Non-Linear ECE Electromagnetism

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Need for Non-Linearity

$$\boldsymbol{E}^a = -\underline{\nabla}\phi^a - \frac{\partial \boldsymbol{A}^a}{\partial t} - \omega^a_{0b}\boldsymbol{A}^b + \boldsymbol{\omega}^a_b\phi^b$$

$$\boldsymbol{B}^a = \underline{\nabla} \times \boldsymbol{A}^a - \boldsymbol{\omega}^a_b \times \boldsymbol{A}^b$$

$$\underline{\nabla} \cdot \boldsymbol{B}^a = 0$$

$$\underline{\nabla} \times \boldsymbol{E}^a + \frac{\partial \boldsymbol{B}^a}{\partial t} = 0$$

Need for Non-Linearity

From Gauss's Law

$$\underline{\nabla} \cdot (\boldsymbol{\omega}_b^a \times \boldsymbol{A}^b) = \boldsymbol{0}$$

$$\boldsymbol{\omega}_b^a \times \boldsymbol{A}^b = \underline{\nabla} \times \boldsymbol{F}^a$$

Note that if

$$F^a = -A^a$$

This is the "Lindstrom Constraint" for magnetic antisymmetry.

Need for Non-Linearity

Substituting these into the Faraday Equation

$$\underline{\nabla} \times \left(-\omega^a_{0b} \boldsymbol{A}^b + \boldsymbol{\omega}^a_b \phi^b - \frac{\partial \boldsymbol{F}^a}{\partial t} \right) = 0$$

$$-\omega^a_{0b} \boldsymbol{A}^b + \boldsymbol{\omega}^a_b \phi^b - \frac{\partial \boldsymbol{F}^a}{\partial t} = \underline{\nabla} \psi^a$$

Write

$$\Phi^a = \phi^a - \psi^a \qquad\qquad \boldsymbol{\mathcal{A}}^a = \boldsymbol{A}^a - \boldsymbol{F}^a$$

Then a Maxwell-Heaviside theory emerges, i.e.

$$\boldsymbol{E}^a = -\underline{\nabla} \Phi^a - \frac{\partial \boldsymbol{\mathcal{A}}^a}{\partial t} \qquad \boldsymbol{B}^a = \underline{\nabla} \times \boldsymbol{\mathcal{A}}^a$$

Non-Linear Field Equations

From the first Bianchi Identity

$$\partial_\mu \tilde{T}^{a\mu\nu} + \omega^a{}_{\mu b} \tilde{T}^{b\mu\nu} = \tilde{R}_\mu{}^{a\mu\nu}$$

$$\partial_\mu T^{a\mu\nu} + \omega^a{}_{\mu b} T^{b\mu\nu} = R_\mu{}^{a\mu\nu}$$

Until now

$$j_H^{a\nu} = \tilde{R}_\mu{}^{a\mu\nu} - \omega^a{}_{\mu b} \tilde{T}^{a\mu\nu} \approx 0$$

$$j_I^{a\nu} = R_\mu{}^{a\mu\nu} - \omega^a{}_{\mu b} T^{a\mu\nu}$$

Giving

$$\partial_\mu \tilde{T}^{a\mu\nu} \approx 0 \qquad \partial_\mu T^{a\mu\nu} = j_I^{a\nu}$$

Non-Linear Field Equations

If we re-define the 4-current densities as

$$j_I^{a\nu} = R_\mu{}^{a\mu\nu}$$

$$j_H^{a\nu} = \tilde{R}_\mu{}^{a\mu\nu} \approx 0$$

Then new non-linear field equations emerge

$$\partial_\mu T^{a\mu\nu} + \omega^a{}_{\mu b} T^{b\mu\nu} = R_\mu{}^{a\mu\nu} = j_I^{a\nu}$$

$$\partial_\mu \tilde{T}^{a\mu\nu} + \omega^a{}_{\mu b} \tilde{T}^{b\mu\nu} = \tilde{R}_\mu{}^{a\mu\nu} = j_H^{a\nu} \approx 0$$

Non-Linear Field Equations

In vector notation

$$\underline{\nabla} \cdot \boldsymbol{B}^a - \boldsymbol{\omega}^a_{\ b} \cdot \boldsymbol{B}^b = 0$$

$$\underline{\nabla} \times \boldsymbol{E}^a + \frac{\partial \boldsymbol{B}^a}{\partial t} + \omega^a_{0\ b} \boldsymbol{B}^b - \boldsymbol{\omega}^a_{\ b} \times \boldsymbol{E}^b = 0$$

$$\underline{\nabla} \cdot \boldsymbol{D}^a - \boldsymbol{\omega}^a_{\ b} \cdot \boldsymbol{D}^b = \rho^a$$

$$\underline{\nabla} \times \boldsymbol{H}^a - \frac{\partial \boldsymbol{D}^a}{\partial t} - \omega^a_{0\ b} \boldsymbol{D}^b - \boldsymbol{\omega}^a_{\ b} \times \boldsymbol{H}^b = \boldsymbol{J}^a$$