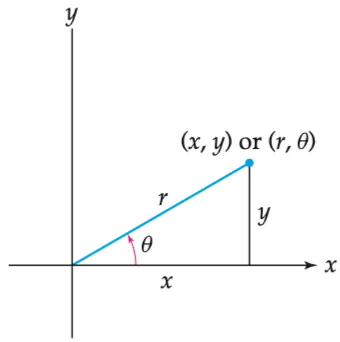
Day 30

1. Integration in polar coordinates

Last time we got practice using polar coordinates. Each point (x, y) can be represented in polar coordinates (r, θ) where $x = r\cos\theta$ and $y = r\sin\theta$. Conventionally, we choose r to be the distance between (x, y) and the origin, and we choose θ to be the angle between the position vector $\langle x, y\rangle$ and $\langle 1, 0\rangle$ (i.e. the angle between $\langle x, y\rangle$ and the positive x-axis):



We saw that many regions can be expressed more easily in polar coordinates than in Cartesian coordinates. We would like to be able to evaluate double integrals over these regions using iterated integrals expressed in polar coordinates. How does this work?

Suppose R is a region which, when expressed in polar coordinates, takes the form

$$R = \{(r, \theta)\colon \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq h_2(\theta)\}.$$

Then

$$\iint_R f(x, y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)r\,dr\,d\theta.$$

Similarly, suppose S is a region which, in polar coordinates, takes the form

$$S = \{(r, \theta)\colon a \leq r \leq b,\ g_1(r) \leq \theta \leq g_2(r)\}.$$

Then

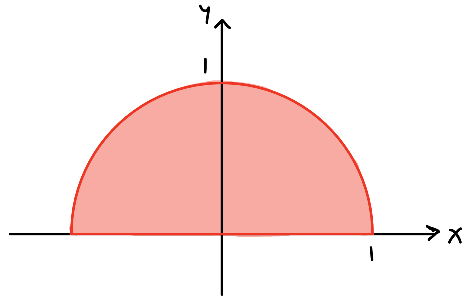
$$\iint_S f(x, y)dA = \int_a^b \int_{g_1(r)}^{g_2(r)} f(r\cos\theta, r\sin\theta)r\,d\theta\,dr.$$

Notice that in passing from the double integral to the iterated integral, dA becomes $rdrd\theta$ or $rd\theta dr$. We'll see later where this extra r comes from. For now, think of it as a "correction" factor for the change of variables $(x, y) \to (r, \theta)$.

Let's see a few examples. First, let's evaluate

$$\iint_R e^{x^2+y^2}dA,$$

where R is the following region:



In polar coordinates, $R = \{(r,\theta)\colon 0 \le \theta \le \pi,\ 0 \le r \le 1\}$. Thus

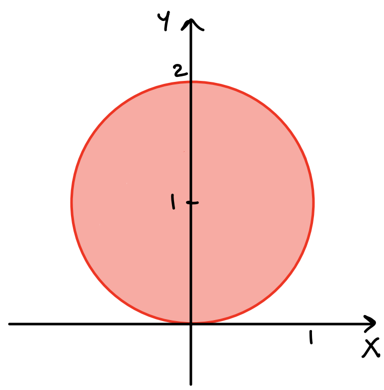
$$\iint_R e^{x^2+y^2}dA = \int_0^\pi \int_0^1 e^{(r\cos\theta)^2+(r\sin\theta)^2} r dr d\theta = \int_0^\pi \int_0^1 e^{r^2} r dr d\theta.$$

From here it's just a matter of evaluating the iterated integral, which turns out to be $\frac{\pi}{2}(e-1)$.

