UNIVERSITY COMPLUTENSE OF MADRID

PHYSICS FACULTY

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SPACETIME ENTANGLEMENT IN QUANTUM FIELD THEORY

BACHELOR'S THESIS

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B.S. Double Degree in Physics and Mathematics

Academic year 2024-25

Abstract:

In this work, we introduce the foundational elements of the algebraic formalism of quantum field theory. A key result, the Reeh-Schlieder theorem, is presented as a pivotal motivation for investigating entanglement between causally disconnected regions of spacetime. To this end, we introduce quantum information theoretic tools for quantifying entanglement and discuss the challenges of applying these techniques within the framework of quantum field theory. Then, we employ the algebraic formalism to study a free scalar quantum field theory, for which we encounter a paradox involving an elusive yet pervasive entanglement structure. Finally, we show the solution to this paradox by resorting to the multimode nature of spacetime entanglement in quantum field theory.

Contents

1 Introduction

Quantum field theory (QFT) stands as the foundation for describing the fundamental interactions of nature. Several attempts to reconcile this theory with gravity have been proposed but the quantization of gravity remains an open issue [1, 2, 3]. A semi-classical approach, however, allows us to study how quantum phenomena differ within different gravitational settings. In this framework, algebraic QFT provides a mathematically rigorous approach that emphasizes the algebraic structures underlying quantum fields [4]. A cornerstone of the algebraic formalism is the Gelfand–Naimark–Segal (GNS) representation [5, 6], which establishes a systematic method to represent algebraic states as vectors in a Hilbert space. This representation not only provides a bridge between algebraic and Hilbert space formulations of QFT but also serves as a tool for exploring deeper properties of quantum states, such as entanglement. One particularly significant result in this context is the Reeh-Schlieder theorem [7, 8], which demonstrates the ubiquity of entanglement in quantum field states, even between causally disconnected regions. This theorem underscores the non-locality inherent in QFT and sets the stage for studying the pervasive and multimode nature of quantum entanglement [9].

Entanglement is a hallmark of quantum mechanics, which manifests as correlations between parts of a quantum system that cannot be explained classically. It is not only a key resource for quantum information processing but also a fundamental concept for understanding quantum correlations in many-body systems, quantum gravity, and the structure of spacetime. For instance, in holographic theories entanglement is directly related to the curvature of the spacetime in the bulk [10]. Despite its conceptual significance, the quantification of entanglement remains a challenging problem, particularly in systems beyond the bipartite case [11]. Entanglement entropy computed using the Von Neumann entropy fails to fully reflect the entanglement structure in mixed states [12]. Therefore, several other techniques such as the positive partial transpose (PPT) criterion [13], and measures like the (logarithmic) negativity [11] have been developed to assess entanglement. These tools are relatively well-suited to quantum systems with a finite number of degrees of freedom, yet their application to QFT is fraught with additional difficulties (see §3.2). When attempting to compute entanglement between two non-overlapping spacetime regions, the Hilbert space of a quantum field does not factor as a tensor product of two Hilbert subspaces associated with the two regions. Thus, one cannot formally define reduced density matrices of a quantum field associated with finite spacetime regions.

One innovative approach to circumvent these challenges is entanglement harvesting, which was first introduced in [14, 15, 16]. This technique uses localized quantum systems, often modeled as particle detectors [17, 18], to probe the entanglement structure of quantum fields. The detectors couple to the field at different spacetime regions, effectively extracting information about the field's entanglement. Entanglement harvesting is particularly advantageous in QFT because it translates the problem of analyzing the field's non-local correlations into a more tractable study of the detectors' quantum states. By strategically configuring the detectors' spatial and temporal parameters, it becomes possible to explore both the presence and the distribution of entanglement in the vacuum state of quantum fields.

This work is organized as follows. We introduce the algebraic QFT formalism and the Reeh-Schlieder theorem in §2. In §3, we discuss our current knowledge about entanglement relevant to this work. We also analyze the difficulties of entanglement computation in general and, specifically, in QFT. Then, we review two different methods for computing entanglement of a quantum field between two spacelike separated regions in §4, using the vacuum state of a free scalar field in Minkowski spacetime as an example. Through entanglement harvesting, we examine the multimode nature of spacetime entanglement [9], confirming its pervasive character as suggested by the Reeh–Schlieder theorem. Interestingly, while multimode entanglement is ubiquitous in the field, the extraction of bimodal entanglement proves to be significantly more challenging [8]. The difficulty in isolating bimodal entanglement reflects the subtle and intricate structure of quantum correlations in QFT, which often defy straightforward characterization. Finally, we provide our conclusion in §5.

Throughout this work, we adopt the metric signature $(+,-,-,-)$. Also, our analysis is restricted to globally hyperbolic spacetimes, where the existence of a Cauchy hypersurface is guaranteed. Last, while the discussion focuses on bosonic fields, the results can be generalized to fermionic fields following standard procedures.

2 Algebraic quantum field theory

The algebraic formalism of QFT is a mathematical framework that emphasizes the algebraic structures underlying physical observables and their relationships. Realized through the Haag-Kastler axioms [4], the algebraic formulation provides a foundation for QFT by abstracting away from specific fields and representations and focusing on the algebraic properties of observables.

In algebraic QFT, the central objects are algebras of observables associated with regions of spacetime. The theory formalizes the concepts of locality (the idea that observables in spacelike-separated regions commute), covariance under spacetime symmetries, and the existence of a vacuum state. By avoiding the reliance on specific Hilbert spaces, algebraic QFT offers a robust approach to handling both free and interacting quantum fields, particularly in curved spacetimes. This framework has led to significant insights into the structural aspects of QFT, such as the formulation of fundamental theorems, including the Reeh-Schlieder theorem [7, 8].

In the context of QFT and, more importantly, QFT in curved spacetimes, we find various reasons for employing this formalism. The Stone-von Neumann theorem states that all (weakly continuous) representations of the canonical commutation relations are unitarily equivalent [19]. This theorem relies heavily on the assumption that the system is described by a finite number degrees of freedom and thus, it applies to quantum mechanics. In QFT, rather, there are infinitely many degrees of freedom, leading to a problem of representation as this theorem no longer holds true. With algebraic QFT we avoid this issue from its building blocks, since no representation is required nor preferred. Furthermore, QFTs in curved backgrounds generally do not exhibit a clear definition of vacuum or particle. Using the algebraic formalism, we need not a particular definition of the vacuum, although symmetries allow for a preferred notion consistent with standard QFT. For instance, in Minkowski spacetime, one can leverage Poincaré invariance to easily and unambiguously define a vacuum state.

2.1 Wightman axioms

In this subsection we present the Wightman axiomatic formulation of QFT [20]. They comprise the general properties that a quantum field must verify and provide a consistent framework. In the following subsection, we will introduce the algebraic axioms, drawing a comparison with those discussed here.

1. Every QFT in Minkowski spacetime, $\mathcal{M}$, consists of:

    - A Hilbert space $\mathcal{H}$ where there exists a unitary representation of the Poincaré group, $\mathcal{P}$[1]. The generators of $\mathcal{P}$ allow defining the energy operator, and the *spectral condition* is postulated as $P^\mu P_\mu \geq 0, P^0 \geq 0$, ensuring that the energy spectrum is positive and that

[1]In the case of fermionic fields, we would need to use the universal cover of the Poincaré group $\mathcal{P}$. Spinors do not transform properly under $\mathcal{P}$ but they do under the universal cover.

causal propagation is to be respected. Associated with this representation of $\mathcal{P}$, there must exist a unique state invariant under the action of $\mathcal{P}$. This state is the eigenstate of P^0 with the minimum eigenvalue and will be referred to as the *vacuum state.*

