Introduction to Probability & Statistics

Assignment 3, 2024/25

Practice Questions

PQ1. Each morning a (fictional) student rolls a die and starts studying if she throws 6. Otherwise, she stays in bed. However, during the four months of exams, she tosses a coin instead of rolling a die and studies if she tosses Heads. On a (randomly chosen) morning, the student is studying. What is the probability that it is exam period? (Assume for simplicity that each month has the same number of days!)

> **Answer**
>
> Let $W = \{$student is working$\}$ and$\}$ $E = \{$exam period$\}$. Then Bayes' theorem tells us that
> $$\mathbb{P}\left(E \mid W\right) = \frac{\mathbb{P}\left(W \mid E\right)\mathbb{P}\left(E\right)}{\mathbb{P}\left(W \mid E\right)\mathbb{P}\left(E\right) + \mathbb{P}\left(W \mid E^c\right)\mathbb{P}\left(E^c\right)}$$
> $$= \frac{\dfrac{1}{2} \cdot \dfrac{1}{3}}{\dfrac{1}{2} \cdot \dfrac{1}{3} + \dfrac{1}{6} \cdot \dfrac{2}{3}} = \frac{3}{5}.$$

PQ2. Consider a discrete random variable V taking values in $\{1, 2, 3, 4\}$ with mass function p_V given by

$$p_V(x) = \begin{cases} xc^x & \text{if } x = 1, 2 \\ c(5 - x) & \text{if } x = 3, 4. \end{cases}$$

What value of c makes p_V a mass function?

> **Answer**
>
> To be a mass function, we need $p_V(x) \geq 0$ for all x and $\sum_x p_V(x) = 1$. For the second of these constraints we need $c + 2c^2 + 2c + c = 1$, and so $c = -1 \pm \sqrt{6}/2$. But if we take $c = -1 - \sqrt{6}/2$ then we get (for example) $p_V(3) < 0$, which isn't allowed. Therefore, we have to take $c = -1 + \sqrt{6}/2 > 0$.

PQ3. What is the probability that exactly 3 heads are obtained in 5 tosses of a fair coin?

> **Answer**
>
> Let X denote the number of heads obtained. Then $X \sim \text{Bin}(5, 1/2)$, and
> $$\mathbb{P}\left(X = 3\right) = \binom{5}{3}\left(\frac{1}{2}\right)^5 = 0.3125\,.$$

PQ4. Let $Y \sim \text{Poisson}(2)$. Write down the mass function for Y. What is $\mathbb{P}\left(Y \geq 2\right)$?

PQ5. Recall the Chevalier de Méré's problem: two dice are thrown 24 times – you win £1 if at least one double 6 is thrown, otherwise you lose £1. How much do you expect to win on average? Is this a good bet?

PQ6. The probability density function f_X of a continuous random variable X is given by

$$f_X(x) = \begin{cases} x(3 - x) + c & \text{if } 1 \leq x \leq 2 \\ 0 & \text{otherwise}\,. \end{cases}$$

Compute c. Give the distribution function of X.

PQ7. A continuous random variable X has a distribution function F which satisfies

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{4x^2 + 3x}{7} & \text{for } 0 \leq x \leq 1 \\ 1 & \text{for } x > 1\,. \end{cases}$$

Determine $\mathbb{E}[X]$.

The probability density function of X is given by

$$f(x) = \frac{d}{dx}F(x) = \frac{8x+3}{7}$$

for $0 \leq x \leq 1$, and $f(x) = 0$ elsewhere. So

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 \left(\frac{8x^2 + 3x}{7}\right)dx = \frac{25}{42}.$$

Assignment Questions – ☁ answers to be uploaded

AQ1. Consider the probability space corresponding to throwing a fair die. Give an example of a random variable X whose image is $\{1, 2, 3, 4\}$ and for which $F_X(\pi) = 2/3$.

