IPS: Assignment 4

Due date: 10am, 23/11/23

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Starter

These questions should help you to gain confidence with the basics.

S1. Let X be a random variable with $\mathbb{E}[X] = 5$. What is the expectation of $3X + 5$? If furthermore $\mathbb{E}[X^2] = 30$, what is the variance of X?

> Answer
>
> We can use the linearity of expectation to find that $\mathbb{E}[3X + 5] = 3\mathbb{E}[X] + 5 = 20$. The variance is $\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 30 - 5^2 = 5$.

S2. Hand-in

Buses leave campus for the train station every 20 minutes, at 0, 20 and 40 minutes past the hour. If a student arrives at the bus stop at a time that is uniformly distributed between 10.00 and 10.45, find the probability that they wait

a) less than 5 minutes for a bus;
b) more than 10 minutes for a bus.

> Answer
>
> Let Y denote the number of minutes past 10.00 that the student arrives at the bus stop: $Y \sim$ Uniform$[0, 45]$. **[1]**
>
> a) They will wait less than 5 minutes if and only $15 \leq Y \leq 20$ or $35 \leq Y \leq 40$. This occurs with probability
>
> $$\mathbb{P}(15 \leq Y \leq 20) + \mathbb{P}(35 \leq Y \leq 40) = \int_{15}^{20} \frac{1}{45} dy + \int_{35}^{40} \frac{1}{45} dy = \frac{2}{9}.$$
>
> **[2]**
>
> b) Similarly, they will wait for more than 10 minutes if they arrive between 10.00 and 10.10, between 10.20 and 10.30, or between 10.40 and 10.45. This has probability $5/9$. **[2]**

S3. A random variable Z has probability density function

$$f_Z(x) = \begin{cases} \frac{6}{5675}(5x^2 + 3x + 11) & \text{for } 3 \leq x \leq 8 \\ 0 & \text{otherwise.} \end{cases}$$

Would you expect $\mathbb{E}[Z]$ to lie closer to 3 or to 8? Calculate $\mathbb{E}[Z]$ and check whether your intuition was correct.

Answer

Since f_Z is increasing on the interval $[3,8]$ we know from the interpretation of expectation as centre of gravity that the expectation should lie closer to 8 than to 3. The computation:

$$\mathbb{E}[Z] = \int_3^8 x f_Z(x)dx = \frac{6}{5675}\int_3^8 \left(5x^3 + 3x^2 + 11x\right)dx = \frac{2787}{454} = 6.14.$$

S4. **Hand-in**
Let $X \sim$ Geom(p). Calculate $\mathbb{E}[h(X)]$, where $h(x) = e^{tx}$ for some $t > 0$. For what values of t is $\mathbb{E}[h(X)] < \infty$?
Hint: use the result for infinite geometric series.

Answer

We use the formula for the expectation of a function of a discrete random variable:

$$\begin{aligned}\mathbb{E}[h(X)] &= \sum_{k=1}^{\infty} h(k)p(1-p)^{k-1} = \sum_{k=1}^{\infty} e^{tk}p(1-p)^{k-1} \\ &= pe^t\sum_{k=1}^{\infty}[e^t(1-p)]^{k-1} = pe^t\sum_{k=0}^{\infty}[e^t(1-p)]^k \\ &= \frac{pe^t}{1-e^t(1-p)}.\end{aligned}$$

[4]
This final step requires $e^t(1-p) < 1$. (Otherwise the geometric sum does not converge to a finite limit.) **[1]**

S5. Suppose that you have a lecture at 14.00, and that the time taken to travel from your room to the lecture theatre is normally distributed with mean 30 minutes and standard deviation 4 minutes. What is the latest time you should leave your room if you want to be 99% certain that you will not miss the start of the lecture? (Hint: if $Z \sim$ N$(0,1)$ then the R function `qnorm(p)` returns the value $z \in \mathbb{R}$ such that $\mathbb{P}(Z \leq z) = p$.)

Answer

Let X denote the travel time to the lecture: $X \sim$ N$(30,16)$. We wish to find x such that $\mathbb{P}(X \leq x) = 0.99$. Now,

$$\mathbb{P}(X \leq x) = \mathbb{P}\left(\frac{X-30}{4} \leq \frac{x-30}{4}\right) = \mathbb{P}\left(Z \leq \frac{x-30}{4}\right)$$

where $Z \sim$ N$(0,1)$.
We can get hold of this value of x by using R (or by consulting statistical tables): `qnorm(0.99)` gives the value 2.326, meaning that $\mathbb{P}(Z \leq 2.326) = 0.99$. Thus we require $(x-30)/4 = 2.326 \iff x = 39.3$. Thus the latest you should leave your room is 39.3 minutes before the start of the lecture: i.e. at 13:20.

