Math 3310
Module on Logic

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Math 3310 Handouts and Homework

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Chapter 4

Logic

4.1 Introduction to Propositional Logic

If we want to develop all of mathematics from a few principles of logic (which would be really cool and is something we can do), then we need to develop propositional logic. Propositions can be combined to create more complicated or complex propositions through the use of *connectives*. The following is a list of connectives:

$\neg$: negation or "not"

$\wedge$: conjunction, or "and"

$\vee$: disjunction , or "or"

$\implies$: implication, or "if-then"

$\equiv$: equivalence, "if and only if", or "iff"

$\neg$

For any proposition p the negation is $\neg p$. So if p is true, then $\neg p$ is false. It is helpful to illustrate this in a *truth table*.

p	$\neg p$
T	**F**
F	**T**

The first row says that if p is true (**T**), then $\neg p$ is false (**F**). The revese hold for the second line. We will construct truth tables for all of the connectives.

$\bigwedge$

Practice reading through, line by line, to make sure the truth table makes sense. In this case, we have two propositions, p and q.

p	q	$p \wedge q$
T		
T		
F		
F		

Reading through this table we see that $p \wedge q$ is true only in the first case, when both are true. Otherwise the statement is logically false.

$\bigvee$

Again we have two propositions, p and q.

p	q	$p \vee q$
T	**T**	
T	**F**	
F	**T**	
F	**F**	

In the table we have $p \vee q$ yields false only when both propositions are false, otherwise it is true.

$\implies$

p	q	$p \implies q$
T	**T**	
T	**F**	
F	**T**	
F	**F**	

In the table we have $p \vee q$ yields false only when both propositions are false, otherwise it is true. Here the only case where the statement is false is when p is true and q is false.

$\equiv$

For equivalence, each proposition, p and q, implies the other, so in a sense, they must agree. If they don't agree, then the proposition $\equiv$ is false

p	q	$p \equiv q$
T	**T**	
T	**F**	
F	**T**	
F	**F**	

We can build formulations of these propositions with connectives. Commonly we employ parenthesis to group notions together, just as with algebra. We will refer to p, q, and r as *propositional variables*, and we will refer to *formulas* as any expression constructed according to the following rules:

1. Each propositional variable is a formula.
2. Given any formulas, X and Y (already constructed), the expressions $\neg X, (X \wedge Y), (X \vee Y), (X \implies Y)$, and $(X \equiv Y)$ are also formulas.

I know it seems geeky, but formally defining things is a big part of mathematics. But essentially I think I have listed what you already intuitively understood a propositional logic formula to mean. ☺

Compound Truth Tables

Now it starts to get interresting. We can combine propositional expressions composed of p and q and determine the resulting truth value of the formulation.

Example 4.1.1 ($p \equiv (q \vee \neg(p \wedge q))$):
Once we know the values of p and q, we can determine $p \equiv (q \vee \neg(p \wedge q))$. Again we will turn to the associated truth table, building it step by step as we assemble the formula.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$p \vee \neg(p \wedge q)$	$p \equiv p \vee \neg(p \wedge q)$
T	**T**				
T	**F**				
F	**T**				
F	**F**				

So X is true in the first two cases, and false in the last two cases.

Example 4.1.2 ($(p \vee \neg q) \implies (r \equiv (p \wedge q))$):
Once we know the values of p, q and r, we can determine $p \equiv (q \vee \neg(p \wedge q))$. Again we will turn to the associated truth table, building it step by step as we assemble the formula. With three variables we have $2^3 = 8$ possible values for X.

p	q	r	$\neg q$	$p \vee \neg q$	$p \wedge q$	$r \equiv (p \wedge q)$	$(p \vee \neg q) \implies (r \equiv (p \wedge q))$
T	**T**	**T**					
T	**T**	**F**					
T	**F**	**T**					
T	**F**	**F**					
F	**T**	**T**					
F	**T**	**F**					
F	**F**	**T**					
F	**F**	**F**					

Tautology

Tautologies are essentially statements that are always true, no matter what. Consider the following example.

Example 4.1.3 ($(p \wedge (q \vee r)) \implies ((p \wedge q) \vee (p \wedge r))$):
Let's build the truth table

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$	$(p \wedge (q \vee r)) \implies ((p \wedge q) \vee (p \wedge r))$
T								
T								
T								
T								
F								
F								
F								
F								

Here the last column is true for all possible values of the variables. This is an example of a tautology.

