MATH 231-01: Homework Assignment 1

8 September 2025

Due: 15 September 2025 by 10:00pm Eastern time, submitted on Moodle as a single PDF.

Instructions: Write your solutions on the following pages. If you need more space, you may add pages, but make sure they are in order and label the problem number(s) clearly. You should attempt each problem on scrap paper first, before writing your solution here. Excessively messy or illegible work will not be graded. You must show your work/reasoning to receive credit. You do not need to include every minute detail; however the process by which you reached your answer should be evident. You may work with other students, but please write your solutions in your own words.

Name: Sean Balbale

Score:

1. A *diameter* of a sphere is any line segment that passes through the center of the sphere and whose endpoints lie on the sphere. Find the equation of the sphere that has a diameter with endpoints $(2, 3, 1)$ and $(8, -1, 3)$.

Sphere equation: $(x-h)^2+(y-k)^2+(z-l)^2=r^2$ where (h,k,l) is the center and r is the radius.

$C=\left(\frac{2+8}{2}, \frac{3+(-1)}{2}, \frac{1+3}{2}\right)=(5,1,2)$ $\quad r=\sqrt{(5-2)^2+(1-3)^2+(2-1)^2}=\sqrt{14} \therefore r^2=14$

$$(x-5)^2+(y-1)^2+(z-2)^2=14$$

2. Does the equation

$$x^2 + y^2 + z^2 - 2x + 4y - 8z - 28 = 0$$

represent a sphere? If so, find its center and radius. If not, explain why not.

$$(x^2-2x)+(y^2+4y)+(z^2-8z)=28$$

Complete the square:

$x: \left(\frac{-2}{2}\right)^2=1$

$y: \left(\frac{4}{2}\right)^2=4$

$z: \left(\frac{-8}{2}\right)^2=16$

$$(x^2-2x+1)+(y^2+4y+4)+(z^2-8z+16)=28+1+4+16$$

$$(x-1)^2+(y+2)^2+(z-4)^2=49$$

yes, this equation represents a sphere.

Center: $(1,-2,4)$ $\quad$ Radius: $r=\sqrt{49}=7$

3. Find a vector of length 5 that points in the opposite direction to $\langle 3, -2, 6\rangle$.

Let $\vec{V} = \langle 3,-2,6\rangle$, $\|\vec{V}\| = \sqrt{3^2+(-2)^2+6^2} = 7$

$\hat{u} = \frac{\vec{V}}{\|\vec{V}\|} = \frac{1}{7}\langle 3,-2,6\rangle = \langle \frac{3}{7}, \frac{-2}{7}, \frac{6}{7}\rangle$

$-5\cdot\hat{u} = -5\langle \frac{3}{7}, \frac{-2}{7}, \frac{6}{7}\rangle = \langle \frac{-15}{7}, \frac{10}{7}, \frac{-30}{7}\rangle$

$\langle \frac{-15}{7}, \frac{10}{7}, \frac{-30}{7}\rangle$

4. Find the angles of the triangle with vertices $(4, -2, 3)$, $(5, -1, 3)$, and $(6, -1, 4)$.

Dot product cosine formula: $\vec{U}\cdot\vec{V} = \|\vec{U}\|\|\vec{V}\|\cos\theta \therefore \cos\theta = \frac{\vec{U}\cdot\vec{V}}{\|\vec{U}\|\|\vec{V}\|}$

Let $A = (4,-2,3)$, $B = (5,-1,3)$, $C = (6,-1,4)$.

$\vec{AB} = B-A = \langle 5-4, -1-(-2), 3-3\rangle = \langle 1,1,0\rangle$ $\quad \|\vec{AB}\| = \sqrt{1^2+1^2+0^2} = \sqrt{2}$

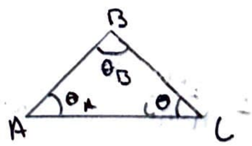
$\vec{AC} = C-A = \langle 6-4, -1-(-2), 4-3\rangle = \langle 2,1,1\rangle$ $\quad \|\vec{AC}\| = \sqrt{2^2+1^2+1^2} = \sqrt{6}$

$\vec{BC} = C-B = \langle 6-5, -1-(-1), 4-3\rangle = \langle 1,0,1\rangle$ $\quad \|\vec{BC}\| = \sqrt{1^2+0^2+1^2} = \sqrt{2}$

$\cos(\theta_A) = \frac{\vec{AB}\cdot\vec{AC}}{\|\vec{AB}\|\|\vec{AC}\|} = \frac{(1)(2)+(1)(1)+(0)(1)}{\sqrt{2}\sqrt{6}} = \frac{\sqrt{3}}{2} \therefore \theta_A = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30^\circ$

