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Exercise 2

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**Problem 1.** Show that the function  $f: \mathbb{R}^2 \to \overline{\mathbb{R}}$  given by

$$f(x) = \begin{cases} \frac{x_1^2}{x_2} & \text{if } x_2 > 0, \\ +\infty & \text{if } x_2 \le 0 \end{cases}$$

is convex. Is it lower semicontinuous?

**Problem 2.** Show that, if  $\mathcal{H}$  and  $\mathcal{G}$  are finite-dimensional inner-product spaces,  $f: \mathcal{H} \to \overline{\mathbb{R}}$  is convex and  $L: \mathcal{H} \to \mathcal{G}$  is linear, the *infimal postcomposition*  $L \triangleright f: \mathcal{G} \to \mathcal{H}$  defined by

$$(L \triangleright f)(x) = \inf \{ f(x) \mid y \in \mathcal{H}, Ly = x \},\$$

is convex.

Hint: Make sure that your proof also works if any of the function values is  $+\infty$  or  $-\infty$ . The strict epigraph might be the most convenient way to do so.

**Problem 3.** For two convex functions  $f_1, f_2 \colon \mathcal{H} \to \overline{\mathbb{R}}$ , show that their infimal convolution  $f_1 \Box f_2 \colon \mathcal{H} \to \overline{\mathbb{R}}$ , which is given by

$$(f_1 \square f_2)(x) = \inf \{ f_1(y) + f_2(x-y) \mid y \in \mathcal{H} \},\$$

is convex. What is the infimal convolution of two indicator functions? What is the infimal convolution of an indicator function and the norm?

Hint: It is possible to use problem 2 here.

**Problem 4.** Let  $f: \mathcal{H} \to \overline{\mathbb{R}}$  be a function which is convex and lower semicontinuous, but not proper. Show that  $f(x) \in \{\pm \infty\}$  for all  $x \in \mathcal{H}$ , i.e., f does not take any real values.

**Problem 5.** Show that the lower level sets of a convex function are convex. Give an example of a non-convex function whose lower level sets are all convex.