

# IPS: Assignment 1

Due date: 10am, 05/10/23

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## Starter

*These questions should help you to gain confidence with the basics.*

### S1. 📁 Hand-in

A bag contains six balls numbered 1–6. Two balls are drawn at random (without replacement), and the outcome is recorded as  $(i, j)$  if ball  $i$  is drawn first, and ball  $j$  drawn second. How many possible outcomes are there?

Now suppose that four of the numbered balls are white, and the other two are black. How many outcomes are there in which the following occur?

- both balls are white;
- both balls have the same colour;
- at least one of the balls is white?

#### Answer

Since we are sampling without replacement there are  $6 \times 5 = 30$  different possible outcomes. [1]

- Here we need to choose a white ball followed by another white ball:  $4 \times 3 = 12$ ; [1]
- As in part (a), the number of ways in which two black balls can be chosen is  $2 \times 1 = 2$ . Hence the total number of outcomes for which both balls have the same colour is  $12 + 2 = 14$ . [1]
- Here we need either both balls to be white (which we've already counted), or exactly one to be white. There are  $4 \times 2 = 8$  ways for the first ball to be white and the second black, and another  $2 \times 4 = 8$  ways for the balls to come out in the opposite order; so there are  $16 + 12 = 28$  possible outcomes altogether. (Alternatively, we know from part (b) that there are only two outcomes where both balls are black, and since there are 30 possible outcomes in total, 28 of these must include at least one white ball.) [2]

### S2. 📁 Hand-in

- Two football teams, each with 11 players, shake hands at the end of a match: every player on Team A shakes hands with every player on Team B. How many handshakes will there be?
- How many will there be if every player shakes hands with every other player (including those on their own team)?

#### Answer

- Each player from Team A has to shake hands with 11 players from Team B. So there are  $11^2 = 121$  handshakes in total. [2]
- If everyone shakes hands with everyone else, then we need to count the number of ways of choosing two players out of 22:  $C_2^{22} = 231$ . (Alternatively, we could count how many

handshakes take place between two members of the same team, and add this to the 121 that we obtained above. With 11 people in Team A, there are  $C_2^{11} = 55$  pairs of players to shake hands, and then there's the same number of handshakes between members of Team B. This gives 110 extra handshakes.) [3]

**S3.** 30 sweets are to be divided amongst 5 children. In how many ways can this be done? What if every child must be given at least two sweets each?

Answer

There are  $C_{5-1}^{30+5-1} = 46,376$  ways of dividing the sweets (allowing for the possibility of some children receiving none). If every child must be given two sweets, then if we do this first of all there are 20 sweets left to be divided: there are  $C_{5-1}^{20+5-1} = 10,626$  ways of doing this.

**S4.** Consider two events  $A$  and  $B$ . Let  $C$  be the event that “ $A$  or  $B$  occurs, but not both”. Express  $C$  in terms of  $A$  and  $B$ , using only the basic operations “union”, “intersection”, and “complement”.

Answer

There are many possible answers. Here are three:

$$\begin{aligned} C &= (A \cup B) \cap (A \cap B)^c \\ &= (A \cup B) \cap (A^c \cup B^c) \\ &= (A \cap B^c) \cup (B \cap A^c). \end{aligned}$$

**S5.** Let  $E$ ,  $F$  and  $G$  be three events. Find expressions for these events:

- Only  $E$  occurs;
- $E$  or  $F$  occurs, but not  $G$ ;
- At least one of the events occurs;
- At most two of the events occur;
- At most one of the events occurs.

Answer

- $E \cap F^c \cap G^c$ ;
- $(E \cup F) \cap G^c$ ;
- $E \cup F \cup G$ ;
- $(E \cap F \cap G)^c$ ;
- $(E \cap F)^c \cap (E \cap G)^c \cap (F \cap G)^c$ .

(You may have slightly different answers to the above, but if so then you should be able to prove that your versions are equivalent to mine!)

**S6.** Two dice are thrown. Let  $A$  be the event that the sum of the dice is even; let  $B$  be the event that the first die shows a higher number than the second; and let  $C$  be the event that the sum of the two dice is 6. List all the outcomes belonging to the events  $A \cap B$ ,  $A \cup C$ ,  $A \cap C^c$ ,  $A^c \cap C$ ,  $A \cap B \cap C$ .

