IPS: Assignment 2

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Starter

These questions should help you to gain confidence with the basics.

S1. Let L, F and G be three events. Here are five events built from these:

a) $L \cap F^c \cap G^c$
b) $(L \cup F) \cap G^c$
c) $L \cup F \cup G$
d) $(L \cap F \cap G)^c$
e) $(L \cap F)^c \cap (L \cap G)^c \cap (F \cap G)^c$.

And here are verbal descriptions of the same five events, but in different order:

1) L or F occurs, but not G
2) At most two of the events occur
3) At most one of the events occurs
4) At least one of the events occurs
5) Only L occurs.

Match the expressions to the verbal descriptions.

Answer

a5, b1, c4, d2, e3

S2. Hand-in
When $\mathbb{P}(A) = 2/3$, $\mathbb{P}(B) = 1/4$, and $\mathbb{P}(A \cup B) = 3/4$, calculate $\mathbb{P}(A \cap B)$ and $\mathbb{P}(A^c \cup B^c)$.

Answer

Solving the relation $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$ (property (P6)) for $\mathbb{P}(A \cap B)$ gives

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) = 2/3 + 1/4 - 3/4 = 1/6$$

[2]. Using De Morgan's laws we get

$$\mathbb{P}(A^c \cup B^c) = \mathbb{P}((A \cap B)^c) = 1 - \mathbb{P}(A \cap B) = 1 - 1/6 = 5/6.$$

[3]

S3. Hand-in
Let A and B be two events. Prove that if $A \subseteq B$ then

$$\mathbb{P}(B \cap A^c) = \mathbb{P}(B) - \mathbb{P}(A).$$

(Make sure to justify your reasoning by making reference to any of the properties (P1)–(P7) covered in lectures, where you are using them in your answer.)

Answer

Since $A \subseteq B$, we can write B as the *disjoint* union $B = (B \cap A^c) \cup A$. **[2]** This gives

$$\mathbb{P}(B) = \mathbb{P}((B \cap A^c) \cup A) \overset{(P3)}{=} \mathbb{P}(B \cap A^c) + \mathbb{P}(A)\,.$$

Rearranging gives the required result. **[3]**

S4. Let C and D be events. Express the probability $\mathbb{P}(C^c \cap D)$ in terms of $\mathbb{P}(D)$ and $\mathbb{P}(C \cap D)$.

Answer

We note that we can write the event D as the disjoint union of the two events $C^c \cap D$ and $C \cap D$. Therefore by additivity (P3) we have

$$\mathbb{P}(D) = \mathbb{P}(C^c \cap D) + \mathbb{P}(C \cap D)\,.$$

Hence $\mathbb{P}(C^c \cap D) = \mathbb{P}(D) - \mathbb{P}(C \cap D)$.

S5. Let L and F be two events for which one knows that the probability that at least one of them occurs is 0.4. What is the probability that neither L nor F occurs?

Answer

The event "at least one of L and F occurs" is the event $L \cup F$. We want the probability of the event that neither L nor F occurs, which is the event $L^c \cap F^c$. According to the second De Morgan's law this event can also be written as $(L \cup F)^c$. To calculate this we can use the property (P4) that for any event A we have $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$. This gives us

$$\mathbb{P}((L \cup F)^c) = 1 - \mathbb{P}(L \cup F) = 1 - 0.4 = 0.6.$$

Main course

These are important, and cover some of the most substantial parts of the course.

M1. Hand-in
Calculate $\mathbb{P}(A \cup B)$ if it is given that $\mathbb{P}(A) = 1/3$ and $\mathbb{P}(B \,|\, A^c) = 2/3$.

