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An Investigation into Local Entropy Generation in Shock Waves

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Abstract

Shock waves are usually modeled as discontinuities in the flow field. For this paper the shock was modeled as a continuous flow field using the one dimensional version of the Navier-Stokes equations. In order to model the flow field as a continuum, it is assumed that the flow properties are averaged over distances much greater than the mean free path of the molecules. In order to test this assumption, the characteristic length scale of the shock wave, or the shock thickness, was compared to the mean free path of the molecules far upstream of the shock. The shock thickness for weak shocks satisfies this requirement, but as the shock strength increases, the shock thickness quickly drops to a couple of mean free paths. However, the ability of the Navier-Stokes equations to model the flow field inside the shock wave can only be truly evaluated by comparison to experimental data. From the scare data that was available, the continuum model was extremely accurate even for a shock thickness of about two mean free paths.

The purpose of the model was to assess the relative importance of viscous forces and heat conduction in generating entropy within the shock wave. The relative contributions of each were calculated for various shock strengths. The calculations show that for weak shocks viscosity generates more entropy than the heat conduction, but as the shock strength is increased, the relative importance of thermal conduction increases and overtakes the relative importance of viscosity.

Mathematical Modeling of the Shock Profile

Equations of Motion

The steady one-dimensional equations of motion are:

$$\frac{d}{dx}(\rho u) = 0$$

$$\rho u \frac{du}{dx} = \frac{d\tau}{dx}$$

$$\rho u \frac{d}{dx}\left(e + \frac{1}{2}u^2\right) = \frac{d}{dx}(\tau u) - \frac{dq}{dx}$$

The heat flow due to thermal conduction is expressed by the Fourier equation.

$$q = -\lambda \frac{dT}{dx}$$

The stress term is a linear function of the pressure and viscous stresses.

$$\tau = -p + \mu \frac{du}{dx}$$

Substituting for the stress and thermal conduction terms and integrating the equations of motion, one can define the following quantities.

$$\rho u = m$$

$$\rho u^2 + p - \mu \frac{du}{dx} = P$$

$$\rho u\left(e + \frac{1}{2}u^2 + \frac{p}{\rho}\right) - \mu u \frac{du}{dx} - \lambda \frac{dT}{dx} = E$$

The constants m, P, and E are determined by either the conditions far upstream or far downstream of the shock, where the gradients in the flow field are zero.

$$m = \rho_0 u_0 = \rho_1 u_1$$
$$P = \rho_0 u_0^2 + p_0 = \rho_1 u_1^2 + p_1$$
$$E = m\left(e_0 + \frac{1}{2}u_0^2 + \frac{p_0}{\rho_0}\right) = m\left(e_1 + \frac{1}{2}u_1^2 + \frac{p_1}{\rho_1}\right)$$

These equations represent the "jump conditions" across the shock.

By applying the continuity equation, the density can be eliminated and the following system of two equations is obtained.

$$\mu \frac{du}{dx} = p + b(u - a)$$
$$\lambda \frac{dT}{dx} = b\left(e - \frac{1}{2}(u-a)^2 - c\right)$$

The terms a, b, c are given by:

$$a = \frac{P}{m}$$
$$b = m$$
$$c = \frac{E}{m} - \frac{P^2}{2m^2}$$

In addition, if the fluid is assumed to be an ideal gas, the following relationships hold.

$$p = \rho R T$$
$$e = c_v T$$
$$R = c_p - c_v$$

Once the velocity and temperature distribution is obtained by integrating the pair of differential equations, all fluid properties can be expressed in terms of u and T.

<u>Viscosity</u>

The temperature dependence of viscosity is approximated by a power law relationship.

$$\mu = \mu_o \left(\frac{T}{T_o}\right)^s$$

According to kinetic theory, the two most frequently used values of s are one-half and one (corresponding to the elastic sphere and Maxwellian molecule model respectively). Experimentally determined values of s are 0.647 for Helium and 0.816 for Argon. Based on tabulated values of the viscosity, the value of s for air is approximately 0.768. The values of μ_o and T_o for air were taken as 1.7285e-5 Pa•s and 273 K respectively.