Second, let's consider the double integral

$$\iint_R xy dA,$$

where R is the disc given by $x^2 + (y-1)^2 \le 1$. Here is a sketch:



Substituting $x = r\cos\theta$ and $y = r\sin\theta$ and simplifying, the inequality $x^2 + (y-1)^2 \le 1$ becomes $0 \le r \le 2\sin\theta$. Thus $R = \{(r,\theta)\colon 0 \le \theta \le \pi,\ 0 \le r \le 2\sin\theta\}$. Therefore

$$\iint_R xy dA = \int_0^\pi \int_0^{2\sin\theta} \cos(\theta)\sin(\theta) r^3 dr d\theta,$$

and this iterated integral works out to be 0.

2. Mass and center of mass

Next we consider an application of double integration. Let R be a region in the plane. We will think of R as being solid, perhaps cut out from a sheet of metal. Such a region is called a *lamina*. Suppose the density of the lamina R at the point (x,y) is given by $\rho(x,y)$. Then the mass of R is

$$\text{mass}(R) = \iint_R \rho(x,y) dA.$$

We won't see a formal proof of this statement, but here's the idea. Divide R into small rectangles $R_1, \ldots, R_N$ with horizontal side-length Δx and vertical side-length Δy. If Δx

and Δy are sufficiently small, then on each R_i the density ρ will be roughly constant, say $\rho(x, y) \approx \rho(x_i, y_i)$ where (x_i, y_i) is the center of R_i. The mass of R_i is therefore approximately $\rho(x_i, y_i)\Delta x\Delta y$, as mass = density $\times$ area when the density is constant. Thus

$$\text{mass}(R) = \sum_{i=1}^{N} \text{mass}(R_i) \approx \sum_{i=1}^{N} \rho(x_i, y_i)\Delta x\Delta y.$$

The right-hand sum is a Riemann sum corresponding to the double integral of ρ over R. Thus, taking $\Delta x, \Delta y \to 0$, we get the formula claimed above.

The *center of mass* of a lamina R is the point $(\overline{x}, \overline{y})$ defined by

$$\overline{x} = \frac{1}{\text{mass}(R)} \iint_R x\rho(x, y)dA, \qquad \overline{y} = \frac{1}{\text{mass}(R)} \iint_R y\rho(x, y)dA.$$

We can think of $\overline{x}$ as being the average x coordinate over R weighted by density; similarly for $\overline{y}$.

As an example, let R be the lamina that occupies the first quadrant of the disc given by $x^2 + y^2 \leq 1$, and assume R has density $\rho(x, y) = k(x^2 + y^2)$ for some constant $k > 0$. What is the center of mass of R? To answer this, we first need to find the mass of R. Using polar coordinates, we get

$$\text{mass}(R) = \iint_R \rho(x, y)dA = \int_0^{\pi/2} \int_0^1 kr^3 dr d\theta = \frac{k\pi}{8}.$$

Now we can calculate $\overline{x}$ and $\overline{y}$. We have

$$\overline{x} = \frac{1}{\text{mass}(R)} \iint_R x\rho(x, y)dA = \frac{8}{k\pi} \int_0^{\pi/2} \int_0^1 k\cos(\theta)r^4 dr d\theta = \frac{8}{5\pi}.$$

The same calculation (or symmetry of the problem relative to x and y) leads to $\overline{y} = \frac{8}{5\pi}$. Thus the center of mass is $(\overline{x}, \overline{y}) = (\frac{8}{5\pi}, \frac{8}{5\pi})$. Notice that the mass of R depends on the constant k, but the center of mass does not.

Day 31

1. Practice with center of mass

In-class exercise: Let R the the lamina that occupies the region bounded by $y = 0$ and $y = \sin(x)$ between $x = 0$ and $x = \pi$. Assuming R has density $\rho(x, y) = y$, what is its center of mass? (Hint: Recall that $\sin(x)^2 = \frac{1}{2}(1 - \cos(2x))$.)

