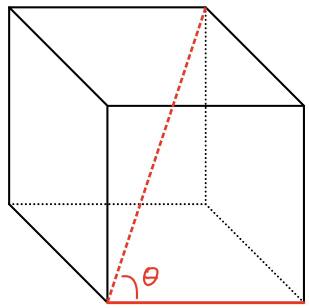
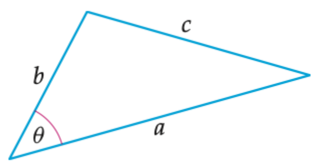
Day 4

1. Cosine formula for the dot product

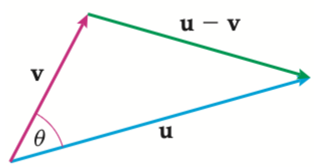
In-class exercise[1]. Use vectors to find the angle between an edge and a diagonal of a cube:



I mentioned last time that the cosine formula for the dot product can be derived using the Law of Cosines. Let's see the proof. The Law of Cosines states that if θ is an angle in a triangle with side-lengths a, b, c labeled as follows



then $a^2 + b^2 - 2ab\cos\theta = c^2$. Now, let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors. Together with $\mathbf{u} - \mathbf{v}$, they form a triangle:



The Law of Cosines tells us that

$$\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta = \|\mathbf{u} - \mathbf{v}\|^{\mathbf{2}}.$$

Using the algebraic properties of the dot product (from last class), we also know that

$$\begin{aligned}\|\mathbf{u} - \mathbf{v}\|^{\mathbf{2}} &= (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} - 2(\mathbf{u} \cdot \mathbf{v}) + \mathbf{v} \cdot \mathbf{v} \\ &= \|\mathbf{u}\|^2 - 2(\mathbf{u} \cdot \mathbf{v}) + \|\mathbf{v}\|^{\mathbf{2}}.\end{aligned}$$

So altogether, we have

$$\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta = \|\mathbf{u}\|^{\mathbf{2}} - \mathbf{2}(\mathbf{u} \cdot \mathbf{v}) + \|\mathbf{v}\|^{\mathbf{2}}$$

and cancelation leads to

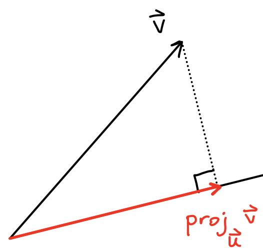
$$\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta = \mathbf{u} \cdot \mathbf{v}.$$

This is what we wanted to show. □

[1] About 0.96 radians or 55.7 degrees

2. Projections

Another important use of the dot product is in defining the concept of a vector projection. I think this is easiest to understand visually:



If we imagine moving along the vector $\mathbf{v}$, then the projection of $\mathbf{v}$ onto $\mathbf{u}$ represents how far we've moved in the direction of $\mathbf{u}$. Here's the formal definition: Let $\mathbf{v}$ and $\mathbf{u}$ be vectors with $\mathbf{u} \neq \mathbf{0}$. Then the *projection of* $\mathbf{v}$ *onto* $\mathbf{u}$ is the vector

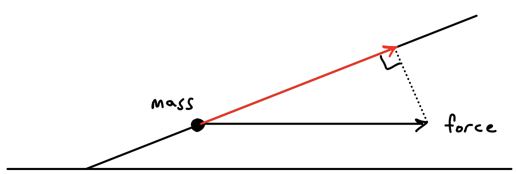
$$\text{proj}_{\mathbf{u}}\,\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{u}}{\|\mathbf{u}\|^2}\mathbf{u}.$$

(Why does this formula correspond to the picture above? You'll answer that in your first homework assignment!) Notice that $\text{proj}_{\mathbf{u}}\,\mathbf{v}$ is always parallel to $\mathbf{u}$ (i.e. a scalar multiple of $\mathbf{u}$). However, if $\mathbf{v}$ and $\mathbf{u}$ form an obtuse angle, then $\text{proj}_{\mathbf{u}}\,\mathbf{v}$ will point in the opposite direction of $\mathbf{u}$.

