SAMPLING AND QUANTIZATION

EECE 2150 Circuits and Signals, Biomedical Engineering

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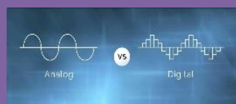
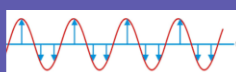
Learning Objectives

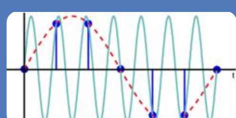


Learn the basics of analog to digital conversion

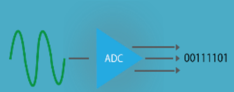


The Nyquist criteria and sampling a continuous time signal

Understanding the difference between continuous and discrete time frequency



Under-sampling and Aliasing



Quantization and mapping the discrete time signal into discrete levels

Analog to Digital Conversion, A/D

- To generate, store or process a continuous time signal on a computer, the signal must first be converted from analog to digital form.
- Samples are first taken from the continuous time signal at a given rate called the sampling rate or sampling frequency. The continuous time signal becomes a discrete time signal, but with continuous amplitude
- The next step is to convert the amplitude of the discrete time signal to a discrete set of levels, this process is referred to as quantization

Sampling

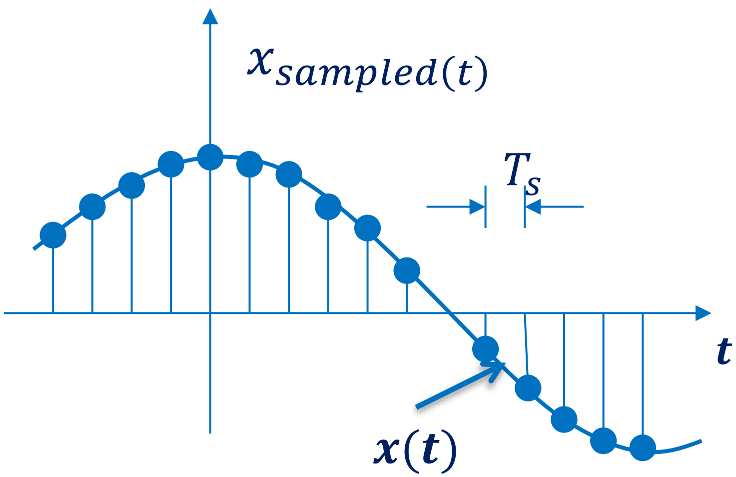
Sampling: Time Domain

- A continuous-time signal is sampled at equally-spaced intervals in time,

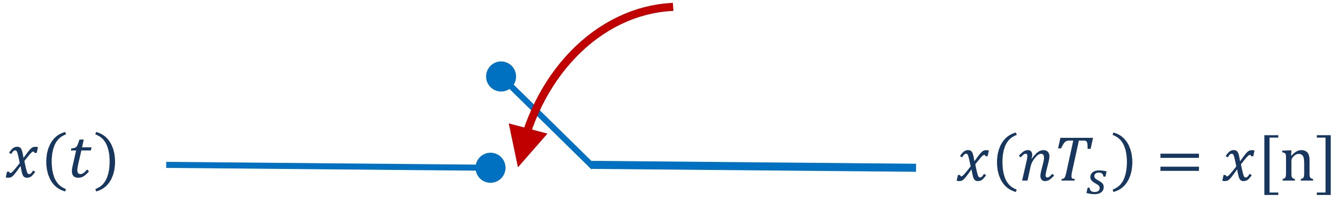
  $x[n] = x(nT_s)$

  $n \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

  T_s is the sampling interval



$$T_s = \frac{1}{f_s}$$

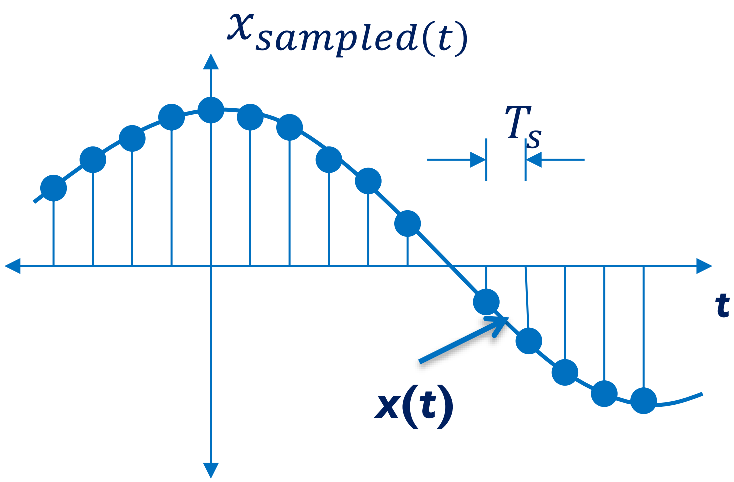


Switch closes momentarily every T_s to take a sample

How to choose the sampling rate

- For a given continuous-time signal, what is the minimum sampling rate? Or what is the largest sampling interval T_s?

$$T_s = \frac{1}{f_s}$$



$$x(nT_s) = x[n]$$

Example: Sampling a Sine wave, choice of f_s

- $x(t) = Asin(\omega_0 t + \phi) = Asin(2\pi f_0 t + \phi)$
- The sampled version of the signal is given by =

$$x_{sampled}(nT_s) = x(nT_s) = Asin(2\pi f_0 nT_s + \phi) = Asin\left(2\pi \frac{f_0}{f_s} n + \phi\right)$$

The continuous time frequency f_0 is in $\frac{cycles}{second}$ $and\ \omega_0$ is in $\frac{radians}{second}$

- The discrete signal can be expressed as

$$x[n] = x_{sampled}(nT_s) = Asin(2\pi \mathrm{F} n + \phi)$$

where $\mathrm{F} = \frac{\mathrm{f_0}}{\mathrm{f_s}} = \frac{\mathrm{T_s}}{\mathrm{T_0}} = \text{discrete time frequency}, \mathit{in\ cylces/sample}$

$\frac{T_0}{T_s} =$ number of samples per one cycle of the signal

$x[n] = Asin(\Omega n + \phi)$, Ω is in $\frac{\text{radians}}{\text{sample}}$ $\quad \Omega = 2\pi F$

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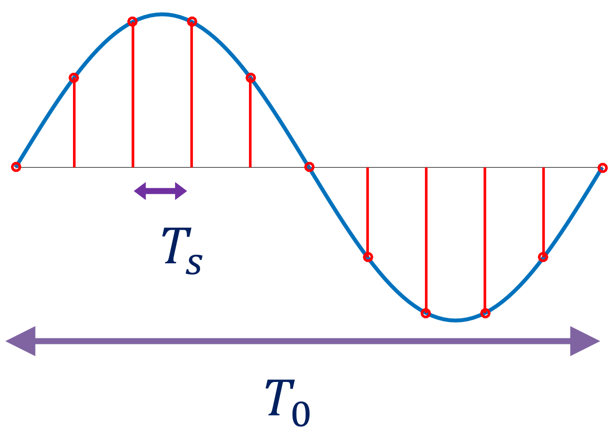
Continuous time versus discrete time signals



Continuous time frequency

- $x(t) = Asin(\omega_o t + \phi)$
- $x(t) = Asin(2\pi f_o t + \phi)$
- f_0 is in $\frac{cycles}{second}$
- ω_0 is in $\frac{radians}{second}$

