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REGULARIZATION BY EARLY STOPPING FOR ONLINE LEARNING ALGORITHMS

LORENZO ROSASCO ^{†*}, ANDREA TACCHETTI[†], AND SILVIA VILLA [†]

Abstract. We study the learning algorithm corresponding to the incremental gradient descent defined by the empirical risk over an infinite dimensional hypotheses space. We consider a statistical learning setting and show that, provided with a universal step-size and a suitable early stopping rule, the learning algorithm thus obtained is universally consistent and derive finite sample bounds. Our results provide a theoretical foundation for considering early stopping in online learning algorithms and shed light on the effect of allowing for multiple passes over the data.

Key words. Online learning, incremental gradient descent, consistency

1. Introduction. Early stopping is a widely used heuristic to achieve regularization in online algorithms for supervised learning, see e.g. [22]. However, its theoretical foundation is still poorly understood. While early stopping has been recently studied for classical gradient descent learning and related algorithms [41, 37, 3, 6, 8, 29], we are not aware of similar studies for online learning. Here, with online learning algorithms, we refer broadly to iterative procedures that have access, at each iteration, to the gradient of a loss function at a single input-output pair.

We consider the statistical learning theory framework and are interested in online learning algorithms for expected risk minimization. The situation typically analyzed in this context is the one where each iteration of the algorithm corresponds to a new input-output pair [20]. In this paper, we are interested in allowing the algorithm to do multiple passes over the data (epochs). Indeed, the effect of the number of epochs to the generalization performance of the algorithm is the subject of our study. The procedure we consider can be seen as the incremental gradient descent (IGD) defined by the empirical risk. However, unlike typical analyses of this method, see e.g. [25], we are not interested in convergence of the algorithm to the minimum of the empirical risk, but rather to the one of the *expected* risk. More precisely, we consider the least squares loss and a hypothesis space which is a (reproducing kernel) Hilbert space. In this setting the minimization of the expected risk is a convex, but not strongly convex, problem and the domain where we minimize is neither compact nor bounded. In particular, the gradients are not bounded, unlike in most studies in stochastic optimization [26]. The instance of IGD we consider is defined by an *unpenalized* empirical risk and the step size is fixed to a universal constant, so that the only free parameter in the algorithm is effectively the number of epochs. Indeed, we show that better generalization is achieved if multiple, although finitely many, passes over the data are considered. Our main results show that: 1) provided with a suitable stopping rule, the IGD algorithm is universally consistent, 2) finite sample bounds can be derived under a suitable smoothness assumption, and 3) model selection via hold-out cross validation, which is typically used in practice, can adaptively achieve the same generalization guarantees. A detailed discussion of previous works is deferred to Section 3.2. Here we note that, the results more closely related to the analysis we present are those in [2, 34, 38] where consistency and finite sample bounds are derived. In [2] a single pass over the data is shown to suffice, however the analysis is restricted to a finite dimensional setting, and the generalization performance is empirically shown to be still increasing after the first epoch. In [34] and [38] the step size and/or a penalization parameter need to be chosen in a distribution dependent way (or by cross validation) and multiple passes over

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the data are *not* allowed. When compared to these methods, we note that the algorithm we consider naturally computes the whole regularization path. While it is possible to use a penalization parameter, or even the step size γ , as a regularization parameter, with these choices computing the regularization path incurs in an extra computational cost. This cost is avoided when considering early stopping by exploiting a built-in warm-restart property.

The rest of the paper is organized as follows. In Section 2, we discuss the setting we consider and in Section 3 we introduce and discuss the IGD algorithm. The main results are presented in Section 4, while their proofs are discussed in Section 6. Section 5 discusses future work. Preliminary results and the rest of the proofs are given in the Appendices A, B, and C. Finally, Appendix D studies finite sample bounds in the hypotheses space norm.

Notation We denote by $\mathbb{R}_+ = [0, +\infty[$ and $\mathbb{R}_{++} =]0, +\infty[$. Given a normed space \mathcal{B} and linear operators $(A_i)_{1 \leq i \leq m}$, $A_i: \mathcal{B} \rightarrow \mathcal{B}$ for every i , their composition $A_m \circ \dots \circ A_1$ will be denoted as $\prod_{i=1}^m A_i$. By convention, we set $\prod_{i=m+1}^m A_i = I$, where I is the identity of \mathcal{B} . The operator norm will be denoted by $\|\cdot\|$ and the Hilbert-Schmidt norm by $\|\cdot\|_{HS}$.

2. Set-Up: Supervised Learning. We study the supervised learning problem. Consider an input space \mathcal{X} , which is a measurable space, and an output space¹ $\mathcal{Y} = \mathbb{R}$. Let $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$ be endowed with a probability measure ρ , $\rho_{\mathcal{X}}$ denote the marginal measure on \mathcal{X} , and $\rho(\cdot|x)$ the conditional measure on \mathcal{Y} given $x \in \mathcal{X}$. The conditional probability is well defined (see e.g. [32] Lemma A.3.16) and we denote by $f_{\rho} = \int_{\mathcal{Y}} y d\rho(y|\cdot)$ the conditional expectation, namely the regression function. We assume the distribution to be fixed, but known only through a set $\mathbf{z} = \{(x_1, y_1), \dots, (x_m, y_m)\}$ of m independent samples identically distributed according to ρ . The set \mathbf{z} is called a training set and, for technical reasons, we assume $m \geq 2$. With an abuse of notation, $(\forall m \in \mathbb{N})$, the measure ρ^m will be denoted by \mathbb{P} .

Further, consider a separable reproducing kernel Hilbert space (RKHS) \mathcal{H} [1], with inner product (norm) denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ ($\|\cdot\|_{\mathcal{H}}$), and a measurable reproducing kernel $K: \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$. The function $t \mapsto K(x, t) \in \mathcal{H}$ is denoted by K_x . Throughout the paper we make the following boundedness assumptions.

ASSUMPTION 1. Assume that there exist $M > 0$ and $\kappa > 0$ such that $\text{supp}\{\rho(\cdot|x)\} \subseteq [-M, M]$, and $\sqrt{K(x, x)} < \kappa$, for almost all $x \in \mathcal{X}$. In the above setting, we are interested in solving the expected risk minimization problem,

$$\inf_{f \in \mathcal{H}} \mathcal{E}(f), \quad \mathcal{E}(f) = \int (y - f(x))^2 d\rho(x, y). \quad (2.1)$$

Note that the corresponding optimization problem is convex (but in general not strongly convex), the domain of optimization is neither compact nor bounded, and, especially, the gradient of the objective function is unbounded. Moreover, we do not assume the infimum to be achieved.

We are interested in approximate solutions \hat{f} which satisfy the following requirement

$$(\forall \rho), (\forall \epsilon > 0), \quad \lim_{m \rightarrow \infty} \mathbb{P} \left(\mathcal{E}(\hat{f}) - \inf_{f \in \mathcal{H}} \mathcal{E}(f) > \epsilon \right) = 0. \quad (2.2)$$

When \mathcal{H} is universal [32] this is exactly universal consistency, see for example [15]. In the following, with a slight abuse of terminology, we will still refer to the condition in equation (2.2) as to universal consistency, even when the kernel is not universal. Moreover, we are interested in strengthening the above requirement to replace convergence in probability with

¹Most results extend naturally to \mathcal{Y} being a separable Hilbert space, e.g. \mathbb{R}^T .

almost sure convergence, as well as to prove convergence in probability uniformly with respect to ρ . As it is known, the latter result can only be derived under suitable assumptions on ρ and it is equivalent to considering learning rates [16, 31].

REMARK 2.1. *When the set of minimizers of \mathcal{E} is not empty, one could also consider consistency in the RKHS norm, that is*

$$(\forall \rho), (\forall \epsilon > 0), \quad \lim_{m \rightarrow \infty} \mathbb{P} \left(\|\hat{f} - f_{\mathcal{H}}\|_{\mathcal{H}} > \epsilon \right) = 0, \quad (2.3)$$

where $f_{\mathcal{H}}$ is the minimum norm minimizer of \mathcal{E} on \mathcal{H} . The analysis of this case is analogous to the one required to prove (2.2), therefore we leave the results and the corresponding proofs in Appendix D.

EXAMPLE 2.2 (The Linear Case). *A particular case of the above setting is the one where \mathcal{X} is a Euclidean space and K is taken to be the associated inner product. If \mathcal{H} is the corresponding RKHS, we have that $f \in \mathcal{H}$ if and only if $(\forall x \in \mathcal{X}) f(x) = \langle w, x \rangle$, for some $w \in \mathcal{X}$ and $\|f\|_{\mathcal{H}} = \|w\|$. Assumption 1 is satisfied if the marginal distribution $\rho_{\mathcal{X}}$ is supported in a ball of radius κ .*

REMARK 2.3 (Stochastic Optimization (SO)). *In SO given a probability space (\mathcal{Z}, ρ) , a separable Hilbert space \mathcal{B} , and a loss function $L : \mathcal{Z} \times \mathcal{B} \rightarrow [0, \infty)$, we are interested in solving*

$$\inf_{h \in \mathcal{B}} \int L(z, h) d\rho(z).$$

In this view, supervised learning corresponds to setting $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$, $\mathcal{B} = \mathcal{H}$ and $L(z, f) = (y - f(x))^2$. Note however that due to the lack of strong convexity, unboundedness of \mathcal{X} and unboundedness of the gradients, the classical assumptions required to apply stochastic gradient descent are not satisfied [26].

3. Early Stopping for Online Learning Algorithms. We consider the estimator obtained applying the incremental gradient descent (IGD) algorithm [5, 4] to empirical risk minimization,

$$\inf_{f \in \mathcal{H}} \hat{\mathcal{E}}(f), \quad \hat{\mathcal{E}}(f) = \frac{1}{m} \sum_{i=1}^m (y_i - f(x_i))^2. \quad (3.1)$$

Given $\hat{f}_t \in \mathcal{H}$ an iteration of IGD generates \hat{f}_{t+1} according to the recursion,

$$\hat{f}_{t+1} = \hat{\psi}_t^m, \quad (3.2)$$

where $\hat{\psi}_t^m$ is obtained at the end of one cycle, namely as the last step of the recursion

$$\hat{\psi}_t^0 = \hat{f}_t; \quad \hat{\psi}_t^i = \hat{\psi}_t^{i-1} - \frac{\gamma_t}{m} (\hat{\psi}_t^{i-1}(x_i) - y_i) K_{x_i}, \quad i = 1, \dots, m \quad (3.3)$$

for a suitable sequence of stepsizes $\{\gamma_t\}_{t \in \mathbb{N}}$, $\gamma_t \in \mathbb{R}_{++}$. In this paper, we consider a sequential approach, where each point of the training set is selected exactly once within each cycle. Each iteration, called epoch, corresponds to one pass over data. In practice, other approaches, e.g. stochastic [25], can be considered and might lead to different behaviors, but we leave the analysis of these latter cases for future study.

