

Introduction to Probability & Statistics

Assignment 5, 2024/25

Assignment Questions – answers to be uploaded

AQ1. Let X_1, \dots, X_{16} be an i.i.d. sample from a $N(3, 1)$ distribution, and let $S = X_1 + X_2 + \dots + X_{16}$. Express $\mathbb{P}(S < 52)$ in terms of the distribution function Φ of the standard normal distribution.

Answer

We know that $\mathbb{E}[S] = 16\mathbb{E}[X] = 48$, and $\text{Var}(S) = 16\text{Var}(X) = 16$ (because the X_i are independent, and hence uncorrelated).

Furthermore, we know that the sum of independent normal distributions has a normal distribution. Thus

$$S \sim N(48, 16).$$

[3]

We can normalize to obtain a standard normal random variable by subtracting the mean and dividing by the standard deviation. Thus

$$\mathbb{P}(S < 52) = \mathbb{P}\left(\frac{S - 48}{\sqrt{16}} < \frac{52 - 48}{\sqrt{16}}\right) = \Phi(1).$$

[2]

AQ2. Let Y_1, Y_2, \dots be an i.i.d. sequence of random variables, each with a $\text{Uniform}(0, 3)$ distribution. Define a new sequence of random variables X_1, X_2, \dots by

$$X_n = \frac{1}{n} \sum_{i=1}^n Y_i^2.$$

Using the Law of Large Numbers, determine the value of $a \in \mathbb{R}$ for which $\mathbb{P}(\lim_{n \rightarrow \infty} X_n = a) = 1$.

Answer

The random variable X_n is just the sample mean of the random variables Y_1^2, \dots, Y_n^2 . These are i.i.d. and clearly have finite mean and variance (since Y_i^2 can only take values in the finite set $[0, 9]$).

The (strong) law of large numbers says that, with probability one, X_n will converge to $\mathbb{E}[Y^2]$.

[3]

Finally, we calculate

$$\mathbb{E}[Y^2] = \int_0^3 y^2 \frac{1}{3} dy = 3,$$

and so the required answer is $a = 3$. [2]

AQ3. Suppose the random variables X_1, X_2 and X_3 all have the same expectation μ . For what values of a and b is

$$M = -4(X_1 - 2) + 9(X_2 - 1) + aX_3 + b$$

an unbiased estimator for μ ?

Answer

M is an unbiased estimator for μ if $\mathbb{E}[M] = \mu$ for any value of μ . [1] We find

$$\begin{aligned}\mathbb{E}[M] &= \mathbb{E}[-4(X_1 - 2) + 9X_2 + a(X_3 - 1) + b] \\ &= -4(\mathbb{E}[X_1] - 2) + 9\mathbb{E}[X_2] + a(\mathbb{E}[X_3] - 1) + b \\ &= -4(\mu - 2) + 9\mu + a(\mu - 1) + b \\ &= (-4 + 9 + a)\mu + (8 - a + b).\end{aligned}$$

[2]

Thus $\mathbb{E}[M] = \mu$ if and only if $a = -4$ and $b = -12$. [2]

AQ4. From a dataset x_1, \dots, x_{10} it has been calculated that

$$\sum_{i=1}^{10} x_i = 491, \quad \sum_{i=1}^{10} (x_i - \bar{x}_{10})^2 = 41.$$

You model the dataset as a random sample from the normal distribution with mean μ and variance σ^2 .

- a) Assume that both μ and σ^2 are unknown. Determine a 95% confidence interval for the mean μ . You can use that $t_{9,0.025} \approx 2.26$.
- b) Now assume that it is known that the variance is $\sigma^2 = 5$. Give a 95% confidence interval for the mean μ in this case. You can use that $z_{0.025} \approx 1.96$.

Answer

- a) Since the variance is unknown, we use the interval

$$\left[\bar{x}_n - t_{n-1, \alpha/2} \frac{s_n}{\sqrt{n}}, \bar{x}_n + t_{n-1, \alpha/2} \frac{s_n}{\sqrt{n}} \right]$$

We first calculate $\bar{x}_n = 491/10 = 49.1$ and

$$s_n^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x}_n)^2 = \frac{1}{10-1} 41 = \frac{41}{9}.$$

We're told that $t_{n-1, \alpha/2} = t_{9,0.025} = 2.26$ and

$$\begin{aligned}\left[\bar{x}_n - t_{n-1, \alpha/2} \frac{s_n}{\sqrt{n}}, \bar{x}_n + t_{n-1, \alpha/2} \frac{s_n}{\sqrt{n}} \right] &\approx \left[49.1 - 2.26 \frac{\sqrt{41}}{3\sqrt{10}}, 49.1 + 2.26 \frac{\sqrt{41}}{3\sqrt{10}} \right] \\ &\approx [47.57, 50.63].\end{aligned}$$

[3]

- b) When σ^2 is known, we use the confidence interval

$$\begin{aligned}\left[\bar{x}_n - z_{\alpha/2} \frac{\sigma}{\sqrt{n}}, \bar{x}_n + z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \right] &\approx \left[49.1 - 1.96 \frac{\sqrt{5}}{\sqrt{10}}, 49.1 + 1.96 \frac{\sqrt{5}}{\sqrt{10}} \right] \\ &\approx [47.71, 50.49].\end{aligned}$$

[2]