Definition 4.1.4 (Tautology):
A formula is a tautology if it is always true.

Definition 4.1.5 (Contradiction):
A formula contradictoy if it is always false.

Definition 4.1.6 (Contingent):
A formula contingent if it is neither a tautology or a contradiction.

Problem 4.1.1 (Which of the following are tautologies)
Which of the following are tautologies, contradictions, and contingencies.

A. $(p \implies q) \implies (q \implies p)$

Pay attention to the T/F values of each proposition and just follow the logic table,...

p	q	$(p \implies q)$	$(q \implies p)$	$(p \implies q) \implies (q \implies p)$
T	**T**			
T	**F**			
F	**T**			
F	**F**			

B. $(p \implies q) \implies (\neg p \implies \neg q)$

p	q	$(p \implies q)$	$\neg p$	$\neg q$	$(\neg p \implies \neg q)$	$(p \implies q) \implies (\neg p \implies \neg q)$
T	**T**					
T	**F**					
F	**T**					
F	**F**					

C. $(p \implies q) \implies (\neg q \implies \neg p)$

p	q	$(p \implies q)$	$\neg p$	$\neg q$	$(\neg q \implies \neg p)$	$(p \implies q) \implies (\neg p \implies \neg q)$
T	**T**					
T	**F**					
F	**T**					
F	**F**					

D. $p \implies \neg p$

E. $p \equiv \neg p$

F. $(p \equiv q) \equiv (\neg p \equiv \neg q)$

G. $\neg(p \wedge q) \equiv (\neg p \wedge \neg q)$

H. $\neg(p \wedge q) \equiv (\neg p \vee \neg q)$

I. $(\neg p \vee \neg q) \equiv \neg(p \vee q)$

J. $\neg(p \vee q) \equiv (\neg p \wedge \neg q)$

K. $(p \equiv (p \wedge q)) \equiv (q \equiv (p \vee q))$

Logial Implication and Equivalence

We say that X logically implies Y if Y is true in all the cases when X true, (i.e. $X \implies Y$ is a tautology!) Moreover, if we have a set of formulas and some formula X, we say that X is logically implied by S when X is true in all cases which all elements of S are true. Finally, two formulas are logically *equivalent* if they are true in the same cases, and false in the same cases (i.e. $X \equiv Y$ is a tautology!)

Working Backwards

Suppose we have a table of values, but we don't know what the formula is. We can actually determine the formula from the truth table. Consdier the following example.

Problem 4.1.2 (Truth Tables)

p	q	r	?
T	**T**	**T**	**T**
T	**T**	**F**	**F**
T	**F**	**T**	**T**
T	**F**	**F**	**F**
F	**T**	**T**	**T**
F	**T**	**F**	**F**
F	**F**	**T**	**F**
F	**F**	**F**	**F**

If we look at the situations when the formula is true, rows one, three, and five, we can assemble a formula that is logically equivalent. Look row one. It says when p, q, and r are true, then the formula is true. In other words, $p \wedge q \wedge r$ must hold. Using this idea, we can build the rest of the formula for the other cases, and then put everything together.

Problem 4.1.3 (Another Truth Table)

p	q	?
T	**T**	**F**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

Use the preceding technique to arrive at the expression.

- Begin by considering all of the cases resulting in truth.

- Now build a expression that includes them all.

- Verify that you have the correct answer by constructing a truth table and filling it in. The resulting table of values must match the given table with the mysterious expression.

- Realize that this is not the only way of expressing this statement. In fact, construct a truth table for the following expression: Build the table for $\neg p \wedge (p \implies q)$

p	q	$\neg p$	$(p \implies q)$	$\neg p \wedge (p \implies q)$
T	**T**			
T	**F**			
F	**T**			
F	**F**			

- We (should) conclude that these two expressions are logically equivalent.

4.2 Interdependence of the Logical Connectives

Something that is curious about the logical connectives, is that they can be expressed in terms of each other. By this I mean that we can use $\to$ and $\neg$ to define $\wedge$. This will give us some greater comfort with these connectives, so together, let's start by working through some derivations of this sort.

Example 4.2.1:
Suppose you are stuck on a dererted island, and suddenly an alien appears. You wish to establish a way to communicate logic with this alien. The alien shows you and $\wedge$ and negation $\neg$. The alien does this my waving its tenticles and piling up sand, then pointing with a stick (I don't know, just predend it fugures out how to show you these two connectives) You need to understand "or" $\vee$ in terms of the alien's $\wedge$ and $\neg$. Mainly because you are sharing dinner and can offer the alien either a coconut or a papaya, but you want something to eat too.
...Anyway, let's defing $\vee$ in terms of $\wedge$ and $\neg$.

