MATH 231-01: Homework Assignment 10

2 December 2025

Due: 9 December 2025 by 10:00pm Eastern time, submitted on Moodle as a single PDF.

Instructions: Write your solutions on the following pages. If you need more space, you may add pages, but make sure they are in order and label the problem number(s) clearly. You should attempt each problem on scrap paper first, before writing your solution here. Excessively messy or illegible work will not be graded. You must show your work/reasoning to receive credit. You do not need to include every minute detail; however the process by which you reached your answer should be evident. You may work with other students, but please write your solutions in your own words.

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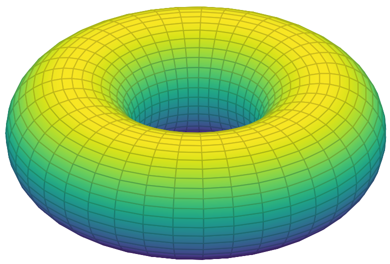
Score:

1. Let Σ be the surface parametrized by

$$\mathbf{r}(s,t) = \langle (2+\cos t)\cos s, (2+\cos t)\sin s, \sin t\rangle, \qquad s,t \in [0, 2\pi].$$

Use GeoGebra (or another online tool) to visualize Σ, then find its surface area. (Hint: If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ and $\mathbf{u}\cdot\mathbf{v} = 0$, then $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|$.)

Visualization of Σ



(1) Compute $\mathbf{r}_s$ and $\mathbf{r}_t$:

$$\mathbf{r}_s = \langle -(2+\cos t)\sin s, (2+\cos t)\cos s, 0\rangle$$
$$\mathbf{r}_t = \langle -\sin t\cos s, -\sin t \sin s, \cos t\rangle$$

(2) Compute the cross product $\mathbf{r}_s \times \mathbf{r}_t$:

$$\mathbf{r}_s \times \mathbf{r}_t = \langle (2+\cos t)\cos s\cos t, (2+\cos t)\sin s\cos t, (2+\cos t)\sin t\rangle$$

(3) Compute the magnitude $\|\mathbf{r}_s \times \mathbf{r}_t\|$:

$$\|\mathbf{r}_s \times \mathbf{r}_t\| = (2+\cos t)\sqrt{\cos^2 t + \sin^2 t} = (2+\cos t)$$

(4) Set up the surface area integral:

$$A = \int_0^{2\pi}\int_0^{2\pi} (2+\cos t)\, ds\, dt$$

(5) Evaluate the integral:

$$A = \int_0^{2\pi}\int_0^{2\pi} (2+\cos t)\, ds\, dt = \int_0^{2\pi} (2+\cos t)(2\pi)\, dt = 2\pi\int_0^{2\pi}(2+\cos t)\, dt$$
$$= 2\pi\left[2t + \sin t\right]_0^{2\pi} = 2\pi(4\pi) = 8\pi^2$$

(6) Final answer:

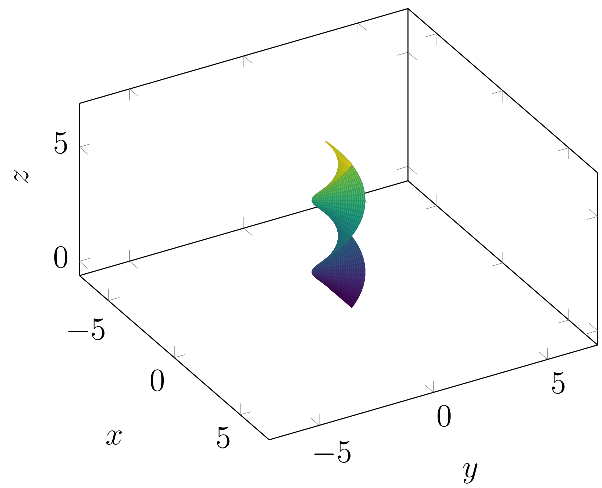
$$\boxed{8\pi^2}$$

2. Let Σ be the surface parametrized by

$$\mathbf{r}(s,t) = \langle s\cos t, s\sin t, t\rangle, \qquad s \in [-1,1],\ t \in [0, 2\pi].$$

Use GeoGebra (or another online tool) to visualize Σ, then write down an iterated integral whose value is the surface area of Σ. (You do not need to evaluate the integral, but you can try to do so as a challenge, or look it up.)

