\{(x, y) \mid (x - x_0)^2 + (y - y_0)^2 < \epsilon\}$$

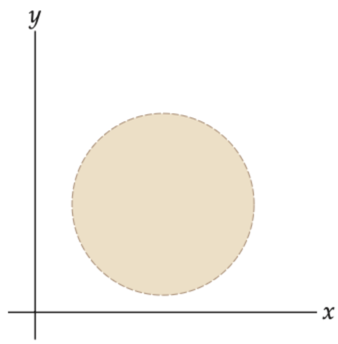
is said to be an ***open disk*** in $\mathbb{R}^2$.

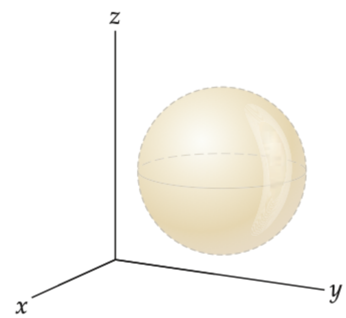
(b) Let $(x_0, y_0, z_0) \in \mathbb{R}^3$. A subset of $\mathbb{R}^3$ of the form

$$\{(x, y, z) \mid (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < \epsilon\}$$

is said to be an ***open ball*** in $\mathbb{R}^3$.

Note that an open disc does not contain its boundary circle, and an open sphere does not contain its boundary sphere. This is precisely what makes these sets "open", as we'll see.





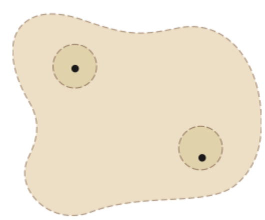
Using open discs and open balls, we can define open sets:

Open Sets in $\mathbb{R}^2$ and $\mathbb{R}^3$

(a) A subset S of $\mathbb{R}^2$ is said to be ***open*** if, for every point $(x, y) \in S$, there is an open disk D such that $(x, y) \in D \subseteq S$.

(b) A subset S of $\mathbb{R}^3$ is said to be ***open*** if, for every point $(x, y, z) \in S$, there is an open ball B such that $(x, y, z) \in B \subseteq S$.

Here is the picture you should have in mind:



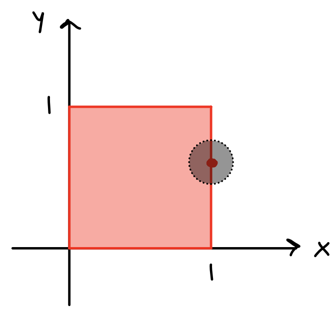
Notice that for each point in the set, we can find an open disc that contains the point and lies entirely within the set.

Day 15

1. Open sets and closed sets

Last time we saw the definition of *open set*, a concept that that's necessary for bringing results of ordinary calculus to the multivariable setting. Recall that a subset S of $\mathbb{R}^2$ is *open* if for every $(x, y) \in S$ there exists an open disc D such that $(x, y) \in D$ and $D \subseteq S$. (The analogous definition of an open subset of $\mathbb{R}^3$ uses balls instead of discs.) Here's a good way to think about open sets: A set S is open if it has the property that whenever (x, y) belongs to S, all points sufficiently close to (x, y) also belong to S.

What's an example of a set that is not open? Consider a square in the plane, including its interior points and boundary, for example $S = \{(x, y) \colon 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$:



If we consider any point on the boundary of the square, we see that any open disc containing that point will necessarily poke out of the square. So S is not an open set. (However, if we were to exclude the boundary, then it would be an open set.)

