MATH 231-01: Homework Assignment 8

3 November 2025

Due: 10 November 2025 by 10:00pm Eastern time, submitted on Moodle as a single PDF.

Instructions: Write your solutions on the following pages. If you need more space, you may add pages, but make sure they are in order and label the problem number(s) clearly. You should attempt each problem on scrap paper first, before writing your solution here. Excessively messy or illegible work will not be graded. You must show your work/reasoning to receive credit. You do not need to include every minute detail; however the process by which you reached your answer should be evident. You may work with other students, but please write your solutions in your own words.

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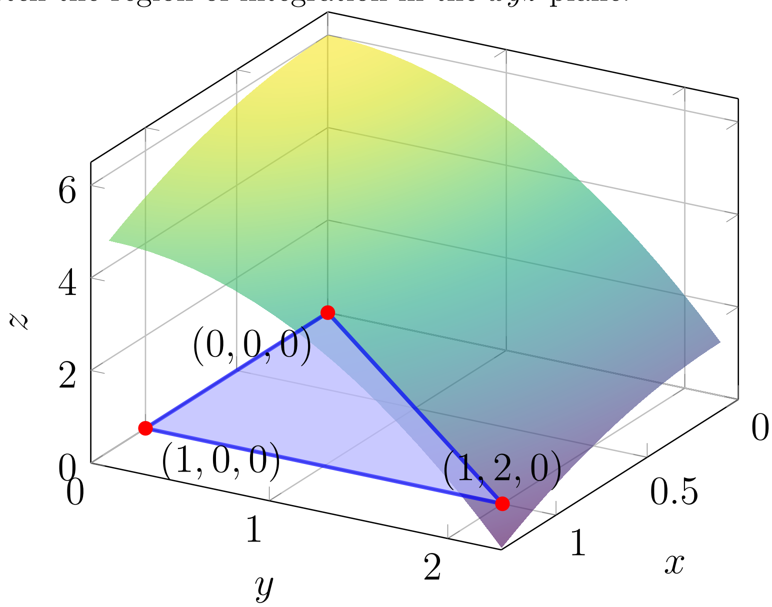
Score:

1. Find the volume of the region that lies above the solid triangle with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(1, 2, 0)$ and below the paraboloid $z = 6 - x^2 - y^2$.

(1) First, sketch the region of integration in the xyz-plane.



(2) Next, define the region of integration in the xy-plane.

$$0 \leq x \leq 1, \quad 0 \leq y \leq 2x$$

(3) Finally, set up and evaluate the double integral to find the volume.

$$\begin{aligned}
V &= \iint_R (6 - x^2 - y^2)\, dA \\
&= \int_0^1 \int_0^{2x} (6 - x^2 - y^2)\, dy\, dx \\
&= \int_0^1 \left[6y - x^2 y - \frac{y^3}{3} \right]_0^{2x} dx \\
&= \int_0^1 \left(12x - 2x^3 - \frac{8x^3}{3} \right) dx \\
&= \int_0^1 \left(12x - \frac{14x^3}{3} \right) dx \\
&= \left[6x^2 - \frac{14x^4}{12} \right]_0^1 \\
&= 6 - \frac{14}{12} = 6 - \frac{7}{6} = \frac{36}{6} - \frac{7}{6} = \frac{29}{6}.
\end{aligned}$$

Thus, the volume of the region is $\boxed{\frac{29}{6}}$.

2. Complete Problem 42 in Section 13.6 of the textbook (p. 1066).

   Describe the solid determined by the limits of integration:

$$\int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{\sqrt{1-r^2}} f(r, \theta, z)\, r\, dz\, dr\, d\theta.$$

The given limits of integration describe a solid in cylindrical coordinates. Let's analyze each limit:

- θ: $0 \le \theta \le \frac{\pi}{2}$ restricts the solid to the first octant (where $x \ge 0$ and $y \ge 0$).
- r: $0 \le r \le 1$ restricts the solid to be within a cylinder of radius 1 centered on the z-axis.
- z: $0 \le z \le \sqrt{1-r^2}$ bounds the solid between the xy-plane ($z = 0$) below and the surface $z = \sqrt{1-r^2}$ above.

To identify the upper surface, square both sides of $z = \sqrt{1-r^2}$:

$$z^2 = 1 - r^2 \implies r^2 + z^2 = 1.$$

Since $r^2 = x^2 + y^2$ in cylindrical coordinates, this becomes:

$$x^2 + y^2 + z^2 = 1.$$

This is the equation of a sphere of radius 1 centered at the origin.

