Chapter One: First Order Equations

1.3 Separable Equations & Applications

Key Formula:

$$p(y)\frac{dy}{dx} = q(x) \quad \Rightarrow \quad \int \frac{1}{p(y)}\,dy = \int q(x)\,dx$$

Example: Solve $\frac{dy}{dx} = 2xy^2$, $y(0) = 1$.

$$\int y^{-2}\,dy = \int 2x\,dx \quad \Rightarrow \quad -y^{-1} = x^2 + C$$

Solution: $y(x) = \frac{-1}{x^2+C}$. **Application:** Population growth, cooling laws, mixing problems.

1.2 Equilibrium & Stability

Equilibrium Solutions: $f(y) = 0$ for autonomous equations. **Stability Analysis:**

- **Stable (sink):** Solutions converge to equilibrium.
- **Unstable (source):** Solutions diverge from equilibrium.

Example: $\frac{dy}{dx} = y^2 - 4y + 3$. Equilibria: $y = 1, 3$. Stability: $y = 1$ (sink), $y = 3$ (source).

1.4 Linear Equations & Applications

General Form: $\frac{dy}{dx} + P(x)y = Q(x)$. **Integrating Factor:**

$$\mu(x) = e^{\int P(x)dx}$$

Example: Solve $\frac{dy}{dx} + y = x$.

$$\mu(x) = e^x, \quad y = e^{-x}\int xe^x\,dx$$

Solution: $y = x - 1 + Ce^{-x}$.

1.5 Bernoulli & Exact Equations

Bernoulli Equation: $\frac{dy}{dx} + P(x)y = Q(x)y^n$. **Transformation:** $v = y^{1-n}$, reduces to linear. **Example:** Solve $\frac{dy}{dx} + 2y = 3y^2$. Substitute $v = y^{-1}$:

$$\frac{dv}{dx} - 2v = -3$$

Solution: $v = Ce^{-2x} - \frac{3}{2}$, so $y = \frac{1}{Ce^{-2x} - \frac{3}{2}}$.
Exact Equation:

$$M(x,y) + N(x,y)y' = 0 \quad \text{if } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Example: Solve $(2x + y) + (x + 3y)y' = 0$. Check exactness:

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = 1$$

Integrate $M(x,y)$:

$$\phi(x,y) = x^2 + xy + \frac{y^2}{2} + C$$

Chapter Two: Second Order Equations

2.1-2.3 General Solutions & Constant Coefficients

General Form: $ay'' + by' + cy = 0$. **Characteristic Equation:** $ar^2 + br + c = 0$. **Case 1:** Distinct Roots r_1, r_2:

$$y(t) = C_1e^{r_1t} + C_2e^{r_2t}$$

Example: Solve $y'' - 3y' + 2y = 0$. Roots: $r_1 = 2, r_2 = 1$. Solution: $y(t) = C_1e^{2t} + C_2e^t$.

2.4-2.6 Mechanical Vibrations

Free Vibrations: $mx'' + kx = 0$. Solution: $x(t) = C_1\cos(\omega t) + C_2\sin(\omega t)$, $\omega = \sqrt{\frac{k}{m}}$. **Forced Vibrations:** $mx'' + cx' + kx = F(t)$. Solve using undetermined coefficients or variation of parameters. **Example:** $x'' + 4x = \sin(2t)$. Solution: $x(t) = C_1\cos(2t) + C_2\sin(2t) + \frac{1}{4}\sin(2t)$.

Chapter Three: Laplace Transform

Key Properties:

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0), \quad \mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

Example: Solve $y'' + 2y' + y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$.

$$s^2Y(s) + 2sY(s) + Y(s) = \frac{1}{s+1}$$

Solution: $Y(s) = \frac{1}{(s+1)^3}$. Inverse transform: $y(t) = \frac{t^2}{2}e^{-t}$.

Chapter Four: Systems of Equations & Matrices

Gaussian Elimination & Rank

Steps:

- Forward elimination to achieve row-echelon form.
- Back substitution to solve.

Example:

$$\begin{bmatrix} 1 & 2 & -1 & | & 8 \\ 0 & 1 & 3 & | & -11 \\ 0 & 0 & 1 & | & -3 \end{bmatrix}$$

Solution: $x_3 = -3, x_2 = -2, x_1 = 4$.

Chapter Six: Eigenvalues & Eigenvectors

Characteristic Polynomial:

$$\det(A - \lambda I) = 0$$

Example:

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}, \quad \det(A - \lambda I) = (\lambda - 5)(\lambda - 2)$$

Eigenvalues: $\lambda = 5, 2$. Eigenvectors: Solve $(A - \lambda I)x = 0$.

Chapter Seven: Systems of Differential Equations

Eigenvalue Method: For $\vec{x}' = A\vec{x}$, solve eigenvalue problem for A. **Example:**

$$\vec{x}' = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \vec{x}$$

Eigenvalues: $\lambda = 3, 1$. Solution:

$$\vec{x}(t) = c_1 e^{3t} \vec{v_1} + c_2 e^{t} \vec{v_2}$$