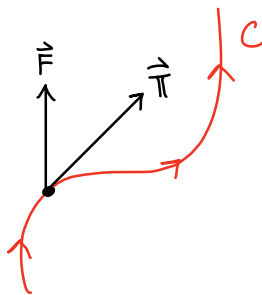


Day 38

1. LINE INTEGRALS OF VECTOR FIELDS

Last week we saw the concept of a line integral of a scalar function along a curve. This had a physical interpretation as the mass of the curve (or wire), with the integrand representing its density. We can also integrate a vector field along a curve. Now, instead of representing mass, the integral will represent *work*. Let C be an *oriented* curve in \mathbb{R}^3 and let \mathbf{F} be a vector field in \mathbb{R}^3 . By “oriented” we mean that there is a specified direction of travel from one end of C to the other. (For example, if C is an arc of a circle, then the orientation would be either clockwise or counterclockwise.) This orientation defines a unique unit tangent vector \mathbf{T} at each point on the curve:



At any given point, the work done by \mathbf{F} along an infinitesimal arc of the curve is given by the dot product $\mathbf{F} \cdot \mathbf{T} ds$. Thus the total work done by \mathbf{F} along C is given by

$$\int_C \mathbf{F} \cdot \mathbf{T} ds.$$

This is called the *line integral of \mathbf{F} along C* . Notice that this integral depends on the orientation of the curve. If we were to reverse the orientation (i.e. traverse the curve in the opposite direction), then the unit tangent \mathbf{T} would become $-\mathbf{T}$ and thus the integral would be multiplied by -1 .

To evaluate this line integral, we first need to fix a parametrization $\mathbf{r}(t)$. Let's choose \mathbf{r} so that it traces out the curve according to its given orientation. Then for each t , the unit tangent at $\mathbf{r}(t)$ is given by $\mathbf{T} = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$. Thus, if $[a, b]$ is the domain of \mathbf{r} , then the line integral can be expressed as

$$\int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \|\mathbf{r}'(t)\| dt = \int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t) dt.$$

This is often abbreviated as

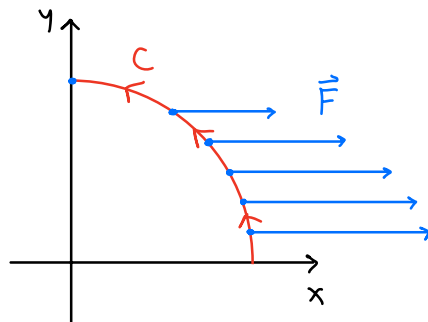
$$\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r}.$$

You can see that there are a few different notations for line integrals of vector fields; this can be confusing at first, but it's useful to recognize them and know that they're equivalent. To summarize: If C is an oriented curve and \mathbf{F} is a vector field, then the line integral of \mathbf{F} along C is

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r} = \int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t) dt,$$

where $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, $t \in [a, b]$, is any parametrization of C in the direction of its orientation.

As an example, let C be the first quadrant of the unit circle $x^2 + y^2 = 1$ with counterclockwise orientation, and let $\mathbf{F}(x, y) = \langle x, 0 \rangle$. Here's a sketch of what \mathbf{F} looks like along C :



To evaluate the line integral of \mathbf{F} along C , we first need to choose a parametrization $\mathbf{r}(t) = \langle x(t), y(t) \rangle$. We can use the usual unit circle parametrization $x(t) = \cos(t)$, $y(t) = \sin(t)$, with $t \in [0, \pi/2]$. Notice that this parametrization agrees with the orientation of C . Now we calculate

$$\begin{aligned} \int_C \mathbf{F}(x, y) \cdot d\mathbf{r} &= \int_0^{\pi/2} \langle x(t), 0 \rangle \cdot \langle x'(t), y'(t) \rangle dt \\ &= \int_0^{\pi/2} \cos(t)(-\sin(t)) dt \\ &= \frac{1}{2} \cos(t)^2 \Big|_{t=0}^{t=\pi/2} \\ &= -\frac{1}{2}. \end{aligned}$$

The negative answer makes sense: We can see that \mathbf{F} opposes the direction of C , so it's doing negative work.

