CSE 321 - Introduction to Algorithm Design Homework 01

1) a) It is false.

For example $f(n) = n$ and $g(n) = \begin{cases} 1, & \text{if } n \text{ is odd} \\ n^2, & \text{if } n \text{ is even} \end{cases}$

functions create contrary example.

b) For $f(n) \in O(g(n))$ there is c and n_0

$$f(n) \leq c \cdot g(n) \quad \forall n \geq n_0 \quad (1)$$

For $h(n)$ is a runtime function $h(n) > 0 \;\; \forall n$

If we divide both sides of equation (1) by $h(n)$

$$\frac{f(n)}{h(n)} \leq c \cdot \frac{g(n)}{h(n)} \quad \forall n \geq n_0$$

So $\dfrac{f(n)}{h(n)} \in O\left(\dfrac{g(n)}{h(n)}\right)$

c) $f(n) \in O(g(n)) \Rightarrow \exists c, n_0 \;\; \forall n \geq n_0 \;\; f(n) \leq c \cdot g(n)$ (1).

If we power both sides of equation 1 ~~by~~ to the k

$$f(n)^k \leq c^k \cdot g(n)^k \quad \forall n \geq n_0$$

$$f(n)^k \in O(g(n)^k)$$

For this solution we assume k is positive. If k is negative this solution is wrong. For this situation $f(n)=1$ $g(n)=n$ $k=-1$ is a contrary example.

2) a) Exponential function > Polynomial function, it is TRUE

b) $\lim_{n \to \infty} \frac{3^n}{2^n} = \lim_{n \to \infty} \left(\frac{3}{2}\right)^n = \infty$, it is TRUE

c) Factorial function > Exponential function, it is FALSE

3) Average Complexity of Linear Search with Repeated Elements

n: number of elements in the list

m: number of distinct elements in the list $\in S = \{s_1, s_2, \ldots, s_m\}$

Searched element $X \in S$

Assume: $L[i]$ has an equal prob. $\frac{1}{m}$ of being any element in S; for $i = 1, 2, 3, \ldots, n$

$$\text{Prob}\{L[i] = s_j\} = \frac{1}{m}; \quad \text{for } \begin{matrix} j = 1, \ldots, m \\ i = 1, \ldots, n \end{matrix}$$

- Prob. that X does not occur in position i $\overset{\Rightarrow \text{Prob.}[X \neq L[i]]}{=} 1 - \frac{1}{m} = \frac{m-1}{m}$; $\forall i = 1, \ldots, n$
- Prob. that X is not in the list $= \prod_{i=1}^{n} [\text{Prob. } X = L[i]] = \prod_{i=1}^{n} \frac{m-1}{m} = \left(\frac{m-1}{m}\right)^n$
- Prob. that X is in the list $= 1 - \left(\frac{m-1}{m}\right)^n$

P_i = Prob. that the algorithm performs exactly i basic operation

$\Downarrow$

Prob. that the 1st occurence of the searched element X is in position i

$$P_i = \text{Prob}(T_i = i) \Rightarrow A(n) = E(T_i) = \sum_{i=1}^{n} i \cdot P_i$$

P_n: Pro. that X does not occur in the first $n-1$ positions

$\Uparrow$

$$P_i = \begin{cases} \left(\frac{m-1}{m}\right)^{i-1} \cdot \frac{1}{m} & ; \text{ if } 1 \leq i \leq n-1 \\ \left(\frac{m-1}{m}\right)^{n-1} & ; \text{ if } i = n \end{cases}$$

$\left(\frac{m-1}{m}\right)^{i-1}$ → Prob. that X does not occur in the 1st $i-1$ position

$\frac{1}{m}$ → Prob. that X occurs in the ith position

$$A(n,m) = \sum_{i=1}^{W(n)} i \cdot P_i = \sum_{i=1}^{n-1} i \cdot \left(\left(\frac{m-1}{m}\right)^{i-1} \cdot \frac{1}{m}\right) + \left(\frac{m-1}{m}\right)^{n-1}$$

