Day 24

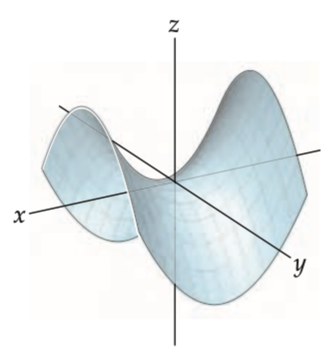
1. Classifying critical points

Last time we saw the definition of the local extrema (i.e. local minima and maxima) of a function. We saw that local extrema only occur at critical points. If we've found a critical point, how can we tell whether it's a local minimum or local maximum (or both or neither)? The *second derivative test* will provide a criterion that answers this question in most cases. We will only state the second derivative test for functions of two variables. (There is a second derivative test for functions with three or more variables, but it requires some knowledge of linear algebra.) First we need a few more definitions:

Saddle Points of a Function of Two Variables

A point (x_0, y_0) in the domain of a function $f(x, y)$ is called a ***saddle point*** of f if (x_0, y_0) is a stationary point of f at which there is neither a maximum nor a minimum.

The prototypical example of a function with a saddle point is the hyperbolic paraboloid $f(x, y) = x^2 - y^2$, whose graph looks like this:



This function has a saddle point at the origin $(0, 0)$. There the tangent plane is horizontal (i.e. the gradient is zero), but the function has neither a local maximum nor a local minimum.

Recall that a function of x and y has four second partial derivatives: f_{xx}, f_{xy}, f_{yx}, and f_{yy}. The second derivative test will involve all of these. It's convenient to organize these derivatives into a 2×2 matrix, know as the Hessian matrix.

The Hessian and the Discriminant of a Function of Two Variables

Let $f(x, y)$ be a function with continuous second-order partial derivatives on some open set S.

(a) The ***Hessian*** of f is the 2×2 matrix of second-order partial derivatives:

$$H_f = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial y \partial x} \\ \dfrac{\partial^2 f}{\partial x \partial y} & \dfrac{\partial^2 f}{\partial y^2} \end{bmatrix}.$$

(b) The ***discriminant*** of f is the determinant of the Hessian. That is,

$$\det(H_f) = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2.$$

In-class exercise[1]**:** Let $f(x, y) = x^2 + xy - y^2$

(a) Find the critical point(s) of f.

(b) At each critical point, find the Hessian matrix H_f and the discriminant $\det(H_f)$.

THEOREM 12.45

The Second-Order Partial-Derivative Test for Classifying Stationary Points

Let $f(x, y)$ be a function with continuous second-order partial derivatives on some open disk containing the point at (x_0, y_0). If f has a stationary point at (x_0, y_0), then

(a) f has a relative maximum at (x_0, y_0) if $\det(H_f(x_0, y_0)) > 0$ with $f_{xx}(x_0, y_0) < 0$ or $f_{yy}(x_0, y_0) < 0$.

(b) f has a relative minimum at (x_0, y_0) if $\det(H_f(x_0, y_0)) > 0$ with $f_{xx}(x_0, y_0) > 0$ or $f_{yy}(x_0, y_0) > 0$.

(c) f has a saddle point at (x_0, y_0) if $\det(H_f(x_0, y_0)) < 0$.

(d) If $\det(H_f(x_0, y_0)) = 0$, no conclusion may be drawn about the behavior of f at (x_0, y_0).

[1] (a) $(0, 0)$
(b) -5