IPS: Assignment 4

Due date: 10am, 23/11/23

Stephen Connor

> **Instructions**
>
> Submit your answers to all questions marked **Hand-in**. You should upload to Moodle your solutions as a *pdf file*.
> Marks will be awarded for clear, logical explanations, as well as for correctness of solutions.
> You should also look at the other questions in preparation for your Week 9 seminar.

Starter

These questions should help you to gain confidence with the basics.

S1. Let X be a random variable with $\mathbb{E}[X] = 5$. What is the expectation of $3X + 5$? If furthermore $\mathbb{E}[X^2] = 30$, what is the variance of X?

S2. Hand-in
Buses leave campus for the train station every 20 minutes, at 0, 20 and 40 minutes past the hour. If a student arrives at the bus stop at a time that is uniformly distributed between 10.00 and 10.45, find the probability that they wait

a) less than 5 minutes for a bus;
b) more than 10 minutes for a bus.

S3. A random variable Z has probability density function

$$f_Z(x) = \begin{cases} \frac{6}{5675}(5x^2 + 3x + 11) & \text{for } 3 \leq x \leq 8 \\ 0 & \text{otherwise.} \end{cases}$$

Would you expect $\mathbb{E}[Z]$ to lie closer to 3 or to 8? Calculate $\mathbb{E}[Z]$ and check whether your intuition was correct.

S4. Hand-in
Let $X \sim \text{Geom}(p)$. Calculate $\mathbb{E}[h(X)]$, where $h(x) = e^{tx}$ for some $t > 0$. For what values of t is $\mathbb{E}[h(X)] < \infty$?
Hint: use the result for infinite geometric series.

S5. Suppose that you have a lecture at 14.00, and that the time taken to travel from your room to the lecture theatre is normally distributed with mean 30 minutes and standard deviation 4 minutes. What is the latest time you should leave your room if you want to be 99% certain that you will not miss the start of the lecture? (Hint: if $Z \sim \text{N}(0, 1)$ then the R function `qnorm(p)` returns the value $z \in \mathbb{R}$ such that $\mathbb{P}(Z \leq z) = p$.)

S6. Give an example of a joint probability table for two discrete random variables X and Y, each having only two possible values, so that $F_{X,Y}(5,6) = 0.4, F_X(5) = 0.5, F_Y(6) = 0.6$ and $\mathbb{E}[X] = 10, \mathbb{E}[Y] = 4$.

S7. Let $X : \Omega \to \{1,2\}$ and $Y : \Omega \to \{0,1\}$ be two discrete random variables. The following is a partial table of their joint and their marginal mass functions:

$y\backslash x$	1	2	$p_Y(y)$
0	$1/6$	$1/2$	
1			
$p_X(x)$	$5/12$		1

a) Fill in the missing values.
b) Determine the joint distribution function of X and Y.
c) Calculate $\mathbb{P}(X + Y = 2)$.
d) Calculate $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.
e) Let $Z = XY$. Calculate $\mathbb{E}[Z]$.
f) Are X and Y independent?

S8. Let X and Y be random variables. Show that Cov$(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$.

Main course

These are important, and cover some of the most substantial parts of the course.

M1. Hand-in (worth 10 marks)
A married couple decide to have children until they have at least one child of each sex: let X denote the total number of children that they have. The probability of any one child being a boy is p (with the sex of each child being independent of all the others).

a) What is the mass function of X? (I.e. write down $\mathbb{P}(X = n)$ for all $n \in X(\Omega)$.)

b) Show that

$$\mathbb{E}[X] = \frac{1 - p(1-p)}{p(1-p)}.$$

Hint: you may find it useful to refer to the result from lectures that if $Y \sim$ Geom(p) then $\mathbb{E}[Y] = 1/p$.

c) For what value of p is $\mathbb{E}[X]$ minimised?

M2. Let $X \sim$ Exp(λ). Use proof by induction to show that

$$\mathbb{E}[X^m] = \frac{m!}{\lambda^m}$$

for all $m \in \mathbb{N} \cup \{0\}$.