Formula: $$\sum_{i=1}^{n-1} i \cdot x^{i-1} = \sum_{i=1}^{n-1} (x^i)' = \left(\sum_{i=1}^{n-1} x^i\right)' = (x + x^2 + \cdots + x^{n-1})'$$

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1}$$

$$= \left(\frac{x^n - 1}{x - 1} - 1\right)' = \frac{(x^n - 1)'(x-1) - (x^n - 1)(x-1)'}{(x-1)^2}$$

$$= \frac{n \cdot x^{n-1}(x-1) - (x^n - 1)}{(x-1)^2} = \frac{n x^n - n x^{n-1} - x^n + 1}{(x-1)^2} = \frac{x^n(n-1) - n x^{n-1} + 1}{(x-1)^2}$$

Employ this formula

$$\frac{1}{m} \cdot \sum_{i=1}^{n-1} i \cdot \left(\frac{m-1}{m}\right)^i \quad ; \quad x = \frac{m-1}{m} \Rightarrow A(n,m) = m \cdot \left[1 - \left(\frac{m-1}{m}\right)^n\right] + \left[\frac{m-1}{m}\right]^{n-1}$$

If m is constant; then $\left(\frac{m-1}{m}\right)^n \to 0$ as $n \to \infty$

$\Downarrow$

number of distinct elements

$\Downarrow$

$A(n,m) \sim m$ constant

$$\lim_{n \to \infty} \left(\frac{m-1}{m}\right)^n = \lim_{n \to \infty} \left(1 - \frac{1}{m}\right)^n$$

$$= \lim_{n \to \infty} c^n \quad ; \quad c: \text{constant}$$

$$A(n, m_{fixed}) \in \Theta(m_{fixed}) = \Theta(1)$$

4) a) $n \times n$ matrix

b) making comparisons between two real numbers

c) Input matrix is symmetrical or not? If symmetrical returns TRUE otherwise FALSE

d) $$\sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1 = \sum_{i=0}^{n-2} [n-1-(i+1)+1] = \sum_{i=0}^{n-2} (n-1-i)$$

$$= n-1 + n-2 + n-3 + \dots + 1 = \frac{(n-1) \cdot n}{2} = \frac{n^2-n}{2}$$

Worst case: $T(n) = \frac{n^2-n}{2}$ $\quad T(n) \in \Theta(n^2)$

Best case: first check return when matrix is not symmetrical

$T(n) \in \Theta(1)$

Average case: When we analyze average case, we compute weighted probabilities. Making 1 comparison to get result equals to probability of first compared real numbers are different. Computation goes on like this for $2, 3, 4, \dots, \frac{n^2-n}{2}$ comparisons and probabilities multiplied by comparisons. But this kind of analyze is not proper for this example. Because there is no limit of elements in this matrix. Probability of any two real numbers' equal to each other theoretically is zero. So average runtime is same as worst case.

$$T(n) = \frac{n^2-n}{2} \in \Theta(n^2)$$

5) a) n (array size)

b) $T(n) = 2T(n/3) + 1 \qquad T(1) = 1$

c) $T(n) = 2T\left(\frac{n}{3}\right) + 1$

$= 2\left(2T\left(\frac{n}{3}\right) + 1\right) + 1$

$= 2\left(2\left(2T\left(\frac{n}{3}\right) + 1\right) + 1\right) + 1$

$\vdots$

$= 2^i\, T\left(\frac{n}{3^i}\right) + 2^i + 2^{i-1} + \dots + 2 + 1$

For $i = \log_3 n$

$T(n) = 2^{\log_3 n} \cdot T\left(\frac{n}{3^{\log_3 n}}\right) + \sum_{k=0}^{\log_3 n} 2^i$

$= n^{\log_3 2} \cdot T(1) + n^{\log_3 2} - 1$

$= 2 \cdot \left(n^{\log_3 2}\right) - 1$

d) $T(n) \in \Theta\left(n^{\log_3 2}\right)$

6) Run mergesort on A set $\Theta(n \log n)$
Add first $n/2$ elements to A_1 $O(n)$
Add last $n/2$ elements to A_2 $O(n)$