The iteration in (3.2)-(3.3) can be readily implemented if we consider a linear kernel, see Example 2.2, or any kernel $K(x, x') = \langle \Phi(x), \Phi(x') \rangle_{\mathbb{R}^D}$ defined by a finite dimensional feature map $\Phi : \mathcal{X} \rightarrow \mathbb{R}^D$. In the latter case we can simply identify the function $\hat{f}_{t+1} : x \mapsto$

$\langle \hat{w}_{t+1}, \Phi(x) \rangle_{\mathbb{R}^D}$ with $\hat{w}_{t+1} \in \mathbb{R}^D$. For a general kernel, it can be easily seen that, for $\hat{f}_0 = 0$, the solution after t epochs can be written as $\hat{f}_t(\cdot) = \sum_{k=1}^m (\alpha_t)_k K_{x_k}$, for suitable coefficients $\alpha_t = ((\alpha_t)_1, \dots, (\alpha_t)_m) \in \mathbb{R}^m$, given by the following recursion,

$$\alpha_{t+1} = c_t^m$$

$$c_t^0 = \alpha_t, \quad (c_t^i)_k = \begin{cases} (c_t^{i-1})_k - \frac{\gamma_t}{m} \left(\sum_{j=1}^m K(x_i, x_j) (c_t^{i-1})_j - y_i \right), & k = i \\ (c_t^{i-1})_k, & k \neq i \end{cases}$$

The above algorithm is closely related, yet different, to the one discussed in [19], where an explicit regularization is considered (choose $\lambda_k > 0$ in the following Equation (3.4)). More precisely, consider the following online learning algorithm

$$\hat{h}_{k+1} = \hat{h}_k - \frac{\gamma_k}{m} (\hat{h}_k(x_{i_k}) - y_{i_k}) K_{x_{i_k}} + \lambda_k \hat{h}_k, \quad k \in \mathbb{N} \quad (3.4)$$

where $\{\lambda_k\}_{k \in \mathbb{N}}$ and $\{\gamma_k\}_{k \in \mathbb{N}}$ are suitable sequences such that $(\forall k \in \mathbb{N}) \lambda_k \in \mathbb{R}_+$ and $\gamma_k \in \mathbb{R}_{++}$, and $i_k \in \{1, \dots, m\}$. We note that different kind of analyses can be carried out when considering the iteration (3.4).

The incremental gradient descent algorithm (3.2)-(3.3) is an instance of the general iterative procedure (3.4). Indeed, it corresponds to the case where the gradient descent step is taken cyclically more than once with respect to the same point ($i_k = k \bmod m$), we do not impose any explicit regularization setting $\lambda_k = 0$, and we keep γ_k constant on each pass over the training set. With this choice, for a given common initialization, we have $\hat{f}_t = \hat{h}_{mt}$, for all t . In the following we are specifically interested to study the effect of multiple passes over the data, when the goal is to minimize the expected, rather than the empirical, risk (see discussion in Section 3.2). Towards this end, we precisely consider the situation where the step size is chosen a priori in a distribution independent way and investigate the regularization effect of the number of epochs. Indeed, we show that early stopping, that is considering multiple – yet finitely many – epochs, is useful. Our results extend to incremental gradient the analysis of early stopping carried out in [41, 37, 3, 6, 29] for the classical (batch) gradient descent and shed light on the effect of considering multiple passes over the data in online learning algorithms.

3.1. Early stopping, Learning and Computations. Recent interest in providing a better understanding of regularization by early stopping is largely motivated by its practical use. Indeed, early stopping has long been applied as a heuristic to achieve regularization, especially in the context of neural networks [22]. In particular, it is natural to consider early stopping when dealing with big data-sets, since in this context algorithms should ideally have computational requirements tailored to the generalization properties allowed by the data. This is exactly the main feature of early stopping regularization. Figure 1 gives an illustration of this fact in a numerical simulation. Here, we added an explicit regularization parameter λ as in [19], see also (3.4), and plot the error of the estimator on a hold-out set, against the number of epochs for a few values of λ . At least two interesting observations can be made. First, the smaller is λ the fewer epochs are needed to achieve the best generalization. Among all choices, fixing $\lambda = 0$ achieves comparable generalization with the minimal amount of computation (iterations). Second, the algorithm we consider naturally computes the whole regularization path. While it is possible to use λ , or even the step size γ , as a regularization parameter (see below), computing the regularization path incurs in an extra cost. This cost is avoided by the iterative procedure in (3.2),(3.3) by virtue of the natural warm-restart property which is built-in.

Before discussing our main results, we briefly compare IGD with other online learning schemes.

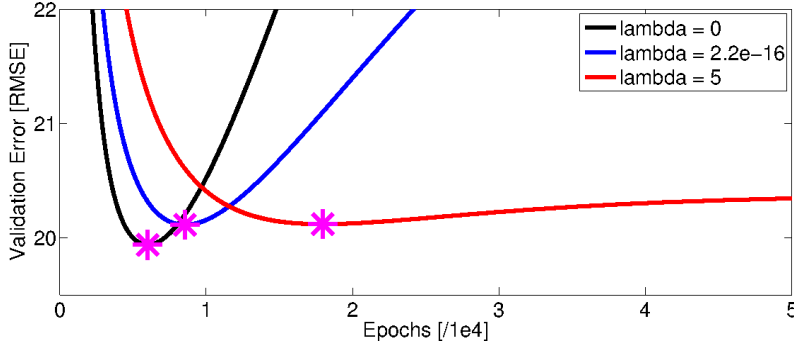


FIG. 3.1. Plots of the error of the estimator of the IGD algorithm on a hold-out set, against the number of epochs for a few values of λ .

3.2. Previous Work. Online learning algorithms have been analyzed within different settings. Since our perspective is perhaps somewhat different (we use IGD to solve expected, rather than empirical, risk minimization), we provide a brief overview of previous theoretical studies.

Minimization of the expected risk.. The question of the consistency of online learning algorithms when \mathcal{H} can be infinite dimensional has been considered in [34, 38] (see also references therein). Both papers deal with an infinite sequence of i.i.d. training points ($m = +\infty$) and fix $i_k = k$: *no epochs* are considered and the data points are seen only once (each iteration correspond to a point). In [34] it is shown that if the step size γ_k and the regularization parameter λ_k are chosen as suitable functions of the number of points, then the corresponding algorithm can be universally consistent with probability one. Moreover, under suitable smoothness assumption, see (4.2), learning rates are derived if *both* $(\gamma_k)_{k \in \mathbb{N}}$ and $(\lambda_k)_{k \in \mathbb{N}}$ are chosen depending on smoothness. In [38] it is shown that indeed similar results can be derived for $\lambda_k = 0$ for all $k \in \mathbb{N}$, but an horizon, that is the total number of points to be considered, needs to be known a priori to appropriately choose the step-size. Recently, a similar analysis, $\lambda_k = 0$ for all $k \in \mathbb{N}$, is developed in [2] in a finite dimensional setting. In this case it is shown that a suitable fixed step-size suffices to ensure convergence (as well as convergence rates) only with one pass over the data, however this is a by-product of considering a finite dimensional setting. Finally we note that another possibility is to consider the case where λ is kept fixed (and different from zero). In this case, convergence of algorithm (3.4) to the solution of the regularized expected risk minimization problem,

$$\min_{f \in \mathcal{H}} \mathcal{E}(f) + \lambda \|f\|_{\mathcal{H}}^2.$$

is derived. These latter results could be coupled with an analysis of the (approximation) error $\min_{f \in \mathcal{H}} \mathcal{E}(f) + \lambda \|f\|_{\mathcal{H}}^2 - \inf_{\mathcal{H}} \mathcal{E}(f)$ to derive convergence for the expected risk.

Minimization of the empirical risk.. In optimization theory, the IGD iteration is used to minimize objective functions which are the sum of m functions. In our case, by definition, we have

$$\hat{\mathcal{E}}(f) = \sum_{i=1}^m V_i(f), \quad V_i(f) = \frac{1}{m} (y_i - f(x_i))^2.$$

therefore the IGD could be applied to minimize the empirical risk, i.e. problem (3.1). Results in this context focus on showing that $\hat{\mathcal{E}}(f^t) - \inf_{f \in \mathcal{H}} \hat{\mathcal{E}}(f)$ converges to zero when t

increases, see [5] and references therein. Related results consider the case where the empirical risk is replaced by a regularized empirical risk, see e.g. [21]. In this paper, we propose a novel perspective showing how IGD can be used to approximately solve the *expected* risk minimization, i.e. problem (2.1) rather than the empirical risk.

Sequential Prediction. Finally, we note that iterative schemes as in (3.4) have been recently extensively studied in online learning (a.k.a. sequential prediction problems), see e.g. [10]. In this context, the data are not stochastic, and the goal is to control the so called regret. If the data are indeed generated by a stochastic process, then there is a classic approach, sometimes called online-to-batch conversion [9], to convert regret bounds to expected risk bounds. However, this latter approach does not seem to be appropriate for studying the effect of multiple passes over the data.

4. Main Results. We begin by stating the universal consistency of (cyclic) IGD with early stopping.

THEOREM 4.1. *Let $\theta \in [0, 1[$, and $(\forall t \in \mathbb{N}), \gamma_t = \kappa^{-2}(t+1)^{-\theta}$. Assume that $(\hat{f}_t)_{t \in \mathbb{N}}$ is defined in (3.2)-(3.3). If we choose a stopping rule $t^* = t^*(m)$ so that*

$$\lim_{m \rightarrow +\infty} t^*(m) = +\infty \quad \text{and} \quad \lim_{m \rightarrow +\infty} t^*(m)^{4(1-\theta)}/m = 0 \quad (4.1)$$

then

$$\lim_{m \rightarrow \infty} \mathcal{E}(\hat{f}_{t^*}) = \inf_{f \in \mathcal{H}} \mathcal{E}(f) \quad \text{almost surely.}$$

The above result shows that consistency is achieved computing a suitable number $t^*(m)$ of iterations of IGD given m points. The number of required iterations tends to infinity as the number of available training points increases. In particular, this excludes the choice $t^*(m) = 1$ for all m , namely considering only one pass over the data. Condition (4.1) can be interpreted as an early stopping rule, since it requires the number of epochs not to grow too fast. Below, we further discuss how a suitable stopping rule can be derived if the problem satisfies certain prior assumptions. We note that several choices of the step-size are allowed in Theorem 4.1. In particular it is possible to choose a constant step-size, see also the discussion below.