The main use of the projection $\text{proj}_{\mathbf{u}}\,\mathbf{v}$ is to decompose $\mathbf{v}$ into parallel and perpendicular components with respect to $\mathbf{u}$, namely

$$\mathbf{v} = \text{proj}_{\mathbf{u}}\,\mathbf{v} + (\mathbf{v} - \text{proj}_{\mathbf{u}}\,\mathbf{v}).$$

This can be very useful. For example, imagine you're pushing some mass up a ramp. If you exert a horizontal force, then only part of that force is actually moving the mass *along* the ramp. (Namely, the projection of the force onto the ramp.) The rest of the force is pushing the mass *into* the ramp, which is wasted effort.



In-class exercise[2]: Use a vector projection to find the shortest distance between the point $(3,5)$ and the line $y = \frac{1}{2}x + 1$.

[2] $\sqrt{5}$

Day 5

1. Cross product

Yesterday we worked with the dot product, an algebraic operation on two vectors that returns a scalar related to the lengths of the vectors and the angle between them. Our next topic is the cross product, another algebraic operation on two vectors (in $\mathbb{R}^3$) that provides useful geometric information. To motivate the cross product, let's first solve the following problem:

Problem. Let $\mathbf{u} = \langle 1, 0, 3\rangle$ and $\mathbf{v} = \langle 0, 3, -2\rangle$. Find a nonzero vector $\mathbf{w}$ that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

How might we approach this? Let's write $\mathbf{w} = \langle x, y, z\rangle$, where x, y, z are unknown. We know $\mathbf{w}$ needs to be orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. So by the dot product orthogonality test, we must have $\mathbf{u} \cdot \mathbf{w} = 0$ and $\mathbf{v} \cdot \mathbf{w} = 0$. This gives a system of equations:

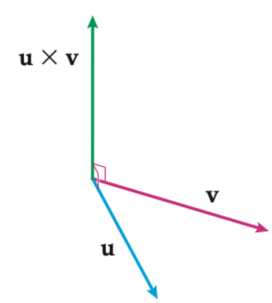
$$\begin{cases} 1x + 0y + 3z = 0, \\ 0x + 3y - 2z = 0. \end{cases}$$

If we set $z = t$, where t can be any real number, then $x = -3t$ and $y = \frac{2}{3}t$. Thus

$$\mathbf{w} = \langle -3t, \frac{2}{3}t, t\rangle \qquad \text{for any } t.$$

If we now choose any $t \neq 0$, we get a valid solution to the problem. For example, if we choose $t = 1$, then $\mathbf{w} = \langle -3, \frac{2}{3}, 1\rangle$. (Double-check for yourself that $\mathbf{w}$ is actually orthogonal to $\mathbf{u}$ and $\mathbf{v}$.)

The cross product will allow us to solve this kind of problem without having to use a system of equations: Give two vectors, the cross product will allow us to quickly come up with a third vector that is orthogonal to both of them.



Here's the definition: Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ be vectors in $\mathbb{R}^3$. The *cross product* of $\mathbf{u}$ and $\mathbf{v}$ is the vector

$$\mathbf{u} \times \mathbf{v} = \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1\rangle.$$

This is kind of a mess. Don't worry, there is a better, equivalent definition of the cross product that is (somewhat) easier to remember.

Note that $\mathbf{u} \times \mathbf{v}$ is indeed orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. To prove this, use algebra to show that $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0$ and $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$. (Ugly I know, but it can be done!)

