

# Introduction to Probability & Statistics

Assignment 2, 2024/25

## Practice Questions

**PQ1.** Let  $E$ ,  $F$  and  $G$  be three events. Here are five events built from these:

- a)  $E \cap F^c \cap G^c$
- b)  $(E \cup F) \cap G^c$
- c)  $E \cup F \cup G$
- d)  $(E \cap F \cap G)^c$
- e)  $(E \cap F)^c \cap (E \cap G)^c \cap (F \cap G)^c$ .

And here are verbal descriptions of the same five events, but in different order:

- 1)  $E$  or  $F$  occurs, but not  $G$
- 2) At most two of the events occur
- 3) At most one of the events occurs
- 4) At least one of the events occurs
- 5) Only  $E$  occurs.

Match the expressions to the verbal descriptions.

Answer

a5, b1, c4, d2, e3

**PQ2.** Let  $C$  and  $D$  be events. Express the probability  $\mathbb{P}(C^c \cap D)$  in terms of  $\mathbb{P}(D)$  and  $\mathbb{P}(C \cap D)$ .

Answer

We note that we can write the event  $D$  as the disjoint union of the two events  $C^c \cap D$  and  $C \cap D$ . Therefore by additivity (P3) we have

$$\mathbb{P}(D) = \mathbb{P}(C^c \cap D) + \mathbb{P}(C \cap D).$$

Hence  $\mathbb{P}(C^c \cap D) = \mathbb{P}(D) - \mathbb{P}(C \cap D)$ .

**PQ3.** Let  $E$  and  $F$  be two events for which one knows that the probability that at least one of them occurs is 0.4. What is the probability that neither  $E$  nor  $F$  occurs?

Answer

The event “at least one of  $E$  and  $F$  occurs” is the event  $E \cup F$ . We want the probability of the event that neither  $E$  nor  $F$  occurs, which is the event  $E^c \cap F^c$ . According to the second De Morgan’s law this event can also be written as  $(E \cup F)^c$ . To calculate this we can use the property

(P4) that for any event  $A$  we have  $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$ . This gives us

$$\mathbb{P}((E \cup F)^c) = 1 - \mathbb{P}(E \cup F) = 1 - 0.4 = 0.6.$$

**PQ4.** Show that if an event occurs with probability 0 then it is independent of all other events.

Answer

Suppose that  $A$  satisfies  $\mathbb{P}(A) = 0$ , and let  $B$  be any other event. Then  $\mathbb{P}(A \cap B) \leq \mathbb{P}(A) = 0$ , since  $A \cap B \subseteq A$  (P7). So,  $0 = \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ , and hence  $A$  is independent of  $B$ .

**PQ5.** Calculate  $\mathbb{P}(B)$  if it is given that  $\mathbb{P}(A \cup B) = 3/7$  and  $\mathbb{P}(A^c | B^c) = 9/14$ .

Answer

From the multiplication rule we know that

$$\mathbb{P}(A^c \cap B^c) = \mathbb{P}(A^c | B^c) \mathbb{P}(B^c).$$

Recalling De Morgan's law we know that  $A^c \cap B^c = (A \cup B)^c$  and hence

$$\mathbb{P}(A^c \cap B^c) = \mathbb{P}((A \cup B)^c) = 1 - \mathbb{P}(A \cup B).$$

Combined this yields

$$\begin{aligned} \mathbb{P}(B) &= 1 - \mathbb{P}(B^c) = 1 - \frac{1 - \mathbb{P}(A \cup B)}{\mathbb{P}(A^c | B^c)} \\ &= 1 - \frac{4/7}{9/14} = \frac{1}{9}. \end{aligned}$$

## Assignment Questions – answers to be uploaded

Make sure to justify your reasoning by making reference to any of the properties (P1)–(P7) covered in lectures, where you are using them in your answers.

**AQ1.** When  $\mathbb{P}(A) = 1/2$ ,  $\mathbb{P}(B) = 1/3$ , and  $\mathbb{P}(A \cup B) = 3/4$ , calculate  $\mathbb{P}(A \cap B)$  and  $\mathbb{P}(A^c \cup B^c)$ .

