Introduction to Probability & Statistics

Assignment 1, 2024/25

Practice Questions

PQ1. A bag contains six balls numbered 1–6. Two balls are drawn at random (without replacement), and the outcome is recorded as (i, j) if ball i is drawn first, and ball j drawn second. How many possible outcomes are there?
Now suppose that four of the numbered balls are white, and the other two are black. How many outcomes are there in which the following occur?

a. both balls are white;
b. both balls have the same colour;
c. at least one of the balls is white?

Answer

Since we are sampling without replacement there are $6 \times 5 = 30$ different possible outcomes.

a. Here we need to choose a white ball followed by another white ball: $4 \times 3 = 12$;
b. As in part (a), the number of ways in which two black balls can be chosen is $2 \times 1 = 2$. Hence the total number of outcomes for which both balls have the same colour is $12 + 2 = 14$.
c. Here we need either both balls to be white (which we've already counted), or exactly one to be white. There are $4 \times 2 = 8$ ways for the first ball to be white and the second black, and another $2 \times 4 = 8$ ways for the balls to come out in the opposite order; so there are $16 + 12 = 28$ possible outcomes altogether. (Alternatively, we know from part (b) that there are only two outcomes where both balls are black, and since there are 30 possible outcomes in total, 28 of these must include at least one white ball.)

PQ2. Two football teams, each with 11 players, shake hands at the end of a match: every player on Team A shakes hands with every player on Team B. How many handshakes will there be?
How many will there be if every player shakes hands with every other player (including those on their own team)?

Answer

Each player from Team A has to shake hands with 11 players from Team B. So there are $11^2 = 121$ handshakes in total.
If everyone shakes hands with everyone else, then we need to count the number of ways of choosing two players out of 22: $C_2^{22} = 231$. (Alternatively, we could count how many handshakes take place between two members of the same team, and add this to the 121 that we obtained above. With 11 people in Team A, there are $C_2^{11} = 55$ pairs of players to shake hands, and then there's the same number of handshakes between members of Team B. This gives 110 extra handshakes.)

PQ3. 30 sweets are to be divided amongst 5 children. In how many ways can this be done? What if every child must be given at least two sweets each?

Answer

There are $C_{5-1}^{30+5-1} = 46,376$ ways of dividing the sweets (allowing for the possibility of some children receiving none). If every child must be given two sweets, then if we do this first of all there are 20 sweets left to be divided: there are $C_{5-1}^{20+5-1} = 10,626$ ways of doing this.

PQ4. Consider two events A and B. Let C be the event that "A or B occurs, but not both". Express C in terms of A and B, using only the basic operations "union", "intersection", and "complement".

Answer

There are many possible answers. Here are three:

$$\begin{aligned} C &= (A \cup B) \cap (A \cap B)^c \\ &= (A \cup B) \cap (A^c \cup B^c) \\ &= (A \cap B^c) \cup (B \cap A^c). \end{aligned}$$

PQ5. Prove, by expanding in terms of factorials, that

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}, \qquad 1 \leq r \leq n\,.$$

Answer

$$\begin{aligned} \binom{n-1}{r-1} + \binom{n-1}{r} &= \frac{(n-1)!}{(n-r)!(r-1)!} + \frac{(n-1)!}{(n-1-r)!r!} \\ &= \frac{(n-1)!}{(n-r)!r!}(r + (n-r)) \\ &= \frac{n!}{(n-r)!r!} = \binom{n}{r}. \end{aligned}$$

PQ6. Consider a group of n objects, labelled $1, \ldots, n$. How many (unordered) sets of size r are there that contain object 1? How many are there that *don't* contain object 1? Use these observations to re-prove the identity in question **PQ5**.

Answer

There are $\binom{n-1}{r-1}$ sets of size r that contain object 1. (First put object 1 in the set, and then choose another $r-1$ objects from the remaining $n-1$ objects available.) There are $\binom{n-1}{r}$ sets that don't contain object 1. There are a total of $\binom{n}{r}$ unordered sets of size r possible with n objects, and since all of these must either contain object 1 or not contain it, we see that

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}, \qquad 1 \leq r \leq n\,.$$

PQ7. Two dice are thrown. Let A be the event that the sum of the dice is even; let B be the event that the first die shows a higher number than the second; and let C be the event that the sum of the two dice is 6. List all the outcomes belonging to the events $A \cap B$, $A \cup C$, $A \cap C^c$, $A^c \cap C$, $A \cap B \cap C$.