We next consider finite sample bounds that can be derived if the regression function satisfies the following smoothness assumption,

$$\|L^{-r} f_\rho\|_\rho \leq R, \quad \text{for some } r > 0. \quad (4.2)$$

where $L: L^2(\mathcal{X}, \rho_{\mathcal{X}}) \rightarrow L^2(\mathcal{X}, \rho_{\mathcal{X}})$ is such that $Lf(x) = \int_{\mathcal{X}} f(x')K(x, x')d\rho_{\mathcal{X}}$ and $\|\cdot\|_\rho$ denotes the norm in $L^2(\mathcal{X}, \rho_{\mathcal{X}}) = \{f: \mathcal{X} \rightarrow \mathbb{R} : \int d\rho_{\mathcal{X}} |f|^2 < \infty\}$. Assumption (4.2) is fairly standard (see [13], and [12, Section 4] for a discussion).

THEOREM 4.2. *Let $(\forall t \in \mathbb{N}), \gamma_t = \kappa^{-2}(t+1)^{-\theta}$, for some $\theta \in [0, 1[$. Assume that (4.2) holds, and fix $\delta \in]0, 1[$. Then, with probability at least $1 - \delta$,*

$$(\forall t \in \mathbb{N}) \quad \mathcal{E}(\hat{f}_t) - \inf_{\mathcal{H}} \mathcal{E} \leq C_\theta \log^2\left(\frac{2}{\delta}\right) \frac{(t+1)^{4(1-\theta)}}{m} + C_r t^{-2r(1-\theta)} \quad (4.3)$$

where $C_\theta = C^2/2\kappa^6(1-\theta)^4$ for some $C \in \mathbb{R}_{++}$ which does not depend on m, t, δ , and $C_r = R^2(2r/e)^{4r}/\kappa^4(1-\theta)^2$. Moreover, if we choose the stopping rule

$$t^*(m) = \lceil m^{\frac{1}{(4+2r)(1-\theta)}} \rceil \quad (4.4)$$

then, with probability at least $1 - \delta$,

$$\mathcal{E}(\hat{f}_{t^*(m)}) - \inf_{\mathcal{H}} \mathcal{E} \leq D \left(\log \frac{2}{\delta} \right)^2 m^{-\frac{r}{r+2}}, \quad (4.5)$$

where the constant $D \in \mathbb{R}_{++}$ can be explicitly given.

Equation (4.3) arises from a form of bias-variance (sample-approximation) decomposition of the error. Choosing the number of epochs that optimize the bound (4.3), we derive an a priori stopping rule (4.4) and a corresponding bound (4.5). Again, this result confirms that the number of epochs acts as a regularization parameter and the best choice following from Equation (4.3) suggests multiple passes over the data to be beneficial. Interestingly, the best distribution independent choice of the step-size is a constant with respect to t , that is $\theta = 0$. Such a bound can be compared to known lower bounds as well as previous results for least squares algorithms that can be studied under the same prior. Lower bounds are known [7, 33] under assumption (4.2) and further assuming that the eigenvalues of L have a polynomial decay, that is

$$(\sigma_i)_{i \in \mathbb{N}} \sim i^{-b}, \quad \text{for some } b \in [1, \infty[. \quad (4.6)$$

This latter property can be interpreted as a measure of the effective dimensionality of the hypotheses space [40, 7]. There are a few special regimes of interest; assuming $r \geq 1/2$ in (4.2) implies that the infimum of the expected risk over \mathcal{H} is achieved [11], and it is known that sharp bounds are harder to get if $r < 1/2$ (see discussion in [33]). For $b = 1$, the condition $(\sigma_i)_{i \in \mathbb{N}} = O(i^{-b})$ always holds. This is the situation we consider in this paper, and it is sometimes called the *capacity independent setting*. The lower bound, under assumptions (4.2), (4.6), are of order $O(n^{-\frac{2rb}{2rb+1}})$, and $O(n^{-\frac{2r}{2r+1}})$ in the capacity independent setting. The bound in (4.3) is not optimal as a consequence of a bad dependence of a suitably defined sample error estimate on the number of epochs. Deriving this bound is the main technical contribution of the paper and the proof is rather involved. While the dependence of the bound on the number of points is optimal, the dependence on the number of epochs can be improved. A comparison with the analysis of batch gradient descent suggests that the optimal dependence should be $\sim t^{(1-\theta)}$. The bound is proved generalizing classical results in inverse problems and is known to be essentially sharp [17]. We also note that several least squares estimator have been considered in this setting. Regularized least squares (a.k.a. kernel ridge regression) has been considered in [40, 7, 33, 24, 18], and iterative methods, as well as a larger class of so called spectral filtering methods [23] have been analyzed in [41, 37, 3, 6, 8, 29]. As we mentioned before, online learning algorithms are considered in [34, 38] where optimal bounds are derived in the capacity independent setting. The bound in the finite dimensional setting derived in [2] are also optimal. An important point here is that, while different least squares methods are based on the same class of priors and likely to have essentially the same statistical guarantees, their computational properties are different. In this view it is interesting to develop efficient least square estimators and early stopping provides a natural approach (see [42] for an interesting idea based on a *divide and conquer* approach).

Finally, we present an adaptive early stopping rule. The stopping rule (4.4) depends on the smoothness parameter r which is typically unknown, and hold-out cross validation is often used in practice. Here we show that this procedure allows to adaptively achieve the same convergence rate as in (4.5). Our analysis follows the one in [8].

Given a training set \mathbf{z} of cardinality m , we define

$$\mathbf{z}_1 = \{(x_1, y_1), \dots, (x_n, y_n)\}, \quad \mathbf{z}_2 = \{(x_{n+1}, y_{n+1}), \dots, (x_m, y_m)\}, \quad (4.7)$$

where $n = \lfloor m/2 \rfloor$ (note that other splits are possible). Fix $t_{\max} \in \mathbb{N}$, and use the training set \mathbf{z}_1 to compute the sequence $(\hat{f}_t)_{t \in \{1, \dots, t_{\max}\}}$. Then, set $\hat{\mathcal{E}}_2$ to be the empirical error corresponding to the training set \mathbf{z}_2 , and

$$\hat{t} = \operatorname{argmin}\{\hat{\mathcal{E}}_2(T_M \hat{f}_t) : t \in \{1, \dots, t_{\max}\}\}. \quad (4.8)$$

where $T_M: \mathcal{H} \rightarrow \mathcal{H}$: $T_M f(x) = \min\{|f(x)|, M\}f(x)/|f(x)|$ is the truncation operator.

THEOREM 4.3. *Let $(\forall t \in \mathbb{N}) \gamma_t = \kappa^{-2}(t+1)^{-\theta}$ for some $\theta \in [0, 1[$. If $t_{\max} > m^{1/(4(1-\theta))}$, with probability greater than $1 - 2\delta$,*

$$\mathcal{E}(T_M \hat{f}_{\hat{t}}) - \inf_{\mathcal{H}} \mathcal{E} \leq 2D \left(\log \frac{2}{\delta} \right)^2 m^{-\frac{r}{r+2}} + \frac{40M^2}{3(1-\theta)} \left(\log \frac{m}{2\delta} \right) m^{-1} \quad (4.9)$$

where D is the constant appearing in equation (4.5).

Noting that for every $r > 0$, $(\log(m/2\delta))m^{-1}$ goes to zero faster than $m^{-r/(r+2)}$, we see that the estimator obtained with hold-out cross-validation achieves the same learning rate as the one corresponding to the optimal choice of the number of epochs. We end noting that, while an hold-out procedure requires splitting the data, it does not worsen the computational complexity of the algorithm, unlike other model selection criterions [29].

5. Some Future Directions. The analysis presented in the paper is a first step towards exploring the properties of early stopping for online learning algorithms. We mention a few directions for future work.

- **Sharpen the bounds.** In particular incorporating kernel dependent assumption such as (4.6), quantifying the *effective dimension* of the hypotheses space. This latter point would be particularly interesting to bridge the analysis in the finite and infinite dimensional setting.
- **Consider other iterative procedures.** Particularly, stochastic variants of the IGD algorithms, where the indices in each cycle are randomly selected [5], and also accelerated version as proposed in [27]. Ideas related to this latter point have been considered in [3, 6, 8] for batch gradient techniques (the gradient of the empirical risk is considered in each iteration). In particular in [3] it is shown that a variant of gradient descent, sometimes called the ν -method, can obtain the same generalization guarantees of non accelerated gradient descent learning, but using much fewer iterations.
- **Generalize the analysis.** In particular, consider loss functions other than the least squares loss, and especially priors other than (4.2),(4.6). This latter question seems particularly interesting and possibly challenging.

6. Proofs of the Main Results. In this section we sketch the proofs of the main results. The error analysis is based on the following decomposition into sample error and approximation error

$$\mathcal{E}(\hat{f}_t) - \inf_{\mathcal{H}} \mathcal{E} \leq 2\kappa^2 \|\hat{f}_t - f_t\|_{\mathcal{H}}^2 + 2(\mathcal{E}(f_t) - \inf_{\mathcal{H}} \mathcal{E}) \quad (6.1)$$

where f_t are the iterates of the IGD on the expected error defined in equation (B.9). Theorems 4.1 and 4.2. will follow from a probabilistic upper bound for the sample error and an upper bound for the approximation error.

THEOREM 6.1 (Sample error). *Let $\hat{f}_0 = f_0 = 0$, $(\gamma_t)_{t \in \mathbb{N}}$ be such that $(\forall t \in \mathbb{N}) \gamma_t \in]0, m\kappa^{-2}]$, and \hat{f}_t and f_t be defined in (3.2) and (B.9), respectively. There exists $C > 0$*

such that, for all $t \in \mathbb{N}$ and $\delta \in]0, 1[$ (sufficiently small),

$$\mathbb{P} \left(\|\hat{f}_t - f_t\|_{\mathcal{H}} \leq \frac{C}{\sqrt{m}} \left(\log \frac{2}{\delta} \right) \sum_{k=1}^{t-1} \gamma_k \sum_{i=0}^{k-1} \gamma_i \right) \geq 1 - \delta.$$

In particular, if $\gamma_i = \kappa^{-2}(i+1)^{-\theta}$ with $\theta \in [0, 1[$, with probability at least $1 - \delta$,

$$\|\hat{f}_t - f_t\|_{\mathcal{H}} \leq \frac{C}{2\kappa^4(1-\theta)^2\sqrt{m}} \left(\log \frac{2}{\delta} \right) (t+1)^{2(1-\theta)}$$

The proof of Theorem 6.1 is postponed to Appendix C. We next show that the approximation error $\mathcal{E}(f_t) - \inf_{\mathcal{H}} \mathcal{E}$ go to zero as t goes to infinity, and give a rate under assumption (4.2). For similar results with respect to the norm in \mathcal{H} see Appendix D.

