Day 48

1. Stokes' theorem

Last week we saw Stokes' theorem, a three-dimensional counterpart to Green's theorem.

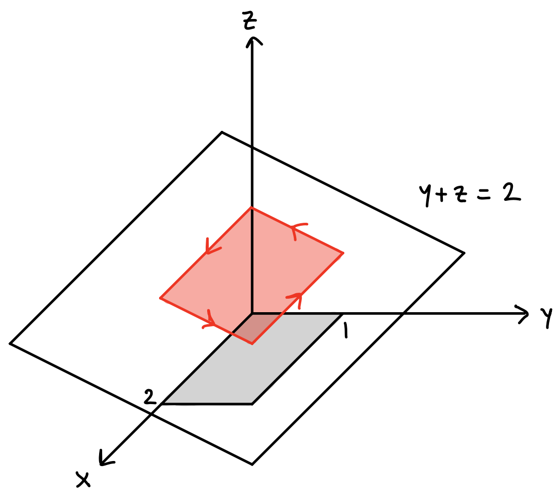
Stokes' Theorem. *Let Σ be an oriented surface with unit normal vector* $\mathbf{n}$ *and boundary curve C oriented counterclockwise relative to* $\mathbf{n}$. *Then*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_\Sigma (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n} dS.$$

One of the main uses of Stokes' theorem is to convert a difficult/tedious line integral into a potentially simpler surface integral. Let's see an example. Let C be the curve in the plane $y + z = 2$ that lies directly above the edges of the rectangle

$$R = \{(x, y) \colon 0 \le x \le 2,\ 0 \le y \le 1\},$$

with counterclockwise orientation when viewed from above. Here's a rough sketch:



Let $\mathbf{F}(x, y, z) = \langle xyz, y-2, yz \rangle$. To evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, we could parametrize the four segments of C and directly compute the line integral along each one. However, Stokes' theorem allows us to convert the line integral along C into a relatively simple surface integral. Let Σ be the part of the plane $y + z = 2$ that is enclosed by C, with upward unit normal vector $\mathbf{n}$. By Stokes' theorem,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_\Sigma (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n} dS.$$

We can calculate that

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & y-2 & yz \end{vmatrix} = \langle z, xy, -xz \rangle.$$

But what is $\mathbf{n}$? The plane $y+z=2$, which Σ lies within, has upward normal vector $\langle 0,1,1\rangle$. Scaling it to a unit vector, we see that

$$\mathbf{n} = \frac{\langle 0,1,1\rangle}{\sqrt{2}}.$$

Thus

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_\Sigma \langle z, xy, -xz\rangle \cdot \frac{\langle 0,1,1\rangle}{\sqrt{2}} dS = \int_\Sigma \frac{x(y-z)}{\sqrt{2}} dS.$$

We can convert the surface integral into a double integral using the fact that Σ is the graph of $g(x,y) = 2-y$ above R. This leads to

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \iint_R \frac{x(y-g(x,y))}{\sqrt{2}} \sqrt{\|\nabla g\|^2+1}\, dA \\ &= \iint_R \frac{x(2y-2)}{\sqrt{2}} \sqrt{2}\, dA \\ &= \int_0^2 \int_0^1 x(2y-2)\, dy\, dx = -2.\end{aligned}$$

Day 49

1. Examples of using Stokes' theorem

The plan for today is to continue working with Stokes' theorem.

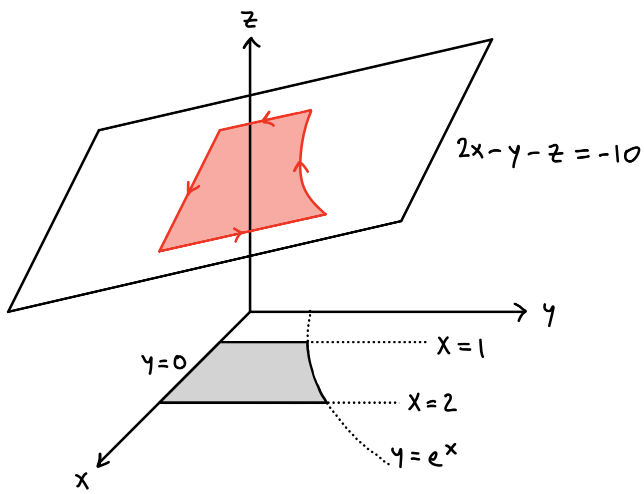
In-class exercise: Find

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where

- $\mathbf{F}(x, y, z) = \langle 3x + y, y - 2z, 2 + 3z \rangle$
- C is the curve in the plane $2x - y - z = -10$ that lies above the curves $x = 1$, $x = 2$, $y = 0$, $y = e^x$ in the xy-plane, oriented counterclockwise when viewed from above.

