Introduction to Probability & Statistics

Assignment 4, 2024/25

Assignment Questions – answers to be uploaded

AQ1. Buses leave campus for the train station every 15 minutes, at 0, 15, 30 and 45 minutes past the hour. If a student arrives at the bus stop at a time that follows a (continuous) uniform distribution on the interval between 10.00 and 10.30, find the probability that they wait

a) less than 5 minutes for a bus;
b) at least 8 minutes for a bus.

> Answer
>
> Let Y denote the number of minutes past 10.00 that the student arrives at the bus stop: $Y \sim \text{Uniform}[0, 30]$. **[1]**
>
> a) They will wait less than 5 minutes if and only if $10 \le Y \le 15$ or $25 \le Y \le 30$. This occurs with probability
>
> $$\mathbb{P}(10 \le Y \le 15) + \mathbb{P}(25 \le Y \le 30) = \int_{10}^{15} \frac{1}{30} dy + \int_{25}^{30} \frac{1}{30} dy = \frac{1}{3}.$$
>
> **[2]**
>
> b) Similarly, they will wait at least 8 minutes if they arrive between 10.00 and 10.07, or between 10.15 and 10.22. This has probability $14/30 = 7/15$. **[2]**

AQ2. Let $X \sim \text{Geom}(p)$. Calculate $\mathbb{E}[h(X)]$, where $h(x) = e^{tx}$ for some $t > 0$. For what values of t is $\mathbb{E}[h(X)] < \infty$?

> Answer
>
> We use the formula for the expectation of a function of a discrete random variable:
>
> $$\begin{aligned} \mathbb{E}[h(X)] &= \sum_{k=1}^{\infty} h(k) p(1-p)^{k-1} = \sum_{k=1}^{\infty} e^{tk} p(1-p)^{k-1} \\ &= pe^t \sum_{k=1}^{\infty} \left[e^t(1-p)\right]^{k-1} = pe^t \sum_{k=0}^{\infty} \left[e^t(1-p)\right]^{k} \\ &= \frac{pe^t}{1 - e^t(1-p)}. \end{aligned}$$
>
> **[4]**
> This final step requires $e^t(1-p) < 1$. (Otherwise the geometric sum does not converge to a finite limit.) **[1]**

AQ3. Show that if Z is a standard normal random variable then, for $x > 0$,

a) $\mathbb{P}(Z > x) = \mathbb{P}(Z < -x)$;
b) $\mathbb{P}(|Z| > x) = 2\mathbb{P}(Z > x)$;
c) $\mathbb{P}(|Z| < x) = 2\mathbb{P}(Z < x) - 1$.

Hint: express the probabilities in terms of integrals over the density function ϕ, and use the fact that ϕ is an even function (i.e. $\phi(z) = \phi(-z)$).

Answer

There are many ways to show these identities. We use the hint about the symmetry of the density function of a standard normal random variable:

$$\phi(-z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(-z)^2}{2}\right) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) = \phi(z).$$

a)
$$\mathbb{P}(Z > x) = \int_x^\infty \phi(z)dz = \int_{-\infty}^{-x} \phi(-u)du = \int_{-\infty}^{-x} \phi(u)du = \mathbb{P}(Z < -x);$$
[1]

b)
$$\mathbb{P}(|Z| > x) = \mathbb{P}(Z > x) + \mathbb{P}(Z < -x) = 2\mathbb{P}(Z > x),$$
where the last equality follows from part (a). **[2]**

c)
$$\begin{aligned}\mathbb{P}(|Z| < x) &= 1 - \mathbb{P}(|Z| > x) = 1 - 2\mathbb{P}(Z > x)\\ &= 1 - 2(1 - \mathbb{P}(Z < x)) = 2\mathbb{P}(Z < x) - 1,\end{aligned}$$
where the second equality follows from part (b). **[2]**

AQ4. Let $X : \Omega \to \{1, 2\}$ and $Y : \Omega \to \{0, 1\}$ be two discrete random variables. The following is a partial table of their joint and their marginal mass functions:

$y\backslash x$	1	2	$p_Y(y)$
0	1/6	1/2	
1			
$p_X(x)$	5/12		1

a) Fill in the missing values.
b) Determine the joint distribution function of X and Y.
c) Calculate $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.
d) Let $Z = XY$. Calculate $\mathbb{E}[Z]$.

Answer

a) The missing entries in the probability table are determined by the requirement that summing the joint probabilities across a row or across a column in the table gives the corresponding marginal probability and by the requirement that the marginal probabilities for

X as well as those for Y have to add up to 1. So first we determine $p_Y(0) = 1/6 + 1/2 = 2/3$. Then we can determine $p_Y(1) = 1 - p_Y(0) = 1 - 2/3 = 1/3$ and $p_X(2) = 1 - p_X(1) = 1 - 5/12 = 7/12$. Finally we determine $p_{X,Y}(1,1) = p_X(1) - p_{X,Y}(1,0) = 5/12 - 1/6 = 1/4$ and $p_{X,Y}(2,1) = p_X(2) - p_{X,Y}(2,0) = 7/12 - 1/2 = 1/12$. **[1]**

$y\backslash x$	1	2	$p_Y(y)$
0	1/6	1/2	2/3
1	1/4	1/12	1/3
$p_X(x)$	5/12	7/12	1

b) The joint distribution function $F_{X,Y}(x,y)$ is by definition given by $\mathbb{P}(X \leq x, Y \leq y)$. So for example

$$F_{X,Y}(1.5, 1.5) = p_{X,Y}(1,0) + p_{X,Y}(1,1) = \frac{1}{6} + \frac{1}{4} = \frac{5}{12}.$$

By doing more such calculations we find that

$$F_{X,Y} = \begin{cases} 0 & \text{if } x < 1 \text{ or } y < 0 \\ 1/6 & \text{if } x \in [1,2) \text{ and } y \in [0,1) \\ 5/12 & \text{if } x \in [1,2) \text{ and } y \geq 1 \\ 2/3 & \text{if } x \geq 2 \text{ and } y \in [0,1) \\ 1 & \text{if } x \geq 2 \text{ and } y \geq 1. \end{cases}$$

[2]

c) For calculating the expectations of X and Y we can use their marginal mass functions:

$$\mathbb{E}[X] = 1 \cdot p_X(1) + 2 \cdot p_X(2) = 1 \cdot \frac{5}{12} + 2 \cdot \frac{7}{12} = \frac{19}{12}$$

and

$$\mathbb{E}[Y] = 0 \cdot p_Y(0) + 1 \cdot p_Y(1) = p_Y(1) = \frac{1}{3}.$$

[1]

d) The random variable $Z = XY$ can take the possible values 0, 1 and 2 with probabilities

$$p_Z(0) = p_{X,Y}(1,0) + p_{X,Y}(2,0) = p_Y(0) = \frac{2}{3}$$
$$p_Z(1) = p_{X,Y}(1,1) = \frac{1}{4}, \quad p_Z(2) = p_{X,Y}(2,1) = \frac{1}{12}.$$

Thus

$$\mathbb{E}[Z] = 1 \cdot P_Z(1) + 2 \cdot P_Z(2) = \frac{1}{4} + 2\frac{1}{12} = \frac{5}{12}.$$

[1]