THEOREM 6.2 (Approximation error). *Let $f_0 = 0$, $(\gamma_t)_{t \in \mathbb{N}}$ be such that $(\forall t \in \mathbb{N}) \gamma_t \in]0, m\kappa^{-2}[$, $\sum_{t \in \mathbb{N}} \gamma_t = +\infty$ and $(f_t)_{t \in \mathbb{N}}$ be defined as in (B.9). Then*

$$\mathcal{E}(f_t) - \inf_{\mathcal{H}} \mathcal{E} \rightarrow 0. \quad (6.2)$$

Moreover, if f_ρ satisfies (4.2), then

$$\mathcal{E}(f_t) - \inf_{\mathcal{H}} \mathcal{E} \leq R^2(r/e)^{2r} \left(\sum_{i=0}^{t-1} \gamma_i \right)^{-2r}.$$

In particular, if $\gamma_i = \kappa^{-2}(i+1)^{-\theta}$ with $\theta \in [0, 1[$,

$$\mathcal{E}(f_t) - \inf_{\mathcal{H}} \mathcal{E} \leq \frac{R^2(r/e)^{2r}}{\kappa^2(1-\theta)} t^{-2r(1-\theta)}.$$

Combining the bounds for the sample and approximation errors, we get a proof of Theorem 4.2 and Theorem 4.1, solving a bias-variance trade-off.

Proof. of Theorem 4.2. Recalling equation (6.1), we have

$$\mathcal{E}(\hat{f}_t) - \inf_{\mathcal{H}} \mathcal{E} \leq 2\kappa^2 \|\hat{f}_t - f_t\|_{\mathcal{H}}^2 + 2(\mathcal{E}(f_t) - \inf_{\mathcal{H}} \mathcal{E}).$$

Applying Theorem 6.1 and Theorem 6.2 we get that with probability at least $1 - \delta$

$$\mathcal{E}(\hat{f}_t) - \inf_{\mathcal{H}} \mathcal{E} \leq \frac{C^2}{2\kappa^6(1-\theta)^2m} \left(\log \frac{2}{\delta} \right)^2 t^{4(1-\theta)} + \frac{R(r/e)^{2r}}{\kappa^2(1-\theta)} t^{-2r(1-\theta)}, \quad (6.3)$$

and (4.3) follows by suitably choosing the constant D .

Next let $t^*(m) = \lceil m^\alpha \rceil$. Substituting this choice of t into the right hand side of (6.3), and minimizing over α , we get the linear equation

$$4\alpha(1-\theta) - 1 = -2\alpha r(1-\theta), \quad (6.4)$$

whose solution is $\alpha = 1/((1-\theta)(4+2r))$. \square

Proof. of Theorem 4.1. Combining Theorem 6.1 and Theorem 6.2 we immediately get convergence in probability of $\mathcal{E}(\hat{f}_t) - \inf_{f \in \mathcal{H}} \mathcal{E}$. Almost sure convergence follows applying the Borel-Cantelli lemma. \square

Proof. of Theorem 4.3 We define

$$t_\rho = \operatorname{argmin}\{\mathcal{E}(T_M \hat{f}_t) : t \in \{1, \dots, T\}\}. \quad (6.5)$$

The latter definition is equivalent to

$$t_\rho = \operatorname{argmin}_{t \in \{1, \dots, t_{\max}\}} \left\| S(T_M \hat{f}_t) - f_\rho \right\|_\rho^2 \quad (6.6)$$

where S is the operator defined in Appendix B. Since by assumption the support of $\rho(y|x)$ is contained in $[-M, M]$, it follows that $f_\rho(x) \in [-M, M]$ almost surely. Therefore, from the definition of t_ρ ,

$$\left\| S(T_M \hat{f}_{t_\rho}) - f_\rho \right\|_\rho \leq \left\| S(T_M \hat{f}_{t^*(m)}) - f_\rho \right\|_\rho \leq \left\| S \hat{f}_{t^*(m)} - f_\rho \right\|_\rho. \quad (6.7)$$

If we denote by Pf_ρ the projection onto the closure of \mathcal{H} in $L^2(\mathcal{X}, \rho_{\mathcal{X}})$, from the chain of inequalities in (6.7), noting that $(\forall g \in \mathcal{H}) \|Sg - Pf_\rho\|_\rho^2 = \|Sg - f_\rho\|_\rho^2 - \|Pf_\rho - f_\rho\|_\rho^2$ we get

$$\mathcal{E}(T_M \hat{f}_{t_\rho}) - \inf_{\mathcal{H}} \mathcal{E} = \left\| S(T_M \hat{f}_{t_\rho}) - Pf_\rho \right\|_\rho^2 \leq \left\| S \hat{f}_{t^*(m)} - Pf_\rho \right\|_\rho^2 = \mathcal{E}(\hat{f}_{t^*(m)}) - \inf_{\mathcal{H}} \mathcal{E}. \quad (6.8)$$

Finally, we claim that with probability greater than $1 - \delta$, if $t_{\max} = m^{1/(4(1-\theta))}$,

$$\left\| S(T_M \hat{f}_{t_\rho}) - Pf_\rho \right\|_\rho^2 \leq 2 \left\| S(T_M \hat{f}_{t_\rho}) - f_\rho \right\|_\rho^2 + \frac{640M^2(1-\theta)}{m} \log \frac{2m}{\delta} \quad (6.9)$$

First note that adding to both sides $\|Pf_\rho - f_\rho\|_\rho^2$ (6.8) is equivalent to

$$\left\| S(T_M \hat{f}_{t_\rho}) - f_\rho \right\|_\rho^2 \leq 2 \left\| S(T_M \hat{f}_{t_\rho}) - f_\rho \right\|_\rho^2 + \frac{640M^2(1-\theta)}{m} \log \frac{2m}{\delta} \quad (6.10)$$

The statement then follows combining (6.8) and (6.9), and applying Theorem 4.2. To prove (6.10), let $\mathbf{z}_2 = \{(x_{n+1}, y_{n+1}), \dots, (x_m, y_m)\}$, and define the random variables for $i = n+1, \dots, m$:

$$\xi_i^t = (S(T_M \hat{f}_t)(x_i) - y_i)^2 - (f_\rho(x_i) - y_i)^2.$$

We have for every $t \in \{1, \dots, t_{\max}\}$

$$|\xi_i^t| \leq 4M^2 \quad (6.11)$$

$$\mathbb{E}[(\xi_i^t)] = \int_{\mathcal{X} \times \mathcal{Y}} \left((T_M \hat{f}_t(x) - y)^2 - (f_\rho(x) - y)^2 \right) d\rho = \|S(T_M \hat{f}_t) - f_\rho\|_\rho^2 \quad (6.12)$$

$$\mathbb{E}[(\xi_i^t)^2] = \int_{\mathcal{X} \times \mathcal{Y}} (T_M \hat{f}_t(x) - f_\rho(x))^2 (T_M \hat{f}_t(x) + f_\rho(x) - 2y)^2 d\rho \quad (6.13)$$

$$\leq 16M^2 \mathbb{E}[(\xi_i^t)] \quad (6.14)$$

□

Applying Proposition A.2 with $X_i = \xi_i^t$, $\mu = \mathbb{E}[(\xi_i^t)]$, $B = 4M^2$, $\sigma^2 = \mathbb{E}[(\xi_i^t)^2] \leq 16M^2 \mathbb{E}[(\xi_i^t)]$, we obtain for all $t \in \{1, \dots, t_{\max}\}$ with probability greater than $1 - \delta$

$$\frac{1}{m-n} \sum_{i=1}^{m-n} \xi_i^t \leq (1 + 16\alpha M^2) \mu + \epsilon \quad (6.15)$$

and

$$(1 - 16\alpha M^2) \mathbb{E}[\xi_i^t] \leq \frac{1}{m-n} \sum_{i=1}^{m-n} \xi_i^t + \epsilon, \quad (6.16)$$

with $\epsilon = \frac{(3+16\alpha M^2)}{24(1-\theta)(m-n)\alpha} \log \frac{(m-n)}{\delta}$. Therefore, since $t_{\max} \geq t^*(m)$

$$\begin{aligned} \|S\hat{f}_{\hat{t}} - f_{\rho}\|_{\rho}^2 &= \mathbb{E}[\xi_{\hat{t}}] \leq \frac{1}{1-16\alpha M^2} \left(\frac{1}{m-n} \sum_{i=1}^{m-n} \xi_i^{\hat{t}} \right) + \frac{\epsilon}{1-16\alpha M^2} \\ &\leq \frac{1}{1-16\alpha M^2} \left(\frac{1}{m-n} \sum_{i=1}^{m-n} \xi_i^{t^*(m)} \right) + \frac{\epsilon}{1-16\alpha M^2} \\ &\leq \frac{1+16\alpha M^2}{1-16\alpha M^2} \mathbb{E}[\xi_i^{t^*(m)}] + \frac{2\epsilon}{1-16\alpha M^2}. \end{aligned}$$

If we choose $\alpha = 1/(48M^2)$ we get $16\alpha M^2 = 1/3$ and

$$\|S\hat{f}_{\hat{t}} - f_{\rho}\|_{\rho}^2 \leq 2(\mathcal{E}(T_M \hat{f}_{t^*(m)}) - \mathcal{E}(f_{\rho})) + \frac{20M^2}{3(1-\theta)(m-n)} \log \frac{(m-n)}{\delta}. \quad (6.17)$$

Recalling that $m-n \geq m/2$ we obtain the statement.

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Appendix A. Useful Results. In the various proofs we repeatedly use the following well-known equation. Let $(X_r)_{r \in \mathbb{N}}$ be a sequence in \mathcal{H} , defined recursively by

$$X_{r+1} = A_r X_r + B_r, \quad (\text{A.1})$$

with $A_r : \mathcal{H} \rightarrow \mathcal{H}$ a linear operator for all $r \in \mathbb{N}$, and $\{B_r\}_{r \in \mathbb{N}} \subseteq \mathcal{H}$. Then, for every $s \in \mathbb{N}$,

$s < r$,

$$X_r = \left(\prod_{i=s}^{r-1} A_i \right) X_s + \sum_{k=s}^{r-1} \left(\prod_{i=k+1}^{r-1} A_i \right) B_k. \quad (\text{A.2})$$

The following concentration inequality due to [28] (see also [39] and [34, Proposition A.3]) and is the useful to obtain a probabilistic upper bound for the sample error.

