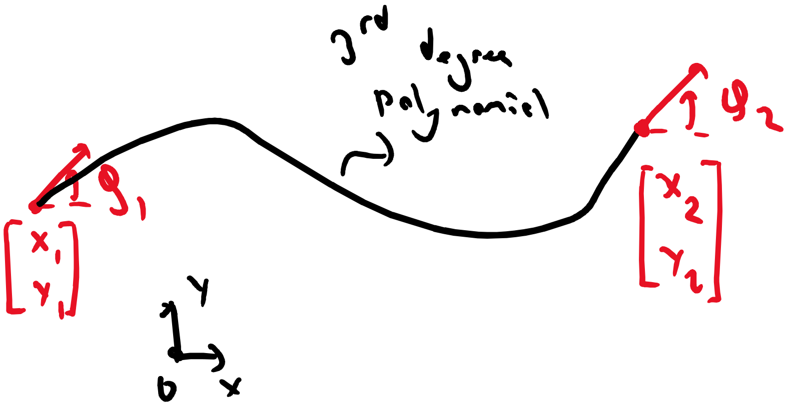
Basic Planar Path Planning (using a 3rd degree polynomial)

Given two planar poses, a path between them can be found using a 3rd degree polynomial.



$$y = a_3 x^3 + a_2 x^2 + a_1 x + a_0$$

$$y' = 3a_3 x^2 + 2a_2 x + a_1$$

We know 4 pieces of information, corresponding to the 4 coefficients of the polynomial.

- $a_3 x_1^3 + a_2 x_1^2 + a_1 x_1 + a_0 = y_1$
- $a_3 x_2^3 + a_2 x_2^2 + a_1 x_2 + a_0 = y_2$
- $3a_3 x_1^2 + 2a_2 x_1 + a_1 = \tan(\theta_1)$
- $3a_3 x_2^2 + 2a_2 x_2 + a_1 = \tan(\theta_3)$

Here, the unknowns are the coefficients a_0, a_1, a_2 and a_3. Organizing the equation gives us a matrix equation of the following form:

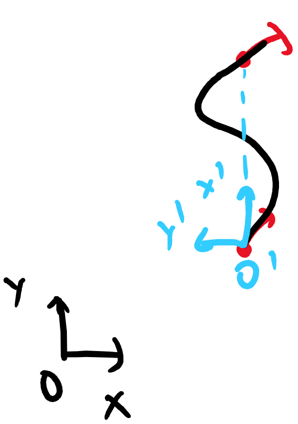
$$\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 0 & 1 & 2x_2 & 3x_2^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \tan(\theta_1) \\ \tan(\theta_2) \end{bmatrix}$$

The solution is then

$$\boldsymbol{a} = A^{-1} b \qquad \text{where } A = \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 0 & 1 & 2x_2 & 3x_2^2 \end{bmatrix}, \ b = \begin{bmatrix} y_1 \\ y_2 \\ \tan(\theta_1) \\ \tan(\theta_2) \end{bmatrix} \text{ and } \boldsymbol{a} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Remark:

For some cases, it may be required to create a temporary coordinate system so that the polynomial is a proper function.
In such cases, it is beneficial to create a coordinate system whose origin is at the first position, and x-axis is aligned with the direction between the locations. For example,



— Temporary frame.

Example implementation in MATLAB:

```
% Target Poses
x1 = 5;
y1 = 25;
theta1 = 0;

x2= 45;
y2= 60;
theta2 = 45;

% Path
A = [1 x1 x1^2 x1^3;
     1 x2 x2^2 x2^3;
     0 1 2*x1 3*x1^2;
     0 1 2*x2 3*x2^2];
b = [y1; y2; tand(theta1); tand(theta2)];

a_coef = inv(A)*b;

a0 = a_coef(1);
a1 = a_coef(2);
a2 = a_coef(3);
a3 = a_coef(4);

y = @(x) a3*x.^3 + a2*x.^2 + a1*x + a0;
yp = @(x) 3*a3*x.^2 + 2*a2*x + a1;
ypp = @(x) 6*a3*x + 2*a2;

X = x1:0.1:x2;
Y = y(X);

% Plot
figure();
hold on
quiver(x1,y1, cosd(theta1),sind(theta1), 'k', 'ShowArrowHead', 'on', 'LineWidth', 5);
quiver(x2,y2, cosd(theta2),sind(theta2), 'k', 'ShowArrowHead', 'on', 'LineWidth', 5);
plot(X,Y, 'r', 'LineWidth', 2);
grid on
hold off
```