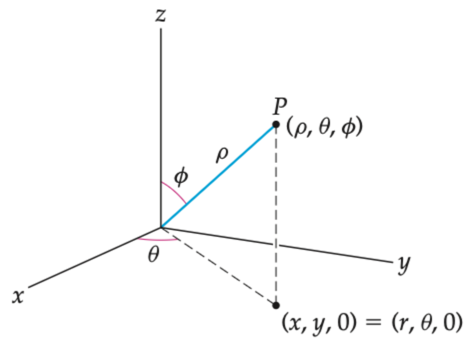
Day 34

1. Integration in spherical coordinates

Last time we began working with spherical coordinates. Recall that we can represent (x, y, z) in spherical coordinates (ρ, θ, ϕ) by taking ρ to be the distance between (x, y, z) and the origin, θ to be the polar angle associated to (x, y), and ϕ to be the angle between the vector $\langle x, y, z\rangle$ and the positive z-axis:



Integration works as follows: If R is a region which, when expressed in spherical coordinates, takes the form

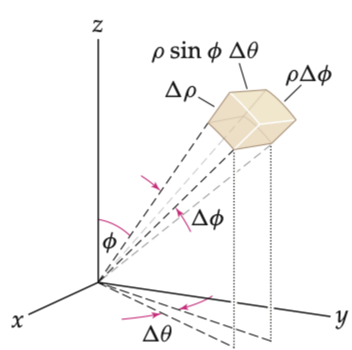
$$R = \{(\rho, \theta, \phi)\colon \alpha \le \theta \le \beta,\ g_1(\theta) \le \phi \le g_2(\theta),\ h_1(\theta, \phi) \le \rho \le h_2(\theta, \phi)\},$$

then

$$\iiint_R f(x, y, z)dV = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(\theta,\phi)}^{h_2(\theta,\phi)} f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\theta))\rho^2\sin(\phi)d\rho d\phi d\theta.$$

As usual, other orders of integration are possible, but this one is the most common.

Notice that dV has become $\rho^2\sin(\phi)d\rho d\phi d\theta$. The reason for this is similar to the reason r appears in $dV = rdzdrd\theta$ (or $dA = rdrd\theta$). Consider a small region of $\mathbb{R}^3$ in which ρ, θ, and ϕ vary by $\Delta\rho$, $\Delta\theta$, and $\Delta\phi$ respectively:

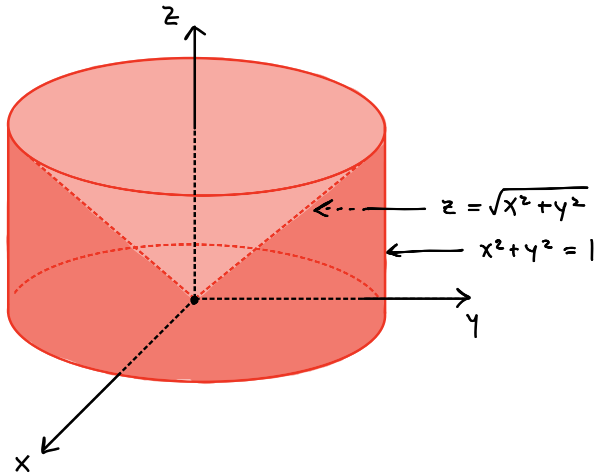


If this region is situated at the point (ρ, θ, ϕ), then its volume in xyz-space works out to be approximately $\rho^2\sin(\phi)\Delta\rho\Delta\phi\Delta\theta$. But when mapped to $\rho\theta\phi$-space, the region becomes a $\Delta\rho \times \Delta\theta \times \Delta\phi$ box with volume $\Delta\rho\Delta\theta\Delta\phi$. So, when integrating in spherical coordinates, we need to multiply our function f by an extra factor of $\rho^2\sin(\phi)$ to correct for this distortion.