Solution. First we need to find the mass of R. We have

$$\begin{aligned}
\text{mass}(R) &= \iint_R \rho(x,y)dA \\
&= \int_0^\pi \int_0^{\sin(x)} y dy dx \\
&= \int_0^\pi \frac{1}{2}\sin(x)^2 dx \\
&= \int_0^\pi \frac{1}{4}(1-\cos(2x))dx \\
&= \left(\frac{1}{4}x - \frac{1}{8}\sin(2x)\right)\Big|_{x=0}^{x=\pi} \\
&= \frac{\pi}{4}.
\end{aligned}$$

Now we can find $(\overline{x}, \overline{y})$, the center of mass. Notice that R (including its density $\rho(x, y)$) is symmetric with respect to the line $x = \frac{\pi}{2}$. Therefore $\overline{x} = \frac{\pi}{2}$. (This can be confirmed using the formula if you're worried.) We calculate $\overline{y}$ as follows:

$$\begin{aligned}
\text{mass}(R)\overline{y} &= \iint_R y\rho(x,y)dA \\
&= \int_0^\pi \int_0^{\sin(x)} y^2 dy dx \\
&= \int_0^\pi \frac{1}{3}\sin(x)^3 dx \\
&= \int_0^\pi \frac{1}{3}(1-\cos(x)^2)\sin(x)dx \\
&= \int_{-1}^1 \frac{1}{3}(1-u^2)du \qquad (u = \cos(x)) \\
&= \frac{4}{9},
\end{aligned}$$

so

$$\overline{y} = \frac{1}{\text{mass}(R)} \cdot \frac{4}{9} = \frac{4}{\pi} \cdot \frac{4}{9} = \frac{16}{9\pi}.$$

Therefore the center of mass is

$$(\overline{x}, \overline{y}) = \left(\frac{\pi}{2}, \frac{16}{9\pi}\right) \approx (1.57, 0.57).$$

2. Triple integrals

Triple integrals work exactly how you would expect. Let R be a solid region in $\mathbb{R}^3$ (think a box, ball, or cylinder) and let $f(x, y, z)$ be a function. The triple integral of f over R is denoted

$$\iiint_R f(x, y, z)dV.$$

As with double integrals (and ordinary integrals), the triple integral is formally defined as a limit of Riemann sums. Each term in the Riemann sum is the signed "volume" of a narrow four-dimensional box extending from a small piece of R up to the graph of f (a hypersurface in $\mathbb{R}^4$). While this is difficult to visualize, the Riemann sums converge nevertheless.

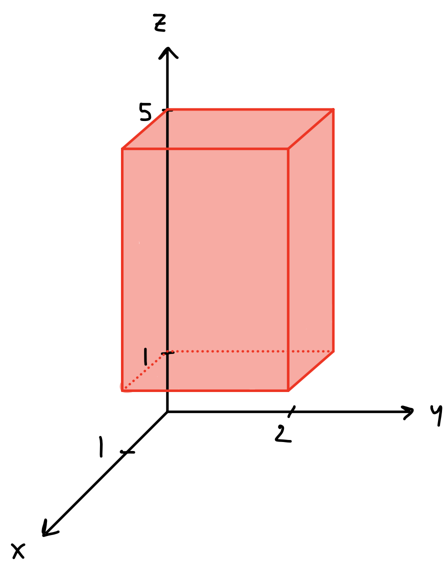
In practice we calculate triple integrals using iterated ordinary integration. Here's a simple example of an iterated integral of a function of three variables:

$$\int_1^5 \int_0^2 \int_0^1 xyz dx dy dz.$$

We evaluate this just as you would expect, namely starting with the inner integral:

$$\int_1^5 \int_0^2 \int_0^1 xyz dx dy dz = \int_1^5 \int_0^2 \left(\frac{1}{2} x^2 yz \Big|_{x=0}^{x=1} \right) dy dz = \int_1^5 \int_0^2 \frac{1}{2} yz dy dz.$$

Now we're left with a two-variable iterated integral, which is familiar territory. Note that the region associated with this iterated integral is a rectangular box in $\mathbb{R}^3$:



The general form of a three-variable iterated integral is

$$\int_a^b \int_{g_1(z)}^{g_2(z)} \int_{h_1(y,z)}^{h_2(y,z)} f(x, y, z) dx dy dz,$$

or the same expression but with the roles of x, y, and z permuted. (Note that there are 6 possible orders of integration!)

