

The Generalization Ability of Online Algorithms for Dependent Data

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Abstract

We study the generalization performance of online learning algorithms trained on samples coming from a **dependent source of data**. We show that the generalization error of any stable online algorithm concentrates around its regret—an easily computable statistic of the online performance of the algorithm—when the underlying ergodic process is β - or ϕ -mixing. **We show high probability error bounds assuming the loss function is convex**, and we also establish sharp convergence rates and deviation bounds for strongly convex losses and several linear prediction problems such as linear and logistic regression, least-squares SVM, and boosting on dependent data. In addition, our results have straightforward applications to stochastic optimization with dependent data, and our analysis requires only martingale convergence arguments; we need not rely on more powerful statistical tools such as empirical process theory.

1 Introduction

Online learning algorithms have the attractive property that regret guarantees—performance of the sequence of points $w(1), \dots, w(n)$ the online algorithm plays measured against a fixed predictor w^* —hold for arbitrary sequences of loss functions, without assuming any statistical regularity of the sequence. It is natural to ask whether one can say something stronger when some probabilistic structure underlies the sequence of examples, or loss functions, presented to the online algorithm. In particular, if the sequence of examples are generated by a stochastic process, can the online learning algorithm output a good predictor for future samples from the same process?

When data is drawn independently and identically distributed from a fixed underlying distribution, Cesa-Bianchi et al. [7] have shown that online learning algorithms can in fact output predictors with good generalization performance. Specifically, they show that for convex loss functions, the average of the n predictors played by the online algorithm has—with high probability—small generalization error on future examples generated i.i.d. from the same distribution. In this paper, we ask the same question when the data is drawn according to a (dependent) ergodic process.

In addition, this paper helps provide justification for the use of regret to a *fixed* comparator w^* as a measure of performance for online learning algorithms. Regret to a fixed predictor is sometimes not a natural metric, which has led several researchers to study online algorithms with performance guarantees for (slowly) changing comparators $w^*(1), w^*(2), \dots$ (see, e.g., Herbster and Warmuth [13, 14]). When data comes i.i.d. from a (unknown) distribution, however, online-to-batch conversions [7] justify computing regret with respect to a fixed w^* . In this paper, we show that

even when data comes from a dependent stochastic process, regret to a fixed comparator is both meaningful and a reasonable evaluation metric.

Though practically, many settings require learning with non-i.i.d. data—examples include time series data from financial problems, meteorological observations, and learning for predictive control—the generalization performance of statistical learning algorithms for non-independent data is perhaps not so well understood as that for the independent scenario. In spite of natural difficulties encountered with dependent data, several researchers have studied the convergence of statistical procedures in non i.i.d. settings [29, 19, 30, 22]. In such scenarios, one generally assumes that the data are drawn from a stationary α -, β -, or ϕ -mixing sequence, which implies that dependence between observations weakens suitably over time. Yu [29] adapts classical empirical process techniques to prove uniform laws of large numbers for dependent data; perhaps a more direct parent to our approach is the work of Mohri and Rostamizadeh [22], who combine algorithmic stability [5] with known concentration inequalities to derive generalization bounds. Steinwart and Christmann [26] show fast rates of convergence for learning from stationary geometrically α -mixing processes, so long as the loss functions satisfy natural localization and self-bounding assumptions. Such assumptions were previously exploited in the machine learning and statistics literature for independent sequences (e.g. [2]), and Steinwart and Christmann extend these results by building off Bernstein-type inequalities for dependent sequences due to Modha and Masry [21].

In this paper, we show that online learning algorithms enjoy guarantees on generalization to unseen data for dependent data sequences from β - and ϕ -mixing sources. In particular, we show that stable online learning algorithms—those that do not change their predictor too aggressively between iterations—also yield predictors with small generalization error. In the most favorable regime of geometric mixing, we demonstrate generalization error on the order of $\mathcal{O}(\log n/\sqrt{n})$ after training on n samples when the loss function is convex and Lipschitz. We also demonstrate faster $\mathcal{O}(\log n/n)$ convergence when the loss function is strongly convex in the hypothesis w , which is the usual case for regularized losses. In addition, we consider linear prediction settings, and show $\mathcal{O}(\log n/n)$ convergence a loss that is strongly convex in its scalar argument (though not in the predictor w) is applied to a linear predictor $\langle w, \cdot \rangle$, which gives fast rates for least squares SVMs, least squares regression, logistic regression, and boosting over bounded sets. We also provide an example and associated learning algorithm for which the expected regret goes to $-\infty$, while any fixed predictor has expected loss zero; this shows that low regret alone is not sufficient to guarantee small expected error when data samples are dependent.

In demonstrating generalization guarantees for online learning algorithms with dependent data, we answer an open problem posed by Cesa-Bianchi et al. [7] on whether online algorithms give good performance on unseen data when said data is drawn from a mixing stationary process. Our results also answer a question posed by Xiao [28] regarding the convergence of the regularized dual averaging algorithm with dependent stochastic gradients. More broadly, our results establish that any suitably stable optimization or online learning algorithm converges in stochastic approximation settings when the noise sequence is mixing. There is a rich history of classical work in this area (see e.g. the book [18] and references therein), but most results for dependent data are asymptotic, and to our knowledge there is a paucity of finite sample and high probability convergence guarantees. The guarantees we provide have applications to, for example, learning from Markov chains, autoregressive processes, or learning complex statistical models for which inference is expensive [27].

Our techniques build off of a recent paper by Duchi et al. [10], where we show high probability bounds on the convergence of the mirror descent algorithm for stochastic optimization even when

the gradients are non-i.i.d. In particular, we build on our earlier martingale techniques, showing concentration inequalities for dependent random variables that are sharper than previously used Bernstein concentration for geometrically α -mixing processes [21, 26] by exploiting recent ideas of Kakade and Tewari [17], though we use weakened versions of ϕ -mixing and β -mixing to prove our high probability results. Further, our proof techniques require only relatively elementary martingale convergence arguments, and we do not require that the input data is stationary but only that it is suitably convergent.

2 Setup, Assumptions, and Notation

We assume that the online algorithm receives n data points x_1, \dots, x_n from a sample space \mathcal{X} , where the data is generated according to a stochastic process P , though the samples x_t are not necessarily i.i.d. or even independent. The online algorithm plays points (hypotheses) $w \in \mathcal{W}$, and at iteration t the algorithm plays the point $w(t)$ and suffers the loss $F(w(t); x_t)$. We assume that the statistical samples x_t have a stationary distribution Π to which they converge (we make this precise shortly), and we measure generalization performance with respect to the expected loss or risk functional

$$f(w) := \mathbb{E}_\Pi[F(w; x)] = \int_{\mathcal{X}} F(w; x) d\Pi(x). \quad (1)$$

Essentially, our goal is to show that after n iterations of any low-regret online algorithm, it is possible to use $w(1), \dots, w(n)$ to output a predictor or hypothesis \hat{w}_n for which $f(\hat{w}_n)$ is guaranteed to be small with respect to any other hypothesis w^* .

Discussion of our statistical assumptions requires a few additional definitions. The total variation distance between distributions P and Q defined on the probability space (S, \mathcal{F}) where \mathcal{F} is a σ -field, each with densities p and q with respect to an underlying measure μ ,¹ is given by

$$d_{\text{TV}}(P, Q) := \sup_{A \in \mathcal{F}} |P(A) - Q(A)| = \frac{1}{2} \int_S |p(s) - q(s)| d\mu(s). \quad (2)$$

Define the σ -field $\mathcal{F}_t = \sigma(x_1, \dots, x_t)$. Let $P_{[s]}^t$ denote the distribution of x_t conditioned on \mathcal{F}_s , that is, given the initial samples x_1, \dots, x_s . Written slightly differently, $P_{[s]}^t = P^t(\cdot \mid \mathcal{F}_s)$ is a version of the conditional probability of x_t given the sigma field $\mathcal{F}_s = \sigma(x_1, \dots, x_s)$. Our main assumption is that the stochastic process is suitably mixing: there is a stationary distribution Π to which the distribution of x_t converges as t grows. We also assume that the distributions $P_{[s]}^t$ and Π are absolutely continuous with respect to an underlying measure μ throughout. We use the following to measure convergence:

Definition 2.1 (Weak β and ϕ -mixing). *The β and ϕ -mixing coefficients of the sampling distribution P are defined, respectively, as*

$$\beta(k) := \sup_{t \in \mathbb{N}} \left\{ 2\mathbb{E}[d_{\text{TV}}(P^{t+k}(\cdot \mid \mathcal{F}_t), \Pi)] \right\} \quad \text{and} \quad \phi(k) := \sup_{t \in \mathbb{N}, B \in \mathcal{F}_t} \left\{ 2d_{\text{TV}}(P^{t+k}(\cdot \mid B), \Pi) \right\}.$$

We say that the process is ϕ -mixing (respectively, β -mixing) if $\phi(k) \rightarrow 0$ ($\beta(k) \rightarrow 0$) as $k \rightarrow \infty$, and we assume without loss that β and ϕ are non-increasing. The above definitions are weaker than

¹This assumption is without loss, since P and Q are each absolutely continuous with respect to the measure $P+Q$.

the standard definitions of mixing [22, 6, 29], which require mixing over the entire future σ -field of the process, that is, $\sigma(x_t, x_{t+1}, x_{t+2}, \dots)$. In contrast, we require mixing over only the single-slice marginal of x_{t+k} . From the definition, we also see that β -mixing is weaker than ϕ -mixing since $\beta(k) \leq \phi(k)$. We state our results in general forms using either the β or ϕ -mixing coefficients of the stochastic process, and we generally use ϕ -mixing results for stronger high-probability guarantees compared to β -mixing. We remark that if the sequence $\{x_t\}$ is i.i.d., then $\phi(1) = \beta(1) = 0$.

Two regimes of β -mixing (and ϕ -mixing) will be of special interest. A process is called geometrically β -mixing (ϕ -mixing) if $\beta(k) \leq \beta_0 \exp(-\beta_1 k^\theta)$ (respectively $\phi(k) \leq \phi_0 \exp(-\phi_1 k^\theta)$) for some $\beta_i, \phi_i, \theta > 0$. Some stochastic processes satisfying geometric mixing include finite-state ergodic Markov chains and a large class of aperiodic, Harris-recurrent Markov processes; see the references [20, 21] for more examples. A process is called algebraically β -mixing (ϕ -mixing) if $\beta(k) \leq \beta_0 k^{-\theta}$ (resp. $\phi(k) \leq \phi_0 k^{-\theta}$) for constants $\beta_0, \phi_0, \theta > 0$. Examples of algebraic mixing arise in certain Metropolis-Hastings samplers when the proposal distribution does not have a lower bounded density [16], some queuing systems, and other unbounded processes.

We now turn to stating the relevant assumptions on the instantaneous loss functions $F(\cdot; x)$ and other quantities relevant to the online learning algorithm. Recall that the algorithm plays points (hypothesis) $w \in \mathcal{W}$. Throughout, we make the following boundedness assumptions on F and the domain \mathcal{W} , which are common in the online learning literature.

Assumption A (Boundedness). *For μ -almost every (henceforth μ -a.e.) x , the function $F(\cdot; x)$ is convex and G -Lipschitz with respect to a norm $\|\cdot\|$ over \mathcal{W} :*

$$|F(w; x) - F(v; x)| \leq G \|w - v\| \quad (3)$$

for all $w, v \in \mathcal{W}$. In addition, \mathcal{W} is compact and has finite radius: for any $w, w^* \in \mathcal{W}$,

$$\|w - w^*\| \leq R. \quad (4)$$

Further, $F(w; x) \in [0, GR]$.