Solution. Let Σ be the part of the plane $2x - y - z = -10$ that is bounded by C, with upward unit normal vector $\mathbf{n}$. Here's a rough sketch:



By Stokes' theorem,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_\Sigma (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n} dS.$$

We can calculate that

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x + y & y - 2z & 2 + 3z \end{vmatrix} = \langle 2, 0, -1 \rangle.$$

To find $\mathbf{n}$, note that the plane $2x - y - z = -10$, which Σ lies within, has normal vector $\langle 2, -1, -1 \rangle$. This is neither an upward normal vector nor a unit vector. But we can convert into such a vector by multiplying by -1 and scaling. This leads to

$$\mathbf{n} = \frac{\langle -2, 1, 1 \rangle}{\sqrt{6}}.$$

Thus

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_\Sigma \langle 2,0,-1\rangle \cdot \frac{\langle -2,1,1\rangle}{\sqrt{6}} dS = \int_\Sigma -\frac{5}{\sqrt{6}} dS.$$

Let R be the region in the xy-plane bounded by the curves $x=1$, $x=2$, $y=0$, and $y=e^x$ (shaded in gray above). Then Σ is the graph of $g(x,y)=2x-y+10$ over R. Thus

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \iint_R -\frac{5}{\sqrt{6}}\sqrt{\|\nabla g\|^2+1}\,dA\\ &= \iint_R -\frac{5}{\sqrt{6}}\sqrt{6}\,dA\\ &= \int_1^2\int_0^{e^x} -5\,dy\,dx\\ &= -5(e^2-e).\end{aligned}$$

Here's another exercise that wasn't stated in class:

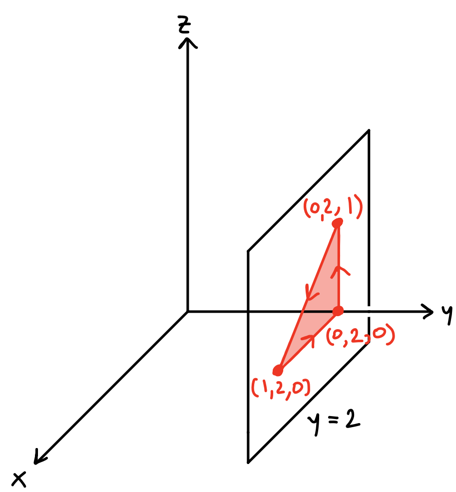
Exercise: Find

$$\int_C \mathbf{F}\cdot d\mathbf{r}$$

where

- $\mathbf{F}(x,y,z) = \langle 3yz, e^x, x^2 z\rangle$
- C is the triangle in the plane $y=2$ with vertices $(1,2,0),(0,2,0),(0,2,1)$, oriented counterclockwise when viewed from the positive y direction.

Solution. Let Σ be the part of the plane $y=2$ enclosed by C, with rightward unit normal vector $\mathbf{n}$. Here's a rough sketch:



By Stokes' theorem,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_\Sigma (\operatorname{curl}\mathbf{F})\cdot \mathbf{n}\,dS.$$

We can calculate that

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3yz & e^x & x^2 z \end{vmatrix} = \langle 0, 3y - 2xz, e^x - 3z \rangle.$$

The rightward unit normal vector to the plane $y = 2$ is $\langle 0,1,0\rangle$; thus $\mathbf{n} = \langle 0,1,0\rangle$. So

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_\Sigma \langle 0, 3y - 2xz, e^x - 3z\rangle \cdot \langle 0,1,0\rangle dS = \int_\Sigma (3y - 2xz) dS.$$

