Appendix A: Multi-Agent Influence Diagrams

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Abstract

Excerpted appendix from Benthall (2018) doctoral dissertation summarizing Koller and Milch's work on Multi-Agent Influence Diagrams and introducing notation.

1 Multi-Agent Influence Diagrams

Multi-Agent Influence Diagrams (MAIDs) are a game-theoretic extension of Bayesian networks developed by Koller and Milch [1]. A MAID is defined by:

1. A set $\mathcal{A}$ of agents
2. A set $\mathcal{X}$ of chance variables
3. A set $\mathcal{D}_a$ of decision variables for each agent $a \in \mathcal{A}$, with $\mathcal{D} = \bigcup_{a \in \mathcal{A}} \mathcal{D}_a$
4. A set $\mathcal{U}_a$ of utility variables for each agent $a \in \mathcal{A}$, with $\mathcal{U} = \bigcup_{a \in \mathcal{A}} \mathcal{U}_a$
5. A directed acyclic graph $\mathcal{G}$ that defines the parent function Pa over $\mathcal{V} = \mathcal{X} \cup \mathcal{D} \cup U$
6. For each chance variable $X \in \mathcal{X}$, a CPD $Pr(X|Pa(X))$
7. For each utility variable $U \in \mathcal{U}$, a CPD $Pr(U|Pa(U))$

The decision variables represent moments where agents can make decisions about how to act given only the information provided by the variable's parents.

Definition 1 (Decision rules)**.** A *decision rule* δ is a function that maps each instantiation $\mathbf{pa}$ of $Pa(D)$ to a probability distribution over $dom(D)$.

Definition 2 (Strategy)**.** An assignment of decision rules to every decision $D \in \mathcal{D}_a$ for a particular agent $a \in \mathcal{D}_a$ for a particular agent $a \in \mathcal{A}$ is called a *strategy*.

Definition 3 (Strategy profile)**.** An assignment σ of decision rules to every decision $D \in \mathcal{D}$ is called a *strategy profile*. A *partial strategy profile* $\sigma_{\mathcal{E}}$ is an assignment of decision rules to a subset $\mathcal{E} \subset \mathcal{D}$. $\sigma_{-\mathcal{E}}$ refers to a restriction of σ to variables not in $\mathcal{E}$.

Decision rules are of the same form as CPDs, and so a MAID can be transformed into a Bayes network by replacing every decision variable with a random variable with the CPD of the decision rule of a strategy profile.

Definition 4. If $\mathcal{M}$ is a MAID and σ is a strategy profile for $\mathcal{M}$, then the *joint distribution for $\mathcal{M}$ induced by σ*, denoted $P_{\mathcal{M}[\sigma]}$, is the joint distribution over $\mathcal{V}$ defined by the Bayes net where:

- the set of variables is $\mathcal{V}$;
- for $X, Y \in \mathcal{V}$, there is an edge $X \to Y$ if and only if $X \in Pa(Y)$;
- for all $X \in \mathcal{X} \cup \mathcal{U}$, the CPD for X is $Pr(X)$;
- for all $D \in \mathcal{D}$, the CPD for D is $\sigma(D)$.

Definition 5. Let $\mathcal{E}$ be a subset of $\mathcal{D}_a$ and let σ be a strategy profile. We say that $\sigma{*}_{\mathcal{E}}$ is *optimal for the strategy profile* σ if, in the induced MAID $\mathcal{M}[\sigma_{-\mathcal{E}}]$, where the only remaining decisions are those in $\mathcal{E}$, the strategy $\sigma{*}_{\mathcal{E}}$ is optimal, i.e., for all strategies $\sigma'_{\mathcal{E}}$:

$$EU_a((\sigma_{-\mathcal{E}}, \sigma{*}_{\mathcal{E}})) \geq EU_a((\sigma_{\mathcal{E}}, \sigma'_{\mathcal{E}}))$$

A major contribution of Koller and Milch [1] is their analysis of how to efficiently discover Nash Equilibrium strategy profiles for MAIDs. Their method involves analyzing the qualitative graphical structure of the MAID to discover the *strategic reliance* of decision variables. When a decision variable D strategically relies on D', then in principle the choice of the optimal decisionr rule for D depends on the choice of the decision rule for D'.

Definition 6 (Strategic reliance)**.** Let D and D' be decision nodes in a MAID $\mathcal{M}$. *D strategically relies on D'* if there exist two strategy profiles σ and σ' and a decision rule δ for D such that:

- δ is optimal for σ;

- σ' differs from σ only at D';

but no decision rule $\delta*$ that agrees with δ on all parent instantiations $\mathbf{pa} \in dom(Pa(D))$ where $P_{\mathcal{M}[\sigma]}(\mathbf{pa}) > 0$ is optimal for σ'.

Definition 7 (s-reachable)**.** A node D' is *s-reachable* from a node D in a MAID $\mathcal{M}$ if there is some utility node $U \in \mathcal{U}_D$ such that if a new parent $\widehat{D'}$ were added to D', there would be an active path in $\mathcal{M}$ from $\widehat{D'}$ to U given $Pa(D) \cup \{D\}$, where a path is active in a MAID if it is active in the same graph, viewed as a BN.

Theorem 8. If D and D' are two decision nodes in a MAID $\mathcal{M}$ and D' is not s-reachable from D in $\mathcal{M}$, then D does not strategically rely on D'.