2. FUNDAMENTAL THEOREM OF LINE INTEGRALS

Recall that a vector field \mathbf{F} is *conservative* if $\mathbf{F} = \nabla f$ for some scalar function f . An important property of conservative vector fields is that their line integrals are easy to compute. This is captured in the *fundamental theorem of line integrals*:

The Fundamental Theorem of Line Integrals

Let C be a smooth curve that is the graph of the vector function $\mathbf{r}(t)$ defined on the interval $[a, b]$ with $P = \mathbf{r}(a)$ and $Q = \mathbf{r}(b)$. If $\mathbf{F}(x, y, z)$ is a conservative vector field with $\mathbf{F}(x, y, z) = \nabla f(x, y, z)$ on an open, connected, and simply connected domain containing the curve C , then

$$\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r} = f(Q) - f(P).$$

Notice that this theorem has a striking similarity to the fundamental theorem of calculus, which states (in part) that

$$\int_a^b g'(x)dx = g(b) - g(a).$$

If $\mathbf{F} = \nabla f$, then we can think of \mathbf{F} as the “derivative” of f or equivalently think of f as an “antiderivative” of \mathbf{F} . The fundamental theorem of line integrals states that the (line) integral of ∇f is determined by the value of f at the two endpoints or “bounds” of the integral.

Let’s revisit the example from above, namely where $\mathbf{F}(x, y) = \langle x, 0 \rangle$ and C is the first quadrant of the unit circle with counterclockwise orientation. It’s easy to see that \mathbf{F} is conservative and that $\mathbf{F} = \nabla f$ for $f(x, y) = \frac{1}{2}x^2$. Since C starts at $(1, 0)$ and ends at $(0, 1)$, the fundamental theorem of line integrals implies that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(0, 1) - f(1, 0) = -\frac{1}{2},$$

as we found before.

Day 39

1. FUNDAMENTAL THEOREM OF LINE INTEGRALS

Yesterday we saw the definition of the line integral of a vector field \mathbf{F} along an oriented curve C , namely

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t) dt,$$

where $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, $t \in [a, b]$, is any parametrization of C in the direction of its orientation. We also saw the fundamental theorem of line integrals, which states that if $\mathbf{F} = \nabla f$ for some f (i.e. if \mathbf{F} is conservative), then for any curve C from P to Q , we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(Q) - f(P).$$

This theorem simplifies the calculation of line integrals of conservative vector fields in a couple of ways:

First, it reduces the integral to the problem of finding a potential function. While the latter problem still involves integration, it's often straightforward and does not require parametrizing a curve. And in some contexts, a potential function might already be available.

Second, the theorem implies that the line integral of a conservative vector field is *path-independent*. It depends on the starting and ending points, but not on the path between them. This can be very convenient: Integration along a complicated path between P and Q is equivalent to integration along, say, the straight line segment joining P and Q .

Let's see the proof of the fundamental theorem of line integrals. Let $\mathbf{F} = \nabla f$ and let $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, $t \in [a, b]$, be a parametrization of C in the direction of its orientation. Abbreviating $x(t), y(t), z(t)$ by x, y, z , we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \nabla f(x, y, z) \cdot \langle x', y', z' \rangle dt.$$

Notice that

$$\nabla f(x, y, z) \cdot \langle x', y', z' \rangle = f_x(x, y, z)x' + f_y(x, y, z)y' + f_z(x, y, z)z' = \frac{d}{dt}f(x, y, z)$$

by the chain rule. Therefore, by the fundamental theorem of calculus, we obtain

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \frac{d}{dt}f(x, y, z) dt = f(x(b), y(b), z(b)) - f(x(a), y(a), z(a)) = f(Q) - f(P),$$

where P and Q are the starting and ending points of C .

