Independence proofs via *forcing*

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Overview

A formula α is said to be *independent* from a theory T if neither α nor $\neg\alpha$ can be proven from T. Forcing is a technique developed by Paul Cohen in the 60s to finish proving the independence of the Continuum Hypothesis (CH) from the Zermelo-Fraenkel axioms of set theory with choice (ZFC), building on previous work by Gödel. Independence results and forcing present a novel way to attack unsolved problems in mathematics.

Model-theoretic independence

A natural way to prove the independence of a formula is adopting a model-theoretic approach – in first-order logic, α is also independent if both $T \cup \{\alpha\}$ and $T \cup \{\neg\alpha\}$ have a model. An useful example to consider are models of geometry: the independence of the parallel postulate from the other Euclidean axioms is made clear by the existence of Euclidean geometry and of non-Euclidean geometries, such as hyperbolic geometry. Forcing provides a method by which to construct these models for ZFC.

Absoluteness

Forcing functions by considering extensions of very specific models of ZFC – those that are standard, transitive and countable. Given one of those models, M, and the set-theoretic universe, V, we say some property is *absolute* if its truth value is the same in V as it is in M – say, "X is the empty set". Countability is not absolute: M contains ω and thus it must contain its powerset, but since M is countable and $2^\omega \in V$ is not, then whatever is considered to be the powerset of ω in M – call that R – is only uncountable *in* M. Thus, "R is uncountable" is true in M, but false in V.

Forcing

Given this, let us consider $\aleph_0$ and that which is considered $\aleph_2$ in M, $\aleph_2^M$. We can consider some sequence of distinct functions F of length $\aleph_2^M$ from $\aleph_0$ to $\{0,1\}$. Since each of these functions defines a subset of $\aleph_0$, then if $F \in M$ we can define a bijection in M from a subset of the powerset of $\aleph_0$ in M and the second uncountable cardinal $\aleph_2$ in M, meaning CH is false in M. Taking advantage of non-absoluteness like this is the main idea behind forcing as a way to prove consistency and independence results – the specifics of how to introduce F to M while keeping it a model of ZFC are what makes forcing strong as a way of constructing new models.

References

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