Answer: The solid is the portion of the unit sphere $x^2 + y^2 + z^2 = 1$ that lies in the first octant (where $x \ge 0$, $y \ge 0$, and $z \ge 0$).

3. Complete Problem 46 in Section 13.6 of the textbook (p. 1067).

Describe the solid determined by the limits of integration:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^{3\csc(\phi)} f(\rho, \theta, \phi)\, \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi.$$

The given limits of integration describe a solid in spherical coordinates. Let's analyze each limit:

- θ: $0 \leq \theta \leq 2\pi$ means the solid is fully rotated around the z-axis.
- ϕ: $\frac{\pi}{4} \leq \phi \leq \frac{\pi}{2}$ restricts the solid to lie between:
  * The cone at $\phi = \frac{\pi}{4}$, which is $z = \sqrt{x^2 + y^2}$ (or $z = r$ in cylindrical coordinates).
  * The xy-plane at $\phi = \frac{\pi}{2}$ (where $z = 0$).
- ρ: $0 \leq \rho \leq 3\csc(\phi)$ bounds the radial distance from the origin to the surface $\rho = 3\csc(\phi)$.

To identify the outer boundary, convert $\rho = 3\csc(\phi)$:

$$\rho = \frac{3}{\sin(\phi)} \implies \rho \sin(\phi) = 3.$$

In spherical coordinates, $\rho \sin(\phi) = r$ where $r = \sqrt{x^2 + y^2}$ is the cylindrical radius. Thus:

$$r = 3 \implies x^2 + y^2 = 9.$$

This is a cylinder of radius 3 centered on the z-axis.

Answer: The solid is the region inside the cylinder $x^2 + y^2 = 9$ that lies between the cone $z = \sqrt{x^2 + y^2}$ and the xy-plane ($z = 0$). This describes a solid with a "bowl" or "washer" shape.

4. Let R be the region bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = y$ and $z = 0$. Evaluate the triple integral

$$\iiint_R z dV.$$

(1) First, convert the given integral to cylindrical coordinates.

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z,$$
$$dV = r\,dz\,dr\,d\theta.$$

(2) Next, determine the limits of integration for r, θ, and z.
   - The cylinder $x^2 + y^2 = 1$ becomes $r = 1$ in cylindrical coordinates. Thus, $0 \le r \le 1$.
   - The region is bounded by planes $z = 0$ and $z = y = r\sin\theta$ over the entire disk.
   - For $0 \le \theta \le \pi$ (where $y \ge 0$): $0 \le z \le r\sin\theta$.
   - For $\pi \le \theta \le 2\pi$ (where $y < 0$): $r\sin\theta \le z \le 0$.

(3) Observe the symmetry of the region and integrand.
   - The region R is symmetric about the xz-plane.
   - The integrand z is an odd function with respect to this symmetry: when $(x, y, z) \in R$ with $y \ge 0$, we have $z \ge 0$, but when $(x, -y, -z) \in R$ with $y < 0$, we have $z \le 0$.
   - By symmetry, the integral over the upper half ($y \ge 0$) cancels with the integral over the lower half ($y < 0$).

(4) We can verify this by computing both halves:

$$\text{Upper half:} \quad \int_0^{\pi}\int_0^1\int_0^{r\sin\theta} z \cdot r\,dz\,dr\,d\theta = \frac{\pi}{16}$$
$$\text{Lower half:} \quad \int_{\pi}^{2\pi}\int_0^1\int_{r\sin\theta}^{0} z \cdot r\,dz\,dr\,d\theta = -\frac{\pi}{16}$$

Therefore, the total integral is:

$$\iiint_R z\,dV = \frac{\pi}{16} + \left(-\frac{\pi}{16}\right) = 0.$$

Thus, the value of the integral is $\boxed{0}$.

5. Find the volume of the region that lies outside the cylinder $x^2 + y^2 = 1$ and inside the sphere $x^2 + y^2 + z^2 = 4$.

(1) First, convert the equations to cylindrical coordinates.

Cylinder: $x^2 + y^2 = 1 \implies r = 1$,

Sphere: $x^2 + y^2 + z^2 = 4 \implies r^2 + z^2 = 4 \implies z = \pm\sqrt{4 - r^2}$.

(2) Next, determine the limits of integration.

