Day 26

1. Applying the second derivative test

Last week we saw the statement of the second derivative test, which provides sufficient conditions for a stationary point of a function $f(x, y)$ to be a local maximum, local minimum, or saddle point. Let's recall the statement:

The Hessian and the Discriminant of a Function of Two Variables

Let $f(x, y)$ be a function with continuous second-order partial derivatives on some open set S.

(a) The ***Hessian*** of f is the 2×2 matrix of second-order partial derivatives:

$$H_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}.$$

(b) The ***discriminant*** of f is the determinant of the Hessian. That is,

$$\det(H_f) = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2.$$

THEOREM 12.45 **The Second-Order Partial-Derivative Test for Classifying Stationary Points**

Let $f(x, y)$ be a function with continuous second-order partial derivatives on some open disk containing the point at (x_0, y_0). If f has a stationary point at (x_0, y_0), then

(a) f has a relative maximum at (x_0, y_0) if $\det(H_f(x_0, y_0)) > 0$ with $f_{xx}(x_0, y_0) < 0$ or $f_{yy}(x_0, y_0) < 0$.

(b) f has a relative minimum at (x_0, y_0) if $\det(H_f(x_0, y_0)) > 0$ with $f_{xx}(x_0, y_0) > 0$ or $f_{yy}(x_0, y_0) > 0$.

(c) f has a saddle point at (x_0, y_0) if $\det(H_f(x_0, y_0)) < 0$.

(d) If $\det(H_f(x_0, y_0)) = 0$, no conclusion may be drawn about the behavior of f at (x_0, y_0).

As an example, let $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 4$. Let's find the critical points of f and, if possible, classify each one as a local maximum, local minimum, or saddle point. Since f is differentiable everywhere, every critical point will be a stationary point, i.e. a solution to $\nabla f(x, y) = \mathbf{0}$. We calculate that

$$\nabla f(x, y) = \langle 6x(y - 1), 3x^2 + 3y^2 - 6y \rangle,$$

and thus we need to solve

$$\begin{cases} 6x(y - 1) = 0, \\ 3x^2 + 3y^2 - 6y = 0. \end{cases}$$

The first equation implies that $x = 0$ or $y = 1$. If $x = 0$, then the second equation becomes $3y(y-2) = 0$ and has solutions $y = 0, 2$. Thus $(0, 0)$ and $(0, 2)$ are critical points. If $y = 1$, then the second equation becomes $3(x^2 - 1) = 0$ and has solutions $x = -1, 1$. Thus $(-1, 1)$ and $(1, 1)$ are also critical points, and there are no others.

Now, for each of the four critical points we've found, we need to decide, if possible, whether it's a local minimum, local maximum, or saddle point. We can use the second derivative test. We calculate that

$$H_f(x,y) = \begin{bmatrix} 6(y-1) & 6x \\ 6x & 6(y-1) \end{bmatrix}$$

and

$$\det(H_f(x,y)) = 36((y-1)^2 - x^2).$$

Therefore, we can conclude:

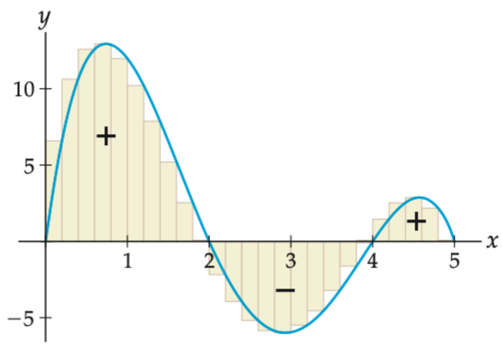
- $\det(H_f(0,0)) = 36 > 0$ and $f_{xx}(0,0) = -6 < 0$, so f has a local maximum at $(0,0)$;
- $\det(H_f(0,2)) = 36 > 0$ and $f_{xx}(0,2) = 6 > 0$, so f has a local minimum at $(0,2)$;
- $\det(H_f(-1,1)) = -36 < 0$, so f has a saddle point at $(-1,1)$;
- $\det(H_f(1,1)) = -36 < 0$, so f has a saddle point at $(1,1)$.