Answer

Writing  $(i, j)$  for the outcome of the two dice, where  $i$  is the score on the first die and  $j$  the score on

the second, we have:

$$A \cap B = \{(3, 1), (4, 2), (5, 1), (5, 3), (6, 2), (6, 4)\};$$

$$A \cup C = A = \{(1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (2, 6), (3, 1), (3, 3), (3, 5), (4, 2), (4, 4), (4, 6), (5, 1), (5, 3), (5, 5), (6, 2), (6, 4), (6, 6)\};$$

$$A \cap C^c = \{(1, 1), (1, 3), (2, 2), (2, 6), (3, 1), (3, 5), (4, 4), (4, 6), (5, 3), (5, 5), (6, 2), (6, 4), (6, 6)\};$$

$$A^c \cap C = \emptyset;$$

$$A \cap B \cap C = \{(4, 2), (5, 1)\}.$$

## Main course

*These are important, and cover some of the most substantial parts of the course.*

**M1.** In how many ways can three men and three women be seated in a row if

- there are no restrictions on the order;
- no two members of the same sex can sit next to one another;
- there are three married (male-female) couples, and each husband and wife must sit next to one another?

Answer

- With no restrictions there are  $6! = 720$  ways.
- There are  $3! = 6$  ways of arranging the men, and 6 ways of arranging the women: this gives 36 arrangements. However, for every ordering of the men and women, we can choose whether to put a man or a woman in the first position – this means that the required number of arrangements is 72.
- There are  $3! = 6$  ways of arranging the three couples in order. But each husband and wife can be arranged in  $2! = 2$  ways, and so there is a total of  $3!2!2! = 48$  arrangements.

**M2.** A student has 1 bookshelf, 12 novels (among which 3 titles appear twice each), 5 books on probability theory, 4 knitting books and 2 dictionaries. In how many ways can he put the books on the shelf such that books in the same category are next to each other? Possibilities where the doubles are switched are considered to be the same and are not counted separately. (Hint: approx.  $8.3 \times 10^{12}$ .)

Answer

The 4 categories can be placed in  $4! = 24$  ways. The 5 books on probability theory can be placed in  $5! = 120$  ways, the 4 knitting books in  $4! = 24$  and the 2 dictionaries in  $2! = 2$  ways. For the novels, by Proposition 1.4 there are  $12!/(2!2!2!)$  ways of arranging them. This gives a total of

$$4!5!4!2! \frac{12!}{2!2!2!} \approx 8.3 \times 10^{12}$$

ways to order all books.

## M3. 📁 Hand-in

Prove, by expanding in terms of factorials, that

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}, \quad 1 \leq r \leq n.$$

Answer

$$\begin{aligned}\binom{n-1}{r-1} + \binom{n-1}{r} &= \frac{(n-1)!}{(n-r)!(r-1)!} + \frac{(n-1)!}{(n-1-r)!r!} \\ &= \frac{(n-1)!}{(n-r)!r!}(r + (n-r)) \\ &= \frac{n!}{(n-r)!r!} = \binom{n}{r}.\end{aligned}$$

[5]

#### M4. 📁 Hand-in

Consider a group of  $n$  objects, labelled  $1, \dots, n$ . How many (unordered) sets of size  $r$  are there that contain object 1? How many are there that *don't* contain object 1? Use these observations to re-prove the identity in question M3.

Answer

There are  $\binom{n-1}{r-1}$  sets of size  $r$  that contain object 1. (First put object 1 in the set, and then choose another  $r-1$  objects from the remaining  $n-1$  objects available.) [1] There are  $\binom{n-1}{r}$  sets that don't contain object 1. [1] There are a total of  $\binom{n}{r}$  unordered sets of size  $r$  possible with  $n$  objects, and since all of these must either contain object 1 or not contain it, we see that

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}, \quad 1 \leq r \leq n.$$

[3]

M5. Prove that, for sets  $E$  and  $F$ :

- $E \cap F \subseteq E \subseteq E \cup F$
- if  $E \subseteq F$  then  $F^c \subseteq E^c$
- $E = (E \cap F) \cup (E \cap F^c)$ , and show that this union is disjoint.