Visualization of Σ (Helicoid)



(1) Compute $\mathbf{r}_s$ and $\mathbf{r}_t$:

$$\mathbf{r}_s = \langle \cos t, \sin t, 0\rangle$$
$$\mathbf{r}_t = \langle -s\sin t, s\cos t, 1\rangle$$

(2) Compute the cross product $\mathbf{r}_s \times \mathbf{r}_t$:

$$\mathbf{r}_s \times \mathbf{r}_t = \langle \sin t, -\cos t, s\rangle$$

(3) Compute the magnitude $\|\mathbf{r}_s \times \mathbf{r}_t\|$:

$$\|\mathbf{r}_s \times \mathbf{r}_t\| = \sqrt{\sin^2 t + \cos^2 t + s^2} = \sqrt{1+s^2}$$

(4) Set up the surface area integral:

$$A = \int_0^{2\pi}\int_{-1}^{1} \sqrt{1+s^2}\, ds\, dt$$

(5) Final answer:

$$\boxed{\int_0^{2\pi}\int_{-1}^{1} \sqrt{1+s^2}\, ds\, dt}$$

3. Let Σ be the part of the cone $z = \sqrt{x^2+y^2}$ that lies between the planes $z = 1$ and $z = 2$, oriented with upward/inward unit normal vector. Calculate the flux of $\mathbf{F}(x, y, z) = \langle x, y, z\rangle$ through Σ.

(1) Parametrize the surface using cylindrical coordinates:

$$\mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r\rangle, \quad r \in [1,2], \theta \in [0, 2\pi]$$

(2) Compute $\mathbf{r}_r$ and $\mathbf{r}_\theta$:

$$\mathbf{r}_r = \langle \cos\theta, \sin\theta, 1\rangle$$
$$\mathbf{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$$

(3) Compute the cross product $\mathbf{r}_r \times \mathbf{r}_\theta$:

$$\mathbf{r}_r \times \mathbf{r}_\theta = \langle -r\cos\theta, -r\sin\theta, r\rangle$$

(4) Compute the magnitude $\|\mathbf{r}_r \times \mathbf{r}_\theta\|$:

$$\|\mathbf{r}_r \times \mathbf{r}_\theta\| = r\sqrt{2}$$

(5) Set up the flux integral:

$$\Phi = \int_0^{2\pi} \int_1^2 \mathbf{F}(\mathbf{r}(r,\theta)) \cdot \frac{\mathbf{r}_r \times \mathbf{r}_\theta}{\|\mathbf{r}_r \times \mathbf{r}_\theta\|} \|\mathbf{r}_r \times \mathbf{r}_\theta\| \, dr \, d\theta$$

(6) Evaluate the integral:

$$\Phi = \int_0^{2\pi} \int_1^2 \langle r\cos\theta, r\sin\theta, r\rangle \cdot \langle -\cos\theta, -\sin\theta, 1\rangle r\sqrt{2} \, dr \, d\theta$$
$$= \int_0^{2\pi} \int_1^2 (-r^2 + r^2)\sqrt{2} \, dr \, d\theta = 0$$

(7) Final answer:

$$\boxed{0}$$

4. Let C be the curve parametrized by

$$\mathbf{r}(t) = \langle \cos(t)^3, \sin(t)^3 \rangle, \qquad t \in [0, 2\pi],$$

and let R be the region enclosed by C. Use GeoGebra (or another online tool) to visualize R, then find the area of R by applying Green's theorem. (Hint: Use the vector field $\mathbf{F}(x, y) = \frac{1}{2}\langle -y, x \rangle$. The identity $\sin(t)^2 \cos(t)^2 = \frac{1}{8}(1 - \cos(4t))$ will be helpful.)

