Formula Sheet

1. Vectors

$$\begin{aligned}
\|\mathbf{v}\| &= \sqrt{v_1^2 + v_2^2 + v_3^2} \\
\mathbf{u} \cdot \mathbf{v} &= u_1 v_1 + u_2 v_2 + u_3 v_3 = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta \\
\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}, \quad \|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin\theta \\
\operatorname{proj}_{\mathbf{u}} \mathbf{v} &= \frac{\mathbf{v} \cdot \mathbf{u}}{\|\mathbf{u}\|^2} \mathbf{u}
\end{aligned}$$

2. Lines and planes

If $P = (x_0, y_0, z_0)$, then:

- line through P with direction $\mathbf{d}$: $\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\mathbf{d}$
- plane through P with normal $\mathbf{n}$: $\mathbf{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

3. Vector functions

$$\begin{aligned}
\mathbf{r}'(t) &= \frac{d}{dt} \langle x(t), y(t), z(t) \rangle = \langle x'(t), y'(t), z'(t) \rangle \\
\text{arc length } &= \int_a^b \|\mathbf{r}'(t)\| dt \\
\text{curvature } &= \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}
\end{aligned}$$

4. Multivariable functions

Gradient and directional derivatives:

$\nabla f = \langle f_x, f_y, f_z \rangle$

∇f points in direction of fastest increase of f

∇f is orthogonal to the level surface

$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$

Tangent planes:

- graph of $f(x, y)$: $f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) = 0$
- level surface $F(x, y, z) = 0$: $\nabla F(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

Chain rule: If $z = f(x, y)$ and $x = g(s, t)$, $y = h(s, t)$, then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}, \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

5. Optimization

Second derivative test: Let f be a twice-differentiable function, and let

$$H_f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}.$$

If $\nabla f(x,y) = \mathbf{0}$, then:

(a) f has a relative maximum at (x,y) if $\det(H_f(x,y)) > 0$ with $f_{xx}(x,y) < 0$ or $f_{yy}(x,y) < 0$;
(b) f has a relative minimum at (x,y) if $\det(H_f(x,y)) > 0$ with $f_{xx}(x,y) > 0$ or $f_{yy}(x,y) > 0$;
(c) f has a saddle point at (x,y) if $\det(H_f(x,y)) < 0$;
(d) no conclusion may be drawn if $\det(H_f(x,y)) = 0$.

6. Double and triple integration

Polar coordinates:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad dA = r dr d\theta$$

Cylindrical coordinates:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z, \qquad dV = r dz dr d\theta$$

Spherical coordinates:

$$x = \rho\sin\phi\cos\theta, \qquad y = \rho\sin\phi\sin\theta, \qquad z = \rho\cos\phi, \qquad dV = \rho^2\sin(\phi) d\rho d\theta d\phi$$

7. Vector fields

Line integrals:

$$\int_C f ds = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\| dt, \qquad \int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) dt$$

Surface integrals:

If Σ is parametrized by $\mathbf{r}(s,t)$:

$$\int_\Sigma f dS = \iint_R f(\mathbf{r}(s,t))\|\mathbf{r}_s\times\mathbf{r}_t\| dA$$

If Σ is the graph of $g(x,y)$:

$$\int_\Sigma f dS = \iint_R f(x,y,g(x,y))\sqrt{\|\nabla g\|^2+1} dA$$

Flux:

$$\int_\Sigma \mathbf{F}\cdot\mathbf{n} dS$$

Curl and divergence:

$$\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F}, \qquad \operatorname{div} = \nabla\cdot\mathbf{F}$$

Unit normal vector:

- graph of $g(x,y)$: $\quad \mathbf{n} = \pm\dfrac{\langle -g_x, -g_y, 1\rangle}{\sqrt{\|\nabla g\|^2+1}}$
- surface parametrized by $\mathbf{r}(s,t)$: $\quad \mathbf{n} = \pm\dfrac{\mathbf{r}_s\times\mathbf{r}_t}{\|\mathbf{r}_s\times\mathbf{r}_t\|}$