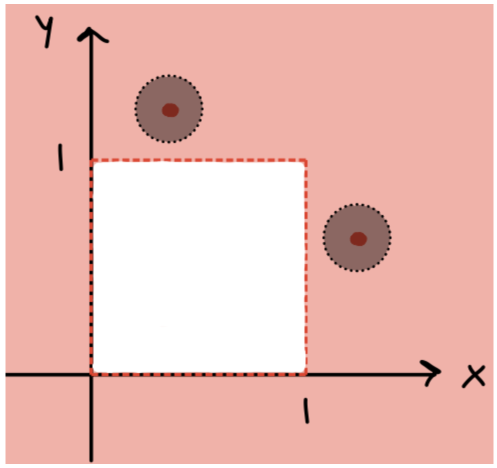
If there are open sets, then there should also be closed sets. To define closed set, we first need to define the complement of a set:

> **The Complement of a Set in $\mathbb{R}^2$ and $\mathbb{R}^3$**
>
> **(a)** Let A be a subset of $\mathbb{R}^2$. The ***complement*** of A, denoted A^c, is the set
> $$A^c = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \notin A\}.$$
> **(b)** Let A be a subset of $\mathbb{R}^3$. The ***complement*** of A, denoted A^c, is the set
> $$A^c = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y, z) \notin A\}.$$

So the complement A^c of a set A is the set of all points that do not belong to A. Note that for any set A, we have $(A^c)^c = A$. The definition of closed set is very simple:

> **Closed Sets in $\mathbb{R}^2$ and $\mathbb{R}^3$**
>
> A subset S of $\mathbb{R}^2$ or $\mathbb{R}^3$ is said to be ***closed*** if its complement, S^c, is open.

Let's check that the square $S = \{(x, y) \colon 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$, that we considered above, is closed. We need to show that its complement is open. The complement of S looks like this:



For any point not in S, we can find an open disc that contains that point and lies entirely outside of S. So S^c is open, meaning that S is closed.

Note that a set can be neither open nor closed. It's also possible for a set to be both open and closed.

2. Limits

Recall the formal definition of the limit of a single-variable function:

The limit of $f(x)$ as $x \to a$ is equal to L if, for every $\varepsilon > 0$, there exists some $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

The intuitive meaning of the statement $\lim_{x \to a} f(x) = L$ is this: We can get $f(x)$ as close to L as we want, provided we make x sufficiently close to a.

By simply converting to vector notation, we get the analogous definition of limit for functions of more than one variable:

The Limit of a Function of Two or More Variables

Let f be a function of two or more variables. The limit of f at $\mathbf{a}$ is L if, for every $\epsilon > 0$, there is a $\delta > 0$ such that $|f(\mathbf{x}) - L| < \epsilon$ whenever $0 < \|\mathbf{x} - \mathbf{a}\| < \delta$. In this case we write $\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = L$.

For example, if f is a function of x and y, then in the definition of limit we would have $\mathbf{x} = \langle x, y \rangle$ and $\mathbf{a}$ would be some position vector in $\mathbb{R}^2$. If f were a function of x, y, and z, then we would have $\mathbf{x} = \langle x, y, z \rangle$ and $\mathbf{a}$ would be a position vector in $\mathbb{R}^3$.

Working with the definition of limit can be challenging at first. Fortunately, once you've shown that a few limits exist, you can find others using limit rules:

Rules for Calculating Limits of Combinations

If $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ and $\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})$ exist, then the following rules hold for their combinations:

Constant Multiple Rule: $\lim_{\mathbf{x}\to\mathbf{a}} kf(\mathbf{x}) = k\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ for any real number k.

Sum Rule: $\lim_{\mathbf{x}\to\mathbf{a}} (f(\mathbf{x}) + g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) + \lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})$.

Difference Rule: $\lim_{\mathbf{x}\to\mathbf{a}} (f(\mathbf{x}) - g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) - \lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})$.

Product Rule: $\lim_{\mathbf{x}\to\mathbf{a}} (f(\mathbf{x})g(\mathbf{x})) = \left(\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})\right)\left(\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})\right)$.

Quotient Rule: $\lim_{\mathbf{x}\to\mathbf{a}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})}{\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})}$, if $\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) \neq 0$.

3. Limit along a path

In ordinary calculus, you saw that $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$. In other words, the limit exists if and only if the limits from the right and from the left exist and are equal. A similar statement holds in the multivariable setting, but it's slightly more complex. In two or more dimensions, there are infinitely many paths along which $\mathbf{x}$ could approach $\mathbf{a}$, not just a path from the right or from the left.