Introduction to Probability & Statistics

Assignment 1, 2025/26

> **ℹ Instructions**
>
> Submit your answers to all questions marked ☁ **Hand-in**. You should upload your solutions to the VLE as a *single pdf file*. Marks will be awarded for clear, logical explanations, as well as for correctness of solutions.
> Solutions to questions marked ✖ have been released at the same time as this assignment, in case you want to check your answers or need a hint.
> You should also look at the other questions in preparation for your Week 3 seminar.

Starters

These questions should help you to gain confidence with the basics.

S1. ✖ A bag contains six balls numbered 1–6. Two balls are pulled out of the bag (without replacement), and the outcome is recorded as (i, j) if ball i is drawn first, and ball j drawn second. How many possible outcomes are there?
Now suppose that balls numbered 1–4 are white, and the other two are black. How many outcomes are there in which the following occur?

  a. both balls are white;
  b. both balls have the same colour;
  c. at least one of the balls is white?

S2. ☁ **Hand-in**
A bag contains ten balls numbered 1–10. Three balls are pulled out of the bag (without replacement), and the outcome is recorded as (i, j, k) if ball i is drawn first, ball j second, and ball k third. By Theorem 1.5, there are $P_3^{10} = 10!/7! = 10 \times 9 \times 8 = 720$ possible outcomes in total. In how many of these outcomes do the following occur:

  a. all three chosen balls are even-numbered?
  b. at least one of the chosen balls shows a number which is a multiple of 5?

> **Answer**
>
>   a. We need to count the number of ordered lists of length three, which can be formed from the set of five even-numbered balls. By Theorem 1.5, this is $P_3^5 = 5!/2! = 60$. **[2 marks]**
>   b. There are eight balls with numbers which are *not* a multiple of 5, so there are $P_3^8 = 8 \times 7 \times 6 = 336$ outcomes in which all of the balls show numbers which are not a multiple of 5. This means that there are $720 - 336 = 384$ outcomes in which at least one of the balls is a multiple of 5. **[3 marks]**

S3. ☁ **Hand-in**

Two rugby union teams, each with 15 players, shake hands at the end of a match: every player on Team A shakes hands with every player on Team B. How many handshakes will there be?

How many will there be if every player shakes hands with every other player (including those on their own team)?

> **Answer**
>
> Each player from Team A has to shake hands with 15 players from Team B. So there are $15^2 = 225$ handshakes in total. **[2 marks]**
>
> If everyone shakes hands with everyone else, then we need to count the number of ways of choosing two players out of 30: $C_2^{30} = 435$. **[3 marks]**
>
> (Alternatively, we could count how many handshakes take place between two members of the same team, and add this to the 225 that we obtained above. With 15 people in Team A, there are $C_2^{15} = 105$ pairs of players to shake hands, and then there's the same number of handshakes between members of Team B. This gives 210 extra handshakes.)

S4. ✂ 30 sweets are to be divided amongst five children. In how many ways can this be done? What if every child must be given at least two sweets each?

S5. ✂ Consider two events A and B. Let C be the event that "A or B occurs, but not both". Express C in terms of A and B, using only the basic operations "union", "intersection", and "complement".

S6. ☁ **Hand-in**

Let E, F and G be three events. Find expressions for these events:

a. Only F occurs;
b. F and G occur;
c. E or F occurs, but not G;
d. At least two of the events occur.

> **Answer**
>
> a. $F \cap E^c \cap G^c$; **[1 mark]**
> b. $F \cap G$; **[1 mark]**
> c. $(E \cup F) \cap G^c$; **[1 mark]**
> d. $(E \cap F) \cup (E \cap G) \cup (F \cap G)$. **[2 marks]**
>
> (You may have slightly different answers to the above, but if so then you should be able to prove that your versions are equivalent to mine!)

S7. Two dice are thrown. Let A be the event that the sum of the dice is even; let B be the event that the first die shows a higher number than the second; and let C be the event that the sum of the two dice is 6. List all the outcomes belonging to the events $A \cap B$, $A \cup C$, $A \cap C^c$, $A^c \cap C$, $A \cap B \cap C$.