Answer

Solving the relation  $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$  (property (P6)) for  $\mathbb{P}(A \cap B)$  gives

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) = 1/2 + 1/3 - 3/4 = 1/12$$

[2]. Using De Morgan's law, and then (P4), we get

$$\mathbb{P}(A^c \cup B^c) = \mathbb{P}((A \cap B)^c) = 1 - \mathbb{P}(A \cap B) = 1 - 1/12 = 11/12.$$

[3]

**AQ2.** Let  $A$  and  $B$  be two events. Prove that if  $A \subseteq B$  then

$$\mathbb{P}(B \cap A^c) = \mathbb{P}(B) - \mathbb{P}(A).$$

Answer

Since  $A \subseteq B$ , we can write  $B$  as the *disjoint* union  $B = (B \cap A^c) \cup A$ . [2] This gives

$$\mathbb{P}(B) = \mathbb{P}((B \cap A^c) \cup A) \stackrel{(P3)}{=} \mathbb{P}(B \cap A^c) + \mathbb{P}(A).$$

Rearranging gives the required result. [3]

**AQ3.** Calculate  $\mathbb{P}(A \cup B)$  if it is given that  $\mathbb{P}(A) = 1/3$  and  $\mathbb{P}(B | A^c) = 1/2$ .

Answer

We use our usual trick of rewriting a union in terms of the union of disjoint events. In this case the best is to write  $A \cup B = A \cup (B \cap A^c)$  and thus, by (P3),  $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B \cap A^c)$ . We can then use that by the multiplication rule

$$\mathbb{P}(B \cap A^c) = \mathbb{P}(B | A^c) \mathbb{P}(A^c)$$

[2]. Furthermore  $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$ , by (P4). Putting everything together gives

$$\begin{aligned} \mathbb{P}(A \cup B) &= \mathbb{P}(A) + \mathbb{P}(B | A^c) (1 - \mathbb{P}(A)) \\ &= \frac{1}{3} + \frac{1}{2}(1 - 1/3) = \frac{2}{3}. \end{aligned}$$

[3]

**AQ4.** Prove that, for any events  $A, B, C$  for which the following probabilities are all defined,

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A) \mathbb{P}(B | A) \mathbb{P}(C | A \cap B).$$

Answer

Expanding the RHS using the definition of conditional probability, we obtain

$$\mathbb{P}(A) \mathbb{P}(B | A) \mathbb{P}(C | A \cap B) = \mathbb{P}(A) \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} \frac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(A \cap B)} = \mathbb{P}(A \cap B \cap C).$$

[5]

## Other Questions (for seminars / extra practice)

**OQ1.** In a card game involving four players, each player is dealt four cards (face down), and the player with the highest Diamond wins (Ace counts high). Upon receiving your four cards you see that your highest Diamond is the Queen of Diamonds. What is the probability that another player has the King or Ace of Diamonds, and hence that you lose the game?

Answer

You can see four cards, meaning that there are 48 cards whose positions are unknown to you. Twelve of these cards have been dealt to the other players. The number of ways of choosing these twelve cards from the remaining 48 is  $C_{12}^{48}$ . The number of ways in which the King and Ace of Diamonds are *not dealt out* is equal to the number of ways in which twelve cards can be chosen from 46 (the 48 unseen cards minus the King and Ace), i.e.  $C_{12}^{46}$ . Therefore the probability that the King and Ace are not dealt (and that you win) is given by  $C_{12}^{46}/C_{12}^{48} = (35 \cdot 36)/(47 \cdot 48) = 0.559$ . So the required probability is 0.441.

**OQ2.** Consider a sample space  $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$  consisting of four equiprobable elements. Let  $A_1 = \{\omega_1, \omega_2\}$ ,  $A_2 = \{\omega_1, \omega_3\}$ ,  $A_3 = \{\omega_1, \omega_4\}$ . Show that  $A_1$  and  $A_2$  are independent,  $A_1$  and  $A_3$  are independent, and  $A_2$  and  $A_3$  are independent. Are  $A_1, A_2, A_3$  independent?