<u>Thermal Conductivity</u>

Given the viscosity, the thermal conductivity can be expressed in terms of the Prandtl number (assumed to be constant for a given fluid).

$$\lambda = \frac{c_p \mu}{Pr}$$

For monatomic gases, the Prandtl number is two-thirds. Based on tabulated values for the viscosity and thermal conductivity, the Prandtl number for air is approximately 0.714.

Initial Perturbation

In order to integrate the pair of differential equations across the shock, it is necessary to introduce an intial perturbation to the flow properties. If the conditions far upstream or downstream were used, all the gradients are zero and no shock occurs. In order to determine the proper perturbation, a stability analysis was performed by Gilbarg in Reference 1. The analysis is omitted here, but two important results were obtained. One, by linearizing the equations, the slope of the solution curve can be found near a nodal point (jump conditions). For an arbitrarily small perturbation in the velocity, the slope gives the appropriate perturbation in the temperature or vice-versa. Two, the stability analysis indicates that the only stable procedure is to perturb the downstream conditions (in the proper direction) and integrate backwards across the shock.

Shock Thickness

Velocity Profile

An important parameter in determining the applicability of the Navier-Stokes equations is the shock thickness. For the purposes of this report, the shock thickness used is the one based on the normalized velocity profile. The normalized velocity changes from a value of one far upstream to zero far downstream.

$$\phi = \frac{u - u_1}{u_0 - u_1}$$

A typical normalized velocity profile is shown in Figure 1. The shock thickness is defined as the transition length over which the normalized velocity ranges from 0.95 to 0.05. This is an identical definition of the velocity thickness used in shear layer analysis.

Another possible definition for the shock thickness is equivalent to the vorticity thickness used in shear layer analysis. One drawback to this definition is that it is particularly sensitive to parameters used to characterize the viscosity, since this definition depends on local gradients.

$$\varepsilon = \frac{u_o - u_1}{\left|\frac{du}{dx}\right|_{max}}$$

The shock thickness should be compared to the Maxwell mean free path of a molecule.

$$l_o = \frac{\mu_o \sqrt{\pi}}{\rho_o \sqrt{2RT_o}}$$

Dependence on Shock Strength

The shock thickness normalized by the mean free path of the molecules far upstream of the shock are displayed in Figure 2 for a range of incoming Mach numbers from 1.04 to 5. Three conclusions can be drawn from these calculations. First, as the shock strength increases, the shock thickness decreases. Second, the shock thickness quickly decreases to the same order of magnitude as the mean free path for incoming Mach numbers greater than approximately 1.5. Third, the shock thickness normalized by the upstream mean free path does not depend on the total temperature or total pressure of the flow. (Both were varied by a factor of two without changing the curve in Figure 2.)

Validity of Model

The reference mean free path is taken far upstream of the shock. The assumptions implicit in the Navier-Stokes assumptions start to break down when the shock thickness is on the same order as the mean free path. The continuum model assumes the flow properties are averaged over distances much greater than the mean free path. However, a comparison with experimental data in Figure 3 shows exact agreement even when the shock thickness is about two mean free paths. As Gilbarg so eloquently said, "Equations have often been successful beyond the limits of their original derivation, and indeed this type of success is one of the hallmarks of a great theory." The continuum model even does better than the kinetic theory of Mott-Smith.

Relative Importance of Viscosity and Thermal Conduction in Generating Entropy across a Shock Wave

Entropy Generation

The rate of change of entropy across the shock can be expressed as:

$$m\frac{ds}{dx}=\frac{\mu}{T}\left(\frac{du}{dx}\right)^2+\frac{\lambda}{T^2}\left(\frac{dT}{dx}\right)^2$$

The first term of the right hand side accounts for entropy generation due to viscous forces and the second term accounts for entropy generation due to thermal conduction. By integrating each term separately, the total entropy rise due to each effect can be calculated and compared.

Dependence on Shock Strength

The ratio of the entropy rise due to viscosity to the entropy rise due to conduction is displayed in Figure 5 for a range of incoming Mach numbers from 1.04 to 5. The calculations show that viscosity generates more entropy than the heat conduction for weak shocks, but as the shock strength is increased, the relative importance of thermal conduction increases and overtakes the relative importance of viscosity.

