Introduction to Probability & Statistics

Assignment 3, 2024/25

Assignment Questions – answers to be uploaded

AQ1. Consider the probability space corresponding to throwing a fair die. Give an example of a random variable X whose image is $\{1, 2, 3, 4\}$ and for which $F_X(\pi) = 2/3$.

> Answer
>
> There are lots of possibilities. The only important thing is that exactly two of the outcomes ($\Omega = \{1, 2, 3, 4, 5, 6\}$) are mapped to 4 so that $F_X(\pi) = \mathbb{P}(X \leq \pi) = 1 - \mathbb{P}(X = 4) = 1 - 2/6 = 2/3$. One possibility is $X(1) = 1, X(2) = 2, X(3) = 3, X(4) = 3, X(5) = 4, X(6) = 4$. **[3 for a correct example; 2 for some sort of reasoning]**

AQ2. Let $\lambda > 0$ be some positive number. Find c such that the following function is a density function:

$$f(x) = ce^{-\lambda|x-1|}, \quad x \in \mathbb{R}.$$

> Answer
>
> For f to be a density, it must be non-negative on $\mathbb{R}$ and satisfy $\int_{-\infty}^{\infty} f(x)dx = 1$. Since the exponential function is non-negative, f is non-negative everywhere as long as $c \geq 0$ **[1]**.
> Note that
>
> $$\begin{aligned}\int_{-\infty}^{\infty} e^{-\lambda|x-1|}dx &= \int_{-\infty}^{1} e^{-\lambda(1-x)}dx + \int_{1}^{\infty} e^{-\lambda(x-1)}dx \\ &= \frac{1}{\lambda}\left[e^{-\lambda(1-x)}\right]_{-\infty}^{1} - \frac{1}{\lambda}\left[e^{-\lambda(x-1)}\right]_{1}^{\infty} = \frac{2}{\lambda}.\end{aligned}$$
>
> **[3]**
> So $f(\cdot)$ is a density if and only if $c = \lambda/2$. **[1]**

AQ3. Suppose that A and B are independent events. Show that A^c and B^c are independent.

Answer

We need to show that $\mathbb{P}(A^c \cap B^c) = \mathbb{P}(A^c)\,\mathbb{P}(B^c)$ **[1]**.

$$\begin{aligned}
\mathbb{P}(A^c \cap B^c) &= \mathbb{P}((A \cup B)^c) \qquad \text{(De Morgan)} \\
&= 1 - \mathbb{P}(A \cup B) \qquad \text{(P4)} \\
&= 1 - (\mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)) \qquad \text{(P6)} \\
&= (1 - \mathbb{P}(A)) - \mathbb{P}(B) + \mathbb{P}(A)\,\mathbb{P}(B) \qquad \text{(independence of } A \text{ and } B\text{)} \\
&= \mathbb{P}(A^c) - \mathbb{P}(B)\,[1 - \mathbb{P}(A)] \qquad \text{(P4)} \\
&= \mathbb{P}(A^c) - \mathbb{P}(B)\,\mathbb{P}(A^c) \qquad \text{(P4)} \\
&= \mathbb{P}(A^c)\,[1 - \mathbb{P}(B)] \\
&= \mathbb{P}(A^c)\,\mathbb{P}(B^c)\,. \qquad \text{(P4)}
\end{aligned}$$

[4; clear justification of reasoning needs to be given for high marks]

AQ4. A box contains three coins: one has a Head on both sides; one has a Tail on both sides; the other coin is normal and fair. A coin is chosen at random from the box and tossed four times. What is the probability that the two-Headed coin was chosen, given that all four tosses are Heads?

Answer

Let $A_1 = \{\text{two-Headed coin chosen}\}$, $A_2 = \{\text{two-Tailed coin chosen}\}$, $A_3 = \{\text{fair coin chosen}\}$, and $B = \{\text{obtain four Heads}\}$ **[1 for setting up some notation]**.
Then

$$\mathbb{P}(B) = \sum_{i=1}^{3} \mathbb{P}(B \mid A_i)\,\mathbb{P}(A_i) = 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + \left(\frac{1}{2}\right)^4 \cdot \frac{1}{3} = \frac{17}{48}\,.$$

[2] Finally, we can use Bayes' theorem:

$$\mathbb{P}(A_1 \mid B) = \frac{\mathbb{P}(B \mid A_1)\,\mathbb{P}(A_1)}{\mathbb{P}(B)} = \frac{1 \cdot 1/3}{17/48} = \frac{16}{17}.$$

[2]