2. SURFACE AREA

Given a curve C , we know how to find its arc length, namely it's the line integral

$$\int_C 1 ds.$$

The analogous concept for surfaces is *surface area*, which we will calculate using a *surface integral*.

The surfaces we're most familiar with are graphs, but as you know, not all surfaces are graphs. To define surface integration for general surfaces, we'll need to use parametrizations (just like for curves). A *surface parametrization* is function \mathbf{r} of the form

$$\mathbf{r}(s, t) = \langle x(s, t), y(s, t), z(s, t) \rangle,$$

or in other words, a function from \mathbb{R}^2 to \mathbb{R}^3 . (Here we're using s as a variable; it does not refer to arc length.)

As an example, let Σ denote the sphere of radius 3 centered at the origin. We normally express Σ using a level surface equation: $x^2 + y^2 + z^2 = 9$. But let's try to find a function \mathbf{r} that parametrizes Σ . Using spherical coordinates, we write

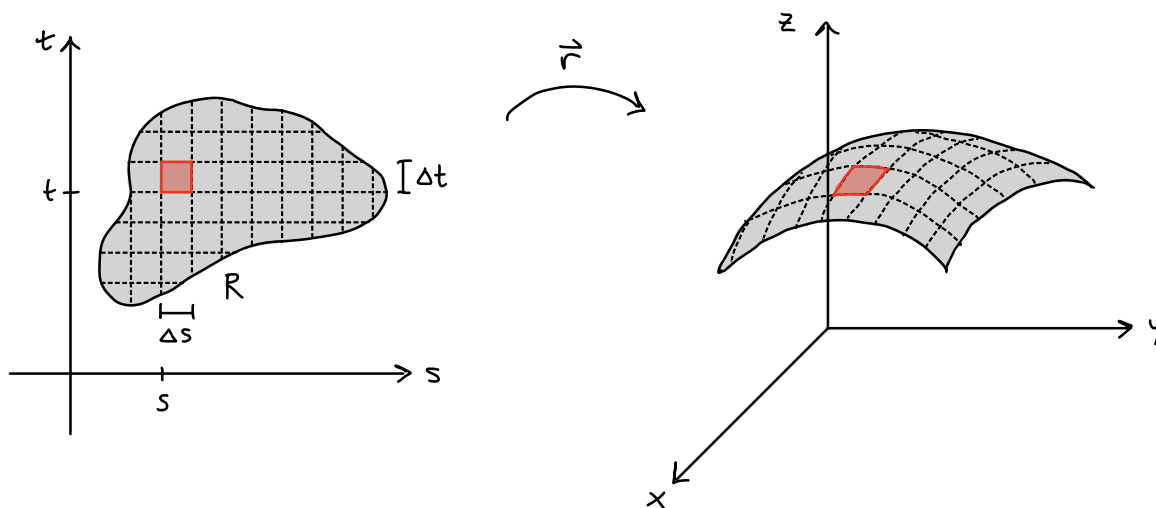
$$x = \rho \sin(\phi) \cos(\theta), \quad y = \rho \sin(\phi) \sin(\theta), \quad z = \rho \cos(\phi),$$

where $\phi \in [0, \pi]$ and $\theta \in [0, 2\pi]$. The point (x, y, z) belongs to Σ if and only if $\rho = 3$. Thus, Σ is parametrized by the function

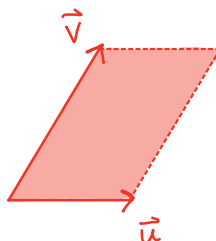
$$\mathbf{r}(\phi, \theta) = \langle 3 \sin(\phi) \cos(\theta), 3 \sin(\phi) \sin(\theta), 3 \cos(\phi) \rangle$$

for $\phi \in [0, \pi]$ and $\theta \in [0, 2\pi]$.