- The region lies outside the cylinder, so r ranges from 1 to 2 (the radius of the sphere at $z = 0$).
- The angle θ ranges from 0 to 2π.
- The height z ranges from the bottom of the sphere to the top: $-\sqrt{4 - r^2} \leq z \leq \sqrt{4 - r^2}$.

(3) Set up the triple integral for the volume.

$$V = \int_0^{2\pi} \int_1^2 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta.$$

(4) Evaluate the integral step-by-step.

$$\begin{aligned} V &= \int_0^{2\pi} \int_1^2 [rz]_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} \, dr \, d\theta \\ &= \int_0^{2\pi} \int_1^2 r \left(2\sqrt{4 - r^2}\right) \, dr \, d\theta \\ &= \int_0^{2\pi} \left(2 \int_1^2 r\sqrt{4 - r^2} \, dr\right) d\theta. \end{aligned}$$

To evaluate the inner integral, use the substitution $u = 4 - r^2$, $du = -2r \, dr$:

$$\begin{aligned} \int_1^2 r\sqrt{4 - r^2} \, dr &= -\frac{1}{2} \int_3^0 \sqrt{u} \, du \\ &= \frac{1}{2} \int_0^3 u^{1/2} \, du \\ &= \frac{1}{2} \left[\frac{2}{3} u^{3/2}\right]_0^3 = \frac{1}{3}(3^{3/2}) = \sqrt{3}. \end{aligned}$$

Thus,

$$V = \int_0^{2\pi} 2\sqrt{3} \, d\theta = 2\sqrt{3} \cdot 2\pi = 4\pi\sqrt{3}.$$

Therefore, the volume of the region is $\boxed{4\pi\sqrt{3}}$.

6. Let R be the region that lies above the plane $z = 1$, below the sphere $x^2+y^2+z^2 = 4$, and within the first octant. Evaluate the triple integral

$$\iiint_R \frac{1}{x^2 + y^2 + z^2} dV.$$

(1) First, convert the equations to spherical coordinates.

$$\text{Sphere: } x^2 + y^2 + z^2 = 4 \implies \rho = 2,$$
$$\text{Plane: } z = 1 \implies \rho \cos\phi = 1 \implies \rho = \sec\phi.$$

(2) Next, determine the limits of integration.

- The radius ρ ranges from the plane to the sphere: $\sec\phi \le \rho \le 2$.
- The angle ϕ ranges from 0 to $\frac{\pi}{3}$ (since $\cos\phi \ge \frac{1}{2}$ in the first octant).
- The angle θ ranges from 0 to $\frac{\pi}{2}$ (first octant).

(3) Set up the triple integral in spherical coordinates.

$$\begin{aligned}\iiint_R \frac{1}{x^2 + y^2 + z^2}\, dV &= \int_0^{\pi/2} \int_0^{\pi/3} \int_{\sec\phi}^{2} \frac{1}{\rho^2} \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\ &= \int_0^{\pi/2} \int_0^{\pi/3} \int_{\sec\phi}^{2} \sin\phi \, d\rho \, d\phi \, d\theta.\end{aligned}$$

(4) Evaluate the integral step-by-step.

$$\begin{aligned} &= \int_0^{\pi/2} \int_0^{\pi/3} [\rho \sin\phi]_{\sec\phi}^{2} \, d\phi \, d\theta \\ &= \int_0^{\pi/2} \int_0^{\pi/3} (2\sin\phi - \sin\phi \sec\phi) \, d\phi \, d\theta \\ &= \int_0^{\pi/2} \int_0^{\pi/3} (2\sin\phi - \tan\phi) \, d\phi \, d\theta.\end{aligned}$$

Now, evaluate the inner integral:

$$\begin{aligned}\int_0^{\pi/3} (2\sin\phi - \tan\phi) \, d\phi &= [-2\cos\phi - \ln|\cos\phi|]_0^{\pi/3} \\ &= \left(-2\cos\frac{\pi}{3} - \ln\left|\cos\frac{\pi}{3}\right|\right) - (-2\cos 0 - \ln|\cos 0|) \\ &= \left(-2 \cdot \frac{1}{2} - \ln\left(\frac{1}{2}\right)\right) - (-2 \cdot 1 - \ln(1)) \\ &= (-1 + \ln 2) - (-2) = 1 + \ln 2.\end{aligned}$$

Finally, evaluate the outer integral:

$$\int_0^{\pi/2} (1 + \ln 2) \, d\theta = (1 + \ln 2) \cdot \frac{\pi}{2} = \frac{\pi}{2}(1 + \ln 2).$$

Therefore, the value of the integral is $\boxed{\frac{\pi}{2}(1 + \ln 2)}$.