Discrete time frequency,

- $x[n] = Asin(\Omega n + \phi)$
- $x[n] = Asin(2\pi F n + \phi)$
- $\mathrm{F} = \frac{\mathrm{f_0}}{\mathrm{f_s}} = \frac{\mathrm{T_s}}{\mathrm{T_0}}$ $\frac{cylces}{sample}$
- $\Omega = 2\pi F$ $\frac{radians}{sample}$

- $\frac{T_0}{T_s} = 10 \rightarrow \frac{f_s}{f_0} = 10 \rightarrow 10\ samples\ per\ cycle$, the sampling rate is 10 times the sinusoidal signal frequency
- The discrete time frequency is defined as
- $F = \frac{f_0}{f_s} = \frac{1}{10}\ cycles\ per\ sample, \Omega = 2\pi F = 0.2\pi$

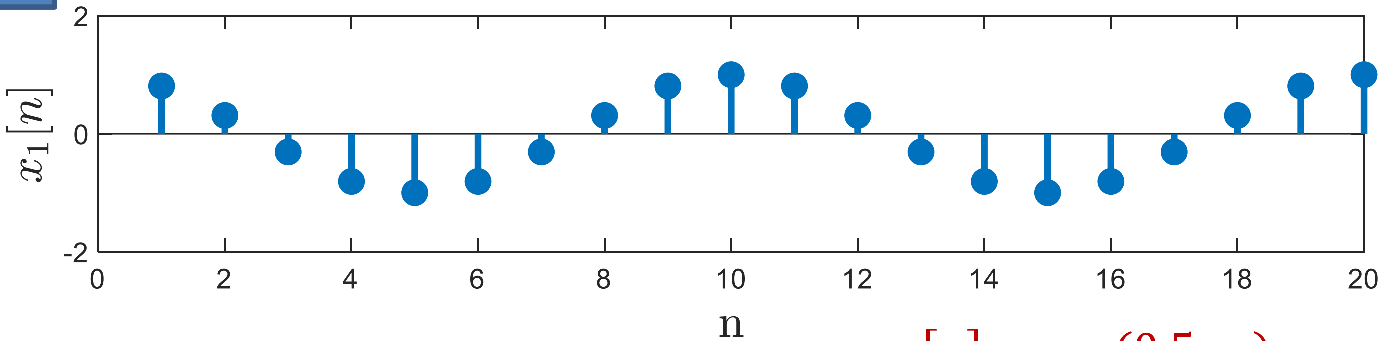
Discrete Time Signals at Different Frequencies

$$\Omega = 2\pi \frac{f_o}{f_s}$$

$x_1[n] = \cos(0.2\pi n)$

$\Omega = 0.2\pi$

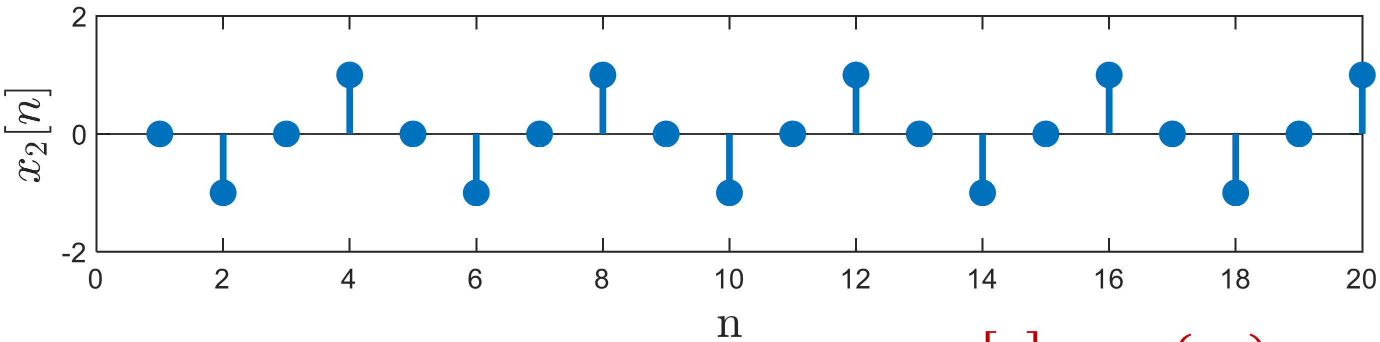
$$\frac{f_o}{f_s} = \frac{1}{10}$$



$x_2[n] = \cos(0.5\pi n)$

$\Omega = 0.5\pi$

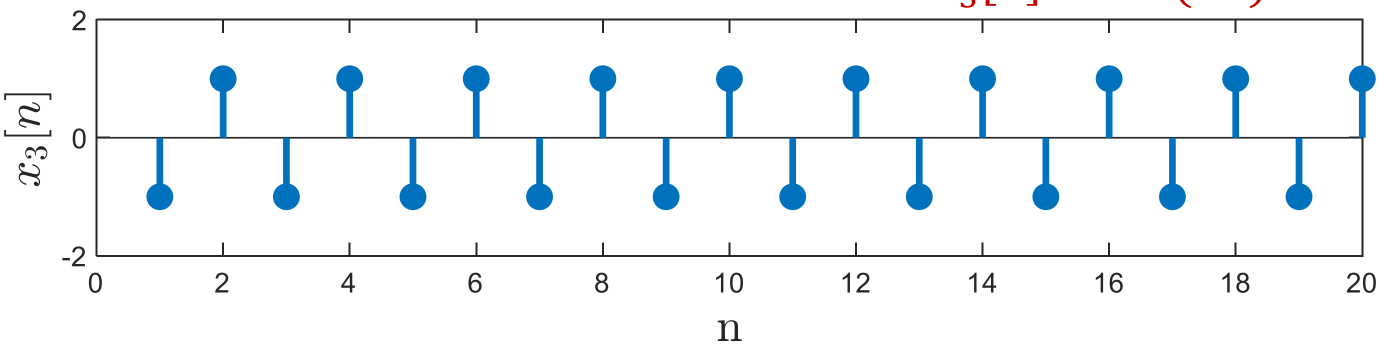
$$\frac{f_o}{f_s} = \frac{1}{4}$$



$x_3[n] = \cos(\pi n)$

$\Omega = \pi$

$$\frac{f_o}{fs} = \frac{1}{2}$$



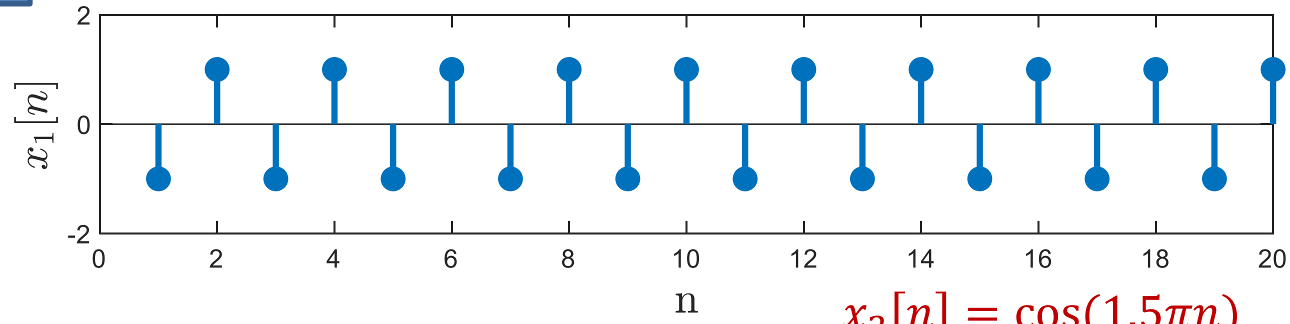
Discrete Time Signals at Different Frequencies