Suppose $S = \Sigma$ is a surface parametrized by $\mathbf{r}(s, t)$ for (s, t) belong to some region R in the plane. To find the surface area of Σ , we can use the usual strategy of breaking up R into small pieces and estimating the surface area corresponding to each piece:



Let's consider one rectangle and the bit of surface associated with it, as shaded in red above. This bit of surface is approximately a parallelogram:



The sides of the parallelogram are related to the partial derivatives of \mathbf{r} . Specifically, by linear approximation,

$$\mathbf{u} \approx \mathbf{r}_s(s, t)\Delta s = \langle x_s(s, t), y_s(s, t), z_s(s, t) \rangle \Delta s, \quad (2.1)$$

$$\mathbf{v} \approx \mathbf{r}_t(s, t)\Delta t = \langle x_t(s, t), y_t(s, t), z_t(s, t) \rangle \Delta t. \quad (2.2)$$

So the area of the parallelogram is

$$\|\mathbf{u} \times \mathbf{v}\| \approx \|\mathbf{r}_s \times \mathbf{r}_t\|.$$

Thus, after summing over all rectangles that make up R and then taking the limit as $\Delta s, \Delta t \rightarrow 0$, we see that

$$\text{surface area of } \Sigma = \iint_R \|\mathbf{r}_s \times \mathbf{r}_t\| dA.$$

If Σ happens to be a graph of a function g on the domain R , then it's parametrized by $\mathbf{r}(x, y) = \langle x, y, g(x, y) \rangle$ for (x, y) belonging to R . This leads to the formula

$$\text{surface area of } \Sigma = \iint_R \sqrt{\|\nabla g\|^2 + 1} dA.$$

Day 40

1. SURFACE INTEGRALS OF SCALAR FUNCTIONS

Yesterday we saw that if Σ is a surface parametrized by $\mathbf{r}(s, t)$, where (s, t) belongs to a region R in \mathbb{R}^2 , then

$$\text{surface area of } \Sigma = \iint_R \|\mathbf{r}_s \times \mathbf{r}_t\| dA.$$

This double integral is often abbreviated as

$$\int_{\Sigma} 1 dS.$$

We interpret $dS = \|\mathbf{r}_s \times \mathbf{r}_t\| dA$ as “integration with respect to surface area” (much like how $ds = \|\mathbf{r}'(t)\| dt$ is integration with respect to arc length). This is an example of a *surface integral*.

More generally, the surface integral of a function f over Σ is defined as

$$\begin{aligned} \int_{\Sigma} f(x, y, z) dS &= \iint_R f(\mathbf{r}(s, t)) \|\mathbf{r}_s \times \mathbf{r}_t\| dA \\ &= \iint_R f(x(s, t), y(s, t), z(s, t)) \|\mathbf{r}_s(s, t) \times \mathbf{r}_t(s, t)\| dA, \end{aligned}$$

where $\mathbf{r}(s, t) = \langle x(s, t), y(s, t), z(s, t) \rangle$ is a parametrization of Σ with domain R . As with the special case of surface area, if Σ is a graph, then we get a simpler formula. Suppose Σ is the graph of the function g over the region R . Then

$$\int_{\Sigma} f(x, y, z) dS = \iint_R f(x, y, g(x, y)) \sqrt{\|\nabla g(x, y)\|^2 + 1} dA.$$

2. SURFACE AREA EXAMPLES

Let's see a couple of examples of calculating surface area. First, let Σ be the sphere of radius ρ centered at the origin. You might be aware that the surface area of Σ is $4\pi\rho^2$. We will confirm this using surface integration. For our parametrization of Σ , we can use

$$\mathbf{r}(\phi, \theta) = \langle \rho \sin(\phi) \cos(\theta), \rho \sin(\phi) \sin(\theta), \rho \cos(\phi) \rangle,$$

where $\phi \in [0, \pi]$ and $\theta \in [0, 2\pi]$. Now we need to calculate

$$\int_{\Sigma} 1 dS = \int_0^{\pi} \int_0^{2\pi} \|\mathbf{r}_{\phi} \times \mathbf{r}_{\theta}\| d\theta d\phi.$$