As a consequence of Assumption A f is also G -Lipschitz. Given the first two bounds (3) and (4) of Assumption A, the final condition can be assumed without loss; we make it explicit to avoid centering issues later. In the sequel, we give somewhat stronger results in the presence of the following additional assumption, which lower bounds the curvature of the expected function f :

Assumption B (Strong convexity). *The expected function f is λ -strongly convex with respect to the norm $\|\cdot\|$, that is,*

$$f(v) \geq f(w) + \langle g, v - w \rangle + \frac{\lambda}{2} \|w - v\|^2 \quad \text{for } w, v \in \mathcal{W} \text{ and for all } g \in \partial f(w). \quad (5)$$

Lastly, to prove generalization error bounds for online learning algorithms, we require them to be appropriately stable, as described in the next assumption.

Assumption C. *There is a non-increasing sequence $\kappa(t)$ such that if $w(t)$ and $w(t+1)$ are successive iterates of the online algorithm, then $\|w(t) - w(t+1)\| \leq \kappa(t)$.*

Here $\|\cdot\|$ is the same norm as that used in Assumption A. We observe that this stability assumption is different from the stability condition of Mohri and Rostamizadeh [22] and neither one implies the

other. It is common (or at least straightforward) to establish bounds $\kappa(t)$ as a part of the regret analysis of online algorithms (e.g. [28]), which motivates our assumption here.

What remains to complete our setup is to quantify our assumptions on the performance of the online learning algorithm. We assume access to an online algorithm whose regret is bounded by (the possibly random quantity) \mathfrak{R}_n for the sequence of points $x_1, \dots, x_n \in \mathcal{X}$, that is, the online algorithm produces a sequence of iterates $w(1), \dots, w(n)$ such that for any fixed $w^* \in \mathcal{W}$,

$$\sum_{t=1}^n F(w(t); x_t) - F(w^*, x_t) \leq \mathfrak{R}_n. \quad (6)$$

Our goal is to use the sequence $w(1), \dots, w(n)$ to construct an estimator \hat{w}_n that performs well on unseen data. Since our samples are dependent, we measure the generalization error on future test samples drawn from the same sample path as the training data [22]. That is, we measure performance on the m samples x_{n+1}, \dots, x_{n+m} drawn from the process $P_{[n]}$, and we would like to bound the future risk of \hat{w}_n , defined as

$$\frac{1}{m} \sum_{t=1}^m \mathbb{E}[F(\hat{w}_n; x_{n+t}) - F(w^*; x_{n+t}) \mid \mathcal{F}_n], \quad (7)$$

the conditional expectation of the losses $F(\hat{w}_n; x)$ given the first n samples. Note that in the i.i.d. setting [7], the expectation above is the excess risk $f(\hat{w}_n) - f(w^*)$ of \hat{w}_n against w^* , because x_{n+t} is independent of x_1, \dots, x_n . Of course, we are in the dependent setting, so the generalization measure (7) requires slightly more care.

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3 Generalization bounds for convex functions

Our definitions and assumptions in place, we show in this section that any suitably stable online learning algorithm enjoys a high-probability generalization guarantee for convex loss functions F . The main results of this section are Theorems 2 and 3, which give high probability convergence of any *stable online learning algorithm* under ϕ - and β -mixing, respectively. Following Theorem 2, we also present an example illustrating that low regret is by itself insufficient to guarantee good generalization performance, which is distinct from i.i.d. settings [7].

Before proceeding with our technical development, we describe the high-level structure and intuition underlying our proofs. The technical insight underpinning many of our results is that under our mixing assumptions, the distribution of the random instance $x_{t+\tau}$ is close to the stationary distribution conditioned on \mathcal{F}_t . That is, looking some number of steps τ into the future from a time t is almost as good as obtaining an unbiased sample from the stationary distribution Π . As a result, the loss $F(w(t); x_{t+\tau})$ is a good proxy for $f(w(t))$, since $w(t)$ only depends on x_1, \dots, x_{t-1} . Lemma 1 formalizes this intuition. (Duchi et al. [10] use a similar technique as a building block.) Under our stability condition, we can further demonstrate that $F(w(t); x_{t+\tau})$ is close to $F(w(t + \tau); x_{t+\tau})$, and the behavior of the latter sequence is nearly the same as the sequence $F(w(t); x_t)$ with respect to which the regret \mathfrak{R}_n is measured. We make these ideas formal in Propositions 1 and 2. We then combine our intermediate results (including bounds on the regret \mathfrak{R}_n), applying relevant martingale concentration inequalities, to obtain the main theorems of this and later sections.

Our starting point is the above-mentioned technical lemma that underlies many of our results.

Lemma 1. *Let $w, v \in \mathcal{W}$ be measurable with respect to the σ -field \mathcal{F}_t and Assumption A hold. Then for any $\tau \in \mathbb{N}$,*

$$\mathbb{E}[F(w; x_{t+\tau}) - F(v; x_{t+\tau}) \mid \mathcal{F}_t] \leq f(w) - f(v) + GR\phi(\tau).$$

and

$$\mathbb{E}\left[\left|\mathbb{E}[F(w; x_{t+\tau}) - F(v; x_{t+\tau}) \mid \mathcal{F}_t] - (f(w) - f(v))\right|\right] \leq GR\beta(\tau).$$

Proof We first prove the result for the ϕ -mixing bound. Recalling that $f(w) = \mathbb{E}_\Pi[F(w; x)]$ and the definition of the underlying measure μ and the densities π and p ,

$$\begin{aligned} \mathbb{E}[F(w; x_{t+\tau}) - F(v; x_{t+\tau}) \mid \mathcal{F}_t] &= \mathbb{E}[F(w; x_{t+\tau}) - f(w) + f(v) - F(v; x_{t+\tau}) \mid \mathcal{F}_t] + f(w) - f(v) \\ &= \int_{\mathcal{X}} [F(w; x) - F(v; x)](p_{[t]}^{t+\tau}(x) - \pi(x))d\mu(x) + f(w) - f(v) \\ &\leq \int_{\mathcal{X}} |F(w; x) - F(v; x)| \left| p_{[t]}^{t+\tau}(x) - \pi(x) \right| d\mu(x) + f(w) - f(v) \\ &\leq GR \int_{\mathcal{X}} \left| p_{[t]}^{t+\tau}(x) - \pi(x) \right| d\mu(x) + f(w) - f(v) \\ &= 2GR \cdot d_{\text{TV}}(P_{[t]}^{t+\tau}, \Pi) + f(w) - f(v), \end{aligned}$$

where for the second inequality we used the Lipschitz assumption A and the compactness assumption on \mathcal{W} . Noting that $2d_{\text{TV}}(P_{[t]}^{t+\tau}, \Pi) \leq \phi(\tau)$ by the definition 2.1 completes the proof of the first part.

To see the second inequality using β -mixing coefficients, we begin by noting that as a consequence of the proof of the first inequality,

$$\mathbb{E}[F(w; x_{t+\tau}) - F(v; x_{t+\tau}) \mid \mathcal{F}_t] - (f(w) - f(v)) \leq 2GRd_{\text{TV}}(P_{[t]}^{t+\tau}, \Pi),$$

and the inequality holds with w and v switched:

$$\mathbb{E}[F(v; x_{t+\tau}) - F(w; x_{t+\tau}) \mid \mathcal{F}_t] - (f(v) - f(w)) \leq 2GRd_{\text{TV}}(P_{[t]}^{t+\tau}, \Pi).$$

Combining the two inequalities and taking expectations, we have

$$\mathbb{E}\left[\left|\mathbb{E}[F(w; x_{t+\tau}) - F(v; x_{t+\tau}) \mid \mathcal{F}_t] - (f(w) - f(v))\right|\right] \leq 2GR\mathbb{E}[d_{\text{TV}}(P^{t+\tau}(\cdot \mid \mathcal{F}_t), \Pi)] \leq GR\beta(\tau)$$

by the definition 2.1 of the mixing coefficients. \square

Using Lemma 1, we can give a proposition that relates the risk on the test sequence to the expected error of a predictor w under the stationary distribution. The result shows that for any w measurable with respect to the σ -field \mathcal{F}_n —we use $\hat{w}_n \in \mathcal{F}_n$, the (unspecified as yet) output of the online learning algorithm—we can prove generalization bounds by showing that w has small risk under the stationary distribution Π .

Proposition 1. *Under the Lipschitz assumption A, for any $w \in \mathcal{W}$ measurable with respect to \mathcal{F}_n , any $w^* \in \mathcal{W}$, and any $\tau \in \mathbb{N}$,*

$$\frac{1}{m} \sum_{t=n+1}^{n+m} \mathbb{E}[F(w; x_t) - F(w^*; x_t) \mid \mathcal{F}_n] \leq f(w) - f(w^*) + \phi(\tau)GR + \frac{(\tau-1)GR}{m}$$

and

$$\mathbb{E} \left[\frac{1}{m} \sum_{t=n+1}^{n+m} \mathbb{E}[F(w; x_t) - F(w^*; x_t) \mid \mathcal{F}_n] \right] \leq \mathbb{E}[f(w)] - f(w^*) + \beta(\tau)GR + \frac{(\tau-1)GR}{m}.$$

Proof The proof follows from the definition 2.1 of mixing. The key idea is to give up on the first $\tau - 1$ test samples and use the mixing assumption to control the loss on the remainder. We have

$$\begin{aligned} & \sum_{t=n+1}^{n+m} \mathbb{E}[F(w; x_t) - F(w^*; x_t) \mid \mathcal{F}_n] \\ &= \sum_{t=n+1}^{n+\tau-1} \mathbb{E}[F(w; x_t) - F(w^*; x_t) \mid \mathcal{F}_n] + \sum_{t=n+\tau}^{n+m} \mathbb{E}[F(w; x_t) - F(w^*; x_t) \mid \mathcal{F}_{n-1}] \\ &\leq (\tau-1)GR + \sum_{t=n+\tau}^{n+m} \mathbb{E}[F(w; x_t) - F(w^*; x_t) \mid \mathcal{F}_n] \end{aligned}$$

since by the Lipschitz assumption A and compactness $F(w; x) - F(w^*; x) \leq GR$. Now, we apply Lemma 1 to the summation, which completes the proof. \square

Proposition 1 allows us to focus on controlling the error on the expected function f under the stationary distribution Π , which is a natural convergence guarantee. Indeed, the function f is the risk functional with respect to which convergence is measured in the standard i.i.d. case, and applying Proposition 1 with $\tau = 1$ and $\phi(1) = 0$ (or $\beta(1) = 0$) confirms that the bound is equal to $f(w) - f(w^*)$. We now turn to controlling the error under f , beginning with a result that relates risk performance of the sequence of hypotheses $w(1), \dots, w(n)$ output by the online learning algorithm to the algorithm's regret, a term dependent on the stability of the algorithm, and an additional random term. This proposition is the starting point for the remainder of our results in this section.

