Part II: Hashing and Hash Tables

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Part (a): Quadratic Probing

We are given a 10-slot closed hash table (slots 0 through 9) and the hash function

$$h(k) = k \mod 10.$$

We wish to insert the following keys using quadratic probing:

$$5, \quad 12, \quad 8, \quad 20, \quad 14, \quad 7.$$

Quadratic probing uses the probe sequence:

$$h(k), \quad h(k) + 1^2, \quad h(k) + 2^2, \quad h(k) + 3^2, \quad \dots \quad \text{(all mod 10)}.$$

However, notice that for these keys the initial hash locations are all distinct:

Step 1: Insert 5:

$$h(5) = 5 \mod 10 = 5.$$

Slot 5 is empty, so place 5 in slot 5.

Step 2: Insert 12:

$$h(12) = 12 \mod 10 = 2.$$

Slot 2 is empty, so place 12 in slot 2.

Step 3: Insert 8:

$$h(8) = 8 \mod 10 = 8.$$

Slot 8 is empty, so place 8 in slot 8.

Step 4: Insert 20:

$$h(20) = 20 \mod 10 = 0.$$

Slot 0 is empty, so place 20 in slot 0.

Step 5: Insert 14:

$$h(14) = 14 \mod 10 = 4.$$

Slot 4 is empty, so place 14 in slot 4.

Step 6: Insert 7:

$$h(7) = 7 \mod 10 = 7.$$

Slot 7 is empty, so place 7 in slot 7.

Thus, the final hash table is:

Slot	Key
0	20
1	—
2	12
3	—
4	14
5	5
6	—
7	7
8	8
9	—

Part (b): Double Hashing

Now we use a 10-slot hash table with the following two hash functions:

$$\text{Primary: } h(k) = k \mod 10, \quad \text{Secondary: } h_2(k) = 7 - (k \mod 7).$$

The keys to insert are the same:

$$5, \quad 12, \quad 8, \quad 20, \quad 14, \quad 7.$$

Double hashing uses the probe sequence:

$$h(k), \quad h(k) + h_2(k), \quad h(k) + 2 \cdot h_2(k), \quad \dots \quad \text{(all mod 10)}.$$

We insert the keys as follows:

Step 1: Insert 5:

$$h(5) = 5 \mod 10 = 5.$$

Slot 5 is empty; insert 5.

Step 2: Insert 12:

$$h(12) = 12 \mod 10 = 2.$$

Slot 2 is empty; insert 12.

Step 3: Insert 8:

$$h(8) = 8 \mod 10 = 8.$$

Slot 8 is empty; insert 8.

Step 4: Insert 20:

$$h(20) = 20 \mod 10 = 0.$$

Slot 0 is empty; insert 20.

Step 5: Insert 14:

$$h(14) = 14 \mod 10 = 4.$$

Slot 4 is empty; insert 14.

Step 6: Insert 7:

$$h(7) = 7 \mod 10 = 7.$$

Slot 7 is empty; insert 7.

No collisions occurred in the primary hash positions; hence the final table is identical to part (a):

Slot	Key
0	20
1	—
2	12
3	—
4	14
5	5
6	—
7	7
8	8
9	—

Part (c): Double Hashing with Custom Hash Functions in a 13-Slot Table

We now have a 13-slot hash table (slots 0 through 12). The keys to insert are:

$$2, \quad 8, \quad 31, \quad 20, \quad 19, \quad 18, \quad 53, \quad 27.$$

The hash functions are defined as:

$$H_1(k) = k \mod 13,$$

$$H_2(k) = \text{Rev}(k+1) \mod 11,$$

where $\text{Rev}(n)$ reverses the decimal digits of n. (For example, $\text{Rev}(37) = 73$, and $\text{Rev}(7) = 7$.)

Step 1: Key 2:

$$H_1(2) = 2 \mod 13 = 2.$$

Compute

$$H_2(2) = \text{Rev}(2+1) = \text{Rev}(3) = 3 \quad \Rightarrow \quad 3 \mod 11 = 3.$$

Slot 2 is empty; insert 2 at slot 2.