As an example, let's evaluate the triple integral

$$\iiint_R z\sqrt{x^2+y^2+z^2}dV,$$

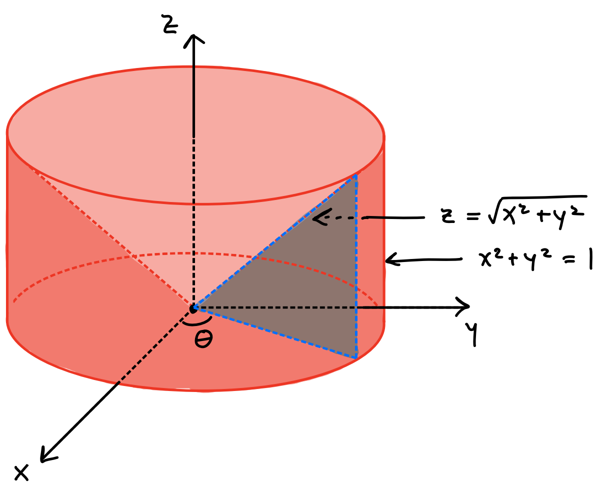
where R is the region bounded by the plane $z=0$, the cylinder $x^2+y^2=1$, and the cone $z=\sqrt{x^2+y^2}$. Here's what the region looks like:



We will convert the triple integral into a $d\rho d\phi d\theta$ iterated integral. Because the region is rotationally symmetric about the z-axis, we have no restrictions on θ, i.e.

$$0 \leq \theta \leq 2\pi.$$

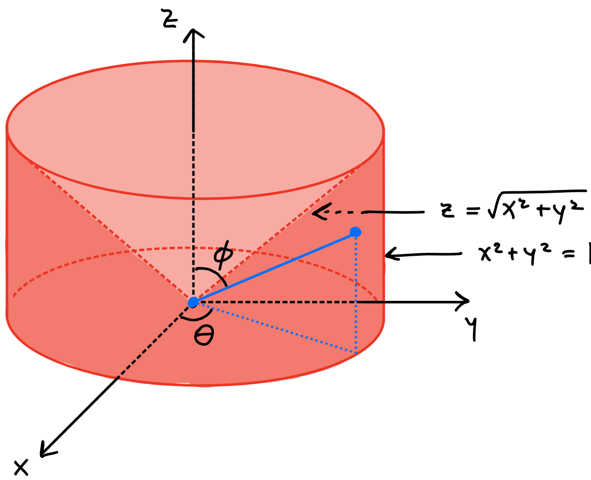
For fixed θ, the set of points in R of the form (ρ, θ, ϕ) forms a triangle:



Within this triangle, the minimum and maximum values of ϕ are $\pi/4$ (along the hypotenuse) and $\pi/2$ (along the horizontal leg). Note that the lower bound of $\pi/4$ can be obtained through geometry or by converting the equation $z=\sqrt{x^2+y^2}$ into spherical coordinates and simplifying. Thus

$$\frac{\pi}{4} \leq \phi \leq \frac{\pi}{2}.$$

For fixed θ and ϕ, the set of points in R of the form (ρ, θ, ϕ) forms a line segment:



Within this segment, the minimum and maximum values of ρ are 0 (at the origin) and $1/\sin(\phi)$ (at the cylindrical boundary). Note that the upper bound of $1/\sin(\phi)$ can be found using trigonometry or by converting the equation $x^2 + y^2 = 1$ into spherical coordinates and simplifying. Thus

$$0 \leq \rho \leq \frac{1}{\sin(\phi)}.$$

Using these bounds, we can now set up and evaluate the triple integral. Be aware that $x^2 + y^2 + z^2 = \rho^2$; this will greatly simplify the integrand. We have

$$\begin{aligned}
\iiint_R z\sqrt{x^2+y^2+z^2}dV &= \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{1/\sin(\phi)} \rho^4 \cos(\phi)\sin(\phi) d\rho d\phi d\theta \\
&= \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \frac{\cos(\phi)}{5\sin(\phi)^4} d\phi d\theta \\
&= \int_0^{2\pi} \left[-\frac{1}{15\sin(\phi)^3} \Bigg|_{\phi=\pi/4}^{\phi=\pi/2} \right] d\theta \\
&= \int_0^{2\pi} \frac{2\sqrt{2}-1}{15} d\theta \\
&= \frac{2\pi(2\sqrt{2}-1)}{15}.
\end{aligned}$$

Day 35

1. Area and volume

There is still one loose end to tie up related to integration. As you know, there is a fundamental connection between double integration and area and triple integration and volume.

