

IPS: Assignment 3

Due date: 10am, 09/11/23

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Starter

These questions should help you to gain confidence with the basics.

S1. Each morning a (fictional) student rolls a die and starts studying if she throws 6. Otherwise, she stays in bed. However, during the four months of exams, she tosses a coin instead of rolling a die and studies if she tosses Heads. On a (randomly chosen) morning, the student is studying. What is the probability that it is exam period? (Assume for simplicity that each month has the same number of days!)

Answer

Let $W = \{\text{student is working}\}$ and $E = \{\text{exam period}\}$. Then Bayes' theorem tells us that

$$\begin{aligned} P(E|W) &= \frac{P(W|E)P(E)}{P(W|E)P(E) + P(W|E^c)P(E^c)} \\ &= \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{2} \cdot \frac{1}{3} + \frac{1}{6} \cdot \frac{2}{3}} = \frac{3}{5}. \end{aligned}$$

S2. Consider a discrete random variable V taking values in $\{1, 2, 3, 4\}$ with mass function p_V given by

$$p_V(x) = \begin{cases} xc^x & \text{if } x = 1, 2 \\ c(5-x) & \text{if } x = 3, 4. \end{cases}$$

What value of c makes p_V a mass function?

Answer

To be a mass function, we need $p_V(x) \geq 0$ for all x and $\sum_x p_V(x) = 1$. For the second of these constraints we need $c + 2c^2 + 2c + c = 1$, and so $c = -1 \pm \sqrt{6}/2$. But if we take $c = -1 - \sqrt{6}/2$ then we get (for example) $p_V(3) < 0$, which isn't allowed. Therefore, we have to take $c = -1 + \sqrt{6}/2 > 0$.

S3. Hand-in

Consider the probability space corresponding to throwing a fair die. Give an example of a random variable X whose image is $\{1, 2, 3\}$ and for which $F_X(\pi/2) = 1/2$.

Answer

There are lots of possibilities. The only important thing is that exactly three outcomes are mapped to 1 so that $F_X(\pi/2) = P(X \leq \pi/2) = P(X = 1) = 1/2$. One possibility is

$X(1) = X(2) = X(3) = 1, X(4) = X(5) = 2, X(6) = 3$. [3 for a correct example; 2 for some sort of reasoning]

S4. What is the probability that exactly 3 heads are obtained in 5 tosses of a fair coin?

Answer

Let X denote the number of heads obtained. Then $X \sim \text{Bin}(5, 1/2)$, and

$$\mathbb{P}(X = 3) = \binom{5}{3} \left(\frac{1}{2}\right)^5 = 0.3125.$$

S5.  Hand-in

Let $\lambda > 0$ be some positive number. Find c such that the following function is a density function:

$$f(x) = ce^{-\lambda|x|}, \quad x \in \mathbb{R}.$$

Answer

For f to be a density, it must be non-negative on \mathbb{R} and satisfy $\int_{-\infty}^{\infty} f(x)dx = 1$. Since the exponential function is non-negative, f is non-negative everywhere as long as $c \geq 0$ [1].

Note that

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-\lambda|x|} dx &= \int_{-\infty}^0 e^{\lambda x} dx + \int_0^{\infty} e^{-\lambda x} dx \\ &= \frac{1}{\lambda} [e^{\lambda x}]_{-\infty}^0 - \frac{1}{\lambda} [e^{-\lambda x}]_0^{\infty} = \frac{2}{\lambda}. \end{aligned}$$

[3]

So $f(\cdot)$ is a density if and only if $c = \lambda/2$. [1]

S6. Let $Y \sim \text{Poisson}(2)$. Write down the mass function for Y . What is $\mathbb{P}(Y \geq 2)$?