$\cos(\theta_B) = \frac{\vec{BA}\cdot\vec{BC}}{\|\vec{BA}\|\|\vec{BC}\|} = \frac{(-1)(1)+(-1)(0)+(0)(1)}{\sqrt{2}\sqrt{2}} = \frac{-1}{2} \therefore \theta_B = \cos^{-1}\left(\frac{-1}{2}\right) = 120^\circ$

$\theta_C = 180 - \theta_A - \theta_B = 180 - 30 - 120 = 30^\circ$



The angles of the triangle are 30°, 120°, 30°

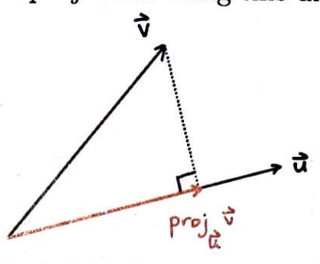
5. Let $\mathbf{v} = \langle 5, 7, 9\rangle$ and $\mathbf{u} = \langle 2, 3, 5\rangle$. Find vectors $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}$ is parallel to $\mathbf{u}$, $\mathbf{b}$ is perpendicular to $\mathbf{u}$, and $\mathbf{v} = \mathbf{a} + \mathbf{b}$.

$$\vec{a} = \text{proj}_{\vec{u}}\vec{v} = \left(\frac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|^2}\right)\vec{u} = \frac{(5)(2)+(7)(3)+(9)(5)}{2^2+3^2+5^2} = \frac{76}{38}\langle 2,3,5\rangle = \langle 4,6,10\rangle$$

$$\vec{b} = \vec{v} - \vec{a} = \langle 5,7,9\rangle - \langle 4,6,10\rangle = \langle 1,1,-1\rangle$$

$\vec{a} = \langle 4,6,10\rangle$ and $\vec{b} = \langle 1,1,-1\rangle$

6. In class we initially defined the projection using this diagram:



(a) Without using the formula for projection, show that

$$\|\operatorname{proj}_{\mathbf{u}} \mathbf{v}\| = \frac{\mathbf{v}\cdot\mathbf{u}}{\|\mathbf{u}\|}.$$

(Hint: Start with trigonometry.)

Let θ be the angle between vectors $\vec{v}$ and $\vec{u}$. From trigonometry tells us that the length of the projection (the adjacent) is the length of the hypotenuse ($\|\vec{v}\|\cos\theta$)

$\|\text{proj}_{\vec{u}}\vec{v}\| = \|\vec{v}\|\cos\theta$ The dot product is $v\cdot u = \|\vec{v}\|\|\vec{u}\|\cos\theta$.

$\|\vec{v}\|\cos\theta = \frac{v\cdot u}{\|u\|}$ via substitution: $\|\text{proj}_{\vec{u}}\vec{v}\| = \frac{v\cdot u}{\|u\|}$

(b) Explain why

$$\frac{1}{\|\operatorname{proj}_{\mathbf{u}} \mathbf{v}\|} \operatorname{proj}_{\mathbf{u}} \mathbf{v} = \frac{1}{\|\mathbf{u}\|} \mathbf{u}$$

must hold. (You should assume still that **u**, **v**, and $\operatorname{proj}_{\mathbf{u}} \mathbf{v}$ come from the diagram above.)

$\frac{1}{\|\operatorname{proj}_{\vec{u}} \vec{v}\|} \operatorname{Proj}_{\vec{u}} \vec{v}$ is the unit vector in the direction of $\operatorname{proj}_{\vec{u}} \vec{v}$

$\frac{1}{\|\vec{u}\|} \vec{u}$ is the unit vector in the direction of $\vec{u}$

$\operatorname{Proj}_{\vec{u}} \vec{v}$ lies on the same line and points in the exact same direction as the vector $\vec{u}$. Since two vectors point in the same direction must have the same unit vector. Thus, the equality must be true.

(c) Combine parts (a) and (b) to conclude that

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{u}}{\|\mathbf{u}\|^2} \mathbf{u}.$$

$$\frac{1}{\|\operatorname{proj}_{\vec{u}} \vec{v}\|} \operatorname{Proj}_{\vec{u}} \vec{v} = \frac{1}{\|\vec{u}\|} \vec{u}$$

$$\operatorname{Proj}_{\vec{u}} \vec{v} = \frac{\|\operatorname{proj}_{\vec{u}} \vec{v}\|}{\|\vec{u}\|} \vec{u}$$

$$\operatorname{Proj}_{\vec{u}} \vec{v} = \frac{\left(\frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|}\right)}{\|\vec{u}\|} \vec{u}$$

$$\operatorname{Proj}_{\vec{u}} \vec{v} = \frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|^2} \vec{u}$$