Introduction to Probability & Statistics

Assignment 2, 2024/25

Instructions

Submit your answers to all of the **Assignment Questions (AQ)**. You should upload your solutions to the VLE as a *pdf file*.
Marks will be awarded for clear, logical explanations, as well as for correctness of solutions.
You should also look at the other questions in preparation for your Week 5 seminar.

Practice Questions

PQ1. Let E, F and G be three events. Here are five events built from these:

a) $E \cap F^c \cap G^c$
b) $(E \cup F) \cap G^c$
c) $E \cup F \cup G$
d) $(E \cap F \cap G)^c$
e) $(E \cap F)^c \cap (E \cap G)^c \cap (F \cap G)^c$.

And here are verbal descriptions of the same five events, but in different order:

1) E or F occurs, but not G
2) At most two of the events occur
3) At most one of the events occurs
4) At least one of the events occurs
5) Only E occurs.

Match the expressions to the verbal descriptions.

Answer

a5, b1, c4, d2, e3

PQ2. Let C and D be events. Express the probability $\mathbb{P}(C^c \cap D)$ in terms of $\mathbb{P}(D)$ and $\mathbb{P}(C \cap D)$.

Answer

We note that we can write the event D as the disjoint union of the two events $C^c \cap D$ and $C \cap D$. Therefore by additivity (P3) we have

$$\mathbb{P}(D) = \mathbb{P}(C^c \cap D) + \mathbb{P}(C \cap D) .$$

Hence $\mathbb{P}(C^c \cap D) = \mathbb{P}(D) - \mathbb{P}(C \cap D)$.

PQ3. Let E and F be two events for which one knows that the probability that at least one of them occurs is 0.4. What is the probability that neither E nor F occurs?

Answer

The event "at least one of E and F occurs" is the event $E \cup F$. We want the probability of the event that neither E nor F occurs, which is the event $E^c \cap F^c$. According to the second De Morgan's law this event can also be written as $(E \cup F)^c$. To calculate this we can use the property (P4) that for any event A we have $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$. This gives us

$$\mathbb{P}((E \cup F)^c) = 1 - \mathbb{P}(E \cup F) = 1 - 0.4 = 0.6.$$

PQ4. Show that if an event occurs with probability 0 then it is independent of all other events.

Answer

Suppose that A satisfies $\mathbb{P}(A) = 0$, and let B be any other event. Then $\mathbb{P}(A \cap B) \leq \mathbb{P}(A) = 0$, since $A \cap B \subseteq A$ (P7). So, $0 = \mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$, and hence A is independent of B.

PQ5. Calculate $\mathbb{P}(B)$ if it is given that $\mathbb{P}(A \cup B) = 3/7$ and $\mathbb{P}(A^c \mid B^c) = 9/14$.

Answer

From the multiplication rule we know that

$$\mathbb{P}(A^c \cap B^c) = \mathbb{P}(A^c \mid B^c)\,\mathbb{P}(B^c).$$

Recalling De Morgan's law we know that $A^c \cap B^c = (A \cup B)^c$ and hence

$$\mathbb{P}(A^c \cap B^c) = \mathbb{P}((A \cup B)^c) = 1 - \mathbb{P}(A \cup B).$$

Combined this yields

$$\begin{aligned}\mathbb{P}(B) = 1 - \mathbb{P}(B^c) &= 1 - \frac{1 - \mathbb{P}(A \cup B)}{\mathbb{P}(A^c \mid B^c)} \\ &= 1 - \frac{4/7}{9/14} = \frac{1}{9}.\end{aligned}$$

Assignment Questions – answers to be uploaded

Make sure to justify your reasoning by making reference to any of the properties (P1)–(P7) covered in lectures, where you are using them in your answers.

AQ1. When $\mathbb{P}(A) = 1/2, \mathbb{P}(B) = 1/3$, and $\mathbb{P}(A \cup B) = 3/4$, calculate $\mathbb{P}(A \cap B)$ and $\mathbb{P}(A^c \cup B^c)$.

AQ2. Let A and B be two events. Prove that if $A \subseteq B$ then

$$\mathbb{P}(B \cap A^c) = \mathbb{P}(B) - \mathbb{P}(A).$$

AQ3. Calculate $\mathbb{P}(A \cup B)$ if it is given that $\mathbb{P}(A) = 1/3$ and $\mathbb{P}(B \mid A^c) = 1/2$.

AQ4. Prove that, for any events A, B, C for which the following probabilities are all defined,

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\,\mathbb{P}(B \mid A)\,\mathbb{P}(C \mid A \cap B).$$

Other Questions (for seminars / extra practice)

OQ1. In a card game involving four players, each player is dealt four cards (face down), and the player with the highest Diamond wins (Ace counts high). Upon receiving your four cards you see that your highest Diamond is the Queen of Diamonds. What is the probability that another player has the King or Ace of Diamonds, and hence that you lose the game?

OQ2. Consider a sample space $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ consisting of four equiprobable elements. Let $A_1 = \{\omega_1, \omega_2\}$, $A_2 = \{\omega_1, \omega_3\}$, $A_3 = \{\omega_1, \omega_4\}$. Show that A_1 and A_2 are independent, A_1 and A_3 are independent, and A_2 and A_3 are independent. Are A_1, A_2, A_3 independent?

OQ3. Show that if an event A is independent of itself, then either $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$.

OQ4. Let A, B, and C be events. Prove that if

$$\mathbb{P}(A \mid C) > \mathbb{P}(B \mid C) \text{ and } \mathbb{P}(A \mid C^c) > \mathbb{P}(B \mid C^c)$$

then $\mathbb{P}(A) > \mathbb{P}(B)$.

OQ5. Given a sample space Ω, let $\mathbb{Q}$ be some function satisfying

$$\mathbb{Q}(A_1 \cup A_2) = \mathbb{Q}(A_1) + \mathbb{Q}(A_2)$$

for any two disjoint subsets A_1 and A_2 of Ω.

Show by induction that if $A_1, ..., A_n$ are $n \geq 2$ disjoint subsets of Ω then

$$\mathbb{Q}(A_1 \cup ... \cup A_n) = \mathbb{Q}(A_1) + ... + \mathbb{Q}(A_n) .$$

! Challenge question

Prove the **law of inclusion-exclusion**: for possibly overlapping sets $A_1, \ldots, A_n$,

$$\begin{aligned}\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i_1<i_2} \mathbb{P}(A_{i_1} \cap A_{i_2}) + \ldots \\ + (-1)^{k+1} \sum_{i_1<i_2<\ldots<i_k} \mathbb{P}(A_{i_1} \cap A_{i_2} \ldots \cap A_{i_k}) \\ + \ldots + (-1)^{n+1} \mathbb{P}(A_1 \cap A_2 \cap \ldots \cap A_n) .\end{aligned}$$

(So, for example, $\mathbb{P}(A_1 \cup A_2 \cup A_3) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) - \mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1 \cap A_3) - \mathbb{P}(A_2 \cap A_3) + \mathbb{P}(A_1 \cap A_2 \cap A_3)$.)