Problem 4.2.1 (Define $\vee$ in terms of $\neg$ and $\wedge$)
To say that $(p \vee q)$ is true, it is equivalent to saying that they are both not simultaneously false. That should help you construct the formula that is logically equivalent to $(p \vee q)$
Let's do this one together,...

Proof. For $p \vee q$ to be true, then one of them must be true. In other words, BOTH are not false $(\neg p \wedge \neg q)$. Now to say that this is not the case, simply negate $(\neg(\neg p \wedge \neg q))$. Therefore

$$p \vee q \equiv \neg(\neg p \wedge \neg q)$$

□

Now let's keep building the logical connectives.

Problem 4.2.2 (Define $\to$ in terms of $\neg$ and $\vee$)
Say $(p \to q)$ is true using $\neg$ and $\vee$.

Problem 4.2.3 (Define $\to$ in terms of $\neg$ and $\wedge$)
Define $\to$ in terms of $\neg$ and $\wedge$

Problem 4.2.4 (Define $\to$ in terms of $\neg$ and $\vee$)
Define $\to$ in terms of $\neg$ and $\vee$

Problem 4.2.5 (Define $\wedge$ in terms of $\neg$ and $\rightarrow$)
Define $\wedge$ in terms of $\neg$ and $\rightarrow$

Problem 4.2.6 (Define $\vee$ in terms of $\neg$ and $\rightarrow$)
Define $\vee$ in terms of $\neg$ and $\rightarrow$

Problem 4.2.7 (Define $\vee$ in terms of only $\rightarrow$)
Define $\vee$ in terms of only $\rightarrow$

Problem 4.2.8 (Define $\equiv$ in terms of $\wedge$ and $\rightarrow$)
Define $\equiv$ in terms of $\wedge$ and $\rightarrow$

Problem 4.2.9 (Define $\equiv$ in terms of $\neg$, $\wedge$, and $\vee$)
Define $\equiv$ in terms of $\neg$, $\wedge$, and $\vee$

Joint Denial

We have seen that the connectives can be expressed in terms of each other, and in fact all of them can be generated from just two, $\neg$ and $\wedge$. We could say that $\neg$ and $\wedge$ form a *basis* for the connectives. (in fact, I am going to say that, right now!) Even more interesting is that there is a connective that, all by itself, forms all the other connectives. The *joint denial* $\downarrow$ will create all five connectives ($\wedge, \vee, \neg, \rightarrow$, and $\equiv$). Joint denial means "p and q are BOTH false". Here is the table.

p	q	$p \downarrow q$
T	**T**	**F**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

Problem 4.2.10 (Derive all five connectives from $\downarrow$ (hint: start with $\neg$))
DERIVE ALL FIVE CONNECTIVES FROM $\downarrow$ (HINT: START WITH $\neg$)

Alternative Denial

There is a second connective that can also generate all five connectives, the *alternative denial*. The alternative denial say that at least p or q is false, $p|q$.

p	q	$p\|q$
T	**T**	**F**
T	**F**	**T**
F	**T**	**T**
F	**F**	**T**

Problem 4.2.11 (Derive all five connectives from $|$)
DERIVE ALL FIVE CONNECTIVES FROM $|$.

The Whole Enchelada

Stop and think about the truth tables we have been making. For propositions p and q, each connective has a unique truth table, otherwise they would be logically equivalent. In other words $\vee$ not logically equivalent to $\wedge$, and so on. So how many possible connectives can there be for p and q? This is a combinatorics (counting) quesiton.

Problem 4.2.12 (How many connectives could there possibly be for p and q?)
We might then ask ourselves, how many connectives could there possibly be for p and q? Thinking binary might help. Often zero and one are used for true and false. For example,

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

The quesiton is then, how many four digit strings of 0 and 1 are there? We need a "symbol" for every possible table entry, and every entry will have a unique combination of true and false. This is a table we can now fill in completely.

