Key Topics

The exam covers sections from Chapters 1-7.

Always Included Questions

- Mixing Problems (1.4)
- Exact Equations (1.5)
- Solving Equations with Laplace Transform (3.2-3.4)
- Undetermined Coefficients (2.5)
- Spring Problems (2.4, 2.6)
- Eigenvalues and Eigenvectors (6.1)
- Solving Systems of Differential Equations (7.3)

Mixing Problems (1.4)

Differential Equation Form:

$$\frac{dA}{dt} = \text{rate in} - \text{rate out}$$

Example 1: Tank with 200 gallons, 50 lbs of salt initially. Brine with 1 lb/gal flows in at 5 gal/min, same outflow rate.

$$\frac{dA}{dt} = 5(1) - \frac{5A}{200}, \quad A(t) = 200 - 150e^{-t/40}$$

Example 2: Tank starts with 500 L water and 5 kg salt. Brine with 0.5 kg/L enters at 10 L/min, 5 L/min outflow.

$$\frac{dA}{dt} = 10(0.5) - \frac{5A}{500+t}$$

Exact Equations (1.5)

Form: $M(x,y) + N(x,y)y' = 0$ **Test for Exactness:**

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Example:

$$(2x + y^2) + (2y + xy)y' = 0$$

Integrate $M(x,y)$:

$$\phi(x,y) = x^2 + xy^2 + g(y)$$

Substitute and solve: $g(y) = C$.

Laplace Transform (3.2-3.4)

Laplace Transform of Derivatives:

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0), \quad \mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

Example: Solve $y'' + 2y' + y = e^{-t}$, $y(0) = 0, y'(0) = 1$.

$$s^2Y(s) + 2sY(s) + Y(s) = \frac{1}{s+1}$$

Solution:

$$Y(s) = \frac{1}{(s+1)^2(s+1)}$$

Undetermined Coefficients (2.5)

Example: Solve $y'' + 3y' + 2y = e^x$.

$$\text{Characteristic roots: } r_1 = -1, r_2 = -2$$

Guess $y_p = Ae^x$, substitute to find A.

Spring Problems (2.4, 2.6)

Form: $mx'' + cx' + kx = F(t)$ Cases:

- Free Oscillation: $c = 0, F(t) = 0$

$$x(t) = C_1\cos(\omega t) + C_2\sin(\omega t)$$

- Damped Oscillation ($c > 0$): Analyze overdamped, underdamped, or critically damped cases.

Eigenvalues and Eigenvectors (6.1)

Characteristic Polynomial:

$$\det(A - \lambda I) = 0$$

Example:

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}, \quad \det(A - \lambda I) = (\lambda - 5)(\lambda - 2)$$

Eigenvalues: $\lambda = 5, 2$.

Solving Systems of Equations (7.3)

Example:

$$\vec{x}' = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\vec{x}$$

Eigenvalues: $\lambda = 3, 1$. Solutions:

$$\vec{x}(t) = c_1e^{3t}\vec{v_1} + c_2e^t\vec{v_2}$$