Answer

- If  $\omega \in E \cap F$ , then  $\omega \in E$  (and  $\omega \in F$ ), and so  $E \cap F \subseteq E$ . Similarly for  $E \subseteq E \cup F$ .
- Let  $\omega \in F^c$ , and suppose that  $\omega \in E$  also. Then, since  $E \subseteq F$ , we would have  $\omega \in F$ . But this would imply that  $\omega \in F$  and  $\omega \in F^c$ , i.e. that  $\omega \in F \cap F^c = \emptyset$ , which is a contradiction. Thus  $\omega \notin E$ , i.e.  $F^c \subseteq E^c$ .
- We can write

$$(E \cap F) \cup (E \cap F^c) = E \cap (F \cup F^c) = E \cap \Omega = E.$$

Furthermore, we know that  $F \cap F^c = \emptyset$ , and so  $(E \cap F) \cap (E \cap F^c) = E \cap (F \cap F^c) = \emptyset$  (by part (a)). Thus the union is certainly disjoint.

M6. Prove the second of De Morgan's laws: for sets  $E_1, \dots, E_n$ :

$$\left( \bigcap_{i=1}^n E_i \right)^c = \bigcup_{i=1}^n E_i^c.$$

### Answer

As usual, we need to show that any element  $\omega$  belonging to the first set belongs to the second, and vice versa.

Suppose first that  $\omega \in (\bigcap_{i=1}^n E_i)^c$ . Then  $\omega$  does not belong to *all* of the sets  $E_i$ , and so there must be at least one set, say  $E_j$ , with  $\omega \notin E_j$  (i.e. with  $\omega \in E_j^c$ ). But this implies that  $\omega \in \bigcup_{i=1}^n E_i^c$ .

Working in the other direction, suppose that  $\omega$  belongs to the set on the RHS. Then there is at least one set  $E_k$  with  $\omega \in E_k^c$ . Thus  $\omega \notin \bigcap_{i=1}^n E_i$ , since

$$E_k^c \cap \left( \bigcap_{i=1}^n E_i \right) \subseteq (E_k^c \cap E_k) = \emptyset.$$

Therefore  $\omega \in (\bigcap_{i=1}^n E_i)^c$ , as required.

**M7.** Let  $\mathbb{P}$  be a probability function and  $A$  and  $B$  events. Show that

$$\mathbb{P}(A^c \cap B^c) = 1 - \mathbb{P}(A) - \mathbb{P}(B) + \mathbb{P}(A \cap B).$$

### Answer

$$\begin{aligned} \mathbb{P}(A^c \cap B^c) &= \mathbb{P}((A \cup B)^c) && \text{by De Morgan's law} \\ &= 1 - \mathbb{P}(A \cup B) && \text{by (P4)} \\ &= 1 - \mathbb{P}(A) - \mathbb{P}(B) + \mathbb{P}(A \cap B) && \text{by (P6)}. \end{aligned}$$

**M8.** Show that if  $B$  is an event such that  $\mathbb{P}(B) = 1$  (but not necessarily  $B = \Omega$ ), then for all events  $A$ ,  $\mathbb{P}(A \cap B) = \mathbb{P}(A)$ .

### Answer

Since  $A \cup B \supseteq B$  for any event  $A$ , it holds that  $\mathbb{P}(A \cup B) \geq \mathbb{P}(B) = 1$ . Hence,  $\mathbb{P}(A \cup B) = 1$  by axiom (P1) and

$$\begin{aligned} \mathbb{P}(A \cap B) &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) \text{ by property (P6)} \\ &= \mathbb{P}(A) + 1 - 1 = \mathbb{P}(A). \end{aligned}$$

**M9.** Consider events  $A$ ,  $B$ , and  $C$ , which can occur in some experiment. Show that the probability that only  $A$  occurs (and not  $B$  or  $C$ ) is equal to  $\mathbb{P}(A \cup B \cup C) - \mathbb{P}(B) - \mathbb{P}(C) + \mathbb{P}(B \cap C)$ .