S6. Give an example of a joint probability table for two discrete random variables X and Y, each having only two possible values, so that $F_{X,Y}(5,6) = 0.4$, $F_X(5) = 0.5$, $F_Y(6) = 0.6$ and $\mathbb{E}[X] = 10$, $\mathbb{E}[Y] = 4$.

Answer

One possible example would be

$y\backslash x$	0	20	$p_Y(y)$
0	0.4	0.2	0.6
10	0.1	0.3	0.4
$p_X(x)$	0.5	0.5	1

S7. Let $X : \Omega \to \{1, 2\}$ and $Y : \Omega \to \{0, 1\}$ be two discrete random variables. The following is a partial table of their joint and their marginal mass functions:

$y\backslash x$	1	2	$p_Y(y)$
0	$1/6$	$1/2$	
1			
$p_X(x)$	$5/12$		1

a) Fill in the missing values.
b) Determine the joint distribution function of X and Y.
c) Calculate $\mathbb{P}(X + Y = 2)$.
d) Calculate $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.
e) Let $Z = XY$. Calculate $\mathbb{E}[Z]$.
f) Are X and Y independent?

Answer

a) The missing entries in the probability table are determined by the requirement that summing the joint probabilities across a row or across a column in the table gives the corresponding marginal probability and by the requirement that the marginal probabilities for X as well as those for Y have to add up to 1. So first we determine $p_Y(0) = 1/6 + 1/2 = 2/3$. Then we can determine $p_Y(1) = 1 - p_Y(0) = 1 - 2/3 = 1/3$ and $p_X(2) = 1 - p_X(1) = 1 - 5/12 = 7/12$. Finally we determine $p_{X,Y}(1, 1) = p_X(1) - p_{X,Y}(1, 0) = 5/12 - 1/6 = 1/4$ and $p_{X,Y}(2, 1) = p_X(2) - p_{X,Y}(2, 0) = 7/12 - 1/2 = 1/12$.

$y\backslash x$	1	2	$p_Y(y)$
0	$1/6$	$1/2$	$2/3$
1	$1/4$	$1/12$	$1/3$
$p_X(x)$	$5/12$	$7/12$	1

b) The joint distribution function $F_{X,Y}(x,y)$ is by definition given by $\mathbb{P}(X \leq x, Y \leq y)$. So for example

$$F_{X,Y}(1.5, 1.5) = p_{X,Y}(1,0) + p_{X,Y}(1,1) = \frac{1}{6} + \frac{1}{4} = \frac{5}{12}.$$

By doing more such calculations we find that

$$F_{X,Y} = \begin{cases} 0 & \text{if } x < 1 \text{ or } y < 0 \\ 1/6 & \text{if } x \in [1,2) \text{ and } y \in [0,1) \\ 5/12 & \text{if } x \in [1,2) \text{ and } y \geq 1 \\ 2/3 & \text{if } x \geq 2 \text{ and } y \in [0,1) \\ 1 & \text{if } x \geq 2 \text{ or } y \geq 1. \end{cases}$$

c) $\mathbb{P}(X+Y=2) = \mathbb{P}(X=1, Y=1) + \mathbb{P}(X=2, Y=0) = 1/4 + 1/2 = 3/4$.

d) For calculating the expectations of X and Y we can use their marginal mass functions:

$$\mathbb{E}[X] = 1 \cdot p_X(1) + 2 \cdot p_X(2) = 1 \cdot \frac{5}{12} + 2 \cdot \frac{7}{12} = \frac{19}{12}$$

and

$$\mathbb{E}[Y] = 0 \cdot p_Y(0) + 1 \cdot p_Y(1) = p_Y(1) = \frac{1}{3}.$$

e) The random variable $Z = XY$ can take the possible values 0, 1 and 2 with probabilities

$$p_Z(0) = p_{X,Y}(1,0) + p_{X,Y}(2,0) = p_Y(0) = \frac{2}{3}$$
$$p_Z(1) = p_{X,Y}(1,1) = \frac{1}{4}, \quad p_Z(2) = p_{X,Y}(2,1) = \frac{1}{12}.$$

Thus

$$\mathbb{E}[Z] = 1 \cdot P_Z(1) + 2 \cdot P_Z(2) = \frac{1}{4} + 2\frac{1}{12} = \frac{5}{12}.$$

f) X and Y are independent if and only if $p_{X,Y}(x,y) = p_X(x)p_Y(y)$ for all x, y. Here however for example

$$p_{X,Y}(1,0) = \frac{1}{6} \neq p_X(1)p_Y(0) = \frac{5}{12} \cdot \frac{2}{3} = \frac{5}{18}.$$

So X and Y are not independent.

