1. Introduction

Link

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1 - Introduction

Q: Why study multi-step imperfect-information games?

A: Most real-world games are incomplete-information games with sequential (or simultaneous) moves.

Examples:

- Negotiation
- Multi-stage auctions
- Sequential auctions of multiple items
- Card games
- etc.

> Techniques for perfect-information games (ex. checkers, chess, and go) don't apply to incomplete-information games.

Why?

- Private information
- Need to understand signals and how other players will interpret signals
- Need to understand deception

2 - Terminology

- *Agent* (or *player*)
- *Action* (or *move*) - choice that the agent can make at a point in the game.
- *Strategy* (s_i) - mapping from history to actions.
  (to the extent the agent can distinguish a history)
- *Strategy set* (S_i) - strategies available to the agent.
- *Strategy profile* ($s_1, s_2, \ldots, s_{|A|}$) - one strategy for each agent.
- *Utility* ($u_i = u_i(s_1, s_2, \ldots, s_{|A|})$) -
  Represents the motivations of the agents, maps outcomes to reals with the property that a higher number implies that outcome is more preferred.

  Note - Utility seems to be closely related to *reward* in RL?

  An agent's utility is only determined after all agents, including nature, have chosen their strategy.

- *Nature* - pseudo agent that is used to model uncertainty.
  Note - Nature seems to be closely related to *environment* in RL.

3 - Agenthood

> Agents seek to maximize their expected utility:
>
> $\max_{\text{strategy}} \sum_{\text{outcome}} p(\text{outcome} \mid \text{strategy})\, u_i(\text{outcome})$

- Utility functions are scale-invariant.

  Note - We are only interested in the relative utility between strategies, so scaling them all by a linear transform does not change our rankings of actions.

- The utility function is up to the agent and can be used to model an agent's risk attitude.

  Example:
  Consider the two lotteries:
  Lottery 1 - 100% chance of winning \$0.5M
  Lottery 2 - 50% chance off winning \$1M

  The lotteries have equal expected values however agents may prefer one to the other depending on their willingness to take risk.

- Often times, in game theory, only expected payoff or expected value (EV) is considered.

4 - Game representations

- *Extensive form* - game tree form in which agents are nodes and their actions are edges.
- *Matrix form* - Player 1's strategy is along the rows, Player 2's strategy is along the columns, each cell denotes the utility for each player given the action pair.

5 - Dominant strategy "equilibrium"

- *Best response* (s_i^*) - $u_i(s_i^*, s_{-i}) \geq u_i(s_i', s_{-i}), \ \forall s_i'$.
  The strategy that yields the highest utility in response to the opposing player's strategy, s_{-i} .

- *Dominant strategy* (s_i^*) - s_i^* is the best response for all opponent strategies, s_{-i}
  The agent's best strategy does not depend on its opponent's strategy.

- *Dominant strategy "equilibrium"* -
  Strategy profile in which each agent has chosen its dominant strategy.
  Therefore, no agent has motivation to change their strategy.

Note - dominant strategies don't always exist.

6 - Nash equilibrium

- *Nash equilibrium* -
  No player has incentive to deviate from their strategy so long as their opponents do not deviate either. For every agent i, $u_i(s_i^*, s_{-i}) \geq u_i(s_i', s_{-i}), \ \forall s_i'$

> Nash equilibrium is a subset of the dominant strategy equilibrium. Dominant strategy equilibrium holds for all opponent strategies; whereas, Nash equilibrium only holds for a fixed opponent strategy.
>
> *i.e.* I don't change if you don't change, and vice versa.

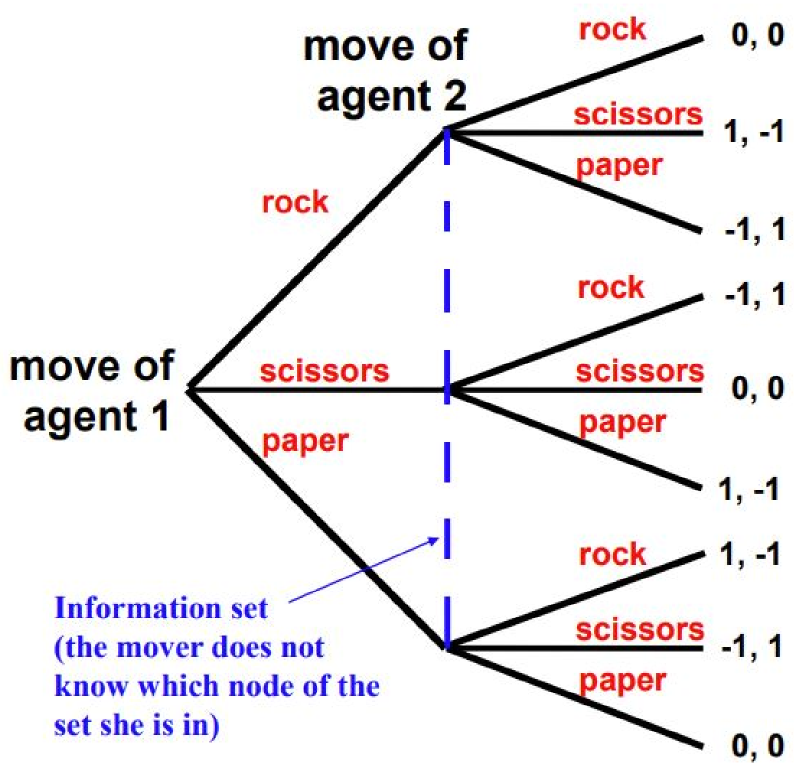
Nash equilibrium may not exist or be unique for all games. (ex. Battle of the Sexes)

- **Theorem: Existence of (pure-strategy) Nash equilibria**
  Any finite game, where each action node is alone in its information set is dominance solvable by backward induction.

  Note - an agent is "alone in its information set" if at every point in the game, the agent knows what moves have been played so far.

  Proof: Multi-player minimax search

7 - Mixed-strategy Nash equilibrium (rock-paper-scissors)



- *Bayesian game* - game in which players have incomplete information.
  In the case of rock-paper-scissors, the agents don't have knowledge of the other's move.

- *Bayes-Nash equilibrium* - Nash equilibrium of a Bayesian game.
  A strategy profile that maximizes the *expected* payoff for each player given their beliefs and strategies played by others.

- *Mixed strategy* - the agent's strategy is a weighted mixture of pure strategies.
  Rock-paper-scissors has a *symmetric* mixed strategy Nash equilibrium where all the agents have the same strategy.

- Existence:
  Theorem: Every finite player, finite strategy game has at least one Nash equilibrium if we admit mixed-strategy equilibria as well as pure. [1950-Nash].

- Complexity of finding a Nash equilibrium in a normal form game:
  - 2-player 0-sum: polytime with LP
  - 2-player games:
    - PPAD-complete [2009-Chen] [2005-Abbot] [2006-Daskalakis]
    - NP-complete to find an approx. *good* Nash equilibrium [2008-Conitzer]
  - 3-player games are FIXP-complete [2007-Etessami]