Day 42

(Note that we spent most of this class completing the example from Day 41.)

1. Another example of calculating flux

Let Σ be the plane parametrized by $\mathbf{r}(s,t) = \langle t, s+t, 2s+t\rangle$ for $0 \le s \le 1$ and $0 \le t \le 2$. Let $\mathbf{n}$ denote the upward unit normal vector on Σ (meaning that the z component of $\mathbf{n}$ is positive). Let $\mathbf{F}(x,y,z) = \langle 0, y, 0\rangle$. Our goal, as before, is to calculate the flux integral

$$\int_\Sigma \mathbf{F}\cdot\mathbf{n}dS.$$

In this example, unlike the previous one, Σ is expressed as a parametrized surface rather than a graph. So the conversion from a surface integral to a double integral will look slightly different. This time, we need to understand how the unit normal $\mathbf{n}$ depends on s and t. For any s and t, the vectors $\mathbf{r}_s$ and $\mathbf{r}_t$ are tangent to Σ at the point $\mathbf{r}(s,t)$. (See the picture from Day 39.) This means that $\mathbf{r}_s \times \mathbf{r}_t$ will be normal to Σ, and thus

$$\mathbf{n} = \pm\frac{\mathbf{r}_s\times\mathbf{r}_t}{\|\mathbf{r}_s\times\mathbf{r}_t\|}.$$

To determine whether $+$ or $-$ should be used, let's calculate $\mathbf{r}_s\times\mathbf{r}_t$ and see if it points upwards (parallel to $\mathbf{n}$) or downwards (antiparallel to $\mathbf{n}$). We have $\mathbf{r}_s = \langle 0,1,2\rangle$ and $\mathbf{r}_t = \langle 1,1,1\rangle$, so $\mathbf{r}_s\times\mathbf{r}_t = \langle -1, 2, -1\rangle$. Since the z component of $\mathbf{r}_s\times\mathbf{r}_t$ is negative, it points in the opposite direction of $\mathbf{n}$; thus

$$\mathbf{n} = -\frac{\mathbf{r}_s\times\mathbf{r}_t}{\|\mathbf{r}_s\times\mathbf{r}_t\|} = \frac{\langle 1,-2,1\rangle}{\|\mathbf{r}_s\times\mathbf{r}_t\|}.$$

The reason we haven't calculated $\|\mathbf{r}_s\times\mathbf{r}_t\|$ is because it will cancel momentarily when we convert the surface integral into a double integral. Alright, so now we know what $\mathbf{n}$ is. What about $\mathbf{F}$? We're given that $\mathbf{F} = \langle 0, y, 0\rangle$. When we plug in $\mathbf{r}(s,t)$, we'll get $s+t$ in place of y. Thus

$$\mathbf{F}\cdot\mathbf{n} = \langle 0, s+t, 0\rangle\cdot\frac{\langle 1,-2,1\rangle}{\|\mathbf{r}_s\times\mathbf{r}_t\|} = \frac{-2(s+t)}{\|\mathbf{r}_s\times\mathbf{r}_t\|}.$$

So

$$\int_\Sigma \mathbf{F}\cdot\mathbf{n}dS = \int_0^1\int_0^2 \frac{-2(s+t)}{\|\mathbf{r}_s\times\mathbf{r}_t\|}\|\mathbf{r}_s\times\mathbf{r}_t\|dtds = \int_0^1\int_0^2 -2(s+t)dtds = -6.$$

Day 43

1. The del operator

For the remainder of the course we will study three major theorems in multivariable calculus: Green's theorem, Stokes' theorem, and the divergence theorem. We will see that each of these theorems generalizes the fundamental theorem of calculus by stating, roughly, that the integral of the "derivative" of a vector field over a region is determined by the values of the vector field the boundary of that region.

To state these theorems efficiently, we need a new piece of notation:

The Del Operator, ∇

The ***del operator***, ∇, is the vector of operations

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

or, in $\mathbb{R}^2$,

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}.$$

It's a bit weird to think about a vector ∇ that just contains partial derivative operators but no functions. However, it allows us to denote certain derivative-related quantities very neatly. Here's a typical example of how ∇ is used: Let $\mathbf{F}(x, y, z) = \langle x^2y, 3y^2z, z^4\rangle$. Then taking the "dot product" of ∇ and $\mathbf{F}$ yields

$$\begin{aligned}\nabla \cdot \mathbf{F}(x, y, z) &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle x^2y, 3y^2z, z^4\rangle \\ &= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(3y^2z) + \frac{\partial}{\partial z}(z^4) \\ &= 2xy + 6yz + 4z^4.\end{aligned}$$

Divergence of a Vector Field

The ***divergence*** of vector field $\mathbf{F}$ is the dot product of ∇ and $\mathbf{F}$.

