Chapter 1: First-Order Equations

1.3 Separable Equations & Applications

Steps to Solve Separable Equations:

1. **Separate variables:** Rewrite the differential equation in the form $\frac{dy}{dx} = g(x)h(y)$, then separate the variables as $\frac{1}{h(y)}dy = g(x)dx$.
2. **Integrate both sides:** Integrate both sides separately. The result will typically involve a constant of integration C.
3. **Solve for y:** If possible, solve the resulting equation for y.
4. **Apply initial condition (if provided):** Use the initial condition to solve for the constant C.

Example Problem: Solve $\frac{dy}{dx} = 2xy$ with initial condition $y(0) = 1$.
Solution:

1. Rewrite the equation: $\frac{1}{y}dy = 2xdx$.
2. Integrate: $\int \frac{1}{y}dy = \int 2xdx$, so $\ln|y| = x^2 + C$.
3. Exponentiate both sides: $y = e^{x^2+C} = Ce^{x^2}$.
4. Use the initial condition $y(0) = 1$: $1 = Ce^0$, so $C = 1$.
5. Final solution: $y = e^{x^2}$.

1.2 Equilibrium and Stability

Equilibrium Points: Equilibrium points are the values where $\frac{dy}{dx} = 0$. These points are classified based on the behavior of nearby solutions:

- **Source:** If solutions move away from the equilibrium point as time increases, the point is called a *source*.
- **Sink:** If solutions move toward the equilibrium point, it is a *sink*.
- **Node:** Solutions either approach or move away in a non-oscillatory manner, depending on whether the node is stable (sink) or unstable (source).

Stability Classification:

- **Stable:** Nearby solutions approach the equilibrium point.
- **Unstable:** Nearby solutions move away from the equilibrium.
- **Semi-stable:** Solutions may approach from one side and move away from the other.

Steps to Determine Stability:

1. **Find equilibrium points:** Set $\frac{dy}{dx} = 0$ and solve for y.
2. **Analyze the sign of $\frac{dy}{dx}$:** Determine how the solutions behave near each equilibrium point by analyzing the sign of $\frac{dy}{dx}$ on either side of the equilibrium.

Example Problem: Find and classify the equilibrium points for $\frac{dy}{dx} = y(2 - y)$.
Solution:

1. Set $y(2 - y) = 0$, giving equilibrium points at $y = 0$ and $y = 2$.
2. Analyze $\frac{dy}{dx}$:
   - For $y < 0$, $\frac{dy}{dx} > 0$, so solutions move toward $y = 0$.
   - For $0 < y < 2$, $\frac{dy}{dx} > 0$, so solutions move toward $y = 2$.
   - For $y > 2$, $\frac{dy}{dx} < 0$, so solutions move toward $y = 2$.
3. Conclusion: $y = 0$ is unstable, and $y = 2$ is stable (sink).

1.4 Linear Equations & Applications

Steps to Solve First-Order Linear Equations:

1. **Standard form:** Write the equation as $y' + p(x)y = q(x)$.
2. **Find the integrating factor:** Calculate the integrating factor $I(x) = e^{\int p(x)dx}$.
3. **Multiply through by the integrating factor:** Multiply both sides of the equation by $I(x)$.
4. **Integrate:** The left-hand side becomes $(I(x)y)'$, allowing you to integrate both sides.
5. **Solve for y:** Solve for y, and apply initial condition if given.

Example Problem: Solve $y' + 3y = xe^{3x}$.
Solution:

1. Already in standard form: $y' + 3y = xe^{3x}$.
2. Find the integrating factor: $I(x) = e^{\int 3dx} = e^{3x}$.
3. Multiply by $I(x)$: $e^{3x}y' + 3e^{3x}y = xe^{6x}$.
4. The left side becomes $(e^{3x}y)'$, so integrate both sides:
$$\int (e^{3x}y)'dx = \int xe^{6x}dx.$$
5. Solve the integral by parts and write the general solution.

1.5 Bernoulli and Exact Equations

Steps to Solve Bernoulli Equations:

1. **Standard form:** Write the equation as $y' + p(x)y = q(x)y^n$.
2. **Substitution:** Let $v = y^{1-n}$, and differentiate to get v'.
3. **Transform into a linear equation:** Substitute v and v' into the original equation, turning it into a first-order linear equation in v.
4. **Solve for v:** Use the integrating factor method.
5. **Back-substitute for y:** Solve for y in terms of v.