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
p	q	$p \wedge q$	$p \vee q$	$p \to q$	$p \equiv q$	$p \downarrow q$	$p \mid q$	$p \not\to q$	$p \not\equiv q$	$q \leftarrow p$	$q \not\leftarrow p$	p	q	$\neg p$	$\neg q$	t	f
1	1																
1	0																
0	1																
0	0																

One way of phrasing the final six is:

11. p, regardless of q
12. q, regardless of p
13. not p, regardless of q
14. not q, regardless of p
15. true regardless of p and q
16. false regardless of p and q

4.3 Introduction to Propositional Tableaux

For the following two topics, **Propositional Tableaux** and **First Order Logic**, I would like us to explore examples to gain an understanding of the potential of these logic methods. We will focus even less on theorems. But these ideas are so powerful and provide such a nice way of translating logic questions into a fomat that a computer can solve that it would be a shame to come this far in the topic of set theory and logic without offering a glimps into how these topics are leading the way in **Formal Systems** and systems thinking.
Let's recap before we proceed. There are eight facts that hold for any formula X and Y (for example, X could be $p \wedge q \vee (\neg p)$):

1. If $\neg X$ is true, then X is false.
2. If $\neg X$ is false, then X is true.
3. If $X \wedge Y$ is true, then X and Y are both true.
4. If $X \wedge Y$ is false, then either X is false or Y is false.
5. If $X \vee Y$ is true, then either X is true or Y is true.
6. If $X \vee Y$ is false, then both X and Y are false.
7. If $X \rightarrow Y$ is true, then either X is false or Y is true.
8. If $X \rightarrow Y$ is false, then X is true and Y is false.

These facts are what we need to build the *tableaux* method. There is one other thing to add. We will use T and F to indicate true and false, respectively. In such cases as TX we will read as, "X is true." Conversely we read FX as, "X is false". But keep in mind we are talking about statements, so TX could be true or false, i.e. if X is false, then TX is false, and if X is true, then TX is true. Keep in mind that if X is false then FX is a true statement.
Let's work through an example.

Example 4.3.1 (Prove $p \vee (q \wedge r) \rightarrow ((p \vee q) \wedge (p \vee r))$):
In this example, we already have the table contructed for us. We will walk through line by line to see what this table is telling us.

$p \vee (q \wedge r) \rightarrow ((p \vee q) \wedge (p \vee r))$

1. $F(p \vee (q \wedge r)) \rightarrow (((p \vee q) \wedge (p \vee r)))$
2. $Tp \vee (q \wedge r)$
3. $F(p \vee q) \wedge (p \vee r)$
4. Tp $\quad$ $Tq \wedge r$
5. $Fp \vee q$ $\quad$ $Fp \vee r$
6. Fp $\quad$ Fp
7. Fq $\quad$ Fr
8. $\times$ $\quad$ $\times$ $\quad$ Tq
9. Tr
10. $Fp \vee q$ $\quad$ $Fp \vee r$
11. Fp $\quad$ Fp
12. Fq $\quad$ Fr

$\times$ $\quad$ $\times$

Let's look more closely at how we construct a tableau. We will consider each case of logical connective, $\wedge, \vee, \rightarrow$ and $\neg$. For each of these there are two cases, T and F

Example 4.3.2 ($T\ \neg X$)**:**
In the tableau this takes the form: $\dfrac{T\ \neg X}{F\ X}$

Example 4.3.3 ($F\ \neg X$)**:**
In the tableau this takes the form: $\dfrac{F\ \neg X}{T\ X}$

Example 4.3.4 ($T\ X \wedge Y$)**:**
In the tableau this takes the form: $\dfrac{T\ X \wedge Y}{T\ X}$ and $\dfrac{T\ X \wedge Y}{T\ Y}$

Example 4.3.5 ($F\ X \vee Y$)**:**
In the tableau this takes the form: $\dfrac{F\ X \vee Y}{F\ X}$ and $\dfrac{F\ X \vee Y}{F\ Y}$

Example 4.3.6 ($F\ X \rightarrow Y$)**:**
In the tableau this takes the form: $\dfrac{F\ X \rightarrow Y}{T\ X}$ and $\dfrac{F\ X \rightarrow Y}{F\ Y}$

Example 4.3.7 ($T\ X \vee Y$):
In the tableau this takes the form:

$$\begin{array}{lccc} 1. & & T\ X \vee Y & \\ & \swarrow & & \searrow \\ 2. & T\ X & & T\ Y \end{array}$$

Example 4.3.8 ($T\ X \to Y$):
In the tableau this takes the form:

$$\begin{array}{lccc} 1. & & T\ X \to Y & \\ & \swarrow & & \searrow \\ 2. & F\ X & & T\ Y \end{array}$$

Example 4.3.9 ($F\ X \wedge Y$):
In the tableau this takes the form:

$$\begin{array}{lccc} 1. & & F\ X \wedge Y & \\ & \swarrow & & \searrow \\ 2. & F\ X & & F\ Y \end{array}$$

Problem 4.3.1 (Prove the following formulas by the tableaux method.)
Prove the following formulas by the tableaux method.