Entropy Thickness

The entropy thickness is defined as the length over which the entropy changes from 5% to 95% of the total entropy rise across the shock. The ratio of the entropy thickness to the velocity thickness is displayed in Figure 6 for a range of incoming Mach numbers from 1.04 to 3. The entropy thickness decreases even more rapidly than the velocity thickness and is always smaller. This means that the entropy change occurs over a smaller distance than the velocity change.

Conclusions

The typical length scale of a shock wave decreases with increasing Mach number. Although the length scale rapidly decreases to the same magnitude as the mean free path, available data shows that the continuum model still accurately predicts fluid properties for shock thickness as low as two mean free paths.

Although the total entropy generation across a shock wave can be obtained from the jump conditions, the only way to obtain the relative contributions is to integrate across the shock wave. The calculations show that for weak shocks viscosity generates more entropy than the heat conduction, but as the shock strength is increased, the relative importance of thermal conduction increases and overtakes the relative importance of viscosity.

References

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Liepman, H., and A. Roshko, 1957, *Elements of Gas Dynamics*, Wiley.

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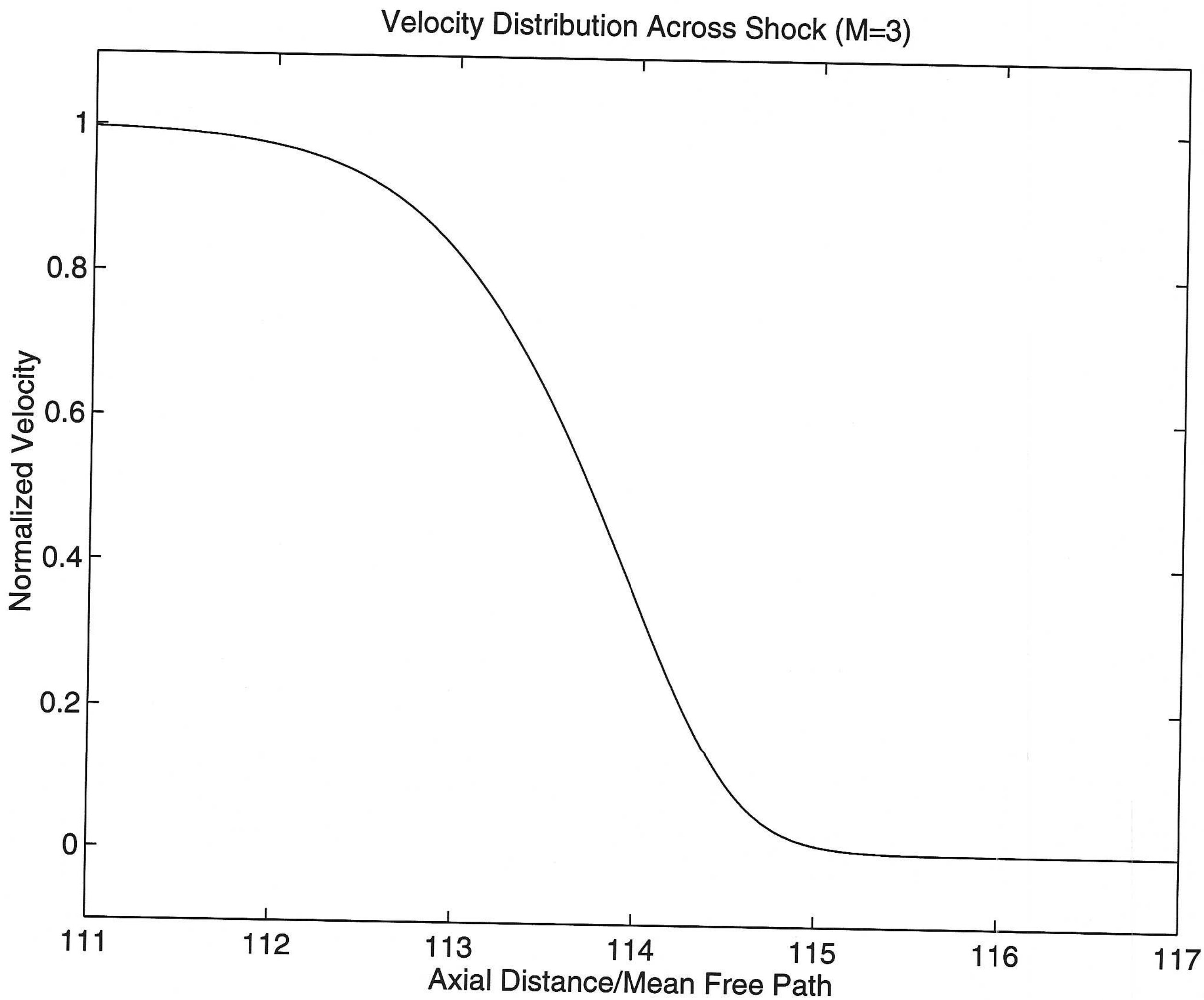