Here's another example:

$$\int_1^5 \int_z^{1+z} \int_y^{y+z} xyz\,dx\,dy\,dz = \int_1^5 \int_z^{1+z} \left(\frac{1}{2} x^2 yz \Big|_{x=y}^{x=y+z} \right) dy\,dz$$
$$= \int_1^5 \int_z^{1+z} \left(\frac{1}{2}(y+z)^2 yz - \frac{1}{2} y^2 yz \right) dy\,dz = \cdots$$

3. Mass and center of mass

The mass and center of mass of a three-dimensional region can be calculated in much the same way as for two-dimensional regions. Let R be a solid three-dimensional region with density function $\rho(x, y, z)$. Then the mass of R is

$$\text{mass}(R) = \iiint_R \rho(x, y, z) dV,$$

and the center of mass of R is $(\overline{x}, \overline{y}, \overline{z})$ where

$$\overline{x} = \frac{1}{\text{mass}(R)} \iiint_R x\rho(x, y, z) dV,$$
$$\overline{y} = \frac{1}{\text{mass}(R)} \iiint_R y\rho(x, y, z) dV,$$
$$\overline{z} = \frac{1}{\text{mass}(R)} \iiint_R z\rho(x, y, z) dV.$$

Day 32

1. Practice with triple integration

Yesterday we started working with triple integrals. We saw that in many ways triple integrals are similar to double integrals. The main challenge with triple integrals is visualizing and/or describing the region of integration, which is a subset of $\mathbb{R}^3$.

Yesterday it was implied but not clearly stated that if R is a region of the form

$$R = \{(x, y, z) \colon a \le z \le b,\ g_1(z) \le y \le g_2(z),\ h_1(y, z) \le x \le h_2(y, z)\},$$

then

$$\iiint_R f(x, y, z)dV = \int_a^b \int_{g_1(z)}^{g_2(z)} \int_{h_1(y,z)}^{h_2(y,z)} f(x, y, z)dxdydz.$$

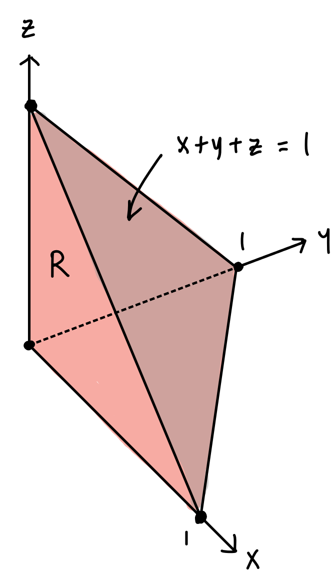
In other words, we can express a triple integral as an iterated integral, provided the region of integration can be expressed in the above form. The roles of x, y, and z can be also be permuted. For example, if

$$R = \{(x, y, z) \colon 1 \le y \le 4,\ y \le z \le y^2,\ y - z \le x \le y + 2z\},$$

then

$$\iiint_R f(x, y, z)dV = \int_1^4 \int_y^{y^2} \int_{y-z}^{y+2z} f(x, y, z)dxdzdy.$$

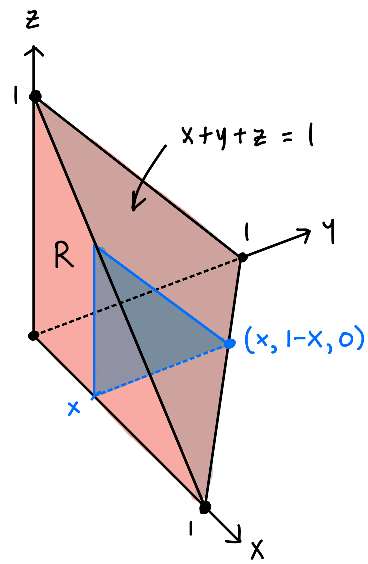
To get some practice with evaluating triple integrals, let's find the center of mass of the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and uniform density $\rho(x, y, z) = 1$.