(a) In $\mathbb{R}^2$, if $\mathbf{F}(x, y) = F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}$, then

$$\operatorname{div} \mathbf{F}(x, y) = \nabla \cdot \mathbf{F}(x, y) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}.$$

(b) In $\mathbb{R}^3$, if $\mathbf{F}(x, y, z) = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}$, then

$$\operatorname{div} \mathbf{F}(x, y, z) = \nabla \cdot \mathbf{F}(x, y, z) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Similarly, we can take the "cross product" of ∇ and a vector field $\mathbf{F}$. Let $\mathbf{F}(x,y,z) = \langle x^2y, 3y^2z, z^4\rangle$ as above. Then

$$\begin{aligned}\nabla\times\mathbf{F}(x,y,z) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & 3y^2z & z^4 \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(z^4) - \frac{\partial}{\partial z}(3y^2z)\right)\mathbf{i} - \left(\frac{\partial}{\partial x}(z^4) - \frac{\partial}{\partial z}(x^2y)\right)\mathbf{j} + \left(\frac{\partial}{\partial x}(3y^2z) - \frac{\partial}{\partial y}(x^2y)\right)\mathbf{k} \\ &= \langle -3y^2, 0, -x^2\rangle.\end{aligned}$$

Curl of a Vector Field

The ***curl*** of a vector field $\mathbf{F}(x,y,z) = F_1(x,y,z)\mathbf{i} + F_2(x,y,z)\mathbf{j} + F_3(x,y,z)\mathbf{k}$ is the cross product of ∇ with $\mathbf{F}(x,y,z)$:

$$\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F}(x,y,z) = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

If $\mathbf{F}(x,y) = (F_1(x,y), F_2(x,y))$ is a vector field in $\mathbb{R}^2$, we define the curl of $\mathbf{F}$ to be curl $(F_1(x,y), F_2(x,y), 0)$. So,

$$\begin{aligned}\operatorname{curl}\mathbf{F} &= \operatorname{curl}\,(F_1(x,y), F_2(x,y), 0) = \nabla\times\langle F_1(x,y), F_2(x,y), 0\rangle \\ &= \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.\end{aligned}$$

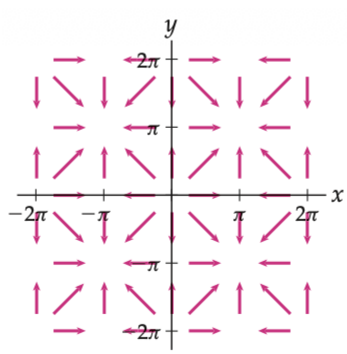
The components of $\operatorname{curl}\mathbf{F}$ may look familiar, as they appear in our criteria for determining whether $\mathbf{F}$ is conservative. If you look back at those criteria, you'll see that they can now be rephrased as follows:

$$\mathbf{F}\text{ is conservative} \qquad \Leftrightarrow \qquad \operatorname{curl}\mathbf{F} = \mathbf{0}.$$

This simple criterion conveniently applies to both two- and three-dimensional vector fields.