$$\Omega = 2\pi \frac{f_o}{f_s}$$

$x_1[n] = \cos(\pi n)$

$\Omega = \pi$

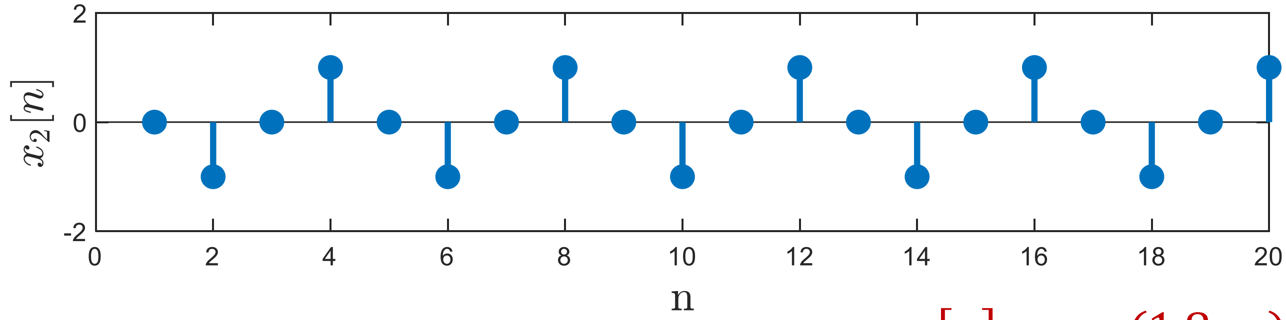
$$\frac{f_o}{fs} = \frac{1}{2}$$



$x_2[n] = \cos(1.5\pi n)$

$\Omega = 1.5\pi$

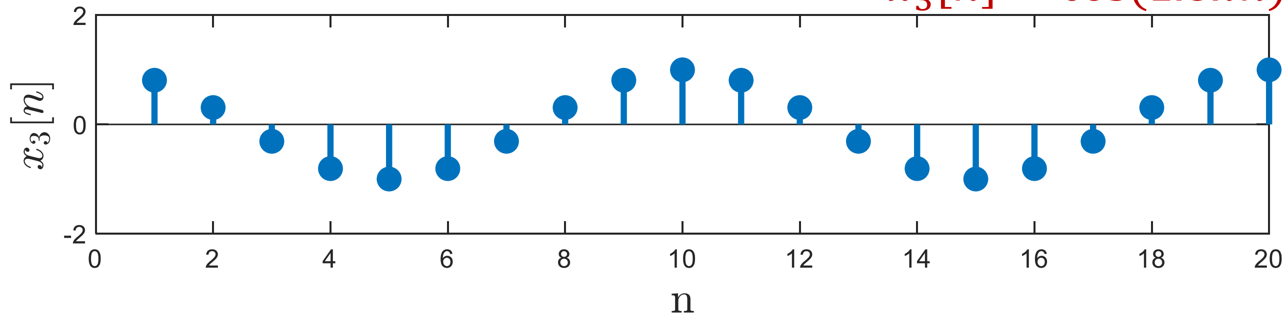
$$\frac{f_o}{f_s} = \frac{3}{4}$$



$x_3[n] = \cos(1.8\pi n)$

$\Omega = 1.8\pi$

$$\frac{f_o}{f_s} = \frac{9}{10}$$



The discrete time Frequency Ω defined in the range $-\pi \le \Omega \le \pi$

- $\cos(2.5\pi n) = \cos((2.5\pi - 2\pi)n) = \cos(0.5\pi n)$
- $\cos(1.5\pi n) = \cos\big((1.5\pi - 2\pi)n\big) = \cos(-0.5\pi n)$

Summary of the Difference between continuous time Frequency, ω, and discrete time frequency, Ω

Continuous time frequency, ω

- $x(t) = A\sin(\omega t + \phi)$
- The units of ω: rad/s,
- ω has a distinct value. As the magnitude of ω increases, the signal fluctuates at a higher rate

Discrete time frequency,, Ω

- $x[n] = A\sin(\Omega n + \phi)$
- $\Omega = 2\pi F \ \frac{radians}{sample}$
- Ω has units of an angle(radians) and therefore it repeats after 2π.
- The discrete time frequency Ω increases in the range from $0 - \pi$ and then decreases from $\pi - 2\pi$ and then repeats.
- Discrete time frequencies near $0, 2\pi, 4\pi \ldots .2n\pi$ are low frequencies , while frequencies in the vicinity of $\pi, 3\pi, \ldots . (2n + 1)\pi$ are high frequencies

The concept of Aliasing, definition of Alias (Oxford Living Dictionary)

- A false or assumed identity
- In Computing : an alternative name or label that refers to a file, command, address, or other item, and can be used to locate or access it.
- In Physics Telecommunications : Each of a set of signal frequencies which, when sampled at a given uniform rate, would give the same set of sampled values, and thus might be incorrectly substituted for one another when reconstructing the original signal.

Under-Sampling and Aliasing

- The higher the frequency content of the signal, the higher the sampling rate should be to preserve the full information in the signal.
- The Sampling Theorem states that a signal can be exactly reproduced if it is sampled at a frequency f_s (Hz) where f_s (Hz) is greater than twice the maximum frequency in the signal, which is known as the Nyquist rate..
- If the signal is sampled at a frequency that is lower that the Nyquist rate, when converted back into a continuous time signal, it will exhibit some form of distortion known as ***aliasing***.
- Aliasing is the presence of unwanted frequency components in the reconstructed signal which were not present when the original signal was sampled.
- The process of aliasing describes the phenomenon in which components of the signal at high frequencies are mistaken for components at lower frequencies.

Frequency Limit

- The sampling theory says that we must capture at least two samples per cycle of a sinusoidal input waveform

$$f_s \geq 2f_0 \quad \rightarrow T_s \leq \frac{T}{2} \qquad \text{where} \quad T = \frac{1}{f_0}$$

where f_s is the sampling frequency and f_0 is bandwidth or highest frequency component of the sampled signal

Sampling: Example

- Consider a signal $x(t) = 5\cos(200\pi t)$

is sampled at a rate of $500\frac{samples}{sec} \rightarrow f_s = 500\ Hz \quad T_s = \frac{1}{f_s}$

$f_0 = \frac{200\pi}{2\pi} = 100Hz,\ x[n] = 5\cos(200\pi n T_s) =$

$5\cos\left(\frac{200\pi n}{f_s}\right) =$

$x[n] = 5\cos\left(\frac{200\pi n}{500}\right) = 5\cos(0.4\pi n)$

$f_s = 500\ Hz, \qquad F = \frac{f_o}{f_s} = \frac{100}{500} = 0.2$

Number of samples in one cycle $\frac{f_s}{f_0} = \frac{500}{100} = 5 = \frac{T_0}{T_s}$

Duration of 40 samples $= 40\mathrm{T_s} = \frac{40}{\mathrm{f_s}} = \frac{40sample}{500\frac{samples}{s}} = 0.08\ sec$