Runtime of algorithm is $\Theta(n \log n)$

7)
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Function NumberOf Zeros( A[1, ..., n]) {
// input : an array of integers
// output : an integer
if (n == 1)
{    if ( A[0] == 0)
          return 1;
     else
          return 0;
}
else
     return NumberOfZeros(A[1,..., n/2]) + NumberOfZeros(A[n/2, ..., n]);
}
```

$T(n) = 2T\left(\frac{n}{2}\right) + 1$ solve using master theorem ....

$T(n) \in \Theta(n)$

8) a) $T(n) = -4T(n-1) - 4T(n-2)$

$$= (-4)^2 (T(n-2) + 2T(n-3) + T(n-4))$$

$$= (-4)^3 (T(n-3) + 3T(n-4) + 3T(n-5) + T(n-6))$$

$$\vdots$$

$$= (-4)^i \left(\binom{i}{0} T(n-i) + \binom{i}{1} T(n-i-1) + \dots + \binom{i}{i} T(-2i) \right)$$

b) $T(n) = T(n-1) + 6T(n-2)$ $\quad T(0)=3 \quad T(1)=6$

$r^2 - r - 6 = 0$ $\qquad A \cdot 3^n + B \cdot (-2)^n$

$(r-3)(r+2)=0$ $\qquad T(0)=3 \quad A + B = 3$

$r=3 \quad r=-2$ $\qquad T(1)=6 \quad 3A - 2B = 6$

$A = \frac{12}{5} \quad B = \frac{3}{5}$

$T(n) = \frac{12}{5} 3^n + \frac{3}{5} (-2)^n \qquad T(n) \in \Theta(3^n)$

c) $r^2 + 5r + 6 = 0$ $\qquad A \cdot (-3)^n + B(-2)^n \Rightarrow$ Sol. of hom. part

$(r+3)(r+2)$ $\qquad$ Non hom. part $\quad c \cdot 4^n = -5c\,4^{n-1} - 6c\,4^{n-2} + 42 \cdot 4^n$

$r=-3 \quad r=-2$

$16c\,4^{n-2} = -20c\,4^{n-2} - 6c\,4^{n-2} + 16 \cdot 42 \cdot 4^{n-2}$

$42c = 16 \cdot 42$

$c = 16$

$A(-3)^n + B(-2)^n + 16 \cdot 4^n$

$T(1) \Rightarrow -3A - 2B + 64 = 56$

$T(2) \Rightarrow 9A + 4B + 256 = 278$

$A = 2 \quad B = 1$

$T(n) = 2(-3)^n + (-2)^n + 16 \cdot 4^n$

9) a) $T(n) = T(n-1) + (n^2+1) \qquad T(0) = 3$

$T(n) = T(n-2) + (n-1)^2 + 1 + n^2 + 1$

$T(n) = T(n-3) + (n-2)^2 + (n-1)^2 + 1 + n^2 + 1 + 1$

$\vdots$

$T(n) = T(n-k) + \sum_{k=1}^{n} (n-k+1)^2 + \sum_{k=1}^{n} 1$

$n-k=0 \qquad n=k$