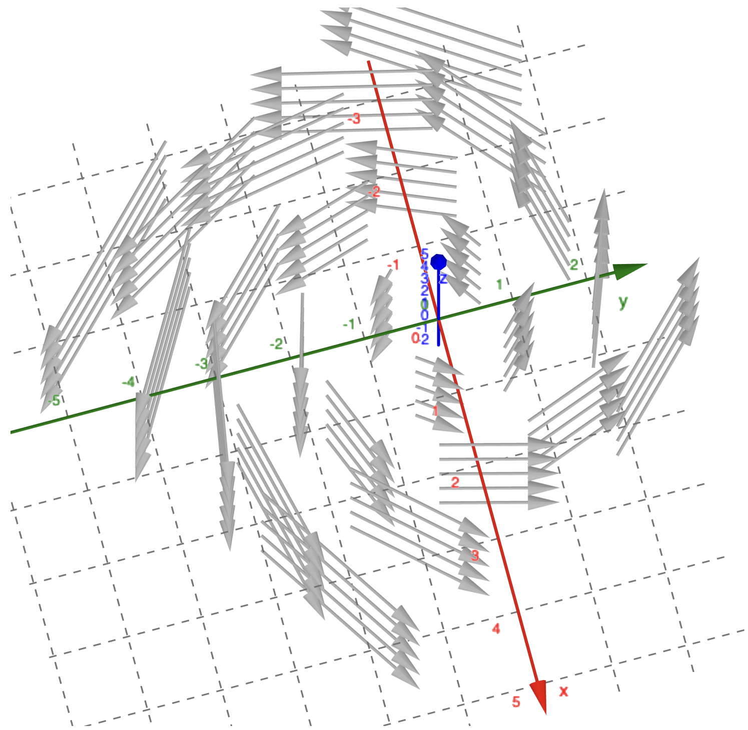
2. What do divergence and curl measure?

The divergence of a vector field measures the degree to which it's expanding or contracting. If $\operatorname{div}\mathbf{F}(x,y,z) > 0$, then $\mathbf{F}$ is expanding at (x,y,z) and we say that the point (x,y,z) is a *source*. If $\operatorname{div}\mathbf{F}(x,y,z) < 0$, then $\mathbf{F}$ is contracting at (x,y,z) and we say that (x,y,z) is a *sink*. Here's a vector plot of the vector field $\mathbf{F}(x,y) = \langle\sin x, \sin y\rangle$:



The origin $(0,0)$ is source and the point (π,π) is a sink. (Of course, by periodicity of sine, this vector field has infinitely many sources and sinks.)

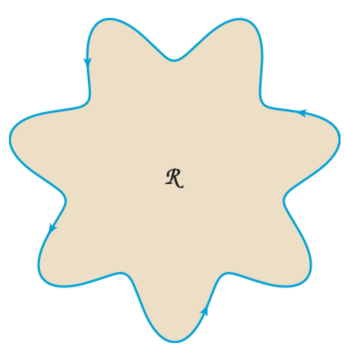
The curl of a vector field tells us how it's rotating. Specifically, the magnitude of $\operatorname{curl}\mathbf{F}$ is the speed of rotation and the direction of $\operatorname{curl}\mathbf{F}$ is the axis of rotation. Note that curl obeys a right-hand rule: Using your right hand, curl your fingers in the direction in which $\mathbf{F}$ is rotating. Then your thumb will point in the direction of $\operatorname{curl}\mathbf{F}$. Here's a vector plot of the vector field $\mathbf{F}(x,y,z)=\langle -y,x,0\rangle$:



In this vantage point, you're looking down at the xy-plane. You can see that $\mathbf{F}$ rotates counterclockwise around the z-axis, so $\operatorname{curl}\mathbf{F}$ will point in the positive z direction.

3. Green's theorem

We now turn to Green's theorem, the first of the three major theorems mentioned earlier. Let R be a bounded region in $\mathbb{R}^2$ whose boundary is simple closed curve C oriented counterclockwise. (Here "simple" means that C has no self-intersections.) The region could look something like this:



Green's theorem establishes a relationship between the work done by $\mathbf{F}$ along C and double integral of (the $\mathbf{k}$-component of) curl $\mathbf{F}$ over R.

THEOREM 14.13 **Green's Theorem**

Let $\mathbf{F}(x, y) = \langle F_1(x, y), F_2(x, y)\rangle$ be a vector field defined on a region $\mathcal{R}$ in the plane whose boundary is a smooth or piecewise-smooth, simple closed curve C. If $\mathbf{r}(t)$ is a parametrization of C in the counterclockwise direction (as viewed from the positive z-axis), then

$$\int_C \mathbf{F}(x, y) \cdot d\mathbf{r} = \int_C F_1(x, y)\, dx + F_2(x, y)\, dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA.$$

Note that $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$ is exactly the $\mathbf{k}$-component of curl $\mathbf{F}$. The conclusion of Green's theorem can also be written as

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} dA.$$

In-class exercise: Let R be the unit disc $x^2+y^2 \leq 1$, and let C be the unit circle $x^2+y^2 = 1$ oriented counterclockwise. Let $\mathbf{F}(x, y) = \langle -y, x\rangle$. Green's theorem tells us that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} dA.$$

Calculate both sides and confirm that they're equal in this case.