Answer

Writing (i, j) for the outcome of the two dice, where i is the score on the first die and j the score on the second, we have:

$$\begin{aligned}
A \cap B &= \{(3,1),(4,2),(5,1),(5,3),(6,2),(6,4)\}\,;\\
A \cup C = A &= \{(1,1),(1,3),(1,5),(2,2),(2,4),(2,6),(3,1),(3,3),(3,5),(4,2),(4,4),\\
&\qquad (4,6),(5,1),(5,3),(5,5),(6,2),(6,4),(6,6)\}\,;\\
A \cap C^c &= \{(1,1),(1,3),(2,2),(2,6),(3,1),(3,5),(4,4),(4,6),(5,3),(5,5),\\
&\qquad (6,2),(6,4),(6,6)\}\,;\\
A^c \cap C &= \emptyset\,;\\
A \cap B \cap C &= \{(4,2),(5,1)\}\,.
\end{aligned}$$

Assignment Questions – answers to be uploaded

AQ1. Let E, F and G be three events. Find expressions for these events:

a. Only F occurs;
b. F and G occur;
c. E or F occurs, but not G;
d. At least two of the events occur.

Answer

a. $F \cap E^c \cap G^c$; **[1]**
b. $F \cap G$; **[1]**
c. $(E \cup F) \cap G^c$; **[1]**
d. $(E \cap F) \cup (E \cap G) \cup (F \cap G)$. **[2]**

(You may have slightly different answers to the above, but if so then you should be able to prove that your versions are equivalent to mine!)

AQ2. In how many ways can three men and four women be seated in a row if

a. there are no restrictions on the order;
b. no two members of the same sex can sit next to one another?

Answer

a. With no restrictions there are $7! = 5040$ ways. **[2]**
b. Any arrangement satisfying this constraint must take the form $WMWMWMW$ (where W is a woman, and M a man). There are $3! = 6$ ways of arranging the men, and $4! = 24$ ways of arranging the women: this gives 144 arrangements. **[3]**

AQ3. A student has 1 bookshelf, 9 novels (among which 2 titles appear twice each), 4 textbooks on probability theory and 6 knitting books. In how many ways can he put the books on the shelf such that books in the same category (novels, textbooks, knitting books) are next to each other? Possibilities where the doubles are switched are considered to be the same and are not counted separately. (Hint: approx. 9.4×10^9.)

Answer

The 3 categories can be placed in $3! = 6$ ways. The 4 textbooks can be placed in $4! = 24$ ways, and the 6 knitting books in $6! = 720$. **[2]** For the novels, by Proposition 1.4 there are $9!/(2!2!)$ ways of arranging them. **[2]** This gives a total of $3!4!6!\frac{9!}{2!2!} = 9,405,849,600$ ways to order all books. **[1]**

AQ4. Three friends want to place a pizza order (one pizza each), from a restaurant which offers three types of pizza (A, B and C). How many different possible pizza orders are there? (E.g. one possible order is that two people choose A, and one person B.)

Answer

(I realised after setting this question that there was the possibility of confusion here, caused by my use of the word "order". When referring to a "pizza order" I was meaning the "request" that was sent to the restaurant, as indicated in the final line of the question. So an order was meant to be of the sort "We'd like two of pizza A and one of pizza B", and the restaurant wouldn't care which person wanted which pizza. The solution below uses that assumption. ***However****, if you thought that the exact permutation of pizzas (corresponding to which person wanted which pizza) had to be taken into account, then you will still have received marks for this provided that (as always) you clearly explained what you were doing.)*

A general pizza order can be written as a vector (n_A, n_B, n_C), where n_A is the number of people ordering pizza A, etc. We require the number of such vectors for which

