Large-scale convex optimisation

Sebastian Banert **Exercise 5** 11 December 2020

Problem 1. Let $f\colon \mathcal{H} \to \overline{\mathbb{R}}$ be convex, and let $x \in \operatorname{dom} f$. Show that

$$\partial f(x) = \left\{ a \in \mathcal{H} \,\middle|\, \forall d \in \mathcal{H} : f'(x; d) \geq \langle a, d \rangle \right\}.$$

Problem 2. Let $f\colon \mathcal{H} \to \overline{\mathbb{R}}$ be proper and convex, let $\bar{x} \in \operatorname{int}(\operatorname{dom} f)$ and $d \in \mathcal{H}$. Show that

$$f'(\bar{x}; d) = \max \{ \langle a, d \rangle \mid a \in \partial f(\bar{x}) \}.$$

Hint: The "max" here means that there is an $a \in \partial f(\bar{x})$ such that $f'(\bar{x}; d) = \langle a, d \rangle$. This problem needs more creativity than usual. Show and use that $f'(\bar{x}; d)$ is a convex function in the variable d, and that it only takes real values. You know that the subdifferential of such a function is non-empty everywhere …

Problem 3. Let $C \subset \mathcal{H}$ be open and convex, and let $f\colon C \to \mathbb{R}$ be convex and differentiable. Show that $\partial f(\bar{x}) = \{\nabla f(\bar{x})\}$ for all $\bar{x} \in C$.

Hint: Use problem 2.

Note: From this exercise, you should see how the subdifferential and the directional derivative relate to each other. Just like the derivative is a linearisation of the function at a point, the subdifferential can be seen as a *sublinearisation*: It determines (by the formula in problem 2) the directional derivative, which is a sublinear function in the direction variable d and describes the local behaviour of the function around the point in question.