CS1800 Homework 7 Solutions

1 Upper Triangular Matrix

Problem Statement:
Use induction to show that an $n \times n$ matrix has $\frac{n(n+1)}{2}$ upper-diagonal entries.

I Solution

I.I Base Case

For $n = 1$:

$$\frac{1 \times (1+1)}{2} = \frac{2}{2} = 1$$

which is correct because a 1×1 matrix has only one entry, which is on the diagonal.

I.II Inductive Hypothesis

Assume for some integer k that a $k \times k$ matrix has $\frac{k(k+1)}{2}$ upper-diagonal entries.

I.III Inductive Step

Consider a $(k+1) \times (k+1)$ matrix. This matrix contains all the entries of the $k \times k$ matrix plus an additional row and column, which contribute $k+1$ new upper-diagonal entries.
So, the total number of upper-diagonal entries in a $(k+1) \times (k+1)$ matrix is:

$$\frac{k(k+1)}{2} + (k+1) = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

Replacing $k + 1 = n$ in the formula, we get:

$$\frac{n(n+1)}{2}$$

Thus, by induction, the formula holds for all $n \geq 1$.

2 A Partial Series $\frac{1}{k(k+1)}$

Problem Statement:
Show that:

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}$$

I Solution

I.I Base Case

For $n = 1$:

$$\sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{1 \cdot 2} = \frac{1}{2}$$

and the formula gives $1 - \frac{1}{2} = \frac{1}{2}$, which is correct.

I.II Inductive Hypothesis

Assume that for some integer k,

$$\sum_{j=1}^{k} \frac{1}{j(j+1)} = 1 - \frac{1}{k+1}$$

I.III Inductive Step

For $n = k + 1$:

$$\sum_{j=1}^{k+1} \frac{1}{j(j+1)} = \left(1 - \frac{1}{k+1}\right) + \frac{1}{(k+1)(k+2)}$$

Simplifying, we get:

$$= 1 - \frac{1}{k+1} + \frac{1}{(k+1)(k+2)} = 1 - \frac{(k+2)-1}{(k+1)(k+2)} = 1 - \frac{1}{k+2}$$

Replacing $k + 1 = n$ in the formula, we get:

$$1 - \frac{1}{n+1}$$

Thus, by induction, the formula holds for all $n \geq 1$.

3 A Partial Series k^3

Problem Statement:
Prove that:

$$\sum_{k=0}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

I Solution

I.I Base Case

For $n = 0$:

$$\sum_{k=0}^{0} k^3 = 0$$

The formula gives $\left(\frac{0\cdot(0+1)}{2}\right)^2 = 0$, which is correct.

I.II Inductive Hypothesis

Assume that for some k,

$$\sum_{j=0}^{k} j^3 = \left(\frac{k(k+1)}{2}\right)^2$$

I.III Inductive Step

For $n = k+1$:

$$\sum_{j=0}^{k+1} j^3 = \sum_{j=0}^{k} j^3 + (k+1)^3$$

Using the inductive hypothesis:

$$= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$$

Simplifying both sides confirms that:

$$\sum_{j=0}^{k+1} j^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$$

Replacing $k+1 = n$ in the formula, we get:

$$\left(\frac{n(n+1)}{2}\right)^2$$

Hence, by induction, the formula holds for all $n \geq 0$.

4 Function Growth ($n < 2^n$)

Problem Statement:
Prove that $n < 2^n$ for all $n \in \mathbb{Z}^+$.

I Solution

I.I Base Case

For $n = 1$:

$$1 < 2^1$$

which is true.

I.II Inductive Hypothesis

Assume that $k < 2^k$ for some $k \geq 1$.

I.III Inductive Step

We need to show that $k+1 < 2^{k+1}$. Since $k < 2^k$ by the hypothesis, add $1 < 2^k$:

$$k + 1 < 2^k + 2^k = 2^{k+1}$$

Thus, by induction, $n < 2^n$ holds for all $n \in \mathbb{Z}^+$.