- each entry is a non-negative integer
- the entries sum to three. **[3]**

From Proposition 1.10 we know that this number is given by $C_{3-1}^{3+3-1} = C_2^5 = 10$. **[2]**
Another possible answer: n_A people choose pizza A, for some $0 \leq n_A \leq 3$. Once these n_A people are sorted out, we have $3 - n_A$ people left, so n_B can take the values $0, \ldots, 3 - n_A$ (i.e. there are $4 - n_A$ possible values for n_B). So we have to sum

$$\sum_{n_A=0}^{3} (4 - n_A) = 4 + 3 + 2 + 1 = 10\,.$$

Another method (though less elegant) is to simply list all of the possibilities:

$$(3,0,0),\ (2,1,0),\ (2,0,1),\ (1,2,0),\ (1,0,2),\ (1,1,1),\ (0,3,0),\ (0,0,3),\ (0,1,2),\ (0,2,1)\,.$$

(This would not be a good idea if the numbers were much bigger, of course!)

Other Questions (for seminars / extra practice)

OQ1. Prove that, for sets E and F:

a. $E \cap F \subseteq E \subseteq E \cup F$
b. if $E \subseteq F$ then $F^c \subseteq E^c$
c. $E = (E \cap F) \cup (E \cap F^c)$, and show that this union is disjoint.

Answer

a. If $\omega \in E \cap F$, then $\omega \in E$ (and $\omega \in F$), and so $E \cap F \subseteq E$. Similarly for $E \subseteq E \cup F$.

b. Let $\omega \in F^c$, and suppose that $\omega \in E$ also. Then, since $E \subseteq F$, we would have $\omega \in F$. But this would imply that $\omega \in F$ *and* $\omega \in F^c$, i.e. that $\omega \in F \cap F^c = \emptyset$, which is a contradiction. Thus $\omega \notin E$, i.e. $F^c \subseteq E^c$.

c. We can write

$$(E \cap F) \cup (E \cap F^c) = E \cap (F \cup F^c) = E \cap \Omega = E\,.$$

Furthermore, we know that $F \cap F^c = \emptyset$, and so $(E \cap F) \cap (E \cap F^c) = E \cap (F \cap F^c) = \emptyset$ (by part (a)). Thus the union is certainly disjoint.

OQ2. Prove the second of De Morgan's laws: for sets $E_1, \ldots, E_n$:

$$\left(\bigcap_{i=1}^{n} E_i\right)^c = \bigcup_{i=1}^{n} E_i^c\,.$$

Answer

As usual, we need to show that any element ω belonging to the first set belongs to the second, and vice versa.
Suppose first that $\omega \in (\bigcap_{i=1}^n E_i)^c$. Then ω does not belong to *all* of the sets E_i, and so there must be at least one set, say E_j, with $\omega \notin E_j$ (i.e. with $\omega \in E_j^c$). But this implies that $\omega \in \bigcup_{i=1}^n E_i'$.
Working in the other direction, suppose that ω belongs to the set on the RHS. Then there is at least one set E_k with $\omega \in E_k^c$. Thus $\omega \notin \bigcap_{i=1}^n E_i$, since

$$E_k^c \cap \left(\bigcap_{i=1}^{n} E_i\right) \subseteq (E_k^c \cap E_k) = \emptyset\,.$$

Therefore $\omega \in (\bigcap_{i=1}^n E_i)^c$, as required.

OQ3. Show that for $n > 0$,

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0\,.$$

Answer

We know from the binomial theorem that for $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}\,.$$

Substituting $a = -1$ and $b = 1$ gives the required result.

OQ4. Let $\mathbb{P}$ be a probability function and A and B events. Show that

$$\mathbb{P}(A^c \cap B^c) = 1 - \mathbb{P}(A) - \mathbb{P}(B) + \mathbb{P}(A \cap B)\,.$$

Answer

$$
\begin{aligned}
\mathbb{P}(A^c \cap B^c) &= \mathbb{P}((A \cup B)^c) && \text{by De Morgan's law} \\
&= 1 - \mathbb{P}(A \cup B) && \text{by (P4)} \\
&= 1 - \mathbb{P}(A) - \mathbb{P}(B) + \mathbb{P}(A \cap B) && \text{by (P6).}
\end{aligned}
$$

OQ5. Show that if B is an event such that $\mathbb{P}(B) = 1$ (but not necessarily $B = \Omega$), then for all events A, $\mathbb{P}(A \cap B) = \mathbb{P}(A)$.