Let's consider area. Let R be a region in $\mathbb{R}^2$. The double integral $\iint_R f(x,y)dA$ is a limit of Riemann sums, each term of which is of the form $f(x_i, y_i)\,\text{Area}(R_i)$, where $R_1, \dots, R_N$ are a partition of R into small rectangles and $(x_1, y_1), \dots, (x_N, y_N)$ are points such that (x_i, y_i) belongs to R_i. If we take f to be the function $f(x,y) = 1$, then the Riemann sum will simply add up the areas of $R_1, \dots, R_N$ and thus converge to the area of R. In other words,

$$\iint_R 1dA = \text{Area}(R).$$

There are other was to see this as well. For example, the integral above represents the mass of R if R has density 1. But the mass of a two-dimensional solid with uniform density is given by mass = density $\times$ area and thus coincides with area when the density is 1.

Similar reasoning applies to triple integrals. If R is a region in $\mathbb{R}^3$, then

$$\iiint_R 1dV = \text{Volume}(R).$$

2. Vector fields

Now we transition into the next, and final, unit of this course: vector fields. So far we've studied the calculus of functions of the following forms:

(1) $\mathbf{r}\colon \mathbb{R} \to \mathbb{R}^2$ or $\mathbf{r}\colon \mathbb{R} \to \mathbb{R}^3$ (scalar input, vector output)
(2) $f\colon \mathbb{R}^2 \to \mathbb{R}$ or $f\colon \mathbb{R}^3 \to \mathbb{R}$ (vector input, scalar output)

A vector field is a function that has both a vector input (which we think of as a point) and a vector output. Specifically:

Vector Field

A ***vector field in*** $\mathbb{R}^2$ is a function $\mathbf{F}(x,y)$ with domain $D \subseteq \mathbb{R}^2$ and whose outputs are vectors in $\mathbb{R}^2$ of the form

$$\mathbf{F}(x,y) = \langle F_1(x,y), F_2(x,y)\rangle$$

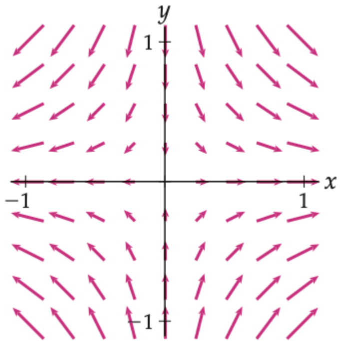
for each point (x,y) in D.

Similarly, a ***vector field in*** $\mathbb{R}^3$ is a function $\mathbf{G}(x,y,z)$ with domain $D \subseteq \mathbb{R}^3$ and whose outputs are vectors in $\mathbb{R}^3$ of the form

$$\mathbf{G}(x,y,z) = \langle G_1(x,y,z), G_2(x,y,z), G_3(x,y,z)\rangle$$

for each point (x,y,z) in D.

A vector field can be visualized using a *vector plot*. For example, if $\mathbf{F}(x,y) = \langle x, -y\rangle$, then its vector plot looks like this:

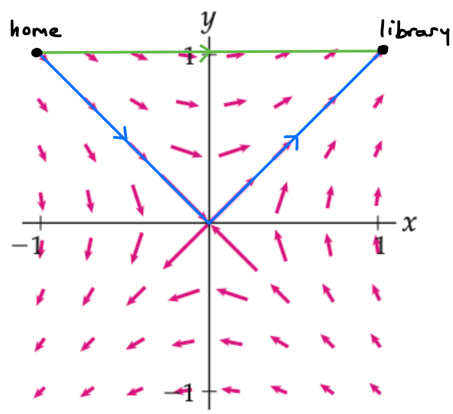


This diagram only shows a sample of the values of **F**, but it's detailed enough to indicate the general shape and behavior of the vector field.