Proposition 2. *Let Assumptions A and C hold and let $w(t)$ denote the sequence of outputs of the online algorithm. Then for any $\tau \in \mathbb{N}$,*

$$\begin{aligned} \sum_{t=1}^n f(w(t)) - f(w^*) &\leq \mathfrak{R}_n + G\tau \sum_{t=1}^n \kappa(t) + 2\tau GR \\ &\quad + \sum_{t=1}^n [f(w(t)) - F(w(t); x_{t+\tau}) + F(w^*; x_{t+\tau}) - f(w^*)]. \end{aligned} \quad (8)$$

Proof We begin by expanding the regret of $w(t)$ on sequence f via

$$\begin{aligned} & \sum_{t=1}^n [f(w(t)) - f(w^*)] \\ &= \sum_{t=1}^n [f(w(t)) - F(w(t); x_{t+\tau}) + F(w(t); x_{t+\tau}) - f(w^*)] \\ &= \sum_{t=1}^n [f(w(t)) - F(w(t); x_{t+\tau}) + F(w^*; x_{t+\tau}) - f(w^*) + F(w(t); x_{t+\tau}) - F(w^*; x_{t+\tau})]. \end{aligned} \quad (9)$$

Now we use stability and the regret guarantee (6) to bound the last two terms of the summation (9). To that end, note that

$$\begin{aligned}
& \sum_{t=1}^n [F(w(t); x_{t+\tau}) - F(w^*; x_{t+\tau})] \\
&= \underbrace{\sum_{t=1}^n [F(w(t); x_t) - F(w^*; x_t)]}_{S_1} + \underbrace{\sum_{t=1}^{n-\tau} [F(w(t); x_{t+\tau}) - F(w(t+\tau); x_{t+\tau})]}_{S_2} \\
&+ \underbrace{\sum_{t=n-\tau+1}^n F(w(t); x_{t+\tau}) - \sum_{t=1}^{\tau} F(w(t); x_t) + \sum_{t=1}^{\tau} F(w^*; x_t) - \sum_{t=n+1}^{n+\tau} F(w^*; x_t)}_{S_3}.
\end{aligned}$$

We now bound the three terms in the summation. S_3 is bounded by $2\tau GR$ under the boundedness assumption A, and the regret bound (6) guarantees that $S_1 \leq \mathfrak{R}_n$. Using the stability assumption C, we can bound S_2 by noting

$$F(w(t); x_{t+\tau}) - F(w(t+\tau); x_{t+\tau}) \leq G \|w(t) - w(t+\tau)\| \leq G \sum_{s=0}^{\tau-1} \kappa(t+s) \leq G\tau\kappa(t),$$

where the last step uses the non-increasing property of the coefficients $\kappa(t)$. Substituting the bounds on S_1 , S_2 , and S_3 into Eq. (9) completes the proposition. \square

The remaining development of this section consists of using the key inequality (8) in Proposition 2 to give expected and high-probability convergence guarantees for the online learning algorithm. Throughout, we define the output of the online algorithm to be the averaged predictor

$$\hat{w}_n = \frac{1}{n} \sum_{t=1}^n w(t). \quad (10)$$

We begin with results giving convergence in expectation for stable online algorithms.

Theorem 1. *Under Assumptions A and C, for any $\tau \in \mathbb{N}$ the predictor \hat{w}_n satisfies the guarantee*

$$\mathbb{E}[f(\hat{w}_n)] - f(w^*) \leq \frac{1}{n} \mathbb{E}[\mathfrak{R}_n] + \beta(\tau)GR + \frac{(\tau-1)G}{n} \left(2R + \sum_{t=1}^n \kappa(t) \right),$$

for any $w^* \in \mathcal{W}$.

Proof From the inequality (8) in Proposition 2, what remains is to take the expectation of the random quantities. To that end, we note that $w(t)$ is measurable with respect to \mathcal{F}_{t-1} (since the iterate at time t depends only on first $t-1$ samples) and apply Lemma 1, which gives

$$\mathbb{E}[\mathbb{E}[F(w^*; x_{t+\tau-1}) - F(w(t); x_{t+\tau-1}) \mid \mathcal{F}_{t-1}]] \leq f(w^*) - f(w(t)) + GR\beta(\tau).$$

Adding the difference to the sum (8) with the setting $\tau \mapsto (\tau - 1)$ gives

$$\mathbb{E} \left[\sum_{t=1}^n f(w(t)) - f(w^*) \right] \leq \mathbb{E}[\mathfrak{R}_n] + G(\tau - 1) \sum_{t=1}^n \kappa(t) + 2(\tau - 1)GR + nGR\beta(\tau).$$

Dividing by n and observing that $f(\hat{w}_n) \leq \frac{1}{n} \sum_{t=1}^n f(w(t))$ by Jensen's inequality completes the proof. \square

We observe that setting $\tau = 1$ and $\beta(1) = 0$ recovers an expected version of the results of Cesa-Bianchi et al. [7, Corollary 2] for i.i.d. samples. Theorem 1 combined with Proposition 1 immediately yields the following generalization bound. Our other results can be similarly extended, but we leave such development to the reader.

Corollary 2. *Under Assumptions A and C, for any $\tau \in \mathbb{N}$ the predictor \hat{w}_n satisfies the guarantee*

$$\frac{1}{m} \mathbb{E} \left[\sum_{t=n+1}^{n+m} F(\hat{w}_n; x_t) - F(w^*; x_t) \right] \leq \frac{1}{n} \mathbb{E}[\mathfrak{R}_n] + 2\beta(\tau)GR + (\tau - 1)GR \left(\frac{2}{n} + \frac{1}{m} + \frac{1}{n} \sum_{t=1}^n \kappa(t) \right).$$

It is clear that the stability assumption we make on the online algorithm plays a key role in our results whenever $\tau > 1$, that is, the samples are indeed dependent. It is natural to ask whether this additional term is just an artifact of our analysis, or whether low-regret by itself ensures a small error under the stationary distribution even for dependent data. The next example shows that low regret—by itself—is insufficient for generalization guarantees, so some additional assumption on the online algorithm is necessary to guarantee small error under the stationary distribution.

Example 1 (Low-regret does not imply convergence). In 1-dimension, define the linear loss $F(w; x) = \langle w, x \rangle$, where $x \in \{-1, 1\}$ and the set $\mathcal{W} = [-1, 1]$. Let $p > 0$ and define following dependent sampling process: at each time t , set

$$x_t = \begin{cases} 1 & \text{with probability } p/2 \\ -1 & \text{with probability } p/2 \\ x_{t-1} & \text{with probability } 1 - p. \end{cases}$$

The stationary distribution Π is uniform on $\{-1, 1\}$, so the expected error $\mathbb{E}_{\Pi}[\langle w, x \rangle] = 0$ for any $w \in \mathcal{W}$. However, we can demonstrate an update rule with negative expected regret as follows. Consider the algorithm which sets $w(t) = -x_{t-1}$, implementing a trivial so-called *follow the leader* strategy. With probability $1 - p/2$, the value $\langle w(t), x_t \rangle = -1$, while $\langle w(t), x_t \rangle = 1$ with probability $p/2$. Consequently, the expectation of the cumulative sum $\sum_{t=1}^n F(w(t); x_t)$ is $-(1 - p)n$. Using standard results on the expected deviation of the simple random walk (e.g. [4]), we know that

$$\mathbb{E} \left[\inf_{w \in \mathcal{W}} \sum_{t=1}^n \langle w, x_t \rangle \right] = -\mathbb{E} \left[\sum_{t=1}^n x_t \right] = \Theta(-\sqrt{n}).$$

We are thus guaranteed that the expected regret of the update rule is $-\Omega((1 - p)n)$.

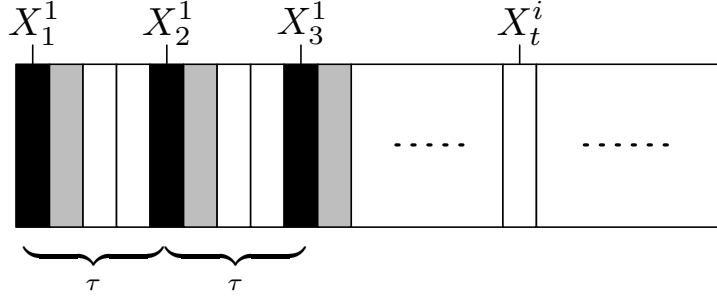


Figure 1. The τ different blocks of near-martingales used in the proof of Theorem 2. Black boxes represent elements in the same index set $\mathcal{I}(1)$, gray in $\mathcal{I}(2)$, and so on.

We have now seen that it is possible to achieve guarantees on the generalization properties of an online learning algorithm by taking expectation over both the training and test samples. We would like to prove stronger results that hold with high probability over the training data, as is possible in i.i.d. settings [7]. The next theorem applies martingale concentration arguments using the Hoeffding-Azuma inequality [1] to give high-probability concentration for the random quantities remaining in Proposition 2's bound.

Theorem 2. *Under Assumptions A and C, with probability at least $1 - \delta$, for any $\tau \in \mathbb{N}$ and any $w^* \in \mathcal{W}$ the predictor \hat{w}_n satisfies the guarantee*

$$f(\hat{w}_n) - f(w^*) \leq \frac{1}{n} \mathfrak{R}_n + \frac{(\tau - 1)G}{n} \sum_{t=1}^n \kappa(t) + 2GR \sqrt{\frac{2\tau}{n} \log \frac{\tau}{\delta}} + \phi(\tau)GR + \frac{2(\tau - 1)GR}{n}.$$

Proof Inspecting the inequality (8) from Proposition 2, we observe that it suffices to bound

$$Z_n := \sum_{t=1}^n [f(w(t)) - f(w^*) - F(w(t); x_{t+\tau-1}) + F(w^*; x_{t+\tau-1})] \quad (11)$$

This is analogous to the term that arises in the i.i.d. case [7], where Z_n is a bounded martingale sequence and hence concentrates around its expectation. Our proof that the sum (11) concentrates is similar to the argument Duchi et al. [10] use to prove concentration for the ergodic mirror descent algorithm. The idea is that though Z_n is not quite a martingale in the general ergodic case, it is in fact a sum of τ near-martingales. This technique of using blocks of random variables in dependent settings has also been used in previous work to directly bound the moment generating function of sums of dependent variables [21], though our approach is different. See Fig. 1 for a graphical representation of our choice (12) of the martingale sequences.

For $i \in \{1, \dots, \tau\}$ and $t \in \{1, \dots, \lceil n/\tau \rceil\}$, define the random variables

$$X_t^i := f(w((t-1)\tau + i)) - f(w^*) + F(w^*; x_{t\tau+i-1}) - F(w((t-1)\tau + i); x_{t\tau+i-1}). \quad (12)$$

In addition, define the associated σ -fields $\mathcal{F}_t^i := \mathcal{F}_{t\tau+i-1} = \sigma(x_1, \dots, x_{t\tau+i-1})$. Then it is clear that X_t^i is measurable with respect to \mathcal{F}_t^i (recall that $w(t)$ is measurable with respect to \mathcal{F}_{t-1}), so the sequence $X_t^i - \mathbb{E}[X_t^i \mid \mathcal{F}_{t-1}^i]$ defines a martingale difference sequence adapted to the filtration \mathcal{F}_t^i , $t = 1, 2, \dots$. Following previous subsampling techniques [21, 10], we define the index set $\mathcal{I}(i)$ to

be the indices $\{1, \dots, \lfloor n/\tau \rfloor + 1\}$ for $i \leq n - \tau \lfloor n/\tau \rfloor$ and $\{1, \dots, \lfloor n/\tau \rfloor\}$ otherwise. Then a bit of algebra shows that

$$Z_n = \sum_{i=1}^{\tau} \sum_{t \in \mathcal{I}(i)} [X_t^i - \mathbb{E}[X_t^i \mid \mathcal{F}_{t-1}^i]] + \sum_{i=1}^{\tau} \sum_{t \in \mathcal{I}(i)} \mathbb{E}[X_t^i \mid \mathcal{F}_{t-1}^i]. \quad (13)$$