Velocity Distribution Across Shock (M=3)
Normalized Velocity
Axial Distance/Mean Free Path
1
0.8
0.6
0.4
0.2
0
111
112
113
114
115
116
117

Figure 1.

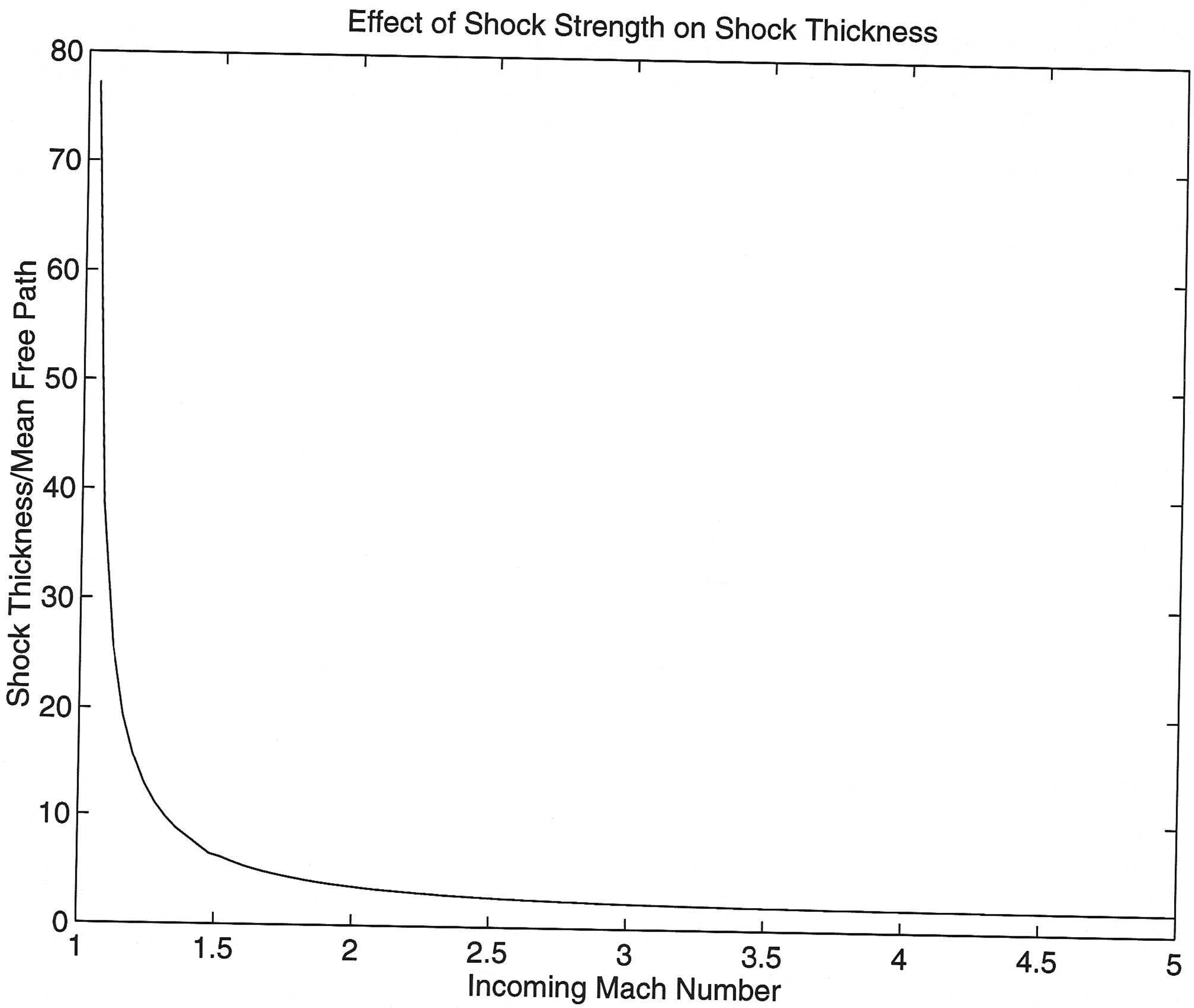

Effect of Shock Strength on Shock Thickness
Shock Thickness/Mean Free Path
Incoming Mach Number
0
10
20
30
40
50
60
70
80
1
1.5
2
2.5
3
3.5
4
4.5
5