For another example, consider the function $f(x,y) = x^4 + y^4$. The only critical point of f is $(0,0)$. To classify this point, we could try applying the second derivative test. Observe, however, that all second partial derivatives of f are equal to 0 at $(0,0)$, and therefore $\det H_f(0,0) = 0$. This means the second derivative test cannot provide any conclusion. So what should we do? If we graph this function, we see that it looks similar to an upward-opening paraboloid with vertex at the origin. Thus $(0,0)$ should be a local minimum. Indeed, because a sum of fourth powers is always nonnegative, we have $f(0,0) = 0 \leq x^4 + y^4 = f(x,y)$ for all (x,y), which confirms that $(0,0)$ is a local (and global) minimum. This example demonstrates that even when the second derivative test is inconclusive, it may be possible to classify a critical point using other techniques.

Day 27

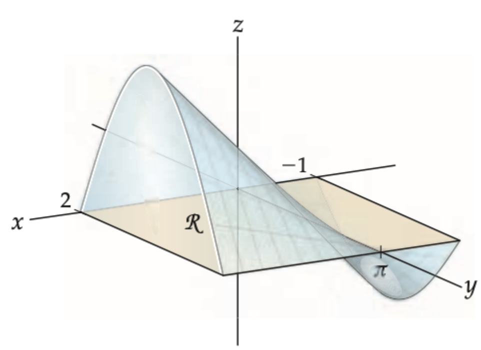
1. Double integrals

At this point we transition into the next unit of this course: integration of multivariable functions. Recall that for a function $f(x)$ and real numbers $a < b$, the definite integral $\int_a^b f(x)dx$ represents the signed area bounded between the graph of f and the interval $[a, b]$ within the x-axis:

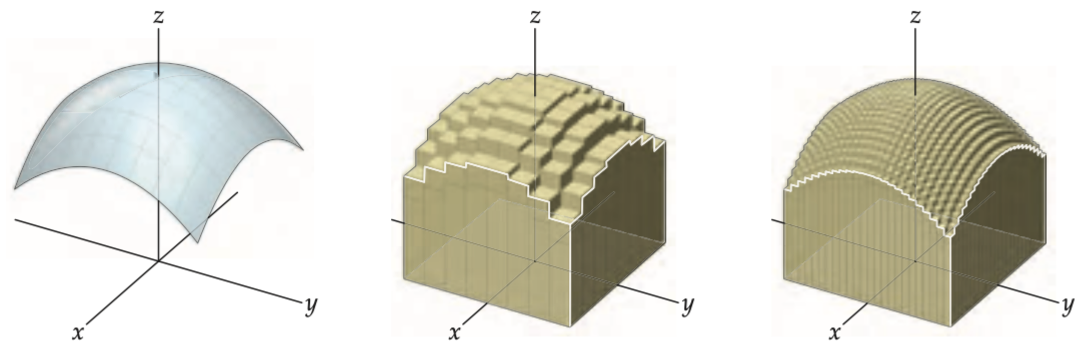


The integral is formally defined as a limit of Riemann sums, but in practice we usually calculate its value using antiderivatives via the fundamental theorem of calculus.

The corresponding concept for a two-variable function $f(x, y)$ is called a *double integral*. Let R be a region of the plane $\mathbb{R}^2$ (think a rectangle or disc). The double integral $\iint_R f(x, y)dA$ is the signed volume bounded between the graph of f and the region R within the xy-plane:



Like the ordinary integral, the double integral is defined as a limit of (two-variable) Riemann sums. Each term in the Riemann sum is the signed volume of a thin rectangular box bounded between the graph of f and the region R. The sum of these volumes converges to a fixed value as the number of boxes tends to infinity:



This limiting value is the definition of the double integral and can be interpreted as the signed volume under the graph.

2. Iterated integrals

Also like with the ordinary integral, we do not typically use the Riemann sum definition to evaluate a double integral. Instead, we calculate it using a closely related concept called an *iterated integral*. This will allow us to represent a double integral as a sequence two ordinary integrals.