The first term in the decomposition (13) is a sum of τ different martingale difference sequences. In addition, the boundedness assumption A guarantees that $|X_t^i - \mathbb{E}[X_t^i \mid \mathcal{F}_{t-1}^i]| \leq 2GR$, so each of the sequences is a bounded difference sequence. The Hoeffding-Azuma inequality [1] then guarantees

$$\mathbb{P} \left[\sum_{t \in \mathcal{I}(i)} [X_t^i - \mathbb{E}[X_t^i \mid \mathcal{F}_{t-1}^i]] \geq \gamma \right] \leq \exp \left(-\frac{\tau \gamma^2}{8nG^2R^2} \right). \quad (14)$$

To control the expectation term from the second sum in the representation (13), we use mixing. Indeed, Lemma 1 immediately implies that $\mathbb{E}[X_t^i \mid \mathcal{F}_{t-1}^i] \leq GR\phi(\tau)$. Combining these bounds with the application (14) of Hoeffding-Azuma inequality, we see by a union bound that

$$\mathbb{P}(Z_n > nGR\phi(\tau) + \gamma) \leq \sum_{i=1}^{\tau} \mathbb{P} \left[\sum_{t \in \mathcal{I}(i)} [X_t^i - \mathbb{E}[X_t^i \mid \mathcal{F}_{t-1}^i]] \geq \gamma/\tau \right] \leq \tau \exp \left(-\frac{\gamma^2}{8\tau nG^2R^2} \right).$$

Equivalently, by setting $\gamma = 2GR\sqrt{2n\tau \log(\tau/\delta)}$, we obtain that with probability at least $1 - \delta$,

$$Z_n \leq GR \left(n\phi(\tau) + 2\sqrt{2n\tau \log \frac{\tau}{\delta}} \right).$$

Dividing by n and using the convexity of f as in the proof of Theorem 1 completes the proof. \square

To better illustrate our results, we now specialize them under concrete mixing assumptions in several corollaries, which should make clearer the rates of convergence of the procedures. We begin with two corollaries giving generalization error bounds for geometrically and algebraically ϕ -mixing processes (defined in Section 2).

Corollary 3. *Under the assumptions of Theorem 2, assume further that $\phi(k) \leq c \exp(-\phi_1 k^\theta)$ for some universal constant c . There exists a finite universal constant C such that with probability at least $1 - \delta$, for any $w^* \in \mathcal{W}$*

$$f(\hat{w}_n) - f(w^*) \leq \frac{1}{n} \mathfrak{R}_n + C \cdot \left[\frac{(\log n)^{1/\theta} G}{n\phi_1^{1/\theta}} \sum_{t=1}^n \kappa(t) + GR \sqrt{\frac{(\log n)^{1/\theta}}{n\phi_1^{1/\theta}} \log \frac{(\log n)^{1/\theta}}{\delta}} \right].$$

The corollary follows from Theorem 2 by taking $\tau = (\log n / (2\phi_1))^{1/\theta}$. When the samples x_t come from a geometrically ϕ -mixing process, Corollary 3 yields a high-probability generalization bound of the same order as that in the i.i.d. setting [7] up to poly-logarithmic factors. Algebraic mixing gives somewhat slower rates:

Corollary 4. *Under the assumptions of Theorem 2, assume further that $\phi(k) \leq \phi_0 k^{-\theta}$. Define $K_n = \sum_{t=1}^n \kappa(t)/R$. There exists a finite universal constant C such that with probability at least $1 - \delta$, for any $w^* \in \mathcal{W}$*

$$f(\hat{w}_n) - f(w^*) \leq \frac{1}{n} \mathfrak{R}_n + C \cdot \left[GR \phi_0^{\frac{1}{1+\theta}} \left(\frac{K_n}{n} \right)^{\frac{\theta}{1+\theta}} + GR \phi_0^{\frac{1}{\theta+1}} \left(K_n n^\theta \right)^{\frac{-1}{2\theta+2}} \sqrt{\frac{1}{\theta+1} \cdot \log \frac{n}{K_n \delta}} \right].$$

The corollary follows by setting $\tau = \phi_0^{1/(\theta+1)} (n/K_n)^{1/(\theta+1)}$. So long as the sum of the stability constants $\sum_{t=1}^n \kappa(t) = o(n)$, the bound in Corollary 4 converges to 0. In addition, we remark that under the same condition on the stability, an argument similar to that for Corollary 7 of Duchi et al. [10] implies $f(\hat{w}_n) - f(w^*) \rightarrow 0$ almost surely whenever $\phi(k) \rightarrow 0$ as $k \rightarrow \infty$.

To obtain concrete generalization error rates from our results, one must know bounds on the stability sequence $\kappa(t)$ (and the regret \mathfrak{R}_n). For many online algorithms, the stability sequence satisfies $\kappa(t) \propto 1/\sqrt{t}$, including online gradient and mirror descent [9]. As a more concrete example, consider Nesterov's dual averaging algorithm [23], which Xiao extends to regularized settings [28]. For convex, G -Lipschitz functions, the dual averaging algorithm satisfies $\mathfrak{R}_n = \mathcal{O}(GR\sqrt{n})$, and with appropriate stepsize choice [28, Lemma 10] proportional to \sqrt{t} , one has $\kappa(t) \leq R/\sqrt{t}$. Noting that $\sum_{t=1}^n t^{-1/2} \leq 2\sqrt{n}$, substituting the stability bound into the result of Theorem 2 immediately yields the following: there exists a universal constant C such that with probability at least $1 - \delta$,

$$f(\hat{w}_n) - f(w^*) \leq \frac{1}{n} \mathfrak{R}_n + C \cdot \inf_{\tau \in \mathbb{N}} \left[\frac{GR(\tau-1)}{\sqrt{n}} + \frac{GR}{\sqrt{n}} \sqrt{\tau \log \frac{\tau}{\delta}} + \phi(\tau)GR \right]. \quad (15)$$

The bound (15) captures the known convergence rates for i.i.d. sequences [7, 28] by taking $\tau = 1$, since $\phi(1) = 0$ in i.i.d. settings. In addition, specializing to the geometric mixing rate of Corollary 3 one obtains a generalization error bound of $\mathcal{O}\left(\left(1 + \frac{1}{\phi_1}\right) \frac{1}{\sqrt{n}}\right)$ to poly-logarithmic factors.

Theorem 2 and the corollaries following require ϕ -mixing of the stochastic sequence x_1, x_2, \dots , which is perhaps an undesirably strong assumption in some situations (for example, when the sample space \mathcal{X} is unbounded). To mitigate this, we now give high-probability convergence results under the weaker assumption that the stochastic process P is β -mixing. These results are (unsurprisingly) weaker than those for ϕ -mixing; nonetheless, there is no significant loss in rates of convergence as long as the process P mixes quickly enough.

Theorem 3. *Under Assumptions A and C, with probability at least $1 - 2\delta$, for any $\tau \in \mathbb{N}$ and for all $w^* \in \mathcal{W}$ the predictor \hat{w}_n satisfies the guarantee*

$$f(\hat{w}_n) - f(w^*) \leq \frac{1}{n} \mathfrak{R}_n + \frac{(\tau-1)G}{n} \sum_{t=1}^n \kappa(t) + 2GR \sqrt{\frac{2\tau}{n} \log \frac{2\tau}{\delta}} + \frac{2\beta(\tau)GR}{\delta} + \frac{2(\tau-1)GR}{n}.$$

Proof Following the proof of Theorem 2, we construct the random variables Z_n and X_t^i as in the definitions (11) and (12). Decomposing Z_n into the two part sum (13), we similarly apply the Hoeffding-Azuma inequality (as in the proof of Theorem 2) to the first term. The treatment of the second piece requires more care.

Observe that for any fixed i, t , the fact that $w((t-1)\tau + i)$ and w^* are measurable with respect to \mathcal{F}_{t-1}^i guarantees via Lemma 1 that

$$\mathbb{E} \left[\left| \mathbb{E} [X_t^i \mid \mathcal{F}_{t-1}^i] \right| \right] \leq GR\beta(\tau).$$

Applying Markov's inequality, we see that with probability at least $1 - \delta$,

$$\sum_{i=1}^{\tau} \sum_{t \in \mathcal{I}(i)} \mathbb{E}[X_t^i | \mathcal{F}_{t-1}^i] \leq \frac{nGR\beta(\tau)}{\delta}.$$

Continuing as in the proof of Theorem 2 yields the result of the theorem. \square

Though the $1/\delta$ factor in Theorem 3 may be large, we now show that things are not so difficult as they seem. Indeed, let us now make the additional assumption that the stochastic process x_1, x_2, \dots is geometrically β -mixing. We have the following corollary.

Corollary 5. *Under the assumptions of Theorem 3, assume further that $\beta(k) \leq \beta_0 \exp(-\beta_1 k^\theta)$. There exists finite universal constant C such that with probability at least $1 - 1/n$ for any $w^* \in \mathcal{W}$*

$$f(\hat{w}_n) - f(w^*) \leq \frac{1}{n} \mathfrak{R}_n + C \cdot \left[\frac{(1.5 \log n)^{1/\theta} G}{n\beta_1^{1/\theta}} \sum_{t=1}^n \kappa(t) + GR \sqrt{\frac{(1.5 \log n)^{1/\theta}}{n\beta_1^{1/\theta}} \log(n(\log n)^{1/\theta})} + \frac{\beta_0 GR}{\sqrt{n}} \right].$$

The corollary follows from Theorem 3 by setting $\tau = (1.5 \log n / \beta_1)^{1/\theta}$ and a few algebraic manipulations. Corollary 5 shows that under geometric β -mixing, we have essentially identical high-probability generalization guarantees as we had for ϕ -mixing (cf. Corollary 3), unless the desired error probability or the mixing constant θ is extremely small. We can make similar arguments for polynomially β -mixing stochastic processes, though the associated weakening of the bound is somewhat more pronounced.

4 Generalization error bounds for strongly convex functions

It is by now well-known that the regret of online learning algorithms scales as $\mathcal{O}(\log n)$ for strongly convex functions, results which are due to work of Hazan et al. [12]. To remind the reader, we recall Assumption B, which states that a function f is λ -strongly convex with respect to the norm $\|\cdot\|$ if for all $g \in \partial f(w)$,

$$f(v) \geq f(w) + \langle g, v - w \rangle + \frac{\lambda}{2} \|w - v\|^2 \quad \text{for } w, v \in \mathcal{W}.$$

For many online algorithms, including online gradient and mirror descent [3, 12, 24, 9] and dual averaging [28, Lemma 11], the iterates satisfy the stability bound $\|w(t) - w(t+1)\| \leq G/(\lambda t)$ when the loss functions $F(\cdot, x)$ are λ -strongly convex. Under these conditions, Corollary 2 gives expected generalization error bound of $\mathcal{O}(\inf_{\tau \in \mathbb{N}} \{\beta(\tau) + \tau \log n/n\})$ as compared to $\mathcal{O}(\inf_{\tau \in \mathbb{N}} \{\beta(\tau) + \sqrt{\tau/n}\})$ for non-strongly convex problems. The improvement in rates, however, does not apply to Theorem 2's high probability results, since the term controlling the fluctuations around the expectation of the martingale we construct scales as $\tilde{\mathcal{O}}(\sqrt{\tau/n})$. That said, when the samples x_t are drawn i.i.d. from the distribution Π , Kakade and Tewari [17] show a generalization error bound of $\mathcal{O}(\log n/n)$ with high probability by using self-bounding properties of an appropriately constructed martingale. In the next theorem, we combine the techniques used to prove our previous results with a self-bounding martingale argument to derive sharper generalization guarantees when the expected function f is strongly convex. Throughout this section, we will focus on error to the minimum of the expected function: $w^* \in \arg \min_{w \in \mathcal{W}} f(w)$.

