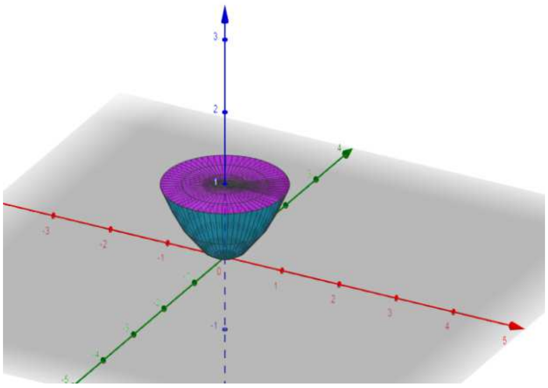
Resolución TP10:

Ejercicio 3 - a

Calcular el área de la superficie del volumen encerrado por paraboloide de ecuación $z = x^2 + y^2$, y el plano $z = 1$.

Resolviendo:



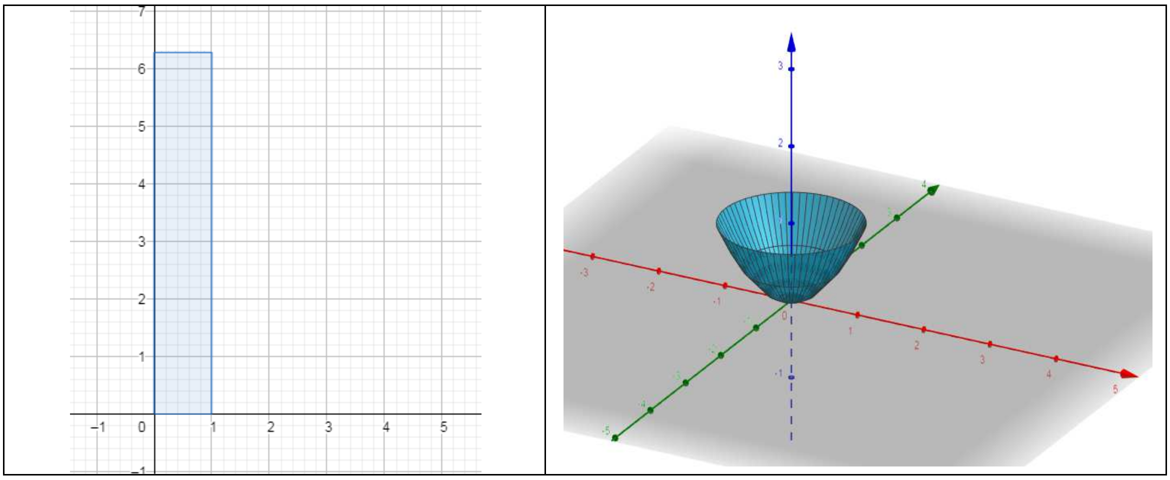
Sabemos que una superficie no suave se puede describir por la unión de superficies suaves de a trozos.

Por lo que se describe el área del paraboloide A_P y el área de la tapa A_T

$$Area(V) = A_P + A_T = \iint\limits_{R_P} ||P_u X P_v|| dudv + \iint\limits_{R_T} ||T_u X T_v|| dudv$$

Usando coordenadas polares:

$$A_P : \begin{cases} P(r,\alpha) = (rcos(\alpha), rsen(\alpha), r^2) \\ DomP = [0,2\pi]X[0,1] \end{cases} \rightarrow Area(P) = \iint_{[0,2\pi]X[0,1]} \left|\left|P_r X P_\alpha\right|\right| drd\alpha$$



$$P(r,\alpha) = (rcos(\alpha), rsen(\alpha), r^2)$$

$$P_r(r,\alpha) = (\cos(\alpha)\,, \operatorname{sen}(\alpha)\,, 2r)$$

$$P_\alpha(r,\alpha) = (-r\operatorname{sen}(\alpha)\,, r\cos(\alpha)\,, 0)$$

$$|P_r X P_\alpha| = \begin{bmatrix} i & j & k \\ \cos(\alpha) & \operatorname{sen}(\alpha) & 2r \\ -r\operatorname{sen}(\alpha) & r\cos(\alpha) & 0 \end{bmatrix}$$

$$|P_r X P_\alpha| = \left(\begin{bmatrix} \operatorname{sen}(\alpha) & 2r \\ r\cos(\alpha) & 0 \end{bmatrix}, - \begin{bmatrix} \cos(\alpha) & 2r \\ -r\operatorname{sen}(\alpha) & 0 \end{bmatrix}, \begin{bmatrix} \cos(\alpha) & \operatorname{sen}(\alpha) \\ -r\operatorname{sen}(\alpha) & r\cos(\alpha) \end{bmatrix} \right)$$

$$|P_r X P_\alpha| = \left(0 - 2r^2\cos(\alpha)\,, -\left(0 - (-2r^2\operatorname{sen}(\alpha))\right), r^2\cos(\alpha) - \ (-rsen(\alpha))\right)$$