- A set of operators acting on $\mathcal{H}$, which are interpreted as the amplitudes of the fields of the theory. While these operators can be defined as solutions to a field equation, $\hat{\phi}(x)$, (e.g. Klein-Gordon equation) they are typically not well defined as point-like operators. Rather, they are to be interpreted as operator-valued distributions. In other words, given a smooth function with compact support, $f \in C_0^\infty(\mathcal{M})$, called *test function*, we define the operator $\hat{\phi}$ by its mapping on f,

$$\hat{\phi}(f) = \int_{\mathcal{M}} \hat{\phi}(x) f(x)\, d^4x. \tag{1}$$

2. The following properties must be satisfied:

a) If there exists the operator $\hat{\phi}(f)$, there needs to exist $\hat{\phi}^\dagger(f) \equiv \hat{\phi}(f)^\dagger$.

b) Let $(a, \Lambda) \in \mathcal{P}$ be an element of Poincaré group, where a is a translation and Λ is a pure Lorentz transformation, $U(a, \Lambda)$ be its unitary representation on $\mathcal{H}$ and M its matrix representation. The operators must transform properly under the action of $\mathcal{P}$,

$$U(a, \Lambda)\hat{\phi}_\mu(x)U(a, \Lambda)^{-1} = M_\mu{}^\nu(\Lambda)\hat{\phi}_\nu(\Lambda x + a). \tag{2}$$

c) The operators must respect causality, i.e., if $f, g \in C_0^\infty(\mathcal{M})$ have space-like separated supports, then $[\hat{\phi}(f), \hat{\phi}(g)] = 0$.[2]

d) The representation of the algebraic relations of the field operators in $\mathcal{H}$ is irreducible.

e) The theory has a dynamic law associated with the propagation of fields.

2.2 Haag-Kastler axioms

The Wightman axioms do not abandon the structure of a Hilbert space and thereby continue to exhibit a strong dependence on Poincaré symmetry, which determines the vacuum state and selects a privileged representation of the commutation relations of the field operators. This difficulty can be overcome by considering that the intrinsic structure of the theory lies in the algebraic relations. It is within this context that algebraic QFT emerges, seeking to establish a formalism independent of any particular representation, formalized through the Haag-Kastler axioms [4].

The algebraic axioms are constructed upon the following assignment: given a manifold $\mathcal{M}$, we associate a *unital algebra*, denoted $\mathcal{A}(\mathcal{O})$, to each bounded open set $\mathcal{O} \subset \mathcal{M}$. A unital algebra in this context can be understood as a complex vector space equipped with an associative operation and a unit element. These algebras are interpreted as the algebras of observables localized in $\mathcal{O}$, i.e., whose support is contained within $\mathcal{O}$; and they are generated by the field operators defined in Eq. (1).

To ensure consistency, the structure of these algebras must respect inclusion relations. Specifically, if $\mathcal{O}_1 \subset \mathcal{O}_2 \subset \mathcal{M}$, there must exist an injective homomorphism, $j_{12} : \mathcal{A}(\mathcal{O}_1) \to \mathcal{A}(\mathcal{O}_2)$, seen as the inclusion $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$. This assignment must satisfy the following consistency condition: if $\mathcal{O}_1 \subset \mathcal{O}_2 \subset \mathcal{O}_3$, then $j_{23} \circ j_{12} = j_{13}$. This condition is commonly referred to as the *isotony condition.*

The collection of algebras $\{\mathcal{A}(\mathcal{O})\}$ is referred to as a *net of local algebras.* The isotony condition enables these algebras to be combined through a direct limit process [21], yielding a global algebra

[2] For a fermionic field: $\{\hat{\phi}(f), \hat{\phi}(g)\} = 0$.

$\mathcal{A}$, known as the *quasilocal algebra*. This global algebra can be understood as the "union" of all the local algebras.

Property **a)** of Wightman axioms is implemented by introducing an involution operation $* : \mathcal{A} \to \mathcal{A}$, which is identified as the adjunction operation[3]. Since the support of an operator remains invariant under adjunction, the operation $*$ can be naturally restricted to the local algebras. Moreover, the adjunction must satisfy the following properties:

$$(AB)^* = B^* A^*, \quad (\lambda A)^* = \bar{\lambda} A^*, \quad \forall \lambda \in \mathbb{C},\ \forall A, B \in \mathcal{A}. \tag{3}$$

With such operation, we refer to the global algebra $\mathcal{A}$ and its local restrictions as *(unital) $*$-algebras.*

Wightman's postulate **c)** suggests that if $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated, their operators must commute, i.e.,

$$[A_1, A_2] = 0, \quad \forall A_1 \in \mathcal{A}(\mathcal{O}_1),\ A_2 \in \mathcal{A}(\mathcal{O}_2). \tag{4}$$

This axiom ensures causality is to be preserved and is commonly known as the *microcausality principle.*

Wightman's property **e)** can also be reformulated as an algebraic postulate: if $\mathcal{O}_1$ contains a Cauchy hypersurface of $\mathcal{O}_2$ (and therefore $\mathcal{O}_1 \subset \mathcal{O}_2$), then the algebras $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$ are isomorphic. The motivation for this postulate stems from the following reasoning. If $\mathcal{O}_1$ contains a Cauchy hypersurface of $\mathcal{O}_2$, any operator whose support lies within $\mathcal{O}_2$ can, by virtue of the dynamical law, evolve to an operator whose support is restricted to the Cauchy hypersurface of $\mathcal{O}_2$, which in turn is contained in $\mathcal{O}_1$. Consequently, the two algebras must be isomorphic.

These conditions constitute the Haag-Kastler axioms. Although inspired by the Wightman axioms, the two sets of postulates are not equivalent. Notably, the Haag-Kastler axioms do not impose the spectral condition or any requirements related to Poincaré invariance. This distinction highlights a key advantage of the Haag-Kastler formulation: it is well-suited for extending QFT to curved gravitational backgrounds. The formulation avoids the choice of specific vacuum states, or Hilbert space constructions, making it independent of any representation.

2.3 Algebraic states and GNS construction

To effectively extract physical results from the algebraic formalism, we need to establish a connection with the formulation of QFT based on Hilbert spaces. For this purpose, we introduce the concept of algebraic states, motivated by the notion of density matrices. An algebraic state $\omega : \mathcal{A} \to \mathbb{C}$, is a functional that maps operators on the $*$-algebra to the field of complex numbers, such that it is a positive semi-definite and normalized map, i.e., $\omega(A^*A) \geq 0$ and $\omega(\mathbb{I}) = 1$. The connection between the two axiomatic systems is then realized through the GNS construction.

Theorem (GNS Construction)**.** Let $\mathcal{A}$ be a unital $*$-algebra and ω an algebraic state. Then, there exists a Hilbert space $\mathcal{H}_\omega$, a dense subset $\mathcal{D}_\omega \subset \mathcal{H}_\omega$, a $*$-representation π_ω of $\mathcal{B}(\mathcal{D}_\omega)$ (the set of bounded linear operators acting on $\mathcal{D}_\omega$), and a (vector) state $|\Psi_\omega\rangle \in \mathcal{H}_\omega$ such that,

$$\pi_\omega(\mathcal{A})|\Psi_\omega\rangle = \mathcal{D}_\omega, \quad \omega(A) = \langle \Psi_\omega | \pi_\omega(A) | \Psi_\omega \rangle, \quad \forall A \in \mathcal{A}. \tag{5}$$

Then, the state $|\Psi_\omega\rangle$ is said to be cyclic. Moreover, this construction is unique up to unitary transformations.

The complete proof of this theorem is beyond the scope of this work, but its essence sheds light into how the Hilbert space is constructed. First, we define a quotient space $\mathcal{D}_\omega$, where any element A satisfying $\omega(A^*A) = 0$ is identified with the trivial state. This quotient space allows ω to define

[3]Notice the change of notation from † to $*$ for historical reasons.

a scalar product, obtaining a pre-Hilbert space, which can then be completed to form a Hilbert space $\mathcal{H}_\omega$. By definition of completing a pre-Hilbert space, where the Hilbert space is constructed precisely so that it is the adherence of the pre-Hilbert space, it follows that $\mathcal{D}_\omega$ is dense in the Hilbert space.