1. $q \to (p \to q)$
2. $[(p \to q) \wedge (q \to r)] \to (p \to r)$
3. $[p \to (q \to r)] \to [(p \to q) \to (q \to r)]$
4. $[(p \to r) \wedge (q \to r)] \to [(p \vee q) \to r]$
5. $[(p \to q) \wedge (p \to r)] \to [p \to (q \wedge r)]$
6. $\neg(p \vee q) \to (\neg p \wedge \neg q)$

7. $(\neg p \wedge \neg q) \rightarrow \neg(p \vee q)$

8. $[p \wedge (q \vee r)] \rightarrow [(p \wedge q) \vee (p \wedge r)]$

9. $[(p \rightarrow q) \wedge (p \rightarrow \neg q)] \rightarrow p$

4.4 Introduction to First Order Logic

Now we connect propositional logic to the notion of *all* and *some*. In day-to-day speaking, the idea of "some" does not always imply a plural, but rather means "at least one, not two or more." It means "one or more" in logic and mathematics.

Example 4.4.1 (Some people are good.):
This statement only means that there is at least one good person (Noah, for example) and everyone else is trash.

As for the notion of "all" also has a bit of a twist that comes from the idea of *vacuously true*. Vacuous truth is sort of irritating but useful. Actually, it is more a technicality than anything else, but it helps us pin down some sticky problems. The best way to explore this is with an example.

Example 4.4.2 (All Hobbits have five legs.):
This statement is true since there are no Hobbits (sorry ☹), and therefore all statements about Hobbits are true.

Now let's get straight to playing around with these ideas.

Problem 4.4.1 (Fantasy Island.)
We are visiting another island. In speaking with *all* the people on this island each one says the same thing, "Some of use are Knights and some of us are Knaves." Who lives on the island?

Problem 4.4.2 (Back to Fantasy Island.)
We are visiting an island where people are all knights or knaves. In speaking with the people on this island each one says the same thing, "All of is here are of the same type." Who lives on the island?

Problem 4.4.3 (On to another Fantasy Island.)
We are visiting another island where people are all knights or knaves. In speaking with the people on this island each one says the same thing, "Some of is here are knights and some of us are knaves." Who lives on the island?

Problem 4.4.4 (More Fantasy Island.)
At this island. This time we are wondering if being a vegan has something to do with begin a knight or knave. All the people say, "Every knight here is a vegan." What can we conclude about the type of people on the island (knights or knaves) and can we conclude anything about the vegan issue?

Problem 4.4.5 (Even More Fantasy Island.)
..Another island. Now each person says, "Some of us here are knaves and vegan." What can we conclude?

Problem 4.4.6 (Still Even More Fantasy Island.)
Now all the people here are the same, and all the people say, "If I am a vegan, then everyone is a vegan." What can we conclude?

Problem 4.4.7 (OMG Still Even More Fantasy Island.)
Again, all the people are the same type, and each person says, "Some of us are vegans, but I'm not." What can we conclude?

Problem 4.4.8 (Finally, the last Fantasy Island.)
Just like the last problem, all the people are the same type, and each person says something just slightly different than on the last island, "Some of us are vegans. I'm not." What can we conclude?

4.4.1 Knights and Knaves

Now for some fun and games. A popular logic puzzle format is structured aound determining the nature of a person who makes a statement as a truth telling knight or a consistent lying knave. In the following problems, we assume that the persons involved are all knights or knaves. Our job is to determine which.

Problem 4.4.9 (You encounter Adalbert and Phelony eating peeled grapes from a bowl made of unobtainium.)
You encounter Adalbert and Phelony eating peeled grapes from a bowl made of unobtainium. Adalbert says to you, "Both of us are knaves." Which is Adalbert and which is Phelony, a knight or a knave?

- Suppose Adalbert says, "At least one of us in a knave." Now what can we conclude?
- Phelony says, "Adalbert and I are the same type, both knights or both knaves". Now what?

