



# Introduction to Probability & Statistics

Assignment 1, 2025/26

## Instructions


Submit your answers to all questions marked  **Hand-in**. You should upload your solutions to the VLE as a *single pdf file*. Marks will be awarded for clear, logical explanations, as well as for correctness of solutions.

Solutions to questions marked  have been released at the same time as this assignment, in case you want to check your answers or need a hint.

You should also look at the other questions in preparation for your Week 3 seminar.

## Starters

*These questions should help you to gain confidence with the basics.*

**S1.**  A bag contains six balls numbered 1–6. Two balls are pulled out of the bag (without replacement), and the outcome is recorded as  $(i, j)$  if ball  $i$  is drawn first, and ball  $j$  drawn second.

How many possible outcomes are there?

Now suppose that balls numbered 1–4 are white, and the other two are black. How many outcomes are there in which the following occur?

- a. both balls are white;
- b. both balls have the same colour;
- c. at least one of the balls is white?

## Answer

Since we are sampling without replacement there are  $6 \times 5 = 30$  different possible outcomes:  $\{(1, 2), (1, 3), (1, 4), \dots, (2, 1), (2, 3), \dots, (6, 5)\}$ .

- a. Here we need to choose a white ball followed by another white ball:  $4 \times 3 = 12$ ;
- b. As in part (a), the number of ways in which two black balls can be chosen is  $2 \times 1 = 2$ . Hence the total number of outcomes for which both balls have the same colour is  $12 + 2 = 14$ .
- c. Here we need either both balls to be white (which we've already counted), or exactly one to be white. There are  $4 \times 2 = 8$  ways for the first ball to be white and the second black, and another  $2 \times 4 = 8$  ways for the balls to come out in the opposite order; so there are  $16 + 12 = 28$  possible outcomes altogether. (Alternatively, we know from part (b) that there are only two outcomes where both balls are black, and since there are 30 possible outcomes in total, 28 of these must include at least one white ball.)

## **S2.** Hand-in

A bag contains ten balls numbered 1–10. Three balls are pulled out of the bag (without replacement), and the outcome is recorded as  $(i, j, k)$  if ball  $i$  is drawn first, ball  $j$  second, and ball  $k$  third. By Theorem 1.5, there are  $P_3^{10} = 10!/7! = 10 \times 9 \times 8 = 720$  possible outcomes in total. In how many of these outcomes do the following occur:

- all three chosen balls are even-numbered?
- at least one of the chosen balls shows a number which is a multiple of 5?

### S3. 📁 Hand-in

Two rugby union teams, each with 15 players, shake hands at the end of a match: every player on Team A shakes hands with every player on Team B. How many handshakes will there be?

How many will there be if every player shakes hands with every other player (including those on their own team)?

S4. ✂ 30 sweets are to be divided amongst five children. In how many ways can this be done? What if every child must be given at least two sweets each?

Answer

There are  $C_{5-1}^{30+5-1} = 46,376$  ways of dividing the sweets (allowing for the possibility of some children receiving none). If every child must be given two sweets, then if we do this first of all there are 20 sweets left to be divided: there are  $C_{5-1}^{20+5-1} = 10,626$  ways of doing this.

S5. ✂ Consider two events  $A$  and  $B$ . Let  $C$  be the event that “ $A$  or  $B$  occurs, but not both”. Express  $C$  in terms of  $A$  and  $B$ , using only the basic operations “union”, “intersection”, and “complement”.

Answer

There are many possible answers. Here are three:

$$\begin{aligned} C &= (A \cup B) \cap (A \cap B)^c \\ &= (A \cup B) \cap (A^c \cup B^c) \\ &= (A \cap B^c) \cup (B \cap A^c). \end{aligned}$$

### S6. 📁 Hand-in

Let  $E$ ,  $F$  and  $G$  be three events. Find expressions for these events:

- Only  $F$  occurs;
- $F$  and  $G$  occur;
- $E$  or  $F$  occurs, but not  $G$ ;
- At least two of the events occur.

S7. Two dice are thrown. Let  $A$  be the event that the sum of the dice is even; let  $B$  be the event that the first die shows a higher number than the second; and let  $C$  be the event that the sum of the two dice is 6. List all the outcomes belonging to the events  $A \cap B$ ,  $A \cup C$ ,  $A \cap C^c$ ,  $A^c \cap C$ ,  $A \cap B \cap C$ .

## Mains

*These are a bit harder: you should aim to be able to do all of these by the end of the course.*

### M1. 📁 Hand-in

In how many ways can four men and four women (all distinguishable!) be seated in a row if

- there are no restrictions on the order?
- no members of the same sex can sit next to one another?
- there are four married couples, and each couple must sit together?

**M2.** ✂ A student has 1 bookshelf, 10 novels (among which 3 titles appear twice each), 4 textbooks on probability theory and 6 knitting books. In how many ways can he put the books on the shelf such that books in the same category (novels, textbooks, knitting books) are next to each other? Possibilities where the doubles are switched are considered to be the same and are not counted separately.