### Answer

The event “only  $A$  occurs and not  $B$  or  $C$ ” is the event  $A \cap (B \cup C)^c$ . The trick in the calculation below is a common one: any event  $D$  can be split into two disjoint components by choosing an arbitrary other event  $A$  and writing

$$D = \Omega \cap D = (A \cap D) \cup (A^c \cap D).$$

We use this below with the choice  $D = (B \cup C)^c$ :

$$\begin{aligned} \mathbb{P}(A \cap (B \cup C)^c) &= \mathbb{P}((B \cup C)^c) - \mathbb{P}(A^c \cap (B \cup C)^c) \\ &= \mathbb{P}((B \cup C)^c) - \mathbb{P}((A \cup B \cup C)^c) \text{ by De Morgan's law} \\ &= 1 - \mathbb{P}(B \cup C) - 1 + \mathbb{P}(A \cup B \cup C) \text{ by (P4)}. \end{aligned}$$

We cancel the ones and use that  $\mathbb{P}(B \cup C) = \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(B \cap C)$  according to (P6) to obtain the

desired result.

## Dessert

*Still hungry for more? Try these if you want to push yourself further.*

**D1.** Show that for  $n > 0$ ,

$$\sum_{i=0}^n (-1)^i \binom{n}{i} = 0.$$

Answer

We know from the binomial theorem that for  $a, b \in \mathbb{R}$  and  $n \in \mathbb{N}$ ,

$$(a + b)^n = \sum_{i=0}^n \binom{n}{i} a^i b^{n-i}.$$

Substituting  $a = -1$  and  $b = 1$  gives the required result.

**D2.** The famous **Birthday Problem**. If there are  $n$  people in a room, what is the probability that at least two people have a common birthday? (Ignore leap years for simplicity!)

Answer

As with many problems of this sort, it is easier to calculate the probability of the complementary event: that is, the probability that everybody has a distinct birthday. With  $n$  people and 365 days in the year, there is a total of  $365^n$  possible outcomes,  $|\Omega| = 365^n$ . All outcomes have equal probability. How many of these have all  $n$  birthdays distinct? There are

$$365 \cdot 364 \cdot \dots \cdot (365 - n + 1) = \frac{365!}{(365 - n)!}$$

outcomes of this kind because we have 365 possible choices for the first person, then 364 for the second, and so on, and  $365 - (n - 1)$  choices for the  $n$ -th person. Thus the probability that at least two people have a common birthday is given by

$$1 - \frac{365!}{(365 - n)! 365^n}.$$

**D3.** Let  $k$  be a non-negative integer. How many distinct integer-valued vectors  $(n_1, n_2, \dots, n_r)$  are there which satisfy both of the following constraints?

- i)  $n_j \geq k$  for all  $j = 1, 2, \dots, r$
- ii)  $n_1 + n_2 + \dots + n_r = n$ .

Answer

Let  $m_j = n_j - (k - 1) \geq 1$ . The number of solutions of

$$n_1 + n_2 + \dots + n_r = n, \quad \text{with } n_j \geq k \text{ for all } j$$

is equal to the number of solutions to

$$m_1 + m_2 + \dots + m_r = n - r(k - 1), \quad \text{with } m_j \geq 1 \text{ for all } j.$$

The number of such solutions is given by

$$C_{r-1}^{n-r(k-1)-1} = \binom{n-r(k-1)-1}{r-1}.$$

Alternatively, this problem is equivalent to having to divide  $n$  sweets among  $r$  children, such that each child gets at least  $k$  sweets each. We can do this by first of all giving each child  $k$  sweets, meaning that there are now  $n - rk$  sweets left to divide (where we are allowed to give some children zero extra sweets). The number of ways of doing this is

$$C_{r-1}^{(n-rk)+r-1} = \binom{n-r(k-1)-1}{r-1}.$$

### ! Challenge question

How many vectors  $(x_1, x_2, \dots, x_n)$  are there, such that each  $x_i$  is a non-negative integer and

$$\sum_{i=1}^n x_i \leq k,$$

where  $k \geq 0$  is an integer?

Can you find a closed form expression for this number (one not involving any summation symbols)?

Does the number of such vectors increase faster as we change  $n \rightarrow n + 1$  or  $k \rightarrow k + 1$ ?