S8. Let X and Y be random variables. Show that $\text{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$.

Answer

We start from the definition of covariance, and use linearity of expectation:

$$\begin{aligned} \text{Cov}(X,Y) &= \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] \\ &= \mathbb{E}[XY - X\mathbb{E}[Y] - \mathbb{E}[X]Y + \mathbb{E}[X]\mathbb{E}[Y]] \\ &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[X]\mathbb{E}[Y] + \mathbb{E}[X]\mathbb{E}[Y] \\ &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]. \end{aligned}$$

Main course

These are important, and cover some of the most substantial parts of the course.

M1. Hand-in (worth 10 marks)

A married couple decide to have children until they have at least one child of each sex: let X denote the total number of children that they have. The probability of any one child being a boy is p (with the sex of each child being independent of all the others).

a) What is the mass function of X? (I.e. write down $\mathbb{P}(X = n)$ for all $n \in X(\Omega)$.)

b) Show that

$$\mathbb{E}[X] = \frac{1 - p(1-p)}{p(1-p)}\,.$$

Hint: you may find it useful to refer to the result from lectures that if $Y \sim \text{Geom}(p)$ then $\mathbb{E}[Y] = 1/p$.

c) For what value of p is $\mathbb{E}[X]$ minimised?

Answer

a) Clearly the couple need to have at least two children, so $X(\Omega) = \{2, 3, 4, \dots\}$. For $n \geq 2$, there are two ways in which the couple can have exactly n children: either they have $n - 1$ boys in a row, and then a girl; or they have $n - 1$ girls and then a boy. Thus

$$\mathbb{P}(X = n) = p^{n-1}(1-p) + (1-p)^{n-1}p\,, \qquad n \geq 2\,.$$

[3]

b) To calculate $\mathbb{E}[X]$ we use the usual formula for the expectation of a discrete random variable:

$$\begin{aligned}\mathbb{E}[X] &= \sum_{n=2}^{\infty} n\mathbb{P}(X = n)\\ &= \sum_{n=2}^{\infty} n[p^{n-1}(1-p) + (1-p)^{n-1}p] \qquad (1)\\ &= \sum_{n=2}^{\infty} np^{n-1}(1-p) + \sum_{n=2}^{\infty} n(1-p)^{n-1}p\,.\end{aligned}$$

[1]

Using the hint, we know that if $Y \sim \text{Geom}(p)$ then $\mathbb{E}[Y] = 1/p$. That is,

$$\sum_{n=1}^{\infty} np(1-p)^{n-1} = \frac{1}{p}.$$

We can use that result in the second sum in Equation 1 by adding and subtracting the missing $n = 1$ term in the sum:

$$\sum_{n=2}^{\infty} np(1-p)^{n-1} = \sum_{n=1}^{\infty} np(1-p)^{n-1} - p = \frac{1}{p} - p.$$

[2]

The first sum in Equation 1 is the same as the second, just with p and $1 - p$ interchanged, so

$$\sum_{n=2}^{\infty} np^{n-1}(1-p) = \frac{1}{1-p} - (1-p).$$

[1]

Adding these two results together we find

$$\mathbb{E}[X] = \frac{1}{1-p} - (1-p) + \frac{1}{p} - p = \frac{1 - p(1-p)}{p(1-p)}.$$

[1]

c) Differentiating $\mathbb{E}[X]$ with respect to p we get $(2p-1)/(p^2(1-p)^2)$. This is equal to zero when $p = 1/2$. (This is clearly where the minimum is obtained, since it is the only turning point and $\mathbb{E}[X] \to \infty$ as $p \to 0$ or $p \to 1$.) **[2]**

M2. Let $X \sim \text{Exp}(\lambda)$. Use proof by induction to show that

$$\mathbb{E}[X^m] = \frac{m!}{\lambda^m}$$

for all $m \in \mathbb{N} \cup \{0\}$.