M3. The joint probability mass function $p_{X,Y}(x,y)$ of two random variables X and Y is summarised by the following table:

$x\backslash y$	-1	0	1
4	$\eta - 1/16$	$1/4 - \eta$	0
5	$1/8$	$3/16$	$1/8$

$x\backslash y$	-1	0	1
6	$\eta + 1/16$	$1/16$	$1/4 - \eta$

where η is a real number.

a) Extend the table by including also the marginal probabilities, i.e., the values of the probability mass functions p_X and p_Y.

b) Which are the valid choices for η?

c) Is there a value of η for which X and Y are independent?

M4. Prove that binomial coefficients satisfy the identity

$$n\binom{n-1}{r-1} = r\binom{n}{r}.$$

Use this to find $\mathbb{E}[X]$ and Var(X), where $X \sim$ Bin(n, p).

M5. Show that if Z is a standard normal random variable then, for $x > 0$,

a) $\mathbb{P}(Z > x) = \mathbb{P}(Z < -x)$;
b) $\mathbb{P}(|Z| > x) = 2\mathbb{P}(Z > x)$;
c) $\mathbb{P}(|Z| < x) = 2\mathbb{P}(Z < x) - 1$.

Hint: express the probabilities in terms of integrals over the density function ϕ, and use the fact that ϕ is a symmetric function (i.e. $\phi(z) = \phi(-z)$).

M6. Let X be a discrete random variable. Show that for all functions $h_1, h_2 : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}[h_1(X) + h_2(X)] = \mathbb{E}[h_1(X)] + \mathbb{E}[h_2(X)]\,.$$

M7. Let X and Y be random variables and let $r, s, t, u \in \mathbb{R}$. Show that

$$\rho(rX + s, tY + u) = \begin{cases} \rho(X, Y) & \text{if } rt > 0 \\ 0 & \text{if } rt = 0 \\ -\rho(X, Y) & \text{if } rt < 0 \end{cases}$$

where $\rho(X, Y)$ denotes the correlation coefficient of X and Y.

M8. Let $X \sim$ Uniform$(0, a)$ for some $a > 0$. Show that for any $n \in \mathbb{N}$,

$$\mathbb{E}[X^n] = \frac{a^n}{n+1}.$$

Use this to determine $\rho(X, X^2)$, and show that this does not depend upon the value of a.

M9. A bag contains 3 cubes, 4 pyramids and 7 spheres. An object is drawn randomly from the bag and its type is recorded. Then the object is replaced. This is repeated 20 times.

a. Let C_i be the indicator random variable for the event that the i-th draw gives a cube, for $i = 1, \dots, 20$. Calculate $\mathbb{E}[C_i]$, $\mathbb{E}[C_i^2]$ and $\mathbb{E}[C_iC_j]$ for $i \neq j$.

b. Let C be the number of times a cube was drawn, Use that $C = \sum_{i=1}^{20} C_i$ to calculate $\mathbb{E}[C]$ and Var(C).

c. Let S_i be the indicator random variable for the event that the i-th draw gives a sphere. Calculate $\mathbb{E}[C_i S_i]$ and $\mathbb{E}[C_i S_j]$ for $i \neq j$.

d. Let S be the number of times a sphere was drawn. Use the above results to calculate $\mathbb{E}[CS]$, $\text{Cov}(C, S)$, $\rho(C, S)$.

Dessert

Still hungry for more? Try these if you want to push yourself further.

D1. Consider a random variable $X \sim \text{Uniform}[a, b]$, where a and b are unknown. You are told that

$$\mathbb{P}(X < 2) = 1/3 \quad \text{and} \quad \mathbb{P}(1 < X \leq 3) = 1/2\,.$$

Given this information, find a and b.

D2. Let X and Y be two independent geometrically distributed random variables with parameter p, i.e., $X \sim \text{Geom}(p)$ and $Y \sim \text{Geom}(p)$. For any natural numbers i and n with $i < n$ calculate the conditional probability $\mathbb{P}(X = i \,|\, X + Y = n)$. Describe in words the meaning in terms of Bernoulli trials of what you just calculated.

> **! Challenge question**
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> A stick of length 1 is snapped into two at a point $U \sim \text{Uniform}(0, 1)$. What is the expected length of the piece containing the point s, where s is some fixed number between 0 and 1? For what values of s is this expected length maximised/minimised? How does the variance of this length depend upon s?