We may also impose some physical constraints to restrict the space of valid algebraic states. At a minimum, for an algebraic state to be physically reasonable, we expect that observables have finite expectation values. It turns out that this is a very restrictive requirement [22]. For example, consider Minkowski spacetime. Requiring the vacuum state to be invariant under Poincaré and scale transformations (for massless fields) fixes the functional form of its two-point correlation function to $\omega(\hat{\phi}(x), \hat{\phi}(y)) \propto \frac{1}{(x-y)^2/2}$, which diverges in the coincidence limit.

Now, consider states whose two-point correlation functions take on the so-called Hadamard form:

$$\omega(\hat{\phi}(x), \hat{\phi}(y)) = \frac{U(x,y)}{\sigma(x,y)} + V(x,y)\log[\sigma(x,y)] + H_\omega(x,y), \tag{6}$$

where $\sigma(x,y)$ is Synge's world function, defined as half the squared geodesic distance between the spacetime points x and y. The functions $U(x,y)$ and $V(x,y)$ are determined entirely by the background metric and the Klein-Gordon equation, while $H_\omega(x,y)$ is a smooth function encoding all state dependence. In any curved spacetime, the $1/\sigma(x,y)$ term dominates the behavior of any Hadamard two-point function in the limit where $y \to x$, regardless of the curvature. Furthermore, if x and y lie within a sufficiently small neighborhood, Synge's world function approximates that of Minkowski spacetime. Hence, a state of this form behaves locally as the Minkowski vacuum, and thus its stress-energy density is guaranteed to be renormalizable via local and covariant methods (often called a "finite-energy" state). Thus, any state of a QFT satisfying the Hadamard condition (Eq. (6)) is a finite-energy state and its two-point correlation function must exhibit the same singularity structure at short distances as the vacuum state in Minkowski spacetime, up to logarithmic corrections. Specifically, the correlation function diverges polynomially as the two points approach each other. This singular behavior at short distances will help in understanding the difficulty of computing entanglement in QFT (see §3.2).

2.4 Restrictions to local algebras and the Reeh-Schlieder theorem

Not every algebra is suitable to serve as a local algebra in QFT. The following classification of algebras, based on the projectors they contain, provides a systematic means to distinguish between different types of algebras and imposes constraints on the properties that a local algebra must satisfy under specific conditions in QFT.

- **Type I.** These algebras contain projectors with a minimal dimension, i.e., there exists no projectors within this algebra with dimension lower than the minimal.
- **Type II.** These algebras do not contain any non-zero projectors with a minimal dimension, that is, there is no lower bound to the dimension of projectors. Moreover, their dimensions are parametrized by a continuum.
- **Type III.** In these algebras, all non-zero projectors have infinite dimension.

One of the results that follow from this classification is that the local algebra associated with the causal region of a given event is constrained to a specific type of algebra.

Theorem. Given a sufficiently small bounded region $\mathcal{R} \subset \mathcal{M}$, defined as the set of points causally connected to a certain spacetime point, then its associated local algebra $\mathcal{A}(\mathcal{R})$ is a Type III algebra [1].

An important corollary of this theorem is that the algebra $\mathcal{A}(\mathcal{R})$ cannot contain non-zero projectors of finite dimension. Provided that performing a measurement involves projecting into a specific finite subset[4], this implies that it is impossible to perform local measurements with arbitrary precision. In particular, one cannot determine the presence of particles or distinguish the vacuum state via local measurements.

One of the most striking results of algebraic QFT, and the main reason why we have introduced this formalism, is the Reeh-Schlieder theorem. It is crucial to our analysis of spacetime entanglement, since it relates any state of the global Hilbert space with local operations performed in the vacuum state.

Theorem (Reeh-Schlieder)**.** Let $\mathcal{R}$ be a bounded open region consisting of all points causally connected to a given spacetime point. Let ω be an algebraic vacuum state that is translationally invariant, and let $|\Omega\rangle$ denote the translationally invariant (vector) state in the GNS representation π_ω. Then, $\pi_\omega(\mathcal{A}(\mathcal{R}))|\Omega\rangle$ is dense in $\mathcal{H}_\omega$.

It follows from the Reeh-Schlieder theorem that not only is the vacuum state $|\Omega\rangle$ cyclic for the global algebra but also for the local algebras. Remarkably, given any state $|\psi\rangle \in \mathcal{H}_\omega$ and any $\epsilon > 0$, there exists a local operator $A \in \mathcal{A}(\mathcal{R})$ such that $\|\pi_\omega(A)|\Omega\rangle - |\psi\rangle\| < \epsilon$. Put more simply, **any state** in the global Hilbert space of the QFT can be approximated with arbitrary precision by acting with local operators on the vacuum state.

3 Methods for computing entanglement

In this section, we explore the general methods for entanglement computation, emphasizing the foundational concepts, definitions, and challenges. We introduce several properties and techniques, aiming to provide a cohesive understanding of how entanglement is quantified in diverse frameworks. We give particular attention to the specific issues encountered when attempting to compute entanglement in a QFT.

Even though the algebraic formalism of QFT allows the derivation of fundamental results, the following analysis does not require the powerful, yet cumbersome, tools of algebraic QFT. The usual Hilbert space formulation is sufficient to show the current methods used for entanglement calculations not only in quantum mechanical systems but also in QFT, with the procedures we will lay out in §4.

3.1 General methods in quantum mechanics

Quantum states can be broadly classified as either pure or mixed. *Pure states* reflect no classical uncertainty, and are characterized by $\mathrm{Tr}(\rho^2) = 1$. *Mixed states*, on the other hand, can be thought of as a classical probability distribution over an ensemble of density matrices and satisfy $\mathrm{Tr}(\rho^2) < 1$. For states comprising multiple subsystems, the nature of correlations between the subsystems further distinguishes a quantum state. A bipartite quantum state ρ in a Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ is called *separable* if it can be expressed as a convex combination of product states, i.e.,

$$\rho = \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)}, \tag{7}$$

where $\{p_i\}$ are probabilities with $\sum_i p_i = 1$, and $\rho_A^{(i)}$ and $\rho_B^{(i)}$ are some density matrices for subsystems A and B, respectively. If no such decomposition exists, the state is called *entangled*.

[4] In this work, we will not study any generalizations of the projection-based measurements.

For bipartite pure states, the process of obtaining entanglement is widely known. Let $\rho = |\psi\rangle_{AB}\langle\psi|_{AB} \in \mathcal{H}_{AB}$ be a pure state of the composite system. The reduced density matrix of subsystem A can be obtained by tracing out the degrees of freedom of subsystem B, $\rho_A = \mathrm{Tr}_B(\rho)$. Then, the entanglement entropy is defined as the von Neumann entropy of the reduced density matrix: $S(\rho_A) = -\mathrm{Tr}(\rho_A \log \rho_A)$. For pure states, the entanglement entropy satisfies the symmetry property $S(\rho_A) = S(\rho_B)$.

For mixed states, the entanglement entropy fails to fully capture the entanglement, essentially due to its inability to distinguish between classical and quantum correlations [12]. Mixed states, which involve a combination of pure states, may exhibit both types of correlations, requiring more sophisticated measures of entanglement. With this aim, let us introduce the partial transpose, which will allow us to derive a necessary condition of separability. Given a bipartite (mixed) state $\rho_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$, the *partial transpose* of ρ_{AB} with respect to the subsystem B, denoted as $\rho_{AB}^{T_B}$, is defined by transposing the elements corresponding to B while leaving the elements of A unchanged. In terms of its components,

$$\langle i_A j_B|\rho_{AB}^{T_B}|k_A l_B\rangle = \langle i_A l_B|\rho_{AB}|k_A j_B\rangle. \tag{8}$$

This definition enables us to derive a separability criterion, crucial for detecting entanglement in a quantum state, known as the *Positive Partial Transpose (PPT) criterion*[5]. The proof for this theorem is provided in detail below, since it is reflective of how basic properties of density matrices can lead to very effective criteria of separability.

Theorem (PPT Criterion)**.** Given a bipartite (mixed) state ρ, if at least one eigenvalue of ρ^{T_B} is negative, then the state ρ is entangled.