THEOREM A.1 (Bernstein-Pinelis). *Let ξ_i be a martingale difference sequence taking values in a Hilbert space. Suppose that $\|\xi_i\| \leq M$. Then for every $\delta \in]0, 1[$ the following holds*

$$\mathbb{P} \left(\left\{ \left\| \frac{1}{m} \sum_{i=1}^m \xi_i \right\| \leq \frac{8M}{3\sqrt{m}} \log \frac{2}{\delta} \right\} \right) \geq 1 - \delta.$$

While more refined concentration inequality could be considered [36], this would affect only the constants in the bound which are not the main focus of this paper. In the proof of Theorem 4.3, we will also use a variant of the previous Bernstein concentration inequality, which is taken from [8].

PROPOSITION A.2. *Let $(\xi_i)_{1 \leq i \leq m}$ be real valued i.i.d. random variables with mean μ , $|\xi_i| \leq B$ and $\mathbb{E}[(\xi_i - \mu)^2] \leq \sigma^2$, for all $i \in \{1, \dots, m\}$. Then for arbitrary $\alpha \in \mathbb{R}_{++}$, $\delta \in]0, 1[$,*

$$\mathbb{P} \left[\left| \frac{1}{m} \sum_{i=1}^m \xi_i - \mu \right| \geq \alpha \sigma^2 + \frac{3 + 4\alpha B}{6m\alpha} \log 2\delta \right] \leq \delta. \quad (\text{A.3})$$

Appendix B. Recursive Expressions and Error Decomposition.

We introduce some linear operators that will be useful in the following. Let $S: \mathcal{H} \rightarrow L^2(\mathcal{X}, \rho_{\mathcal{X}})$ be the embedding operator. S is well defined and $\|S\| \leq \kappa$. If we set $T = S^*S$ and $L = SS^*$, then $T: \mathcal{H} \rightarrow \mathcal{H}$, $L: L^2(\mathcal{X}, \rho_{\mathcal{X}}) \rightarrow L^2(\mathcal{X}, \rho_{\mathcal{X}}): Lf(x) = \int_{\mathcal{X}} f(x') K(x, x') d\rho_{\mathcal{X}}$ with $\|T\| \leq \kappa^2$ and $\|L\| \leq \kappa^2$. Fix $x \in \mathcal{X}$. Then $S_x: \mathcal{H} \rightarrow \mathbb{R}: f \mapsto f(x)$ is well defined and $\|S_x\| \leq \kappa$. Its adjoint is $S_x^*: \mathbb{R} \rightarrow \mathcal{H}: a \mapsto aK_x$. The operator $S_x^*S_x$ is denoted by T_x . Clearly $\|T_x\| \leq \kappa^2$.

Finally, given the training set \mathbf{z} , the operator $\sum_{i=1}^m T_{x_i}/m$ is denoted by \hat{T} . Note that, using these linear operators, we can write the empirical error and the risk as

$$\hat{\mathcal{E}}(f) = \frac{1}{m} \sum_{i=1}^m (S_{x_i} f - y_i)^2, \quad \mathcal{E}(f) = \|Sf - f_{\rho}\|_{\rho}^2 + \mathcal{E}(f_{\rho}), \quad (\text{B.1})$$

for all $f \in \mathcal{H}$. We describe some useful recursive expressions for the IGD and the gradient descent (GD) iteration. These will be used in the error analysis and provide some useful comparison between these two methods.

LEMMA B.1. *Let $\hat{f}_0 \in \mathcal{H}$ and for every $t \in \mathbb{N}$, let \hat{f}_t be defined as in equations (3.2) and (3.3). Then*

$$\hat{f}_{t+1} = \prod_{i=1}^m \left(I - \frac{\gamma_t}{m} T_{x_i} \right) \hat{f}_t + \frac{\gamma_t}{m} \sum_{i=1}^m \prod_{k=i+1}^m \left(I - \frac{\gamma_t}{m} T_{x_k} \right) S_{x_i}^* y_i \quad (\text{B.2})$$

Proof. The update of $\hat{\psi}_t^i$ in (3.3) is of the form (A.1), with $A_r = I - \frac{\gamma_t}{m} T_{x_{r+1}}$, and $B_r = \frac{\gamma_t}{m} S_{x_{r+1}}^* y_{r+1}$, and $X_0 = \hat{f}_t$. Equation (B.2) follows by writing (A.2) for $r = m$. \square

PROPOSITION B.2. Assume that $m \geq 2$. The iteration of the incremental gradient descent can be written as

$$\hat{f}_{t+1} = \left(I - \frac{\gamma_t}{m} \sum_{j=1}^m T_{x_j} \right) \hat{f}_t + \gamma_t \left(\frac{1}{m} \sum_{j=1}^m S_{x_j}^* y_j \right) + \gamma_t^2 \left(\hat{A}_t \hat{f}_t - \hat{b}_t \right) \quad (\text{B.3})$$

with

$$\hat{A}_t = \frac{1}{m^2} \sum_{k=2}^m \prod_{i=k+1}^m \left(I - \frac{\gamma_t}{m} T_{x_i} \right) T_{x_k} \sum_{j=1}^{k-1} T_{x_j}, \quad \hat{b}_t = \frac{1}{m^2} \sum_{k=2}^m \prod_{i=k+1}^m \left(I - \frac{\gamma_t}{m} T_{x_i} \right) T_{x_k} \sum_{j=1}^{k-1} S_{x_j}^* y_j \quad (\text{B.4})$$

Proof. In order to prove the equality in (B.3), we first derive a equation for the quantity $h_i = \sum_{j=1}^i (T_{x_j} \hat{\psi}_t^{j-1} - S_{x_j}^* y_j)$. We show by induction that for all $i = 1, \dots, m$

$$h_i = \sum_{j=1}^i T_{x_j} \hat{f}_t - \sum_{j=1}^i S_{x_j}^* y_j + \frac{\gamma_t}{m} \hat{R}_t^i \hat{f}_t + \frac{\gamma_t}{m} \hat{r}_t^i \quad (\text{B.5})$$

for a linear operator $\hat{R}_t^i : \mathcal{H} \rightarrow \mathcal{H}$ and an element $\hat{r}_t^i \in \mathcal{H}$. Since

$$h_1 = T_{x_1} \hat{\psi}_t^0 - S_{x_1}^* y_1 = T_{x_1} \hat{f}_t - S_{x_1}^* y_1,$$

clearly (B.5) holds for $i = 1$ with $\hat{R}_t^1 = 0$ and $\hat{r}_t^1 = 0$. Now we suppose by inductive hypothesis that (B.5) holds for i , and we prove it for $i+1$. First note that $\hat{\psi}_t^i = \hat{f}_t - (\gamma_t/m) h_i$, and then by inductive hypothesis,

$$\begin{aligned} h_{i+1} &= h_i + T_{x_{i+1}} \hat{\psi}_t^i - S_{x_{i+1}}^* y_{i+1} \\ &= h_i + T_{x_{i+1}} \left(\hat{f}_t - \frac{\gamma_t}{m} h_i \right) - S_{x_{i+1}}^* y_{i+1} \\ &= \left(I - \frac{\gamma_t}{m} T_{x_{i+1}} \right) h_i + T_{x_{i+1}} \hat{f}_t - S_{x_{i+1}}^* y_{i+1} \\ &= \left(I - \frac{\gamma_t}{m} T_{x_{i+1}} \right) \left(\sum_{j=1}^i T_{x_j} \hat{f}_t - \sum_{j=1}^i S_{x_j}^* y_j + \frac{\gamma_t}{m} \hat{R}_t^i \hat{f}_t + \frac{\gamma_t}{m} \hat{r}_t^i \right) + T_{x_{i+1}} \hat{f}_t - S_{x_{i+1}}^* y_{i+1} \\ &= \sum_{j=1}^{i+1} T_{x_j} \hat{f}_t - \sum_{j=1}^{i+1} S_{x_j}^* y_j + \frac{\gamma_t}{m} \left(\left(I - \frac{\gamma_t}{m} T_{x_{i+1}} \right) \hat{R}_t^i - T_{x_{i+1}} \sum_{j=1}^i T_{x_j} \right) \hat{f}_t \\ &\quad + \frac{\gamma_t}{m} \left(\left(1 - \frac{\gamma_t}{m} T_{x_{i+1}} \right) \hat{r}_t^i + T_{x_{i+1}} \sum_{j=1}^i S_{x_j}^* y_j \right). \end{aligned} \quad (\text{B.6})$$

By setting for $i = 1, \dots, m-1$

$$\hat{R}_t^{i+1} = \left(1 - \frac{\gamma_t}{m} T_{x_{i+1}} \right) \hat{R}_t^i - T_{x_{i+1}} \sum_{j=1}^i T_{x_j}, \quad \hat{r}_t^{i+1} = \left(I - \frac{\gamma_t}{m} T_{x_{i+1}} \right) \hat{r}_t^i + T_{x_{i+1}} \sum_{j=1}^i S_{x_j}^* y_j$$

we get (B.5) for $i + 1$. From (B.5) we derive

$$\hat{f}_{t+1} = \hat{f}_t - \frac{\gamma_t}{m} h_m = \left(I - \frac{\gamma_t}{m} \sum_{j=1}^m T_{x_j} - \left(\frac{\gamma_t}{m} \right)^2 \hat{R}_t^m \right) \hat{f}_t + \frac{\gamma_t}{m} \sum_{j=1}^m S_{x_j}^* y_j - \left(\frac{\gamma_t}{m} \right)^2 \hat{r}_t^i \quad (\text{B.7})$$

Equation (B.3) then follows by defining $\hat{A}_t = -\hat{R}_t^m/m^2$ and $\hat{b}_t = \hat{r}_t^m/m^2$ and recalling (3.3). Moreover, applying (A.2), we have

$$\hat{A}_t = \frac{1}{m^2} \sum_{k=2}^m \prod_{i=k+1}^m \left(I - \frac{\gamma_t}{m} T_{x_i} \right) T_{x_k} \sum_{j=1}^{k-1} T_{x_j}, \quad \hat{b}_t = \frac{1}{m^2} \sum_{k=2}^m \prod_{i=k+1}^m \left(I - \frac{\gamma_t}{m} T_{x_i} \right) T_{x_k} \sum_{j=1}^{k-1} S_{x_j}^* y_j$$