Figure 2

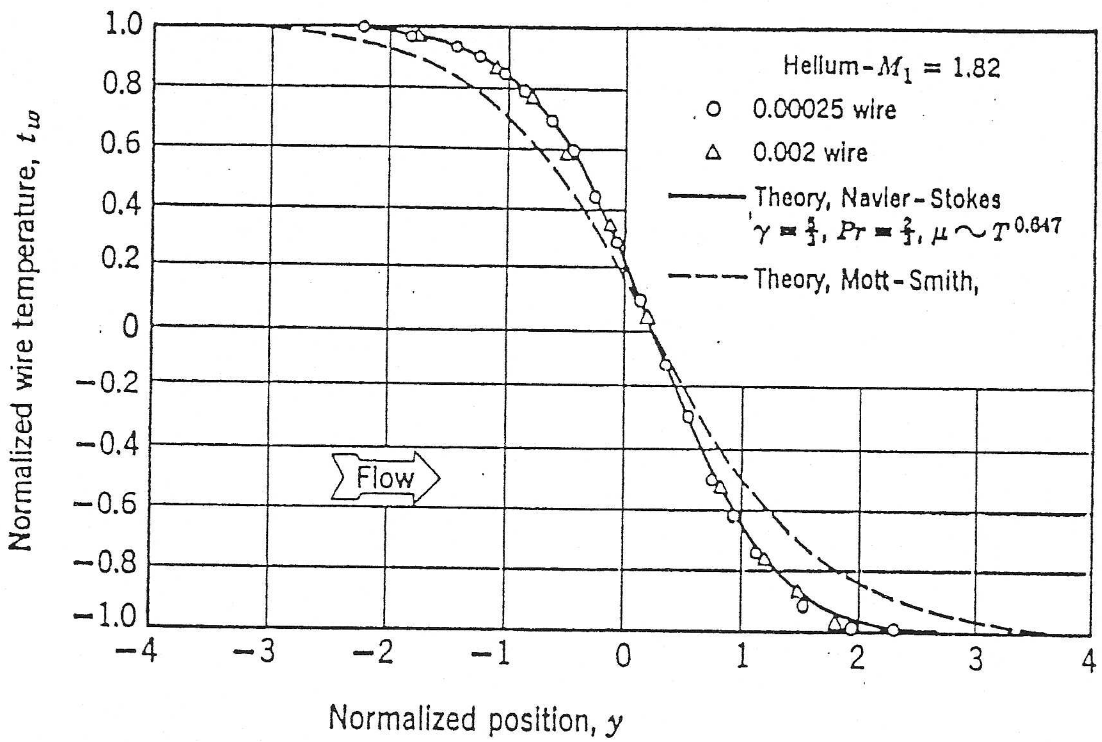