Proof. Let us show the contrapositive statement. Suppose ρ is separable. Then, it can be written as $\rho = \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)}$, as defined above. Taking the partial transpose with respect to B, we have

$$\rho^{T_B} = (\mathbb{I} \otimes T)\left(\sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)}\right) = \sum_i p_i \,(\mathbb{I} \otimes T)\left(|\psi_i\rangle\langle\psi_i|_A \otimes |\phi_i\rangle\langle\phi_i|_B\right), \tag{9}$$

where T denotes the transpose operation. Simplifying further,

$$\rho^{T_B} = \sum_i p_i |\psi_i\rangle\langle\psi_i|_A \otimes T(|\phi_i\rangle\langle\phi_i|_B) = \sum_i p_i \rho_A^{(i)} \otimes (\rho_B^{(i)})^T. \tag{10}$$

Since transposing a matrix preserves its eigenvalues, $(\rho_B^{(i)})^T$ has the same eigenvalues as $\rho_B^{(i)}$ for all i. Thus, ρ^{T_B} has the same eigenvalues as ρ. In particular, all eigenvalues of ρ^{T_B} are non-negative. $\square$

In the special cases where $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$ or $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^3$, the PPT criterion is both a necessary and sufficient condition for separability [23]. Consequently, in these cases, ρ is entangled if and only if ρ^{T_B} has at least one negative eigenvalue.

We now introduce two different but equivalent measures to quantify the degree of violation of the PPT criterion for a given state. First, the *logarithmic negativity* of a bipartite state $\rho_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ is defined as the logarithm of the trace norm of the partial transpose of ρ_{AB},

$$E_{\mathcal{N}}(\rho_{AB}) = \log_2(\|\rho_{AB}^{T_B}\|_1) = \log_2(\sum_i |\lambda_i|), \quad \lambda_i \in \mathrm{Sp}(\rho_{AB}^{T_B}), \tag{11}$$

where we have used that for a Hermitian operator M, the trace norm simplifies to the sum of the absolute value of its eigenvalues[6]. If the partial transpose of ρ_{AB} is a density matrix (i.e., all its

[5]Also referred to as the Peres-Horodecki criterion, after its discoverers [13].

[6]In general, $\|M\|_1 = \mathrm{Tr}(\sqrt{M^\dagger M})$

eigenvalues are non-negative), the logarithmic negativity is exactly 0, and greater than 0 otherwise. Second, the *negativity* of the state ρ_{AB} is defined as the absolute value of the sum of all negative eigenvalues of the partial transpose of ρ_{AB},

$$\mathcal{N}(\rho_{AB}) = \left| \sum_{\lambda_i < 0} \lambda_i \right|, \tag{12}$$

with $\lambda_i \in \mathrm{Sp}(\rho_{AB}^{T_B})$. The two metrics are equivalent, as seen in the following expression:

$$\mathcal{N}(\rho_{AB}) = \left| \sum_{\lambda_i < 0} \lambda_i \right| = \frac{1}{2} \sum_i (|\lambda_i| - \lambda_i) = \frac{1}{2} (\sum_i |\lambda_i| - \sum_i \lambda_i) = \frac{1}{2} (\|\rho_{AB}^{T_B}\|_1 - 1), \tag{13}$$

where we have used that the sum of the eigenvalues of $\rho_{AB}^{T_B}$ equals 1 because the partial transpose leaves the diagonal terms invariant. Reordering, we finally arrive at

$$E_{\mathcal{N}}(\rho_{AB}) = \log_2(1 + 2\mathcal{N}(\rho_{AB})). \tag{14}$$

In terms of these quantities, we can simply reformulate the PPT criterion as the two equivalent statements:

- If ρ_{AB} is separable, then $E_{\mathcal{N}}(\rho_{AB}) = 0$ (resp. $\mathcal{N}(\rho_{AB}) = 0$).
- If $E_{\mathcal{N}}(\rho_{AB}) > 0$ (resp. $\mathcal{N}(\rho_{AB}) > 0$), then ρ_{AB} is entangled.

Finally, let us consider continuous-variable quantum systems, which are characterized by variables that can vary continuously, in contrast to discrete-variable systems. One example of such systems is the quantum harmonic oscillator, which can be described in terms of position x and momentum p. These variables correspond to operators that act on an infinite-dimensional Hilbert space and satisfy the canonical commutation relation $[\hat{x}, \hat{p}] = i\hbar$. These variables or, in fact, any set of canonically conjugate pairs, can be used to conveniently represent a continuous-variable system. Each of these pairs of canonical variables is referred to as a *mode* (further details are provided in §4.2). This phase-space representation is often realized through the *Wigner function* [24], which is a quasi-probability distribution[7] that combines all phase-space variables into a single representation.

For continuous-variable quantum systems, an important class of states are *Gaussian states*. We can define Gaussian states as quantum states whose Wigner function are Gaussian distributions in phase space. Equivalently, these states are fully characterized by their first moments (mean values of canonical variables) and second moments (covariance matrix of the canonical variables). Basically, Gaussian states reduce the complexity of describing continuous-variable quantum systems which, in general, requires the complete density matrix. Furthermore, they are quite convenient for entanglement computations, as all necessary calculations can be performed using the covariance matrix alone. An example of a Gaussian state is the *vacuum state* (ground state) of a free scalar quantum field in Minkowski spacetime. Intuitively, in a free field theory, the fluctuations in the vacuum state are expected to be Gaussian, as a direct consequence of the harmonic oscillator basis that underpins a free quantum field. In this case, the vacuum state is a zero mean state and all the information about the system is contained in the two-point correlation function, or equivalently, the covariance matrix of the canonical operators.

Gaussian states satisfy additional mathematical properties that make them particularly powerful for entanglement analysis [8]. One of these properties is the direct applicability of separability criteria to Gaussian bipartite systems, as demonstrated in the following theorem.

[7] A quasi-probability distribution is similar to a probability distribution but relaxes one of Kolmogorov's axioms of probability: probability densities need not be non-negative.

Theorem. Consider a bipartite (mixed) Gaussian state where one of the subsystems corresponds to a single mode, that is, its associated Hilbert space is equivalent to that of a single harmonic oscillator. For such systems, the Positive Partial Transpose (PPT) criterion is both a necessary and sufficient condition for separability [25, 26]. In this case, negativity is said to be a faithful quantifier of entanglement, as it provides a complete characterization of separable and entangled states.

3.2 Entanglement computation in QFT

We have shown that standard methods for computing entanglement rely fundamentally on partitioning a given Hilbert space into two subspaces and defining a subsystem in one subspace by tracing out the degrees of freedom in the complementary subspace. In QFT, however, the Hilbert space comprises an infinite number of degrees of freedom. As a result, the trace of operators is typically not well-defined, as the infinite sum of eigenvalues may fail to converge. Without these constructs, the techniques outlined in the previous subsection are inapplicable.

Given a region of spacetime $\mathcal{R} \subset \mathcal{M}$, where $\mathcal{M}$ denotes the spacetime manifold, we consider the local algebra of observables whose support is contained within $\mathcal{R}$. Using the GNS construction, this algebra is represented as a Hilbert subspace, denoted $\mathcal{H}_{\mathcal{R}}$. One might conjecture that the Hilbert space of the QFT factors as $\mathcal{H} = \mathcal{H}_{\mathcal{R}} \otimes \mathcal{H}_{\bar{\mathcal{R}}}$, where $\bar{\mathcal{R}}$ represents the complement of $\mathcal{R}$ in spacetime. However, this factorization does not hold, nor is it possible to associate a density submatrix with either of these subspaces. In addition, both $\mathcal{H}_{\mathcal{R}}$ and $\mathcal{H}_{\bar{\mathcal{R}}}$ have infinitely many degrees of freedom and, as noted earlier, the trace is not well-defined. More formally, as discussed in §2.4, the local algebra associated with any causally connected, finite subregion of spacetime is of type III, and type III algebras cannot contain operators like the trace[8] [27].