□

Equation (B.3) allows to compare the resulting update of one epoch of the incremental gradient descent with the one of a standard gradient descent on the empirical error with stepsize γ_t , which is given by

$$\hat{\varphi}_{t+1} = \left(I - \frac{\gamma_t}{m} \sum_{j=1}^m T_{x_j} \right) \hat{\varphi}_t + \frac{\gamma_t}{m} \sum_{j=1}^m S_{x_j}^* y_j \quad (\text{B.8})$$

for an arbitrary $\hat{\varphi}_0 \in \mathcal{H}$. As can be seen directly comparing (B.3) and (B.8), the incremental gradient descent can be interpreted as a perturbed gradient descent step, with perturbation

$$\hat{e}_t = \gamma_t^2 \left(\hat{A}_t \hat{f}_t - \hat{b}_t \right),$$

which is proportional to γ_t^2 . To analyze consistency properties of IGD applied to the empirical risk, we need to introduce an auxiliary iteration, obtained by applying the incremental gradient descent algorithm to the expected loss, which clearly, for fixed m , can be written as

$$f \mapsto \sum_{i=1}^m \frac{\mathcal{E}(f)}{m}.$$

Reasoning as in Proposition B.2, simply replacing T_{x_i} by T and S_{x_i} by S , we get

$$f_{t+1} = (I - \gamma_t T) f_t + \gamma_t S^* f_\rho + \gamma_t^2 (A_t f_t - b_t) \quad (\text{B.9})$$

with

$$A_t = \frac{1}{m^2} \sum_{k=2}^m \left[\prod_{i=k+1}^m \left(I - \frac{\gamma_t}{m} T \right) \right] T \sum_{j=1}^{k-1} T f_t, \quad b_t = \frac{1}{m^2} \sum_{k=2}^m \left[\prod_{i=k+1}^m \left(I - \frac{\gamma_t}{m} T \right) \right] T \sum_{j=1}^{k-1} S^* f_\rho \quad (\text{B.10})$$

REMARK B.3. First note that, although not explicitly specified, the operator A_t and the element $b_t \in \mathcal{H}$ depend on m . Moreover, since in this case the gradient of each summand coincides with the gradient of the function \mathcal{E} , one epoch of the incremental gradient method in (B.9) corresponds to m steps of a gradient descent with stepsize γ_t/m (where the stepsize is fixed across m iterations).

Appendix C. Proof of Theorems 6.1 and 6.2. To prove Theorem 6.1 we need some auxiliary results. The proof will be given at the end of the section.

LEMMA C.1. For all $t \in \mathbb{N}$,

$$\hat{f}_t - f_t = \left[\prod_{k=0}^{t-1} (I - \gamma_k T + \gamma_k^2 A_k) \right] (\hat{f}_0 - f_0) + \sum_{k=0}^{t-1} \gamma_k \left[\prod_{j=k+1}^{t-1} (I - \gamma_j T + \gamma_j^2 A_j) \right] \chi_k \quad (\text{C.1})$$

with

$$\chi_k = (T - \hat{T})\hat{f}_k + \gamma_k(\hat{A}_k - A_k)\hat{f}_k + \left(\frac{1}{m} \sum_{i=1}^m \hat{S}_{x_i}^* y_i - S^* f_\rho\right) + \gamma_k(b_k - \hat{b}_k), \quad (\text{C.2})$$

where \hat{A}_k and \hat{b}_k are defined in (B.4), and A_k and b_k in (B.10). *Proof.* Fix $t \in \mathbb{N}$. We have

$$\hat{f}_{t+1} = (I - \gamma_t \hat{T} + \gamma_t^2 \hat{A}_t) \hat{f}_t + \gamma_t \sum_{i=1}^m \hat{S}_{x_i}^* y_i - \gamma_t^2 \hat{b}_t.$$

Adding and subtracting $(-\gamma_t T + \gamma_t^2 A_t) \hat{f}_t$ we get

$$\hat{f}_{t+1} = (I - \gamma_t T + \gamma_t^2 A_t) \hat{f}_t + \gamma_t (T - \hat{T}) \hat{f}_t + \gamma_t^2 (\hat{A}_t - A_t) \hat{f}_t + \gamma_t \frac{1}{m} \sum_{i=1}^m \hat{S}_{x_i}^* y_i - \gamma_t^2 \hat{b}_t.$$

Therefore

$$\begin{aligned} \hat{f}_{t+1} - f_{t+1} &= (I - \gamma_t T + \gamma_t^2 A_t)(\hat{f}_t - f_t) + \gamma_t (T - \hat{T}) \hat{f}_t + \gamma_t^2 (\hat{A}_t - A_t) \hat{f}_t \\ &\quad + \gamma_t \left(\frac{1}{m} \sum_{i=1}^m \hat{S}_{x_i}^* y_i - S^* f_\rho \right) + \gamma_t^2 (b_t - \hat{b}_t) \end{aligned}$$

Relying on equation (A.1) we get (C.1). \square

We next state other lemmas which are needed to bound the various terms appearing in the decomposition derived in Lemma C.1. First we bound the norm of the operator which is applied to the random variable χ_k .

LEMMA C.2. Fix $m \in \mathbb{N}$, $m \geq 1$, $j \geq 1$, and let $\gamma_j \in]0, m\kappa^{-2}]$. Then

$$\|I - \gamma_j T + \gamma_j^2 A_j\| \leq 1. \quad (\text{C.3})$$

Proof. By definition of A_j

$$I - \gamma_j T + \gamma_j^2 A_j = \left(I - \frac{\gamma_j}{m} T \right)^m. \quad (\text{C.4})$$

Since $\|T\| \leq \kappa^2$ and by assumption $\gamma_j/m \leq \kappa^{-2}$, $\|I - \frac{\gamma_j}{m} T\| \leq 1$ and the statement follows. \square

The longest part of the proof is the one required to get a bound on the norm of the random variable χ_k . We will proceed bounding each summand separately.

LEMMA C.3. Assume $(\gamma_k)_{k \in \mathbb{N}}$ to be a sequence in $]0, +\infty[$, and $\hat{f}_0 = 0$. Then, for all $t \in \mathbb{N}$, $\|\hat{f}_t\|_K \leq \kappa M \sum_{k=0}^{t-1} \gamma_k$. The following lemma is a direct consequence of Bernstein-Pinelis inequality A.1, taking into account that $\|T\|_{HS} \leq \kappa^2$ and $\|T_{x_i}\|_{HS} \leq \kappa^2$ (see also [14])

LEMMA C.4. For every $\delta \in]0, 1[$

$$\mathbb{P} \left(\left\| \frac{1}{m} \sum_{i=1}^m T_{x_i} - T \right\|_{HS} \leq \frac{16\kappa^2}{3\sqrt{m}} \log \frac{2}{\delta} \right) \geq 1 - \delta \quad (\text{C.5})$$

and

$$\mathbb{P} \left(\left\| \frac{1}{m} \sum_{i=1}^m S_{x_i}^* y_i - S^* f_\rho \right\|_{\mathcal{H}} \leq \frac{16\kappa M}{3\sqrt{m}} \log \frac{2}{\delta} \right) \geq 1 - \delta \quad (\text{C.6})$$

LEMMA C.5. Let $(\gamma_k)_{k \in \mathbb{N}}$ be a sequence in $]0, m\kappa^{-2}[$. For any $\delta \in]0, 1[$ and $k \in \mathbb{N}$,

$$\mathbb{P} \left(\|\hat{A}_k - A_k\|_{HS} \leq \frac{2\kappa^4}{m^2} + \frac{64\kappa^4}{9\sqrt{m}} \log \frac{2}{\delta} \right) \geq 1 - \delta \quad (\text{C.7})$$

Proof. We first show a useful decomposition. Recall that

$$\hat{A}_k = \frac{1}{m} \sum_{j=2}^m \frac{1}{m} \prod_{i=j+1}^m \left(I - \frac{\gamma_k}{m} T_{x_i} \right) T_{x_j} \sum_{l=1}^{j-1} T_{x_l} \quad A_k = \frac{1}{m} \sum_{j=2}^m \prod_{i=j+1}^m \left(I - \frac{\gamma_k}{m} T \right) T \frac{1}{m} \sum_{l=1}^{j-1} T.$$

If we set for $j \in \{2, \dots, m\}$

$$\hat{B}_{k,j} = \left[\prod_{i=j+1}^m \left(I - \frac{\gamma_k}{m} T_{x_i} \right) \right] T_{x_j}, \quad B_{k,j} = \left[\prod_{i=j+1}^m \left(I - \frac{\gamma_k}{m} T \right) \right] T$$

we have

$$\begin{aligned} \hat{A}_k - A_k &= \frac{1}{m} \sum_{j=2}^m \frac{j-1}{m} \hat{B}_{k,j} \left(\frac{1}{j-1} \sum_{l=1}^{j-1} T_{x_l} \right) - \frac{1}{m} \sum_{j=2}^m B_{k,j} \frac{1}{m} \sum_{l=1}^{j-1} T \\ &= \frac{1}{m} \left[\sum_{j=2}^m \frac{j-1}{m} \hat{B}_{k,j} \left(\frac{1}{j-1} \sum_{l=1}^{j-1} T_{x_l} - T \right) + Q_m T \right], \end{aligned} \quad (\text{C.8})$$

with

$$Q_m = \sum_{j=2}^m \frac{j-1}{m} (\hat{B}_{k,j} - B_{k,j}). \quad (\text{C.9})$$

We next bound each term appearing in the decomposition in equation (C.8). Let $(\gamma_k)_{k \in \mathbb{N}}$ be a sequence in $]0, m\kappa^{-2}[$. By Lemma C.4, with probability greater than $1 - \delta$,

$$\left\| \frac{1}{j-1} \sum_{l=1}^{j-1} T_{x_l} - T \right\|_{HS} \leq \frac{8\kappa^2}{3\sqrt{j-1}} \log \frac{2}{\delta}. \quad (\text{C.10})$$

On the other hand

$$\|\hat{B}_{k,j}\| \leq \prod_{i=j+1}^m \left\| I - \frac{\gamma_k}{m} T_{x_i} \right\| \|T_{x_j}\| \leq \kappa^2. \quad (\text{C.11})$$

Note that $\sum_{j=2}^m \frac{j-1}{m} \hat{B}_{k,j} \left(\frac{1}{j-1} \sum_{l=1}^{j-1} T_{x_l} - T \right)$ is Hilbert-Schmidt, for T_{x_l} and T are Hilbert-Schmidt operators, with $\|T_{x_l}\| \leq \kappa^2$ and $\|T\|_{HS} \leq \kappa^2$, and the family of Hilbert-Schmidt

operators is an ideal with respect to the composition in $L(\mathcal{H})$ (see Theorem VI.22 and Exercise 28 in [30]). Therefore, combining (C.10) and (C.11), and by noting that $\sum_{j=1}^m \sqrt{j-1} \leq 2m^{3/2}/3$

$$\frac{1}{m} \left\| \sum_{j=2}^m \frac{j-1}{m} \hat{B}_{k,j} \left(\frac{1}{j-1} \sum_{l=1}^{j-1} T_{x_l} - T \right) \right\|_{HS} \leq \frac{8\kappa^4}{3m^2} \log \frac{2}{\delta} \sum_{j=2}^m \sqrt{j-1} \leq \frac{16\kappa^4}{9} \frac{1}{\sqrt{m}} \log \frac{2}{\delta} \quad (\text{C.12})$$

holds with probability greater than $1 - \delta$, for any $\delta \in]0, 1]$.