7. Let R be the region that lies outside the sphere $x^2 + y^2 + (z-1)^2 = 1$, inside the sphere $x^2 + y^2 + z^2 = 4$, and above the plane $z = 0$. Suppose R has uniform density $\rho(x, y, z) = 1$ and center of mass $(\overline{x}, \overline{y}, \overline{z})$. Find $\overline{z}$.

(1) First, convert the equations to spherical coordinates.

Outer sphere: $x^2 + y^2 + z^2 = 4 \implies \rho = 2$,

Inner sphere: $x^2 + y^2 + (z-1)^2 = 1 \implies x^2 + y^2 + z^2 - 2z = 0 \implies \rho = 2\cos\phi$.

(2) Next, determine the limits of integration.

- The radius ρ ranges from the inner sphere to the outer sphere: $2\cos\phi \leq \rho \leq 2$.
- The angle ϕ ranges from 0 to $\frac{\pi}{2}$ (above the plane $z = 0$).
- The angle θ ranges from 0 to 2π (full rotation around z-axis).

(3) Set up the triple integrals for mass and moment about the xy-plane. For uniform density $\rho(x, y, z) = 1$:

$$m = \iiint_R 1\, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_{2\cos\phi}^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta,$$

$$M_{xy} = \iiint_R z\, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_{2\cos\phi}^{2} (\rho\cos\phi) \cdot (\rho^2 \sin\phi) \, d\rho \, d\phi \, d\theta.$$

(4) Evaluate the mass integral m:

$$\begin{aligned} m &= \int_0^{2\pi} \int_0^{\pi/2} \left[\frac{\rho^3}{3} \sin\phi \right]_{2\cos\phi}^{2} d\phi \, d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/2} \frac{\sin\phi}{3} \left(8 - 8\cos^3\phi \right) d\phi \, d\theta \\ &= \int_0^{2\pi} \frac{8}{3} \int_0^{\pi/2} \sin\phi (1 - \cos^3\phi) \, d\phi \, d\theta. \end{aligned}$$

Evaluating the inner integral using substitution $u = \cos\phi$, $du = -\sin\phi \, d\phi$:

$$\begin{aligned} \int_0^{\pi/2} \sin\phi (1 - \cos^3\phi) \, d\phi &= \int_1^0 -(1 - u^3) \, du = \int_0^1 (1 - u^3) \, du \\ &= \left[u - \frac{u^4}{4} \right]_0^1 = 1 - \frac{1}{4} = \frac{3}{4}. \end{aligned}$$

Thus,

$$m = \int_0^{2\pi} \frac{8}{3} \cdot \frac{3}{4} \, d\theta = \int_0^{2\pi} 2 \, d\theta = 4\pi.$$

(5) Now, evaluate the moment integral M_{xy}:

$$M_{xy} = \int_0^{2\pi} \int_0^{\pi/2} \left[\frac{\rho^4}{4} \cos\phi \sin\phi \right]_{2\cos\phi}^{2} d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \frac{\cos\phi \sin\phi}{4} \left(16 - 16\cos^4\phi\right) d\phi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \left(4\cos\phi\sin\phi - 4\cos^5\phi\sin\phi\right) d\phi\, d\theta.$$

Using substitution $u = \cos\phi$, $du = -\sin\phi\, d\phi$:

$$\int_0^{\pi/2} \left(4\cos\phi\sin\phi - 4\cos^5\phi\sin\phi\right) d\phi = \int_1^0 (-4u + 4u^5)\, du$$

$$= \int_0^1 (4u - 4u^5)\, du = \left[2u^2 - \frac{2u^6}{3}\right]_0^1$$

$$= 2 - \frac{2}{3} = \frac{4}{3}.$$

Thus,

$$M_{xy} = \int_0^{2\pi} \frac{4}{3}\, d\theta = \frac{8\pi}{3}.$$

(6) Finally, compute $\overline{z}$:

$$\overline{z} = \frac{M_{xy}}{m} = \frac{8\pi/3}{4\pi} = \frac{8\pi}{3} \cdot \frac{1}{4\pi} = \frac{8}{12} = \frac{2}{3}.$$

Therefore, the z-coordinate of the center of mass is $\boxed{\frac{2}{3}}$.