An iterated integral is an expression of the form

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx = \int_a^b \Big(\int_{g_1(x)}^{g_2(x)} f(x,y)dy \Big) dx$$

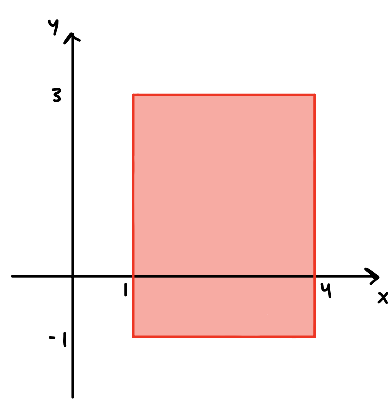
or

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy = \int_c^d \Big(\int_{h_1(y)}^{h_2(y)} f(x,y)dx \Big) dy.$$

As a first example, let's consider the iterated integral

$$\int_1^4 \int_{-1}^3 x^2 y dydx.$$

In the outer integral, x ranges between 1 and 4. Then, with x fixed, the inner integral allows y to range between -1 and 3. The set of points (x, y) that satisfy $1 \leq x \leq 4$ and $-1 \leq y \leq 3$ is a rectangle in the plane:



As we will soon see, the iterated integral above is equal to the corresponding double integral over this rectangle. That is, if R is the rectangle $R = [1,4] \times [-1,3]$, then

$$\iint_R x^2 y dA = \int_1^4 \int_{-1}^3 x^2 y dy dx.$$

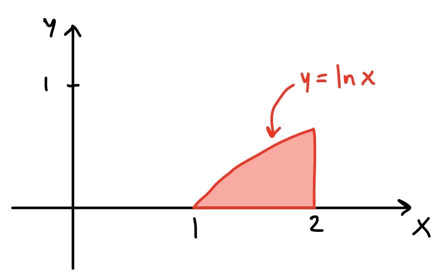
Now let's calculate the iterated integral:

$$\begin{aligned}
\int_1^4 \int_{-1}^3 x^2 y dy dx &= \int_1^4 \left[\frac{x^2 y^2}{2} \Big|_{y=-1}^{y=3} \right] dx \\
&= \int_1^4 4x^2 dx \\
&= \frac{4}{3} x^3 \Big|_{x=1}^{x=4} \\
&= 84.
\end{aligned}$$

As a second example, let's consider

$$\int_1^2 \int_0^{\ln x} x e^y dy dx.$$

In the outer integral, x ranges between 1 and 2. With x fixed, the inner integral has y ranging between 0 and $\ln x$. The set of points (x, y) that satisfy $1 \le x \le 2$ and $0 \le y \le \ln x$ looks like this:



Again, if we call this region R, then we'll soon see that

$$\iint_R x e^y dA = \int_1^2 \int_0^{\ln x} x e^y dy dx.$$

Now let's calculate the iterated integral:

$$\begin{aligned}
\int_1^2 \int_0^{\ln x} x e^y dy dx &= \int_1^2 \left[x e^y \Big|_{y=0}^{y=\ln x} \right] dx \\
&= \int_1^2 (x^2 - x) dx \\
&= \left(\frac{x^3}{3} - \frac{x^2}{2} \right) \Big|_{x=1}^{x=2} \\
&= \frac{5}{6}.
\end{aligned}$$

In-class exercise[1]**:** Consider the iterated integral

$$\int_0^2 \int_x^{2x} (x^2 - 3y^2)\,dy\,dx.$$

(a) Sketch the region of integration.

(b) Evaluate the iterated integral.

3. Fubini's theorem

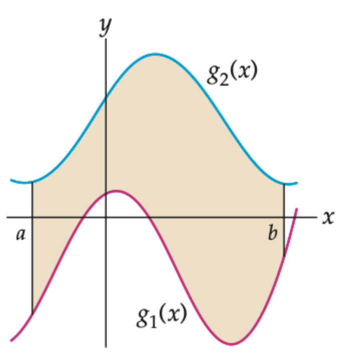
So what exactly is the relationship between double integrals and iterated integrals? Fubini's theorem addresses this question. Before stating this theorem, we need some terminology:

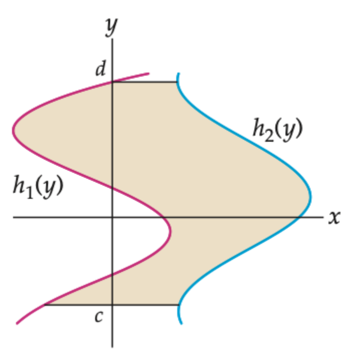
Type I and Type II Regions

(a) Let $y = g_1(x)$ and $y = g_2(x)$ be two functions defined on the interval $[a, b]$ such that $g_1(x) \le g_2(x)$ for every $x \in [a, b]$. The region Ω bounded above by $g_2(x)$, below by $g_1(x)$, on the left by the line $x = a$, and on the right by the line $x = b$ is said to be a ***type I region***.

