Large-scale convex optimisation

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Problem 1. You are given a noisy greyscale image y of size $M \times N$ pixels (denoted by $y_{i,j}$, $i = 1, \dots, M$, $j = 1, \dots, N$). We want to find a less noisy image x of the same size by solving the optimisation problem

$$\begin{aligned}
&\min_x \frac{\lambda}{2} \sum_{i=1}^{M} \sum_{j=1}^{M} |x_{i,j} - y_{i,j}|^2 + \sum_{i=1}^{M} \sum_{j=1}^{N-1} |x_{i,j} - x_{i,j+1}| + \sum_{i=1}^{M-1} \sum_{j=1}^{N} |x_{i,j} - x_{i+1,j}| \\
&= \min_x \frac{\lambda}{2} \|x - y\|_2^2 + \|Lx\|_1 \qquad (1)
\end{aligned}$$

for some $\lambda > 0$ and an appropriate linear operator L.

It turns out that it is not straightforward to apply the forward-backward method to this problem, but that (see the coming videos on duality or [1] for more details) that the problem

$$\min_v \frac{1}{2\lambda} \|L^* v - \lambda y\|_2^2 \qquad \text{s.t. } \|v\|_\infty \leq 1 \qquad (2)$$

can be used to solve eq. (1). If $\bar{v}$ is a solution of eq. (2), then $\bar{x} = y - \frac{1}{\lambda} L^* \bar{v}$ is a solution of eq. (1). (For the definition of $\|\cdot\|_1$ and $\|\cdot\|_\infty$, see Exercise 6.)

Determine the necessary objects (in particular L and L^* and a Lipschitz constant of the gradient of the smooth part of (2)) to solve eq. (2) with the help of the forward-backward and the accelerated forward-backward algorithm as well as the projected subgradient method. Plot the values of the objective function of eqs. (1) and (2). Determine a value for λ such that the resulting image makes a reasonable visual impression. A bigger λ gives more weight to the term which enforces x to be close to y, and a smaller λ gives more weight to the term which enforces x to be less noisy.

References

[1] Amir Beck and Marc Teboulle, *A fast dual proximal gradient algorithm for convex minimization and applications*, Operations Research Letters 42(1), pp. 1–6, 1987, `https://doi.org/10.1016/j.orl.2013.10.007`.