Math 2341 Exam 1 Notesheet Sean Balbale October 2024

Laplace Transform

3.1 Laplace Transform and Inverse

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt, \quad \mathcal{L}^{-1}\{F(s)\} = f(t)$$

3.2 Transforms of Derivatives and IVPs

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0), \quad \mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

3.3 Shifting Theorems

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a), \quad \mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$$

3.4 Discontinuous Inputs

Heaviside function: $u_c(t)$,

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}$$

3.5 Convolution

Convolution Theorem:

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau, \quad \mathcal{L}\{f * g\} = F(s)G(s)$$

Systems of Equations and Matrices (Chapter 4)

4.1 Systems and Matrices

Representation: A system of equations can be written as $Ax = b$, where:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

4.2 Gaussian Elimination

Steps:

- **Forward elimination:** Reduce the system to an upper triangular matrix.
- **Back substitution:** Solve for variables starting from the last row.

Example:

$$\begin{bmatrix} 2 & 1 & -1 & | & 8 \\ -3 & -1 & 2 & | & -11 \\ -2 & 1 & 2 & | & -3 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & -1 & | & 8 \\ 0 & -0.5 & 0.5 & | & -1 \\ 0 & 0 & 1 & | & -2 \end{bmatrix}$$

4.3 Row-Echelon Form and Rank

Definitions:

- **Row-echelon form:** Leading coefficients (pivots) are to the right of the pivots in the rows above.
- **Rank:** The number of pivot positions in the matrix.

Example (Row-Echelon Form):

$$\begin{bmatrix} 1 & 2 & 3 & | & 4 \\ 0 & 1 & 4 & | & 5 \\ 0 & 0 & 1 & | & 6 \end{bmatrix}$$

This is in row-echelon form because each leading entry is to the right of the leading entry in the row above, and all rows of zeros are at the bottom.

4.6 Cofactor Expansions

Determinant:

$$\det(A) = \sum (-1)^{i+j} a_{ij} \det(A_{ij})$$

Example:

For $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$, $\det(A) = 1(24-0) - 2(0-5) + 3(0-4) = 22$

Eigenvalues and Eigenvectors (Chapter 6)

6.1 Eigenvalues and Eigenvectors

Definition: $Ax = \lambda x$, where λ is an eigenvalue and x is an eigenvector.
Characteristic Polynomial:

$$\det(A - \lambda I) = 0$$

Example:

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}, \quad \det(A - \lambda I) = \det \begin{bmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix} = (\lambda - 5)(\lambda - 2)$$

Eigenvalues: $\lambda = 5, 2$.

Diagonalization

Matrix Diagonalization:

$$A = PDP^{-1}, \quad P: \text{Matrix of eigenvectors}$$
$$, \quad D: \text{Diagonal matrix of eigenvalues.}$$

Steps:

- Find eigenvalues λ from $\det(A - \lambda I) = 0$.
- Solve $(A - \lambda I)x = 0$ for eigenvectors.
- Construct P and D.