

## Day 46

### 1. DERIVING GREEN'S THEOREM

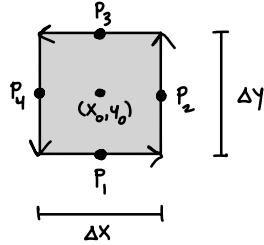
Last week we saw Green's theorem, which states that if  $R$  is a bounded region in  $\mathbb{R}^2$  whose boundary  $C$  is a simple closed curve with counterclockwise orientation, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA,$$

for any vector field  $\mathbf{F} = \langle F_1, F_2 \rangle$  on  $\mathbb{R}^2$ . Our goal for today is to get a sense for why Green's theorem is true. The argument will have two parts:

- (1) Confirming that Green's theorem holds (at least approximately) when  $R$  is a very small rectangle.
- (2) Confirming that Green's theorem holds (at least approximately) for a general region  $R$  by decomposing  $R$  into many small rectangles.

Let's begin with the first part. Consider the following small rectangle  $R$ , with boundary  $C$  oriented counterclockwise:



The line integral  $\int_C \mathbf{F} \cdot d\mathbf{r}$  is the total work done by  $\mathbf{F}$  along each side of the rectangle  $R$ . If  $R$  is very small, then  $\mathbf{F}$  will be roughly constant along each side. We can therefore approximate the total work by choosing one point from each side and adding up the work done by  $\mathbf{F}$  at each of those points. The bottom side of the  $R$  is described by the vector  $\Delta x \mathbf{i}$ , and thus the work done by  $\mathbf{F}$  along that side is

$$\mathbf{F}(P_1) \cdot \Delta x \mathbf{i}.$$

We can write down similar expressions for the work done along the other three sides, leading to

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &\approx \mathbf{F}(P_1) \cdot \Delta x \mathbf{i} + \mathbf{F}(P_2) \cdot \Delta y \mathbf{j} + \mathbf{F}(P_3) \cdot (-\Delta x \mathbf{i}) + \mathbf{F}(P_4) \cdot (-\Delta y \mathbf{j}) \\ &= F_1(P_1)\Delta x + F_2(P_2)\Delta y - F_1(P_3)\Delta x - F_2(P_4)\Delta y \\ &= \left( \frac{F_2(P_2) - F_2(P_4)}{\Delta x} - \frac{F_1(P_3) - F_1(P_1)}{\Delta y} \right) \Delta x \Delta y. \end{aligned}$$

If  $\Delta x$  and  $\Delta y$  are sufficiently small, then

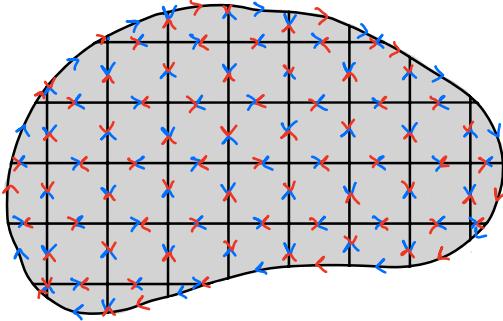
$$\frac{F_2(P_2) - F_2(P_4)}{\Delta x} \approx \frac{\partial}{\partial x} F_2(x_0, y_0) \quad \text{and} \quad \frac{F_1(P_3) - F_1(P_1)}{\Delta y} \approx \frac{\partial}{\partial y} F_1(x_0, y_0).$$

Thus

$$\int_C \mathbf{F} \cdot d\mathbf{r} \approx \left( \frac{\partial}{\partial x} F_2(x_0, y_0) - \frac{\partial}{\partial y} F_1(x_0, y_0) \right) \text{Area}(R) \approx \iint_R \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA,$$

which is what we were aiming to show.

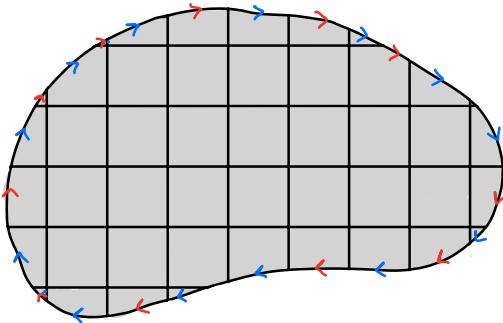
For the second part of our argument, we take  $R$  to be a generic region whose boundary  $C$  is a simple closed curve. We break up  $R$  into many small rectangles  $R_1, \dots, R_N$  with boundaries  $C_1, \dots, C_N$ , each oriented counterclockwise:



By the first part of our argument, we have

$$\iint_R \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = \sum_{i=1}^N \iint_{R_i} \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA \approx \sum_{i=1}^N \int_{C_i} \mathbf{F} \cdot d\mathbf{r}.$$

In the right-hand sum, the line integral along  $C_i$  can be further decomposed into a sum of line integrals along each of the four segments of  $C_i$ . Thus the right-hand sum is adding up the line integrals along all of the labeled segments in picture above. Any interior segment will be traversed twice, in opposite directions (represented by opposing red and blue arrows). The corresponding line integrals will cancel. Thus we're left with the sum of the line integrals along boundary segments:



But this is just  $\int_C \mathbf{F} \cdot d\mathbf{r}$ . Therefore

$$\iint_R \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA \approx \int_C \mathbf{F} \cdot d\mathbf{r},$$

which is what we were aiming to show.

**In-class exercise** Use Green's theorem to evaluate  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where  $\mathbf{F}(x, y) = \langle xy, xy^3 \rangle$  and  $C$  is the boundary of the rectangle  $R = \{(x, y) : 0 \leq x \leq 3, 0 \leq y \leq 2\}$  oriented counterclockwise.