(b) Let $x = h_1(y)$ and $x = h_2(y)$ be two functions defined on the interval $[c, d]$ such that $h_1(y) \le h_2(y)$ for every $y \in [c, d]$. The region Ω bounded on the left by $h_1(x)$, on the right by $h_2(y)$, below by the line $y = c$, and above by the line $y = d$ is said to be a ***type II region***.

In other words, a region that's bounded above and below by two functions of x is a type I region, and a region that's bounded on either side by two functions of y is a type II region. Here's the picture to have in mind:





Many regions are both of type I and type II. In fact, all of the regions we've considered so far are of both types. Let's look at the region $R = \{(x, y)\colon 1 \le x \le 2,\ 0 \le y \le \ln x\}$ from our second example above. It looked like this:

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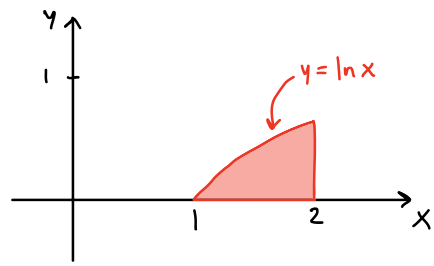
[1] (a) Triangle bounded by the lines $y = x$, $y = 2x$, and $x = 2$
(b) -24



We initially expressed R as a type I region (bounded between $y = 0$ and $y = \ln x$). But we could equally well express R as a type II region by observing that, within R, y ranges between 0 and $\ln 2$, and if y is fixed then x ranges between e^y and 2. In other words, $R = \{(x, y) \colon 0 \le y \le \ln 2,\ e^y \le x \le 2\}$.

Here is the version of Fubini's theorem that we will use. Note that the textbook states this theorem only in the case where R is a rectangular region.

Fubini's theorem. *Let R be a type I region given by*

$$R = \{(x, y) \colon a \le x \le b,\ g_1(x) \le y \le g_2(x)\}.$$

Let S be a type II region given by

$$S = \{(x, y) \colon c \le y \le d,\ h_1(y) \le x \le h_2(y)\}.$$

Then

$$\iint_R f(x, y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dydx \qquad \textit{and} \qquad \iint_S f(x, y)dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x, y)dxdy.$$

Day 28

1. Using Fubini's theorem

Yesterday we learned about the double integral of a function $f(x, y)$ over region R in the xy-plane. This was denoted $\iint_R f(x, y)dA$. While the double integral is formally defined as a limit of Riemann sums, Fubini's theorem provides an easier method of calculation. Specifically, Fubini's theorem tells us that for "nice" regions (of type I or type II), we can express $\iint_R f(x, y)dA$ as an iterated integral, a composition of two single-variable integrals.

In-class exercise[1]**:** Let $R = \{(x, y)\colon -2 \leq x \leq 2,\ 0 \leq y \leq 4 - x^2\}$. Evaluate the double integral

$$\iint_R x dA.$$

Besides providing a method of calculating double integrals, Fubini's theorem also allows us to reverse the order of integration in an iterated integral, provided the region of integration is both of type I and type II. This can be extremely helpful.

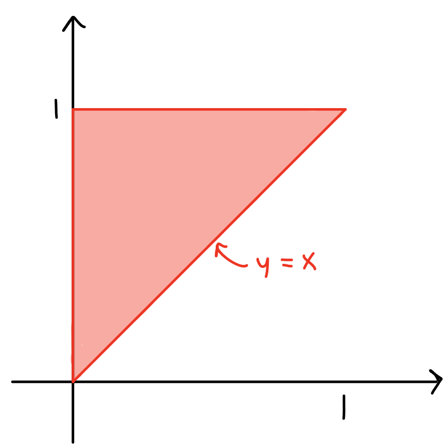
As an example, let's consider the iterated integral

$$\int_0^1 \int_x^1 e^{y^2} dy dx.$$

The function e^{y^2} does not have a nice antiderivative, so we have no method for evaluating the inner integral. Let's take a look at the region of integration and see if we can express the iterated integral in a different way. The region of integration is given by

$$R = \{(x, y)\colon 0 \leq x \leq 1,\ x \leq y \leq 1\}$$

and looks like this:



While we expressed R as a type I region, we can see visually that it's also of type II: Observe that, within R, y varies between 0 and 1, and if y is fixed then x ranges between 0 and y.