Answer

The statement is true for $m = 0$:

$$\mathbb{E}[X^0] = \mathbb{E}[1] = 1 = \frac{0!}{\lambda^0}.$$

We next show that if the statement holds for some $k \in \mathbb{N} \cup \{0\}$ then it holds for $k+1$:

$$\begin{aligned}
\mathbb{E}[X^{k+1}] &= \int_{-\infty}^{\infty} x^{k+1} f_X(x)dx = \int_0^{\infty} x^{k+1} \lambda e^{-\lambda x} dx \\
&= \int_0^{\infty} x^{k+1} \frac{d}{dx}\left(-e^{-\lambda x}\right) dx \\
&= -\left[x^{k+1} e^{-\lambda x}\right]_0^{\infty} + \int_0^{\infty} (k+1) x^k e^{-\lambda x} dx \\
&= 0 + \frac{k+1}{\lambda} \int_0^{\infty} x^k \lambda e^{-\lambda x} dx \\
&= \frac{k+1}{\lambda} \mathbb{E}[X^k] = \frac{k+1}{\lambda} \frac{k!}{\lambda^k} \quad \text{by our induction hypothesis} \\
&= \frac{(k+1)!}{\lambda^{k+1}}.
\end{aligned}$$

Thus the statement holds for all $m \in \mathbb{N} \cup \{0\}$ by induction.

M3. The joint probability mass function $p_{X,Y}(x, y)$ of two random variables X and Y is summarised by the following table:

$x \backslash y$	-1	0	1
4	$\eta - 1/16$	$1/4 - \eta$	0
5	$1/8$	$3/16$	$1/8$
6	$\eta + 1/16$	$1/16$	$1/4 - \eta$

where η is a real number.

a) Extend the table by including also the marginal probabilities, i.e., the values of the probability mass functions p_X and p_Y.

b) Which are the valid choices for η?

c) Is there a value of η for which X and Y are independent?

Answer

a) We extend the probability table to also include the marginal probability mass functions p_X and p_Y:

$x\backslash y$	-1	0	1	$p_X(x)$
4	$\eta - 1/16$	$1/4 - \eta$	0	$3/16$
5	$1/8$	$3/16$	$1/8$	$7/16$
6	$\eta + 1/16$	$1/16$	$1/4 - \eta$	$3/8$
$p_Y(y)$	$2\eta + 1/8$	$1/2 - \eta$	$3/8 - \eta$	1

b) All entries of the probability table must be non-negative and they must sum up to 1. In order for $p_{X,Y}(4, -1)$ to be non-negative we need $\eta \geq 1/16$. In order for $p_{X,Y}(4, 0)$ and $p_{X,Y}(6, 1)$ to be non-negative we need $\eta \leq 1/4$. The sum over all entries is not affected by the value of η, so does not give any additional constraints. Therefore any $\eta \in [1/16, 1/4]$ is a valid choice.

c) It is easy to find counterexamples to the factorisation of the joint probability mass function that would have to hold if X and Y were independent. For example

$$p_X(4)p_Y(1) = \frac{3}{16}\left(\frac{3}{8} - \eta\right) \neq 0 = p_{X,Y}(4, 1)$$

unless $\eta = 3/8$. However the value $\eta = 3/8$ is not allowed, and hence X and Y can never be independent.

M4. Prove that binomial coefficients satisfy the identity

$$n\binom{n-1}{r-1} = r\binom{n}{r}.$$

Use this to find $\mathbb{E}[X]$ and Var(X), where $X \sim$ Bin(n, p).

Answer

First we prove the identity:

$$n\binom{n-1}{r-1} = n\,\frac{(n-1)!}{(r-1)!(n-r)!} = r\,\frac{n!}{r!(n-r)!} = r\binom{n}{r}.$$

For the mean and variance, remember that, since $p_X(\cdot)$ is a mass function, it must sum to one. That is,

$$\sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k} = 1\,. \qquad (2)$$

Now,

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{k=0}^{n} k\binom{n}{k}p^k(1-p)^{n-k} \\
&= \sum_{k=1}^{n} n\binom{n-1}{k-1}p^k(1-p)^{n-k} \quad \text{(by our identity)} \\
&= np\sum_{k=1}^{n} \binom{n-1}{k-1}p^{k-1}(1-p)^{n-k} \\
&= np\sum_{j=0}^{n-1} \binom{n-1}{j}p^{j}(1-p)^{(n-1)-j} \quad \text{(putting } j=k-1\text{)} \\
&= np,
\end{aligned}$$