1.1 Tactical independence

This dissertation introduces a new concept related to Multi-Agent Influence Diagrams: tactical independence.

Definition 9 (Tactical independence)**.** For decision variables D and D' in MAID $\mathcal{M}$, D and D' are *tactically independent* for conditioning set $\mathcal{C}$ iff for all strategy profiles σ on $\mathcal{M}$, in $P_{\mathcal{M}[\sigma]}$, the joint distribution for $\mathcal{M}$ induced by σ,

$$D \perp\!\!\!\perp D'|C$$

Because tactical independence depends on the independence of variables on an induced probability distribution that is representable by a Bayesian network, the d-separation tests for independence apply readily.

Theorem 10. For decision variables D and D' in MAID $\mathcal{M}$, and for conditioning set $\mathcal{C}$, if D and D' are d-separated given $\mathcal{C}$ on $\mathcal{M}$ considered as a Bayesian network, then D and D' are tactically independent given $\mathcal{C}$.

Proof. Suppose D and D' are d-separated given $\mathcal{C}$ on $\mathcal{M}$ considered as a Bayesian network.

For any strategy profile σ, the joint distribution for $\mathcal{M}$ induced by σ, $P_{\mathcal{M}[\sigma]}$ has the same graphical structure as $\mathcal{M}$ considered as a Bayesian network.

Therefore, D and D' are d-separated given $\mathcal{C}$ in the graph corresponding to $P_{\mathcal{M}[\sigma]}$ for all σ.

Because D and D' are d-separated given $\mathcal{C}$ in the Bayesian network, $D \perp\!\!\!\perp D'|C$. □

1.2 Notation

We will use a slightly different graphical notation than that used by Koller and Milch [1].

In the models in this paper, we will denote random variables with undecorated capital letters, e.g. A, B, C. I will denote strategic nodes with a tilde over a capital letter, e.g. $\tilde{A}, \tilde{B}, \tilde{C}$. The random variable defined by the optimal strategy at a decision node, when such a variable is well-defined, will be denoted with a hat, e.g. $\hat{A}, \hat{B}, \hat{C}$. Nodes that represent the payoff or utility to an agent will be denoted with a breve, e.g. $\breve{A}, \breve{B}, \breve{C}$. Particular agents will be identified by a lower case letter and the assignment of strategic and utility nodes to them will be denoted by subscript. E.g., $\tilde{A}_q$ and $\breve{U}_q$ denote an action taken by agent q and a payoff awarded to q, respectively.

2 Data Games

What distinguishes a data game from a MAID is the use of optional arrows to support mechanism design. A dotted arrow in a data game an optional arrow. The diagram defines two separate models, one including the arrow and one without. When considering an instantiation of the model with the dotted edge present, we will say the model or edge is *open*. When the edge is absent, we'll say it's *closed*.

As we have distinguished between strategic reliance and tactical independence, we can distinguish between the strategic and tactical value of information.

The strategic value of an information flow to an agent is the difference in utility to that agent in the open and closed conditions of the game, given each game is at strategic equilibrium for all players.

Definition 11 (Strategic value of information)**.** Given two MAID diagrams $\mathcal{M}_o$ and $\mathcal{M}_c$ that differ only by a single edge, e, and a strategic profile solution for each diagram, $\hat{\sigma}_o$ and $\hat{\sigma}_c$, the *strategic value of* e *to* a is the difference in expected utility to a under the two respective induced joint distributions:

$$E(P_{\mathcal{M}_o[\hat{\sigma}_o]}(U_a)) - E(P_{\mathcal{M}_c[\hat{\sigma}_c]}(U_a))$$

Definition 11 is an incomplete definition because it leaves open what *solution concept* is used to determine the strategic profile solutions. For the purpose of the results in this paper, we use Nash Equilibrium as the solution concept for determining strategic value of information.

In contrast with the strategic value of information, the tactical value of information is the value of the information to an agent given an otherwise fixed strategy profile. We allow the agent receiving the data to make a tactical adjustment to their strategy at the decision variable at the head of the new information flow.

Definition 12 (Best tactical response to information)**.** Given two MAID diagrams $\mathcal{M}_o$ and $\mathcal{M}_c$ differing only in optional edge e with head in decision variable D_a, the *best tactical response to* e given strategy profile solution $\hat{\sigma}$, $\hat{\delta}_{\sigma,e}$ is the decision rule δ for D such that δ is optimal for $\hat{\sigma}$ for player a.

Definition 13 (Tactical value of information)**.** Given two MAID diagrams $\mathcal{M}_o$ and $\mathcal{M}_c$ differing only in optional edge e with head in decision variable D, the *tactical value of* e to agent a given strategy profile solution $\hat{\sigma}$ is the difference in expected utility of the open condition with the best tactical response to e and the closed condition using the original strategy:

$$EU_a((\hat{\sigma}_{-D}, \hat{\delta}_{\hat{\sigma},e}) - EU_a(\hat{\sigma})$$

Note that the uniqueness of a best tactical response has not yet been proven. However, if the best tactical response is not unique, then the tactical value of the information will be the same for any best tactical response. This definition, like Definition 11, depends on an implicit solution concept.

References

[1] Daphne Koller and Brian Milch. Multi-agent influence diagrams for representing and solving games. *Games and economic behavior*, 45(1):181–221, 2003.