$T(n) = T(0) + \sum_{k=1}^{n} (n-k+1)^2 + \sum_{k=1}^{n} 1$

$T(n) = 3 + \frac{n(n+1)(2n+1)}{6} + n = \frac{2n^3+n^2+6n+19}{6}$

b) $T(n) = \sum_{i=1}^{n-1} T(i) + n^2 \qquad T(1) = 1$

$T(n-1) = \sum_{i=1}^{(n-1)-1} T(i) + (n-1)^2$

$T(n) - T(n-1) = T(n-1) + n^2 - (n^2 - 2n + 1)$

$T(n) = 2T(n-1) + 2n - 1$

c) $T(n) = 2T(n-1) - T(n-2) + n \qquad T(0) = T(1) = 0$

$p(r) = r^2 - 2r + 1 = (r-1)^2$

$\wp(r) = (r-1)^2$

$p(r) \cdot \wp(r) = (r-1)^2 (r-1)^2 = (r-1)^4$

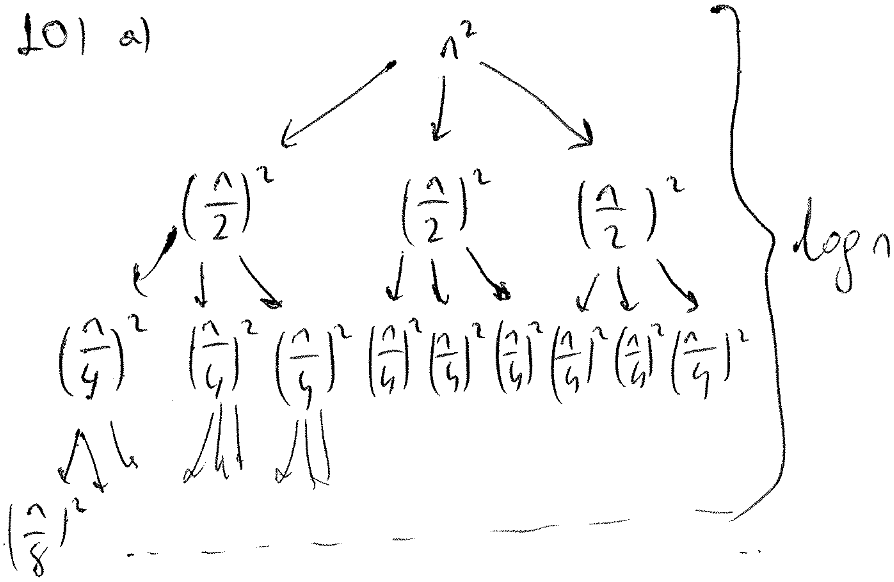
$T(n) = a + bn + cn^2 + dn^3 = (r-1)^4$

$a = 0 \qquad c = 1/2$

$b = -2/3 \qquad d = 1/6$

$T(n) = \frac{n^3+3n^2-4n}{6}$

10) a)



$$n^2 = \left(\frac{3}{4}\right)^0 n^2$$

$$\frac{3}{4} n^2 = \left(\frac{3}{4}\right)^1 n^2$$

$$\frac{9 n^2}{16} = \left(\frac{3}{4}\right)^2 n^2$$

$$\frac{27 n^2}{64} = \left(\frac{3}{4}\right)^3 n^2$$

$$\underbrace{\sum_{i=0}^{\log n - 1} \left(\frac{3}{4}\right)^i \cdot n^2}_{\text{other rows}} + \underbrace{3^{\log n} \cdot f(1)}_{\text{last row (leaves)}}$$

$$= n^2 + \frac{3}{4} n^2 + \frac{9}{16} n^2 + \dots + \frac{3}{4}^{\log n - 1} \cdot n^2 + 4 \cdot 3^{\log n}$$

$$= n^2 \left(1 + \frac{3}{4} + \frac{9}{16} + \dots + \frac{3}{4}^{\log n - 1} \right) + 4 \cdot 3^{\log n}$$

$$= n^2 \left(\frac{1 - \frac{3}{4}^{\log n}}{1 - \frac{3}{4}} \right) + 4 \cdot n^{\log_2 3}$$

$$= 4 n^2 \left(1 - \frac{n^{\log_2 3}}{n^2} \right) + 4 \cdot n^{\log_2 3}$$

$$= 4n^2 - 4 n^{\log_2 3} + 4 n^{\log_2 3} = 4n^2$$

$$f(n) = 4n^2$$

b) $T(n) = 3T(n/2) + n^2 \log n \qquad T(1) = 1$

$a = 3$
$b = 2$
$f(n) = n^2 \log n$

$$n^2 \log n \in \Omega\left(n^{\log_2 3 + \varepsilon}\right), \quad \varepsilon > 0 \quad ✓$$

$$\frac{3}{4} n^2 \log n/2 \leq c \cdot n^2 \log n \quad c = 1, n > 0 \quad ✓$$

$$T(n) \in \Theta(n^2 \log n)$$