Next we write the quantity $\sum_{j=2}^m \frac{j-1}{m} (\hat{B}_{k,j} - B_{k,j})$ appearing in the second term in (C.8) as the sum of a martingale and a bounded term. For short, we set $\gamma = \frac{\gamma_k}{m}$ and for all $j \in \{2, \dots, m\}$ we denote

$$\hat{\Pi}_{j+1}^m = \frac{l-1}{m} \prod_{i=j+1}^m (I - \gamma T_{x_i}), \quad \Pi_{j+1}^m = \frac{j-1}{m} \prod_{i=j+1}^{m-1} (I - \gamma T),$$

so that from the definition of Q_m in (C.9),

$$Q_m = \sum_{j=2}^m (\hat{\Pi}_{j+1}^m T_{x_j} - \Pi_{j+1}^m T),$$

for all $m > 1$. We can derive a recursive update for the quantity Q_m as follows

$$\begin{aligned} Q_{m+1} &= \sum_{j=2}^{m+1} (\hat{\Pi}_{j+1}^{m+1} T_{x_j} - \Pi_{j+1}^{m+1} T) \\ &= \frac{m}{m+1} \left[(T_{x_{m+1}} - T) + \sum_{j=2}^m (\hat{\Pi}_{j+1}^{m+1} T_{x_j} - \Pi_{j+1}^{m+1} T) \right] \\ &= \frac{m}{m+1} \left[(T_{x_{m+1}} - T) + \sum_{j=2}^m ((I - \gamma T_{x_{m+1}}) \hat{\Pi}_{j+1}^m T_{x_j} - (I - \gamma T) \Pi_{j+1}^m T) \right] \\ &= \frac{m}{m+1} \left[(T_{x_{m+1}} - T) + (I - \gamma T_{x_{m+1}}) \sum_{j=2}^m (\hat{\Pi}_{j+1}^m T_{x_j} - \Pi_{j+1}^m T) \right. \\ &\quad \left. + \gamma (T - T_{x_{m+1}}) \sum_{j=2}^m \Pi_{j+1}^m T \right] \\ &= \frac{m}{m+1} (I - \gamma T_{x_{m+1}}) Q_m + \frac{m}{m+1} (T_{x_{m+1}} - T) \left(I - \gamma \sum_{j=2}^m \Pi_{j+1}^m T \right). \end{aligned}$$

Applying equation (A.1), we get

$$Q_m = \prod_{l=3}^m \frac{l-1}{l} \left(I - \frac{\gamma_k}{m} T_{x_l} \right) \frac{1}{2} (T_{x_2} - T) + \xi_{k,l} \quad (\text{C.13})$$

where

$$\xi_{k,l} = \sum_{l=3}^m \prod_{i=l+1}^m \frac{i-1}{i} \left(I - \frac{\gamma_k}{m} T_{x_i} \right) \frac{l-1}{l} (T_{x_l} - T) \left(I - \frac{\gamma_k}{m} \sum_{j=2}^{l-1} \frac{j-1}{l-1} \prod_{i=j+1}^{l-1} \left(I - \frac{\gamma_k}{m} T \right) T \right).$$

For all $l = 3, \dots, m$

$$\mathbb{E}[\xi_{k,l}] = 0,$$

being T_{x_3}, \dots, T_{x_m} independent and $\mathbb{E}[(T_{x_l} - T)] = 0$ for all $l = 3, \dots, m$. Moreover the conditional expectation

$$\mathbb{E}[\xi_{k,l} \mid \xi_{k,l+1}, \dots, \xi_{k,m}] = 0,$$

since T_{x_l} is independent from $T_{x_{l+1}}, \dots, T_{x_m}$. Therefore the sequence $(\xi_{k,l})$ for $l = 3, \dots, m$ is a martingale difference sequence for all k .

The operator $\xi_{k,l}$ is Hilbert-Schmidt, since it is the composition of a Hilbert-Schmidt operator with a continuous one. Next, since the operator T is compact and self-adjoint and $0 \leq \gamma_k/m \leq 1/\|T\|$, from the spectral mapping theorem, for every $l \in \{3, \dots, m\}$

$$\|I - \frac{\gamma_k}{m} \sum_{j=2}^{l-1} \frac{j-1}{l-1} \prod_{i=j+1}^{l-1} \left(I - \frac{\gamma_k}{m} T\right) T\| = \sup_{x \in [0,1]} \left| 1 - \frac{1}{l-1} \sum_{j=2}^{l-1} (j-1) (1-x)^{l-j-1} x \right|.$$

We have

$$\begin{aligned} 0 &\leq \frac{1}{l-1} \sum_{j=2}^{l-1} (j-1) (1-x)^{l-j-1} x \leq \sum_{j=2}^{l-1} (1-x)^{l-j-1} x \\ &= \sum_{j=2}^{l-1} (1-x)^{l-j-1} - (1-x)^{l-j} = 1 - (1-x)^{l-2} \leq 1 \end{aligned}$$

Therefore

$$\|I - \frac{\gamma_k}{m} \sum_{j=2}^{l-1} \frac{j-1}{l-1} \prod_{i=j+1}^{l-1} \left(I - \frac{\gamma_k}{m} T\right) T\| \leq 1.$$

Using the last inequality, we derive

$$\begin{aligned} \|\xi_{k,l}\|_{HS} &\leq \frac{l-1}{l} \left\| \prod_{i=l+1}^m \frac{i-1}{i} \left(I - \frac{\gamma_k}{m} T_{x_i}\right) \right\| \|T_{x_l} - T\|_{HS} \|I - \frac{\gamma_k}{m} \sum_{j=2}^{l-1} \frac{j-1}{l-1} \prod_{i=j+1}^{l-1} \left(I - \frac{\gamma_k}{m} T\right) T\| \\ &\leq \frac{l-1}{l} \|T_{x_l} - T\|_{HS} \prod_{i=l+1}^m \frac{i-1}{i} \\ &\leq \frac{l-1}{m} (\|T_{x_j}\|_{HS} + \|T\|_{HS}) \\ &\leq 2\kappa^2. \end{aligned}$$

Then applying Theorem A.1 to $\xi_{k,l}$, with probability greater than $1 - \delta$

$$\left\| \frac{1}{m} \sum_{l=3}^m \xi_{k,l} \right\| \leq \frac{16\kappa^2}{3\sqrt{m}} \log \frac{2}{\delta}. \quad (\text{C.14})$$

On the other hand,

$$\left\| \prod_{l=3}^m \frac{l-1}{l} \left(I - \frac{\gamma_k}{m} T_{x_l}\right) \frac{T_{x_2} - T}{2} \right\| \leq 2\kappa^2/m, \quad (\text{C.15})$$

therefore, combining (C.14) with (C.15), and recalling (C.13), with probability greater than $1 - \delta$

$$\frac{1}{m} \|Q_m T\|_{HS} \leq \left(\frac{2\kappa^2}{m^2} + \frac{16\kappa^2}{3\sqrt{m}} \log \frac{2}{\delta} \right) \kappa^2 \quad (\text{C.16})$$

The statement then follows recalling the decomposition in (C.8), and summing (C.16) with (C.12). \square

LEMMA C.6. *Let $(\gamma_k)_{k \in \mathbb{N}}$ be a sequence in $]0, m\kappa^{-2}[$. For any $\delta \in]0, 1[$ and $k \in \mathbb{N}$,*

$$\mathbb{P} \left(\|\hat{b}_k - b_k\|_{\mathcal{H}} \leq \frac{2\kappa^2 M^2}{m^2} + \frac{64\kappa^2 M^2}{9\sqrt{m}} \log \frac{2}{\delta} \right) \geq 1 - \delta \quad (\text{C.17})$$

Proof. Using the same notation as in Lemma C.5, from the definition of \hat{b}_k and b_k in equations (B.3) and (B.10) respectively, we have

$$\hat{b}_k - b_k = \frac{1}{m^2} \sum_{j=1}^m \hat{B}_{k,j} \sum_{l=1}^{j-1} S_{x_l}^* y_l - \frac{1}{m^2} \sum_{j=1}^m B_{k,j} \sum_{l=1}^{j-1} S^* f_{\rho}. \quad (\text{C.18})$$

Starting from this decomposition, the proof follows the same line as the one of Lemma C.5. \square

We are now ready to prove the probabilistic upper bound on the sample error. *Proof. of Theorem 6.1* By Lemma C.1 we get

$$\hat{f}_t - f_t = \sum_{k=0}^{t-1} \gamma_k \left[\prod_{j=k+1}^{t-1} (I - \gamma_j T + \gamma_j^2 A_j) \right] \chi_k \quad (\text{C.19})$$

with χ_k defined as in (C.2), for $\hat{f}_0 = f_0 = 0$. Hence, applying Lemma C.2,

$$\|\hat{f}_t - f_t\|_{\mathcal{H}} \leq \sum_{k=1}^{t-1} \gamma_k \|\chi_k\|_{\mathcal{H}}. \quad (\text{C.20})$$

Recalling that

$$\chi_k = (T - \hat{T})\hat{f}_k + \gamma_k(\hat{A}_k - A_k)\hat{f}_k + \left(\frac{1}{m} \sum_{i=1}^m \hat{S}_{x_i}^* y_i - S^* f_{\rho} \right) + \gamma_k(b_k - \hat{b}_k).$$

then, by Lemmas C.3, C.4, C.6, and C.5, we get

$$\begin{aligned} \|\chi_k\|_{\mathcal{H}} &\leq \left(\frac{16\kappa^2}{3\sqrt{m}} \log \frac{2}{\delta} + \gamma_k \left(\frac{2\kappa^4}{m^2} + \frac{64\kappa^4}{9\sqrt{m}} \log \frac{2}{\delta} \right) \right) \kappa M \sum_{i=0}^{k-1} \gamma_i \\ &\quad + \frac{16\kappa M}{3\sqrt{m}} \log \frac{2}{\delta} + \gamma_k \left(\frac{2\kappa^2 M^2}{m^2} + \frac{64\kappa^2 M^2}{9\sqrt{m}} \log \frac{2}{\delta} \right) \end{aligned}$$

