Math 2341 Exam 1 Notesheet Sean Balbale October 2024

# 1st Order Equations

## Separable Equations

**Solve:** 1. Separate variables. 2. Integrate both sides. 3. Solve for $y$.
**Example:** $\frac{dy}{dx} = 2xy, y(0) = 1. \Rightarrow y = e^{x^2}$

## Equilibrium & Stability

**Equilibrium:** Set $\frac{dy}{dx} = 0$.
**Stability:**

- **Stable:** Solutions approach equilibrium.
- **Unstable:** Solutions move away.
- **Semi-stable:** Approach one side, move away other.

**Example:** $\frac{dy}{dx} = y(2 - y)$: $y = 0$ (unstable), $y = 2$ (stable).

## Linear Equations

**Solve:** Write $y' + p(x)y = q(x)$. Find integrating factor $I(x) = e^{\int p(x)dx}$, multiply both sides, integrate, solve for $y$.

## Bernoulli Equations

**Solve:** Write $y' + p(x)y = q(x)y^n$. Substitute $v = y^{1-n}$, solve as linear equation.

## Exact Equations

**Solve:** 1. Check $M_y = N_x$. 2. Integrate $M$ and $N$ to find potential function $\Psi(x, y)$.

## Logistic Equations

**Formula:** $p(t) = \frac{MP_0}{P_0 + (M-P_0)e^{-Mkt}}$
**Example:** Suppose that I 00 rabbits are shipwrecked on a deserted island and their population P(t) after t years is determined by a logistic growth model, where the natural growth rate of the rabbits is k = 0.000 I and the carrying capacity of the island is I 000 rabbits.

$$p(t) = \frac{1000(100)}{100 + (1000 - 100)e^{-1000(0.0001)t}}$$

## Mixing Problems

**Formula:** $\frac{dx}{dt} = rate_{in} - rate_{out}$
**Example:** A tank contains 100 L water, 10 g salt. Brine (5 g/L salt) enters at 4 L/min, drained at 2 L/min. Find salt after 10 min:

$$\frac{dx}{dt} = 20 - \frac{2x(t)}{100 + 2t}$$

## Newton's Law of Cooling

**Formula:** $\frac{dT}{dt} = -k(T - T_{\text{ambient}})$
**Example:** Coffee at 90°C cools in room at 20°C, after 10 min it's 70°C. $T(t) = 20 + Ce^{-kt}$.

# 2nd Order Equations

## Homogeneous Equations

Solve $ay'' + by' + cy = 0$:

- Distinct roots: $y = C_1e^{r_1x} + C_2e^{r_2x}$
- Repeated root: $y = (C_1 + C_2x)e^{rx}$
- Complex roots: $y = e^{\alpha x}(C_1 \cos(\beta x) + C_2 \sin(\beta x))$

## Non-Homogeneous Equations

Solve $y'' + by' + cy = f(x)$: 1. Solve complementary solution $y_c$. 2. Guess $y_p$ based on $f(x)$.
**General solution:** $y(x) = y_c(x) + y_p(x)$

## Undetermined Coefficients

The form of $y_p$ depends on $f(x)$:

- Polynomial $f(x)$: $y_p = x^k \times$ poly. k is number of 0 roots.
- $f(x) = e^{ax}$: $y_p = x^ke^{ax}$. k is times a is root.
- $f(x) = e^{ax}\cos(bx)$ or $\sin(bx)$: assume $y_p = x^ke^{ax}(\text{poly}\cos(bx) + \text{poly}\sin(bx))$ k is times $a+bi$ are roots.

**Example:** $y'' + 3y' + 2y = 3x + 5$, $y_c = C_1e^{-x} + C_2e^{-2x}$, guess $y_p = Ax + B$.

## Wronskian & Linearity

Check linear independence:

$$W(y_1, y_2) = y_1y_2' - y_1'y_2 \neq 0 \implies \text{independent.}$$

## Mechanical Vibrations

**Free Vibrations:** $mx'' + cx' + kx = 0$

- Undamped ($c = 0$): $x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)$, $\omega = \sqrt{k/m}$
- Damped: Depends on $c^2 - 4mk$ (over/under/critically damped)

**Forced Vibrations:** Solve $mx'' + cx' + kx = F(t)$ using undetermined coefficients or variation of parameters.

## Variation of Parameters

Solve complementary equation $y_c = C_1y_1 + C_2y_2$. Assume $y_p = v_1(x)y_1 + v_2(x)y_2$, solve for $v_1, v_2$ using:

$$v_1' = \frac{-y_2f(x)}{W(y_1, y_2)}, \quad v_2' = \frac{y_1f(x)}{W(y_1, y_2)}$$