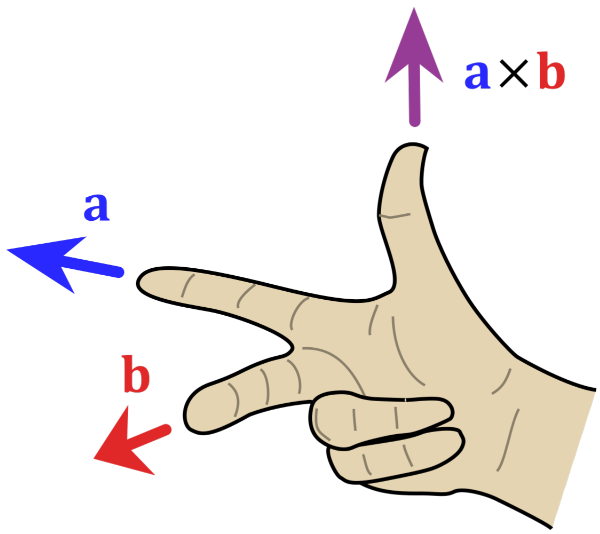
Let's see what we get using the vectors $\mathbf{u} = \langle 1, 0, 3\rangle$ and $\mathbf{v} = \langle 0, 3, -2\rangle$ from above:

$$\mathbf{u} \times \mathbf{v} = \langle 0(-2) - 3 \cdot 3, 3 \cdot 0 - 1(-2), 1 \cdot 3 - 0 \cdot 0\rangle = \langle -9, 2, 3\rangle.$$

Notice that this is $3\mathbf{w}$, where $\mathbf{w}$ was our solution to the problem above. So $\mathbf{u} \times \mathbf{v}$ is indeed orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

2. Geometry of the cross product

Given vectors $\mathbf{a}$ and $\mathbf{b}$, which way does $\mathbf{a} \times \mathbf{b}$ point? (We know it's orthogonal to $\mathbf{a}$ and $\mathbf{b}$, but there are still two possible directions.) Use the right-hand rule:



Using your right hand, point your index finger in the direction of $\mathbf{a}$ and your middle finger in the direction of $\mathbf{b}$. Then your thumb will point in the direction of $\mathbf{a} \times \mathbf{b}$.

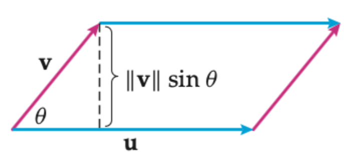
Like the dot product, the cross product of two vectors is related to the angle between them. Specifically, if neither $\mathbf{u}$ nor $\mathbf{v}$ is the zero vector, then

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta,$$

where θ is the smallest angle between $\mathbf{u}$ and $\mathbf{v}$. The proof is based on Lagrange's identity, which relates the cross product to the dot product: $\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$.

In-class exercise[1]**:** Under what conditions on $\mathbf{u}$ and $\mathbf{v}$ do we have $\mathbf{u} \times \mathbf{v} = \mathbf{0}$?

The sine formula above provides a way of relating the cross product to area and volume. Consider the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$:



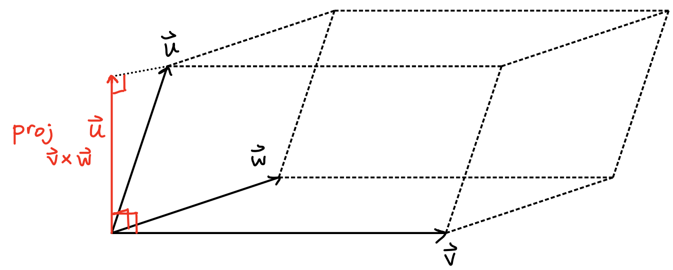
[1]This holds if and only if $\mathbf{u}$ and $\mathbf{v}$ are parallel, meaning that one of them is a scalar multiple of the other.

Its area is the product of the length of its base and its height. This works out to be $\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. Thus, by the sine formula, we have:

The Area of a Parallelogram

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^3$. Then the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$ is given by $\|\mathbf{u} \times \mathbf{v}\|$.