Answer

The probability mass function is given by

$$p_Y(k) = \frac{2^k}{k!} e^{-2}.$$

We can then calculate

$$\mathbb{P}(Y \geq 2) = 1 - \mathbb{P}(Y < 2) = 1 - (\mathbb{P}(Y = 0) + \mathbb{P}(Y = 1)) = 1 - (1 + 2)e^{-2}.$$

S7. Recall the Chevalier de Méré's problem: two dice are thrown 24 times – you win £1 if at least one double 6 is thrown, otherwise you lose £1. How much do you expect to win on average? Is this a good bet?

Answer

The probability of throwing a double 6 with one throw of two dice is $1/36$. The probability of there being *no* occurrences of a double 6 in 24 throws of two dice is therefore $(35/36)^{24} \approx 0.509$. Letting X denote your winnings from the game, this is a random variable

satisfying

$$\mathbb{P}(X = 1) \approx 0.491, \quad \text{and} \quad \mathbb{P}(X = -1) \approx 0.509.$$

Your expected winnings are hence given by

$$\mathbb{E}[X] \approx 0.491 - 0.509 \approx 0.018 < 0.$$

Since you expect to lose on average, this isn't a good bet to make!

S8. The probability density function f_X of a continuous random variable X is given by

$$f_X(x) = \begin{cases} -\frac{6(x^2+7x-c)}{25} & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Compute c . Give the distribution function of X in the interval $[0, 1]$.

Answer

The constant c is determined by the requirement that the integral of the density function over the entire real line is equal to one:

$$1 = \int_{-\infty}^{\infty} f_X(x) dx = \int_0^1 -\frac{6(x^2+7x-c)}{25} dx = \left[-\frac{6(x^3/3 + 7x^2/2 - cx)}{25} \right]_0^1 = \frac{6c-23}{25}.$$

This implies that $c = 8$. Note that for this value of c the function f_X is non-negative everywhere, which is also required in order for it to be a density.

The distribution function is obtained as an integral over the density function, but not over the whole real line but only from $-\infty$ up to x :

$$F_X(x) = \int_{-\infty}^x f_X(s) ds = \int_0^x -\frac{6(s^2+7s-8)}{25} ds = -\frac{6(x^3/3 + 7x^2/2 - 8x)}{25}.$$

S9. A continuous random variable X has a distribution function F which satisfies

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{4x^2+3x}{7} & \text{for } 0 \leq x \leq 1 \\ 1 & \text{for } x > 1. \end{cases}$$

Determine $\mathbb{E}[X]$.

Answer

The probability density function of X is given by

$$f(x) = \frac{d}{dx} F(x) = \frac{8x+3}{7}$$

for $0 \leq x \leq 1$, and $f(x) = 0$ elsewhere. So

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x) dx = \int_0^1 \left(\frac{8x^2+3x}{7} \right) dx = \frac{25}{42}.$$

Main course

These are important, and cover some of the most substantial parts of the course.

M1. 📁 Hand-in

Suppose that A and B are independent events. Show that A^c and B^c are independent.

Answer

We need to show that $\mathbb{P}(A^c \cap B^c) = \mathbb{P}(A^c)\mathbb{P}(B^c)$ **[1]**.

$$\begin{aligned}\mathbb{P}(A^c \cap B^c) &= \mathbb{P}((A \cup B)^c) && \text{(De Morgan)} \\ &= 1 - \mathbb{P}(A \cup B) && \text{(P4)} \\ &= 1 - (\mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)) && \text{(P6)} \\ &= (1 - \mathbb{P}(A)) - \mathbb{P}(B) + \mathbb{P}(A)\mathbb{P}(B) && \text{(independence of } A \text{ and } B) \\ &= \mathbb{P}(A^c) - \mathbb{P}(B)[1 - \mathbb{P}(A)] && \text{(P4)} \\ &= \mathbb{P}(A^c) - \mathbb{P}(B)\mathbb{P}(A^c) && \text{(P4)} \\ &= \mathbb{P}(A^c)[1 - \mathbb{P}(B)] \\ &= \mathbb{P}(A^c)\mathbb{P}(B^c). && \text{(P4)}\end{aligned}$$

[4; some justification of reasoning needs to be given for high marks]

M2. 📁 Hand-in

A box contains four coins: one has a Head on both sides; one has a Tail on both sides; the other two coins are normal and fair. A coin is chosen at random from the box and tossed three times. What is the probability that the two-Headed coin was chosen, given that all three tosses are Heads?