Despite the lack of an obvious factorization, there are strong indications that QFTs must be highly entangled between distinct spacetime regions. One comes from lattice field theory (whose algebras are of type I due to the finite-dimensional nature of a lattice): the entanglement entropy of the vacuum state between a region and its complement is ultraviolet (UV) divergent, $S \sim \dfrac{\text{Area}(\partial\mathcal{R})}{\epsilon^{d-2}}$, where d denotes the spacetime dimension and ϵ is the lattice spacing. This UV divergence arises from short-distance correlations in the vacuum. As argued in §2.3, any finite-energy state of a QFT must exhibit the same short-distance entanglement structures as the vacuum state. Hence, the entanglement between complementary spacetime regions in a QFT is formally infinite for every state in the Hilbert space. This observation provides a heuristic explanation as to why the Hilbert space does not factor as subspaces associated with complementary regions. If such a factorization were possible, there would exist separable (i.e., unentangled) states, which contradicts the fact that the entanglement between complementary regions is (formally) infinite. Nonetheless, the most compelling indication of entanglement in QFT is the Reeh-Schlieder theorem. As discussed in §2.4, any state in the Hilbert space can be approximated with arbitrary precision by the action of an operator associated with an arbitrarily small spacetime region on the vacuum state. This implies that any state is entangled regardless of the region of spacetime with which it is associated, providing yet another piece of evidence that the Hilbert space cannot factor in such a manner.

[8] A more general notion of a trace can be defined as a $[0, \infty]$-valued operator within a type III algebra. However, the conventional finite-valued trace, which is computed by projecting onto a finite-dimensional subspace, has no place in an algebra comprising only infinite-dimensional projectors.

4 Spacetime entanglement of a free scalar quantum field

Computing entanglement in a QFT is quite a challenging task, primarily due to the infinite number of degrees of freedom. This difficulty is pronounced when attempting to calculate entanglement between two spatially separated regions of spacetime. The challenge arises because the Hilbert space in QFT does not factor as the tensor product of Hilbert subspaces associated with each region. The fixation with a bipartite factorization of the Hilbert space is not arbitrary. Bipartite states represent the only partition where entanglement calculations are currently feasible. For multipartite pure states, let alone mixed states, it remains an open question with limited progress to date [11].

Two primary approaches have been developed to address this issue. The first, referred to as mode-wise analysis, involves restricting the global algebra of operators to two subalgebras, each with a finite number of degrees of freedom, which are associated with the respective regions of spacetime [8]. By doing so, one can compute the entanglement between the two finite subsets of degrees of freedom. However, in QFT, even a finite region of spacetime contains an infinite number of those and thus, this method inevitably results in the loss of information. Second, the so-called entanglement harvesting, which involves extracting entanglement between regions by interacting with quantum fields by means of external probes [14, 15, 16].

Previous studies employing the first method have aimed to extract bimodal entanglement between pairs of modes corresponding to two space-like separated regions [8]. These studies concluded that bimodal entanglement is elusive and demonstrated that effective detection of bimodal entanglement requires careful fine-tuning of the definitions of the subregions and the associated modes. However, in [9], the authors utilized the entanglement harvesting protocol in the same subregions as in [8], and observed non-zero entanglement, thereby presenting an apparent paradox. As it turns out, the multimode nature of spacetime entanglement plays a pivotal role in resolving this paradox, offering new insights into the structure of quantum correlations in QFT.

4.1 Algebraic description of a free real scalar field in Minkowski spacetime

Let $(\mathcal{M}, g)$ be Minkowski spacetime. Following the algebraic formalism, we define the algebra of observables $\mathcal{A}(\mathcal{M})$ associated with a collection of subalgebras $\mathcal{A}(\mathcal{R})$, where $\mathcal{R} \subset \mathcal{M}$. These subalgebras represent the degrees of freedom supported within the region $\mathcal{R}$.

Consider a real scalar free field, whose dynamics are determined by the Klein-Gordon equation, $(\Box + m^2)\hat{\phi} = 0$, where $\Box$ is the d'Alembertian operator and m is the mass of the scalar field. The local algebras $\mathcal{A}(\mathcal{R})$ are then generated by the identity operator and the smeared operators $\hat{\phi}(f)$ as defined in Eq. (1), where f has a compact support within $\mathcal{R}$. We now introduce the adjunction operation $*$ and impose that the field operator $\hat{\phi}$ is self-adjoint, $\hat{\phi}(f)^* = \hat{\phi}(f)$, as we are dealing with a real scalar field.

When considering the field operator as a distribution, the Klein-Gordon equation for the field operator can be expressed in a more proper way as $\hat{\phi}[(\Box + m^2)f] = 0$, for any smooth, compactly supported f. Next, we impose the following commutation relations:

$$[\hat{\phi}(f), \hat{\phi}(g)] = iE(f, g)\mathbb{I}, \tag{15}$$

where $E(f, g)$ is defined as

$$E(f, g) = \int d^4x d^4x' \, E(x, x') f(x) g(x'), \tag{16}$$

and $E(x, x')$ represents the *causal propagator*—the difference between the retarded and advanced Green's functions—of the Klein-Gordon equation in Minkowski spacetime. Eq. (15) is equivalent to the standard equal-time canonical commutation relations but is formulated in a covariant manner that does not require the specification of "equal time". Since the causal propagator $E(x, x')$ vanishes whenever x, x' are space-like separated, the canonical commutation relations imply that operators of subalgebras associated with space-like separated regions commute, that is, microcausality is verified (Eq. (4)).

We may define a preferred notion of vacuum state by leveraging the symmetries of our spacetime. As mentioned in §2, by imposing Poincaré invariance and scale invariance (for massless fields) to an algebraic state, the functional form of the two-point correlation function is uniquely fixed to a Hadamard form (see Eq. (6)) with $U =$ const., $V = H_\omega = 0$, and the Synge world function simply given by $\sigma(x, y) = (x - y)^2/2$. Additionally, since a Klein-Gordon field is free, its vacuum state is guaranteed to be a (zero-mean) Gaussian state. Consequently, specifying the two-point correlation function fully determines the vacuum state.

4.2 Mode-wise analysis

The first approach to calculating entanglement between two distinct regions of spacetime in QFT involves a mode-wise analysis [8]. The procedure is as follows. Consider a set of test functions, compactly supported smooth functions on Minkowski spacetime, $\{f_1, g_1, \ldots, f_N, g_N\} \in \mathcal{C}_0^\infty(\mathcal{M})$, such that

$$E(f_j, f_k) = E(g_j, g_k) = 0, \quad E(f_j, g_k) = \delta_{jk}, \tag{17}$$

where E is defined in Eq. (16). The two sets of N smeared field operators $\{\hat{\phi}(f_1), \ldots, \hat{\phi}(f_N)\}$ and $\{\hat{\phi}(g_1), \ldots, \hat{\phi}(g_N)\}$ satisfy canonical commutation relations: $[\hat{\phi}(f_j), \hat{\phi}(g_k)] = i\delta_{jk}\hat{\mathbb{I}}$. These operators define a quantum system comprising N (bosonic) degrees of freedom, where each degree of freedom is associated with the subalgebra generated by the canonically conjugate pair $(\hat{\phi}(f_i), \hat{\phi}(g_i))$, which is referred to as a *field mode*.

When we select a finite subset of the infinitely many degrees of freedom of a quantum field, the study of entanglement reduces to analyzing the entanglement among a finite set of quantum harmonic oscillators. While the entanglement between a finite number of modes does not encompass the full entanglement structure of the QFT, it provides a meaningful approximation. For example, let A, B be two disjoint subregions of a spacelike Cauchy hypersurface. One can study the bipartite entanglement between the two subregions by constructing two sets of modes whose associated smearing functions are supported entirely within A and B, respectively. Such modes can be defined, for instance, in terms of field and momentum operators of the form:

$$\hat{\Phi}(F) = \int_\Sigma \mathrm{d}\Sigma F \hat{\phi}, \quad \hat{\Pi}(F) = \int_\Sigma \mathrm{d}\Sigma F n^\mu \nabla_\mu \hat{\phi}, \tag{18}$$

where F is normalized to satisfy

$$\int_\Sigma \mathrm{d}\Sigma F^2 = 1, \tag{19}$$

so that $[\hat{\Phi}(F), \hat{\Pi}(F)] = i\hat{\mathbb{I}}$. Hence $(\hat{\Phi}(F), \hat{\Pi}(F))$ defines a mode of the field. If the support of F is included in A (resp. B), then the mode belongs to the local algebra of A (resp. B) or, equivalently, to the local algebra of the domain of dependence of A (resp. B).

