# Non-Linear ECE Electromagnetism

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AIAS July 2013

# Need for Non-Linearity

$$\mathbf{E}^a = -\underline{\nabla}\phi^a - \frac{\partial \mathbf{A}^a}{\partial t} - \omega^a_{0b}\mathbf{A}^b + \boldsymbol{\omega}^a_b \phi^b$$

$$\mathbf{B}^a = \underline{\nabla} \times \mathbf{A}^a - \boldsymbol{\omega}^a_b \times \mathbf{A}^b$$

$$\underline{\nabla} \cdot \mathbf{B}^a = 0$$

$$\underline{\nabla} \times \mathbf{E}^a + \frac{\partial \mathbf{B}^a}{\partial t} = 0$$

# Need for Non-Linearity

From Gauss's Law

$$\underline{\nabla} \cdot (\boldsymbol{\omega}_b^a \times \mathbf{A}^b) = 0$$
$$\boldsymbol{\omega}_b^a \times \mathbf{A}^b = \underline{\nabla} \times \mathbf{F}^a$$

Note that if

$$F^a = -A^a$$

This is the "Lindstrom Constraint" for magnetic antisymmetry.

# Need for Non-Linearity

Substituting these into the Faraday Equation

$$\underline{\nabla} \times \left( -\omega^a_{0b} \mathbf{A}^b + \boldsymbol{\omega}^a_b \phi^b - \frac{\partial \mathbf{F}^a}{\partial t} \right) = 0$$

$$-\omega^a_{0b} \mathbf{A}^b + \boldsymbol{\omega}^a_b \phi^b - \frac{\partial \mathbf{F}^a}{\partial t} = \underline{\nabla} \psi^a$$

Write

$$\Phi^a = \phi^a - \psi^a \qquad \mathcal{A}^a = \mathbf{A}^a - \mathbf{F}^a$$

Then a Maxwell-Heaviside theory emerges, i.e.

$$\mathbf{E}^a = -\underline{\nabla} \Phi^a - \frac{\partial \mathcal{A}^a}{\partial t} \qquad \mathbf{B}^a = \underline{\nabla} \times \mathcal{A}^a$$

# Non-Linear Field Equations

From the first Bianchi Identity

$$\partial_\mu \widetilde{T}^{\,a\mu\nu} + \omega^a{}_{\mu b} \widetilde{T}^{\,b\mu\nu} = \widetilde{R}_\mu{}^{a\mu\nu}$$
$$\partial_\mu T^{\,a\mu\nu} + \omega^a{}_{\mu b} T^{\,b\mu\nu} = R_\mu{}^{a\mu\nu}$$

Until now

$$j_H^{a\nu} = \widetilde{R}_\mu{}^{a\mu\nu} - \omega^a{}_{\mu b} \widetilde{T}^{\,a\mu\nu} \approx 0$$
$$j_I^{a\nu} = R_\mu{}^{a\mu\nu} - \omega^a{}_{\mu b} T^{\,a\mu\nu}$$

Giving

$$\partial_\mu \widetilde{T}^{\,a\mu\nu} \approx 0 \qquad\qquad \partial_\mu T^{\,a\mu\nu} = j_I^{a\nu}$$

# Non-Linear Field Equations

If we redefine the 4-current densities as

$$j_I^{a\nu} = R_\mu{}^{a\mu\nu}$$
$$j_H^{a\nu} = \widetilde{R}_\mu{}^{a\mu\nu} \approx 0$$

Then new non-linear field equations emerge

$$\partial_\mu T^{\,a\mu\nu} + \omega^a{}_{\mu b} T^{\,b\mu\nu} = R_\mu{}^{a\mu\nu} = j_I^{a\nu}$$
$$\partial_\mu \widetilde{T}^{\,a\mu\nu} + \omega^a{}_{\mu b} \widetilde{T}^{\,b\mu\nu} = \widetilde{R}_\mu{}^{a\mu\nu} = j_H^{a\nu} \approx 0$$

# Non-Linear Field Equations

In vector notation

$$\underline{\nabla} \cdot \mathbf{B}^a - \boldsymbol{\omega}^a_b \cdot \mathbf{B}^b = 0$$

$$\underline{\nabla} \times \mathbf{E}^a + \frac{\partial \mathbf{B}^a}{\partial t} + \omega^a_{0b} \mathbf{B}^b - \boldsymbol{\omega}^a_b \times \mathbf{E}^b = 0$$

$$\underline{\nabla} \cdot \mathbf{D}^a - \boldsymbol{\omega}^a_b \cdot \mathbf{D}^b = \rho^a$$

$$\underline{\nabla} \times \mathbf{H}^a + \frac{\partial \mathbf{D}^a}{\partial t} + \omega^a_{0b} \mathbf{D}^b - \boldsymbol{\omega}^a_b \times \mathbf{H}^b = \mathbf{J}^a$$