Example Problem (Bernoulli): Solve $y' + y = y^2$.
Solution:

1. Standard form: $y' + y = y^2$, where $n = 2$.
2. Substitution: Let $v = y^{-1}$, then $v' = -y^{-2}y'$.
3. Transform into the linear equation: $v' - v = -1$.
4. Solve for v, then substitute back $y = \frac{1}{v}$.

Steps to Solve Exact Equations:

1. **Check for exactness:** For an equation of the form $M(x,y)dx + N(x,y)dy = 0$, the equation is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.
2. **Find the potential function $\Psi(x,y)$:** Solve $\frac{\partial \Psi}{\partial x} = M(x,y)$ and $\frac{\partial \Psi}{\partial y} = N(x,y)$.
3. **Solve for the solution:** The solution is $\Psi(x,y) = C$, where C is a constant.

Example Problem (Exact Equation): Solve $(2xy + y^2)dx + (x^2 + 2xy)dy = 0$.
Solution:

1. Check for exactness: $M(x,y) = 2xy + y^2$ and $N(x,y) = x^2 + 2xy$. We have:
$$\frac{\partial M}{\partial y} = 2x + 2y, \quad \frac{\partial N}{\partial x} = 2x + 2y$$
Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.
2. Find the potential function $\Psi(x,y)$:
$$\frac{\partial \Psi}{\partial x} = M(x,y) = 2xy + y^2 \Rightarrow \Psi(x,y) = x^2y + xy^2 + h(y)$$
3. Differentiate with respect to y:
$$\frac{\partial \Psi}{\partial y} = x^2 + 2xy + h'(y)$$
Equate with $N(x,y) = x^2 + 2xy$, so $h'(y) = 0$, which implies $h(y) = C$.
4. Final solution: $x^2y + xy^2 = C$.

1.6 Logistic Equation

The logistic equation models population growth with limited resources, introducing a maximum population, called the carrying capacity M.
Logistic Differential Equation:
$$\frac{dP}{dt} = kP(M - P)$$
Where:

- $P(t)$ is the population at time t,
- k is the growth rate constant,
- M is the carrying capacity (maximum population).

Key Characteristics:

- $P(t) < M$: Population grows ($\frac{dP}{dt} > 0$).
- $P(t) > M$: Population decreases ($\frac{dP}{dt} < 0$).
- Equilibrium points: $P = 0$ (unstable) and $P = M$ (stable).

Solution of the Logistic Equation:

1. Separate variables:
$$\frac{dP}{P(M-P)} = k\,dt$$
2. Use partial fractions and integrate:
$$\frac{1}{M}\left(\ln P - \ln|M - P|\right) = kt + C$$
3. Solve for $P(t)$:
$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-Mkt}}$$
Where P_0 is the initial population.

Example 1: Bacterial Population Growth A bacterial population grows from 100 to 200 in 1 hour. The environment supports a maximum population of 1000.
$$\frac{dP}{dt} = kP(1000 - P)$$
Using the data $P(1) = 200$, find k and predict the population at future times.
Example 2: Wildlife Population A wildlife preserve can support 500 deer. Initially, there are 50 deer, and the population grows to 150 after 2 years.
$$\frac{dP}{dt} = rP(500 - P)$$
Solve the logistic equation to find r and predict the population over time.

Mixing Problems

Steps to Solve Mixing Problems:

1. Let $x(t)$ denote the amount of solute in the tank at time t.
2. The rate of change of the solute is given by the equation:
$$\frac{dx}{dt} = \text{Rate of solute in} - \text{Rate of solute out}.$$
3. Set up the equation for $\frac{dx}{dt}$ based on the inflow and outflow rates and solve the resulting differential equation.

Example Problem: A tank initially contains 100 liters of water with 10 grams of salt. Brine containing 5 g/L of salt enters the tank at 4 L/min, and the well-mixed solution is drained at 2 L/min. Find the amount of salt after 10 minutes.
Solution:

1. Let $x(t)$ be the amount of salt in the tank at time t.
2. Rate in: $4 \times 5 = 20$ g/min. Rate out: $\frac{x(t)}{V(t)} \times 2$, where $V(t) = 100 + 2t$.
3. Set up the equation:
$$\frac{dx}{dt} = 20 - \frac{2x(t)}{100 + 2t}.$$
4. Solve the differential equation using an integrating factor.