[1] 0

Thus we also have the type II expression $R = \{(x, y)\colon 0 \le y \le 1,\ 0 \le x \le y\}$. This means (by Fubini's theorem) that the iterated integral above can be rewritten as

$$\int_0^1 \int_x^1 e^{y^2} dy dx = \int_0^1 \int_0^y e^{y^2} dx dy.$$

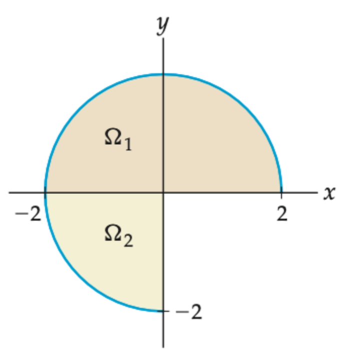
Notice that right-hand side can be evaluated:

$$\begin{aligned}
\int_0^1 \int_0^y e^{y^2} dx dy &= \int_0^1 \left[e^{y^2} x \Big|_{x=0}^{x=y} \right] dy \\
&= \int_0^1 e^{y^2} y dy \\
&= \frac{1}{2} e^{y^2} \Big|_{y=0}^{y=1} \\
&= \frac{1}{2}(e-1).
\end{aligned}$$

To summarize what we've learned so far: If R is a region of type I or type II, then we can calculate the double integral $\iint_R f(x, y) dA$ by expressing it as an iterated integral. Many regions are of both type I and type II, and utilizing one type over the other may result in an easier iterated integral.

2. Decomposing regions

What do we do for regions that are neither of type I nor type II? In general, there's no easy answer; regions can be incredibly complicated. But most reasonable regions that you will encounter can be decomposed into nonoverlapping subregions of type I or type II. For example, consider this region:



It's neither of type I nor type II. However, if we let Ω_1 be the subregion where y is nonnegative and let Ω_2 be the subregion where y is negative, then Ω_1 and Ω_2 will each be of type I (in fact they're of both types).

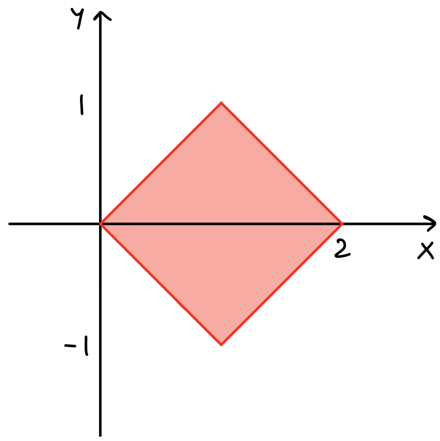
Double integrals are well behaved with respect to these kinds of decompositions: If R is a union of two subregions R_1 and R_2 that do not overlap (except possibly at their boundaries), then

$$\iint_R f(x, y) dA = \iint_{R_1} f(x, y) dA + \iint_{R_2} f(x, y) dA.$$

As an example, let's evaluate

$$\iint_R \frac{1}{(1+x+y)^2} dA,$$

where R is the square region with vertices $(0,0)$, $(1,1)$, $(2,0)$, $(1,-1)$. Here's a sketch of R:



It's neither of type I nor type II, but we can decompose it into two such regions. Let R_1 be the upper half of the square, and let R_2 be the lower half. Visually, we can see that R_1 and R_2 are of type II. The upper edges of the square lie within the lines $x = y$ and $x = 2 - y$. Thus

$$R_1 = \{(x, y) \colon 0 \le y \le 1,\ y \le x \le 2 - y\}.$$

The lower edges of the square lie within the lines $x = -y$ and $x = y + 2$. Thus

$$R_2 = \{(x, y) \colon -1 \le y \le 0,\ -y \le x \le y + 2\}.$$

Therefore, we have

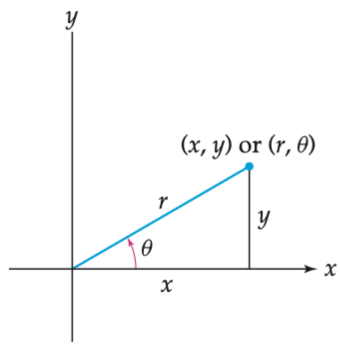
$$\begin{aligned}\iint_R \frac{1}{(1+x+y)^2} dA &= \iint_{R_1} \frac{1}{(1+x+y)^2} dA + \iint_{R_2} \frac{1}{(1+x+y)^2} dA \\ &= \int_0^1 \int_y^{2-y} \frac{1}{(1+x+y)^2} dx dy + \int_{-1}^0 \int_{-y}^{y+2} \frac{1}{(1+x+y)^2} dx dy.\end{aligned}$$

From here it's just a matter of evaluating the iterated integrals. The first one works out to be $\frac{1}{2}\ln(3) - \frac{1}{3}$, and the second is $1 - \frac{1}{2}\ln(3)$. Therefore the final answer is $2/3$.