Note that in this example $f_s = 5f_o$ is larger than the Nyquist rate and the signal can be accurately recovered

Aliasing Example

- Consider a signal $x_1(t) = 5\,cos(200\pi t)$ is sampled at a rate of $500\ samples/sec.$ Here $f_s = 5f_0$

$$f_0 = \frac{200\pi}{2\pi} = 100Hz,\ x_1[n] = 5\cos\left(\frac{200\pi n}{f_s}\right) =$$

$$x_1[n] = 5\cos\left(\frac{200\pi n}{500}\right) = 5\cos(0.4\pi n)$$

- Now another signal $x_2(t) = 5\cos(800\pi t)$ signal sampled at the same rate
- Here $\frac{\mathrm{f_s}}{\mathrm{f_o}} = \frac{500}{400} = 1.25 \quad x_2[n] = 5\cos\left(\frac{800\pi n}{f_s}\right) =$

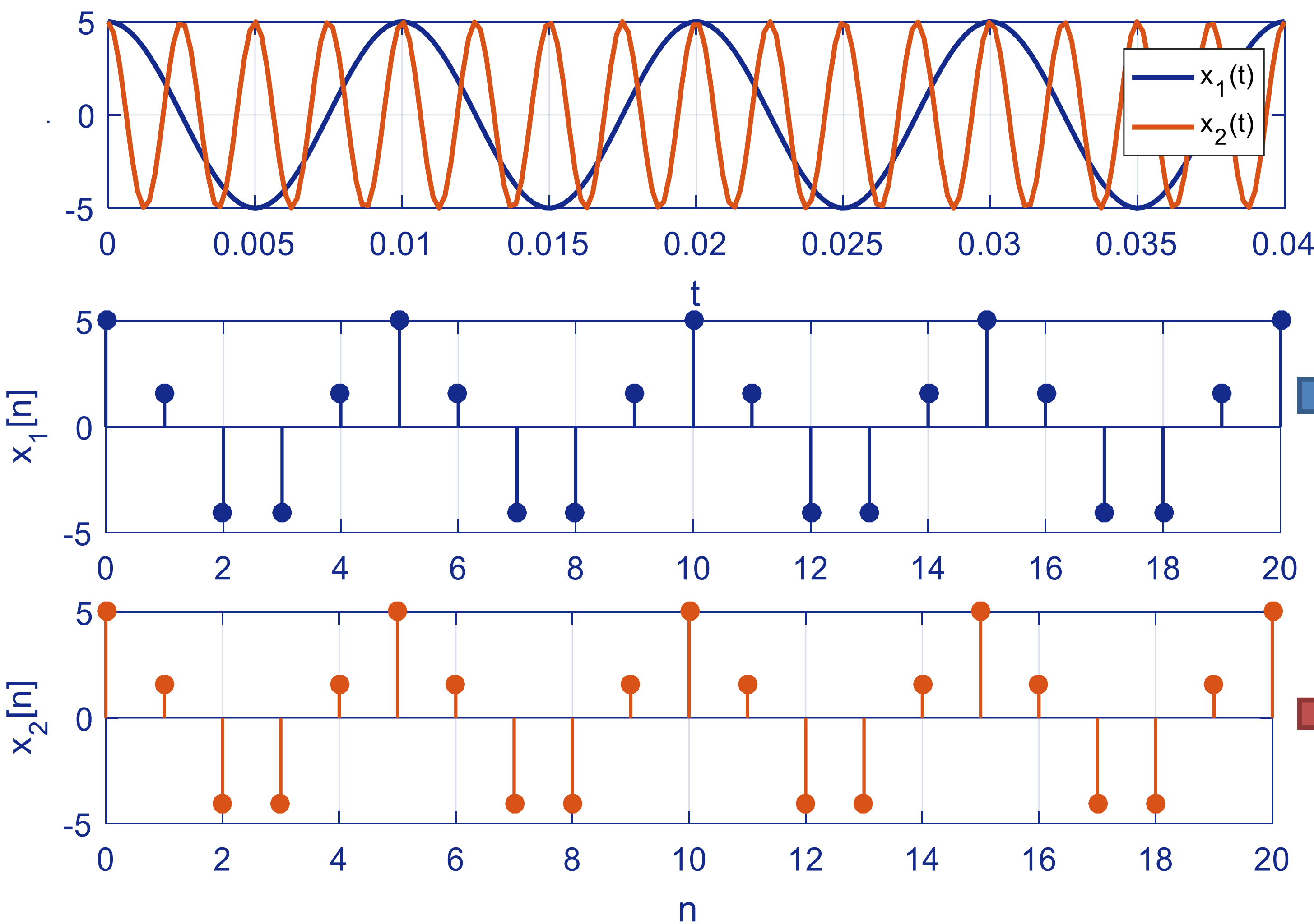
$$x_2[n] = 5\cos\left(\frac{800\pi n}{500}\right) = 5\cos(1.6\pi n) = 5\cos(1.6\pi n - 2\pi n)$$

$x_2[n] = 5\cos(-0.4\pi n) \quad = 5cos(0.4\pi n) = \mathrm{x_1[n]} \Rightarrow$ even function

- The frequency $0.4\pi\ \frac{rad}{sample}$ is the alias frequency of the frequency $1.6\pi\ rad/sample$
- The signal $x_2(t)$ which is under sampled will be incorrectly recovered as $x_1(t)$

Aliasing Example:

$x_1(t) = 5\cos(200\pi t) \quad x_2(t) = 5\cos(800\pi t) \quad f_s = 500Hz$



$f_s = 5f_{01}$

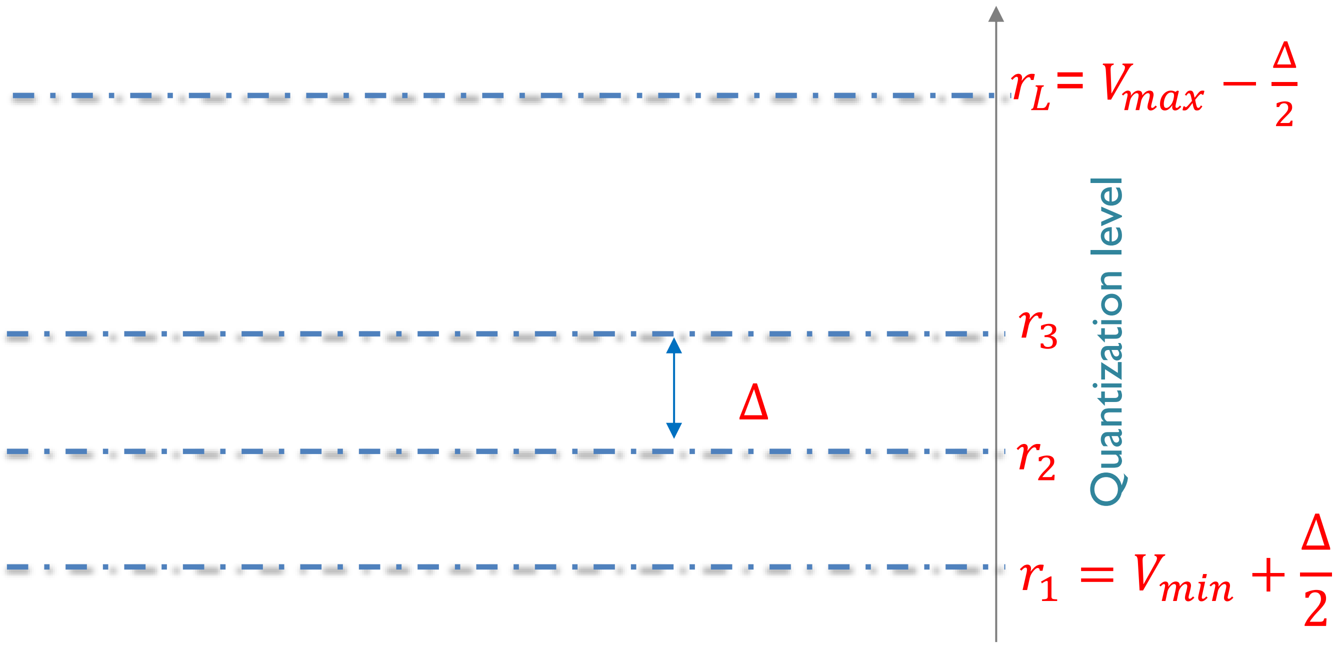
$f_s = 1.25f_{02}$

Quantization

Quantization

- Quantization is the process of mapping continuous signal values to a discrete set of quantization levels.
- A quantized signal can only take on a discrete, usually finite, set of values.
- Unlike sampling, where exact reconstruction is possible under suitable conditions, quantization introduces irreversible errors, resulting in information loss.
- Quantization distortion is introduced into the quantized signal and cannot be completely eliminated.
- Increasing the number of discrete levels in an analog-to-digital converter (ADC) reduces quantization error but increases the required data storage.
- For an ADC system with N bits, the number of discrete levels L is given by $L = 2^N$

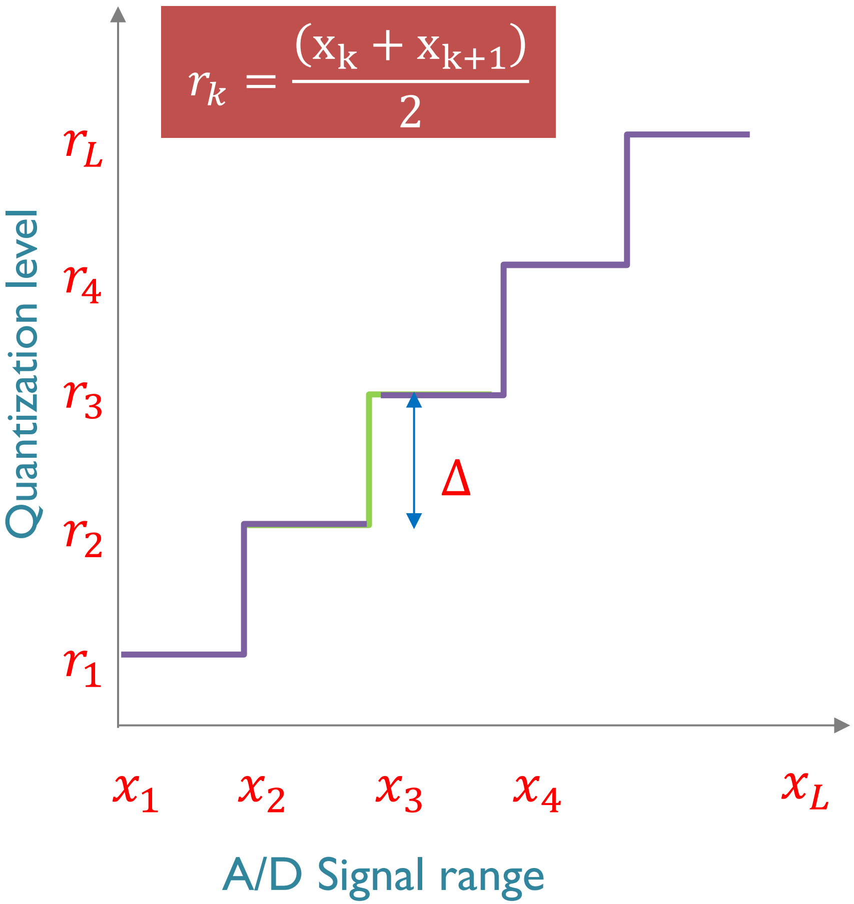
Uniform Quantization



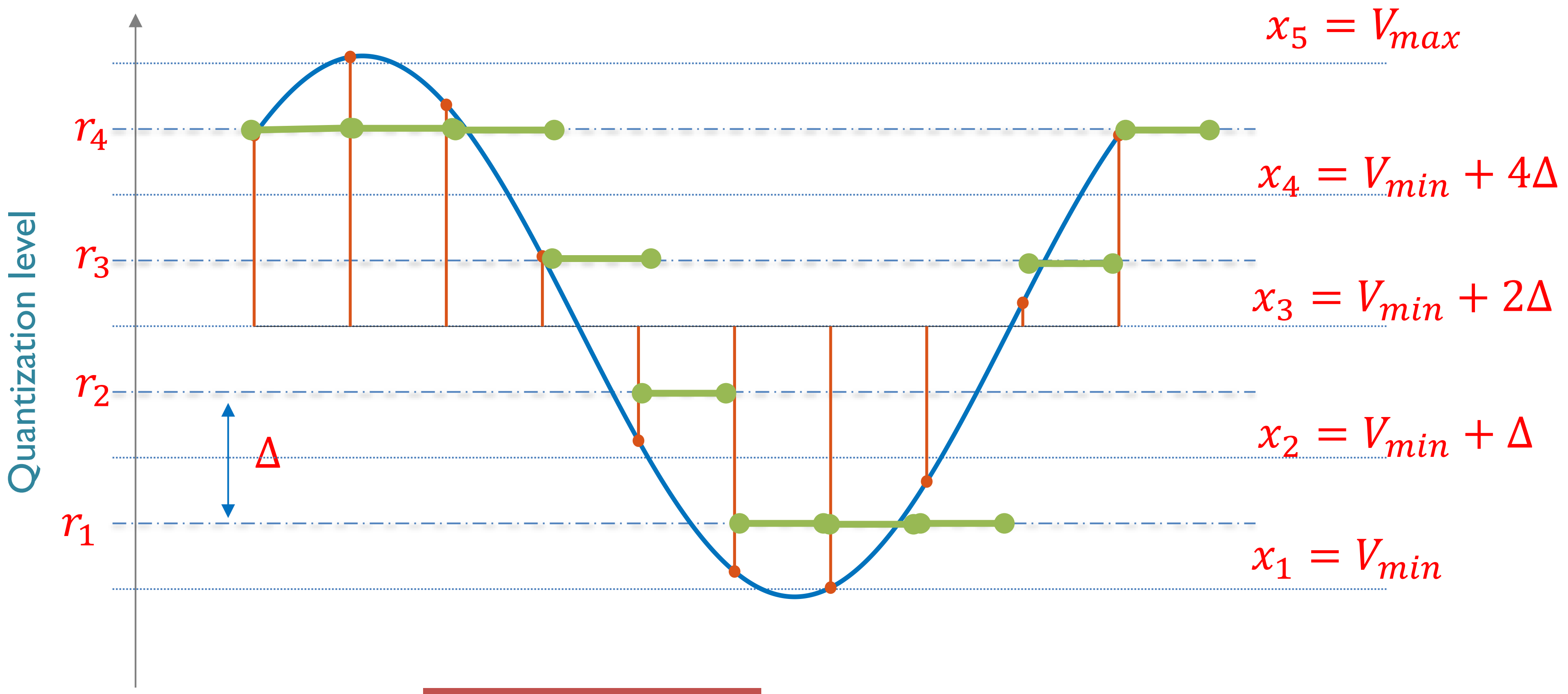
- **Uniform quantization** maps continuous signal values to a discrete set of quantization levels that are evenly spaced across the signal range
- V_{FS} is the full-scale voltage range of the A/D, $V_{FS} = V_{max} - V_{min}$
- V_{min} =minimum voltage level of A/D, V_{max} = maximum voltage level of A/D
- For a system of N bits, the number of quantization levels is given by $L = 2^N$
- The quantization step, also defined as the system resolution is given by $\Delta = \frac{V_{FS}}{2^N}$

