Large-scale convex optimisation

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Problem 1. Let $f\colon \mathcal{H} \to \overline{\mathbb{R}}$ be a function (not necessarily convex or lower semicontinuous), and let $x \in \mathcal{H}$. Show that $\partial f(x)$ is closed and convex.

Problem 2. Show that, for a function $f\colon \mathcal{G} \to \overline{\mathbb{R}}$ and a linear operator $L\colon \mathcal{H} \to \mathcal{G}$, the subdifferential satisfies

$$L^*\partial f(Lx) := \{L^*a \mid a \in \partial f(Lx)\} \subseteq \partial(f \circ L)(x)$$

for all $x \in \mathcal{H}$. Show that, for two functions $f_1, f_2\colon \mathcal{H} \to \overline{\mathbb{R}}$, the subdifferential satisfies

$$\partial f_1(x) + \partial f_2(x) \subseteq \partial(f_1 + f_2)(x)$$

for all $x \in \mathcal{H}$.

Problem 3. Find two sets $C_1, C_2 \subseteq \mathbb{R}^2$ (where the inner product on $\mathbb{R}^2$ is given by $\langle x, y\rangle = x^\top y$ for all $x, y \in \mathbb{R}^2$) and a point $x \in \mathbb{R}^2$ such that $N_{C_1}(x) + N_{C_2}(x) \neq N_{C_1 \cap C_2}(x)$. Visualise the normal cones. Conclude that the inclusions in problem 2 can be strict.

Problem 4. Let $m \geq 1$, and let $f_1, \ldots, f_m\colon \mathcal{H} \to \overline{\mathbb{R}}$ be functions, and let $f\colon \mathcal{H} \to \overline{\mathbb{R}}$ be defined by $f(x) = \max\{f_1(x), \ldots, f_m(x)\}$ for $x \in \mathcal{H}$. For $x \in \mathcal{H}$, let $I(x)$ be the set of *active indices*, i.e.,

$$I(x) = \{i \in \{1, \ldots, m\} \mid f_i(x) = f(x)\}.$$

Show that

$$\operatorname{conv}\left(\bigcup_{i \in I(x)} \partial f_i(x)\right) \subseteq \partial f(x)$$

for all $x \in \mathcal{H}$, the left-hand side being the convex hull of all active subgradients.

Note: Sufficient conditions for equality (also in problem 2) will be discussed later in the part on duality.

Problem 5. Let $f\colon \mathcal{H} \to \overline{\mathbb{R}}$ be convex, let $x, d \in \mathcal{H}$. Show that the mapping

$$t \mapsto \frac{f(x + td) - f(x)}{t}$$

is monotonically non-increasing for $t > 0$. Conclude that the *directional derivative*

$$f'(x; d) := \lim_{t \downarrow 0} \frac{f(x + td) - f(x)}{t}$$

exists as a value in $\overline{\mathbb{R}}$ and satisfies

$$f'(x; d) = \inf_{t > 0} \frac{f(x + td) - f(x)}{t}.$$

Hint: You can use that the above-mentioned limit exists if the values

$$\limsup_{t \downarrow 0} \frac{f(x+td)-f(x)}{t} := \inf_{\varepsilon>0} \sup_{0<t<\varepsilon} \frac{f(x+td)-f(x)}{t},$$
$$\liminf_{t \downarrow 0} \frac{f(x+td)-f(x)}{t} := \sup_{\varepsilon>0} \inf_{0<t<\varepsilon} \frac{f(x+td)-f(x)}{t}$$

are equal.