We have

$$\begin{aligned} \mathbf{r}_{\phi}(\phi, \theta) &= \langle \rho \cos(\phi) \cos(\theta), \rho \cos(\phi) \sin(\theta), -\rho \sin(\phi) \rangle, \\ \mathbf{r}_{\theta}(\phi, \theta) &= \langle -\rho \sin(\phi) \sin(\theta), \rho \sin(\phi) \cos(\theta), 0 \rangle. \end{aligned}$$

We could now proceed to calculate $\|\mathbf{r}_{\phi} \times \mathbf{r}_{\theta}\|$ by applying the definition of cross product and then taking the magnitude. If we're persistent, everything will simplify nicely (via repeated use of the Pythagorean identity). But let's try to be more clever. Recall that

$$\|\mathbf{r}_{\phi} \times \mathbf{r}_{\theta}\| = \|\mathbf{r}_{\phi}\| \|\mathbf{r}_{\theta}\| \sin(\vartheta)$$

where ϑ is the smallest angle between \mathbf{r}_ϕ and \mathbf{r}_θ . By calculating their dot product, we can see that \mathbf{r}_ϕ and \mathbf{r}_θ are orthogonal, so $\sin(\vartheta) = 1$. Thus

$$\|\mathbf{r}_\phi \times \mathbf{r}_\theta\| = \|\mathbf{r}_\phi\| \|\mathbf{r}_\theta\|.$$

Using the Pythagorean identity, it's straightforward to check that

$$\|\mathbf{r}_\phi\| = \rho \quad \text{and} \quad \|\mathbf{r}_\theta\| = \rho |\sin(\phi)| = \rho \sin(\phi).$$

So altogether, the surface area is

$$\int_{\Sigma} 1 dS = \int_0^\pi \int_0^{2\pi} \rho^2 \sin(\phi) d\theta d\phi = \int_0^\pi 2\pi \rho^2 \sin(\phi) d\phi = 4\pi \rho^2,$$

as we expected.

For our second example, let Σ be the part of the hyperbolic paraboloid $z = x^2 - y^2$ that lies inside the cylinder $x^2 + y^2 = 1$. In this case, Σ is the graph of $g(x, y) = x^2 - y^2$ with (x, y) belonging to the unit disc $R = \{(x, y) : x^2 + y^2 \leq 1\}$. So the surface area is given by

$$\int_{\Sigma} 1 dS = \iint_R \sqrt{\|\nabla g\|^2 + 1} dA = \iint_R \sqrt{4(x^2 + y^2) + 1} dA.$$

Using polar coordinates, we see that

$$\int_{\Sigma} 1 dS = \int_0^{2\pi} \int_0^1 \sqrt{4r^2 + 1} r dr d\theta = \int_0^{2\pi} \frac{1}{12} (4r^2 + 1)^{3/2} \Big|_{r=0}^{r=1} d\theta = \frac{\pi}{6} (5\sqrt{5} - 1).$$

(Side observation: The elliptic paraboloid $z = x^2 + y^2$ is the graph of the function $h(x, y) = x^2 + y^2$. Notice that $\|\nabla h\| = \|\nabla g\|$. This implies that the elliptic paraboloid will have the same surface area as the hyperbolic paraboloid above any given region R . This doesn't seem obvious by just looking at these surfaces!)

Day 41

1. ANOTHER SURFACE INTEGRATION EXAMPLE

Last time we saw a couple of examples of calculating surface area, but we haven't worked out a general surface integral yet.

In-class exercise: Let Σ be the part of the paraboloid $z = x^2 + y^2$ that lies above the unit square $R = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$. Evaluate

$$\int_{\Sigma} \frac{x+y}{\sqrt{4z+1}} dS.$$

Solution. Notice that Σ is the graph of $g(x, y) = x^2 + y^2$ above R . Therefore

$$\int_{\Sigma} \frac{x+y}{\sqrt{4z+1}} dS = \iint_R \frac{x+y}{\sqrt{4g(x, y)+1}} \sqrt{\|\nabla g(x, y)\|^2 + 1} dA.$$