Uniform Quantization

- $L = 2^N$
- $\Delta = \frac{V_{FS}}{2^N} = \text{system resolution}$
- $V_{FS} = V_{max} - V_{min}$
- V_{min} minimum voltage level of A/D
- V_{max} =maximum voltage level of A/D
- If the signal amplitude is between x_k and x_{k+1}, it's mapped into r_k
- $r_k = V_{min} + \frac{\Delta}{2} + (k-1)\Delta\,, \mathrm{k} = 1,2,..\mathrm{L}$
- $x_k = V_{min} + (k-1)\Delta.\ k = 1,2,.L+1$



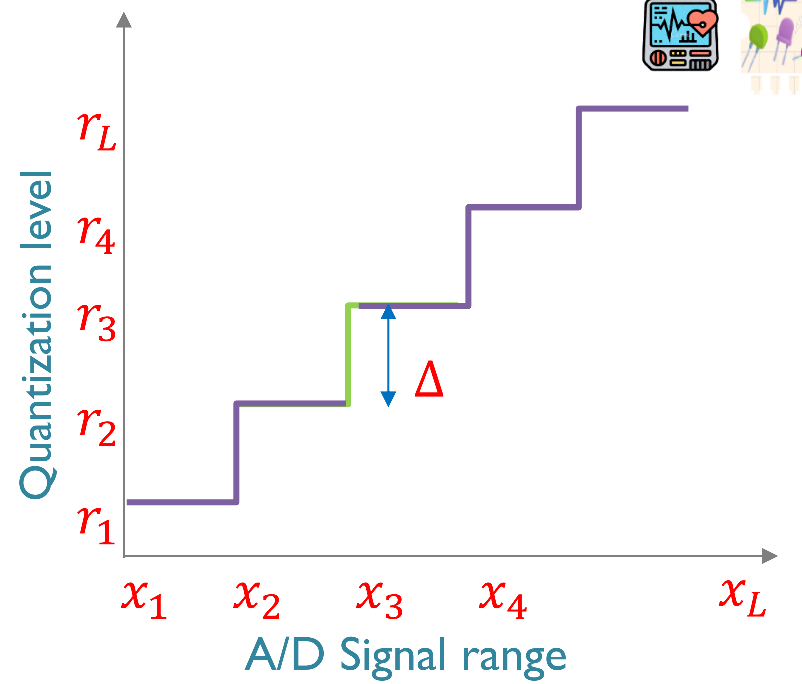
Quantization



$$r_k = \frac{(x_k + x_{k+1})}{2}$$

Quantization Error

- If the signal falls within the full-scale range of the A/D converter, the maximum absolute value of the quantization error is half the distance between two adjacent quantization levels, q_e is $\frac{\Delta}{2}$
- $|q_e| \leq \Delta/2$
- If the signal exceeds the full-scale range, it will be clipped to the nearest quantization level within that range, and the resulting quantization error will depend on the extent of the clipping.



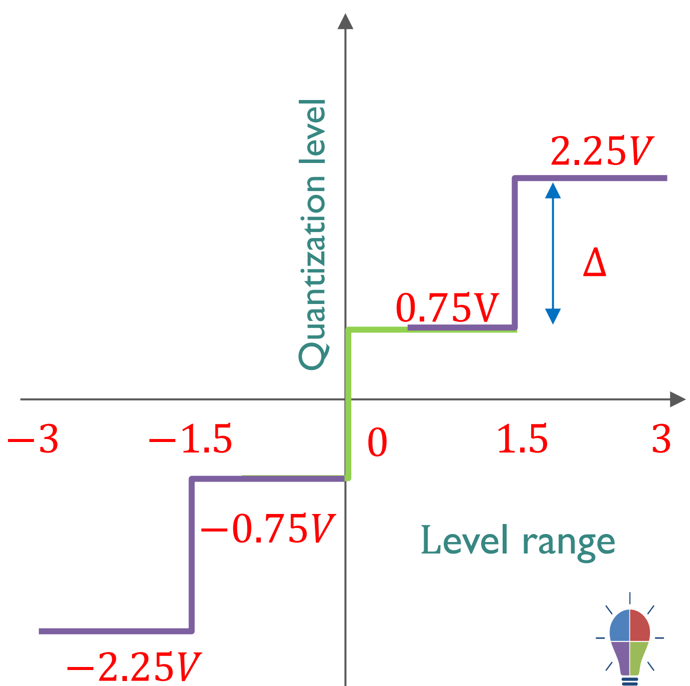
A/D quantizer specifications

- $\Delta = \frac{V_{FS}}{2^N}$ = system resolution
- $L = 2^N$
- $V_{FS} = V_{max} - V_{min}$
- V_{min} minimum voltage level of A/D
- V_{max} =maximum voltage level of A/D

Simple Example

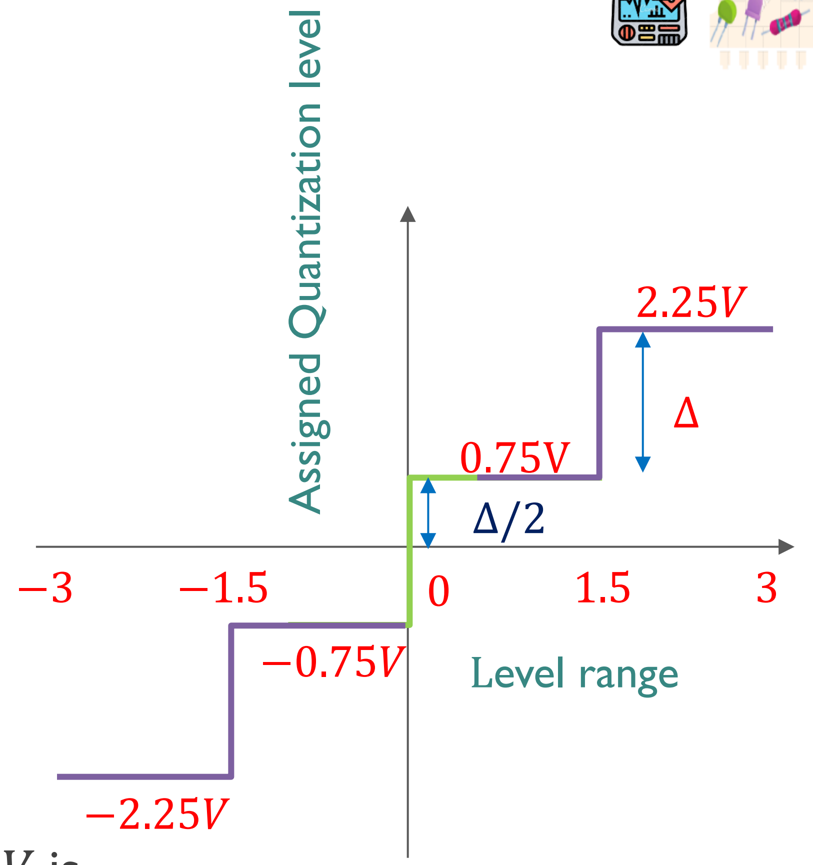
- Consider a 2-bit system, $N = 2,\ L = 2^N = 2^2 = 4$ Levels
- Consider the voltage range of the A/D is $\pm 3V$
- $V_{min} = -3V, V_{max} = 3V \qquad V_{FS} = 3 - (-3) = 6V \qquad \Delta = \frac{V_{FS}}{2^N} = \frac{6}{4} = 1.5V$
- $r_k = V_{min} + \frac{\Delta}{2} + (k-1)\Delta$, $\mathrm{k} = 1,2,..\mathrm{L}$ $\quad x_k = V_{min} + (k-1)\Delta.\ k = 1,2,.L+1$