In-class exercise: Sketch a vector plot for $\mathbf{F}(x, y) = \dfrac{\langle y, -x\rangle}{\|\langle y, -x\rangle\|}$.

3. Motivation and plan for the next few weeks

Vector fields are interesting both from a mathematical and a practical point of view. They are very useful for modeling anything that flows through space. For example, suppose it's a windy day and you wanted to plan a bike ride. You have access to lots of weather data, including a vector field $\mathbf{F}(x, y)$ that represents the velocity of the wind at the point (x, y). You're considering two possible routes: (1) Longer distance but with the wind at your back the whole time; (2) Shorter distance but with less wind assistance:



Which route would be easier? The main tool needed to answer this question is the *line integral*. This will be a new form of integration that allows us to calculate the total work done by a vector field along a given curve.

After studying line integrals and work, we'll consider higher dimensional variants: *surface integrals* and "flux" (which quantifies how a vector field flows through a surface).

Finally we'll see a few theorems that can be understood as generalizations of the fundamental theorem of calculus in the vector field setting.

4. Conservative vector fields

Recall that the gradient of a function $f(x, y)$ is the vector $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$. We can see now that ∇f is actually vector field in $\mathbb{R}^2$; it assigns the vector $\langle f_x(x, y), f_y(x, y)\rangle$ to each point (x, y). Similarly, the gradient $\nabla f(x, y, z)$ of a three-variable function $f(x, y, z)$ is a vector field in $\mathbb{R}^3$.

Conservative Vector Field

A ***conservative vector field*** $\mathbf{F}$ is a vector field that can be written as the gradient of some function f. That is,

$$\mathbf{F}(x, y) = \nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

if $f(x, y)$ is a function of two variables, or

$$\mathbf{F}(x, y, z) = \nabla f(x, y, z) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

if $f(x, y, z)$ is a function of three variables.

In either case, any function f whose gradient is equal to $\mathbf{F}$ is called a ***potential function*** for $\mathbf{F}$.

Conservative vector fields have many nice properties. One of the most important properties is that a line integral of a conservative vector field is path-independent; we'll see this soon.

Given a vector field $\mathbf{F}(x, y) = \langle F_1(x, y), F_2(x, y)\rangle$, how can we tell whether it's conservative? We're looking for a function $f(x, y)$ such that $f_x = F_1$ and $f_y = F_2$. Recall Clairaut's theorem: If f is twice continuously differentiable, then $f_{xy} = f_{yx}$. Therefore, if there exists a (twice continuously differentiable) function f such that $f_x = F_1$ and $f_y = F_2$, then we must have

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}.$$

The converse turns out to be true as well. If the above relation holds, then a potential function f can be found. So to summarize: If $\mathbf{F} = \langle F_1, F_2\rangle$ (and F_1 and F_2 are continuously differentiable), then $\mathbf{F}$ is conservative if and only if

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}.$$

A similar statement holds for vector fields on $\mathbb{R}^3$, but there are three pairs of mixed partial derivatives to check.

Day 36

1. Conservative vector fields

Yesterday we saw the definition of a *vector field*, a function

$$\mathbf{F}(x, y) = \langle F_1(x, y), F_2(x, y)\rangle$$

from $\mathbb{R}^2$ to $\mathbb{R}^2$ or

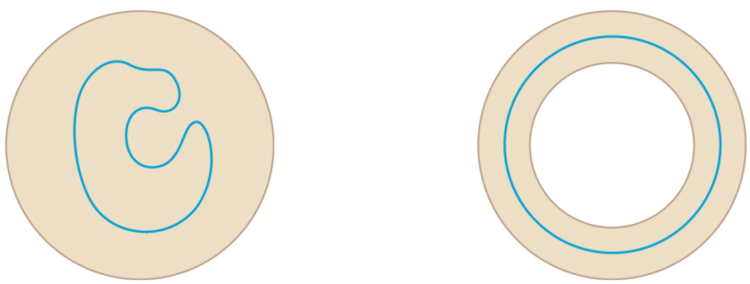
$$\mathbf{F}(x, y, z) = \langle F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)\rangle$$