*Solution.* By Green's theorem, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = \int_0^3 \int_0^2 (y^3 - x) dy dx = \int_0^3 (4 - 2x) dx = 3.$$

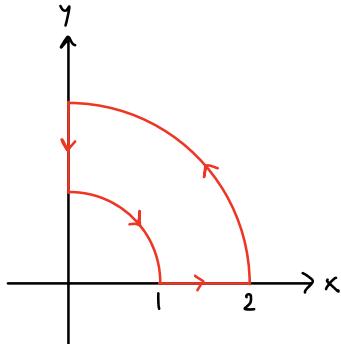
Notice that if we were to evaluate the line integral directly (rather than using Green's theorem), we would need to break it up as a sum of line integrals along each of the four sides of the rectangle. This is doable but rather tedious.

## Day 47

## 1. MORE EXAMPLES OF GREEN'S THEOREM

Yesterday, we saw a (rough) proof of Green's theorem, as well as an example that illustrated how useful the theorem can be. The plan for today is to see a couple more examples, then move on to Stokes' theorem, a three-dimensional analogue of Green's theorem. Here's a warm-up problem:

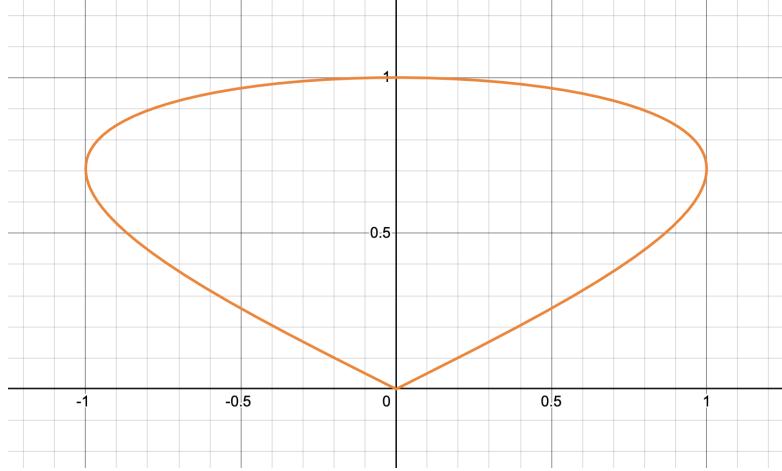
**In-class exercise:** Evaluate  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where  $\mathbf{F}(x, y) = \langle -y, x^2 \rangle$  and  $C$  is the oriented curve sketched below:



*Solution.* Let  $R$  be the region bounded by  $C$ . By Green's theorem, we have

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA \\ &= \iint_R (2x + 1) dA \\ &= \int_0^{\pi/2} \int_1^2 (2r \cos \theta + 1) r dr d\theta \\ &= \int_0^{\pi/2} \left( \frac{14}{3} \cos \theta + \frac{3}{2} \right) d\theta \\ &= \frac{14}{3} + \frac{3\pi}{4}.\end{aligned}$$

For our second example, we'll use Green's theorem to calculate an area. Let  $R$  be the region enclosed by the curve  $C$  with parametrization  $\mathbf{r}(t) = \langle \sin(2t), \sin(t) \rangle$  for  $t \in [0, \pi]$ . The curve looks like this:



To find the area of  $R$  we could, in principle, describe  $R$  in terms of  $x$  and  $y$  and set up an iterated integral. However, this would be tedious and ugly. Instead we can use Green's theorem. Let  $\mathbf{F} = \langle F_1, F_2 \rangle$  be a vector field such that  $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$ . Notice that  $\mathbf{r}$  parametrizes  $C$  counterclockwise. Thus, by Green's theorem, we have

$$\text{Area}(R) = \iint_R 1 dA = \iint_R \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

All we have to do now is choose  $\mathbf{F}$  and evaluate the line integral along  $C$ . There are many possible choices for  $\mathbf{F}$ ; let's use  $\mathbf{F}(x, y) = \langle 0, x \rangle$ . Then

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^\pi \langle 0, \sin(2t) \rangle \cdot \langle 2 \cos(2t), \cos(t) \rangle dt \\ &= \int_0^\pi \sin(2t) \cos(t) dt \\ &= \int_0^\pi 2 \sin(t) \cos(t)^2 dt \quad (\text{double-angle formula for sine}) \\ &= -\frac{2}{3} \cos(t)^3 \Big|_{t=0}^{t=\pi} \\ &= \frac{4}{3}. \end{aligned}$$

## 2. STOKES' THEOREM

As mentioned above, the three-dimensional counterpart to Green's theorem is called Stokes' theorem. Here's the statement:

**Stokes' Theorem.** *Let  $\Sigma$  be an oriented surface with unit normal vector  $\mathbf{n}$  and boundary curve  $C$  oriented counterclockwise relative to  $\mathbf{n}$ . Then*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_\Sigma (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n} dS.$$

In other words, if  $\Sigma$  and its boundary  $C$  have compatible orientations, then the line integral of  $\mathbf{F}$  along  $C$  is equal to the flux of  $\operatorname{curl} \mathbf{F}$  through  $\Sigma$ .

Suppose  $\Sigma$  lies within the  $xy$ -plane and has upward unit normal vector  $\mathbf{n} = \mathbf{k}$ . Then Stokes' theorem tells us that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{\Sigma} (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} dS = \int_{\Sigma} \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dS.$$

This looks a lot like the conclusion of Green's theorem. In fact, Green's theorem is a logical consequence of Stokes' theorem. (The proof is not too difficult; ask me about it if you're curious!)