There are lots of possibilities. The only important thing is that exactly two of the outcomes ($\Omega = \{1, 2, 3, 4, 5, 6\}$) are mapped to 4 so that $F_X(\pi) = \mathbb{P}(X \leq \pi) = 1 - \mathbb{P}(X = 4) = 1 - 2/6 = 2/3$. One possibility is $X(1) = 1, X(2) = 2, X(3) = 3, X(4) = 3, X(5) = 4, X(6) = 4$. **[3 for a correct example; 2 for some sort of reasoning]**

AQ2. Let $\lambda > 0$ be some positive number. Find c such that the following function is a density function:

$$f(x) = ce^{-\lambda|x-1|}, \quad x \in \mathbb{R}.$$

For f to be a density, it must be non-negative on $\mathbb{R}$ and satisfy $\int_{-\infty}^{\infty} f(x)dx = 1$. Since the exponential function is non-negative, f is non-negative everywhere as long as $c \geq 0$ **[1]**.
Note that

$$\int_{-\infty}^{\infty} e^{-\lambda|x-1|}dx = \int_{-\infty}^1 e^{-\lambda(1-x)}dx + \int_1^{\infty} e^{-\lambda(x-1)}dx$$
$$= \frac{1}{\lambda}\left[e^{-\lambda(1-x)}\right]_{-\infty}^1 - \frac{1}{\lambda}\left[e^{-\lambda(x-1)}\right]_1^{\infty} = \frac{2}{\lambda}.$$

[3]
So $f(\cdot)$ is a density if and only if $c = \lambda/2$. **[1]**

AQ3. Suppose that A and B are independent events. Show that A^c and B^c are independent.

AQ4. A box contains three coins: one has a Head on both sides; one has a Tail on both sides; the other coin is normal and fair. A coin is chosen at random from the box and tossed four times. What is the probability that the two-Headed coin was chosen, given that all four tosses are Heads?

Other Questions (for seminars / extra practice)

OQ1. Consider a discrete random variable Y taking values in the set $\{0, 1, \ldots, 8\}$ and with probability mass function p_Y given by

$$
p_Y(k) = \begin{cases}
c\,a & \text{if } k = 0, 1, 2, 3, 4, 5\\
c\,a^2 & \text{if } k = 6, 7\\
c(1-a)^2 & \text{if } k = 8,
\end{cases}
$$

where a is some fixed number between 0 and 1 and c is a constant to be determined. What value of c makes p_Y a mass function? (The answer is a function of a.)

and so the required value of c is

$$c = \frac{1}{(3a+1)(a+1)}.$$

OQ2 Let $X \sim \text{Bern}(p)$. Let $Y = 1 - X$ and $V = X^2$. Show that

a) $Y \sim \text{Bern}(1-p)$;
b) $V \sim \text{Bern}(p)$.

Answer

The mass function of X is

$$p_X(x) = \begin{cases} 1-p & \text{if } x = 0 \\ p & \text{if } x = 1 \\ 0 & \text{if } x \neq 0, 1. \end{cases}$$

a) Thus we see that

$$\mathbb{P}(Y = 0) = \mathbb{P}(1 - X = 0) = \mathbb{P}(X = 1) = p$$
$$\text{and} \quad \mathbb{P}(Y = 1) = \mathbb{P}(1 - X = 1) = \mathbb{P}(X = 0) = 1 - p.$$

Furthermore, $\mathbb{P}(Y = x) = 0$ if $x \neq 0, 1$. Thus $Y \sim \text{Bern}(1-p)$, as required.

b) Similarly,

$$\mathbb{P}(V = 0) = \mathbb{P}\left(X^2 = 0\right) = \mathbb{P}(X = 0) = 1 - p$$
$$\text{and} \quad \mathbb{P}(V = 1) = \mathbb{P}\left(X^2 = 1\right) = \mathbb{P}(X = 1) = p,$$

and so $V \sim \text{Bern}(p)$.

OQ3. A fair die is thrown until the sum of the results of the throws exceeds 6. The random variable X is the number of throws needed for this. Let F_X be the distribution function of X. Determine $F_X(1)$, $F_X(2)$ and $F_X(7)$.