Answer

The 3 categories can be placed in  $3! = 6$  ways. The 4 textbooks can be placed in  $4! = 24$  ways, and the 6 knitting books in  $6! = 720$ . For the novels, by Proposition 1.4 there are  $10!/(2!2!2!)$  ways of arranging them. This gives a total of  $3!4!6!\frac{10!}{2!2!2!} = 47,029,248,000$  ways to order all books.

**M3.** Prove, by expanding in terms of factorials, that

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}, \quad 1 \leq r \leq n.$$

**M4.** Consider a group of  $n$  objects, labelled  $1, \dots, n$ . How many (unordered) sets of size  $r$  are there that contain object 1? How many are there that *don't* contain object 1? Use these observations to re-prove the identity in the previous question.

**M5.** ✂ Prove that, for sets  $E$  and  $F$ :

- $E \cap F \subseteq E \subseteq E \cup F$
- if  $E \subseteq F$  then  $F^c \subseteq E^c$
- $E = (E \cap F) \cup (E \cap F^c)$ , and show that this union is disjoint.

Answer

- If  $\omega \in E \cap F$ , then  $\omega \in E$  (and  $\omega \in F$ ), and so  $E \cap F \subseteq E$ . Similarly for  $E \subseteq E \cup F$ .
- Let  $\omega \in F^c$ , and suppose that  $\omega \in E$  also. Then, since  $E \subseteq F$ , we would have  $\omega \in F$ . But this would imply that  $\omega \in F$  and  $\omega \in F^c$ , i.e. that  $\omega \in F \cap F^c = \emptyset$ , which is a contradiction. Thus  $\omega \notin E$ , i.e.  $F^c \subseteq E^c$ .
- We can write
$$(E \cap F) \cup (E \cap F^c) = E \cap (F \cup F^c) = E \cap \Omega = E.$$
Furthermore, we know that  $F \cap F^c = \emptyset$ , and so  $(E \cap F) \cap (E \cap F^c) = E \cap (F \cap F^c) = \emptyset$  (by part (a)). Thus the union is certainly disjoint.

**M6.** Prove the second of De Morgan's laws: for sets  $E_1, \dots, E_n$ :

$$\left( \bigcap_{i=1}^n E_i \right)^c = \bigcup_{i=1}^n E_i^c.$$

**M7.** Let  $\mathbb{P}$  be a probability function and  $A$  and  $B$  events. Show that

$$\mathbb{P}(A^c \cap B^c) = 1 - \mathbb{P}(A) - \mathbb{P}(B) + \mathbb{P}(A \cap B).$$

**M8.** Show that if  $B$  is an event such that  $\mathbb{P}(B) = 1$ , then for all events  $A$ ,  $\mathbb{P}(A \cap B) = \mathbb{P}(A)$ . (Hint: your answer *should not* assume that  $B = \Omega$ ! It's certainly possible to have  $\mathbb{P}(B) = 1$  for a set  $B \neq \Omega$ .)

**M9.** Consider events  $A$ ,  $B$ , and  $C$ , which can occur in some experiment. Show that the probability that only  $A$  occurs (and not  $B$  or  $C$ ) is equal to  $\mathbb{P}(A \cup B \cup C) - \mathbb{P}(B) - \mathbb{P}(C) + \mathbb{P}(B \cap C)$ .

## Desserts

*Still hungry for more? Try these if you want to push yourself further. (These are mostly harder than I'd expect you to answer in an exam, or involve non-examinable material.)*

**D1.** Show that for  $n > 0$ ,

$$\sum_{i=0}^n (-1)^i \binom{n}{i} = 0.$$

**D2.** The famous **Birthday Problem**. If there are  $n$  people in a room, what is the probability that at least two people have a common birthday? (Ignore leap years for simplicity!)

**D3.** Let  $k$  be a non-negative integer. How many distinct integer-valued vectors  $(n_1, n_2, \dots, n_r)$  are there which satisfy both of the following constraints?

- i)  $n_j \geq k$  for all  $j = 1, 2, \dots, r$
- ii)  $n_1 + n_2 + \dots + n_r = n$ .

**D4.** Calculating combinations for large integers can get a bit messy, but **Stirling's Approximation** can sometimes be very useful: this says that as  $n$  gets large,  $n!$  can be well-approximated by

$$\sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$$

Use Stirling's approximation to show that, for large  $n$ , the number of ways of choosing  $n$  out of  $2n$  distinct items is approximately

$$\frac{4^n}{\sqrt{\pi n}}.$$

### ! Challenge question

How many vectors  $(x_1, x_2, \dots, x_n)$  are there, such that each  $x_i$  is a non-negative integer and

$$\sum_{i=1}^n x_i \leq k,$$

where  $k \geq 0$  is an integer?

Can you find a closed form expression for this number (one not involving any summation symbols)?

Does the number of such vectors increase faster as we change  $n \rightarrow n + 1$  or  $k \rightarrow k + 1$ ?