Asymptotic Complexity Proofs

Part II: Asymptotic Notation Proofs

1. Finding the Complexity of the Given Equation

Equation:

$$10n^3 + 24n^2 + 3n \log n + 144$$

Analysis:

- The highest order term dominates the function as n grows large.
- The dominant term is $10n^3$.

Conclusion:

$$10n^3 + 24n^2 + 3n \log n + 144 = \Theta(n^3)$$

2. Proof: $n^3 + 20n + 1 = O(n^3)$

Definition of Big-O: A function $f(n)$ is $O(g(n))$ if there exist positive constants c and n_0 such that:

$$f(n) \leq c \cdot g(n) \quad \text{for all } n \geq n_0$$

Step-by-Step Analysis:

$$f(n) = n^3 + 20n + 1, \quad g(n) = \frac{n^3}{6}$$

For large n, we approximate:

$$n^3 + 20n + 1 \leq c \cdot \frac{n^3}{6}$$

Ignoring lower-order terms:

$$n^3 \leq c \cdot \frac{n^3}{6}$$

Choosing $c = 6$, we get:

$$n^3 \leq 6 \cdot \frac{n^3}{6} = n^3$$

Thus, the inequality holds for all $n \geq 1$, proving:

$$n^3 + 20n + 1 = O(n^3)$$

3. Proof: $7n^2 + 5 = O(n^3)$

We need to show that:

$$7n^2 + 5 \leq c \cdot n^3$$

For large n, the dominant term is $7n^2$, so:

$$7n^2 \leq c \cdot n^3$$

Dividing by n^2:

$$7 \leq c \cdot n$$

Choosing $c = 7$ and $n_0 = 1$, the inequality holds.
Conclusion:

$$7n^2 + 5 = O(n^3)$$

4. Proof: $n^3 + 20n = \Omega(n^2)$

Definition of Big-Omega: A function $f(n)$ is $\Omega(g(n))$ if there exist positive constants c and n_0 such that:

$$f(n) \geq c \cdot g(n) \quad \text{for all } n \geq n_0$$

Step-by-Step Analysis:

$$f(n) = n^3 + 20n, \quad g(n) = n^2$$

For large n, we approximate:

$$n^3 + 20n \geq c \cdot n^2$$

Dividing by n^2:

$$n + \frac{20}{n} \geq c$$

For large n, $\frac{20}{n} \to 0$, so we approximate:

$$n \geq c$$

Choosing $c = 1$ and $n_0 = 1$, the inequality holds.
Conclusion:

$$n^3 + 20n = \Omega(n^2)$$

5. Proof: $3n \log n + 4n + 5n \log n = \Theta(n \log n)$

We rewrite:

$$3n \log n + 5n \log n + 4n = 8n \log n + 4n$$

For large n, the dominant term is $8n \log n$, ignoring $4n$:

$$8n \log n$$

Since $8n \log n$ is a constant multiple of $n \log n$, we conclude:

$$\Theta(n \log n)$$

Conclusion:

$$3n \log n + 4n + 5n \log n = \Theta(n \log n)$$