Note that Σ is not a graph of a function of x and y. However, we can convert the surface integral above into a double integral by parametrizing Σ. Specifically, Σ is the image of the vector function

$$\mathbf{r}(x,z) = \langle x, 2, z\rangle$$

for $0 \le x \le 1$ and $0 \le z \le 1 - x$. This leads to

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \int_0^{1-x} (3 \cdot 2 - 2xz) \|\mathbf{r}_x \times \mathbf{r}_z\| dz dx \\ &= \int_0^1 \int_0^{1-x} (6 - 2xz) dz dx \\ &= \frac{35}{12}. \end{aligned}$$

2. Finding a vector normal to a surface

In the examples we've looked at over the past couple of days, the surface Σ was always contained within a plane. This made it straightforward to find the unit normal vector $\mathbf{n}$. But what do we do when Σ is not part of a plane? There are two cases to consider:

Case 1: Suppose Σ is a **graph**, that is, Σ is described by the equation $z = g(x,y)$ for some function g. Let $F(x,y,z) = z - g(x,y)$. Then Σ is a level surface of F, namely Σ is the set of solutions to the equation $F(x,y,z) = 0$. This implies that the vector

$$\nabla F = \langle -g_x, -g_y, 1\rangle$$

is normal to Σ at every point. Note that this vector is not a unit vector, but it can be turned into a unit vector by scaling. Also note that this vector is upward-pointing, since its third component is positive. If you need a downward-pointing normal vector, just multiply by -1.

As an example, suppose Σ is the graph of $g(x,y) = 4 - x^2 - y^2$ and you need to find the upward unit normal vector $\mathbf{n}$. This is given by

$$\mathbf{n} = \frac{\langle -g_x, -g_y, 1\rangle}{\sqrt{g_x^2 + g_y^2 + 1}} = \frac{\langle 2x, 2y, 1\rangle}{\sqrt{4(x^2 + y^2) + 1}}.$$

Case 2: If Σ is not a graph, we can still treat it as a **parametric surface**, that is, the image of a vector function $\mathbf{r}(s,t)$. At each point, the vectors $\mathbf{r}_s$ and $\mathbf{r}_t$ are tangent[1] to Σ.

[1] To see this, note that when t is fixed and s varies, the function $\mathbf{r}(s,t)$ traces out a curve within Σ. The derivative $\mathbf{r}_s(s,t)$ is a tangent vector to this curve. Similarly, when s is fixed t varies, $\mathbf{r}(s,t)$ again traces out a curve within Σ, and $\mathbf{r}_t(s,t)$ is a tangent vector to this curve.

Thus the vector

$$\mathbf{r}_s \times \mathbf{r}_t$$

is normal to Σ at every point. Again, be aware that this vector is typically not a unit vector and it may not point in the direction you need. But you can always scale it and flip it as necessary.

As an example, suppose Σ is the surface parametrized by $\mathbf{r}(s,t) = \langle st, t, s^2 \rangle$ and you need to find the upward unit normal vector $\mathbf{n}$. The cross product $\mathbf{r}_s \times \mathbf{r}_t$ is given by

$$\mathbf{r}_s \times \mathbf{r}_t = \langle t, 0, 2s \rangle \times \langle s, 1, 0 \rangle = \langle -2s, 2s^2, t \rangle.$$

If $t > 0$, then this vector is upward-pointing and thus

$$\mathbf{n} = \frac{\langle -2s, 2s^2, t \rangle}{\sqrt{4(s^2 + s^4) + t^2}}.$$

But if $t < 0$, then $\mathbf{r}_s \times \mathbf{r}_t$ is downward-pointing, so you need to multiply by -1. This gives

$$\mathbf{n} = \frac{\langle 2s, -2s^2, -t \rangle}{\sqrt{4(s^2 + s^4) + t^2}}.$$

(And if $t = 0$, then Σ does not have an upward normal vector at the point $\mathbf{r}(s,t)$.)

Day 50

1. Another Stokes' example

In-class exercise: Express

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

as an iterated integral, where

- $\mathbf{F}(x, y, z) = \langle z^2, x, y \rangle$
- C is the curve on the paraboloid $z = x^2 + y^2$ that lies above the square with vertices $(1,1), (1,-1), (-1,1), (-1,-1)$, oriented counterclockwise when viewed from above.