(1) Set up the area integral using Green's theorem:

$$A = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

where $\mathbf{F}(x, y) = \frac{1}{2}\langle -y, x \rangle$, so $P = -\frac{y}{2}$ and $Q = \frac{x}{2}$.

(2) Compute the partial derivatives:

$$\frac{\partial Q}{\partial x} = \frac{1}{2}, \quad \frac{\partial P}{\partial y} = -\frac{1}{2}$$

(3) Substitute into the area integral:

$$A = \iint_R (1) dA$$

(4) Parametrize the curve C:

$$x(t) = \cos^3 t, \quad y(t) = \sin^3 t, \quad t \in [0, 2\pi]$$

(5) Compute dx and dy:

$$dx = -3\cos^2 t \sin t dt, \quad dy = 3\sin^2 t \cos t dt$$

(6) Set up the line integral:

$$A = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$

where $\mathbf{r}'(t) = \langle -3\cos^2 t \sin t, 3\sin^2 t \cos t \rangle$.

(7) Evaluate the line integral:

$$\begin{aligned} A &= \int_0^{2\pi} \frac{1}{2}\langle -\sin^3 t, \cos^3 t \rangle \cdot \langle -3\cos^2 t \sin t, 3\sin^2 t \cos t \rangle dt \\ &= \int_0^{2\pi} \frac{1}{2}(3\sin^4 t \cos^2 t + 3\cos^4 t \sin^2 t) dt \\ &= \frac{3}{2} \int_0^{2\pi} \sin^2 t \cos^2 t dt \\ &= \frac{3}{16}\left[t - \frac{\sin(4t)}{4} \right]_0^{2\pi} = \frac{3}{16}(2\pi) = \frac{3\pi}{8} \end{aligned}$$

(8) Final answer:

$$\boxed{\frac{3\pi}{8}}$$

5. Let C be the curve within the surface $z = 2 + \cos x + \sin y$ that lies directly above the square with vertices $(0,0), (0,\pi), (\pi,0), (\pi,\pi)$ and is oriented counterclockwise when viewed from above. Let $\mathbf{F}(x,y,z) = \langle z - 3y, x + z, 2y \rangle$. Evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

(1) By Stokes' Theorem: $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$.

(2) Calculate Curl($\mathbf{F}$):

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ z - 3y & x + z & 2y \end{vmatrix}$$

$$= \mathbf{i}(2 - 1) - \mathbf{j}(0 - 1) + \mathbf{k}(1 - (-3)) = \langle 1, 1, 4 \rangle$$

(3) Find the normal vector $\mathbf{n}\, dS$: For surface $z = 2 + \cos x + \sin y$, oriented upward (consistent with counterclockwise C):

$$\mathbf{n}\, dS = \langle -z_x, -z_y, 1 \rangle \, dA$$

$$z_x = -\sin x, \quad z_y = \cos y$$

$$\mathbf{n}\, dS = \langle \sin x, -\cos y, 1 \rangle \, dA$$

(4) Evaluate the dot product:

$$(\nabla \times \mathbf{F}) \cdot \mathbf{n} = \langle 1, 1, 4 \rangle \cdot \langle \sin x, -\cos y, 1 \rangle = \sin x - \cos y + 4$$

(5) Integrate over the region $[0, \pi] \times [0, \pi]$:

$$\int_0^\pi \int_0^\pi (\sin x - \cos y + 4) \, dy \, dx$$

(6) Inner integral (y):

$$[\sin x \cdot y - \sin y + 4y]_0^\pi = (\pi \sin x - 0 + 4\pi) - (0) = \pi \sin x + 4\pi$$