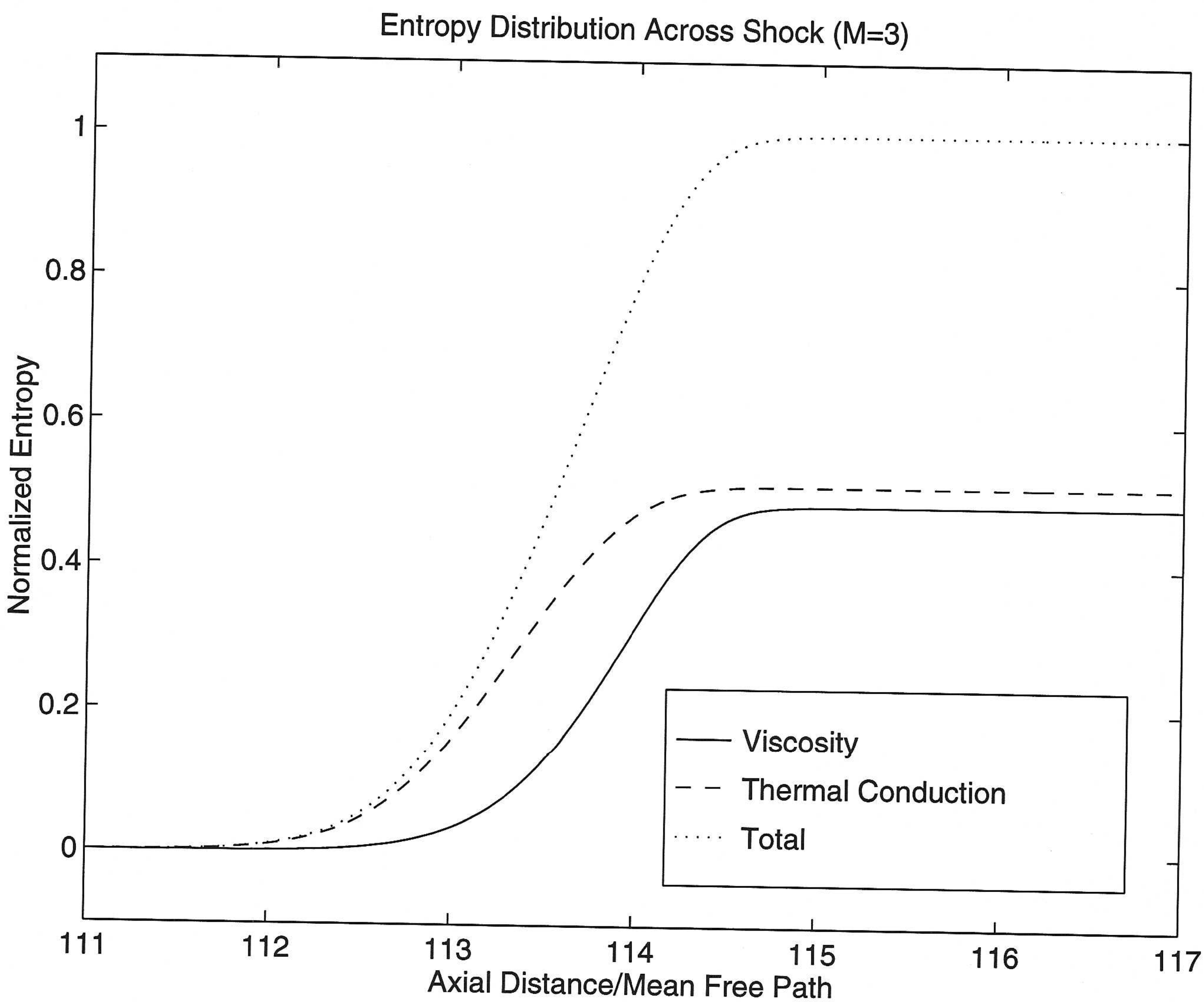


