1st Order Equations

Separable Equations

Solve: 1. Separate variables. 2. Integrate both sides. 3. Solve for y.
Example: $\frac{dy}{dx} = 2xy, y(0) = 1. \Rightarrow y = e^{x^2}$

Equilibrium & Stability

Equilibrium: Set $\frac{dy}{dx} = 0$.
Stability:

- **Stable:** Solutions approach equilibrium.
- **Unstable:** Solutions move away.
- **Semi-stable:** Approach one side, move away other.

Example: $\frac{dy}{dx} = y(2 - y)$: $y = 0$ (unstable), $y = 2$ (stable).

Linear Equations

Solve: Write $y' + p(x)y = q(x)$. Find integrating factor $I(x) = e^{\int p(x)dx}$, multiply both sides, integrate, solve for y.

Bernoulli Equations

Solve: Write $y' + p(x)y = q(x)y^n$. Substitute $v = y^{1-n}$, solve as linear equation.

Exact Equations

Solve: 1. Check $M_y = N_x$. 2. Integrate M and N to find potential function $\Psi(x, y)$.

Mixing Problems

Example: A tank contains 100 L water, 10 g salt. Brine (5 g/L salt) enters at 4 L/min, drained at 2 L/min. Find salt after 10 min:

$$\frac{dx}{dt} = 20 - \frac{2x(t)}{100 + 2t}$$

Newton's Law of Cooling

Formula: $\frac{dT}{dt} = -k(T - T_{\text{ambient}})$
Example: Coffee at 90°C cools in room at 20°C, after 10 min it's 70°C. $T(t) = 20 + Ce^{-kt}$.

2nd Order Equations

Homogeneous Equations

Solve $ay'' + by' + cy = 0$:

- Distinct roots: $y = C_1e^{r_1x} + C_2e^{r_2x}$
- Repeated root: $y = (C_1 + C_2x)e^{rx}$
- Complex roots: $y = e^{\alpha x}(C_1 \cos(\beta x) + C_2 \sin(\beta x))$

Non-Homogeneous Equations

Solve $y'' + by' + cy = f(x)$: 1. Solve complementary solution y_c. 2. Guess y_p based on $f(x)$.
General solution: $y(x) = y_c(x) + y_p(x)$

Undetermined Coefficients

The form of y_p depends on $f(x)$:

- Polynomial $f(x)$: $y_p = x^k \times$ poly. k is number of 0 roots.
- $f(x) = e^{ax}$: $y_p = x^k e^{ax}$. k is times a is root.
- $f(x) = e^{ax}\cos(bx)$ or $\sin(bx)$: assume $y_p = x^k e^{ax}(\text{poly}\cos(bx) + \text{poly}\sin(bx))$ k is times $a+bi$ are roots.

Example: $y'' + 3y' + 2y = 3x + 5$, $y_c = C_1e^{-x} + C_2e^{-2x}$, guess $y_p = Ax + B$.

Wronskian & Linearity

Check linear independence:

$$W(y_1, y_2) = y_1y_2' - y_1'y_2 \neq 0 \implies \text{independent.}$$

Mechanical Vibrations

Free Vibrations: $mx'' + cx' + kx = 0$

- Undamped ($c = 0$): $x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)$, $\omega = \sqrt{k/m}$
- Damped: Depends on $c^2 - 4mk$ (over/under/critically damped)

Forced Vibrations: Solve $mx'' + cx' + kx = F(t)$ using undetermined coefficients or variation of parameters.

Variation of Parameters

Solve complementary equation $y_c = C_1y_1 + C_2y_2$. Assume $y_p = v_1(x)y_1 + v_2(x)y_2$, solve for v_1, v_2 using:

$$v_1' = \frac{-y_2f(x)}{W(y_1, y_2)}, \quad v_2' = \frac{y_1f(x)}{W(y_1, y_2)}$$