Answer

Since $A \cup B \supseteq B$ for any event A, it holds that $\mathbb{P}(A \cup B) \geq \mathbb{P}(B) = 1$. Hence, $\mathbb{P}(A \cup B) = 1$ by axiom (P1) and

$$
\begin{aligned}
\mathbb{P}(A \cap B) &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) \text{ by property (P6)} \\
&= \mathbb{P}(A) + 1 - 1 = \mathbb{P}(A).
\end{aligned}
$$

OQ6. Consider events A, B, and C, which can occur in some experiment. Show that the probability that only A occurs (and not B or C) is equal to $\mathbb{P}(A \cup B \cup C) - \mathbb{P}(B) - \mathbb{P}(C) + \mathbb{P}(B \cap C)$.

Answer

The event "only A occurs and not B or C" is the event $A \cap (B \cup C)^c$. The trick in the calculation below is a common one: any event D can be split into two disjoint components by choosing an arbitrary other event A and writing

$$
D = \Omega \cap D = (A \cap D) \cup (A^c \cap D).
$$

We use this below with the choice $D = (B \cup C)^c$:

$$
\begin{aligned}
\mathbb{P}(A \cap (B \cup C)^c) &= \mathbb{P}((B \cup C)^c) - \mathbb{P}(A^c \cap (B \cup C)^c) \\
&= \mathbb{P}((B \cup C)^c) - \mathbb{P}((A \cup B \cup C)^c) \text{ by De Morgan's law} \\
&= 1 - \mathbb{P}(B \cup C) - 1 + \mathbb{P}(A \cup B \cup C) \text{ by (P4).}
\end{aligned}
$$

We cancel the ones and use that $\mathbb{P}(B \cup C) = \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(B \cap C)$ according to (P6) to obtain the desired result.

OQ7. The famous **Birthday Problem**. If there are n people in a room, what is the probability that at least two people have a common birthday? (Ignore leap years for simplicity!)

Answer

As with many problems of this sort, it is easier to calculate the probability of the complementary event: that is, the probability that everybody has a distinct birthday. With n people and 365 days in the year, there is a total of 365^n possible outcomes, $|\Omega| = 365^n$. All outcomes have equal probability. How many of these have all n birthdays distinct? There are

$$
365 \cdot 364 \cdot \dots \cdot (365 - n + 1) = \frac{365!}{(365 - n)!}
$$

outcomes of this kind because we have 365 possible choices for the first person, then 364 for the second, and so on, and $365 - (n - 1)$ choices for the n-th person. Thus the probability that at least two people have a common birthday is given by

$$1 - \frac{365!}{(365-n)!}\frac{1}{365^n}.$$

OQ8. Let k be a non-negative integer. How many distinct integer-valued vectors $(n_1, n_2, \ldots, n_r)$ are there which satisfy both of the following constraints?

i) $n_j \geq k$ for all $j = 1, 2, \ldots, r$
ii) $n_1 + n_2 + \cdots + n_r = n$.

Answer

Let $m_j = n_j - (k-1) \geq 1$. The number of solutions of

$$n_1 + n_2 + \cdots + n_r = n, \quad \text{with } n_j \geq k \text{ for all } j$$

is equal to the number of solutions to

$$m_1 + m_2 + \cdots + m_r = n - r(k-1), \quad \text{with } m_j \geq 1 \text{ for all } j.$$

The number of such solutions is given by

$$C_{r-1}^{n-r(k-1)-1} = \binom{n - r(k-1) - 1}{r-1}.$$

Alternatively, this problem is equivalent to having to divide n sweets among r children, such that each child gets at least k sweets each. We can do this by first of all giving each child k sweets, meaning that there are now $n-rk$ sweets left to divide (where we are allowed to give some children zero extra sweets). The number of ways of doing this is

$$C_{r-1}^{(n-rk)+r-1} = \binom{n - r(k-1) - 1}{r-1}.$$