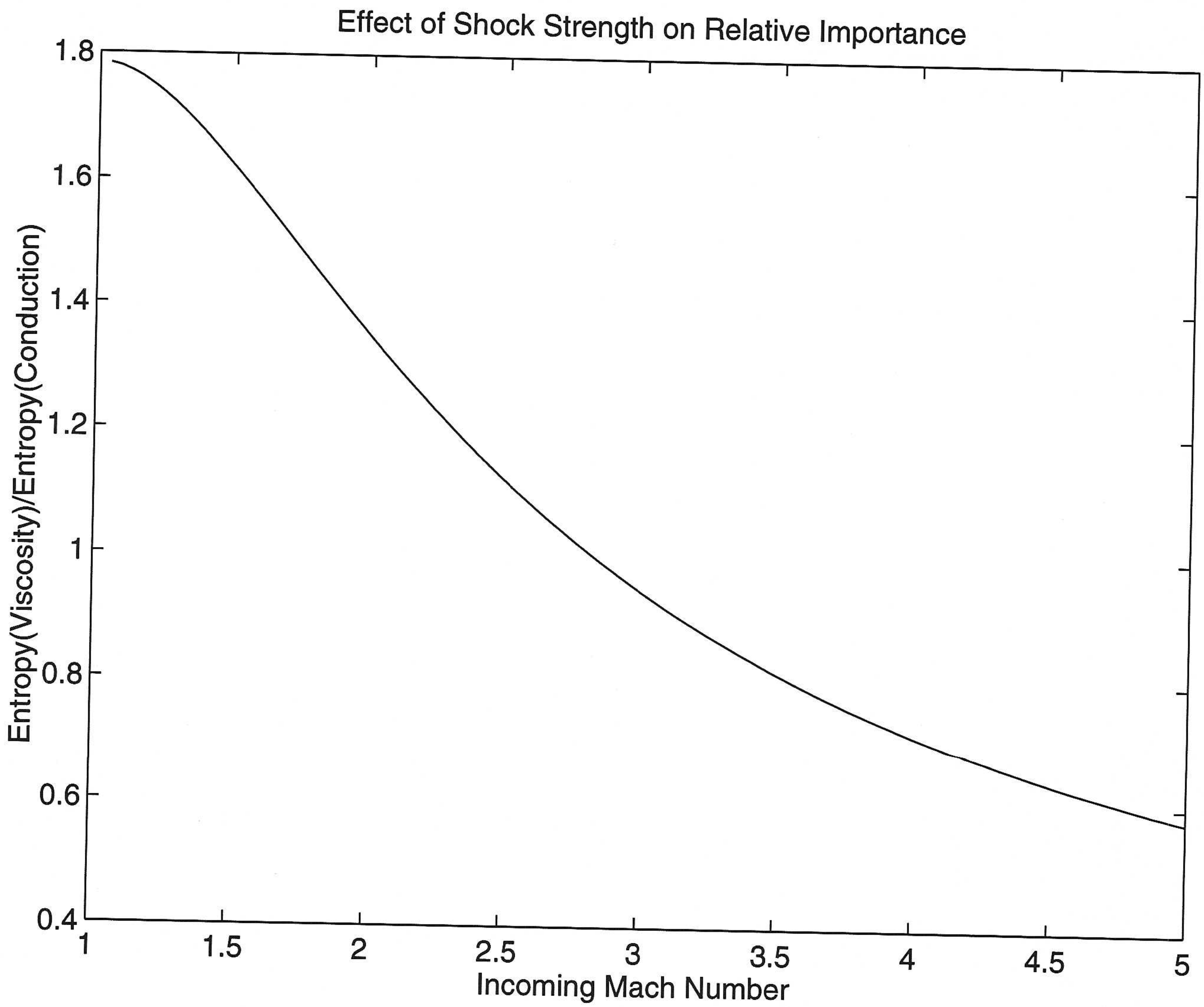
FIG. 13·9 Temperature profile through a normal shock wave. $M_1 = 1.82$ in helium. Temperatures measured with a resistance wire thermometer. [From F. S. Sherman, "A Low-Density Wind Tunnel Study of Shock Wave Structure and Relaxation Phenomena in Gases," *NACA Tech. Note* 3298 (1955).]

Figure 3.



Entropy Distribution Across Shock (M=3)
Normalized Entropy
0
0.2
0.4
0.6
0.8
1
111
112
113
114
115
116
117
Axial Distance/Mean Free Path
Viscosity
Thermal Conduction
Total

Figure 4.



Effect of Shock Strength on Relative Importance
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
Entropy(Viscosity)/Entropy(Conduction)
1
1.5
2
2.5
3
3.5
4
4.5
5
Incoming Mach Number

Figure 5.