First we need to find the mass of R, and before that we need to find a description of R that will allow us to set up an iterated integral. We can see that the minimum and maximum x values within R are 0 and 1, so

$$0 \le x \le 1.$$

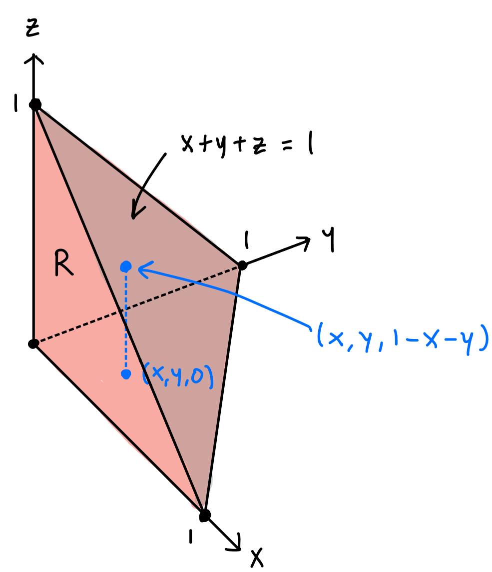
Now let's fix x and think about the range of possible y values. The points (x, y, z) with x fixed form a triangle:



Within this triangle, the minimum and maximum y values are 0 and $1 - x$, so

$$0 \leq y \leq 1 - x.$$

Finally, let's fix x and y and think about the range of possible z values. The set of points (x, y, z) with x and y fixed form a line segment:



Within this segment, the minimum and maximum z values are 0 and $1 - x - y$, so

$$0 \leq z \leq 1 - x - y.$$

We have now shown that[1]

$$R = \{(x, y, z) \colon 0 \le x \le 1,\ 0 \le y \le 1 - x,\ 0 \le z \le 1 - x - y\}.$$

Using this expression for R, we can now set up an iterated integral that tells us the mass:

$$\text{mass}(R) = \iiint_R \rho(x, y, z)dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1dzdydx.$$

This works out to be $1/6$.

Next we calculate the center of mass, $(\overline{x}, \overline{y}, \overline{z})$. Notice that the region R (including its density $\rho(x, y, z) = 1$) is symmetric with respect x, y, and z. Therefore we must have $\overline{x} = \overline{y} = \overline{z}$, which means we only need to compute one such coordinate. (If this worries you, you can calculate each one separately.) Let's find $\overline{x}$. Using the formula, we have

$$\overline{x} = \frac{1}{\text{mass}(R)} \iiint_R x\rho(x, y, z)dV = 6 \int_0^1 \int_0^{1-x} \int_0^{1-x-y} xdzdydx,$$

which works out to be $1/4$. Thus the center of mass of R is

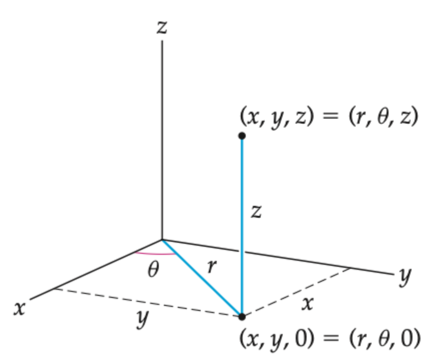
$$(\overline{x}, \overline{y}, \overline{z}) = \Big(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\Big).$$

This makes sense geometrically: Since R has uniform density, its center of mass should be (and is) the average of its four vertices.