Consider constructing truth tables! Let Adalbert be p and Phelony be q. Perhaps you can let knights be **T**, for truth, and knaves by **F**. Once we have a truth table, the truth will be revealed.

4.4.2 A Paper on Knights and Knaves

Now that we see how the logic tables can be used to solve these riddles, the following paper takes things a step further by introducing more personality types. [1]

4.5 Introducing ∀ and ∃

Let us begin by recognizing x, y, and z to represent arbitrary object in some "thing" (say a *domain*). This domain is determined by the context of the question we are dealing with, i.e it should be obvious as we have discussed before.
Given a property P and an object x, we say that x has property P by noting Px.
If we want to say that every object in the domain has property P we use the *universal quantifyer* ∀,

$$\forall x \ Px$$

and read this to be, "For all x, x has property P." or "For all x, Px"
Now suppose we want to say that some x posesses property P (could be more than one, or simply one x). For this we use the *existential quantifyer*. This is expressed

$$\exists x \ Px$$

and we read this, "There exists an x having property P."
Combine these new quantifiers with the logical connectives.

Example 4.5.1 (The Good Guys and Bad Guys):
Let G be the property of being good. Then Gx means x is a good guy. Now $\forall x \ Gx$ says that everyone is good, and $\exists x \ Gx$ means that there is a good person, at least one good person (Noah).
The question is now, how do we say that no one is good? Perhaps $\neg\exists x \ Gx$ might work. Alternatively, $\forall x \ (\neg Gx)$

Example 4.5.2 (Good People Go To Heaven):
Let H be the property of going to heaven. How do we say, "All good people go to heaven."? We can equivalently say, "There is a person who is good, and they go to heaven." Write that using propositional logic.

Example 4.5.3 (Some Good People Go To Heaven):
Let H be the property ofgoing to heaven. How do we say, "Some good people go to heaven."? We can equivalently say, "For some good persons, if that person is good, then they go to heaven." Write that using propositional logic.

Example 4.5.4 (God Helps Those Who Help Themselves):
Curiously, this statement contains some ambiguity. Does God help all people who help themselves, or does God help only people who help themselves? Or even, that God helps all and only those people who help themselves?

Problem 4.5.1 (Sherlock Holmes and Dr. Moriarty)
Let h stand for Sherlock Holmes and m for Dr. Moriarty. Let's denote the relation "x can catch y" as xCy. Now interpret the following statements in symbolic form.

(a) Sherlock can catch anyone who can catch Moriarty.

(b) Sherlock can catch anyone whom Moriarty can catch.

(c) Sherlock can catch anyone who can be caught by Moriarty.

(d) If anyone can catch Moriarty, then Sherlock can.

(e) If everyone can catch Moriarty, then Sherlock can.

(f) Anyone who can catch Sherlock can catch Moriarty.

(g) No one can catch Sherlock unless they can catch Moriarty.

(h) Everyone can catch someone who cannot catch Moriarty.

(i) Anyone who can catch Holmes can catch anyone whom Holmes can catch.

Problem 4.5.2 (Interpret the following statements in symbolic form.)
Let's use x knows y are the relation xKy. Interpret the following statements in symbolic form.

1. Everyone knows someone.

2. Someone knows everyone.

3. Someone is known by everyone.

4. Every person, x, knows someone who doesn't know x.

5. There is someone, x, who knows everyone who knows x.

Problem 4.5.3 (Express the following)
Let's use x and y to be natural numbers $\{0, 1, 2, \cdots\}$. Use the relation $x < y$ to mean "x is less than y." and $x > y$ to mean, "x is greater than y." Express the following

1. For every number x there is a greater number.

2. Every number other than 0 is greater than some number.

3. 0 is the one and only number having the property that no number is less than it.

4. Without using the identity symbol, $=$, but using $<$ and $>$ express, "x is equal to y." and "x is unequal to y."

4.6 First-Order Logic

We will now define tableaux for *First-Order Logic*. We build on the previous eight talueaux rules of Propositional Logic along with fours rules for the quantifiers. Just as in the previous discussion, let's set up the tableau rules for $\forall$ and $\exists$. There are four of these, and none of them introduce branches to the tableau.