Step 2: Key 8:

$$H_1(8) = 8 \mod 13 = 8.$$

Compute

$$H_2(8) = \text{Rev}(8+1) = \text{Rev}(9) = 9 \quad \Rightarrow \quad 9 \mod 11 = 9.$$

Slot 8 is empty; insert 8 at slot 8.

Step 3: Key 31:

$$H_1(31) = 31 \mod 13 = 5.$$

Compute

$$H_2(31) = \text{Rev}(31+1) = \text{Rev}(32) = 23, \quad 23 \mod 11 = 1.$$

Slot 5 is empty; insert 31 at slot 5.

Step 4: Key 20:

$$H_1(20) = 20 \mod 13 = 7.$$

Compute

$$H_2(20) = \text{Rev}(20+1) = \text{Rev}(21) = 12, \quad 12 \mod 11 = 1.$$

Slot 7 is empty; insert 20 at slot 7.

Step 5: Key 19:

$$H_1(19) = 19 \mod 13 = 6.$$

Compute

$$H_2(19) = \text{Rev}(19+1) = \text{Rev}(20) = 2, \quad 2 \mod 11 = 2.$$

Slot 6 is empty; insert 19 at slot 6.

Step 6: Key 18:

$$H_1(18) = 18 \mod 13 = 5.$$

Slot 5 is already occupied (with 31). Now compute

$$H_2(18) = \text{Rev}(18+1) = \text{Rev}(19) = 91, \quad 91 \mod 11 = 91 - 88 = 3.$$

Use the double hashing probe sequence:

- First probe: index 5 (occupied).
- Second probe: index $(5+3) \mod 13 = 8$ (occupied).
- Third probe: index $(5+2\cdot 3) \mod 13 = (5+6) = 11$ (empty).

Insert 18 at slot 11.

Step 7: Key 53:

$$H_1(53) = 53 \mod 13 = 1.$$

Compute

$$H_2(53) = \text{Rev}(53+1) = \text{Rev}(54) = 45, \quad 45 \mod 11 = 1.$$

Slot 1 is empty; insert 53 at slot 1.

Step 8: Key 27:

$$H_1(27) = 27 \mod 13 = 1.$$

Slot 1 is occupied (by 53). Now compute

$$H_2(27) = \text{Rev}(27+1) = \text{Rev}(28) = 82, \quad 82 \mod 11 = 82 - 77 = 5.$$

Probe sequence:

- First probe: index 1 (occupied).
- Second probe: index $(1+5) \mod 13 = 6$ (occupied).
- Third probe: index $(1+2\cdot 5) \mod 13 = 11$ (occupied).
- Fourth probe: index $(1+3\cdot 5) \mod 13 = 16 \mod 13 = 3$ (empty).

Insert 27 at slot 3.

The final hash table is:

Slot	**Key**
0	—
1	53
2	2
3	27
4	—
5	31
6	19
7	20
8	8
9	—
10	—
11	18
12	—

Part (d): Pseudo-Random Probing

We now use a 10-slot hash table with the hash function

$$h(k) = k \mod 10,$$

and pseudo-random probing using the permutation of offsets:

$$5, \quad 9, \quad 2, \quad 1, \quad 4, \quad 8, \quad 6, \quad 3, \quad 7.$$

The keys to insert are:

$$3, \quad 12, \quad 9, \quad 2, \quad 79, \quad 44.$$

Insertion Steps

Step 1: Insert 3:

$$h(3) = 3 \mod 10 = 3.$$

Slot 3 is empty; insert 3 in slot 3.

Step 2: Insert 12:

$$h(12) = 12 \mod 10 = 2.$$

Slot 2 is empty; insert 12 in slot 2.

Step 3: Insert 9:

$$h(9) = 9 \mod 10 = 9.$$

Slot 9 is empty; insert 9 in slot 9.