from $\mathbb{R}^3$ to $\mathbb{R}^3$. Also, recall that a vector field $\mathbf{F}$ is *conservative* if there exists a scalar function f such that $\mathbf{F} = \nabla f$. The function f is called a *potential function* for $\mathbf{F}$. At the end of class, we saw that

$$\mathbf{F} = \langle F_1, F_2\rangle \text{ is conservative} \qquad \Leftrightarrow \qquad \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}.$$

This statement is technically only true for "nice" vector fields $\mathbf{F}$. Specifically, the domain of $\mathbf{F}$ must be an open, simply connected region in which the component functions F_1 and F_2 have continuous partial derivatives. This will be a standing assumption for the remainder of the course.

What does "simply connected" mean? A region R is *simply connected* if any loop in R can be smoothly contracted to a point without leaving R. Below are two regions in $\mathbb{R}^2$. The left-hand region (a disc) is simply connected, while the right-hand region (an annulus) is not:



Essentially, a region in $\mathbb{R}^2$ is simply connected if it doesn't have any holes. (However, regions in $\mathbb{R}^3$ may have holes and still be simply connected.)

If we know that a vector field on $\mathbb{R}^2$ is conservative, how do we find a potential function? Suppose, for example that $\mathbf{F}(x, y) = \langle x, -y\rangle$. We want to find a function f such that $f_x(x, y) = x$ and $f_y(x, y) = -y$. We can work backwards by antidifferentiating:

$$f(x, y) = \int f_x(x, y)dx + C(y) = \frac{x^2}{2} + C(y).$$

Note that since we antidifferentiated with respect to x, the constant of integration $C(y)$ may depend on y; i.e. $C(y)$ is a function of y. Now differentiating with respect to y, we see that $f_y(x, y) = C'(y)$. But we also know that $f_y(x, y) = -y$. Thus $C'(y) = -y$ and consequently

$$C(y) = -\frac{y^2}{2} + A$$

for some constant A. So altogether, we must have

$$f(x,y) = \frac{x^2 - y^2}{2} + A.$$

Technically we've shown that if $f_x(x,y) = x$ and $f_y(x,y) = -y$, then f must be of the form above. It's straightforward to check any such f actually works.

In-class exercise[1]**:** Is the vector field $\mathbf{F}(x,y) = \langle 3x^2\cos(y), -x^3\sin(y)\rangle$ conservative? If so, find a potential function f.

As mentioned last class, there is also a criterion for determining whether a vector field in $\mathbb{R}^3$ is conservative. It again has to do with mixed partial derivatives: If $\mathbf{F} = \langle F_1, F_2, F_3\rangle$ is a vector field in $\mathbb{R}^3$ and there exists a function f such that $\nabla f = \mathbf{F}$, then Clairaut's theorem implies that

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \qquad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}.$$

Again, this condition turns out to be both necessary and sufficient for $\mathbf{F}$ to be conservative (provided $\mathbf{F}$ is "nice" in the sense described earlier). So

$$\mathbf{F} = \langle F_1, F_2, F_3\rangle \text{ is conservative} \qquad \Leftrightarrow \qquad \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}.$$

[1] Yes, $\mathbf{F}$ is conservative and has potential function $f(x,y) = x^3\cos(y) + A$ where A is any constant.