Now, given three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, consider the parallelepiped that they generate:



Its volume is the product of the area of its base and its height. The area of its base is $\|\mathbf{v} \times \mathbf{w}\|$, using the fact we just derived. Its height is $\|\operatorname{proj}_{\mathbf{v}\times\mathbf{w}} \mathbf{u}\|$. Using the definition of projection, this product simplifies:

$$\begin{aligned}
\|\mathbf{v} \times \mathbf{w}\|\|\operatorname{proj}_{\mathbf{v}\times\mathbf{w}} \mathbf{u}\| &= \|\mathbf{v} \times \mathbf{w}\|\left\|\frac{\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})}{\|\mathbf{v} \times \mathbf{w}\|^2}\mathbf{v} \times \mathbf{w}\right\| \\
&= \|\mathbf{v} \times \mathbf{w}\|\frac{|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|}{\|\mathbf{v} \times \mathbf{w}\|^2}\|\mathbf{v} \times \mathbf{w}\| \\
&= |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|.
\end{aligned}$$

The quantity $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ is called the *triple product* (or *triple scalar product*) of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. What we have shown is that

The Triple Scalar Product and the Volume of Parallelepipeds

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^3$. Then $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$ is the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Furthermore, the volume of the parallelepiped is $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ if and only if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ form a right-handed triple.

(You can ignore the third sentence about right-handed triples, or read about it in the textbook.)

One useful consequence of this fact is that three vectors are coplanar (i.e. parallel to the same plane) if and only if their triple product is zero. This is because the vectors are coplanar if and only if the parallelepiped they generate has zero volume!

We've now seen a few different operations that tell us information about a collection of vectors. Let's collect some of the facts we've learned:

- Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.
- If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^3$, then $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

- If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $\mathbb{R}^3$, then $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are coplanar if and only if $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=0$.

3. A different way to compute the cross product

As mentioned earlier, there's an alternate method for computing the cross product. (It's actually exactly the same, but a bit easier to remember if you're familiar with determinants. It's okay if you're not!) Here it is: If $\mathbf{u}=\langle u_1,u_2,u_3\rangle$ and $\mathbf{v}=\langle v_1,v_2,v_3\rangle$, then

$$\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\u_1&u_2&u_3\\v_1&v_2&v_3\end{vmatrix}=\begin{vmatrix}u_2&u_3\\v_2&v_3\end{vmatrix}\mathbf{i}-\begin{vmatrix}u_1&u_3\\v_1&v_3\end{vmatrix}\mathbf{j}+\begin{vmatrix}u_1&u_2\\v_1&v_2\end{vmatrix}\mathbf{k}.$$

Here the vertical lines mean "determinant". The 2×2 determinant is defined by

$$\begin{vmatrix}a&b\\c&d\end{vmatrix}=ad-bc.$$

So you can see that this formula for $\mathbf{u}\times\mathbf{v}$ is equivalent to our original definition.

Let's try it out on an example. Let $\mathbf{u}=\langle 2,0,3\rangle$ and $\mathbf{v}=\langle 4,1,5\rangle$. Then

$$\begin{aligned}\mathbf{u}\times\mathbf{v}&=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&0&3\\4&1&5\end{vmatrix}\\&=(0\cdot5-3\cdot1)\mathbf{i}-(2\cdot5-3\cdot4)\mathbf{j}+(2\cdot1-0\cdot4)\mathbf{k}\\&=-3\mathbf{i}+2\mathbf{j}+2\mathbf{k}\\&=\langle-3,2,2\rangle.\end{aligned}$$

4. Algebraic properties of the cross product

Finally we make note of a few algebraic properties of the cross product:

Multiplication by a Scalar and the Cross Product

For any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ and any scalar c,

$$c(\mathbf{u}\times\mathbf{v})=(c\,\mathbf{u})\times\mathbf{v}=\mathbf{u}\times(c\,\mathbf{v}).$$

Distributive Properties of the Cross Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^3$. Then

(a) $\mathbf{u}\times(\mathbf{v}+\mathbf{w})=\mathbf{u}\times\mathbf{v}+\mathbf{u}\times\mathbf{w}$.

(b) $(\mathbf{u}+\mathbf{v})\times\mathbf{w}=\mathbf{u}\times\mathbf{w}+\mathbf{v}\times\mathbf{w}$.

