Large-scale convex optimisation

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Problem 1. Show that every closed, convex set $C \subseteq \mathcal{H}$ is the intersection of all half-spaces containing it, i.e.,

$$\begin{aligned} C &= \bigcap_{\substack{a \in \mathcal{H} \\ \alpha \in \mathbb{R}}} \{H_{a,\alpha} \,|\, C \subseteq H_{a,\alpha}\} \\ &= \bigcap \{\{x \in \mathcal{H} \,|\, \langle a, x\rangle \leq \alpha\} \,|\, a \in \mathcal{H}, \alpha \in \mathbb{R}, \forall y \in C : \langle a, y\rangle \leq \alpha\}. \end{aligned}$$

Problem 2. Find two closed, convex sets $C_1 \subseteq \mathbb{R}^2$ and $C_2 \subseteq \mathbb{R}^2$ such that $C_1 - C_2$ is not closed and there is no $a \in \mathbb{R}^2$ such that $\inf_{y_1 \in C_1} \langle a, y_1\rangle > \sup_{y_2 \in C_2} \langle a, y_2\rangle$.

Problem 3. Show that $\operatorname{int}(C_1 - C_2) \neq \emptyset$ whenever $\operatorname{int} C_1 \neq \emptyset$ and $C_2 \neq \emptyset$. Find two convex sets C_1 and C_2 with empty interior such that $\operatorname{int}(C_1 - C_2) \neq \emptyset$.

Problem 4. A function $f\colon \mathcal{H} \to \overline{\mathbb{R}}$ is called

- *subadditive* if $f(x + y) \leq f(x) + f(y)$ for all $x, y \in \mathcal{H}$,
- *positively homogeneous* if $f(tx) = tf(x)$ for all $t \geq 0$ and $x \in \mathcal{H}$.

A subadditive and positively homogeneous function is called *sublinear*. Show that a positively homogeneous function is subadditive if and only if it is convex. Show that the *support function* σ_S of a (not necessarily convex) set $S \subseteq \mathcal{H}$, which is defined by

$$\sigma_S(x) = \sup\{\langle x, s\rangle \,|\, s \in S\}$$

is sublinear.