Answer

Let $A_1 = \{\text{two-Headed coin chosen}\}$, $A_2 = \{\text{two-Tailed coin chosen}\}$, $A_3 = \{\text{fair coin chosen}\}$, and $B = \{\text{obtain three Heads}\}$ **[1 for setting up some notation]**.

Then

$$\mathbb{P}(B) = \sum_{i=1}^3 \mathbb{P}(B | A_i) \mathbb{P}(A_i) = 1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} + \left(\frac{1}{2}\right)^3 \cdot \frac{1}{2} = \frac{5}{16}.$$

[2] Finally, we can use Bayes' theorem:

$$\begin{aligned}\mathbb{P}(A_1 | B) &= \frac{\mathbb{P}(A_1 \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A_1 \cap B)}{\mathbb{P}(A_1)} \frac{\mathbb{P}(A_1)}{\mathbb{P}(B)} \\ &= \mathbb{P}(B | A_1) \frac{\mathbb{P}(A_1)}{\mathbb{P}(B)} = 1 \cdot \frac{1/4}{5/16} = \frac{4}{5}.\end{aligned}$$

[2]

M3. Consider a discrete random variable Y taking values in the set $\{0, 1, \dots, 8\}$ and with probability mass function p_Y given by

$$p_Y(k) = \begin{cases} ca & \text{if } k = 0, 1, 2, 3, 4, 5 \\ ca^2 & \text{if } k = 6, 7 \\ c(1-a)^2 & \text{if } k = 8, \end{cases}$$

where a is some fixed number between 0 and 1 and c is a constant to be determined. What value of c makes p_Y a mass function? (The answer is a function of a .)

Answer

To be a mass function, we need $\sum_k p_Y(k) = 1$. That is,

$$c[6a + 2a^2 + (1 - a)^2] = 1,$$

and so the required value of c is

$$c = \frac{1}{(3a + 1)(a + 1)}.$$

M4. Let $X \sim \text{Bern}(p)$. Let $Y = 1 - X$ and $V = X^2$. Show that

- a) $Y \sim \text{Bern}(1 - p)$;
- b) $V \sim \text{Bern}(p)$.

Answer

The mass function of X is

$$p_X(x) = \begin{cases} 1 - p & \text{if } x = 0 \\ p & \text{if } x = 1 \\ 0 & \text{if } x \neq 0, 1. \end{cases}$$

a) Thus we see that

$$\begin{aligned} \mathbb{P}(Y = 0) &= \mathbb{P}(1 - X = 0) = \mathbb{P}(X = 1) = p \\ \text{and } \mathbb{P}(Y = 1) &= \mathbb{P}(1 - X = 1) = \mathbb{P}(X = 0) = 1 - p. \end{aligned}$$

Furthermore, $\mathbb{P}(Y = x) = 0$ if $x \neq 0, 1$. Thus $Y \sim \text{Bern}(1 - p)$, as required.

b) Similarly,

$$\begin{aligned} \mathbb{P}(V = 0) &= \mathbb{P}(X^2 = 0) = \mathbb{P}(X = 0) = 1 - p \\ \text{and } \mathbb{P}(V = 1) &= \mathbb{P}(X^2 = 1) = \mathbb{P}(X = 1) = p, \end{aligned}$$

and so $V \sim \text{Bern}(p)$.

M5. A fair die is thrown until the sum of the results of the throws exceeds 6. The random variable X is the number of throws needed for this. Let F_X be the distribution function of X . Determine $F_X(1)$, $F_X(2)$ and $F_X(7)$.