Solution. Let Σ be the part of the paraboloid $z = x^2 + y^2$ enclosed by C, with upward unit normal vector $\mathbf{n}$. By Stokes' theorem,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_\Sigma (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n} dS.$$

We can calculate that

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & x & y \end{vmatrix} = \langle 1, 2z, 1 \rangle.$$

To find $\mathbf{n}$, we can use the fact that Σ is the graph of $g(x, y) = x^2 + y^2$ (above the given square region). From the end of yesterday's class, we know that

$$\mathbf{n} = \frac{\langle -g_x, -g_y, 1 \rangle}{\sqrt{g_x^2 + g_y^2 + 1}} = \frac{\langle -2x, -2y, 1 \rangle}{\sqrt{\|\nabla g\|^2 + 1}}.$$

(It will be useful to leave the denominator in this unevaluated form, as it will cancel when we convert the surface integral into an iterated integral momentarily.) So far we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_\Sigma \langle 1, 2z, 1 \rangle \cdot \frac{\langle -2x, -2y, 1 \rangle}{\sqrt{\|\nabla g\|^2 + 1}} dS = \int_\Sigma \frac{-2x - 4yz + 1}{\sqrt{\|\nabla g\|^2 + 1}} dS.$$

To turn this into an iterated integral, we again use the fact that Σ is the graph of g over the given square. This leads to

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{-1}^{1} \int_{-1}^{1} \frac{-2x - 4yg(x, y) + 1}{\sqrt{\|\nabla g\|^2 + 1}} \sqrt{\|\nabla g\|^2 + 1} dx dy \\ &= \int_{-1}^{1} \int_{-1}^{1} (-2x - 4y(x^2 + y^2) + 1) dx dy. \end{aligned}$$

2. The divergence theorem

Recall that Green's theorem allows us to convert a line integral along the boundary of a two-dimensional region into a double integral over that region. Similarly, Stokes' theorem converts a line integral along the boundary of a surface into a surface integral over that surface. The divergence theorem provides a similar relationship between a surface integral over the boundary of a solid region and a triple integral over that region.

Divergence Theorem. *Let R be a solid region bounded by a closed oriented surface Σ with outward unit normal vector $\mathbf{n}$. Then*

$$\int_{\Sigma} \mathbf{F} \cdot \mathbf{n} dS = \iiint_R \operatorname{div} \mathbf{F} \, dV.$$

In other words, for a closed surface Σ oriented with outward unit normal, the flux of $\mathbf{F}$ through Σ is the integral of its divergence over the region enclosed by Σ. Recall that the divergence of $\mathbf{F}$ is the scalar quantity defined by

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle F_1, F_2, F_3 \rangle = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Often it is easier to calculate $\operatorname{div} \mathbf{F}$ than $\mathbf{F} \cdot \mathbf{n}$. Thus the divergence theorem simplifies the process of computing flux through a closed surface.

Day 51

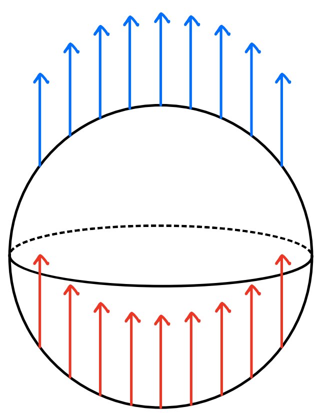
1. The divergence theorem

Last time we saw the divergence theorem, which provides a relationship between the flux of a vector field through a closed surface and the integral of its divergence over the region enclosed by that surface.

Divergence Theorem. *Let R be a solid region bounded by a closed oriented surface Σ with outward unit normal vector $\mathbf{n}$. Then*

$$\int_{\Sigma} \mathbf{F} \cdot \mathbf{n} dS = \iiint_R \operatorname{div} \mathbf{F}\, dV.$$

As a first example, let Σ be the unit sphere with outward unit normal vector $\mathbf{n}$, and let $\mathbf{F}(x, y, z) = \langle 0, 0, 1\rangle$. Without calculating anything, should we expect the flux of $\mathbf{F}$ through Σ to be positive, negative, or zero? Here's a rough sketch of $\mathbf{F}$ along Σ:

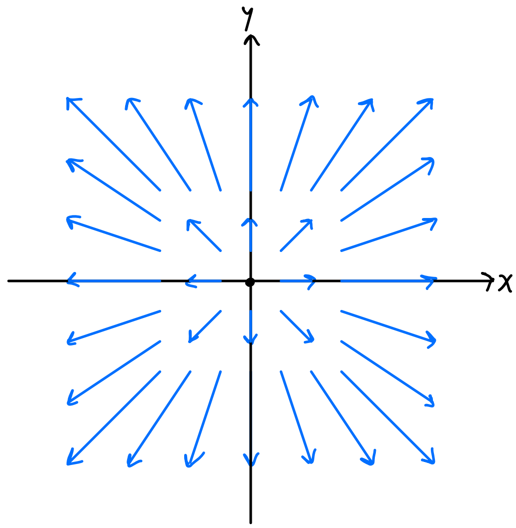


The red vectors correspond to points at which $\mathbf{F}$ is flowing *into* Σ, while the blue vectors correspond to points at which $\mathbf{F}$ is flowing *out of* Σ. By symmetry of the sphere, the total "inflow" is equal to the total "outflow", so we might guess that the flux is zero overall. The divergence theorem confirms this: We have $\operatorname{div} \mathbf{F} = 0$, so

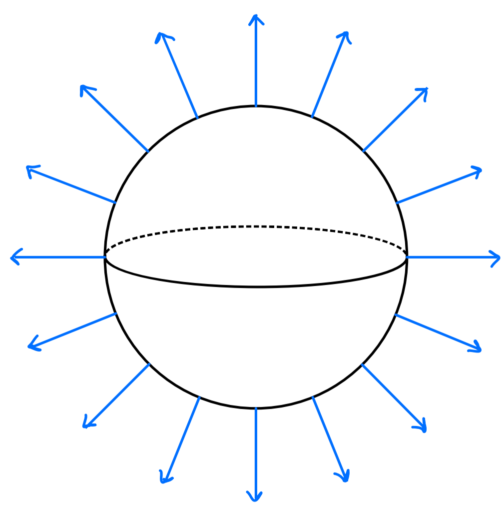
$$\text{flux } = \int_{\Sigma} \mathbf{F} \cdot \mathbf{n} dS = \iiint_R \operatorname{div} \mathbf{F}\, dV = 0,$$

where R denotes the unit ball enclosed by Σ.

Now let's modify this example by considering the vector field $\mathbf{F}(x, y, z) = \langle x, y, z\rangle$. This vector field is harder to sketch (use GeoGebra), but the key feature is that it's "expanding" or "spreading out" at every point (i.e. it has positive divergence). The corresponding vector field in $\mathbb{R}^2$ is $\langle x, y\rangle$, which looks like this:



Let's think about the flux of $\mathbf{F}$ through Σ (still the unit sphere). Since $\mathbf{F}$ is flowing away from the origin at every point, the flux of $\mathbf{F}$ consists entirely of outward flow:



Since the direction of the flow agrees with the unit normal vector $\mathbf{n}$ (which points outward), the flux of $\mathbf{F}$ through Σ must be positive. Again, the divergence theorem confirms this: We have $\operatorname{div} \mathbf{F} = 3$, so

$$\text{flux} = \int_{\Sigma} \mathbf{F} \cdot \mathbf{n} dS = \iiint_{R} \operatorname{div} \mathbf{F}\, dV = 3\,\mathrm{Vol}(R) = 4\pi,$$

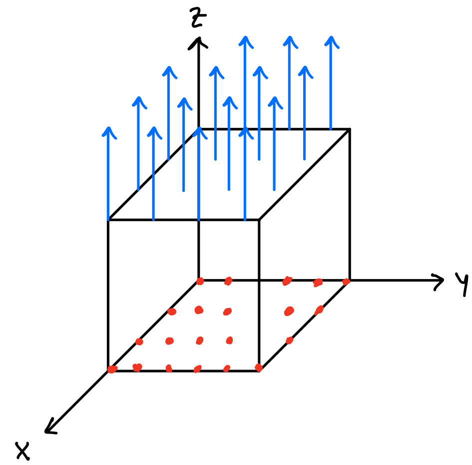
where R is the unit ball enclosed by Σ.

In-class exercise: Let Σ be the surface of the unit cube $R = \{(x, y, z) \colon 0 \leq x, y, z \leq 1\}$, with outward unit normal vector $\mathbf{n}$. Let $\mathbf{F}(x, y, z) = \langle 0, 0, z^2 \rangle$.

(a) Is the flux of $\mathbf{F}$ through Σ positive, negative, or zero?

(b) Calculate the flux of $\mathbf{F}$ through Σ.