Mains

These are a bit harder: you should aim to be able to do all of these by the end of the course.

M1. ☁ **Hand-in**

In how many ways can four men and four women (all distinguishable!) be seated in a row if

a. there are no restrictions on the order?
b. no members of the same sex can sit next to one another?
c. there are four married couples, and each couple must sit together?

M2. ✂ A student has 1 bookshelf, 10 novels (among which 3 titles appear twice each), 4 textbooks on probability theory and 6 knitting books. In how many ways can he put the books on the shelf such that books in the same category (novels, textbooks, knitting books) are next to each other? Possibilities where the doubles are switched are considered to be the same and are not counted separately.

M3. Prove, by expanding in terms of factorials, that

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}, \qquad 1 \leq r \leq n.$$

M4. Consider a group of n objects, labelled $1, \ldots, n$. How many (unordered) sets of size r are there that contain object 1? How many are there that *don't* contain object 1? Use these observations to re-prove the identity in the previous question.

M5. ✂ Prove that, for sets E and F:

   a. $E \cap F \subseteq E \subseteq E \cup F$
   b. if $E \subseteq F$ then $F^c \subseteq E^c$
   c. $E = (E \cap F) \cup (E \cap F^c)$, and show that this union is disjoint.

M6. Prove the second of De Morgan's laws: for sets $E_1, \ldots, E_n$:

$$\left(\bigcap_{i=1}^{n} E_i \right)^c = \bigcup_{i=1}^{n} E_i^c.$$

M7. Let $\mathbb{P}$ be a probability function and A and B events. Show that

$$\mathbb{P}(A^c \cap B^c) = 1 - \mathbb{P}(A) - \mathbb{P}(B) + \mathbb{P}(A \cap B).$$

M8. Show that if B is an event such that $\mathbb{P}(B) = 1$, then for all events A, $\mathbb{P}(A \cap B) = \mathbb{P}(A)$. (Hint: your answer *should not* assume that $B = \Omega$! It's certainly possible to have $\mathbb{P}(B) = 1$ for a set $B \neq \Omega$.)

M9. Consider events A, B, and C, which can occur in some experiment. Show that the probability that only A occurs (and not B or C) is equal to $\mathbb{P}(A \cup B \cup C) - \mathbb{P}(B) - \mathbb{P}(C) + \mathbb{P}(B \cap C)$.

Desserts

Still hungry for more? Try these if you want to push yourself further. (These are mostly harder than I'd expect you to answer in an exam, or involve non-examinable material.)

D1. Show that for $n > 0$,

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0.$$

D2. The famous **Birthday Problem**. If there are n people in a room, what is the probability that at least two people have a common birthday? (Ignore leap years for simplicity!)

D3. Let k be a non-negative integer. How many distinct integer-valued vectors $(n_1, n_2, \ldots, n_r)$ are there which satisfy both of the following constraints?

    i) $n_j \geq k$ for all $j = 1, 2, \ldots, r$
    ii) $n_1 + n_2 + \cdots + n_r = n$.

D4. Calculating combinations for large integers can get a bit messy, but **Stirling's Approximation** can sometimes be very useful: this says that as n gets large, $n!$ can be well-approximated by

$$\sqrt{2\pi n} \left(\frac{n}{e} \right)^n .$$

Use Stirling's approximation to show that, for large n, the number of ways of choosing n out of $2n$ distinct items is approximately

$$\frac{4^n}{\sqrt{\pi n}} .$$

! Challenge question

How many vectors $(x_1, x_2, \ldots, x_n)$ are there, such that each x_i is a non-negative integer and

$$\sum_{i=1}^{n} x_i \leq k ,$$

where $k \geq 0$ is an integer?
Can you find a closed form expression for this number (one not involving any summation symbols)?
Does the number of such vectors increase faster as we change $n \to n + 1$ or $k \to k + 1$?