$r_k = r_{k-1} + \Delta$	$x_k = x_{k-1} + \Delta$
$r_1 = V_{min} + \frac{\Delta}{2} = -2.25V$	$x_1 = V_{\min} = -3V$
$r_2 = \mathrm{r}_1 + 1.5 = -0.75V$	$x_2 = x_1 + 1.5 = -1.5V$
$r_3 = \mathrm{r}_2 + 1.5 = 0.75V$	$x_3 = x_2 + 1.5 = 0V$
$r_4 = \mathrm{r}_3 + 1.5 = 2.25V$	$x_4 = x_3 + 1.5 = 1.5V$
	$x_5 = x_4 + 1,5 = 3V$



Simple Example, continued

- 2-bit A/D system, $N = 2, L = 2^N = 2^2 = 4$ Levels
- The voltage range of the A/D is $\pm 3V$
- $V_{FS} = 3 - (-3) = 6V$
- $\Delta = \frac{V_{FS}}{2^N} = \frac{6}{4} = 1.5V$
- If the signal range is within $\pm 3, q_e = \frac{\Delta}{2} = 0.75V$ is the maximum quantization error.
- If the signal has a range $\pm 5V$, then the maximum quantization error will be $5V - 2.25V = 2.75V$



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Quantization error

- If the signal range is within the A/D full scale range, $q_e = \frac{\Delta}{2}$ is the maximum quantization error.
- If the signal exceeds the A/D full scale range, the error will be given by
-

Largest quantization level

Smallest quantization level

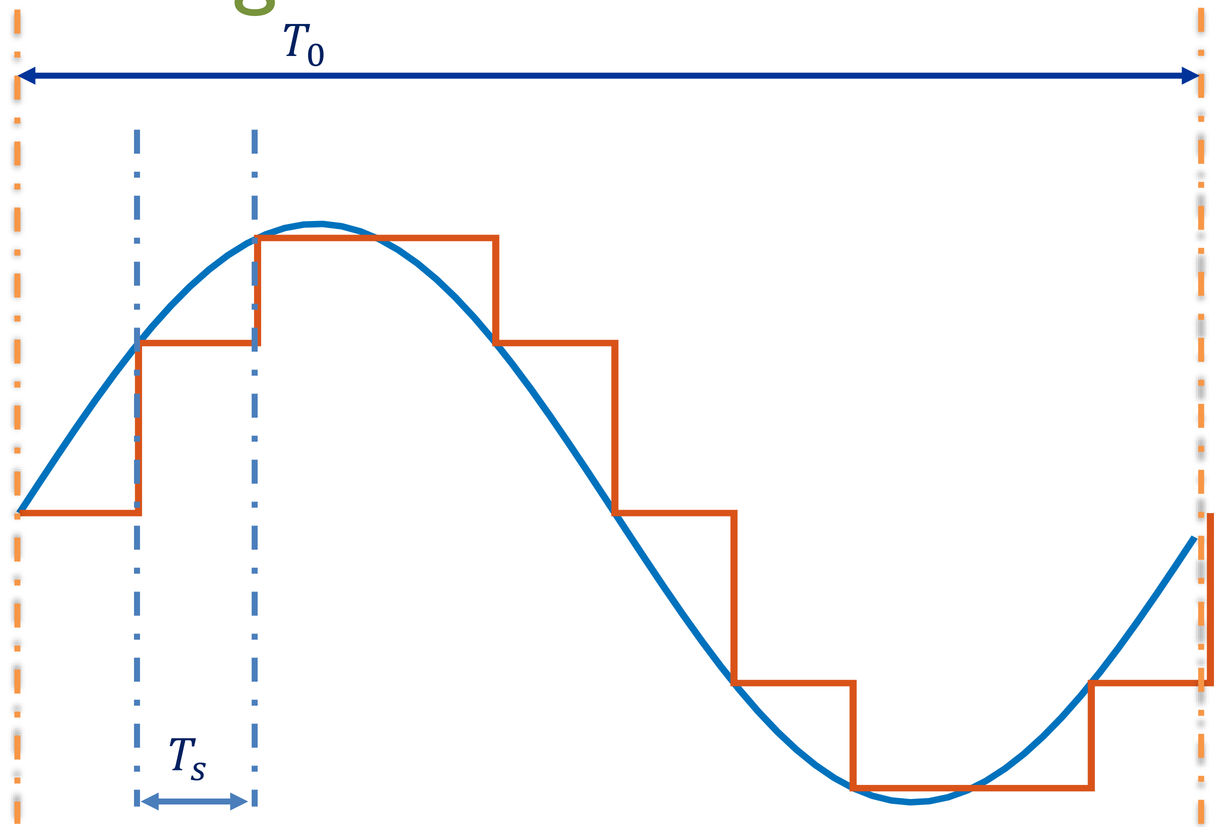
$$q_e = V_{signal} - \left(V_{max} - \frac{\Delta}{2}\right) \quad for\, V_{signal} > V_{max}$$

$$q_e = \left(V_{min} + \frac{\Delta}{2}\right) - V_{signal} \quad for\, V_{signal} < V_{min}$$

