Large-scale convex optimisation

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Problem 1. An alternative approach to the shortest-path problem in $\mathbb{R}^2$ is to see a curve as the graph of a function $f\colon \mathbb{R} \to \mathbb{R}$. Given two points $(a_1, a_2), (b_1, b_2)$ with $a_1 < b_1$, we are looking for functions $f\colon [a_1, b_1] \to \mathbb{R}$ with $f(a_1) = a_2$ and $f(b_1) = b_2$. Then the arc-length of the function is given by

$$L(f) = \int_{a_1}^{b_1} \sqrt{1 + f'(x)^2} \mathrm{d}x$$

(here, f' is the first derivative of f). Discretise this form of the shortest-path problem by restricting f to a finite-dimensional subspace of piecewise linear functions.

Problem 2. For a set $S \subseteq \mathcal{H}$, the *closure* and the *interior* are defined by

$$\begin{aligned} \operatorname{cl} S &= \bigl\{x \in \mathcal{H} \bigm| \forall \varepsilon > 0 \exists x' \in S : \bigl\|x - x'\bigr\| < \varepsilon\bigr\}, \\ \operatorname{int} S &= \bigl\{x \in S \bigm| \exists \varepsilon > 0 : \bigl\{x' \in \mathcal{H} \bigm| \bigl\|x - x'\bigr\| < \varepsilon\bigr\} \subseteq S\bigr\}, \end{aligned}$$

respectively. (You can say that the closure of S is the set of all limit points of sequences in S and that the interior is the set of points in S such that S also contains a ball around them.) Show that the closure and the interior of a convex set are convex.

Hint: You might want to draw pictures to visualise the situation.

Problem 3. Let $C \subseteq \mathcal{H}$ be convex, let $x \in \operatorname{int} C$ and $y \in \operatorname{cl} C$. Show that $(1 - \lambda)x + \lambda y \in \operatorname{int} C$ for all $0 \leq \lambda < 1$.

Problem 4. The *convex hull* $\operatorname{conv} S$ of a set $S \subseteq \mathcal{H}$ is the smallest convex set which contains S, i.e.,

$$\operatorname{conv} S = \bigcap \{C \subseteq \mathcal{H} \mid C \text{ is convex}, C \supseteq S\}.$$

Show that $\operatorname{conv} S$ is the set of all convex combinations of S, i.e.,

$$\operatorname{conv} S = \left\{ \sum_{i=1}^{n} \lambda_i x_i \,\middle|\, n \geq 1, \forall i = 1, \dots, n : \lambda_i \geq 0, x_i \in S, \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

What is the convex hull of the set $S = (\mathbb{R} \times \{1\}) \cup \{(0, 0)\} \subseteq \mathbb{R}^2$?

Problem 5. Show *Jensen's inequality:* Let $f\colon \mathcal{H} \to \overline{\mathbb{R}}$ be a convex function, let $n \geq 1$, let $x_1, \dots, x_n \in \mathcal{H}$ and $\lambda_1, \dots, \lambda_n \geq 0$ such that $\sum_{i=1}^{n} \lambda_i = 1$. Then

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i f(x_i).$$

Hint: Use induction.

Problem 6. Let $f\colon \mathcal{H} \to \overline{\mathbb{R}}$ be a function. The *convex envelope* $\operatorname{conv} f\colon \mathcal{H} \to \overline{\mathbb{R}}$ is the pointwise supremum of all convex functions that are smaller than f:

$$(\operatorname{conv} f)(x) = \sup\left\{g(x) \,\middle|\, g\colon \mathcal{H} \to \overline{\mathbb{R}} \text{ is convex}, g(y) \le f(y) \text{ for all } y \in \mathcal{H}\right\}.$$

Show that, for all $x \in \mathcal{H}$,

$$(\operatorname{conv} f)(x) = \inf\left\{\sum_{i=1}^{n} \lambda_i f(x_i) \,\middle|\, n \ge 1, \forall i = 1, \dots, n : \lambda_i \ge 0, x_i \in \mathcal{H}, \sum_{i=1}^{n} \lambda_i = 1, \sum_{i=1}^{n} \lambda_i x_i = x\right\}.$$

What is the convex envelope of the function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \neq 0? \end{cases}$$