Large-scale convex optimisation

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Problem 1. Show that the function $f\colon \mathbb{R}^2 \to \overline{\mathbb{R}}$ given by

$$f(x) = \begin{cases} \frac{x_1^2}{x_2} & \text{if } x_2 > 0, \\ +\infty & \text{if } x_2 \leq 0 \end{cases}$$

is convex. Is it lower semicontinuous?

Problem 2. Show that, if $\mathcal{H}$ and $\mathcal{G}$ are finite-dimensional inner-product spaces, $f\colon \mathcal{H} \to \overline{\mathbb{R}}$ is convex and $L\colon \mathcal{H} \to \mathcal{G}$ is linear, the *infimal postcomposition* $L \triangleright f\colon \mathcal{G} \to \overline{\mathbb{R}}$ defined by

$$(L \triangleright f)(x) = \inf \{f(y) \mid y \in \mathcal{H}, Ly = x\},$$

is convex.

Hint: Make sure that your proof also works if any of the function values is $+\infty$ or $-\infty$. The strict epigraph might be the most convenient way to do so.

Problem 3. For two convex functions $f_1, f_2\colon \mathcal{H} \to \overline{\mathbb{R}}$, show that their *infimal convolution* $f_1 \,\Box\, f_2\colon \mathcal{H} \to \overline{\mathbb{R}}$, which is given by

$$(f_1 \,\Box\, f_2)(x) = \inf \{f_1(y) + f_2(x - y) \mid y \in \mathcal{H}\},$$

is convex. What is the infimal convolution of two indicator functions? What is the infimal convolution of an indicator function and the norm?

Hint: It is possible to use problem 2 here.

Problem 4. Let $f\colon \mathcal{H} \to \overline{\mathbb{R}}$ be a function which is convex and lower semicontinuous, but not proper. Show that $f(x) \in \{\pm\infty\}$ for all $x \in \mathcal{H}$, i.e., f does not take any real values.

Problem 5. Show that the lower level sets of a convex function are convex. Give an example of a non-convex function whose lower level sets are all convex.