Answer

A trick you can use here, as most of the time when calculating the probability of a union of events, is to rewrite it in terms of the union of disjoint events. In this case the best is to write $A \cup B = A \cup (B \cap A^c)$ and thus $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B \cap A^c)$. We can then use that by the multiplication rule

$$\mathbb{P}(B \cap A^c) = \mathbb{P}(B|A^c)\mathbb{P}(A^c)$$

[2]. Furthermore $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$. Putting everything together gives

$$\begin{aligned}\mathbb{P}(A \cup B) &= \mathbb{P}(A) + \mathbb{P}(B \,|\, A^c)(1 - \mathbb{P}(A)) \\ &= \frac{1}{3} + \frac{2}{3}(1 - 1/3) = \frac{7}{9}.\end{aligned}$$

[3]

M2. **Hand-in**
Prove that, for any events A, B, C for which the following probabilities are all defined,

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B \mid A)\mathbb{P}(C \mid A \cap B).$$

Answer

Expanding the RHS using the definition of conditional probability, we obtain

$$\mathbb{P}(A)\mathbb{P}(B \mid A)\mathbb{P}(C \mid A \cap B) = \mathbb{P}(A)\,\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}\,\frac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(A \cap B)} = \mathbb{P}(A \cap B \cap C).$$

[5]

M3. Calculate $\mathbb{P}(B)$ if it is given that $\mathbb{P}(A \cup B) = 3/7$ and $\mathbb{P}(A^c \mid B^c) = 9/14$.

Answer

From the multiplication rule we know that

$$\mathbb{P}(A^c \cap B^c) = \mathbb{P}(A^c \mid B^c)\mathbb{P}(B^c).$$

Recalling De Morgan's law we know that $A^c \cap B^c = (A \cup B)^c$ and hence

$$\mathbb{P}(A^c \cap B^c) = \mathbb{P}((A \cup B)^c) = 1 - \mathbb{P}(A \cup B).$$

Combined this yields

$$\begin{aligned}\mathbb{P}(B) &= 1 - \mathbb{P}(B^c) = 1 - \frac{1 - \mathbb{P}(A \cup B)}{\mathbb{P}(A^c \mid B^c)} \\ &= 1 - \frac{4/7}{9/14} = \frac{1}{9}.\end{aligned}$$

M4. Show that if an event occurs with probability 0 then it is independent of all other events.

Answer

Suppose that A satisfies $\mathbb{P}(A) = 0$, and let B be any other event. Then $\mathbb{P}(A \cap B) \leq \mathbb{P}(A) = 0$, since $A \cap B \subseteq A$ (P7). So, $0 = \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$, and hence A is independent of B.

M5. Show that if an event A is independent of itself, then either $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$.

Answer

Let $\mathbb{P}(A) = p$. Since $A \cap A = A$, if A is independent of itself we have $\mathbb{P}(A \cap A) = \mathbb{P}(A) = p$ and $\mathbb{P}(A \cap A) = \mathbb{P}(A)\mathbb{P}(A) = p^2$. So,

$$p = p^2 \iff p(p-1) = 0 \iff p = 0 \text{ or } p = 1.$$

M6. Let A, B, and C be events. Prove that if

$$\mathbb{P}(A \mid C) > \mathbb{P}(B \mid C) \text{ and } \mathbb{P}(A \mid C^c) > \mathbb{P}(B \mid C^c)$$

then $\mathbb{P}(A) > \mathbb{P}(B)$.

Answer

We can use the partition theorem:

$$\mathbb{P}(A) = \mathbb{P}(A \mid C)\mathbb{P}(C) + \mathbb{P}(A \mid C^c)\mathbb{P}(C^c) > \mathbb{P}(B \mid C)\mathbb{P}(C) + \mathbb{P}(B \mid C^c)\mathbb{P}(C^c) = \mathbb{P}(B).$$

M7. Given a sample space Ω, let $\mathbb{Q}$ be some function satisfying

$$\mathbb{Q}(A_1 \cup A_2) = \mathbb{Q}(A_1) + \mathbb{Q}(A_2)$$

for any two disjoint subsets A_1 and A_2 of Ω.