The Reeh-Schlieder theorem states that the subregions A and B are sufficiently entangled that the action of any operator of the algebra of A on the vacuum state $|\Omega\rangle$ can be approximated with arbitrary precision by the action of operators from the algebra of B, and vice versa. This result might initially suggest that entanglement is "ubiquitous" in QFT, implying that we could observe entanglement between virtually any pair of modes associated with regions A and B. However, with a more careful interpretation, the theorem merely guarantees that, for a given mode of region A, there exists a subset of modes in region B with which it is entangled. Consequently, the applicability of the mode-wise analysis for retrieving the entanglement structure of quantum fields is limited by the extent to which such modes can capture the entanglement present in the field. Then, one could expect that some fine-tuning is required to effectively detect entanglement between specific pairs of modes.

This belief was examined in detail in [8] for the case of the vacuum of a massless real scalar quantum field in a (D+1) dimensional Minkowski spacetime. In this case, the vacuum is a zero mean Gaussian state that satisfies the conditions of the Reeh-Schlieder theorem. Consider two spacelike separated spherical regions, each with radius R, within the same time slice and centered at x_A and x_B, respectively. We can select modes of the form $(\hat{\Phi}(F), \hat{\Pi}(F))$ with the definitions in Eq. (18), using $F_I(x) = F(x - x_I)$ for $I \in \{A, B\}$. Let F belong to the compactly supported family of smearing functions,

$$F^{(\delta)}(x) = A_\delta \left(1 - \frac{|x|^2}{R^2}\right)^\delta \Theta\left(1 - \frac{|x|}{R}\right), \tag{20}$$

where $\delta \geq 1$, Θ is the Heaviside function, and

$$A_\delta = \sqrt{\frac{\Gamma(1 + 2\delta + D/2)}{\pi^{D/2} R^D \Gamma(1 + \delta)}} \tag{21}$$

is the normalization constant that ensures that $F^{(\delta)}$ satisfies the normalization condition Eq. (19). It is worth remarking that this family of smearing functions is not pathological. These functions are compactly supported within a sphere of radius R, belong to a differentiability class $\delta \geq 1$, are spherically symmetric, and peak at the origin. Such functions are similar to those typically used to model, for instance, the spatial smearing of a compactly supported particle detector [17].

The explicit derivation of the logarithmic negativity, $E_\mathcal{N}$, of such a system is omitted in this text, but can be found in [8] and [9]. Nonetheless, as explained in §3.1, it is fairly straightforward to compute, if the properties of Gaussian states are considered. Specifically, we can make all calculations through the covariance matrix of the modes resulting in a simple form for the negativity. Then, for $|x_a - x_b| \geq 2R$ (so long as the two regions do not overlap[9]),

$$E_\mathcal{N} = 0, \quad \forall \delta \geq 1,\ D \geq 2, \tag{22}$$

where $E_\mathcal{N}$ is the logarithmic negativity between a pair of modes of regions A and B. Notice that in this case the logarithmic negativity is a faithful entanglement quantifier (see end of §3.1), and thus $E_\mathcal{N} = 0$ means that the joint state of the selected modes of regions A and B is separable.

Stressing the preceding discussion, we acknowledged that a certain degree of fine-tuning should be necessary to observe entanglement between two modes. In other words, the Reeh-Schlieder theorem does not necessarily imply the presence of entanglement between any arbitrary pair of modes. What we have found is perhaps more striking: no pair of modes is entangled between the regions A and B. As it turns out, the level of fine-tuning required to detect bimodal entanglement is significantly more stringent than one might initially expect. However, this does not mean that the task is impossible. As demonstrated in [28], there are precedents for detecting bimodal entanglement. In fact, they showed that the spatial smearing functions considered in this analysis fail to capture the entanglement inherent in the field. By selecting a different, much more tailored family of spatial distributions, they were indeed able to detect bimodal entanglement between spacelike separated regions. We conclude, therefore, that a careful choice of smearing functions is crucial for understanding the entanglement structure of the modes under consideration.

[9]If the two regions overlap, the discussion becomes ill-defined, as the focus is on identifying entanglement between spacelike-separated regions, where its presence is counterintuitive and challenges classical intuitions about locality.

4.3 Entanglement harvesting

Entanglement harvesting is a protocol in relativistic quantum information that can be used to study the entanglement structure of a quantum field [14]. It allows the extraction of entanglement between two localized regions of the field using probes that couple to the field in these regions. Analyzing the origin of the entanglement acquired by the probes provides insights into the available entanglement within the regions of interest. This section reviews the framework of entanglement harvesting and examines the conditions necessary for probes to extract entanglement from a quantum field.

4.3.1 Particle detectors coupled to a field

Probes used in entanglement harvesting are typically modeled as particle detectors, i.e., localized quantum systems coupled to a quantum field. There exist a few models of particle detectors, but in this work we will only use those introduced by Unruh [17] and DeWitt [18].

Consider two-level UDW detectors interacting with a real scalar quantum field $\hat{\phi}(t,x)$, in (3+1) dimensional Minkowski spacetime. The detector's internal quantum states reside in a Hilbert space $\mathcal{H}_D \cong \mathbb{C}^2$. The detector is assumed to follow an inertial trajectory $z(t) = (t, x_0)$ in spacetime and its internal dynamics are determined by the free Hamiltonian $\hat{H}_D = \Omega\hat{\sigma}^+\hat{\sigma}^-$, where $\Omega > 0$ is the energy gap between the ground state $|g\rangle$ and excited state $|e\rangle$ of the qubit. The operators $\hat{\sigma}^+$ and $\hat{\sigma}^-$ act as ladder operators, satisfying $\hat{\sigma}^+|g\rangle = |e\rangle$ and $\hat{\sigma}^-|e\rangle = |g\rangle$, with all other applications yielding zero.

The interaction occurs in a localized region of spacetime, described by a compactly supported spacetime smearing function, which we asume factors as $\chi(t)F(x)$, where the switching function $\chi(t)$ controls the interaction duration, and $F(x)$ determines its spatial shape. The detector couples linearly to the quantum field via its monopole moment $\hat{\mu}(t) = e^{i\Omega t}\hat{\sigma}^+ + e^{-i\Omega t}\hat{\sigma}^-$ in the interaction picture. Then, we may write the Hamiltonian of the interaction of the detector and the field as

$$\hat{H}_{\text{int}}(t) = \lambda\chi(t)\hat{\mu}(t)\int d^3x\, F(x)\hat{\phi}(t,x), \tag{23}$$

with λ a dimensionless coupling constant.

To compute the final state of a detector after its interaction with the field, the initial state of the detector-field system is assumed to be unentangled, $\hat{\rho}_D \otimes \hat{\rho}_\phi$, where $\hat{\rho}_D$ represents the detector's initial state and $\hat{\rho}_\phi$ denotes the initial state of the field. The system evolves under the unitary operator $\hat{U}_{\text{int}} = \mathcal{T}\exp\left(-i\int dt\hat{H}_{\text{int}}(t)\right)$, where $\mathcal{T}$ is the time-ordering operator, and the compact support of the switching function $\chi(t)$ makes the integration interval finite. The detector's final state, $\hat{\rho}_D$, is obtained by tracing over the field's degrees of freedom:

$$\hat{\rho}_D = \text{Tr}_\phi\big(\hat{U}_{\text{int}}(\hat{\rho}_D \otimes \hat{\rho}_\phi)\hat{U}^\dagger_{\text{int}}\big). \tag{24}$$