Answer

Note that  $\mathbb{P}(A_1) = \mathbb{P}(A_2) = \mathbb{P}(A_3) = 1/2$  and  $\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1 \cap A_3) = \mathbb{P}(A_2 \cap A_3) = 1/4 = (1/2) \cdot (1/2)$ , and so the three pairs of events ( $A_1$  and  $A_2$ , etc) are independent. However,

$$\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(\{\omega_1\}) = 1/4 \neq \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3) = 1/8.$$

**OQ3.** Show that if an event  $A$  is independent of itself, then either  $\mathbb{P}(A) = 0$  or  $\mathbb{P}(A) = 1$ .

Answer

Let  $\mathbb{P}(A) = p$ . Since  $A \cap A = A$ , if  $A$  is independent of itself we have  $\mathbb{P}(A \cap A) = \mathbb{P}(A) = p$  and  $\mathbb{P}(A \cap A) = \mathbb{P}(A)\mathbb{P}(A) = p^2$ . So,

$$p = p^2 \iff p(p - 1) = 0 \iff p = 0 \text{ or } p = 1.$$

**OQ4.** Let  $A, B$ , and  $C$  be events. Prove that if

$$\mathbb{P}(A|C) > \mathbb{P}(B|C) \text{ and } \mathbb{P}(A|C^c) > \mathbb{P}(B|C^c)$$

then  $\mathbb{P}(A) > \mathbb{P}(B)$ .

Answer

We can use the partition theorem:

$$\mathbb{P}(A) = \mathbb{P}(A|C)\mathbb{P}(C) + \mathbb{P}(A|C^c)\mathbb{P}(C^c) > \mathbb{P}(B|C)\mathbb{P}(C) + \mathbb{P}(B|C^c)\mathbb{P}(C^c) = \mathbb{P}(B).$$

**OQ5.** Given a sample space  $\Omega$ , let  $\mathbb{Q}$  be some function satisfying

$$\mathbb{Q}(A_1 \cup A_2) = \mathbb{Q}(A_1) + \mathbb{Q}(A_2)$$

for any two disjoint subsets  $A_1$  and  $A_2$  of  $\Omega$ .

Show by induction that if  $A_1, \dots, A_n$  are  $n \geq 2$  disjoint subsets of  $\Omega$  then

$$\mathbb{Q}(A_1 \cup \dots \cup A_n) = \mathbb{Q}(A_1) + \dots + \mathbb{Q}(A_n).$$

Answer

The base case is when  $n = 2$ ; this case is clearly true - it's exactly what we've been told in the question about  $\mathbb{Q}$ .

Now suppose that the result holds for some particular value of  $n \geq 2$ , and suppose that we have disjoint sets  $A_1, \dots, A_{n+1}$ . Then we can write

$$A_1 \cup \dots \cup A_{n+1} = (\cup_{i=1}^n A_i) \cup A_{n+1}.$$

This is a disjoint union of two sets:  $(\cup_{i=1}^n A_i)$  and  $A_{n+1}$ , and so we know that

$$\mathbb{Q}((\cup_{i=1}^n A_i) \cup A_{n+1}) = \mathbb{Q}(\cup_{i=1}^n A_i) + \mathbb{Q}(A_{n+1}).$$

But by our inductive hypothesis,

$$\mathbb{Q}(\cup_{i=1}^n A_i) = \mathbb{Q}(A_1) + \mathbb{Q}(A_2) + \dots + \mathbb{Q}(A_n).$$

It follows that

$$\mathbb{Q}(A_1 \cup \dots \cup A_{n+1}) = \mathbb{Q}(A_1) + \dots + \mathbb{Q}(A_n) + \mathbb{Q}(A_{n+1})$$

and so the required result holds for any value of  $n \geq 2$ , by the principle of mathematical induction.