The Cross Product Is Anticommutative

For any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$,

$$\mathbf{v}\times\mathbf{u}=-(\mathbf{u}\times\mathbf{v}).$$

The last of these may be surprising; the cross product is not a commutative operation! When we interchange the roles of $\mathbf{u}$ and $\mathbf{v}$, the direction of their cross product flips. This makes sense from the point of view of the right-hand rule. It's also a fact about determinants that interchanging two rows reverses the sign.

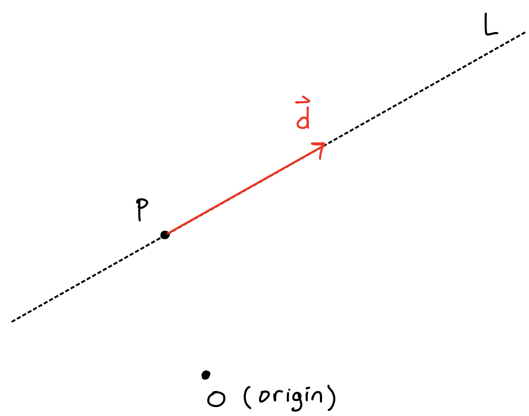
Day 6

1. Looking ahead

We've now worked a lot with vectors and seen a few examples of how they can be used. Over the next couple of days, we'll use what we've learned to describe lines and planes in $\mathbb{R}^3$. You might ask why we need to bother with this in a multivariable calculus course. (When does the calculus start?) Recall that if f is a differentiable function, then locally its graph looks like a line in $\mathbb{R}^2$: at each point on the graph, there is a unique tangent line. Next week, we'll study curves, which are functions that also have tangent lines, but now possibly in $\mathbb{R}^3$. Following that, we'll look at multivariable functions, whose graphs locally look like a plane in $\mathbb{R}^3$ (the tangent plane). Thus, to understand the calculus of these functions, we need to have a solid grasp of lines and planes.

2. Lines in $\mathbb{R}^3$

A line L in $\mathbb{R}^3$ is determined by a point $P = (x_0, y_0, z_0)$ and a nonzero vector $\mathbf{d} = \langle a, b, c, \rangle$:



Every point $Q = (x, y, z)$ on the line L satisfies the *vector equation*

$$\overrightarrow{OQ} = \overrightarrow{OP} + t\mathbf{d}$$

for some real number t. Here, $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ refer to the vectors that extend from the origin to the points P and Q (i.e. the position vectors for P and Q). Written in component form, the vector equation becomes

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle.$$

The vector equation is really saying that we can start at P and move some distance in the direction of $\mathbf{d}$ or $-\mathbf{d}$ to arrive at Q. This is what it means for Q to belong to the line determined by P and $\mathbf{d}$. Note that there are many different vector equations for the line L. If we replace P with any other point on L, we get a vector equation that looks different but still represents L. Similarly, we could replace $\mathbf{d}$ with any nonzero scalar multiple of $\mathbf{d}$.

The right-hand side of the vector equation is an example of a *vector function*: Let

$$\mathbf{r}(t) = \overrightarrow{OP} + t\mathbf{d} = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle.$$

For each input t, the function $\mathbf{r}(t)$ outputs a vector. A point Q belongs to the line L if and only if there exists a value of t such that $\overrightarrow{OQ} = \mathbf{r}(t)$. We'll be working with vector functions in more detail next week.

There are a couple of other ways to represent the line L. First is the set of *parametric equations*

$$\begin{cases} x = x_0 + at, \\ y = y_0 + bt, \\ z = z_0 + ct. \end{cases}$$

This is the same as the component form of the vector equation, except that each component is now represented by a separate equation. For example, if the vector equation is

$$\langle x, y, z\rangle = \langle 2, 3, 1\rangle + t\langle 5, -2, 4\rangle,$$

then the parametric equations are

$$\begin{cases} x = 2 + 5t, \\ y = 3 - 2t, \\ z = 1 + 4t. \end{cases}$$

Conversely, if you know the parametric equations, then you can figure out the vector equation. (The coefficients of t give you the direction vector $\mathbf{d}$, and the other constants give you the point P.)