4.3.2 Entanglement harvesting using UDW detectors

In the entanglement harvesting protocol, two detectors, A and B, with energy gaps Ω_A and Ω_B, follow inertial trajectories $z_A(t) = (t, x_A)$ and $z_B(t) = (t, x_B)$. Their spacetime smearing functions are given by $\chi_A(t)F_A(x)$ and $\chi_B(t)F_B(x)$. The interaction Hamiltonian for the detectors-field coupling is expressed as

$$\hat{H}_{\text{int}}(t) = \lambda(\chi_A(t)\hat{\mu}_A(t)\hat{\Phi}_A(t) + \chi_B(t)\hat{\mu}_B(t)\hat{\Phi}_B(t)), \tag{25}$$

where $\hat{\Phi}_A(t)$ and $\hat{\Phi}_B(t)$ are the smeared field operators probed by each detector at time t, defined by the smearing functions $F_A(x), F_B(x)$:

$$\hat{\Phi}_I(t) = \hat{\Phi}(t, F_I) = \int d^3x\, F_I(x)\hat{\phi}(t,x), \quad I \in \{A, B\}. \tag{26}$$

Assuming the initial state of the system is $\hat{\rho}_{AB}\otimes\hat{\rho}_\phi$, with $\hat{\rho}_{AB}=|g_A,g_B\rangle\langle g_A,g_B|$ (both detectors initially in their ground states) and $\hat{\rho}_\phi$ is a zero mean Gaussian state of the field, the detectors' final state at leading order in the coupling constant λ is given by ([9])

$$\hat{\rho}_{AB}=\begin{pmatrix}1-\mathcal{L}_{AA}-\mathcal{L}_{BB} & 0 & 0 & \mathcal{M}^* \\ 0 & \mathcal{L}_{BB} & \mathcal{L}_{AB}^* & 0 \\ 0 & \mathcal{L}_{AB} & \mathcal{L}_{AA} & 0 \\ \mathcal{M} & 0 & 0 & 0\end{pmatrix}+\mathcal{O}(\lambda^4), \tag{27}$$

in the basis $\{|g_A,g_B\rangle,|g_A,e_B\rangle,|e_A,g_B\rangle,|e_A,e_B\rangle\}$. The coefficients $\mathcal{L}_{IJ}$ and $\mathcal{M}$ are:

$$\mathcal{L}_{IJ}=\lambda^2\int \mathrm{d}t\mathrm{d}t'\chi_I(t)\chi_J(t')e^{-\mathrm{i}(\Omega_I t-\Omega_J t')}\langle\hat{\Phi}_I(t)\hat{\Phi}_J(t')\rangle_{\hat{\rho}_\phi},\quad I,J\in\{A,B\}, \tag{28}$$

$$\mathcal{M}=-\lambda^2\int \mathrm{d}t\mathrm{d}t'\chi_A(t)\chi_B(t')e^{\mathrm{i}(\Omega_A t+\Omega_B t')}\langle\mathcal{T}\hat{\Phi}_A(t)\hat{\Phi}_B(t')\rangle_{\hat{\rho}_\phi}. \tag{29}$$

The state $\hat{\rho}_{AB}$ in Eq. (27) encodes the changes in the detectors' state after their interaction with the field, assuming weak coupling in localized spacetime regions. The terms $\mathcal{L}_{AA}$ and $\mathcal{L}_{BB}$ represent the leading-order excitation probabilities for detectors A and B, respectively. These local contributions contain information about the entanglement between the field and each individual detector, whereas the cross terms $\mathcal{L}_{AB}$ and $\mathcal{M}$ characterize the correlations acquired between the detectors due to the interaction.

Given the final state, we may wonder whether this state has ended up entangled. For a bipartite qubit system ($\mathbb{C}^2\otimes\mathbb{C}^2$), the logarithmic negativity is a faithful quantifier of entanglement as we saw in §3.1. At leading order in λ, the partial transpose of the density matrix in Eq. (27) has only one potentially negative eigenvalue,

$$E=-\sqrt{|\mathcal{M}|^2-\left(\frac{\mathcal{L}_{AA}-\mathcal{L}_{BB}}{2}\right)^2}+\frac{\mathcal{L}_{AA}+\mathcal{L}_{BB}}{2}. \tag{30}$$

The logarithmic negativity ($E_\mathcal{N}=2\mathcal{N}+\mathcal{O}(\lambda^4)$) can then be expressed as,

$$E_\mathcal{N}(\hat{\rho}_{AB})=\max(0,-2E)+\mathcal{O}(\lambda^4). \tag{31}$$

From Eq. (30), we see that the detectors become entangled if the $\mathcal{M}$ term is sufficiently large compared to the average of $\mathcal{L}_{AA}$ and $\mathcal{L}_{BB}$. At leading order, the negativity reflects a competition between the individual excitation probabilities, which are influenced by local noise, and the $\mathcal{M}$ term. The $\mathcal{M}$ term, in turn, involves an integral over the (time-ordered) correlations of the field operators $\hat{\Phi}_A(t)$ and $\hat{\Phi}_B(t)$, smeared with the switching functions $\chi_A(t)$ and $\chi_B(t)$, respectively.

Consider the smearing function $F_I(x)=F^{(2)}(x-x_I)$, with $F^{(2)}(x-x_I)$ defined in Eq. (20), and the switching function given by,

$$\chi(t)=\begin{cases}\left(1-\frac{4t^2}{T^2}\right)^{5/2}, & t\in\left[-\frac{T}{2},\frac{T}{2}\right]\\ 0, & \text{otherwise}\end{cases} \tag{32}$$

which is a C^2 function with compact support in the interval $[-T/2,T/2]$. With the choice $T=40R$, and $|x_A-x_B|=T+2R$, the detectors remain spacelike separated during the entire duration of the interaction with the field. In Figure 1 we show the logarithmic negativity between the two UDW detectors, according to Eq. (30). We observe that they can, indeed, evolve into an entangled state, so long that the product of the energy gap $\Omega=\Omega_1=\Omega_2$ and the radius of the smearing function R, is inside a certain interval.

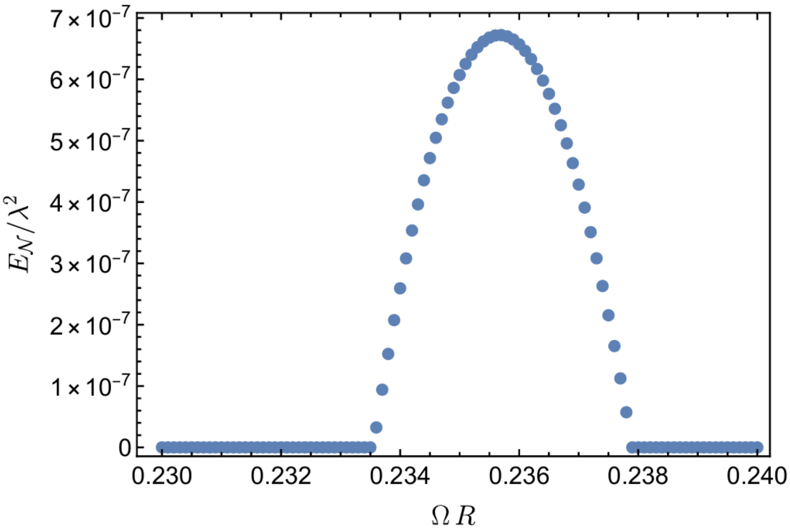

Figure 1: From Figure 1 in [9]. Logarithmic negativity of two UDW particle detectors as a function of the energy gap Ω and the radius of the smearing function, R. Notice that the optimal combination is $\Omega R \simeq 0.236$.

4.3.3 When can two detectors harvest entanglement from a quantum field?

As we have seen, the logarithmic negativity can be non-zero, indicating that the two detectors may become entangled via their interaction with the quantum field. There are two potential mechanisms whereby this entanglement could arise. First, the detectors might communicate through the field. Second, the two regions of the field could have been entangled initially, with the detectors harvesting a portion of this pre-existing entanglement. For this result to hold meaningful implications, we need to ensure that it is the latter mechanism that is responsible for the observed entanglement. Specifically, we must confirm that the detectors cannot communicate via the field, concluding that the harvested entanglement originates solely from the intrinsic entanglement structure of the field.