Example 4.6.1 ($T \ \forall \ x\phi(x)$):
In the tableau this takes the form: $\dfrac{T \ \forall \ x\phi(x)}{T \ \phi(a)}$ where a is any parameter

Example 4.6.2 ($T \ \exists \ x\phi(x)$):
In the tableau this takes the form: $\dfrac{T \ \exists \ x\phi(x)}{T \ \phi(a)}$ where a is a parameter *that has not yet shown up in the tableau.*

Example 4.6.3 ($F \ \forall \ x\phi(x)$):
In the tableau this takes the form: $\dfrac{F \ \forall \ x\phi(x)}{F \ \phi(a)}$ where a is a parameter *that has not yet shown up in the tableau.*

Example 4.6.4 ($F \ \exists \ x\phi(x)$):
In the tableau this takes the form: $\dfrac{F \ \exists \ x\phi(x)}{F \ \phi(a)}$ where a is any parameter

Keeping things real, let's start off with an example.

Example 4.6.5 ($\exists\ x\ Px \to \neg\forall\ x\neg(Px \vee Qx)$):
Suppose we want to prove the formula

$$\exists\ x\ Px \to \neg\forall\ x\neg(Px \vee Qx)$$

We begin as before with the negation, i.e assume false. If we can prove that this statement is never false, then,...it is always true.

$$F\ \exists\ x\ Px \to \neg\forall\ x\neg(Px \vee Qx)$$

1. $F\ \exists x\ Px \to \neg\forall\ x\neg(Px \vee Qx)$
2. $T\ \exists x\ Px, just = by(1)$
3. $F\ \forall x\neg(Px \vee Qx)$
4. $T\ \forall x\ \neg(Px \vee Qx)$
5. $T\ Pa$
6. $T\ \neg(Pa \vee Qa)$
7. $F\ Pa \vee Qa$
8. $F\ Pa$

$F\ \exists\ x\ Px \to \neg\forall\ x\neg(Px \vee Qx)$

1.	$F\ \exists x\ Px \to \neg\forall\ x\neg(Px \vee Qx)$	
2.	$T\ \exists x\ Px$	by (1)
3.	$F\ \forall x\neg(Px \vee Qx)$	by (1)
4.	$T\ \forall x\ \neg(Px \vee Qx)$	by (3)
5.	$T\ Pa$	by (2)
6.	$T\ \neg(Pa \vee Qa)$	by (4)
7.	$F\ Pa \vee Qa$	by (6)
8.	$F\ Pa$	by (7)
	$\times$	

And we can close because line (8) contradicts (5).

Problem 4.6.1 **(**$(\forall x\ Px \vee \exists x(Px \to Qx)) \to \exists x\ Qx.$**)**
PROVE

$$(\forall x\ Px \vee \exists x(Px \to Qx)) \to \exists x\ Qx$$

1. ____________________

2. ____________________

3. ____________________

4. ____________________

5. ____________________

6. ____________________

7. ____________________

8. ____________________

9. ____________________

$F\ (\forall x\ Px \vee \exists x(Px \to Qx)) \to \exists x\ Qx$

1.	$F\ (\forall x\ Px \vee \exists x(Px \to Qx)) \to \exists x\ Qx$	
2.	T ________________	BY (1)
3.	F ________________	BY (1)
4.	T ________________	BY (2)
5.	T ________________	BY (2)
6.	T ________________	BY (5)
7.	T ________________	BY (4)
8.	F ________	BY (3)
9.	$F\ Pa \qquad T\ Qa$	BOTH BY(6)

AND WE CAN CLOSE BECAUSE LINE (9) CONTRADICTS (6).

We end our exploration of sets and logic here, but I want to leave you with a list of topics to which we have journeyed precariously close. At this point, if you find these ideas thus far interesting, you might consider reading further into the following topics. I think you have been given the keys that open the door to get to the next level.

1. Gödel's Incompletness Theorem. — Totally famous!!
2. Elementary Arithmetic. — I kid you not! How do we know it works?
3. Formal Systems. — This converts implicit (recursive) definitions into explicit.
4. Peano Arithmetic. — Who said $1 + 1 = 2$?
5. The Unprovability of Consistency. — More from Gödel.

4.6.1 A Paper on What Lincoln Really Meant to Say

I just thought this was soooo nerdy that I had to include it. Sheldon from the *Big Bang* would probably dig this. [2]

Bibliography

[1] Jason Rosenhouse. Knights, knaves, normals, and neutrals. *The College Mathematics Journal*, 45(4):297–306, 2014.

[2] Paul K. Stockmeyer. What did lincoln really mean? *The College Mathematics Journal*, 35(2):103–104, 2004.