Theorem 4. *Let Assumptions A, B, and C hold, so the expected function f is λ -strongly convex with respect to the norm $\|\cdot\|$ over \mathcal{W} . Then for any $\delta < 1/e$, $n \geq 3$, with probability at least $1 - 4\delta \log n$, for any $\tau \in \mathbb{N}$ the predictor \hat{w}_n satisfies*

$$f(\hat{w}_n) - f(w^*) \leq \frac{2}{n} \mathfrak{R}_n + \frac{2(\tau-1)G}{n} \left(\sum_{t=1}^n \kappa(t) + 2R \right) + \frac{32G^2\tau}{\lambda n} \log \frac{\tau}{\delta} + \frac{12\tau RG}{n} \log \frac{\tau}{\delta} + 2RG\phi(\tau).$$

Before we prove the theorem, we illustrate its use with a simple corollary. We again use Xiao's extension of Nesterov's dual averaging algorithm [23, 28], where for G -Lipschitz λ -strongly convex losses F it is shown that

$$\|x(t) - x(t+1)\| \leq \kappa(t) \leq \frac{G}{\lambda t}.$$

Consequently, Theorem 4 yields the following corollary, applicable to dual averaging, mirror descent, and online gradient descent:

Corollary 6. *In addition to the conditions of Theorem 4, assume the stability bound $\kappa(t) \leq G/\lambda t$. There is a universal constant C such that with probability at least $1 - \delta \log n$,*

$$f(\hat{w}_n) - f(w^*) \leq \frac{2}{n} \mathfrak{R}_n + C \cdot \inf_{\tau \in \mathbb{N}} \left[\frac{(\tau-1)G^2}{\lambda n} \log n + \frac{\tau G^2}{\lambda n} \log \frac{\tau}{\delta} + \frac{G^2}{\lambda} \phi(\tau) \right].$$

Proof The proof follows by noting the following two facts: first, $\sum_{t=1}^n \kappa(t) \leq (G/\lambda)(1 + \log n)$, and secondly, the definition (5) of strong convexity implies

$$G \|w - v\| \geq f(v) - f(w) \geq \langle \nabla f(w), v - w \rangle + \frac{\lambda}{2} \|v - w\|^2.$$

Recalling [15] that $\|\nabla f(w)\|_* \leq G$, we have $\|w - v\| \leq 4G/\lambda$ for all $w, v \in \mathcal{W}$, so $R \leq 2G/\lambda$. \square

We can further extend Corollary 6 using mixing rate assumptions on ϕ as in Corollaries 3 and 4, though this follows the same lines as those. For a few more concrete examples, we note that online gradient and mirror descent as well as dual averaging [12, 9, 24, 28] all have $\mathfrak{R}_n \leq C \cdot (G^2/\lambda) \log n$ when the loss functions $F(\cdot; x)$ are strongly convex (this is stronger than assuming that the expected function f is strongly convex, but it allows sharp logarithmic bounds on the random quantity \mathfrak{R}_n). In this special case, Corollary 6 implies the generalization bound

$$f(\hat{w}_n) - f(w^*) = \mathcal{O} \left(\frac{G^2}{\lambda} \inf_{\tau \in \mathbb{N}} \left[\tau \frac{\log n}{n} + \phi(\tau) \right] \right)$$

with high probability. For example, online algorithms for SVMs (e.g. [25]) and other regularized problems satisfy a sharp high-probability generalization guarantee, even for non-i.i.d. data.

We now turn to proving Theorem 4, beginning with a martingale concentration inequality.

Lemma 7 (Freedman [11], Kakade and Tewari [17]). *Let X_1, \dots, X_n be a martingale difference sequence adapted to the filtration \mathcal{F}_t with $|X_t| \leq b$. Define $V = \sum_{t=1}^n \mathbb{E}[X_t^2 | \mathcal{F}_{t-1}]$. For any $\delta < 1/e$ and $n \geq 3$*

$$\mathbb{P} \left[\sum_{t=1}^n X_t \geq \max\{2\sqrt{V}, 3b\sqrt{\log 1/\delta}\} \sqrt{\log 1/\delta} \right] \leq 4\delta \log n.$$

Proof of Theorem 4 For the proof of this theorem, we do not start from the Proposition 2, as we did for the previous theorems, but begin directly with an appropriate martingale. Recalling the definition (12) of the random variables X_t^i and the σ -fields $\mathcal{F}_t^i = \sigma(x_1, \dots, x_{t\tau+i-1})$ from the proof of Theorem 2, our goal will be to give sharper concentration results for the martingale difference sequence $X_t^i - \mathbb{E}[X_t^i \mid \mathcal{F}_{t-1}^i]$. To apply Lemma 7, we must bound the variance of the difference sequence. To that end, note that the conditional variance is bounded as

$$\begin{aligned} & \mathbb{E}[(X_t^i - \mathbb{E}[X_t^i \mid \mathcal{F}_{t-1}^i])^2 \mid \mathcal{F}_{t-1}^i] \\ & \leq \mathbb{E}[(X_t^i)^2 \mid \mathcal{F}_{t-1}^i] \\ & = \mathbb{E}\left[(f(w((t-1)\tau + i)) - f(w^*) - F(w((t-1)\tau + i); x_{t\tau+i-1}) + F(w^*; x_{t\tau+i-1}))^2 \mid \mathcal{F}_{t-1}^i\right] \\ & \leq 4G^2 \|w((t-1)\tau + i) - w^*\|^2, \end{aligned}$$

where in the last line we used the Lipschitz assumption A and the fact that $w((t-1)\tau + i) \in \mathcal{F}_{t-1}^i$. Of course, since w^* minimizes f , the λ -strong convexity of f implies (see e.g. [15]) that for any $w \in \mathcal{W}$, $f(w) - f(w^*) \geq \frac{\lambda}{2} \|w - w^*\|^2$. Consequently, we see that

$$\mathbb{E}[(X_t^i - \mathbb{E}[X_t^i \mid \mathcal{F}_{t-1}^i])^2 \mid \mathcal{F}_{t-1}^i] \leq \frac{8G^2}{\lambda} [f(w((t-1)\tau + i)) - f(w^*)]. \quad (16)$$

What remains is to use the single term conditional variance bound (16) to achieve deviation control over the entire sequence X_t^i . To that end, recall the index sets $\mathcal{I}(i)$ defined in the proof of Theorem 2, and define the summed variance terms $V_i := \sum_{t \in \mathcal{I}(i)} \mathbb{E}[(X_t^i - \mathbb{E}[X_t^i \mid \mathcal{F}_{t-1}^i])^2 \mid \mathcal{F}_{t-1}^i]$. Then the bound (16) gives

$$V_i \leq \frac{8G^2}{\lambda} \sum_{t \in \mathcal{I}(i)} [f(w(\tau(t-1) + i)) - f(w^*)].$$

Using the preceding variance bound, we can apply Freedman's concentration result (Lemma 7) to see that with probability at least $1 - (4\delta \log n)/\tau$,

$$\sum_{t \in \mathcal{I}(i)} (X_t^i - \mathbb{E}[X_t^i \mid \mathcal{F}_{t-1}^i]) \leq \max \left\{ 2\sqrt{V_i}, 6GR\sqrt{\log(\tau/\delta)} \right\} \sqrt{\log(\tau/\delta)} \quad (17)$$

We can use the inequality (17) to show concentration. Define the summations

$$S_i := \sum_{t \in \mathcal{I}(i)} f(w(\tau(t-1) + i)) - f(w^*) \quad \text{and} \quad \hat{S}_i := \sum_{t \in \mathcal{I}(i)} F(w(\tau(t-1) + i); x_{t\tau+i-1}) - F(w^*; x_{t\tau+i-1}).$$

Then the definition (12) of the random variables X_t^i coupled with the inequality (17) implies that

$$\begin{aligned} S_i & \leq \hat{S}_i + \max \left\{ \sqrt{\frac{32G^2}{\lambda}} \sqrt{S_i}, 6GR\sqrt{\log \frac{\tau}{\delta}} \right\} \sqrt{\log \frac{\tau}{\delta}} + \sum_{t \in \mathcal{I}(i)} \mathbb{E}[X_t^i \mid \mathcal{F}_{t-1}^i] \\ & \leq \hat{S}_i + \sqrt{\frac{32G^2 \log \frac{\tau}{\delta}}{\lambda}} \sqrt{S_i} + 6GR \log \frac{\tau}{\delta} + |\mathcal{I}(i)|\phi(\tau)RG, \end{aligned}$$

where we have applied Lemma 1. Solving the induced quadratic in $\sqrt{S_i}$, we see

$$\sqrt{S_i} \leq \sqrt{\frac{8G^2 \log \frac{\tau}{\delta}}{\lambda}} + \sqrt{\frac{8G^2}{\lambda} \log \frac{\tau}{\delta} + \widehat{S}_i + |\mathcal{I}(i)|\phi(\tau)RG + 6GR \log \frac{\tau}{\delta}}.$$

Squaring both sides and using that $(a+b)^2 \leq 2a^2 + 2b^2$, we find that

$$S_i \leq \frac{32G^2}{\lambda} \log \frac{\tau}{\delta} + 2\widehat{S}_i + 12GR \log \frac{\tau}{\delta} + 2|\mathcal{I}(i)|\phi(\tau)RG \quad (18)$$

with probability at least $1 - 4\delta \log n / \tau$.

We have now nearly completed the proof of the theorem. Our first step for the remainder is to note that

$$\sum_{i=1}^{\tau} S_i = \sum_{t=1}^n f(w(t)) - f(w^*)$$

Applying a union bound, we use the inequality (18) to see that with probability at least $1 - 4\delta \log n$,

$$\sum_{t=1}^n f(w(t)) - f(w^*) \leq 2 \sum_{i=1}^{\tau} \widehat{S}_i + \frac{32G^2\tau}{\lambda} \log \frac{\tau}{\delta} + 12\tau GR \log \frac{\tau}{\delta} + 2n\phi(\tau)RG.$$

All that remains is to use stability to relate the sum $\sum_{i=1}^{\tau} \widehat{S}_i$ to the regret \mathfrak{R}_n , which is similar to what we did in the proof of Proposition 2. Indeed, by the definition of the sums \widehat{S}_i we have

$$\begin{aligned} \sum_{i=1}^{\tau} \widehat{S}_i &= \sum_{t=1}^n F(w(t); x_{t+\tau-1}) - F(w^*; x_{t+\tau-1}) \\ &= \sum_{t=1}^n F(w(t); x_t) - F(w^*; x_t) + \sum_{t=1}^{n-\tau} F(w(t); x_{t+\tau-1}) - F(w(t+\tau-1); x_{t+\tau-1}) \\ &\quad + \sum_{t=1}^{\tau-1} F(w^*; x_t) - \sum_{t=n+1}^{n+\tau-1} F(w^*; x_t) + \sum_{t=n-\tau+1}^n F(w(t); x_{t+\tau-1}) - \sum_{t=1}^{\tau-1} F(w(t); x_t) \\ &\leq \mathfrak{R}_n + 2(\tau-1)GR + (\tau-1)G \sum_{t=1}^n \kappa(t), \end{aligned} \quad (19)$$

where the inequality follows from the definition (6) of the regret, the boundedness assumption A, and the stability assumption C. Applying the final bound, we see that

$$\sum_{t=1}^n f(w(t)) - f(w^*) \leq 2\mathfrak{R}_n + 2(\tau-1)G \sum_{t=1}^n \kappa(t) + \frac{32G^2\tau}{\lambda} \log \frac{\tau}{\delta} + 12\tau GR \log \frac{\tau}{\delta} + 2n\phi(\tau)RG + 4(\tau-1)RG$$

with probability at least $1 - 4\delta \log n$. Dividing by n and applying Jensen's inequality completes the proof. \square

We now turn to the case of β -mixing. As before, the proof largely follows the proof of the ϕ -mixing case, with a suitable application of Markov's inequality being the only difference.