Solution. (a) First notice that if $\mathbf{F}$ is nonzero, then $\mathbf{F}$ is pointing upward. This implies that $\mathbf{F}$ does not flow through any of the vertical faces of the cube; thus we only need to consider the top and bottom faces. Here's a rough sketch:



We have $\mathbf{F} = \mathbf{0}$ along the bottom face, because that face lies within the plane $z = 0$. This means that there is no flow through the bottom face. On the top face, which lies within the plane $z = 1$, we have $\mathbf{F} = \langle 0, 0, 1 \rangle$. So there is outward flow through the top of the cube, and therefore the flux overall will be positive.

(b) By the divergence theorem,

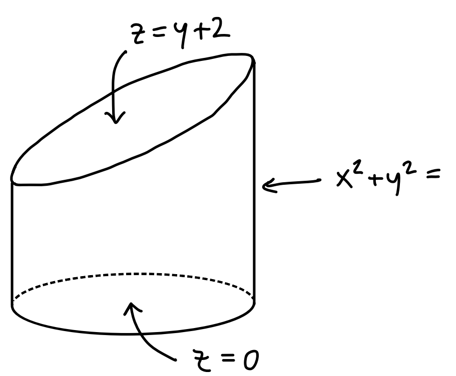
$$\text{flux} \;= \int_{\Sigma} \mathbf{F} \cdot \mathbf{n} dS = \iiint_R \operatorname{div} \mathbf{F}\, dV = \int_0^1 \int_0^1 \int_0^1 2z dV = 1.$$

In-class exercise: (Will be covered in Monday's class.) Let Σ be the surface of the region R bounded by the the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = y + 2$, with outward unit normal vector $\mathbf{n}$. Let $\mathbf{F} = \langle 0, 1 - x^2 - y^2, 0 \rangle$.

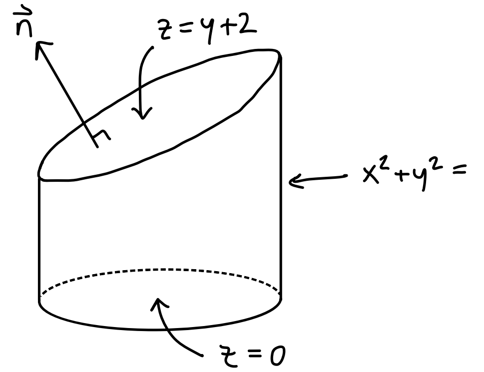
(a) Is the flux of $\mathbf{F}$ through Σ positive, negative, or zero?

(b) Calculate the flux of $\mathbf{F}$ through Σ.

Solution. (a) First let's sketch the surface:



On the cylindrical boundary of Σ, we have $\mathbf{F} = \mathbf{0}$ and therefore no flow through the surface. There is also no flow through the bottom face of the surface. This is because $\mathbf{F}$ is always parallel to the vector $\langle 0,1,0\rangle$ and therefore parallel to the bottom face. So the flux is entirely determined by the top face of Σ. At any point (x,y,z) belonging to the interior the top face, we have $x^2+y^2<1$. This implies that, along the top face, $\mathbf{F}$ points in the positive y direction (rightward). However, the unit normal $\mathbf{n}$ is tilted toward the negative y direction (leftward):



So the flux will be negative overall.

(b) By the divergence theorem,

$$\begin{aligned}
\text{flux} &= \int_\Sigma \mathbf{F}\cdot\mathbf{n}dS = \iiint_R \operatorname{div}\mathbf{F}\,dV \\
&= \iiint_R -2y dV \\
&= \int_0^{2\pi}\int_0^1\int_0^{r\sin\theta+2} -2(r\sin\theta)r dz dr d\theta \qquad \text{(cylindrical coordinates)} \\
&= \int_0^{2\pi}\int_0^1 (-2r^3\sin(\theta)^2 - 4r^2\sin\theta) dr d\theta \\
&= \int_0^{2\pi}\Big(-\frac{1}{2}\sin(\theta)^2 - \frac{4}{3}\sin\theta\Big)d\theta \\
&= \int_0^{2\pi}\Big(-\frac{1}{4}(1-\cos(2\theta)) - \frac{4}{3}\sin\theta\Big)d\theta \qquad \text{(power-reduction formula)} \\
&= -\frac{\pi}{2}.
\end{aligned}$$