Large-scale convex optimisation

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Problem 1. Let $f\colon \mathcal{H} \to \overline{\mathbb{R}}$ be proper, convex, and lower semicontinuous, and let $\gamma > 0$. The function ${}^{\gamma}f\colon \mathcal{H} \to \overline{\mathbb{R}}$ (called the *Moreau envelope*) is defined as

$$ {}^{\gamma}f(x) = \left(f \,\Box\, \frac{1}{2\gamma}\|\cdot\|^2\right)(x) = \inf_{y\in\mathcal{H}} \left\{ f(y) + \frac{1}{2\gamma}\|y-x\|^2 \right\}. $$

Show that ${}^{\gamma}f(x) \in \mathbb{R}$ for all $x \in \mathcal{H}$, that f and ${}^{\gamma}f$ have the same set of minimisers, and that ${}^{\gamma}f$ is differentiable and $\nabla {}^{\gamma}f(x) = \frac{1}{\gamma}\left(x - \operatorname{Prox}_f^{\gamma}(x)\right)$ for all $x \in \mathcal{H}$.

Problem 2. Let $C \subseteq \mathcal{H}$ be nonempty, closed, and convex, and let $\gamma > 0$. Calculate ${}^{\gamma}\delta_C(x)$ and $\operatorname{Prox}_{\delta_C}^{\gamma}(x)$ for $x \in \mathcal{H}$.

Problem 3. For $\gamma > 0$ and $x \in \mathbb{R}$, compute ${}^{\gamma}|\cdot|(x)$ and $\operatorname{Prox}_{|\cdot|}^{\gamma}(x)$ (i.e., the Moreau envelope and proximal points of the absolute value function), the inner product on $\mathbb{R}$ being the usual product of real numbers.