V_{max} is the maximum allowed input level for A/D

V_{min} is the minimum allowed input level for A/D

Stair-case approximation of a sampled signal

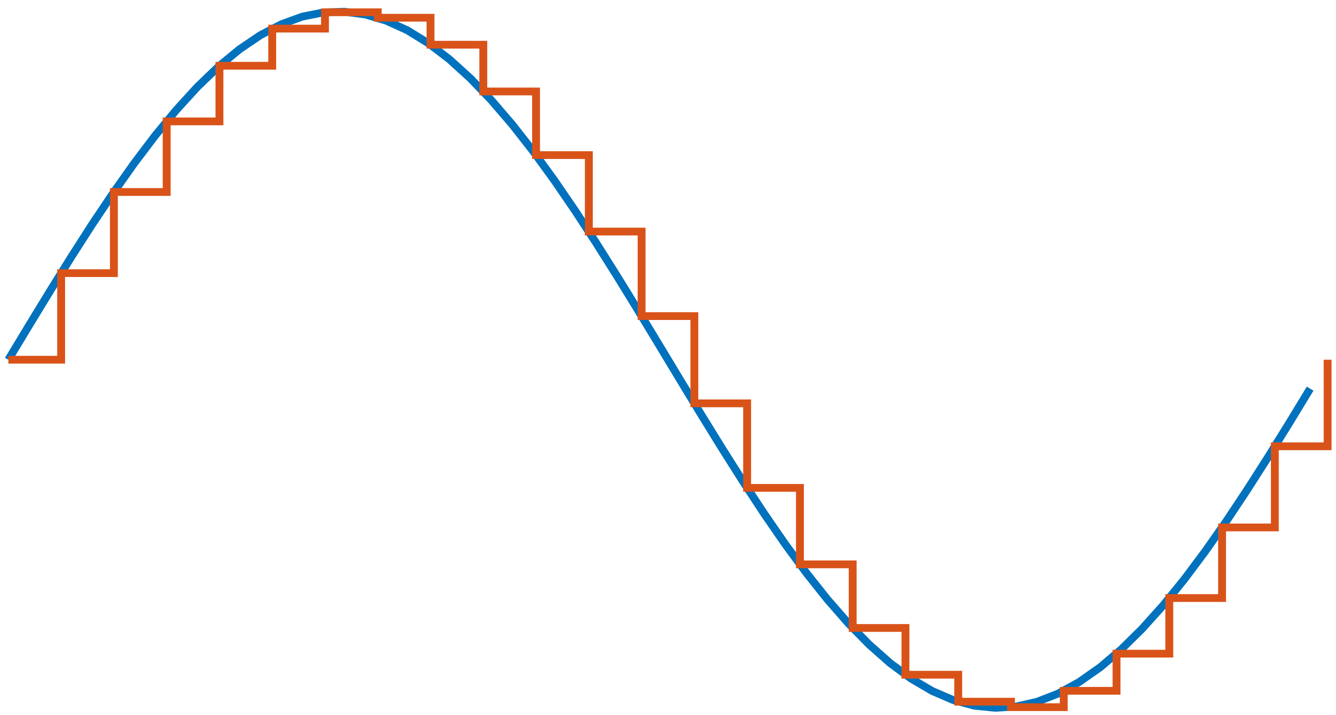


T_s =sampling interval $\quad T_0$ = period of signal

In the figure shown $f_s = 10f_0$

Note that the difference between adjacent steps in this figure is not representative of the quantization step size; instead, it is a function of the sampling frequency

Stair-case Approximation as the Sampling Rate Increases



In the figure shown $f_s = 25f_0$

Notice that the difference between adjacent steps is becoming smaller as the sampling frequency increases. The quantization step is characteristic of the A/D

$$\Delta = \frac{V_{FS}}{2^N}$$

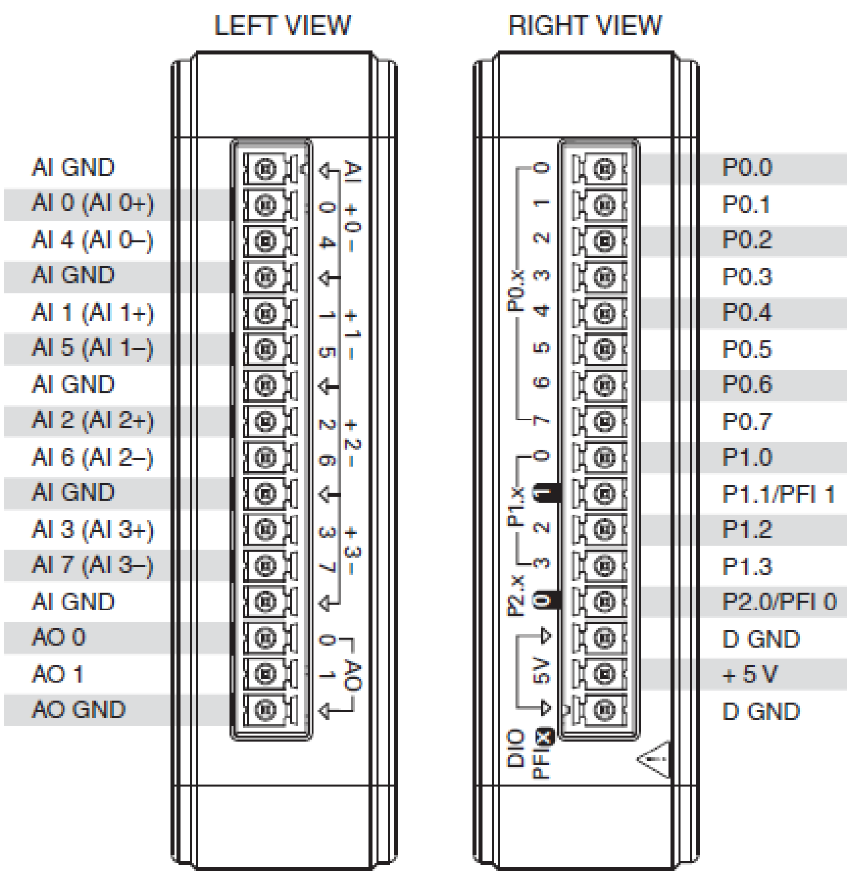
NI USB-6001
Data Acquisition, DAQ, USB device

- Analog-to-digital converter (ADC) with 14-bit Resolution
- Maximum Sample 20 kS/s
- The NI DAQ device has an input signal range of ±10 V.
- Provides eight single-ended analog input (AI) channels, which may also be configured as four differential channels. It also includes two analog output (AO) channels, 13 digital input/output (DIO) channels, and a 32-bit counter.

Pinout and signal description

- AI GND : **Analog Input Ground**—The reference point for single-ended analog input measurements.
- AI <0..7> AI GND Input **Analog Input Channels 0 to 7**—For single-ended measurements, each signal corresponds to one analog input voltage channel.
- For differential measurements, AI 0 and AI 4 are the positive and negative inputs of differential analog input channel 0.
- The following signal pairs also form differential input channels:AI <1,5>, AI <2, 6>, and AI <3, 7>.

Figure 4. NI USB-6001/6002/6003 Pinout



Taking differential measurement

- For differential signals, connect the positive lead of the voltage signal or source to the AI+ terminal, and the negative lead to the AI- terminal.

Figure 6. Connecting a Differential Voltage Signal

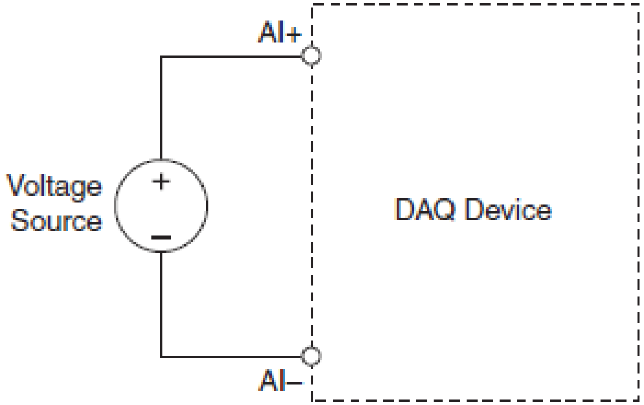


Figure 7. Example of a Differential ±10 V Measurement