

IPS: Assignment 3

Due date: 10am, 09/11/23

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Instructions

Submit your answers to all questions marked  **Hand-in**. You should upload to [Moodle](#) your solutions as a *pdf file*.

Marks will be awarded for clear, logical explanations, as well as for correctness of solutions. You should also look at the other questions in preparation for your Week 7 seminar.

Starter

These questions should help you to gain confidence with the basics.

S1. Each morning a (fictional) student rolls a die and starts studying if she throws 6. Otherwise, she stays in bed. However, during the four months of exams, she tosses a coin instead of rolling a die and studies if she tosses Heads. On a (randomly chosen) morning, the student is studying. What is the probability that it is exam period? (Assume for simplicity that each month has the same number of days!)

S2. Consider a discrete random variable V taking values in $\{1, 2, 3, 4\}$ with mass function p_V given by

$$p_V(x) = \begin{cases} xc^x & \text{if } x = 1, 2 \\ c(5 - x) & \text{if } x = 3, 4. \end{cases}$$

What value of c makes p_V a mass function?

S3. Hand-in

Consider the probability space corresponding to throwing a fair die. Give an example of a random variable X whose image is $\{1, 2, 3\}$ and for which $F_X(\pi/2) = 1/2$.

S4. What is the probability that exactly 3 heads are obtained in 5 tosses of a fair coin?

S5. Hand-in

Let $\lambda > 0$ be some positive number. Find c such that the following function is a density function:

$$f(x) = ce^{-\lambda|x|}, \quad x \in \mathbb{R}.$$

S6. Let $Y \sim \text{Poisson}(2)$. Write down the mass function for Y . What is $\mathbb{P}(Y \geq 2)$?

S7. Recall the Chevalier de Méré's problem: two dice are thrown 24 times – you win £1 if at least one double 6 is thrown, otherwise you lose £1. How much do you expect to win on average? Is this a good bet?

S8. The probability density function f_X of a continuous random variable X is given by

$$f_X(x) = \begin{cases} -\frac{6(x^2+7x-c)}{25} & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Compute c . Give the distribution function of X in the interval $[0, 1]$.

S9. A continuous random variable X has a distribution function F which satisfies

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{4x^2+3x}{7} & \text{for } 0 \leq x \leq 1 \\ 1 & \text{for } x > 1. \end{cases}$$

Determine $E[X]$.

Main course

These are important, and cover some of the most substantial parts of the course.

M1. 📁 Hand-in

Suppose that A and B are independent events. Show that A^c and B^c are independent.

M2. 📁 Hand-in

A box contains four coins: one has a Head on both sides; one has a Tail on both sides; the other two coins are normal and fair. A coin is chosen at random from the box and tossed three times. What is the probability that the two-Headed coin was chosen, given that all three tosses are Heads?

M3. Consider a discrete random variable Y taking values in the set $\{0, 1, \dots, 8\}$ and with probability mass function p_Y given by

$$p_Y(k) = \begin{cases} ca & \text{if } k = 0, 1, 2, 3, 4, 5 \\ ca^2 & \text{if } k = 6, 7 \\ c(1-a)^2 & \text{if } k = 8, \end{cases}$$

where a is some fixed number between 0 and 1 and c is a constant to be determined. What value of c makes p_Y a mass function? (The answer is a function of a .)

M4. Let $X \sim \text{Bern}(p)$. Let $Y = 1 - X$ and $V = X^2$. Show that

- a) $Y \sim \text{Bern}(1 - p)$;
- b) $V \sim \text{Bern}(p)$.

M5. A fair die is thrown until the sum of the results of the throws exceeds 6. The random variable X is the number of throws needed for this. Let F_X be the distribution function of X . Determine $F_X(1)$, $F_X(2)$ and $F_X(7)$.

M6. Consider a roulette wheel, with numbers 00, 0, 1, 2, 3, ..., 36 (38 numbers in total). A ball is thrown onto the wheel as it is spinning, and comes to rest by one of the numbers. You always bet that the ball will stop on one of the numbers 1, 2, ..., 12. Let N be the random variable giving the number of bets that you lose *before* your first win. Give $p_N(5)$ and $F_N(5)$.

M7. $N \geq 3$ people go for coffee. Each person flips a fair coin: if all but one of the coins shows the same face (Head or Tail), then the odd person out pays for all the drinks; if not then the coins are tossed again until this event occurs. How many times on average must each person toss their coin before somebody is selected in this way?

M8. Prove the memoryless property of the exponential distribution which states that if $X \sim \text{Exp}(\lambda)$ then for any $t, s \geq 0$,

$$\mathbb{P}(X > t + s \mid X > s) = \mathbb{P}(X > t).$$

Dessert

Still hungry for more? Try these if you want to push yourself further.

D1. N people go for coffee. Each person flips a coin, of which $N - 1$ are fair, and one has probability q of coming up Heads (for some $q \in [0, 1]$): if all but one of the coins shows the same face (Head or Tail), then the odd person out pays for all the drinks; if not then the coins are tossed again until this event occurs. How does the expected number of coin tosses each person makes vary with q ?

D2. Alice and Bob decide to duel, using just one six-shot revolver, and one bullet, between them. They decide to duel in the following way: with the bullet inserted into the revolver, Alice will spin the cylinder and shoot at Bob (killing him if the gun fires); if Alice misses, Bob will spin the cylinder and shoot at Alice. Assuming there is a $1/6$ probability that the revolver will fire each time (since the revolver has 6 chambers), what is

- the distribution of the number of turns T until the gun fires?
- the probability that Alice wins the duel?

! Challenge question

At time 0, an urn contains 1 Red and 1 Black ball. Just before each time $n = 1, 2, 3, \dots$, a ball is chosen at random from the urn and then replaced, *along with a new ball of the same colour*. For $n = 0, 1, 2, \dots$ and $1 \leq r \leq n + 1$, let

$$p_{n,r} = \mathbb{P}(\text{at time } n, \text{ the urn contains exactly } r \text{ Red balls}).$$

- Write down the values of $p_{0,1}$, $p_{1,1}$ and $p_{1,2}$;
- By conditioning on the possible numbers of red balls at time 1, calculate $p_{2,r}$ for $r = 1, 2, 3$;
- Can you find (and prove!) a general formula for $p_{n,r}$? [HINT: guess at the right answer using your findings from parts (a) and (b). (Calculate $p_{3,r}$ explicitly if you can't see a pattern.) Then use induction.]