Answer

There is no possibility to exceed 6 already on the first throw because the die only has numbers up to 6. So $F_X(1) = \mathbb{P}(X \leq 1) = 0$.

There are, however, 21 outcomes of the throw of two dice that give a sum greater than 6. These can be read off easily from Table 4.1 in the textbook. There are 36 outcomes altogether, all equally likely, so

$$F_X(2) = \mathbb{P}(X \leq 2) = \mathbb{P}(X = 2) = \frac{21}{36} = \frac{7}{12}.$$

Finally, there is no way to avoid getting a sum of more than six once one has more than six throws, because each die will show at least one. So $F_X(7) = \mathbb{P}(X \leq 7) = 1 - \mathbb{P}(X > 7) = 1 - 0 =$

1.

M6. Consider a roulette wheel, with numbers 00, 0, 1, 2, 3, ..., 36 (38 numbers in total). A ball is thrown onto the wheel as it is spinning, and comes to rest by one of the numbers. You always bet that the ball will stop on one of the numbers 1, 2, ..., 12. Let N be the random variable giving the number of bets that you lose *before* your first win. Give $p_N(5)$ and $F_N(5)$.

Answer

The probability of winning on a particular bet is $12/38$ because 12 of the 38 outcomes lead to a win and all outcomes are equally likely. The probability of losing on a particular bet is therefore $26/38$. The individual spins of the roulette wheel are independent. So the probability of losing the first 5 first bets and then winning on the next is $p_N(5) = \mathbb{P}(N = 5) = (26/38)^5 \cdot 12/38 \approx 0.047$. Also $F_N(5) = \mathbb{P}(N \leq 5) = 1 - \mathbb{P}(N \geq 6)$ is the probability of not losing all first 6 bets. So $F_N(5) = 1 - (26/38)^6 \approx 0.90$. We notice that $(N + 1) \sim \text{Geom}(12/38)$.

M7. $N \geq 3$ people go for coffee. Each person flips a fair coin: if all but one of the coins shows the same face (Head or Tail), then the odd person out pays for all the drinks; if not then the coins are tossed again until this event occurs. How many times on average must each person toss their coin before somebody is selected in this way?

Answer

We first calculate the probability that there is an odd person out on the first (or indeed, any) flip of the coins. This happens if

- a) $N - 1$ people get Tails and one person gets a Head; or
- b) $N - 1$ people get Heads and one person gets a Tail.

Clearly these two events have equal probability (since the coins are all fair). Thinking of case (a), there are N ways in which we can choose the person who gets a Head, and the probability that this person gets a Head and the rest get tails equals $(1/2)^N$. So the probability of (a) happening is $N/2^N$, and since (b) has the same probability we obtain

$$p = \mathbb{P}(\text{odd person out on any particular flip}) = \frac{N}{2^{N-1}}.$$

Now, the number of flips needed for there to be an odd person out, X , follows a $\text{Geom}(p)$ distribution, and so we obtain

$$\text{expected number of tosses} = \mathbb{E}[X] = \frac{2^{N-1}}{N}.$$

This gets pretty big as N gets large!

M8. Prove the memoryless property of the exponential distribution which states that if $X \sim \text{Exp}(\lambda)$ then for any $t, s \geq 0$,

$$\mathbb{P}(X > t + s | X > s) = \mathbb{P}(X > t).$$

Answer

$$\begin{aligned}\mathbb{P}(X > t + s | X > s) &= \frac{\mathbb{P}(\{X > t + s\} \cap \{X > s\})}{\mathbb{P}(X > s)} \\&= \frac{\mathbb{P}(X > t + s)}{\mathbb{P}(X > s)} \\&= \frac{1 - \mathbb{P}(X \leq t + s)}{1 - \mathbb{P}(X \leq s)} \\&= \frac{1 - F_X(t + s)}{1 - F_X(s)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda t} \\&= 1 - F_X(t) = \mathbb{P}(X > t).\end{aligned}$$

Dessert

Still hungry for more? Try these if you want to push yourself further.

