Introduction to Probability & Statistics

Assignment 2, 2024/25

Assignment Questions – answers to be uploaded

Make sure to justify your reasoning by making reference to any of the properties (P1)–(P7) covered in lectures, where you are using them in your answers.

AQ1. When $\mathbb{P}(A) = 1/2, \mathbb{P}(B) = 1/3$, and $\mathbb{P}(A \cup B) = 3/4$, calculate $\mathbb{P}(A \cap B)$ and $\mathbb{P}(A^c \cup B^c)$.

Answer

Solving the relation $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$ (property (P6)) for $\mathbb{P}(A \cap B)$ gives

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) = 1/2 + 1/3 - 3/4 = 1/12$$

[2]. Using De Morgan's law, and then (P4), we get

$$\mathbb{P}(A^c \cup B^c) = \mathbb{P}((A \cap B)^c) = 1 - \mathbb{P}(A \cap B) = 1 - 1/12 = 11/12.$$

[3]

AQ2. Let A and B be two events. Prove that if $A \subseteq B$ then

$$\mathbb{P}(B \cap A^c) = \mathbb{P}(B) - \mathbb{P}(A).$$

Answer

Since $A \subseteq B$, we can write B as the *disjoint* union $B = (B \cap A^c) \cup A$. **[2]** This gives

$$\mathbb{P}(B) = \mathbb{P}((B \cap A^c) \cup A) \overset{(P3)}{=} \mathbb{P}(B \cap A^c) + \mathbb{P}(A).$$

Rearranging gives the required result. **[3]**

AQ3. Calculate $\mathbb{P}(A \cup B)$ if it is given that $\mathbb{P}(A) = 1/3$ and $\mathbb{P}(B \mid A^c) = 1/2$.

Answer

We use our usual trick of rewriting a union in terms of the union of disjoint events. In this case the best is to write $A \cup B = A \cup (B \cap A^c)$ and thus, by (P3), $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B \cap A^c)$. We can then use that by the multiplication rule

$$\mathbb{P}(B \cap A^c) = \mathbb{P}(B|A^c)\,\mathbb{P}(A^c)$$

[2]. Furthermore $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$, by (P4). Putting everything together gives

$$\begin{aligned}\mathbb{P}(A \cup B) &= \mathbb{P}(A) + \mathbb{P}(B \mid A^c)(1 - \mathbb{P}(A)) \\ &= \frac{1}{3} + \frac{1}{2}(1 - 1/3) = \frac{2}{3}.\end{aligned}$$

[3]

AQ4. Prove that, for any events A, B, C for which the following probabilities are all defined,

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\,\mathbb{P}(B \mid A)\,\mathbb{P}(C \mid A \cap B)\ .$$

Answer

Expanding the RHS using the definition of conditional probability, we obtain

$$\mathbb{P}(A)\,\mathbb{P}(B \mid A)\,\mathbb{P}(C \mid A \cap B) = \mathbb{P}(A)\,\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}\,\frac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(A \cap B)} = \mathbb{P}(A \cap B \cap C)\ .$$

[5]