Show by induction that if $A_1, ..., A_n$ are $n \geq 2$ disjoint subsets of Ω then

$$\mathbb{Q}(A_1 \cup ... \cup A_n) = \mathbb{Q}(A_1) + ... + \mathbb{Q}(A_n) .$$

Answer

The base case is when $n = 2$; this case is clearly true - it's exactly what we've been told in the question about $\mathbb{Q}$.
Now suppose that the result holds for some particular value of $n \geq 2$, and suppose that we have disjoint sets $A_1, \ldots, A_{n+1}$. Then we can write

$$A_1 \cup \cdots \cup A_{n+1} = (\cup_{i=1}^{n} A_i) \cup A_{n+1} .$$

This is a disjoint union of two sets: $(\cup_{i=1}^{n} A_i)$ and A_{n+1}, and so we know that

$$\mathbb{Q}((\cup_{i=1}^{n} A_i) \cup A_{n+1}) = \mathbb{Q}(\cup_{i=1}^{n} A_i) + \mathbb{Q}(A_{n+1}) .$$

But by our inductive hypothesis,

$$\mathbb{Q}(\cup_{i=1}^{n} A_i)) = \mathbb{Q}(A_1) + \mathbb{Q}(A_2) + \ldots \mathbb{Q}(A_n) .$$

It follows that

$$\mathbb{Q}(A_1 \cup ... \cup A_{n+1}) = \mathbb{Q}(A_1) + ... + \mathbb{Q}(A_n) + \mathbb{Q}(A_{n+1})$$

and so the required result holds for any value of $n \geq 2$, by the principle of mathematical induction.

Dessert

Still hungry for more? Try these if you want to push yourself further.

D1. In a card game involving four players, each player is dealt four cards (face down), and the player with the highest Diamond wins (Ace counts high). Upon receiving your four cards you see that your highest Diamond is the Queen of Diamonds. What is the probability that another player has the King or Ace of Diamonds, and hence that you lose the game?

Answer

You can see four cards, meaning that there are 48 cards whose positions are unknown to you. Twelve of these cards have been dealt to the other players. The number of ways of choosing these twelve cards from the remaining 48 is C_{12}^{48}. The number of ways in which the King and Ace of Diamonds are *not dealt out* is equal to the number of ways in which twelve cards can be chosen from 46 (the 48 unseen cards minus the King and Ace), i.e. C^{46}_{12}. Therefore the probability that the King and Ace are not dealt (and that you win) is given by $C^{46}_{12}/C^{48}_{12} = (35 \cdot 36)/(47 \cdot 48) = 0.559$. So the required probability is 0.441.

D2. Consider a sample space $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ consisting of four equiprobable elements. Let $A_1 = \{\omega_1, \omega_2\}$, $A_2 = \{\omega_1, \omega_3\}$, $A_3 = \{\omega_1, \omega_4\}$. Show that A_1 and A_2 are independent, A_1 and A_3 are independent, and A_2 and A_3 are independent. Are A_1, A_2, A_3 independent?

Answer

Note that $\mathbb{P}(A_1) = \mathbb{P}(A_2) = \mathbb{P}(A_3) = 1/2$ and $\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1 \cap A_3) = \mathbb{P}(A_2 \cap A_3) = 1/4 = (1/2) \cdot (1/2)$, and so the three pairs of events (A_1 and A_2, etc) are independent. However,

$$\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(\{\omega_1\}) = 1/4 \neq \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3) = 1/8.$$

! Challenge question

Prove the **law of inclusion-exclusion**: for possibly overlapping sets $A_1, \dots, A_n$,

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i_1<i_2} \mathbb{P}(A_{i_1} \cap A_{i_2}) + \dots + (-1)^{k+1} \sum_{i_1<i_2<\dots<i_k} \mathbb{P}(A_{i_1} \cap A_{i_2} \dots \cap A_{i_k}) \\ + \dots + (-1)^{n+1}\mathbb{P}(A_1 \cap A_2 \cap \dots \cap A_n)\,.$$

(So, for example, $\mathbb{P}(A_1 \cup A_2 \cup A_3) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) - \mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1 \cap A_3) - \mathbb{P}(A_2 \cap A_3) + \mathbb{P}(A_1 \cap A_2 \cap A_3)$.)