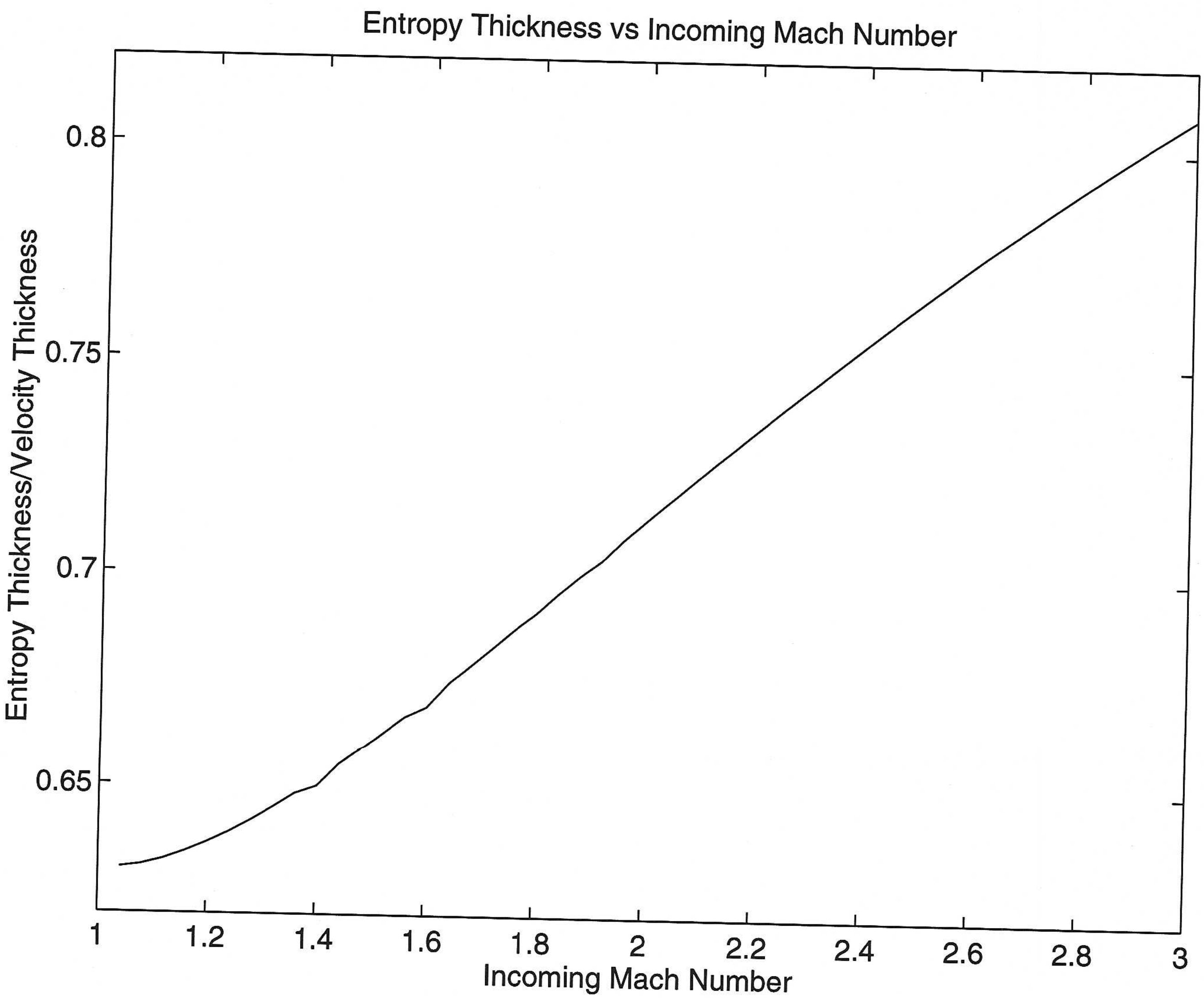

Entropy Thickness vs Incoming Mach Number
0.8
0.75
0.7
0.65
Entropy Thickness/Velocity Thickness
1
1.2
1.4
1.6
1.8
2
2.2
2.4
2.6
2.8
3
Incoming Mach Number

Figure 6.