We calculate that $\|\nabla g(x, y)\|^2 = 4(x^2 + y^2) = 4g(x, y)$. Thus

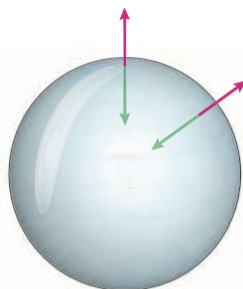
$$\int_{\Sigma} \frac{x+y}{\sqrt{4z+1}} dS = \iint_R (x+y) dA = \int_0^1 \int_0^1 (x+y) dx dy = 1.$$

In this example, the gradient factor canceled nicely, leaving us with a very simple double integral. This will not always be the case. Sometimes additional steps (e.g. u substitutions) may be needed to calculate the final value.

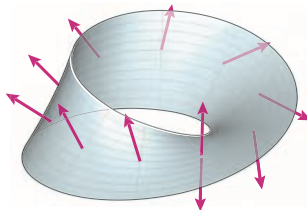
2. SURFACE INTEGRALS OF VECTOR FIELDS

Recall that we can compute line integrals of both scalar functions and vector fields. The latter had a physical interpretation as the work done by the vector field along the curve. Surface integrals also make sense for both scalar functions and vector fields. In the case of vector fields, the quantity we'll consider is not work but *flux*. This quantifies the flow of the vector field through the surface.

Just like with line integrals of vector fields, we'll need to consider *orientation*. In the case of surfaces, an orientation is a consistent choice of unit normal vector \mathbf{n} . An *orientable* surface is a surface for which an orientation can be specified. For example, a sphere is orientable because we can define \mathbf{n} to be the outward (or inward) normal vector:



A Möbius strip is the classic example of a nonorientable surface:



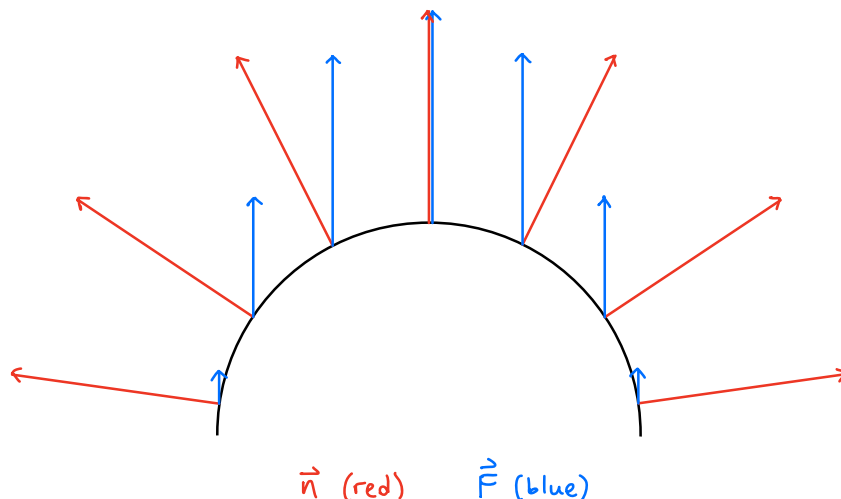
If we start with a “downward” pointing normal vector and move it continuously along this surface, we eventually arrive back at the same point but with an “upward” pointing normal vector! Concepts like *upward*, *downward*, *outward*, and *inward* don’t make sense for the Möbius strip.

Let Σ be an oriented surface with unit normal \mathbf{n} . Let \mathbf{F} be a vector field in \mathbb{R}^3 . Then the *surface integral* (or *flux*) of \mathbf{F} through Σ is defined as

$$\int_{\Sigma} \mathbf{F} \cdot \mathbf{n} dS.$$

Notice that $\mathbf{F} \cdot \mathbf{n}$ is a scalar function, so in theory we already know how to evaluate this.