D1. N people go for coffee. Each person flips a coin, of which $N - 1$ are fair, and one has probability q of coming up Heads (for some $q \in [0, 1]$): if all but one of the coins shows the same face (Head or Tail), then the odd person out pays for all the drinks; if not then the coins are tossed again until this event occurs. How does the expected number of coin tosses each person makes vary with q ?

Answer

It doesn't vary with q – the expected number of tosses is always $2^{N-1}/N$! Why?

$$\begin{aligned}\mathbb{P}(\text{odd person out on any given flip}) &= q \binom{N-1}{N-2} \left(\frac{1}{2}\right)^{N-2} \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^{N-1} \\&\quad + (1-q) \binom{N-1}{N-2} \left(\frac{1}{2}\right)^{N-2} \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^{N-1},\end{aligned}$$

where the first quantity is the probability that the person with the strange coin gets a Head, and either $N - 2$ people also get Heads (with one Tail), or everybody else gets a Tail, etc. So the probability of having one odd person out in any given round is $p = N/2^{N-1}$, irrespective of the value of q . The number of rounds needed until there's one odd person out has a $\text{Geom}(p)$ distribution, with mean $2^{N-1}/N$.

D2. Alice and Bob decide to duel, using just one six-shot revolver, and one bullet, between them. They decide to duel in the following way: with the bullet inserted into the revolver, Alice will spin the cylinder and shoot at Bob (killing him if the gun fires); if Alice misses, Bob will spin the cylinder and shoot at Alice. Assuming there is a $1/6$ probability that the revolver will fire each time (since the revolver has 6 chambers), what is

- the distribution of the number of turns T until the gun fires?
- the probability that Alice wins the duel?

Answer

- The time until the gun fires follows a $\text{Geom}(1/6)$ distribution: $T \sim \text{Geom}(1/6)$.
- Alice wins if:

- the gun fires first time; or
- the gun doesn't fire on the first two goes, but fires on the third; or
- the gun doesn't fire on the first four goes, but fires on the fifth; or...

That is, Alice wins if the gun fires at an *odd* time:

$$\mathbb{P}(\text{Alice wins}) = \mathbb{P}(T \text{ is odd}).$$

Since $\mathbb{P}(T = n) = (5/6)^{n-1}(1/6)$, we require

$$\begin{aligned} \mathbb{P}(T \text{ is odd}) &= \frac{1}{6} \left[1 + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^4 + \dots \right] \\ &= \frac{1}{6} \left[\frac{1}{1 - \left(\frac{5}{6}\right)^2} \right] \\ &= \frac{6}{11}, \end{aligned}$$

where the middle equality follows from the usual formula for a geometric series. So Alice has the greater chance of winning (because she gets to take the first shot!).

! Challenge question

At time 0, an urn contains 1 Red and 1 Black ball. Just before each time $n = 1, 2, 3, \dots$, a ball is chosen at random from the urn and then replaced, *along with a new ball of the same colour*. For $n = 0, 1, 2, \dots$ and $1 \leq r \leq n + 1$, let

$$p_{n,r} = \mathbb{P}(\text{at time } n, \text{ the urn contains exactly } r \text{ Red balls}).$$

- Write down the values of $p_{0,1}$, $p_{1,1}$ and $p_{1,2}$;
- By conditioning on the possible numbers of red balls at time 1, calculate $p_{2,r}$ for $r = 1, 2, 3$;
- Can you find (and prove!) a general formula for $p_{n,r}$? [HINT: guess at the right answer using your findings from parts (a) and (b). (Calculate $p_{3,r}$ explicitly if you can't see a pattern.) Then use induction.]