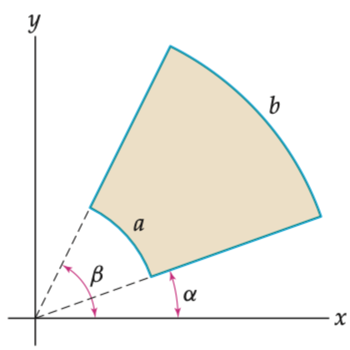
Day 29

1. Polar coordinates

Recall that every point (x, y) in $\mathbb{R}^2$ can be represented in polar coordinates (r, θ), where $x = r\cos\theta$ and $y = r\sin\theta$. This representation is not unique, however we commonly choose r to be the distance between (x, y) and origin and θ to be the angle between the vectors $\langle x, y\rangle$ and $\langle 1, 0\rangle$. We can visualize the relationship like this:



Many regions in $\mathbb{R}^2$ are expressed more easily in polar coordinates than Cartesian coordinates. For example, consider the following region R:



We can see that R is neither of type I nor type II, and even subdividing it into regions of type I or II would be cumbersome. But R can be expressed very simply in polar coordinates:

$$R = \{(r, \theta) : \alpha \leq \theta \leq \beta,\ a \leq r \leq b\}.$$

If we were to draw R in the "$r\theta$-plane" (say with the horizontal axis representing r and the vertical axis representing θ), then R would be a rectangle.

In-class exercise[1]: Express the following regions in polar coordinates. (Hint: For region (c), first find a polar coordinate equation for the boundary circle by converting its Cartesian coordinate equation.)

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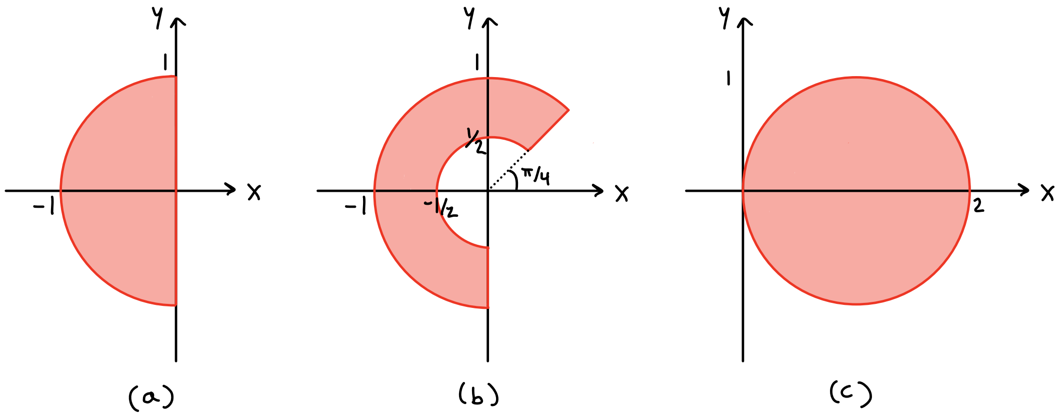
[1] (a) $\{(r, \theta) : \pi/2 \leq \theta \leq 3\pi/2,\ 0 \leq r \leq 1\}$
(b) $\{(r, \theta) : \pi/4 \leq \theta \leq 3\pi/2,\ 1/2 \leq r \leq 1\}$
(c) $\{(r, \theta) : -\pi/2 \leq \theta \leq \pi/2, 0 \leq r \leq 2\cos\theta\}$ (Note that the condition $r \leq 2\cos\theta$ comes from converting the Cartesian formula $(x-1)^2 + y^2 = 1$ to polar form $(r\cos(\theta) - 1)^2 + (r\sin\theta)^2 = 1$ and simplifying.)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x
(a)
y
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-1
-1/2
π/4
x
(b)
y
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x
(c)