Let’s see an example. Let Σ be the upper hemisphere of the unit sphere $x^2 + y^2 + z^2 = 1$ with outward unit normal vector \mathbf{n} , and let $\mathbf{F}(x, y, z) = \langle 0, 0, z \rangle$. Before calculating the surface integral, let’s sketch Σ and \mathbf{F} :



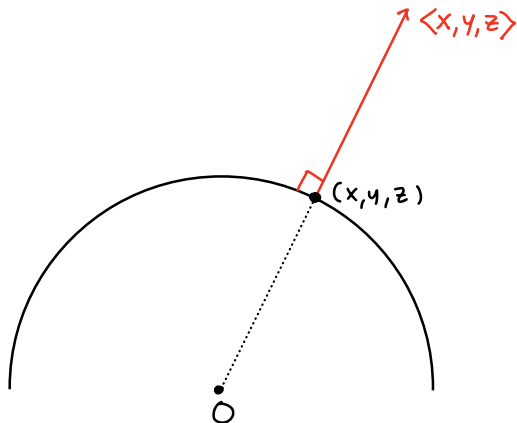
For simplicity, we’ve created a two-dimensional sketch. You can see that along the hemisphere, \mathbf{F} points in the same general direction as \mathbf{n} . (More precisely, \mathbf{F} forms an acute angle with \mathbf{n} at every point on Σ .) Thus, we should expect that the flux of \mathbf{F} through Σ will be positive.

Let’s now calculate the flux integral

$$\int_{\Sigma} \mathbf{F} \cdot \mathbf{n} dS.$$

We know that $\mathbf{F}(x, y, z) = \langle 0, 0, z \rangle$, but we don’t yet have an expression for \mathbf{n} as function of x, y, z . Here’s an important geometric fact that’s worth remembering: If (x, y, z) is a point

on the unit sphere, then the vector $\langle x, y, z \rangle$ is the outward unit normal vector to the sphere at that point. This should be believable based on geometry¹:



So in the flux integral above we have $\mathbf{n} = \langle x, y, z \rangle$. Thus

$$\int_{\Sigma} \mathbf{F} \cdot \mathbf{n} dS = \int_{\Sigma} \langle 0, 0, z \rangle \cdot \langle x, y, z \rangle dS = \int_{\Sigma} z^2 dS.$$

To evaluate this integral, we'll use the fact that Σ is the graph of

$$g(x, y) = \sqrt{1 - x^2 - y^2}$$

over the unit disc $R = \{(x, y) : x^2 + y^2 \leq 1\}$. (Alternatively, we could parametrize Σ using spherical coordinates.) This leads to

$$\int_{\Sigma} \mathbf{F} \cdot \mathbf{n} dS = \iint_R g(x, y)^2 \sqrt{\|\nabla g(x, y)\|^2 + 1} dA.$$

Now we need to calculate ∇g . We find that

$$\nabla g(x, y) = -\frac{\langle x, y \rangle}{\sqrt{1 - x^2 - y^2}}$$

and thus

$$\|\nabla g(x, y)\|^2 + 1 = \frac{x^2 + y^2}{1 - x^2 - y^2} + 1 = \frac{1}{1 - x^2 - y^2}.$$

Substituting this into the double integral above, we get

$$\int_{\Sigma} \mathbf{F} \cdot \mathbf{n} dS = \iint_R (1 - x^2 - y^2) \frac{1}{\sqrt{1 - x^2 - y^2}} dA = \iint_R \sqrt{1 - x^2 - y^2} dA.$$

Now this is starting to look manageable. Remember that R is the unit disc, so we should convert to polar coordinates. This leads to

$$\int_{\Sigma} \mathbf{F} \cdot \mathbf{n} dS = \int_0^{2\pi} \int_0^1 \sqrt{1 - r^2} r dr d\theta = \int_0^{2\pi} \left. -\frac{1}{3}(1 - r^2)^{3/2} \right|_{r=0}^{r=1} d\theta = \frac{2\pi}{3}.$$

Notice that our answer is positive, as we expected.

¹One can confirm this more rigorously by considering the gradient of the function $f(x, y, z) = x^2 + y^2 + z^2$, for which the unit sphere is a level surface.