Answer

There is no possibility to exceed 6 already on the first throw because the die only has numbers up to 6. So $F_X(1) = \mathbb{P}(X \leq 1) = 0$.
There are, however, 21 outcomes of the throw of two dice that give a sum greater than 6. These can be read off easily from Table 4.1 in the textbook. There are 36 outcomes altogether, all equally likely, so

$$F_X(2) = \mathbb{P}(X \leq 2) = \mathbb{P}(X = 2) = \frac{21}{36} = \frac{7}{12}.$$

Finally, there is no way to avoid getting a sum of more than six once one has more than six throws, because each die will show at least one. So $F_X(7) = \mathbb{P}(X \leq 7) = 1 - \mathbb{P}(X > 7) = 1 - 0 = 1$.

OQ4. Consider a roulette wheel, with numbers 00, 0, 1, 2, 3, ..., 36 (38 numbers in total). A ball is thrown onto the wheel as it is spinning, and comes to rest by one of the numbers. You always bet that the ball will stop on one of the numbers 1, 2, ..., 12. Let N be the random variable giving the number of bets that you lose *before* your first win. Calculate $p_N(0)$, $p_N(5)$ and $F_N(5)$.

OQ5. $N \geq 3$ people go for coffee. Each person flips a fair coin: if all but one of the coins shows the same face (Head or Tail), then the odd person out pays for all the drinks; if not then the coins are tossed again until this event occurs. How many times on average must each person toss their coin before somebody is selected in this way?

OQ6. Prove the memoryless property of the exponential distribution which states that if $X \sim \mathsf{Exp}(\lambda)$ then for any $t, s \geq 0$,
$$\mathbb{P}(X > t + s \mid X > s) = \mathbb{P}(X > t).$$

OQ7. [Harder] $N \geq 3$ people go for coffee. Each person flips a coin, of which $N - 1$ are fair, and one has probability q of coming up Heads (for some $q \in [0, 1]$): if all but one of the coins shows the same face (Head or Tail), then the odd person out pays for all the drinks; if not then the coins are tossed again until this event occurs. How does the expected number of coin tosses each person makes vary with q?

Answer

It doesn't vary with q – the expected number of tosses is always $2^{N-1}/N$! Why?

$$\mathbb{P}\left(\text{odd person out on any given flip}\right) = q\left[\binom{N-1}{N-2}\left(\frac{1}{2}\right)^{N-2}\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^{N-1}\right]$$
$$+ (1-q)\left[\binom{N-1}{N-2}\left(\frac{1}{2}\right)^{N-2}\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^{N-1}\right],$$

where the first quantity is the probability that the person with the strange coin gets a Head, and either $N - 2$ people also get Heads (with one Tail), or everybody else gets a Tail, etc. So the probability of having one odd person out in any given round is $p = N/2^{N-1}$, irrespective of the value of q. The number of rounds needed until there's one odd person out has a Geom(p) distribution, with mean $2^{N-1}/N$.

OQ8. [Harder] Alice and Bob decide to duel, using just one six-shot revolver, and one bullet, between them. They decide to duel in the following way: with the bullet inserted into the revolver, Alice will spin the cylinder and shoot at Bob (killing him if the gun fires); if Alice misses, Bob will spin the cylinder and shoot at Alice. Assuming there is a $1/6$ probability that the revolver will fire each time (since the revolver has 6 chambers), what is

  a) the distribution of the number of turns T until the gun fires?
  b) the probability that Alice wins the duel?

Answer

  a) The time until the gun fires follows a Geom($1/6$) distribution: $T \sim \text{Geom}(1/6)$.

  b) Alice wins if:

  - the gun fires first time; or
  - the gun doesn't fire on the first two goes, but fires on the third; or
  - the gun doesn't fire on the first four goes, but fires on the fifth; or...

That is, Alice wins if the gun fires at an *odd* time:

$$\mathbb{P}\left(\text{Alice wins}\right) = \mathbb{P}\left(T \text{ is odd}\right).$$

Since $\mathbb{P}\left(T = n\right) = (5/6)^{n-1}(1/6)$, we require

$$\begin{aligned}
\mathbb{P}\left(T \text{ is odd}\right) &= \frac{1}{6}\left[1 + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^4 + \dots\right] \\
&= \frac{1}{6}\left[\frac{1}{1 - \left(\frac{5}{6}\right)^2}\right] \\
&= \frac{6}{11},
\end{aligned}$$

where the middle equality follows from the usual formula for a geometric series. So Alice has the greater chance of winning (because she gets to take the first shot!).