Ay190 – Worksheet 3
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I worked with Daniel DeFelippis

1 Problem 1

1.1 Part a)

To integrate $f(x) = \sin x$, I used the equations given in the notes for the midpoint rule, trapezoid rule, and Simpson's rule, respectively:

$$Q_i = (b_i - a_i) f\left(\frac{a_i + b_i}{2}\right) \tag{1}$$

$$Q_i = \frac{1}{2}(b_i - a_i)[f(b_i) + f(a_i)] \tag{2}$$

$$Q_i = \frac{b_i - a_i}{6}\left[f(a_i) + 4f\left(\frac{a_i + b_i}{2}\right) + f(b_i)\right] \tag{3}$$

The results are shown in Table 1.

Table 1: Results of Numerical Integration

	$h = \frac{\pi}{10}$	$h = \frac{\pi}{100}$
Midpoint	2.0082484079	2.0000822491
Trapezoidal	1.9835235375	1.9998355039
Simpson's	2.00000678444	2.00000000068
Analytic	2	2

Table 2 shows the errors in these methods.

Table 2: Errors in Numerical Integration

	$h = \frac{\pi}{10}$	$h = \frac{\pi}{100}$
Midpoint	8.25×10^{-3}	8.22×10^{-5}
Trapezoidal	1.65×10^{-2}	1.64×10^{-4}
Simpson's	6.78×10^{-6}	6.76×10^{-10}

The global error in the midpoint and trapezoidal rules goes as h^2, so the error should drop by a factor of 100 when h decreases by a factor of 10. The global error in Simpson's rule goes as h^4, so the error should drop by a factor of 10^4. Table 1 shows that this indeed is the case.

1.2 Part b)

The results of the numerical integration of $f(x) = x \sin x$ are shown in Table 3. The errors are shown in Table 4. As before, the decrease in error behaves as expected.

Table 3: Results of Numerical Integration

	$h = \frac{\pi}{10}$	$h = \frac{\pi}{100}$
Midpoint	3.1545492224	3.1417218501
Trapezoidal	3.1157114868	3.1413342637
Simpson's	3.14160331057	3.14159265465
Analytic	π	π

Table 4: Errors in Numerical Integration

	$h = \frac{\pi}{10}$	$h = \frac{\pi}{100}$
Midpoint	1.29×10^{-2}	1.29×10^{-4}
Trapezoidal	2.59×10^{-2}	2.58×10^{-4}
Simpson's	1.07×10^{-5}	1.06×10^{-9}

2 Problem 2

2.1 Part a)

I used Gauss-Laguerre Quadrature to calculate

$$n_e = \frac{8\pi (k_B T)^3}{(2\pi\hbar c)^3} \int_0^\infty \frac{x^2}{e^x + 1} \, \mathrm{d}x. \tag{4}$$

I rewrote the integral in the form

$$Q = \int_0^\infty f(x) e^{-x} \, \mathrm{d}x \tag{5}$$

where

$$f(x) = \frac{x^2 e^x}{e^x + 1}. \tag{6}$$

I used Python's built-in Gauss-Laguerre roots and weights. My results are shown in Table 5.

Table 5: Gauss-Laguerre Integration

n	Q	$n_e (cm^{-3})$
5	1.802027	1.896×10^{35}
10	1.803095	1.897×10^{35}
20	1.803085	1.897×10^{35}

To check for converge, I used the fact that Gauss-Laguerre quadrature has an error term of

$$\mathcal{O}(n) \propto \frac{(n!)^2}{(2n)!}, \tag{7}$$

which I found on Wolfram Mathworld. Using self-convergence

$$\frac{|Q(20) - Q(10)|}{|Q(10) - Q(5)|} = \frac{\mathcal{O}(20) - \mathcal{O}(10)}{\mathcal{O}(10) - \mathcal{O}(5)} \tag{8}$$

The left hand side evaluates to 9.14×10^{-3}, while the right hand side evaulates to 1.37×10^{-3}. These are a little off, but the error terms also have a factor of the $2n^{th}$ derivative of f which I did not include. The $n = 20$ value of 1.803085 matches what I found for the integral using Mathematica.

2.2 Part b)

I first rewrote the integral over the i^{th} bin:

$$\int_{5i}^{5(i+1)} \frac{x^2}{e^x + 1}\,\mathrm{d}x \tag{9}$$

in a form I could use Gauss-Legendre Quadrature on. I made the substitution

$$u = \frac{x - \frac{5i+5(i+1)}{2}}{5/2} = \frac{2x - 10i - 5}{5} \tag{10}$$

implying

$$x = \frac{5u + 10i + 5}{2} \tag{11}$$

$$\mathrm{d}x = \frac{5}{2}\mathrm{d}u. \tag{12}$$

This substitution changes Eq. 9 to

$$\frac{5}{2}\int_{-1}^{1} \frac{x^2}{e^x + 1}\,\mathrm{d}u \tag{13}$$

with the expression for x in Eq. 11. I then implemented Gauss-Legendre Quadrature, again using Python's built-in weights and roots. I then calculated

$$\left[\frac{\mathrm{d}n_e}{\mathrm{d}E}\right]_i = \frac{[n_e]_i}{\Delta E} \tag{14}$$

for bins of size $\Delta E = 5$ MeV from 0 to 150 MeV. The results are displayed in Table 6.

To check that my code worked, I recalculated the total n_e using

$$n_e = \sum_{i=0}^{\infty} \left[\frac{\mathrm{d}n_e}{\mathrm{d}E}\right]_i \times \Delta E. \tag{15}$$

I found $n_e = 1.897 \times 10^{35}$, matching what I found in Part a).

Table 6: Spectral Distribution

E Range (MeV)	$\frac{dn_e}{dE}$ (MeV/cm^3)	E Range (MeV)	$\frac{dn_e}{dE}$ (MeV/cm^3)	E Range (MeV)	$\frac{dn_e}{dE}$ (MeV/cm^3)
0-5	3.27×10^{34}	50-55	1.04×10^{16}	100-105	7.93×10^{-6}
5-10	5.12×10^{33}	55-60	8.51×10^{13}	105-110	5.88×10^{-8}
10-15	1.15×10^{32}	60-65	6.80×10^{11}	110-115	4.35×10^{-10}
15-20	1.64×10^{30}	65-70	5.37×10^{9}	115-120	3.20×10^{-12}
20-25	1.90×10^{28}	70-75	4.19×10^{7}	120-125	2.35×10^{-14}
25-30	1.96×10^{26}	75-80	3.23×10^{5}	125-130	1.71×10^{-16}
30-35	1.88×10^{24}	80-85	2.47×10^{3}	130-135	1.25×10^{-18}
35-40	1.71×10^{22}	85-90	1.88×10^{1}	135-140	9.06×10^{-21}
40-45	1.49×10^{20}	90-95	1.42×10^{-1}	140-145	6.56×10^{-23}
45-50	1.26×10^{18}	95-100	1.06×10^{-3}	145-150	4.74×10^{-25}