Step 4: Insert 2:

$$h(2) = 2 \mod 10 = 2,$$

but slot 2 is occupied (by 12). Apply the pseudo-random probe offsets:

- First offset: 5. New index: $(2+5) \mod 10 = 7$. Slot 7 is empty; insert 2 at slot 7.

Step 5: Insert 79:

$$h(79) = 79 \mod 10 = 9,$$

but slot 9 is occupied (by 9). Probe:

- First offset: 5. New index: $(9+5) \mod 10 = 4$. Slot 4 is empty; insert 79 at slot 4.

Step 6: Insert 44:

$$h(44) = 44 \mod 10 = 4,$$

but slot 4 is occupied (by 79). Probe:

- First offset: 5. New index: $(4+5) \mod 10 = 9$ (occupied).
- Second offset: 9. New index: $(4+9) \mod 10 = 13 \mod 10 = 3$ (occupied).
- Third offset: 2. New index: $(4+2) \mod 10 = 6$ (empty).

Insert 44 at slot 6.

The final hash table is:

Slot	Key
0	—
1	—
2	12
3	3
4	79
5	—
6	44
7	2
8	—
9	9

Probability That an Empty Slot Will Be Next Filled

After inserting 44, the empty slots are: 0, 1, 5, and 8. Assume that for the next key the initial hash value, $r = h(\text{new key})$, is uniformly distributed over $\{0, 1, \ldots, 9\}$. The probing sequence for a new key (after its hash r) is:

$$r, \quad r+5, \quad r+9, \quad r+2, \quad r+1, \quad r+4, \quad r+8, \quad r+6, \quad r+3, \quad r+7 \pmod{10}.$$

We analyze, for each starting r, which empty slot is encountered first (given our table occupancy):

- $r = 0$**:** Sequence: 0 (empty) $\rightarrow$ **slot 0** is chosen.
- $r = 1$**:** Sequence: 1 (empty) $\rightarrow$ **slot 1** is chosen.
- $r = 2$**:** Sequence: 2 (filled), $2+5=7$ (filled), $2+9=11 \mod 10 = 1$ (empty) $\rightarrow$ **slot 1**.
- $r = 3$**:** Sequence: 3 (filled), $3+5=8$ (empty) $\rightarrow$ **slot 8**.
- $r = 4$**:** Sequence: 4 (filled), $4+5=9$ (filled), $4+9=13 \mod 10 = 3$ (filled), $4+2=6$ (filled), $4+1=5$ (empty) $\rightarrow$ **slot 5**.
- $r = 5$**:** Sequence: 5 (empty) $\rightarrow$ **slot 5**.
- $r = 6$**:** Sequence: 6 (filled), $6+5=11 \mod 10 = 1$ (empty) $\rightarrow$ **slot 1**.
- $r = 7$**:** Sequence: 7 (filled), $7+5=12 \mod 10 = 2$ (filled), $7+9=16 \mod 10 = 6$ (filled), $7+2=9$ (filled), $7+1=8$ (empty) $\rightarrow$ **slot 8**.
- $r = 8$**:** Sequence: 8 (empty) $\rightarrow$ **slot 8**.
- $r = 9$**:** Sequence: 9 (filled), $9+5=14 \mod 10 = 4$ (filled), $9+9=18 \mod 10 = 8$ (empty) $\rightarrow$ **slot 8**.

Since the initial hash r is uniformly distributed, each value occurs with probability 1/10. We tally the outcomes:

- **Slot 0:** Occurs when $r = 0 \Rightarrow$ probability $= \frac{1}{10} = 0.1$.
- **Slot 1:** Occurs when $r = 1, 2, 6 \Rightarrow$ probability $= \frac{3}{10} = 0.3$.
- **Slot 5:** Occurs when $r = 4, 5 \Rightarrow$ probability $= \frac{2}{10} = 0.2$.
- **Slot 8:** Occurs when $r = 3, 7, 8, 9 \Rightarrow$ probability $= \frac{4}{10} = 0.4$.