Consider detectors localized in spacelike-separated regions. In such a scenario, the supports of the smearing functions corresponding to the detectors are causally disconnected. Then, the unitary time evolution operator can factor as $\hat{U}_{\text{int}} = \hat{U}_A\hat{U}_B = \hat{U}_B\hat{U}_A$, where $\hat{U}_A$ acts exclusively on detector A and the field, while $\hat{U}_B$ acts exclusively on detector B and the field. This factorization is valid due to the microcausality condition, which ensures that field operators smeared over regions A and B commute, as well as the commutativity of observables localized in distinct regions. Under this factorization, $\hat{U}_{\text{int}}$ cannot couple detectors A and B during their time evolution but it allows the field modes interacting with the detectors to be entangled, enabling the transfer of entanglement between the field and the detectors.

There are cases, however, in which $\hat{U}_{\text{int}}$ guarantees that the final state is separable, eliminating any initial entanglement present in the field. One notable example are delta-coupled detectors [29], where the switching function is proportional to a Dirac delta function $\delta(t - t_0)$. In this case, the smearing function specifies the field mode to which each detector couples, but the interaction occurs so abruptly that noise effectively destroys any pre-existing entanglement (see Eq. (30)). However, it is possible to harvest entanglement if the coupling is represented by a linear combination of a sufficiently large number of delta couplings. Not only that, any continuous interaction can be approximated by such a linear combination [30]. This can be intuitively understood by considering the time resolution of the detector. A detector, characterized by its energy gap, has a finite temporal resolution, which prevents it from distinguishing between a continuous interaction and a sufficiently large (albeit finite) number of closely spaced delta couplings. As a result, a detector cannot resolve whether the coupling arises from a continuous interaction or a discrete sequence of delta-coupled interactions. This assumption holds under the condition that the detector's resolution is subexponential, which is to be expected for any physically realizable detector.

4.4 The paradox of spacetime entanglement

Let us summarize what we know so far. In (3+1)-Minkowski spacetime, two field modes defined by smearing functions over spacelike-separated, spherically symmetric regions, are not entangled. In contrast, two detectors linearly coupled to the quantum field in the same spacelike-separated regions, and defined by the same smearing functions, do exhibit entanglement.

Selecting the spatial support of a detector through the smearing function F necessarily implies that the detector couples to the field modes defined by the same smearing function. Provided that we found no entanglement between any pair of modes, an apparent contradiction arises: where does the entanglement harvested by the detectors come from? Unlike the mode-wise analysis performed to compute bimodal entanglement, the detectors interact with an infinite number of field degrees of freedom. For UDW detectors, the field modes coupled to each detector are defined by smearing the field operators against the function F at each instant t. Thus, the detectors interact with a one-parameter family of field modes, $(\hat{\Phi}(t,F),\hat{\Pi}(t,F))$, throughout their time evolution. These modes are not, in general, independent. Specifically, if the support of F at times t and t' are causally connected, the corresponding modes typically do not commute. By virtue of the discussion in the last paragraph of §4.3.3, the coupling can be approximated by a finite sum of instantaneous couplings, which in turn define individual field modes. One might initially argue that the entanglement arises from bimodal entanglement between field modes at different times, but this belief was disproven in [8]. Also, the non-relativistic nature of the detectors has raised doubts as to how reliable they are, as entanglement can emerge as an artifact of the model. Nevertheless, it has been shown that the "classical" approximation of the detectors provides a lower bound to the entanglement harvested by fully relativistic detectors [31]. It can also be understood through the finite resolution of the detectors. While in a covariant description, time could be defined by the time coordinate of the inertial reference frame of the detector, classical detectors break covariance. Hence, there may exist observers in inertial reference frames where UV energies appear as infrared energies due to the lack of a preferred notion of time. To avoid it, one can restrict themselves only to those observers who measure the same harvested entanglement, as detectors are limited by their finite energy resolution.

The only plausible explanation is then the following: the entanglement harvested by the detectors must originate from the initial entanglement present in the quantum field and it must be of multimode nature, meaning it is distributed across several field modes.

4.5 The multimode nature of spacetime entanglement in QFT

To further support this claim, we proceed with a mode-wise analysis similar to that of §4.2, to confirm that the entanglement between field modes is indeed multimodal.

Computing the entanglement between two sets of a finite number of degrees of freedom in a Gaussian state is a relatively easy task. However, computing the entanglement between two one-parameter families of field modes is computationally prohibitive. To address this, we select N modes from each family. Fortunately, this will be sufficient to detect entanglement, though the absence of detected entanglement would not have necessarily implied its nonexistence, as additional modes may have been required. We define the field modes to which each detector couples by considering an interaction consisting of N delta couplings, as previously argued. These modes are time-dependent,

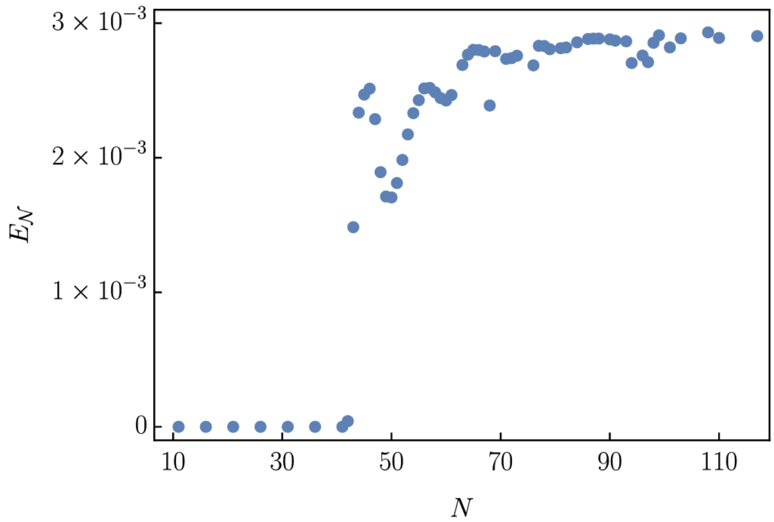

Figure 2: From Figure 3 in [9]. Logarithmic negativity between two sets of N modes associated with two detectors A and B as a function of N. The detectors are located in two spacelike-separated regions of Minkowski spacetime.

and the commutation relations no longer obey the equal-time canonical commutation relations. For each region, it is possible to identify a set of modes that commute with the operators of the other region, as they are causally disconnected. However, the commutation relations among the operators within the same region are not guaranteed to be canonical. Hence, we apply a "symplectic Gram-Schmidt" algorithm, using the commutators as the inner product [9]. This algorithm returns a set of N independent modes for each region and thus, we can employ a similar analysis as in §4.2. Details of the calculation of the logarithmic negativity between N modes of region A and N modes of region B are omitted here, although it is (again) drastically simplified thanks to the fact that the vacuum state of a free scalar field in Minkowski spacetime is a zero-mean Gaussian state, with all information encoded in the covariance matrix of the modes. The results are summarized in Figure 2.

Even though no single mode from region A is entangled with any single mode from region B, the presence of entanglement between field modes associated with causally disconnected regions of spacetime is readily apparent, and it is inherently of multimode nature. Furthermore, Figure 2 reveals a threshold to the number of modes required to detect entanglement, 42 in our scenario. Finally, within the computational constraints, the logarithmic negativity appears to plateau. This observation suggests that the entanglement between the field modes coupling to the detectors might be finite and be well described by a sufficiently large number of modes.

5 Conclusion

In this work, primarily based on the research conducted in [8, 9], we have explored the intricate structure of spacetime entanglement within the framework of QFT. Making use of the algebraic formalism and quantum information techniques, applied to the vacuum state of a free scalar field in Minkowski spacetime, we have seen that entanglement in a QFT exhibits a multimode nature, as suggested by the Reeh-Schlieder theorem. This pervasive entanglement underscores the complex and non-local character of quantum correlations in spacetime.

While bimodal entanglement proves elusive and highly dependent on precise spatial configurations, multimode entanglement offers a robust avenue for capturing the non-locality in QFT. By employing techniques such as entanglement harvesting or a mode-wise analysis, we have highlighted the challenges and nuances associated with quantifying entanglement present even between spacelike-separated regions. Specifically, the logarithmic negativity becomes nonzero only when contributions from multiple modes are included.

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