Day 37

1. Conservative vector fields in $\mathbb{R}^3$

At the end of last class we saw that if $\mathbf{F} = \langle F_1, F_2, F_3\rangle$ is a "nice[1]" vector field in $\mathbb{R}^3$, then

$$\mathbf{F} = \langle F_1, F_2, F_3\rangle \text{ is conservative} \qquad \Leftrightarrow \qquad \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}.$$

As an example, suppose $\mathbf{F}(x,y,z) = \langle y^2, 2xy + z\cos(yz), y\cos(yz)+1\rangle$. One can check that $\mathbf{F}$ satisfies the criterion above, so $\mathbf{F}$ is conservative. Let's find a potential function f. It must satisfy

$$f_x(x,y,z) = y^2, \qquad f_y(x,y,z) = 2xy + z\cos(yz), \qquad f_z(x,y,z) = y\cos(yz)+1.$$

As before, we begin by antidifferentiating. Using f_x, we get

$$f(x,y,z) = \int f_x(x,y,z)dx + C(y,z) = xy^2 + C(y,z),$$

where $C(y,z)$ is a function of y and z (but not x). Next we differentiate with respect to y to get

$$f_y(x,y,z) = 2xy + \frac{\partial}{\partial y}C(y,z)$$

Comparing with our target formula for f_y (above), it follows that

$$\frac{\partial}{\partial y}C(y,z) = z\cos(yz)$$

and thus

$$C(y,z) = \sin(yz) + D(z),$$

where $D(z)$ is a function of z (but not y). So at this point we have

$$f(x,y,z) = xy^2 + \sin(yz) + D(z).$$

We finally differentiate with respect to z to get

$$f_z(x,y,z) = y\cos(yz) + D'(z).$$

Comparing to our target formula for f_z (above), it follows that $D'(z) = 1$ and thus

$$D(z) = z + E,$$

where E is a constant. Putting everything together, we get

$$f(x,y,z) = xy^2 + \sin(yz) + z + E,$$

and it's easy to check that any such f does indeed satisfy $\nabla f = \mathbf{F}$.

[1] i.e. continuously differentiable on an open, simply connected domain

2. Line integral of a scalar function

Suppose we have a wire, represented by a curve C in $\mathbb{R}^3$. Assuming the wire has uniform density 1, what is its mass? It should be equal to the length of the wire, or in other words, the arc length of C. For example, if the wire is 10 cm long and its density is 1 g/cm, then its mass should be 10 g. Recall that the arc length of C is given by

$$\int_a^b \|\mathbf{r}'(t)\|dt,$$

where $\mathbf{r}$ is a vector function that parametrizes C and $[a,b]$ is the domain of $\mathbf{r}$. What if instead the wire had varying density given by some function $\rho(x,y,z)$? In that case, to find the mass we would want to integrate the density "along" C. If $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, such an integral is given by

$$\int_C \rho(x,y,z)ds = \int_a^b \rho(x(t),y(t),z(t))\|\mathbf{r}'(t)\|dt.$$

This called a *line integral* of the function $\rho(x,y,z)$. The concept of a line integral makes sense for any function, not just density functions:

> **Line Integral of a Multivariate Function**
>
> Let C be a curve in $\mathbb{R}^3$ with a smooth parametrization $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ for $t \in [a,b]$. Then the ***integral of*** $f(x,y,z)$ ***along*** C is
>
> $$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\|\mathbf{r}'(t)\|\,dt.$$

The definition for curves and functions in $\mathbb{R}^2$ is analogous.

As an example, let C be the part of the unit circle $x^2+y^2=1$ that lies in the first quadrant of $\mathbb{R}^2$, and let $f(x,y) = x^2y$. Let's calculate

$$\int_C f(x,y)ds.$$

The first step is to parametrize the curve C. We can use the usual unit circle parametrization $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ where $x(t) = \cos(t)$ and $y(t) = \sin(t)$. Because C lies in the first quadrant, we restrict t to the interval $[0, \pi/2]$. The line integral is therefore

$$\begin{aligned}
\int_C f(x,y)ds &= \int_0^{\pi/2} x(t)^2 y(t)\sqrt{x'(t)^2 + y'(t)^2}dt \\
&= \int_0^{\pi/2} \cos(t)^2\sin(t)\sqrt{(-\sin(t))^2+\cos(t)^2}dt \\
&= \int_0^{\pi/2} \cos(t)^2\sin(t)dt \\
&= -\frac{\cos(t)^3}{3}\Bigg|_{t=0}^{t=\pi/2} \\
&= \frac{1}{3}.
\end{aligned}$$