4.a) $U \in R^{n \times r}$

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} & \cdots & u_{1r} \\ u_{21} & u_{22} & u_{23} & u_{24} & \cdots & u_{2r} \\ u_{31} & u_{32} & u_{33} & u_{34} & \cdots & u_{3r} \\ u_{41} & u_{42} & u_{43} & u_{44} & \cdots & u_{4r} \\ \vdots \\ u_{n1} & u_{n2} & u_{n3} & u_{n4} & \cdots & u_{nr} \end{bmatrix} = [u_1 \; u_2 \; u_3 \; u_4 \cdots\cdots u_r]$$

$D \in R^{r \times r}$

$$D = \begin{bmatrix} \sigma_{11} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_{22} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \sigma_{33} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \sigma_{44} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \sigma_{rr} \end{bmatrix}$$

$V \in R^{d \times r}$ $\quad V^T \in R^{r \times d}$

$$V^T = \begin{bmatrix} v_{11} & v_{12} & v_{13} & v_{14} & \cdots & v_{1d} \\ v_{21} & v_{22} & v_{23} & v_{24} & \cdots & v_{2d} \\ v_{31} & v_{32} & v_{33} & v_{34} & \cdots & v_{3d} \\ v_{41} & v_{42} & v_{43} & v_{44} & \cdots & v_{4d} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ v_{r1} & v_{r2} & v_{r3} & v_{r4} & \cdots & v_{rd} \end{bmatrix} = \begin{bmatrix} v_1^T \\ v_2^T \\ v_3^T \\ v_4^T \\ \vdots \\ v_r^T \end{bmatrix}$$

so, $DV^T = \begin{bmatrix} \sigma_1 v_1^T \\ \sigma_2 v_2^T \\ \sigma_3 v_3^T \\ \sigma_4 v_4^T \\ \vdots \\ \sigma_r v_r^T \end{bmatrix}$

so, $A = UDV^T = [u_1 \; u_2 \; u_3 \; u_4 \cdots u_r] \begin{bmatrix} \sigma_1 v_1^T \\ \sigma_2 v_2^T \\ \sigma_3 v_3^T \\ \sigma_4 v_4^T \\ \vdots \\ \sigma_r v_r^T \end{bmatrix}$

$= \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \sigma_3 u_3 v_3^T + \sigma_4 u_4 v_4^T + \cdots + \sigma_r u_r$

$= \sum_{i=1}^{r} \sigma_i u_i v_i^T$

4.b)

$$A = UDV^T$$

Now, $AV = UDV^TV$

We know, that, $V^TV = I$

so, $AV = UD(V^TV) = UDI = UD$

Now, $A \in R^{n\times d}$ and, $V \in R^{d\times r}$

so, we, look at the column representation of AV,

$$A[v_1\ v_2\ v_3\ v_4 \cdots v_r] = UD$$

$$[Av_1\ Av_2\ Av_3\ Av_4 \cdots Av_r] = UD$$

Then, $U \in R^{n\times r}$, and, $D \in R^{r\times r}$

so, $UD = \begin{bmatrix} \sigma_{11}u_{11} & \sigma_{22}u_{12} & \cdots & \sigma_{rr}u_{1r} \\ \sigma_{11}u_{21} & \sigma_{22}u_{22} & \cdots & \sigma_{rr}u_{2r} \\ \sigma_{11}u_{31} & \sigma_{22}u_{32} & \cdots & \sigma_{rr}u_{3r} \\ \sigma_{11}u_{41} & \sigma_{22}u_{42} & \cdots & \sigma_{rr}u_{4r} \\ \vdots & \vdots & & \vdots \\ \sigma_{11}u_{n1} & \sigma_{22}u_{n2} & \cdots & \sigma_{rr}u_{nr} \end{bmatrix} = [\sigma_1 u_1\ \sigma_2 u_2\ \sigma_3 u_3\ \sigma_4 u_4 \cdots \sigma_r u_r]$

so, $[Av_1\ Av_2\ Av_3\ Av_4 \cdots Av_r] = [\sigma_1 u_1\ \sigma_2 u_2\ \sigma_3 u_3\ \sigma_4 u_4 \cdots \sigma_r u_r]$

so, $Av_i = \sigma_i u_i$

so, $u_i = \dfrac{1}{\sigma_i} Av_i$

4.c) As, the projection of a vector a onto a subspace spanned by $v_1, v_2, v_3, v_4, \ldots v_k$ where the v_i are pairwise orthogonal is given by the sum of projections of a onto the individual v_i,

we can represent it as,

The projection of the vector a onto V_k is given by,

$$\sum_{i=1}^{k} (a \cdot v_i)\, v_i^T$$

Thus, the matrix whose rows are the projections of the rows of A onto V_k is given by,

$$\sum_{i=1}^{k} A v_i v_i^T$$

Now, from 4.(b) result,

$$\sum_{i=1}^{k} A v_i v_i^T = \sum_{i=1}^{k} \sigma_i u_i v_i^T = A_k.$$

Thus, the rows of A_k are the projection of the rows of A onto the subspace V_k.