Theorem 5. *In addition to Assumptions A and C, assume further that the expected function f is λ -strongly convex with respect to the norm $\|\cdot\|$ over \mathcal{W} . Then for any $\delta < 1/e$, $n \geq 3$, with probability greater than $1 - 5\delta \log n$, for any $\tau \in \mathbb{N}$ the predictor \hat{w}_n satisfies*

$$f(\hat{w}_n) - f(w^*) \leq \frac{2}{n}\mathfrak{R}_n + \frac{2(\tau-1)G}{n} \left(\sum_{t=1}^n \kappa(t) + 2R \right) + \frac{32G^2\tau}{\lambda n} \log \frac{\tau}{\delta} + \frac{12\tau RG}{n} \log \frac{2\tau}{\delta} + \frac{2RG\beta(\tau)}{\delta}.$$

Proof We closely follow the proof of Theorem 4. Through the bound (17), no step in the proof of Theorem 4 uses ϕ -mixing. The use of ϕ -mixing occurs in bounding terms of the form $\mathbb{E}[X_t^i | \mathcal{F}_{t-1}^i]$. Rather than bounding them immediately (as was done following Eq. (17) in the proof of Theorem 4), we carry them further through the steps of the proof. Using the notation of Theorem 4's proof, in place of the inequality (18), we have

$$S_i \leq \frac{32G^2}{\lambda} \log \frac{\tau}{\delta} + 2\hat{S}_i + 12GR \log \frac{\tau}{\delta} + \sum_{t \in \mathcal{I}(i)} \mathbb{E}[X_t^i | \mathcal{F}_{t-1}^i]$$

with probability at least $1 - 4\delta \log n / \tau$. Paralleling the proof of Theorem 4, we find that with probability at least $1 - 4\delta \log n$,

$$\begin{aligned} & \sum_{t=1}^n f(w(t)) - f(w^*) \\ & \leq 2\mathfrak{R}_n + 2(\tau-1)G \sum_{t=1}^n \kappa(t) + \frac{32G^2\tau}{\lambda} \log \frac{\tau}{\delta} + 12\tau GR \log \frac{\tau}{\delta} + 4(\tau-1)RG + \sum_{i=1}^{\tau} \sum_{t \in \mathcal{I}(i)} \mathbb{E}[X_t^i | \mathcal{F}_{t-1}^i]. \end{aligned} \quad (20)$$

As in the proof of Theorem 3, we apply Markov's inequality to the final term, which gives with probability at least $1 - \delta$

$$\sum_{i=1}^{\tau} \sum_{t \in \mathcal{I}(i)} \mathbb{E}[X_t^i | \mathcal{F}_{t-1}^i] \leq \frac{2nGR\beta(\tau)}{\delta}.$$

Substituting this bound into the inequality (20) and applying a union bound (noting that $\delta < \delta \log n$) completes the proof. \square

As was the case for Theorem 3, when the process x_1, x_2, \dots is geometrically β -mixing, we can obtain a corollary of the above result showing no essential loss of rates with respect to geometrically ϕ -mixing processes. We omit details as the technique is basically identical to that for Corollary 5.

5 Linear Prediction

For this section, we place ourselves in the common statistical prediction setting where the statistical samples come in pairs of the form $(x, y) \in \mathcal{X} \times \mathcal{Y}$, where y is the label or target value of the sample x , and the samples are finite dimensional: $\mathcal{X} \subset \mathbb{R}^d$. Now we measure the goodness of the hypothesis w on the example (x, y) by

$$F(w; (x, y)) = \ell(y, \langle x, w \rangle), \quad \ell : \mathcal{Y} \times \mathbb{R} \rightarrow \mathbb{R}, \quad (21)$$

where the loss function ℓ measures the accuracy of the prediction $\langle x, w \rangle$. An extraordinary number of statistical learning problems fall into the above framework: linear regression, where the loss is of the form $\ell(y, \langle x, w \rangle) = \frac{1}{2}(y - \langle x, w \rangle)^2$; logistic regression, where $\ell(y, \langle x, w \rangle) = \log(1 + \exp(-y \langle x, w \rangle))$; boosting and SVMs all have the form (21).

The loss function (21) makes it clear that individual samples cannot be strongly convex, since the linear operator $\langle x, \cdot \rangle$ has a nontrivial null space. However, in many problems, the expected loss function $f(w) := \mathbb{E}_\Pi[F(w; (x, y))]$ is strongly convex even though individual loss functions $F(w; (x, y))$ are not. To quantify this, we now assume that $\|x\|_2 \leq r$ for μ -a.e. $x \in \mathcal{X}$, and make the following assumption on the loss:

Assumption D (Linear strong convexity). *For fixed y , the loss function $\ell(y, \cdot)$ is a λ -strongly convex and L -Lipschitz scalar function over $[-Rr, Rr]$:*

$$\ell(y, b) \geq \ell(y, a) + \ell'(y, a)(b - a) + \frac{\lambda}{2}(b - a)^2 \quad \text{and} \quad |\ell(y, b) - \ell(y, a)| \leq L|a - b|$$

for any $a, b \in \mathbb{R}$ with $\max\{|a|, |b|\} \leq Rr$.

Our choice of Rr above is intentional, since $\langle x, w \rangle \leq Rr$ by Hölder's inequality and our compactness assumption (4). A few examples of such loss functions include logistic regression and least-squares regression, the latter of which satisfies Assumption D with $\lambda = 1$. To see that the expected loss function satisfying Assumption D is strongly convex, note that²

$$\begin{aligned} f(v) &= \mathbb{E}_\Pi[\ell(y, \langle x, v \rangle)] \\ &\geq \mathbb{E}_\Pi \left[\ell(y, \langle x, w \rangle) + \ell'(y, \langle x, w \rangle)(\langle x, v \rangle - \langle x, w \rangle) + \frac{\lambda}{2}(\langle x, v \rangle - \langle x, w \rangle)^2 \right] \\ &= \mathbb{E}_\Pi[F(w; (x, y)) + \langle \nabla F(w; (x, y)), v - w \rangle] + \frac{\lambda}{2} \mathbb{E}_\Pi[\langle x, v \rangle^2 + \langle x, w \rangle^2 - 2 \langle x, w \rangle \langle x, v \rangle] \\ &= f(w) + \langle \nabla f(w), v - w \rangle + \frac{\lambda}{2} \langle \text{Cov}(x)(w - v), w - v \rangle, \end{aligned} \tag{22}$$

where $\text{Cov}(x)$ is the covariance matrix of x under the stationary distribution Π . So as long as $\lambda_{\min}(\text{Cov}(x)) > 0$, we see that the expected function f is $\lambda \cdot \lambda_{\min}(\text{Cov}(x))$ -strongly convex.

If we had access to a stable online learning algorithm with small (i.e. logarithmic) regret for losses of the form (21) satisfying Assumption D, we could simply apply Theorem 4 and guarantee good generalization properties of the predictor \hat{w}_n the algorithm outputs. The theorem assumes only strong convexity of the expected function f , which—as per our above discussion—is the case for linear prediction, so the sharp generalization guarantee would follow from the inequality (22). However, we found it difficult to show that existing algorithms satisfy our desiderata of logarithmic regret and stability, both of which are crucial requirements for our results. Below, we present a slight modification of Hazan et al.'s follow the approximate leader (FTAL) algorithm [12] to achieve the desired results. Our approach is to essentially combine FTAL with the Vovk-Azoury-Warmuth forecaster [8, Chapter 11.8], where the algorithm uses the sample x to make its prediction. Specifically, our algorithm is as follows. At iteration t of the algorithm, the algorithm receives x_t ,

²For notational convenience we use ∇F to denote either the gradient or a measurable selection from the subgradient set ∂F ; this is no loss of generality.

plays the point $w(t)$, suffers loss $F(w(t); (x_t, y_t))$, then adds $\nabla F(w(t); (x_t, y_t))$ to its collection of observed (sub)gradients. The algorithm's calculation of $w(t)$ at iteration t is

$$w(t) = \operatorname{argmin}_{w \in \mathcal{W}} \left\{ \sum_{i=1}^{t-1} \langle \nabla F(w(i); (x_i, y_i)), w \rangle + \frac{\lambda}{2} \sum_{i=1}^{t-1} \langle w(i) - w, x_i \rangle^2 + \frac{\lambda}{2} w^\top (x_t x_t^\top + \epsilon I) w \right\}. \quad (23)$$

The algorithm above is quite similar to Hazan et al.'s FTAL algorithm [12], and the following proposition shows that the algorithm (23) does in fact have logarithmic regret (we give a proof of the proposition, which is somewhat technical, in Appendix A).

Proposition 3. *Let the sequence $w(t)$ be defined by the update (23) under Assumption D. Then for any $\epsilon > 0$ and any sequence of samples (x_t, y_t) ,*

$$\sum_{t=1}^n F(w(t); (x_t, y_t)) - F(w^*; (x_t, y_t)) \leq \frac{9L^2 d}{2\lambda} \log \left(\frac{r^2 n}{\epsilon} + 1 \right) + \frac{\lambda \epsilon}{2} \|w^*\|_2^2.$$

What remains is to show that a suitable form of stability holds for the algorithm (23) that we have defined. The additional stability provided by using x_t in the update of $w(t)$ appears to be important. In the original version [12] of the FTAL algorithm, the predictor $w(t)$ can change quite drastically if a sample x_t sufficiently different from the past—in the sense that $\langle x_{t'}, x_t \rangle \approx 0$ for $t' < t$ —is encountered. In the presence of dependence between samples, such large updates can be detrimental to performance, since they keep the algorithm from exploiting the mixing of the stochastic process. Returning to our argument on stability, we recall the proof of Theorem 4, specifically the argument leading to the bound (19). We see that the stability bound does not require the full power of Assumption C, but in fact it is sufficient that

$$F(w(t); (x_{t+\tau}, y_{t+\tau})) - F(w(t + \tau); (x_{t+\tau}, y_{t+\tau})) \leq \tau \kappa(t),$$

that is, the differences in loss values are stable. To quantify the stability of the algorithm (23), we require two definitions that will be useful here and in our subsequent proofs. Define the outer product matrices

$$A_t := \sum_{i=1}^t x_i x_i^\top \quad \text{and} \quad A_{t,\epsilon} := A_t + \epsilon I. \quad (24)$$

Given a positive definite matrix A , the associated Mahalanobis norm and its dual are defined as

$$\|w\|_A^2 := \langle Aw, w \rangle \quad \text{and} \quad \|w\|_{A^{-1}}^2 := \langle A^{-1}w, w \rangle.$$

Then the following proposition (whose proof we provide in Appendix A) shows that stability holds for the linear-prediction algorithm (23).

