Introduction to Probability & Statistics

Assignment 3, 2024/25

Instructions

Submit your answers to all of the **Assignment Questions (AQ)**. You should upload your solutions to the VLE as a *pdf file*.
Marks will be awarded for clear, logical explanations, as well as for correctness of solutions.
You should also look at the other questions in preparation for your Week 7 seminar.

Practice Questions

PQ1. Each morning a (fictional) student rolls a die and starts studying if she throws 6. Otherwise, she stays in bed. However, during the four months of exams, she tosses a coin instead of rolling a die and studies if she tosses Heads. On a (randomly chosen) morning, the student is studying. What is the probability that it is exam period? (Assume for simplicity that each month has the same number of days!)

Answer

Let $W = \{\text{student is working}\}$ and$\}$ $E = \{\text{exam period}\}$. Then Bayes' theorem tells us that

$$\begin{aligned}\mathbb{P}(E \mid W) &= \frac{\mathbb{P}(W \mid E)\,\mathbb{P}(E)}{\mathbb{P}(W \mid E)\,\mathbb{P}(E) + \mathbb{P}(W \mid E^c)\,\mathbb{P}(E^c)} \\ &= \frac{\frac{1}{2}\cdot\frac{1}{3}}{\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{6}\cdot\frac{2}{3}} = \frac{3}{5}.\end{aligned}$$

PQ2. Consider a discrete random variable V taking values in $\{1,2,3,4\}$ with mass function p_V given by

$$p_V(x) = \begin{cases} xc^x & \text{if } x = 1,2 \\ c(5-x) & \text{if } x = 3,4. \end{cases}$$

What value of c makes p_V a mass function?

Answer

To be a mass function, we need $p_V(x) \geq 0$ for all x and $\sum_x p_V(x) = 1$. For the second of these constraints we need $c+2c^2+2c+c = 1$, and so $c = -1 \pm \sqrt{6}/2$. But if we take $c = -1-\sqrt{6}/2$ then we get (for example) $p_V(3) < 0$, which isn't allowed. Therefore, we have to take $c = -1+\sqrt{6}/2 > 0$.

PQ3. What is the probability that exactly 3 heads are obtained in 5 tosses of a fair coin?

Answer

Let X denote the number of heads obtained. Then $X \sim \text{Bin}(5, 1/2)$, and

$$\mathbb{P}(X = 3) = \binom{5}{3}\left(\frac{1}{2}\right)^5 = 0.3125\,.$$

PQ4. Let $Y \sim \text{Poisson}(2)$. Write down the mass function for Y. What is $\mathbb{P}(Y \geq 2)$?

Answer

The probability mass function is given by

$$p_Y(k) = \frac{2^k}{k!}e^{-2}\,.$$

We can then calculate

$$\mathbb{P}(Y \geq 2) = 1 - \mathbb{P}(Y < 2) = 1 - (\mathbb{P}(Y = 0) + \mathbb{P}(Y = 1)) = 1 - (1 + 2)\,e^{-2}.$$

PQ5. Recall the Chevalier de Méré's problem: two dice are thrown 24 times – you win £1 if at least one double 6 is thrown, otherwise you lose £1. How much do you expect to win on average? Is this a good bet?

Answer

The probability of throwing a double 6 with one throw of two dice is $1/36$. The probability of there being *no* occurrences of a double 6 in 24 throws of two dice is therefore $(35/36)^{24} \approx 0.509$. Letting X denote your winnings from the game, this is a random variable satisfying

$$\mathbb{P}(X = 1) \approx 0.491\,, \quad \text{and} \quad \mathbb{P}(X = -1) \approx 0.509\,.$$

Your expected winnings are hence given by

$$\mathbb{E}[X] \approx 0.491 - 0.509 \approx -0.018 \;< 0\,.$$

Since you expect to lose on average, this isn't a good bet to make!

PQ6. The probability density function f_X of a continuous random variable X is given by

$$f_X(x) = \begin{cases} x(3-x) + c & \text{if } 1 \leq x \leq 2 \\ 0 & \text{otherwise}\,. \end{cases}$$

Compute c. Give the distribution function of X.

Answer

The constant c is determined by the requirement that the integral of the density function over the entire real line is equal to one:

$$1 = \int_{-\infty}^{\infty} f_X(x)dx = \int_1^2 (x(3-x) + c)dx = 13/6 + c\,.$$

This implies that $c = -7/6$. Note that for this value of c the function f_X is non-negative everywhere, which is also required in order for it to be a density.
The distribution function is obtained as an integral over the density function, but not over the

whole real line but only from $-\infty$ up to x:

$$F_X(x) = \begin{cases} 0 & \text{if } x < 1 \\ \int_1^x f_X(s)ds = \int_1^x (s(3-s) - 7/6)ds = \frac{9x^2 - 2x^3 - 7x}{6} & \text{if } 1 \le x \le 2 \\ 1 & \text{if } 2 < x. \end{cases}$$

PQ7. A continuous random variable X has a distribution function F which satisfies

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{4x^2+3x}{7} & \text{for } 0 \le x \le 1 \\ 1 & \text{for } x > 1\,. \end{cases}$$

Determine $\mathbb{E}[X]$.