The previous bound implies that, (if δ is sufficiently small) there exists a constant $C > 0$ such that

$$\|\chi_k\|_{\mathcal{H}} \leq \frac{C}{\sqrt{m}} \left(\log \frac{2}{\delta} \right) \sum_{i=0}^{k-1} \gamma_i. \quad (\text{C.21})$$

Therefore, from (C.20) we get

$$\|\hat{f}_t - f_t\|_{\mathcal{H}} \leq \frac{C}{\sqrt{m}} \left(\log \frac{2}{\delta} \right) \sum_{k=0}^{t-1} \gamma_k \sum_{i=0}^{k-1} \gamma_i.$$

Next, if $(\forall i \in \mathbb{N})$, $\gamma_i = \kappa^{-2}(i+1)^{-\theta}$, then $\sum_{i=0}^{k-1} \gamma_i \leq \frac{1}{\kappa^2}(1-\theta)k^{1-\theta}$ and thus

$$\sum_{k=0}^{t-1} \gamma_k \sum_{i=0}^{k-1} \gamma_i \leq \frac{1}{\kappa^4(1-\theta)} \sum_{k=0}^{t-1} (k+1)^{1-2\theta}.$$

An analogous computation shows that

$$\frac{1}{\kappa^4(1-\theta)} \sum_{k=0}^{t-1} (k+1)^{1-2\theta} \leq \frac{1}{2\kappa^4(1-\theta)^2} (t+1)^{2(1-\theta)},$$

which gives the statement. \square

Proof. of Theorem 6.2 Fix $m \in \mathbb{N}$, $m \geq 2$. Let Pf_ρ be the projection onto the closure (in $L^2(\mathcal{X}, \rho_{\mathcal{X}})$) of the range of S . By Remark B.3, we know that the t -th epoch of the incremental gradient descent iteration in (B.9) coincides with m steps of gradient descent with stepsize γ_t/m . More precisely, $(\forall t \in \mathbb{N})$, $f_t = h_{mt}$, where $h_0 = 0$ and

$$h_{k+1} = (I - \eta_k T)h_k + \eta_k S^* P f_\rho, \quad \eta_k = \frac{\gamma_t}{m}, \text{ for } k \in [mt, (m+1)t - 1].$$

From equation (A.1), it follows

$$h_{k+1} = \sum_{j=0}^k \prod_{i=j+1}^k (I - \eta_i T) \eta_j S^* P f_\rho.$$

Hence

$$\|Sh_{k+1} - Pf_\rho\|_\rho = \left\| \left(S \sum_{j=0}^k \eta_j \prod_{i=j+1}^k (I - \eta_i T) S^* - I \right) P f_\rho \right\|_\rho \quad (\text{C.22})$$

By the spectral theorem, see e.g. [17, equation (2.43)],

$$\|Sh_{k+1} - Pf_\rho\|_\rho = \left\| \left(L \sum_{j=0}^k \eta_j \prod_{i=j+1}^k (I - \eta_i L) - I \right) P f_\rho \right\|_\rho. \quad (\text{C.23})$$

Define the spectral function $r_t :]0, \|L\|[\rightarrow \mathbb{R}$

$$r_t(\lambda) = \lambda \sum_{j=0}^k \eta_j \prod_{i=j+1}^k (I - \eta_i \lambda) - 1. \quad (\text{C.24})$$

It follows from the definition that $Pf_\rho \in \overline{R(L)}$, for $\overline{R(S)} = \overline{R(L)}$ by [35, Theorem 11.2(b)]. Thus, $Pf_\rho \in N(L)^\perp$. Then, by (C.23) and definition of r_t , if $(\sigma_n)_{n \in \mathbb{N}}$ are the strictly positive eigenvalues of L , and $(v_n)_{n \in \mathbb{N}}$ is the corresponding family of eigenvectors in $L^2(\mathcal{X}, \rho_{\mathcal{X}})$,

$$\begin{aligned} \|Sh_{k+1} - Pf_\rho\|_\rho^2 &= \|r_t(L) P f_\rho\|_\rho^2 = \sum_{n=1}^{+\infty} \left(\sigma_n \sum_{j=0}^k \eta_j \prod_{i=j+1}^k (1 - \eta_i \sigma_n) - 1 \right)^2 |\langle P f_\rho, v_n \rangle|^2 \\ &= \sum_{n=1}^{+\infty} \left(\prod_{i=0}^k (1 - \eta_i \sigma_n) \right)^2 |\langle P f_\rho, v_n \rangle|^2 \end{aligned} \quad (\text{C.25})$$

$$(\text{C.26})$$

Note that $(\forall n \in \mathbb{N}), \eta_i \sigma_n \in]0, 1[$, and

$$0 \leq \prod_{i=0}^k (1 - \eta_i \sigma_n) \leq e^{-\sigma_n \sum_{i=0}^k \eta_i}.$$

Since $\sigma_n > 0$ and $\lim_{k \rightarrow \infty} \sum_{i=0}^k \eta_i = +\infty$, it follows that $\lim_{k \rightarrow +\infty} e^{-\sigma_n \sum_{i=0}^k \eta_i} = 0$. On the other hand,

$$0 \leq \sum_{n=1}^{+\infty} \left(\prod_{i=0}^k (1 - \eta_i \sigma_n) \right)^2 |\langle P f_\rho, v_n \rangle|^2 \leq \|P f_\rho\|_\rho^2. \quad (\text{C.27})$$

Hence, from equation (C.25)

$$\lim_{k \rightarrow +\infty} \|S h_{k+1} - P f_\rho\|_\rho^2 = \lim_{k \rightarrow +\infty} \sum_{n=1}^{+\infty} \left(\prod_{i=0}^k (1 - \eta_i \sigma_n) \right)^2 |\langle P f_\rho, v_n \rangle|^2 = 0$$

where the last equality follows by applying the dominated convergence theorem, which can be used thanks to (C.27). Equation (6.2) then follows recalling that $f_t = h_{mt}$.

If condition (4.2) holds, we can apply [37, Theorem 2.10] to $(h_k)_{k \in \mathbb{N}}$. We have

$$\mathcal{E}(f_t) - \inf_{f \in \mathcal{H}} \mathcal{E} = \mathcal{E}(h_{mt}) - \inf_{f \in \mathcal{H}} \mathcal{E} \leq (r/e)^{2r} \left(\sum_{j=0}^{t-1} \left(\sum_{i=jm}^{(j+1)m-1} \frac{\gamma_j}{m} \right) \right)^{-2r} = (r/e)^{2r} \left(\sum_{j=0}^{t-1} \gamma_j \right)^{-2r}. \quad (\text{C.28})$$

□

Appendix D. Consistency with respect to the norm in \mathcal{H} . In this section we collect two main theorems, which are the analogous of Theorems 4.1 and 4.2, respectively.

THEOREM D.1. *Under the same assumptions as in Theorem 4.1, if \mathcal{E} has minimizers on \mathcal{H} , and $f_{\mathcal{H}}$ is the one of minimal norm,*

$$\lim_{m \rightarrow \infty} \|\hat{f}_{t^*} - f_{\mathcal{H}}\|_{\mathcal{H}} = 0 \quad \text{almost surely.}$$

THEOREM D.2. *Under the same assumptions as in Theorem 4.2, suppose additionally that $r \geq 1/2$. Then \mathcal{E} has minimizers, so that $f_{\mathcal{H}}$ exists. Moreover, for some constant $C > 0$,*

$$\|\hat{f}_t - f_{\mathcal{H}}\|_{\mathcal{H}} \leq \frac{C}{2\kappa^4(1-\theta)^2} \left(\log \frac{2}{\delta} \right) \frac{(t+1)^{2(1-\theta)}}{m} + \frac{R((r-1/2)/e)^{(r-1/2)}}{\kappa^2(1-\theta)} t^{(-r+1/2)(1-\theta)}. \quad (\text{D.1})$$

If we choose the stopping rule $t^(m) = \lceil m^{\frac{1}{(3+2r)(1-\theta)}} \rceil$ then there exists a constant D such that, with probability at least $1 - \delta$,*

$$\|\hat{f}_{t^*(m)} - f_{\mathcal{H}}\|_{\mathcal{H}} \leq D \log \frac{2}{\delta} m^{-\frac{2r-1}{3+2r}}. \quad (\text{D.2})$$

The proof of the previous theorems rely on the following decomposition into sample ad approximation error

$$\|\hat{f}_t - f_{\mathcal{H}}\|_{\mathcal{H}} \leq \|\hat{f}_t - f_t\|_{\mathcal{H}} + \|f_t - f_{\mathcal{H}}\|_{\mathcal{H}}. \quad (\text{D.3})$$

To estimate the sample error, we can rely on Theorem 6.1. An upper bound of the approximation error is given in the following theorem

THEOREM D.3 (Approximation error in \mathcal{H}). *Let $f_0 = 0$, $(\gamma_t)_{t \in \mathbb{N}}$ be such that $(\forall t \in \mathbb{N}) \gamma_t \in]0, m\kappa^{-2}[$, $\sum_{t \in \mathbb{N}} \gamma_t = +\infty$ and $(f_t)_{t \in \mathbb{N}}$ be defined as in (B.9). Assume that (4.2) is satisfied for some $r \geq 1/2$. Then $f_{\mathcal{H}}$ exists and*

$$\|f_t - f_{\mathcal{H}}\|_{\mathcal{H}} \leq R((r - 1/2)/e)^{(r-1/2)} \left(\sum_{i=0}^{t-1} \gamma_i \right)^{-r+1/2}.$$

In addition, if $\gamma_t = \kappa^{-2}(t+1)^{-\theta}$ for some $\theta \in [0, 1[$

$$\|f_t - f_{\mathcal{H}}\|_{\mathcal{H}} \leq \frac{R((r - 1/2)/e)^{(r-1/2)}}{\kappa^2(1 - \theta)} t^{(-r+1/2)(1-\theta)}.$$

The theorem follows from [37, Theorem 2.10].

Proof. of Theorem D.1 Equation (D.1) immediately follows by applying Theorems 6.1 and D.3. Next let $t^*(m) = \lceil m^\alpha \rceil$. substituting this choice of t into the right hand side of (D.1), and minimizing over α , we get the linear equation

$$2\alpha(1 - \theta) - 1 = \alpha(-r + 1/2)(1 - \theta), \quad (\text{D.4})$$

whose solution is $\alpha = 1/((3 + 2r)(1 - \theta))$. \square