IPS: Assignment 1

Due date: 10am, 05/10/23

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Instructions

Submit your answers to all questions marked **Hand-in**. You should upload to Moodle your solutions as a *pdf file*.
Marks will be awarded for clear, logical explanations, as well as for correctness of solutions.
You should also look at the other questions in preparation for your Week 3 seminar.

Starter

These questions should help you to gain confidence with the basics.

S1. Hand-in
A bag contains six balls numbered 1–6. Two balls are drawn at random (without replacement), and the outcome is recorded as (i, j) if ball i is drawn first, and ball j drawn second. How many possible outcomes are there?
Now suppose that four of the numbered balls are white, and the other two are black. How many outcomes are there in which the following occur?

a. both balls are white;
b. both balls have the same colour;
c. at least one of the balls is white?

S2. Hand-in

a. Two football teams, each with 11 players, shake hands at the end of a match: every player on Team A shakes hands with every player on Team B. How many handshakes will there be?
b. How many will there be if every player shakes hands with every other player (including those on their own team)?

S3. 30 sweets are to be divided amongst 5 children. In how many ways can this be done? What if every child must be given at least two sweets each?

S4. Consider two events A and B. Let C be the event that "A or B occurs, but not both". Express C in terms of A and B, using only the basic operations "union", "intersection", and "complement".

S5. Let E, F and G be three events. Find expressions for these events:

a. Only E occurs;
b. E or F occurs, but not G;
c. At least one of the events occurs;
d. At most two of the events occur;
e. At most one of the events occurs.

S6. Two dice are thrown. Let A be the event that the sum of the dice is even; let B be the event that the first die shows a higher number than the second; and let C be the event that the sum of the two dice is 6. List all the outcomes belonging to the events $A \cap B$, $A \cup C$, $A \cap C^c$, $A^c \cap C$, $A \cap B \cap C$.

Main course

These are important, and cover some of the most substantial parts of the course.

M1. In how many ways can three men and three women be seated in a row if

a. there are no restrictions on the order;
b. no two members of the same sex can sit next to one another;
c. there are three married (male-female) couples, and each husband and wife must sit next to one another?

M2. A student has 1 bookshelf, 12 novels (among which 3 titles appear twice each), 5 books on probability theory, 4 knitting books and 2 dictionaries. In how many ways can he put the books on the shelf such that books in the same category are next to each other? Possibilities where the doubles are switched are considered to be the same and are not counted separately. (Hint: approx. 8.3×10^{12}.)

M3. Hand-in
Prove, by expanding in terms of factorials, that

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}, \qquad 1 \leq r \leq n\,.$$

M4. Hand-in
Consider a group of n objects, labelled $1, \dots, n$. How many (unordered) sets of size r are there that contain object 1? How many are there that *don't* contain object 1? Use these observations to re-prove the identity in question **M3**.

M5. Prove that, for sets E and F:

a. $E \cap F \subseteq E \subseteq E \cup F$
b. if $E \subseteq F$ then $F^c \subseteq E^c$
c. $E = (E \cap F) \cup (E \cap F^c)$, and show that this union is disjoint.

M6. Prove the second of De Morgan's laws: for sets $E_1, \dots, E_n$:

$$\left(\bigcap_{i=1}^{n} E_i\right)^c = \bigcup_{i=1}^{n} E_i^c\,.$$

M7. Let $\mathbb{P}$ be a probability function and A and B events. Show that

$$\mathbb{P}(A^c \cap B^c) = 1 - \mathbb{P}(A) - \mathbb{P}(B) + \mathbb{P}(A \cap B).$$

M8. Show that if B is an event such that $\mathbb{P}(B) = 1$ (but not necessarily $B = \Omega$), then for all events A, $\mathbb{P}(A \cap B) = \mathbb{P}(A)$.

M9. Consider events A, B, and C, which can occur in some experiment. Show that the probability that only A occurs (and not B or C) is equal to $\mathbb{P}(A \cup B \cup C) - \mathbb{P}(B) - \mathbb{P}(C) + \mathbb{P}(B \cap C)$.

Dessert

Still hungry for more? Try these if you want to push yourself further.

D1. Show that for $n > 0$,

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0\,.$$

D2. The famous **Birthday Problem**. If there are n people in a room, what is the probability that at least two people have a common birthday? (Ignore leap years for simplicity!)

D3. Let k be a non-negative integer. How many distinct integer-valued vectors $(n_1, n_2, \ldots, n_r)$ are there which satisfy both of the following constraints?

i) $n_j \geq k$ for all $j = 1, 2, \ldots, r$
ii) $n_1 + n_2 + \cdots + n_r = n$.

! Challenge question

How many vectors $(x_1, x_2, \ldots, x_n)$ are there, such that each x_i is a non-negative integer and

$$\sum_{i=1}^{n} x_i \leq k\,,$$

where $k \geq 0$ is an integer?
Can you find a closed form expression for this number (one not involving any summation symbols)?
Does the number of such vectors increase faster as we change $n \to n+1$ or $k \to k+1$?