Proposition 4. *Let $w(t)$ be generated according to the update (23) and let Assumption D hold. Then for any $\tau \in \mathbb{N}$,*

$$\begin{aligned} & F(w(t); (x_{t+\tau}, y_{t+\tau})) - F(w(t + \tau); (x_{t+\tau}, y_{t+\tau})) \\ & \leq \frac{L^2}{2\lambda} \left(6\tau \|x_{t+\tau}\|_{A_{t+\tau,\epsilon}^{-1}}^2 + 5 \sum_{s=1}^{\tau-1} \|x_{t+s}\|_{A_{t+s,\epsilon}^{-1}}^2 + 3 \|x_t\|_{A_{t,\epsilon}^{-1}}^2 \right) \end{aligned}$$

We use one more observation to derive a generalization bound for the approximate follow-the-leader update (23). For any loss ℓ satisfying Assumption D, standard convex analysis gives that $|\ell'(y, a)| \leq L$ so by straightforward algebra (taking $a = -Rr$ and $b = Rr$),

$$2L|a - b| \geq \frac{\lambda}{2}(b - a)^2, \quad \text{implying} \quad \lambda \leq \frac{2L}{Rr}. \quad (25)$$

Now, using Proposition 4 and the regret bound from Proposition 3, we now give a fast high-probability convergence guarantee for online algorithms applied to linear prediction problems, such as linear or logistic regression, satisfying Assumption D. Specifically,

Theorem 6. *Let $w(t)$ be generated according to the update (23) with $\epsilon = 1$. Then with probability at least $1 - 4\delta \log n$, for any $\tau \in \mathbb{N}$,*

$$\begin{aligned} f(\hat{w}_n) - f(x^*) &\leq \frac{L^2 d}{\lambda n} (9 + 14\tau) \log(r^2 n + 1) + \frac{\lambda}{n} \|w^*\|_2^2 + \frac{32L^2 r^2 \tau}{\lambda n \cdot \lambda_{\min}(\text{Cov}(x))} \log \frac{\tau}{\delta} \\ &\quad + \frac{8\tau L^2}{\lambda n} \left(3 \log \frac{\tau}{\delta} + 1\right) + \frac{4L^2}{\lambda} \phi(\tau). \end{aligned}$$

Proof Given the regret bound in Proposition 3, all that remains is to control the stability of the algorithm. To that end, note that

$$\sum_{t=1}^{n-\tau} F(w(t); (x_{t+\tau}, y_{t+\tau})) - F(w(t+\tau); (x_{t+\tau}, y_{t+\tau})) \leq \frac{7L^2 \tau}{\lambda} \sum_{t=1}^n \|x_t\|_{A_{t,\epsilon}^{-1}}^2 \leq \frac{7L^2 \tau d}{\lambda} \log \left(\frac{r^2 n}{\epsilon} + 1 \right), \quad (26)$$

the last inequality following from an application of Hazan et al.'s Lemma 11 [12]. Further, using Assumption D, we know that the Lipschitz constant of F is $G \leq Lr$. We mimic the proof of Theorem 4 for the remainder of the argument. This requires a minor redefinition of our martingale sequence, since $w(t)$ depends on x_t in the update (23), whereas our previous proofs required $w(t)$ to be measurable with respect to \mathcal{F}_{t-1} . As a result, we now define

$$X_t^i := f(w((t-1)\tau + i)) - f(w^*) + F(w^*; x_{t\tau+i}) - F(w((t-1)\tau + i); x_{t\tau+i}),$$

and the associated σ -fields $\mathcal{F}_t^i := \mathcal{F}_{t\tau+i} = \sigma(x_1, \dots, x_{t\tau+i})$. The sequence $X_t^i - \mathbb{E}[X_t^i | \mathcal{F}_{t-1}^i]$ defines a martingale difference sequence adapted to the filtration \mathcal{F}_t^i , $t = 1, 2, \dots$. The remainder of the proof parallels that of Theorem 4, with the modification that terms involving $(\tau-1)G$ are replaced by terms involving τG . Specifically, we use the inequality (19), the regret bound from Proposition 3, and the stability guarantee (26) to see

$$\begin{aligned} f(\hat{w}_n) - f(w^*) &\leq \frac{L^2 d}{\lambda n} (9 + 14\tau) \log \left(\frac{r^2 n}{\epsilon} + 1 \right) + \frac{\lambda \epsilon}{n} \|w^*\|_2^2 + \frac{32L^2 r^2 \tau}{\lambda n \cdot \lambda_{\min}(\text{Cov}(x))} \log \frac{\tau}{\delta} \\ &\quad + \frac{3\tau L R r}{n} \left(4 \log \frac{\tau}{\delta} + 1\right) + 2L R r \phi(\tau). \end{aligned}$$

Noting that $Rr \leq 2L/\lambda$ by the bound (25) completes the proof. \square

To simplify the conclusions of Theorem 6, we can ignore constants and the size of the sample space \mathcal{X} . Doing this, we see that with probability at least $1 - \delta$,

$$f(\hat{w}_n) - f(w^*) \leq \mathcal{O}(1) \cdot \inf_{\tau \in \mathbb{N}} \left[\frac{L^2 d \tau}{\lambda n} \log n + \frac{L^2 \tau}{\lambda n \cdot \lambda_{\min}(\text{Cov}(x))} \log \frac{\tau \log n}{\delta} + \frac{L^2}{\lambda} \phi(\tau) \right].$$

In particular, we can specialize this result in the face of different mixing assumptions on the process. We give the bound only for geometrically mixing processes, that is, when $\phi(k) \leq \phi_0 \exp(-\phi_1 k^\theta)$. Then we have—as in Corollary 3—the following:

Corollary 8. *Let $w(t)$ be generated according to the follow-the-approximate leader update (23) and assume that the process P is geometrically ϕ -mixing. Then with probability at least $1 - \delta$,*

$$f(\hat{w}_n) - f(w^*) \leq \mathcal{O}(1) \cdot \left[\frac{L^2 d (\log n)^{1+\frac{1}{\theta}}}{\phi_1^{1/\theta} \lambda n} + \frac{L^2 (\log n)^{\frac{1}{\theta}}}{\phi_1^{1/\theta} \lambda n \cdot \lambda_{\min}(\text{Cov}(x))} \log \left(\frac{\log n}{\delta} \right) \right].$$

We conclude this section by noting without proof that, since all the results here build on the theorems of Section 4, it is possible to analogously derive corresponding high-probability convergence guarantees when the stochastic process P is β -mixing rather than ϕ -mixing. In this case, we build on Theorem 5 rather than Theorem 4, but the techniques are largely identical.

6 Conclusions

In this paper, we have shown how to obtain high-probability data-dependent bounds on the generalization error, or excess risk, of hypotheses output by online learning algorithms, even when samples are dependent. In doing so, we have extended several known results on the generalization properties of online algorithms with independent data. By using martingale tools, we have given (we hope) direct simple proofs of convergence guarantees for learning algorithms with dependent data without requiring the machinery of empirical process theory. In addition, the results in this paper may be of independent interest for stochastic optimization, since they show both the expected and high-probability convergence of any low-regret stable online algorithm for stochastic approximation problems, even with dependent samples.

We believe there are a few natural open questions this work raises. First, can online algorithms guarantee good generalization performance when the underlying stochastic process is only α -mixing? Our techniques do not seem to extend readily to this more general setting, as it is less natural for measuring convergence of conditional distributions, so we suspect that a different or more careful approach will be necessary. Our second question regards adaptivity: can an online algorithm be more intimately coupled with the data and automatically adapt to the dependence of the sequence of statistical samples x_1, x_2, \dots ? This might allow both stronger regret bounds and better rates of convergence than we have achieved.

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A Technical Proofs

Proof of Proposition 3 We first give an equivalent form of the algorithm (23) for which it is a bit simpler to prove results (though the form is less intuitive). Define the (sub)gradient-like vectors $g(t)$ for all t as

$$g(t) := \nabla F(w(t); (x_t, y_t)) - \lambda x_t x_t^\top w(t). \quad (27)$$

Then a bit of algebra shows that the algorithm (23) is equivalent to

$$w(t) = \operatorname{argmin}_{w \in \mathcal{W}} \left\{ \sum_{i=1}^{t-1} \langle g(i), w \rangle + \frac{\lambda}{2} \langle A_{t,\epsilon} w, w \rangle \right\}. \quad (28)$$

We now turn to the proof of the regret bound in the theorem. Our proof is similar to the proofs of related results of Nesterov [23] and Xiao [28]. We begin by noting that via Assumption D,

$$\begin{aligned} & \sum_{t=1}^n F(w(t); (x_t, y_t)) - F(w^*; (x_t, y_t)) \\ & \leq \sum_{t=1}^n \langle \nabla F(w(t); (x_t, y_t)), w(t) - w^* \rangle - \frac{\lambda}{2} \sum_{t=1}^n (w(t) - w^*)^\top x_t x_t^\top (w(t) - w^*) \\ & = \sum_{t=1}^n \left\langle \nabla F(w(t); (x_t, y_t)) - \lambda x_t x_t^\top w(t), w(t) - w^* \right\rangle + \frac{\lambda}{2} \sum_{t=1}^n \langle x_t x_t^\top w(t), w(t) \rangle - \frac{\lambda}{2} \sum_{t=1}^n \langle x_t x_t^\top w^*, w^* \rangle \\ & = \sum_{t=1}^n \langle g(t), w(t) - w^* \rangle + \frac{\lambda}{2} \sum_{t=1}^n \langle x_t, w(t) \rangle^2 - \frac{\lambda}{2} \langle A_n w^*, w^* \rangle. \end{aligned} \quad (29)$$

Define the proximal function $\psi_t(w) = \frac{\lambda}{2} \langle A_{t,\epsilon} w, w \rangle$ and let $z(t) = \sum_{i=1}^t g(i)$. Then we can bound the regret (29) by taking a supremum and introducing the conjugate to ψ , defined by $\psi_n^*(z) = \sup_{w \in \mathcal{W}} \{ \langle z, w \rangle - \psi_n(w) \}$. In particular, we see that for any $\epsilon \geq 0$

$$\begin{aligned} & \sum_{t=1}^n F(w(t); (x_t, y_t)) - F(w^*; (x_t, y_t)) \\ & \leq \sum_{t=1}^n \langle g(t), w(t) \rangle + \frac{\lambda}{2} \sum_{t=1}^n \langle x_t, w(t) \rangle^2 + \sup_{w \in \mathcal{W}} \left\{ -\langle z(n), w \rangle - \frac{\lambda}{2} \langle A_n w, w \rangle - \frac{\lambda \epsilon}{2} \|w\|_2^2 \right\} + \frac{\lambda \epsilon}{2} \|w^*\|_2^2 \\ & = \sum_{t=1}^n \langle g(t), w(t) \rangle + \frac{\lambda}{2} \sum_{i=1}^n \langle x_i, w(i) \rangle^2 + \psi_n^*(-z(n)) + \frac{\lambda \epsilon}{2} \|w^*\|_2^2. \end{aligned} \quad (30)$$