(7) Outer integral (x):

$$\int_0^\pi (\pi \sin x + 4\pi) \, dx = [-\pi \cos x + 4\pi x]_0^\pi$$

$$= (-\pi(-1) + 4\pi^2) - (-\pi(1) + 0) = \pi + 4\pi^2 + \pi = 2\pi + 4\pi^2$$

(8) Final answer:

$$\boxed{2\pi + 4\pi^2}$$

6. Let Σ be the surface of the solid tetrahedron bounded by the planes

$$x = 0, \quad y = 0, \quad z = 0, \quad x + y + z = 1$$

in the first octant, with outward unit normal vector $\mathbf{n}$. Let

$$\mathbf{F}(x, y, z) = \langle x^2 + y, xy, -2xz - y \rangle.$$

Evaluate

$$\int_{\Sigma} \mathbf{F} \cdot \mathbf{n} dS.$$

(1) Compute the divergence of $\mathbf{F}$:

$$\begin{aligned} \nabla \cdot \mathbf{F} &= \frac{\partial(x^2 + y)}{\partial x} + \frac{\partial(xy)}{\partial y} + \frac{\partial(-2xz - y)}{\partial z} \\ &= 2x + x - 2x = x \end{aligned}$$

(2) Set up the volume integral using the Divergence Theorem:

$$\int_{\Sigma} \mathbf{F} \cdot \mathbf{n} dS = \iiint_V (\nabla \cdot \mathbf{F}) dV$$

where V is the tetrahedron bounded by the planes.

(3) Set up the limits of integration:

$$0 \le x \le 1, \quad 0 \le y \le 1 - x, \quad 0 \le z \le 1 - x - y$$

(4) Evaluate the volume integral:

$$\begin{aligned} \iiint_V x\, dz\, dy\, dx &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x\, dz\, dy\, dx \\ &= \int_0^1 \int_0^{1-x} x(1 - x - y)\, dy\, dx \\ &= \int_0^1 x \left[(1 - x)y - \frac{y^2}{2} \right]_0^{1-x} dx \\ &= \int_0^1 x \left[(1 - x)(1 - x) - \frac{(1 - x)^2}{2} \right] dx \\ &= \int_0^1 x \left[\frac{(1 - x)^2}{2} \right] dx = \frac{1}{2} \int_0^1 x(1 - 2x + x^2) dx \\ &= \frac{1}{2} \left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} \right]_0^1 = \frac{1}{2} \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) \\ &= \frac{1}{2} \left(\frac{6 - 8 + 3}{12} \right) = \frac{1}{2} \cdot \frac{1}{12} = \frac{1}{24} \end{aligned}$$

(5) Final answer:

$$\boxed{\frac{1}{24}}$$

7. Let Σ be the surface of the solid region bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $z = 3$, with outward unit normal vector $\mathbf{n}$. Let

$$\mathbf{F}(x, y, z) = \langle x^3, y^3, z^3 \rangle.$$

Evaluate

$$\int_{\Sigma} \mathbf{F} \cdot \mathbf{n} dS.$$

(1) Compute the divergence of $\mathbf{F}$:

$$\nabla \cdot \mathbf{F} = \frac{\partial(x^3)}{\partial x} + \frac{\partial(y^3)}{\partial y} + \frac{\partial(z^3)}{\partial z} = 3x^2 + 3y^2 + 3z^2$$

(2) Set up the volume integral using the Divergence Theorem:

$$\int_{\Sigma} \mathbf{F} \cdot \mathbf{n} dS = \iiint_V (\nabla \cdot \mathbf{F}) dV$$

where V is the solid region bounded by the cylinder and planes.

(3) Convert to cylindrical coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$
$$dV = r\,dr\,d\theta\,dz$$

The limits are: $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$, $0 \leq z \leq 3$.

(4) Evaluate the volume integral:

$$\iiint_V (3r^2 + 3z^2) r\,dr\,d\theta\,dz = \int_0^{2\pi} \int_0^2 \int_0^3 (3r^3 + 3z^2 r) dz\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^2 [3r^3 z + rz^3]_0^3 dr d\theta = \int_0^{2\pi} \int_0^2 (9r^3 + 27r) dr d\theta$$

$$= \int_0^{2\pi} [\frac{9r^4}{4} + \frac{27r^2}{2}]_0^2 d\theta = \int_0^{2\pi} (36 + 54) d\theta = 90 \int_0^{2\pi} d\theta = 90(2\pi) = 180\pi$$

(5) Final answer:

$$\boxed{180\pi}$$