Answer

The probability density function of X is given by

$$f(x) = \frac{d}{dx} F(x) = \frac{8x+3}{7}$$

for $0 \le x \le 1$, and $f(x) = 0$ elsewhere. So

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x) dx = \int_0^1 \left(\frac{8x^2 + 3x}{7} \right) dx = \frac{25}{42}.$$

Assignment Questions – answers to be uploaded

AQ1. Consider the probability space corresponding to throwing a fair die. Give an example of a random variable X whose image is $\{1, 2, 3, 4\}$ and for which $F_X(\pi) = 2/3$.

AQ2. Let $\lambda > 0$ be some positive number. Find c such that the following function is a density function:

$$f(x) = ce^{-\lambda|x-1|}, \quad x \in \mathbb{R}.$$

AQ3. Suppose that A and B are independent events. Show that A^c and B^c are independent.

AQ4. A box contains three coins: one has a Head on both sides; one has a Tail on both sides; the other coin is normal and fair. A coin is chosen at random from the box and tossed four times. What is the probability that the two-Headed coin was chosen, given that all four tosses are Heads?

Other Questions (for seminars / extra practice)

OQ1. Consider a discrete random variable Y taking values in the set $\{0, 1, \ldots, 8\}$ and with probability mass function p_Y given by

$$p_Y(k) = \begin{cases} c\,a & \text{if } k = 0, 1, 2, 3, 4, 5 \\ c\,a^2 & \text{if } k = 6, 7 \\ c(1-a)^2 & \text{if } k = 8, \end{cases}$$

where a is some fixed number between 0 and 1 and c is a constant to be determined. What value of c makes p_Y a mass function? (The answer is a function of a.)

OQ2 Let $X \sim \text{Bern}(p)$. Let $Y = 1 - X$ and $V = X^2$. Show that

a) $Y \sim$ Bern$(1-p)$;
b) $V \sim$ Bern(p).

OQ3. A fair die is thrown until the sum of the results of the throws exceeds 6. The random variable X is the number of throws needed for this. Let F_X be the distribution function of X. Determine $F_X(1)$, $F_X(2)$ and $F_X(7)$.

OQ4. Consider a roulette wheel, with numbers 00, 0, 1, 2, 3, . . . , 36 (38 numbers in total). A ball is thrown onto the wheel as it is spinning, and comes to rest by one of the numbers. You always bet that the ball will stop on one of the numbers 1, 2, . . . , 12. Let N be the random variable giving the number of bets that you lose *before* your first win. Calculate $p_N(0)$, $p_N(5)$ and $F_N(5)$.

OQ5. $N \geq 3$ people go for coffee. Each person flips a fair coin: if all but one of the coins shows the same face (Head or Tail), then the odd person out pays for all the drinks; if not then the coins are tossed again until this event occurs. How many times on average must each person toss their coin before somebody is selected in this way?

OQ6. Prove the memoryless property of the exponential distribution which states that if $X \sim$ Exp(λ) then for any $t, s \geq 0$,

$$\mathbb{P}\left(X > t + s \,|\, X > s\right) = \mathbb{P}\left(X > t\right).$$

OQ7. [Harder] $N \geq 3$ people go for coffee. Each person flips a coin, of which $N-1$ are fair, and one has probability q of coming up Heads (for some $q \in [0,1]$): if all but one of the coins shows the same face (Head or Tail), then the odd person out pays for all the drinks; if not then the coins are tossed again until this event occurs. How does the expected number of coin tosses each person makes vary with q?

OQ8. [Harder] Alice and Bob decide to duel, using just one six-shot revolver, and one bullet, between them. They decide to duel in the following way: with the bullet inserted into the revolver, Alice will spin the cylinder and shoot at Bob (killing him if the gun fires); if Alice misses, Bob will spin the cylinder and shoot at Alice. Assuming there is a $1/6$ probability that the revolver will fire each time (since the revolver has 6 chambers), what is

a) the distribution of the number of turns T until the gun fires?
b) the probability that Alice wins the duel?

! Challenge question

At time 0, a bag contains 1 Red and 1 Black ball. Just before each time $n = 1, 2, 3, \ldots$, a ball is chosen at random from the bag and then replaced, *along with a new ball of the same colour*. For $n = 0, 1, 2, \ldots$ and $1 \leq r \leq n+1$, let

$$p_{n,r} = \mathbb{P}\left(\text{at time } n\text{, the bag contains exactly } r \text{ Red balls}\right).$$

a) Write down the values of $p_{0,1}$, $p_{1,1}$ and $p_{1,2}$;
b) By conditioning on the possible numbers of red balls at time 1, calculate $p_{2,r}$ for $r = 1, 2, 3$;
c) Can you find (and prove!) a general formula for $p_{n,r}$? [HINT: guess at the right answer using your findings from parts (a) and (b). (Calculate $p_{3,r}$ explicitly if you can't see a pattern.) Then use induction.]