The function ψ_n^* has $(1/\lambda)$ -Lipschitz continuous gradient with respect to the Mahalanobis norm induced by $A_{n,\epsilon}$ (e.g. [15, 23]), and further it is known that $\nabla \psi_n^*(z) = \operatorname{argmin}_{w \in \mathcal{W}} \{ \langle -z, w \rangle + \psi_n(w) \}$ so that $\nabla \psi_n^*(-z(n-1)) = w(n)$ by definition of the update (23). Thus we see

$$\begin{aligned} \psi_n^*(-z(n)) & \leq \psi_n^*(-z(n-1)) + \langle \nabla \psi_n^*(-z(n-1)), z(n-1) - z(n) \rangle + \frac{1}{2\lambda} \|z(n) - z(n-1)\|_{A_{n,\epsilon}^{-1}}^2 \\ & = \psi_n^*(-z(n-1)) - \langle w(n), g(n) \rangle + \frac{1}{2\lambda} \|g(n)\|_{A_{n,\epsilon}^{-1}}^2 \\ & = -\langle z(n-1), w(n) \rangle - \frac{\lambda}{2} \langle A_{n,\epsilon} w(n), w(n) \rangle - \langle w(n), g(n) \rangle + \frac{1}{2\lambda} \|g(n)\|_{A_{n,\epsilon}^{-1}}^2. \end{aligned}$$

since $w(n)$ minimizes $\langle z(n-1), w \rangle + \psi_n(w)$. Plugging the last inequality into the bound (30) yields

$$\begin{aligned}
& \sum_{t=1}^n F(w(t); (x_t, y_t)) - F(w^*; (x_t, y_t)) \\
& \leq \sum_{t=1}^n \langle g(t), w(t) \rangle + \frac{\lambda}{2} \sum_{t=1}^n \langle x_t, w(t) \rangle^2 - \langle z(n-1), w(n) \rangle - \frac{\lambda}{2} \langle A_{n,\epsilon} w(n), w(n) \rangle - \langle w(n), g(n) \rangle \\
& \quad + \frac{\lambda\epsilon}{2} \|w^*\|_2^2 + \frac{1}{2\lambda} \|g(n)\|_{A_{n,\epsilon}^{-1}}^2 \\
& = \sum_{t=1}^{n-1} \langle g(t), w(t) \rangle + \frac{\lambda}{2} \sum_{t=1}^{n-1} \langle x_t, w(t) \rangle^2 - \langle z(n-1), w(n) \rangle - \frac{\lambda}{2} \langle A_{n-1,\epsilon} w(n), w(n) \rangle \\
& \quad + \frac{\lambda\epsilon}{2} \|w^*\|_2^2 + \frac{1}{2\lambda} \|g(n)\|_{A_{n,\epsilon}^{-1}}^2 \\
& \leq \sum_{t=1}^{n-1} \langle g(t), w(t) \rangle + \frac{\lambda}{2} \sum_{t=1}^{n-1} \langle x_t, w(t) \rangle^2 + \psi_{n-1}^*(-z(n-1)) + \frac{\lambda\epsilon}{2} \|w^*\|_2^2 + \frac{1}{2\lambda} \|g(n)\|_{A_{n,\epsilon}^{-1}}^2
\end{aligned}$$

since $A_n = A_{n-1} + x_n x_n^\top$. Repeating the argument inductively down from $n-1$, we find

$$\sum_{t=1}^n F(w(t); (y_t, x_t)) - F(w^*; (y_t, x_t)) \leq \frac{1}{2\lambda} \sum_{t=1}^n \|g(t)\|_{A_{t,\epsilon}^{-1}}^2 + \frac{\lambda\epsilon}{2} \|w^*\|_2^2. \quad (31)$$

The bound (31) nearly completes the proof of the theorem, but we must control the gradient norm $\|g(t)\|_{A_{t,\epsilon}^{-1}}^2$ terms. To that end, let $\alpha_t = \ell'(y_t, \langle x_t, w(t) \rangle) \in \mathbb{R}$ and note that

$$\|g(t)\|_{A_{t,\epsilon}^{-1}}^2 = \left\langle A_{t,\epsilon}^{-1}(\alpha_t x_t - \lambda x_t x_t^\top w(t)), \alpha_t x_t - \lambda x_t x_t^\top w(t) \right\rangle \leq (L + \lambda R r)^2 \|x_t\|_{A_{t,\epsilon}^{-1}}^2$$

since by Assumption D, $|\alpha_t| \leq L$. Now we apply a result of Hazan et al. [12, Lemma 11], giving

$$\sum_{t=1}^n \|g(t)\|_{A_{t,\epsilon}^{-1}}^2 \leq (L + \lambda R r)^2 d \log \left(\frac{r^2 n}{\epsilon} + 1 \right).$$

Using that $\lambda \leq 2L/(Rr)$, we combine this with the bound (31) to get the result of the theorem. \square

Proof of Proposition 4 We begin by noting that any $g \in \partial F(w(t); (x_{t+\tau}, y_{t+\tau}))$ can be written as $\alpha x_{t+\tau}$ for some $\alpha \in [-L, L]$. Thus, using the first-order convexity inequality, we see there is such an α for which

$$F(w(t); x_{t+\tau}) - F(w(t+\tau); x_{t+\tau}) \leq \alpha \langle x_{t+\tau}, w(t) - w(t+\tau) \rangle.$$

Now we apply Hölder's inequality and Lemma 9, which together yield

$$\begin{aligned}
& \langle x_{t+\tau}, w(t) - w(t+\tau) \rangle \\
& \leq \|x_{t+\tau}\|_{A_{t+\tau,\epsilon}^{-1}} \|w(t) - w(t+\tau)\|_{A_{t+\tau,\epsilon}} \\
& \leq \frac{3L}{\lambda} \sum_{s=0}^{\tau-1} \|x_{t+\tau}\|_{A_{t+\tau,\epsilon}^{-1}} \|x_{t+s}\|_{A_{t+s,\epsilon}^{-1}} + \frac{2L}{\lambda} \sum_{s=1}^{\tau} \|x_{t+\tau}\|_{A_{t+\tau,\epsilon}^{-1}} \|x_{t+s}\|_{A_{t+s,\epsilon}^{-1}} \\
& \leq \frac{3L}{2\lambda} \left[\sum_{s=0}^{\tau-1} \|x_{t+s}\|_{A_{t+s,\epsilon}^{-1}}^2 + \tau \|x_{t+\tau}\|_{A_{t+\tau,\epsilon}^{-1}}^2 \right] + \frac{L}{\lambda} \left[\sum_{s=1}^{\tau} \|x_{t+s}\|_{A_{t+s,\epsilon}^{-1}}^2 + \tau \|x_{t+\tau}\|_{A_{t+\tau,\epsilon}^{-1}}^2 \right]
\end{aligned}$$

where we have used the fact that $(a^2 + b^2)/2 \geq ab$ for any $a, b \in \mathbb{R}$. A re-organization of terms and using the fact that $|\alpha| \leq L$ completes the proof. \square

Lemma 9. *Let $w(t)$ be generated according to the update (23). Then for any $\tau \in \mathbb{N}$,*

$$\|w(t) - w(t + \tau)\|_{A_{t+\tau,\epsilon}} \leq \frac{3L}{\lambda} \sum_{s=0}^{\tau-1} \|x_{t+s}\|_{A_{t+s,\epsilon}^{-1}} + \frac{2L}{\lambda} \sum_{s=1}^{\tau} \|x_{t+s}\|_{A_{t+s,\epsilon}^{-1}}.$$

Proof Recall the definition (24) of the outer product matrices A_t and the construction (27) of the subgradient vectors $g(t)$ from the proof of Proposition 3. With the definition $z(t) = \sum_{i=1}^t g(i)$, also as in Proposition 3, the update (23) is equivalent to

$$w(t) = \operatorname{argmin}_{w \in \mathcal{W}} \left\{ \langle z(t-1), w \rangle + \frac{\lambda}{2} \langle A_{t,\epsilon} w, w \rangle \right\}. \quad (32)$$

Now, let us understand the stability of the solutions to the above updates. Fixing $\tau \in \mathbb{N}$, the first order conditions for the optimality of $w(t+1)$ in the update (32) for $w(t)$ and $w(t+\tau)$ imply

$$\langle z(t+\tau-1) + \lambda A_{t+\tau,\epsilon} w(t+\tau), w - w(t+\tau) \rangle \geq 0 \quad \text{and} \quad \langle z(t-1) + \lambda A_{t,\epsilon} w(t), w' - w(t) \rangle \geq 0,$$

for all $w, w' \in \mathcal{W}$. Taking $w = w(t)$ and $w' = w(t+\tau)$, then adding the two inequalities, we see

$$\langle z(t+\tau-1) - z(t-1) + \lambda A_{t+\tau,\epsilon} w(t+\tau) - \lambda A_{t,\epsilon} w(t), w(t) - w(t+\tau) \rangle \geq 0. \quad (33)$$

The remainder of the proof consists of manipulating the inequality (33) to achieve the desired result. To begin, we rearrange Eq. (33) to state

$$\begin{aligned} & \langle z(t+\tau-1) - z(t-1), w(t) - w(t+\tau) \rangle \\ & \geq \lambda \langle A_{t+\tau,\epsilon} (w(t) - w(t+\tau)), w(t) - w(t+\tau) \rangle + \lambda \langle (A_{t,\epsilon} - A_{t+\tau,\epsilon}) w(t), w(t) - w(t+\tau) \rangle \\ & = \lambda \|w(t) - w(t+\tau)\|_{A_{t+\tau,\epsilon}}^2 + \lambda \langle (A_{t,\epsilon} - A_{t+\tau,\epsilon}) w(t), w(t) - w(t+\tau) \rangle. \end{aligned}$$

Using Hölder's inequality applied to the dual norms $\|\cdot\|_A$ and $\|\cdot\|_{A^{-1}}$, we see that

$$\begin{aligned} & \lambda \|w(t) - w(t+\tau)\|_{A_{t+\tau,\epsilon}}^2 \\ & \leq \|z(t+\tau-1) - z(t-1)\|_{A_{t+\tau,\epsilon}^{-1}} \|w(t) - w(t+\tau)\|_{A_{t+\tau,\epsilon}} \\ & \quad + \lambda \|(A_{t+\tau,\epsilon} - A_{t,\epsilon}) w(t)\|_{A_{t+\tau,\epsilon}^{-1}} \|w(t) - w(t+\tau)\|_{A_{t+\tau,\epsilon}} \end{aligned}$$

and dividing by $\lambda \|w(t) - w(t+\tau)\|$ gives

$$\|w(t) - w(t+\tau)\|_{A_{t+\tau,\epsilon}} \leq \frac{1}{\lambda} \|z(t+\tau-1) - z(t-1)\|_{A_{t+\tau,\epsilon}^{-1}} + \|(A_{t+\tau,\epsilon} - A_{t,\epsilon}) w(t)\|_{A_{t+\tau,\epsilon}^{-1}}. \quad (34)$$

Now we note the fact that $A_{t+\tau,\epsilon} - A_{t,\epsilon} = \sum_{s=1}^{\tau} x_{t+s} x_{t+s}^\top$, so

$$\|(A_{t+\tau,\epsilon} - A_{t,\epsilon}) w(t)\|_{A_{t+\tau,\epsilon}^{-1}} \leq \max_{s \in [\tau]} |\langle x_{t+s}, w(t) \rangle| \sum_{s=1}^{\tau} \|x_{t+s}\|_{A_{t+\tau,\epsilon}^{-1}} \leq Rr \sum_{s=1}^{\tau} \|x_{t+s}\|_{A_{t+\tau,\epsilon}^{-1}}.$$

In addition, we have $z(t + \tau - 1) - z(t - 1) = \sum_{s=0}^{\tau-1} g(t + s)$, and as in the proof of Proposition 3,

$$\|z(t + \tau - 1) - z(t - 1)\|_{A_{t+\tau,\epsilon}^{-1}} \leq (L + \lambda Rr) \sum_{s=0}^{\tau-1} \|x_{t+s}\|_{A_{t+\tau,\epsilon}^{-1}} \leq 3L \sum_{s=0}^{\tau-1} \|x_{t+s}\|_{A_{t+\tau,\epsilon}^{-1}},$$

where for the last inequality we used the bound (25), which implies $Rr \leq \frac{2L}{\lambda}$. Thus the inequality (34) yields

$$\|w(t) - w(t + \tau)\|_{A_{t+\tau,\epsilon}} \leq \frac{3L}{\lambda} \sum_{s=0}^{\tau-1} \|x_{t+s}\|_{A_{t+\tau,\epsilon}^{-1}} + \frac{2L}{\lambda} \sum_{s=1}^{\tau} \|x_{t+s}\|_{A_{t+\tau,\epsilon}^{-1}}.$$

Noting that $A_{t+1,\epsilon} \succeq A_{t,\epsilon}$ completes the proof. \square

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