CS1800 Homework 4 Solutions

Problem 1: Survey

Given 20 multiple-choice questions, each with four possible answers (A, B, C, or D):

i. Unique ways to complete the entire survey

Each question has 4 possible answers. Hence, the total number of ways to complete the survey is:

$$4^{20} = 1,099,511,627,776$$

ii. Unique ways to complete the survey if at least one question is unanswered

Each question now has 5 options (A, B, C, D, or no answer). Thus, the total number of ways is:

$$5^{20-1} = 5^{19} = 1.90734863 * 10^{13}$$

iii. How many ways can a participant respond if they select option A for exactly 8 questions and option B for the remaining 12?

This is a combination problem, where we choose 8 questions from 20 for option A:

$$\binom{20}{8} = \frac{20!}{8!12!} = 125,970$$

Problem 2: Soccer Team

i. In how many ways can the coach choose 11 players to play?

The number of ways to choose 11 players from 20 is:

$$\binom{20}{11} = \frac{20!}{11!9!} = 167,960$$

ii. How many ways can the coach choose a lineup?

We need to choose:

- 4 midfielders from 7: $\binom{7}{4}$
- 3 defenders from 6: $\binom{6}{3}$
- 3 attackers from 5: $\binom{5}{3}$
- 1 goalkeeper from 2: $\binom{2}{1}$

The total number of ways is:

$$\binom{7}{4} \times \binom{6}{3} \times \binom{5}{3} \times \binom{2}{1} = 35 \times 20 \times 10 \times 2 = 14,000$$

iii. How many ways can the coach choose a lineup if one defender can also play attack?

In this case, there are 6 potential attackers and 6 defenders. The number of ways is:

$$\binom{7}{4} \times \binom{6}{3} \times \binom{6}{3} \times \binom{2}{1} = 35 \times 20 \times 20 \times 2 = 28,000$$

Problem 3: Sharing Books

i. How many ways can we distribute 15 unique books among 5 friends?

The number of ways to distribute 15 unique books to 5 friends is:

$$5^{15} = 30,517,578,125$$

ii. How many ways to distribute 15 identical books among 5 friends?

Using the stars and bars formula:
Let $k = 15$ and $n = 5$.

$$\binom{n+k-1}{k-1} = \binom{15+5-1}{4} = \binom{19}{4} = 3,876$$

iii. How many ways if one friend does not receive exactly 4 books?

We subtract the number of ways where one friend gets exactly 4 books from the total:

$$\text{Total ways} = \binom{19}{4}, \quad \text{Ways where one friend gets 4 books} = \binom{14}{3}$$

Thus:

$$\binom{19}{4} - \binom{14}{3} = 3,876 - 336 = 3,512$$

Problem 4: Principle of Inclusion-Exclusion Factors

i. How many integers from 1 to 1000 are multiples of 7?

$$\left\lfloor \frac{1000}{7} \right\rfloor = 142$$

ii. How many integers from 1 to 1000 are multiples of 11?

$$\left\lfloor \frac{1000}{11} \right\rfloor = 90$$

iii. How many integers from 1 to 1000 are multiples of both 7 and 11?

Multiples of both 7 and 11 are multiples of the LCM (77):

$$\left\lfloor \frac{1000}{7 \cdot 11} \right\rfloor = 12$$

iv. How many integers from 1 to 1000 are multiples of 7 or 11?

Using inclusion-exclusion:

$$142 + 90 - 12 = 220$$

v. How many integers from 1 to 1000 are multiples of 7 but not 11?

$$142 - 12 = 130$$

vi. How many integers from 1 to 1000 are multiples of neither 7 nor 11?

$$1000 - (142 + 90) = 768$$

Problem 5: Pigeonhole Principle

i. How many pigeons, at least, are guaranteed to share the first nest?

$$\left\lceil \frac{14}{3} \right\rceil = \lceil 4.667 \rceil = 5$$

ii. How many pigeons, at least, are guaranteed to be in the nest with the most pigeons?

$$\left\lceil \frac{14}{3} \right\rceil = 5$$

iii. Minimum number of students to guarantee a table with at least 3 students

By the pigeonhole principle:

$$n = 29 \quad \text{(at least one table will have 3 students)}$$

If you place 2 students at each of the 14 tables, that uses up 28 students. However, when you add a 29th student, they will need to sit at one of the tables that already has 2 students, meaning that table will now have 3 students. Therefore, with 29 students, it's guaranteed that at least one table will have 3 students.

iv. What does the pigeonhole principle imply about 123 students taking an exam with 102 questions?

Since there are 123 students but only 103 possible different scores, by the pigeonhole principle, there must be at least two students with the same score.

Problem 6: Rectangle Extra

i. How many integer rectangles have an area of 720?

The prime factorization of 720 is:

$$720 = 2^4 \times 3^2 \times 5$$

The total number of divisors (and thus possible rectangles) is:

$$(4 + 1)(2 + 1)(1 + 1) = 30$$

Thus, there are 30 distinct integer rectangles.