HW8: Function Growth, Sequences & Series

Due: Nov 26, 2024 @ 11:59 PM

Problem 1 [36 pts]: Sequences & Series

(i) Sequence: $18, 72, 288, 1152, 4608, 18432, \ldots$

Type: Geometric sequence (constant ratio $r = 4$).
Expression for a_k:

$$a_k = 18 \cdot 4^k$$

Sum of the first 11 terms:

$$S_{11} = 18(\frac{1-4^{11}}{1-4}) = 25165818$$

(ii) Sequence: $-1, 1, 7, 17, 31, 49, \ldots$

Type: Quadratic sequence (constant second difference $\Delta^2 = 4$).
Expression for a_k:

$$a_k = 2k^2 - 4k + 1$$

Sum of the first 11 terms:

$$S_{11} = \sum_{k=1}^{11}(2k^2 - 4k + 1) = 759$$

(iii) Sequence: $0, -1, -2, -3, -4, -5, \ldots$

Type: Arithmetic sequence (constant difference $d = -1$).
Expression for a_k:

$$a_k = -k$$

Sum of the first 11 terms:

$$S_{11} = \sum_{k=1}^{11}(-k) = -66$$

Problem 2 [18 pts (3 each)]: Function Growth True/False

1. $5n^3 + 2n = O(n^4)$: **False**
   The growth of $5n^3 + 2n$ is $O(n^3)$, not $O(n^4)$.

2. $n^5 = O(7n + 1)$: **False**
   n^5 grows faster than $O(7n + 1)$.

3. If $2n^3 + 6n = O(h(n))$, then $2n^3 + 6n > h(n)$: **False**
   The statement misinterprets Big-O; $O(h(n))$ does not imply $f(n) > h(n)$ for all n.

4. $4n + 1 = \Omega(n^2)$: **False**
   The growth of $4n + 1$ is $O(n)$, so it cannot be $\Omega(n^2)$.

5. $6n^2 + 4 = O(6n^2 + 4)$: **True**
   A function is always O of itself.

6. $5 \log_2 n = \Theta(7 \log_{10} n + 1)^2$: **False**
   The growth of $\log n$ is much slower than $(\log n)^2$, so this cannot hold.

Problem 3 [22 pts (14, 8 pts)]: Function Growth

(i) Function Columns

Group the following functions based on asymptotic growth rates (O):

Column 1: 10000
Column 2: $\log_2 n,\ 3^{\log_3 n}$
Column 3: n
Column 4: $n \log_2 n,\ n \log_3 n$
Column 5: $n^2,\ 100n$
Column 6: $2^n,\ 3^n$
Column 7: $n!$

(ii) Simplest $f(n)$ for $3n + 4n^2 + 3n! = O(f(n))$:

$$f(n) = n!$$

Problem 4 [24 pts (6 each)]: Demonstrating Function Growth

For each statement, find c, x_0 such that $0 \leq f(x) \leq cg(x)$ for $x \geq x_0$, or provide a justification if the statement is false.

1. $2^x = O(3^x)$: **True**.
   Exponential functions 2^x and 3^x satisfy $2^x \leq c \cdot 3^x$ for $c = 1$ and $x_0 = 1$.
2. $5x^3 + x = O(x^3)$: **True**.
   The cubic term dominates, so $5x^3 + x \leq 6x^3$ for $c = 6$ and $x_0 = 1$.
3. $x^4 = O(\ln x)$: **False**.
   Polynomial growth x^4 is faster than logarithmic growth $\ln x$, so no constants c, x_0 can satisfy the inequality.
4. $4x + 7 = O(x^2)$: **True**.
   For $x \geq 8$, $4x + 7 \leq x^2$ can be satisfied with $c = 1$ and $x_0 = 8$.