Mixing Problems (1.4)

Differential Equation Form:

$$\frac{dA}{dt} = \text{rate in} - \text{rate out}$$

Example 1: Tank with 200 gallons, 50 lbs of salt initially. Brine with 1 lb/gal flows in at 5 gal/min, same outflow rate.

$$\frac{dA}{dt} = 5(1) - \frac{5A}{200}, \quad A(t) = 200 - 150e^{-t/40}$$

Example 2: Tank starts with 500 L water and 5 kg salt. Brine with 0.5 kg/L enters at 10 L/min, 5 L/min outflow.

$$\frac{dA}{dt} = 10(0.5) - \frac{5A}{500 + t}$$

Exact Equations (1.5)

Form: $M(x, y) + N(x, y)y' = 0$ **Test for Exactness:**

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Example:

$$(2x + y^2) + (2y + xy)y' = 0$$

Integrate $M(x, y)$:

$$\phi(x, y) = x^2 + xy^2 + g(y)$$

Substitute and solve: $g(y) = C$.

Laplace Transform (3.2-3.4)

Laplace Transform of Derivatives:

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0), \quad \mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

Example: Solve $y'' + 2y' + y = e^{-t}$, $y(0) = 0, y'(0) = 1$.

$$s^2Y(s) + 2sY(s) + Y(s) = \frac{1}{s+1}$$

Solution:

$$Y(s) = \frac{1}{(s+1)^2(s+1)}$$

Undetermined Coefficients (2.5)

Form: $y'' + ay' + by = f(t)$ where

1. f(x) = a polynomial in x
2. f(x) = e^{ax} (a polynomial in x)
3. f(x) = e^{ax} (cos(bx) or sin(bx)) (a polynomial in x)

then the particular solution y_p is:

1. $y_p = x^k$ (a general polynomial of the same degree) where k is the number of times that 0 is a root of the characteristic equation.
2. $y_p = x^k e^{ax}$ (a general polynomial of the same degree) where k is the number of times that a is a root of the characteristic equation.
3. $y_p = x^k e^{ax}$ [(a general polynomial of the same degree) cos(bx) + (a general polynomial of the same degree) sin(bx)] where k is the number of times that $a + bi$ is a root of the characteristic equation.

Example 1: Solve $y'' + 3y' + 2y = 4e^{-3x}$.

$$\text{Characteristic roots: } r = -1, -2$$

Guess $y_p = x^0 e^{-3x}(A)$, substitute to find A.
Example 2: Solve $y'' + 4y = 2cos(3x)$.

$$\text{Characteristic roots: } r = 0 \pm 2i$$

Guess $y_p = x^0 e^{0t}(A\cos(3t) + B\sin(3t))$, substitute to find A, B.

Spring Problems (2.4, 2.6)

Form: $mx'' + cx' + kx = F(t)$ Cases:

- Free Oscillation: $c = 0, F(t) = 0$

$$x(t) = C_1\cos(\omega t) + C_2\sin(\omega t)$$

- Damped Oscillation ($c > 0$): Analyze overdamped, underdamped, or critically damped cases.

Eigenvalues and Eigenvectors (6.1)

Characteristic Polynomial:

$$\det(A - \lambda I) = 0$$

Example:

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}, \quad \det(A - \lambda I) = (\lambda - 5)(\lambda - 2)$$

Eigenvalues: $\lambda = 5, 2$.

Solving Systems of Equations (7.3)

Example:

$$\vec{x}' = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\vec{x}$$

Eigenvalues: $\lambda = 3, 1$. Solutions:

$$\vec{x}(t) = c_1 e^{3t}\vec{v_1} + c_2 e^t\vec{v_2}$$