2. Integration in cylindrical coordinates

We have seen that there are various different coordinate systems on $\mathbb{R}^2$. The Cartesian and polar coordinate systems are the ones we're most familiar with. In the context of double integrals, Cartesian coordinates were useful for integrating over rectangles, while polar coordinates were more convenient for integrating over discs. (Both systems can handle other types of sets as well.)

There are likewise several commonly used coordinate systems on $\mathbb{R}^3$. One of the simplest is the *cylindrical coordinate system*. To express (x, y, z) in cylindrical coordinates, you just convert (x, y) to polar coordinates and leave z alone! Thus a cylindrical representation of (x, y, z) is simply (r, θ, z) where (r, θ) is a polar representation of (x, y). Visually, it looks like this:



[1]Note that we could have analyzed the variable ranges in a different order (say y, then z, then x). This would lead to a similar and equally valid description of R.

Note that the usual polar conversions still apply here: $x = r\cos\theta$ and $y = r\sin\theta$.

Integration in cylindrical coordinates works in the way you would expect. If R, when expressed in cylindrical coordinates, is of the form

$$R = \{(r,\theta,z)\colon \alpha \le \theta \le \beta,\ g_1(\theta) \le r \le g_2(\theta),\ h_1(r,\theta) \le z \le h_2(r,\theta)\},$$

then

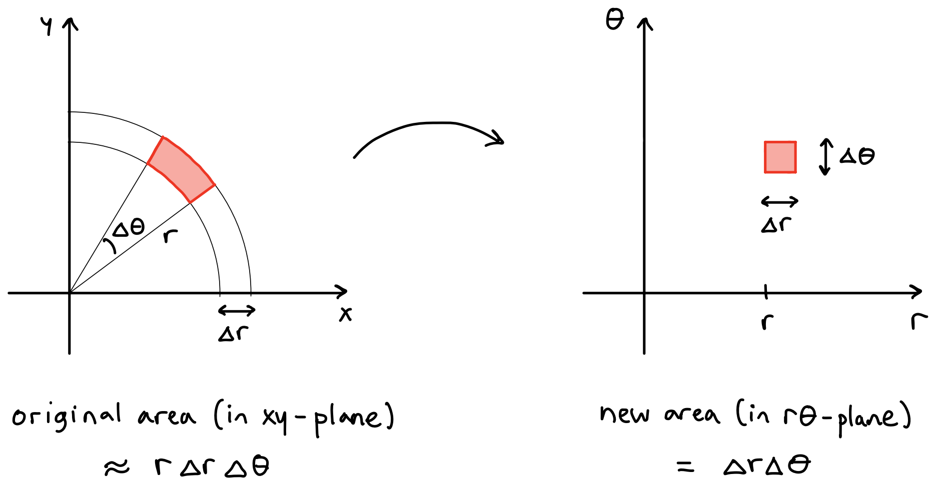
$$\iiint_R f(x,y,z)dV = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(r,\theta)}^{h_2(r,\theta)} f(r\cos\theta, r\sin\theta, z) r dz dr d\theta.$$

Notice that dV has become $rdzdrd\theta$. The factor of r is present for the same reason as in polar integration. However, we never actually saw why that was.

Let's detour for a moment back to polar coordinates. Remember that the double integral is a limit of Riemann sums. In particular,

$$\iint_R f(x,y)dA \approx \sum_{i=1}^N f(x_i,y_i)\,\mathrm{Area}(R_i),$$

where $R_1,\ldots,R_N$ are small subregions whose union is R and $(x_1,y_1),\ldots,(x_N,y_N)$ are points such that (x_i,y_i) belongs to R_i. When we convert to polar coordinates, the area of each piece R_i will change. Specifically, if R_i sits at a distance of r from the origin, then its area will be scaled by $1/r$:



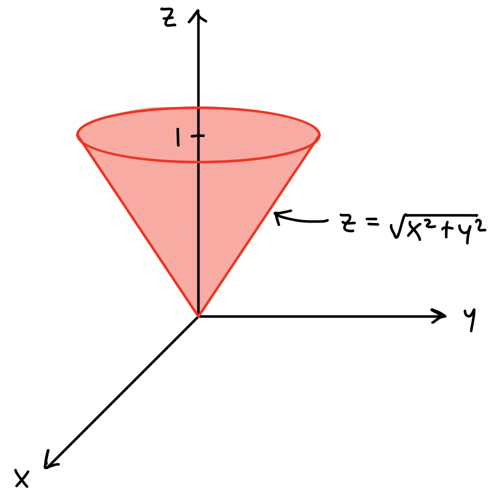
To correct this, we need to multiply the new area by r; this is where the factor of r in the polar integral originates.

Day 33

1. Practice with integration in cylindrical coordinates

Last time we began working with cylindrical coordinates. Recall that a cylindrical coordinate representation of (x, y, z) can be found by converting (x, y) to polar coordinates and leaving z unchanged. Unsurprisingly, integration in cylindrical coordinates closely resembles integration in polar coordinates.

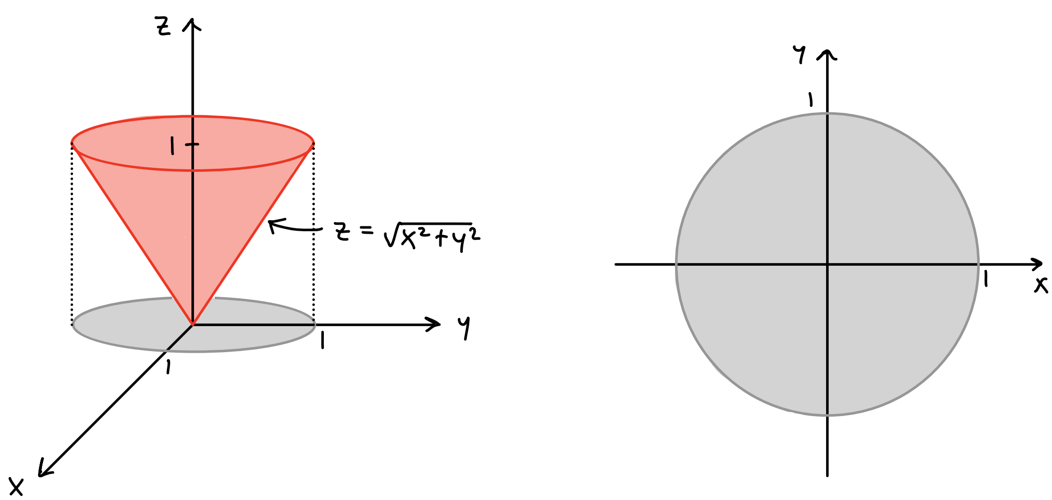
As an example, let R be the following region:



Let's evaluate the triple integral

$$\iiint_R z\,dV.$$

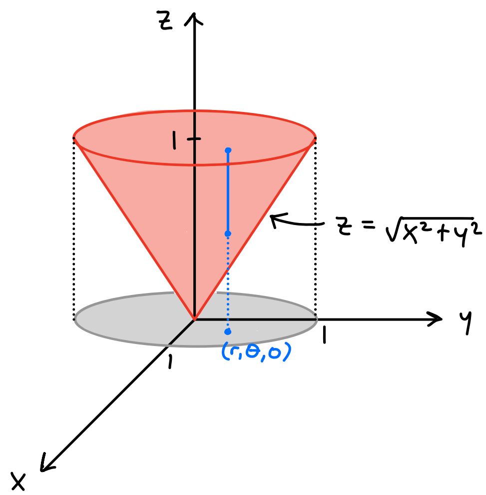
The first step is to express the region R in some coordinate system. Cylindrical coordinates will be convenient because R is rotationally symmetric with respect to the z-axis. First let's consider the possible values of r and θ among all points (r, θ, z) belonging to R. This can be achieved by projecting R onto the xy-plane and describing the resulting set using polar coordinates:



Since the top face of R is a circle of radius 1 with center on the z-axis, the projection of R onto the xy-plane will be the standard unit disc. Its polar coordinate description is

$$0 \leq \theta \leq 2\pi, \qquad 0 \leq r \leq 1.$$

Now suppose r and θ are fixed. What are the possible z values among all points of the form (r, θ, z) that belong to R? These points form a line segment extending from the cone up to its top face:



Notice that when x and y are converted to polar coordinates, the equation $z = \sqrt{x^2 + y^2}$ becomes $z = r$. So the minimum z value on this line segment is r and the maximum z value is 1; thus

$$r \leq z \leq 1.$$

We have now shown that in cylindrical coordinates

$$R = \{(r, \theta, z) \colon 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 1,\ r \leq z \leq 1\}.$$

Thus

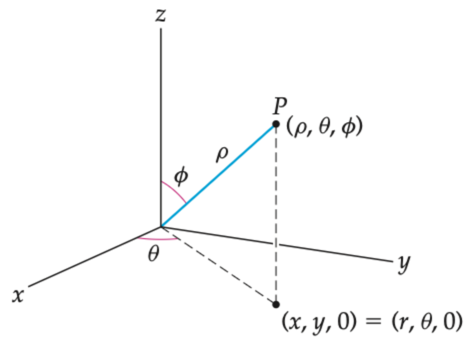
$$\iiint_R z dV = \int_0^{2\pi} \int_0^1 \int_r^1 zrdzdrd\theta = \int_0^{2\pi} \int_0^1 \left(\frac{1}{2} - \frac{r^2}{2}\right) rdrd\theta = \int_0^{2\pi} \frac{1}{8} d\theta = \frac{\pi}{4}.$$

2. Spherical coordinates

Another frequently used coordinate system on $\mathbb{R}^3$ is known as *spherical coordinates*. As their name would imply, spherical coordinates are especially useful for describing regions with spherical symmetry. A spherical coordinate representation of a point (x, y, z) is a triple (ρ, θ, ϕ) such that

$$x = \rho \sin(\phi)\cos(\theta), \qquad y = \rho \sin(\phi)\sin(\theta), \qquad z = \rho\cos(\phi).$$

As with other coordinate systems we've seen, the choice of (ρ, θ, ϕ) is not unique; however there is again a standard way to choose these coordinates geometrically:



Thus:

- ρ is the distance between (x, y, z) and the origin;
- θ is the usual polar angle in the xy-plane (i.e. the angle between the vector $\langle x, y\rangle$ and the positive x-axis);
- ϕ is the angle between the vector $\langle x, y, z\rangle$ and the positive z axis.

We can (and will) always choose $\rho \geq 0$ and[1] $\phi \in [0, \pi]$. Typically we'll choose $\theta \in [0, 2\pi]$, though it's sometimes convenient to allow θ to be negative (e.g. $\theta \in [-\pi, \pi]$), just as in the polar coordinate setting.

Notice that the equation $\rho = R$ represents the sphere of radius R centered at the origin. If $\alpha \in [0, 2\pi]$ and $\beta \in [0, \pi]$ are constants, what do the equations $\theta = \alpha$ and $\phi = \beta$ represent?

[1] To justify choosing $\phi \in [0, \pi]$, note that $\sin(\phi)\cos(\theta) = \sin(2\pi - \phi)\cos(\theta + \pi)$ and $\sin(\phi)\sin(\theta) = \sin(2\pi - \phi)\sin(\theta + \pi)$. This implies that (ρ, θ, ϕ) and $(\rho, \theta + \pi, 2\pi - \phi)$ always represent the same point.