Solving for t in the parametric equations, we get the *symmetric equations*

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c},$$

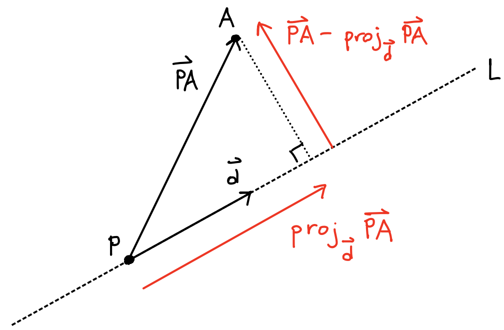
provided a, b, and c are nonzero.

To get a feel for these definitions, let's try this problem:

Problem: Find the distance between the point $A = (3, 5, 9)$ and the line L represented by

$$\frac{x + 2}{2} = \frac{y - 1}{3} = z - 3.$$

Last week we saw a method for finding the distance between a point and a line: We draw a vector from the line to the point, then decompose that vector into parallel and orthogonal components with respect to the line. The length of the orthogonal component is the distance we're looking for. To use the same approach here, we need figure out the direction of L and identify a point on L. The symmetric equations above tell us that L has direction vector $\mathbf{d} = \langle 2, 3, 1\rangle$ and contains the point $P = (-2, 1, 3)$. Therefore, the distance between A and L is equal to $\|\overrightarrow{PA} - \operatorname{proj}_{\mathbf{d}} \overrightarrow{PA}\|$, where $\overrightarrow{PA}$ is the vector from P to A:



Now this a matter of computation. Comparing the points P and Q, we see that $\overrightarrow{PA} = \langle 5, 4, 6 \rangle$. Thus

$$\operatorname{proj}_{\mathbf{d}} \overrightarrow{PA} = \frac{\overrightarrow{PA} \cdot \mathbf{d}}{\|\mathbf{d}\|^2}\mathbf{d} = \frac{28}{14}\langle 2, 3, 1 \rangle = \langle 4, 6, 2 \rangle$$

and

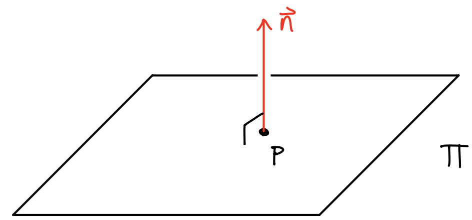
$$\overrightarrow{PA} - \operatorname{proj}_{\mathbf{d}} \overrightarrow{PA} = \langle 1, -2, 4 \rangle,$$

which implies that the distance is $\sqrt{21}$.

Day 7

1. Planes in $\mathbb{R}^3$

Last time we saw that a line is determined by a point and a nonzero vector. A plane is also determined by a point and a nonzero vector, but in different way. Let $P = (x_0, y_0, z_0)$ and let $\mathbf{n} = \langle a, b, c\rangle \neq \mathbf{0}$. Then there is a unique plane Π that contains the point P and is orthogonal to the vector $\mathbf{n}$:



The vector $\mathbf{n}$ is called a *normal vector* for the plane Π. What does it mean for Π to be orthogonal to $\mathbf{n}$? It means that if we take any two distinct points in Π and form the vector between them, then this vector will be orthogonal to $\mathbf{n}$. In particular, a point $Q = (x, y, z)$ belongs to Π if and only if the vector $\overrightarrow{PQ} = \langle x - x_0, y - y_0, z - z_0\rangle$ satisfies

$$\overrightarrow{PQ} \cdot \mathbf{n} = 0,$$

or equivalently

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \tag{1.1}$$

If we set $d = ax_0 + by_0 + cz_0$ (which is the same as $\overrightarrow{OP} \cdot \mathbf{n}$), we get the equation