thanks to Equation 2.
Furthermore,

$$\begin{aligned}
\mathbb{E}[X(X-1)] &= \sum_{k=0}^{n} k(k-1)\binom{n}{k}p^k(1-p)^{n-k} \\
&= \sum_{k=0}^{n} k(k-1)\frac{n!}{k!(n-k)!}p^k(1-p)^{n-k} \\
&= \sum_{k=0}^{n} \frac{n!}{(n-k)!(k-2)!}p^k(1-p)^{n-k} \\
&= n(n-1)p^2\sum_{k=2}^{n} \frac{(n-2)!}{((n-2)-(k-2))!(k-2)!}p^{k-2}(1-p)^{(n-2)-(k-2)} \\
&= n(n-1)p^2\sum_{j=0}^{n-2} \binom{n-2}{j}p^{j}(1-p)^{(n-2)-j} \quad \text{(putting } j=k-2\text{)} \\
&= n(n-1)p^2,
\end{aligned}$$

again thanks to Equation 2. It follows that

$$\mathbb{E}[X^2] = n(n-1)p^2 + np,$$

and so

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = np(1-p).$$

M5. Show that if Z is a standard normal random variable then, for $x > 0$,

a) $\mathbb{P}(Z > x) = \mathbb{P}(Z < -x)$;
b) $\mathbb{P}(|Z| > x) = 2\mathbb{P}(Z > x)$;
c) $\mathbb{P}(|Z| < x) = 2\mathbb{P}(Z < x) - 1$.

Hint: express the probabilities in terms of integrals over the density function ϕ, and use the fact that ϕ is a symmetric function (i.e. $\phi(z) = \phi(-z)$).

Answer

There are many ways to show these identities. We use the hint about the symmetry of the density function of a standard normal random variable:

$$\phi(-z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(-z)^2}{2}\right) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) = \phi(z).$$

a)

$$\mathbb{P}(Z > x) = \int_x^\infty \phi(z)dz = \int_{-\infty}^{-x} \phi(-u)du = \int_{-\infty}^{-x} \phi(u)du = \mathbb{P}(Z < -x);$$

b)

$$\mathbb{P}(|Z| > x) = \mathbb{P}(Z > x) + \mathbb{P}(Z < -x) = 2\mathbb{P}(Z > x),$$

where the last equality follows from part (a).

c)

$$\begin{aligned}\mathbb{P}(|Z| < x) &= 1 - \mathbb{P}(|Z| > x) = 1 - 2\mathbb{P}(Z > x) \\ &= 1 - 2\,(1 - \mathbb{P}(Z < x)) = 2\mathbb{P}(Z < x) - 1\,,\end{aligned}$$

where the second equality follows from part (b).

M6. Let X be a discrete random variable. Show that for all functions $h_1, h_2 : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}[h_1(X) + h_2(X)] = \mathbb{E}[h_1(X)] + \mathbb{E}[h_2(X)]\,.$$

Answer

Let $h(x) = h_1(x) + h_2(x)$. From the formula for the expectation of a function of a discrete random variable it follows that

$$\begin{aligned}\mathbb{E}[h(X)] &= \sum_{k \in X(\Omega)} h(k)p_X(k) \\ &= \sum_{k \in X(\Omega)} (h_1(k) + h_2(k))p_X(k) \\ &= \sum_{k \in X(\Omega)} h_1(k)p_X(k) + \sum_{k \in X(\Omega)} h_2(k)p_X(k) \\ &= \mathbb{E}[h_1(X)] + \mathbb{E}[h_2(X)]\,.\end{aligned}$$

M7. Let X and Y be random variables and let $r, s, t, u \in \mathbb{R}$. Show that

$$\rho(rX + s, tY + u) = \begin{cases} \rho(X, Y) & \text{if } rt > 0 \\ 0 & \text{if } rt = 0 \\ -\rho(X, Y) & \text{if } rt < 0 \end{cases}$$

where $\rho(X, Y)$ denotes the correlation coefficient of X and Y.