Newton's Law of Cooling

Steps to Solve Newton's Law of Cooling:

1. The rate of change of temperature is proportional to the difference between the object's temperature and the ambient temperature:
$$\frac{dT}{dt} = -k(T - T_{\text{ambient}}).$$
2. Separate variables and integrate to solve for $T(t)$.
3. Apply initial conditions to solve for constants.

Example Problem: A cup of coffee at 90°C is left in a room at 20°C. After 10 minutes, the coffee is 70°C. When will it be 50°C?
Solution:

1. Set up the equation:
$$\frac{dT}{dt} = -k(T - 20).$$
2. Separate variables and integrate to get $T(t) = 20 + Ce^{-kt}$.
3. Use the initial condition $T(0) = 90$ to find C, and use $T(10) = 70$ to solve for k.
4. Solve for t when $T(t) = 50$.

Chapter 2: Second-Order Equations

2.1 Introduction

Steps to Solve Second-Order Homogeneous Equations:

1. **Write the characteristic equation:** For $ay'' + by' + cy = 0$, solve the characteristic equation $ar^2 + br + c = 0$.
2. **Find the roots:**
   - Two distinct real roots: $y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$.
   - Repeated root: $y = (C_1 + C_2 x)e^{rx}$.
   - Complex roots: $y = e^{\alpha x}(C_1 \cos(\beta x) + C_2 \sin(\beta x))$, where $r = \alpha \pm i\beta$.

Example Problem: Solve $y'' - 5y' + 6y = 0$.
Solution:

1. Characteristic equation: $r^2 - 5r + 6 = 0$.
2. Solve for r: The roots are $r = 2$ and $r = 3$.
3. General solution: $y = C_1 e^{2x} + C_2 e^{3x}$.

Steps to Solve Non-Homogeneous Equations:

1. **Solve the homogeneous equation:** Solve the homogeneous equation $ay'' + by' + cy = 0$ to find the complementary solution $y_c(x)$.
   - **Write the characteristic equation:** Solve the characteristic equation $ar^2 + br + c = 0$.
   - **Find the roots:**
     - Two distinct real roots: $y_c(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$.
     - Repeated root: $y_c(x) = (C_1 + C_2 x)e^{rx}$.
     - Complex roots: $y_c(x) = e^{\alpha x}(C_1 \cos(\beta x) + C_2 \sin(\beta x))$, where $r = \alpha \pm i\beta$.
2. **Find the particular solution:** Based on the form of the non-homogeneous term $f(x)$, guess the form of the particular solution $y_p(x)$.
   - If $f(x)$ is a polynomial, assume $y_p(x)$ is a polynomial of the same degree.
   - If $f(x) = e^{ax}$, assume $y_p(x) = Ae^{ax}$.
   - If $f(x) = \sin(bx)$ or $\cos(bx)$, assume $y_p(x) = A\cos(bx) + B\sin(bx)$.
   - If $f(x) = e^{ax}\sin(bx)$ or $e^{ax}\cos(bx)$, assume $y_p(x) = e^{ax}\,(A\cos(bx) + B\sin(bx))$.
3. **Substitute $y_p(x)$ into the original equation:** Plug the particular solution into the non-homogeneous equation to solve for the unknown coefficients.
4. **Form the general solution:** The general solution is the sum of the complementary solution and the particular solution:
$$y(x) = y_c(x) + y_p(x)$$
5. **Solve for constants using initial conditions:** If initial conditions are provided, substitute them into the general solution to solve for the constants C_1 and C_2.

Example Problem: Solve $y'' + 3y' + 2y = 3x^2 + 5x + 7$.
Solution:

1. **Solve the homogeneous equation:**
   - Characteristic equation: $r^2 + 3r + 2 = 0$.
   - Solve for r: The roots are $r = -1$ and $r = -2$.
   - Complementary solution: $y_c(x) = C_1 e^{-x} + C_2 e^{-2x}$.
2. **Guess the particular solution:** Since the non-homogeneous term $f(x) = 3x^2 + 5x + 7$ is a polynomial of degree 2, assume:
$$y_p(x) = Ax^2 + Bx + C$$
3. **Substitute $y_p(x)$ into the original equation:**
$$y_p'(x) = 2Ax + B, \quad y_p''(x) = 2A$$
   Substitute into $y'' + 3y' + 2y = 3x^2 + 5x + 7$:
$$2A + 3(2Ax + B) + 2(Ax^2 + Bx + C) = 3x^2 + 5x + 7$$
   Expand and collect like terms:
$$(2A) + (6A + 2B)x + (2A + 3B + 2C) = 3x^2 + 5x + 7$$
   Solve for A, B, and C:

- Coefficient of x^2: $2A = 3 \quad \Rightarrow \quad A = \frac{3}{2}$
- Coefficient of x: $6A + 2B = 5 \quad \Rightarrow \quad 6\left(\frac{3}{2}\right) + 2B = 5 \quad \Rightarrow \quad B = -2$
- Constant term: $2A + 3B + 2C = 7 \quad \Rightarrow \quad 2\left(\frac{3}{2}\right) + 3(-2) + 2C = 7 \quad \Rightarrow \quad C = 5$

Therefore, the particular solution is:

$$y_p(x) = \frac{3}{2}x^2 - 2x + 5$$

4. **General solution:**

$$y(x) = C_1 e^{-x} + C_2 e^{-2x} + \frac{3}{2}x^2 - 2x + 5$$

Wronskian and Linearity

Wronskian Determinant: The Wronskian is used to determine the linear independence of two solutions to a second-order differential equation. For solutions $y_1(x)$ and $y_2(x)$, the Wronskian is given by:

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2.$$

- If $W(y_1, y_2) \neq 0$ for all x, the solutions are linearly independent.
- If $W(y_1, y_2) = 0$, the solutions are linearly dependent.

Steps to Check Linearity:

1. Find the general solutions y_1 and y_2.
2. Compute the Wronskian determinant $W(y_1, y_2)$.
3. If $W \neq 0$, the functions are linearly independent, and the general solution is a linear combination of the two.

Example: Given solutions $y_1 = e^x$ and $y_2 = e^{-x}$ for the equation $y'' - y = 0$, find whether the solutions are linearly independent.
Solution:

$$W(y_1, y_2) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2.$$

Since $W \neq 0$, the solutions are linearly independent.

2.4 Free and Forced Mechanical Vibrations

Free Vibrations: The motion of a system is governed by:

$$mx'' + cx' + kx = 0$$

where m is the mass, c is the damping coefficient, and k is the spring constant.

- **Undamped motion ($c = 0$):**

  $$mx'' + kx = 0$$

  Characteristic equation: $r^2 + \frac{k}{m} = 0$. Solution: $x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)$, where $\omega = \sqrt{\frac{k}{m}}$.

- **Damped motion ($c \neq 0$):** The general solution depends on whether the system is underdamped, overdamped, or critically damped, based on the discriminant $c^2 - 4mk$.

Forced Vibrations: The motion is governed by:

$$mx'' + cx' + kx = F(t)$$

where $F(t)$ is the external force. The solution is the sum of the complementary solution (solving the homogeneous equation) and a particular solution, often found using **undetermined coefficients** or **variation of parameters**.

2.5 Method of Undetermined Coefficients

If

$$ay'' + by' + cy = A(x)$$

where:

1. $A(x)$ is a polynomial in x, then the particular solution y_p is:

$$y_p = x^k \times \text{(a general polynomial of the same degree)}$$

   where k is the number of times that 0 is a root of the characteristic equation.

2. If $A(x) = e^{ax}$, then:

$$y_p = x^k e^{ax} \times \text{(a general polynomial of the same degree)}$$

   where k is the number of times that a is a root of the characteristic equation.

3. If $A(x) = e^{ax}\cos(bx)$ or $A(x) = e^{ax}\sin(bx)$, then:

$$y_p = x^k e^{ax} \left[\text{(polynomial of the same degree)}\cos(bx) + \text{(another polynomial of the same degree)}\sin(bx)\right]$$

   where k is the number of times that $a + bi$ are roots of the characteristic equation.

2.6 Variation of Parameters

Steps for Variation of Parameters:

1. **Solve the complementary equation** to get the general solution $y_c = C_1 y_1(x) + C_2 y_2(x)$.

2. **Assume a particular solution** of the form:

$$y_p = v_1(x)y_1(x) + v_2(x)y_2(x)$$

   where $v_1(x)$ and $v_2(x)$ are functions to be determined.

3. **Solve for** $v_1(x)$ **and** $v_2(x)$ using Wronskian matrix:

$$v_1 = -\int \frac{y_2 f(x)}{W(y_1, y_2)} dx, \quad v_2 = \int \frac{y_1 f(x)}{W(y_1, y_2)} dx$$

4. **Integrate** to find $v_1(x)$ and $v_2(x)$.

5. **Find the particular solution:** Use $y_p = v_1(x)y_1(x) + v_2(x)y_2(x)$.