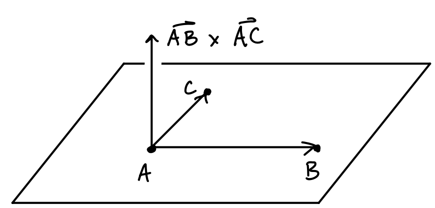
$$ax + by + cz = d. \tag{1.2}$$

Conversely, suppose you're given an equation of the form (1.1) or (1.2). How do you figure out which plane it represents? The coefficients of x, y, z give you a normal vector $\mathbf{n}$. To get a point P that belongs to the plane, find a solution to the equation. Note that a plane equation is very underdetermined. There are three variables but only one equation, so it will never be difficult to find a solution.

Let's work through the following problem:

Problem: Find an equation for the plane containing the points $A = (1, 2, 7)$, $B = (2, 5, 3)$, and $C = (0, 1, 0)$.

Remember that to find the plane we need a normal vector and a point on the plane. We already have a point, actually three of them. To find a normal vector, we can form two non-parallel vectors in the plane, then take their cross product to get a vector that's orthogonal to both of them:



Let's use the vectors $\overrightarrow{AB} = \langle 1, 3, -4\rangle$ and $\overrightarrow{AC} = \langle -1, -1, -7\rangle$. (We could alternatively use either of these with $\overrightarrow{BC}$.) We can see that $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are nonparallel because neither one is a scalar multiple of the other. So we can take as our normal vector

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -4 \\ -1 & -1 & -7 \end{vmatrix} = -25\mathbf{i} - (-11)\mathbf{j} + 2\mathbf{k} = \langle -25, 11, 2\rangle.$$

To write down an equation for the plane, we need a point on the plane. But again we already have three of them; let's just use $A = (1, 2, 7)$. This leads to an answer of

$$-25(x-1) + 11(y-2) + 2(z-7) = 0.$$

Note that there are many equivalent equations for the same plane; this is just one possible answer.

Question[1]: What would go wrong if the points A, B, C were collinear? How could we find a plane that contains them?

In-class exercise[2]: Let Π_1 be the plane with equation $3x + 2y + z = 1$, and let Π_2 be the plane with equation $x + y = 0$.

(a) Do Π_1 and Π_2 intersect? Explain how you know.
(b) If they do intersect, find their line of intersection. If they don't intersect, find the distance between them.

[1] The cross product $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$ would be $\mathbf{0}$, so it wouldn't give us a valid normal vector for a plane. Instead, take $\mathbf{n}$ to be any nonzero vector that is orthogonal to $\overrightarrow{AB}$; for example you could use $\mathbf{n} = \overrightarrow{AB} \times \mathbf{v}$ where $\mathbf{v}$ is any (nonzero) vector that is not parallel to $\overrightarrow{AB}$. The equation $\overrightarrow{AQ} \cdot \mathbf{n} = 0$ would then represent a plane that contains A, B, and C. Note that there are infinitely many such planes!

[2] (a) Yes, they intersect. Their normal vectors $\mathbf{n_1} = \langle 3, 2, 1\rangle$ and $\mathbf{n_2} = \langle 1, 1, 0\rangle$ are not parallel.
(b) The line of intersection is contained in both planes. It's therefore orthogonal to both $\mathbf{n_1}$ and $\mathbf{n_2}$, or in other words, it's parallel to $\mathbf{n_1} \times \mathbf{n_2}$. So $\mathbf{d} = \mathbf{n_1} \times \mathbf{n_2} = \langle -1, 1, 1\rangle$ is a direction vector for the line. To get a point P on the line, find a simultaneous solution to both plane equations, e.g. $P = (0, 0, 1)$. This results in the equation $\overrightarrow{OQ} = \overrightarrow{OP} + t\mathbf{d}$, or $\langle x, y, z\rangle = \langle 0, 0, 1\rangle + t\langle -1, 1, 1\rangle$.