Answer

Let us first assume that $\mathrm{Var}(X)\mathrm{Var}(Y) > 0$ and $rt > 0$. Then the definition of the correlation coefficient gives

$$\rho(rX + s, tY + u) = \frac{\mathrm{Cov}(rX + s, tY + u)}{\sqrt{\mathrm{Var}(rX + s)\mathrm{Var}(tY + u)}}. \tag{3}$$

We already know that

$$\mathsf{Var}(rX + s) = r^2\mathsf{Var}(X), \quad \mathsf{Var}(tY + u) = t^2\mathsf{Var}(Y). \tag{4}$$

We need to derive a similar transformation rule for the covariance.

$$\begin{aligned}
\mathsf{Cov}(rX + s, tY + u) &= \mathbb{E}[(rX + s - \mathbb{E}[rX + s])(tY + u - \mathbb{E}[tY + u])] \\
&= \mathbb{E}[(rX + s - (r\mathbb{E}[X] + s))(tY + u - (t\mathbb{E}[Y] + u))] \\
&= \mathbb{E}[r(X - \mathbb{E}[X])t(Y - \mathbb{E}[Y])] \\
&= rt\mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] \\
&= rt\mathsf{Cov}(X, Y)\,,
\end{aligned} \tag{5}$$

where we repeatedly used the linearity of expectation. Using the transformation rules Equation 4 and Equation 5 in Equation 3 gives

$$\rho(rX + s, tY + u) = \frac{rt}{\sqrt{r^2t^2}} \frac{\mathsf{Cov}(X, Y)}{\sqrt{\mathsf{Var}(X)\mathsf{Var}(Y)}}.$$

The statement now follows from the observation that

$$\frac{rt}{\sqrt{r^2t^2}} = \begin{cases} 1 & \text{if } rt > 0 \\ -1 & \text{if } rt < 0\,. \end{cases}$$

In case $\mathsf{Var}(X)\mathsf{Var}(Y) = 0$ or $rt = 0$ also $\mathsf{Var}(rX + s)\mathsf{Var}(tY + u) = rt\mathsf{Var}(X)\mathsf{Var}(Y) = 0$, and thus $\rho(rX + s, tY + u) = 0$ by definition. This agrees with the statement because when $\mathsf{Var}(X)\mathsf{Var}(Y) = 0$ also $\rho(X, Y) = 0$.

M8. Let $X \sim \mathsf{Uniform}(0, a)$ for some $a > 0$. Show that for any $n \in \mathbb{N}$,

$$\mathbb{E}[X^n] = \frac{a^n}{n + 1}.$$

Use this to determine $\rho(X, X^2)$, and show that this does not depend upon the value of a.

Answer

For $n \in \mathbb{N}$ we calculate

$$\mathbb{E}[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)dx = \int_0^a \frac{x^n}{a}dx = \frac{1}{a}\left[\frac{x^{n+1}}{n + 1}\right]_0^a = \frac{a^n}{n + 1}.$$

Now we calculate the covariance of X and X^2:

$$\mathsf{Cov}(X, X^2) = \mathbb{E}[X^3] - \mathbb{E}[X]\mathbb{E}[X^2] = \frac{a^3}{4} - \frac{a^2}{3}\frac{a}{2} = \frac{a^3}{12}.$$

We also have

$$\mathsf{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{a^2}{3} - \left(\frac{a}{2}\right)^2 = \frac{a^2}{12}$$

and

$$\mathsf{Var}(X^2) = \mathbb{E}[X^4] - \mathbb{E}[X^2]^2 = \frac{a^4}{5} - \left(\frac{a^2}{3}\right)^2 = \frac{4a^4}{45}.$$

Finally, we calculate

$$\rho(X, X^2) = \frac{\text{Cov}(X, X^2)}{\sqrt{\text{Var}(X)\text{Var}(X^2)}} = \frac{a^3/12}{\sqrt{a^6/135}} = \frac{\sqrt{135}}{12} = \frac{\sqrt{15}}{4},$$

which doesn't depend upon a.

M9. A bag contains 3 cubes, 4 pyramids and 7 spheres. An object is drawn randomly from the bag and its type is recorded. Then the object is replaced. This is repeated 20 times.

a. Let C_i be the indicator random variable for the event that the i-th draw gives a cube, for $i = 1, \ldots, 20$. Calculate $\mathbb{E}[C_i]$, $\mathbb{E}[C_i^2]$ and $\mathbb{E}[C_iC_j]$ for $i \neq j$.

b. Let C be the number of times a cube was drawn, Use that $C = \sum_{i=1}^{20} C_i$ to calculate $\mathbb{E}[C]$ and Var(C).

c. Let S_i be the indicator random variable for the event that the i-th draw gives a sphere. Calculate $\mathbb{E}[C_iS_i]$ and $\mathbb{E}[C_iS_j]$ for $i \neq j$.

d. Let S be the number of times a sphere was drawn. Use the above results to calculate $\mathbb{E}[CS]$, Cov(C, S), $\rho(C, S)$.