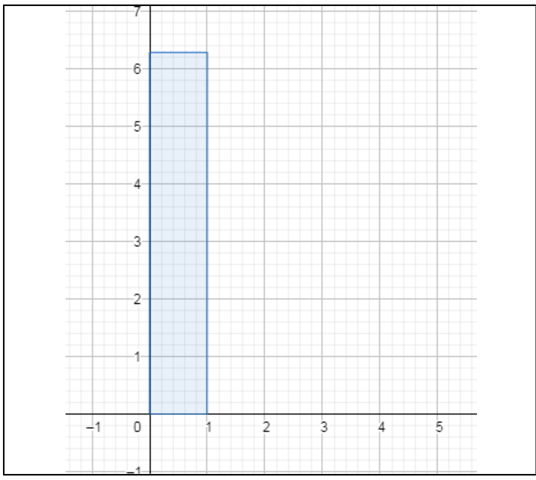
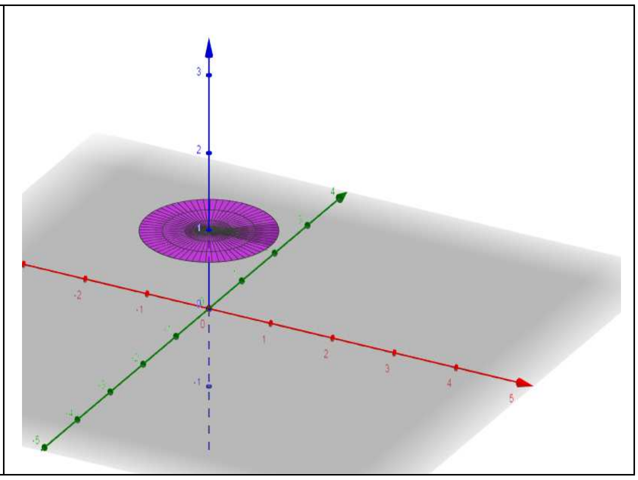
$$|P_r X P_\alpha| = (-2r^2\cos(\alpha)\,, -2r^2\operatorname{sen}(\alpha)\,, r)$$

$$\left|\left|P_r X P_\alpha\right|\right| = \sqrt{(-2r^2\cos(\alpha))^2 + (-2r^2\operatorname{sen}(\alpha))^2 + (r)^2} = r\sqrt{4r^2+1}$$

$$\iint_{[0,2\pi]X[0,1]} \left|\left|P_r X P_\alpha\right|\right| drd\alpha \ = \int_0^{2\pi}\int_0^1 r\sqrt{4r^2+1}\, dr\, d\alpha = \frac{5\sqrt{5}-1}{6}\pi$$

Usando coordenadas polares:

$$A_T : \begin{cases} T(r,\alpha) = (rcos(\alpha), rsen(\alpha), 1) \\ DomP = [0,2\pi]X[0,1] \end{cases} \rightarrow Area(T) = \iint\limits_{[0,2\pi]X[0,1]} \left| |T_r X T_\alpha| \right| dr d\alpha$$

$$T(r,\alpha) = (rcos(\alpha), rsen(\alpha), 1)$$

$$T_r(r,\alpha) = (\cos(\alpha)\,, \operatorname{sen}(\alpha)\,, 0)$$

$$T_\alpha(r,\alpha) = (-r\operatorname{sen}(\alpha)\,, r\cos(\alpha)\,, 0)$$

$$|T_r X T_\alpha| = \begin{bmatrix} i & j & k \\ \cos(\alpha) & \operatorname{sen}(\alpha) & 0 \\ -r\operatorname{sen}(\alpha) & r\cos(\alpha) & 0 \end{bmatrix}$$

$$|T_r X T_\alpha| = \left(\begin{bmatrix} \operatorname{sen}(\alpha) & 0 \\ r\cos(\alpha) & 0 \end{bmatrix}, -\begin{bmatrix} \cos(\alpha) & 0 \\ -r\operatorname{sen}(\alpha) & 0 \end{bmatrix}, \begin{bmatrix} \cos(\alpha) & \operatorname{sen}(\alpha) \\ -r\operatorname{sen}(\alpha) & r\cos(\alpha) \end{bmatrix} \right)$$

$$|P_r X P_\alpha| = (0,0,r)$$

$$\left| |T_r X T_\alpha| \right| = \sqrt{(r)^2} = r$$

$$\iint\limits_{[0,2\pi]X[0,1]} \left| |T_r X T_\alpha| \right| dr d\alpha \; = \int_0^{2\pi} \int_0^1 r\, dr\, d\alpha = \pi$$

Finalmente:

$$Area(V) = A_P + A_T = \frac{5\sqrt{5}-1}{6}\pi + \pi = \frac{5\sqrt{5}-1+6}{6}\pi = \frac{5}{6}\pi(\sqrt{5}+1)$$