Answer

a. As three of the 14 shapes are cubes, the probability to draw a cube is $3/14$. Hence $C_i \sim$ Bern$(3/14)$. This immediately gives

$$\mathbb{E}[C_i] = \mathbb{E}[C_i^2] = \frac{3}{14}.$$

For $i \neq j$ the event that the i-th draw gives a cube and the event that the j-th cube gives a draw are independent (because we put the shape back after each draw). Thus the indicator random variables C_i and C_j for these events are also independent and thus

$$\mathbb{E}[C_iC_j] = \mathbb{E}[C_i]\mathbb{E}[C_j] = \left(\frac{3}{14}\right)^2 = \frac{9}{196}.$$

b. The linearity of expectation gives

$$\mathbb{E}[C] = \mathbb{E}[\sum_{i=1}^{20} C_i] = \sum_{i=1}^{20} \mathbb{E}[C_i] = 20\frac{3}{14} = \frac{30}{7}.$$

Because the C_i are independent of each other, the variance of their sum equals the sum of their variances:

$$\text{Var}(C) = \text{Var}(\sum_{i=1}^{20} C_i) = \sum_{i=1}^{20} \text{Var}(C_i) = 20\frac{3}{14}\frac{11}{14} = \frac{165}{49}.$$

c. We observe that $C_iS_i = 0$ because on the same draw one can not simultaneously have a cube and a sphere. Thus also $\mathbb{E}[C_iS_i] = 0$. If $i \neq j$ we can use independence to factorise the expectation:

$$\mathbb{E}[C_iS_j] = \mathbb{E}[C_i]\mathbb{E}[S_j] = \frac{3}{14}\frac{1}{2} = \frac{3}{28},$$

where we used that the probability of drawing a sphere is $1/2$.

d. We have

$$\mathbb{E}[CS] = \mathbb{E}[\sum_{i=1}^{20} C_i \sum_{j=1}^{20} S_j] = \sum_{i=1}^{20}\sum_{j=1}^{20} \mathbb{E}[C_i S_j].$$

We split the sum over all pairs (i, j) into the pairs where $i \neq j$ and the pairs (i, i), so

$$\mathbb{E}[CS] = \sum_{i=1}^{20}\sum_{\substack{j=1 \\ j\neq i}}^{20} \mathbb{E}[C_i S_j] + \sum_{i=1}^{20} \mathbb{E}[C_i S_i]\,.$$

Using our above results for $\mathbb{E}[C_i S_i]$ and $\mathbb{E}[C_i S_j]$ and recognising that there are $20\cdot 19 = 380$ pairs where $i \neq j$ this gives us

$$\mathbb{E}[CS] = \sum_{i=1}^{20}\sum_{\substack{j=1 \\ j\neq i}}^{20} \frac{3}{28} + \sum_{i=1}^{20} 0 = 380\frac{3}{28} = \frac{285}{7}\,.$$

We also calculate

$$\mathbb{E}[S] = \mathbb{E}[\sum_{i=1}^{20} S_i] = \sum_{i=1}^{20} \mathbb{E}[S_i] = 20\frac{1}{2} = 10.$$

The covariance can then be calculated as

$$\text{Cov}(C, S) = \mathbb{E}[CS] - \mathbb{E}[C]\mathbb{E}[S] = \frac{285}{7} - \frac{30}{7}10 = -\frac{15}{7}.$$

To calculate the correlation coefficient we also need

$$\text{Var}(S) = \text{Var}(\sum_{i=1}^{20} S_i) = \sum_{i=1}^{20} \text{Var}(S_i) = 20\frac{1}{2}\frac{1}{2} = 5.$$

The correlation coefficient is

$$\rho(C, S) = \frac{\text{Cov}(C, S)}{\sqrt{\text{Var}(C)\text{Var}(S)}} = -\sqrt{\frac{3}{11}} \approx -0.5222.$$

Dessert

Still hungry for more? Try these if you want to push yourself further.

D1. Consider a random variable $X \sim \text{Uniform}[a, b]$, where a and b are unknown. You are told that

$$\mathbb{P}(X < 2) = 1/3 \quad \text{and} \quad \mathbb{P}(1 < X \leq 3) = 1/2\,.$$

Given this information, find a and b.

Answer

From the first equation we immediately know that $a < 2 < b$. Now, for a continuous random variable, we obtain the probability that it lies in an interval (c, d) by integrating the density function over that interval, i.e.

$$\mathbb{P}(c \leq X \leq d) = \int_c^d f_X(x)dx\,.$$

Since $X \sim \text{Uniform}[a, b]$, we know that

$$f_X(x) = \begin{cases} 1/(b-a) & x \in [a, b] \\ 0 & \text{otherwise.} \end{cases}$$

Thus we obtain

$$1/3 = \mathbb{P}(X < 2) = \mathbb{P}(a \leq X < 2) = \int_a^2 1/(b-a)dx = (2-a)/(b-a)\,. \tag{6}$$

In order to use the second equation ($\mathbb{P}(1 < X \leq 3) = 1/2$) in the same way, we have two possibilities to consider:

1. $a < 1$
2. $1 \leq a < 2$

Suppose first that $a < 1$. Then

$$1/2 = \mathbb{P}(1 < X \leq 3) = \int_1^3 1/(b-a)dx = 2/(b-a)\,, \tag{7}$$

since the density function f_X is equal to $1/(b-a)$ for all $x \in [1, 3]$ if $a < 1$.
If $1 \leq a$ however, then instead we obtain

$$1/2 = \mathbb{P}(1 < X \leq 3) = \int_1^a 0\,dx + \int_a^3 1/(b-a)dx = (3-a)/(b-a)\,. \tag{8}$$

We now have to solve these simultaneous equations in order to find a and b. If we assume that $1 \leq a < 2$, then we must try to solve Equation 6 and Equation 8 together; but this gives

$$2(3-a) = 3(2-a)\,,$$

resulting in $a = 0$. But this contradicts our assumption that $1 \leq a$!
So it must be the case that $a < 1$: now we must solve Equation 6 and Equation 7, and this *is* possible, with $a = 2/3$ and $b = 14/3$.

D2. Let X and Y be two independent geometrically distributed random variables with parameter p, i.e., $X \sim \text{Geom}(p)$ and $Y \sim \text{Geom}(p)$. For any natural numbers i and n with $i < n$ calculate the conditional probability $\mathbb{P}(X = i \,|\, X + Y = n)$. Describe in words the meaning in terms of Bernoulli trials of what you just calculated.

Answer

According to the definition of conditional probability,

$$\mathbb{P}(X = i \,|\, X + Y = n) = \frac{\mathbb{P}(X = i,\ X + Y = n)}{\mathbb{P}(X + Y = n)}.$$

For the numerator we can use that the event $\{X = i, X + Y = n\}$ is the event $\{X = i, Y = n - i\}$.
We then know that the independence of X and Y implies the factorisation of that probability:

$$\mathbb{P}(X = i,\ X + Y = n) = \mathbb{P}(X = i, Y = n - i) = \mathbb{P}(X = i)\mathbb{P}(Y = n - i).$$

We can now substitute in the probability mass function for the geometric distribution with parameter p:

$$\mathbb{P}(X = i) = (1-p)^{i-1}p$$

and thus

$$\mathbb{P}(Y = n - i) = (1 - p)^{n-i-1}p.$$

This gives

$$\mathbb{P}(X = i,\ X + Y = n) = (1 - p)^{i-1}p(1 - p)^{n-i-1}p = (1 - p)^{n-2}p^2.$$

Note that this is independent of i.
For the denominator we use the partition theorem to write

$$\mathbb{P}(X + Y = n) = \sum_{i=1}^{n-1} \mathbb{P}(X = i, X + Y = n).$$

From our calculation above we see that every term in the sum is the same, so

$$\mathbb{P}(X + Y = n) = (n - 1)\mathbb{P}(X = i, X + Y = n).$$

Putting this all together we finally find that

$$\mathbb{P}(X = i \,|\, X + Y = n) = \frac{\mathbb{P}(X = i, X + Y = n)}{\mathbb{P}(X + Y = n)} = \frac{1}{n - 1}.$$

A geometric random variable counts the number of turns until the first success in repeated Bernoulli trials. Therefore the sum $X + Y$ of two identical and independent geometric random variables counts the number of turns until the *second* success. So the conditional probability we calculated is the probability that the first success happens on a particular trial i given that the second success happens on the n-th trial. The result shows that the first success is then equally likely to occur on any of the $n - 1$ trials before the n-th trial.

! Challenge question

A stick of length 1 is snapped into two at a point $U \sim \text{Uniform}(0, 1)$. What is the expected length of the piece containing the point s, where s is some fixed number between 0 and 1? For what values of s is this expected length maximised/minimised? How does the variance of this length depend upon s?