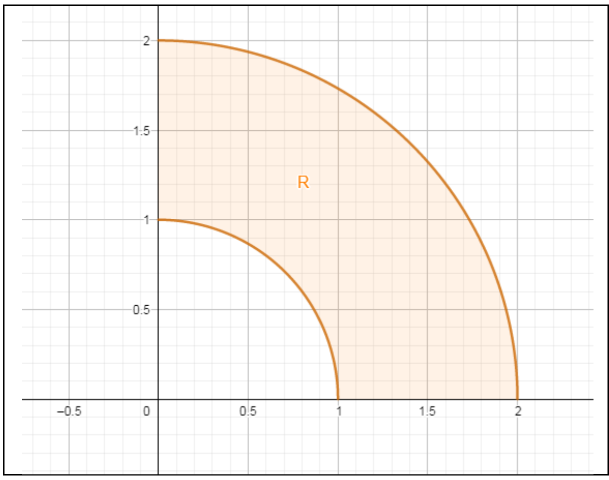
Resolución TP7:

Ejercicio 4 - c

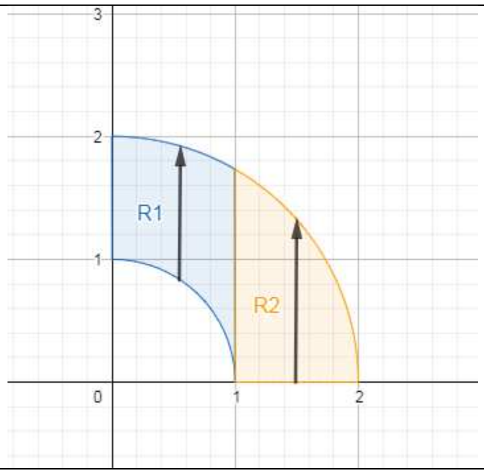
Graficar la región de integración R y resolver la integral I.

$$R: \{(x,y) \epsilon R^2 /\ 1 \ \leq x^2 + y^2 \leq 4 \ \wedge x \geq 0 \ \wedge y \geq 0\}$$

$$I = \iint_R x + y dxdy$$



Dado el Recinto R, vemos que no lo podemos integrar de una sola porcion ya sea usando integracion por tipo 1 o por tipo 2.
En tal caso se puede seleccionar cualquiera de los 2 para operar.



$$R1: \begin{cases} 0 < x < 1 \\ \sqrt{1-x^2} < y < \sqrt{4-x^2} \end{cases}$$

$$R2: \begin{cases} 1 < x < 2 \\ 0 < y < \sqrt{4-x^2} \end{cases}$$

$$I = \iint_R (x+y)dxdy = \iint_{R1} (x+y)dydx + \iint_{R2} (x+y)dydx$$

$$I_1 = \iint_{R1} (x+y)dydx = \int_{x=0}^{x=1} \int_{y=\sqrt{1-x^2}}^{y=\sqrt{4-x^2}} (x+y)dy\,dx$$

$$I_1 = \int_{x=0}^{x=1} \left[xy + \frac{y^2}{2}\right]_{y=\sqrt{1-x^2}}^{y=\sqrt{4-x^2}} \; dx$$

$$I_1 = \int_{x=0}^{x=1} \left[\left(x\sqrt{4-x^2} + \frac{(\sqrt{4-x^2})^2}{2}\right) - \left(x\sqrt{1-x^2} + \frac{(\sqrt{1-x^2})^2}{2}\right)\right] \; dx$$

$$I_1 = \int_{x=0}^{x=1} \left[\left(x\sqrt{4-x^2} + \frac{4-x^2}{2}\right) - \left(x\sqrt{1-x^2} + \frac{1-x^2}{2}\right)\right] \; dx$$

$$I_1 = \int_{x=0}^{x=1} x\sqrt{4-x^2} - x\sqrt{1-x^2} + \frac{4-x^2}{2} - \frac{1-x^2}{2} \; dx$$

$$I_1 = \int_{x=0}^{x=1} x\sqrt{4-x^2} - x\sqrt{1-x^2} + \frac{3}{2} \; dx$$

$$I_1 = \int_{x=0}^{x=1} x\sqrt{4-x^2}\, dx - \int_{x=0}^{x=1} x\sqrt{1-x^2} \; dx + \int_{x=0}^{x=1} \frac{3}{2} \; dx$$

Ver cálculos auxiliares

$$I_1 = \left(\frac{8}{3} - \sqrt{3}\right) - \left(\frac{1}{3}\right) + \left(\frac{3}{2}\right) = \frac{23}{6} - \sqrt{3}$$

$$I_2 = \iint_{R2} (x+y)dydx = \int_{x=1}^{x=2} \int_{y=0}^{y=\sqrt{4-x^2}} (x+y)dy\,dx$$

$$I_2 = \int_{x=1}^{x=2} \left[xy + \frac{y^2}{2}\right]_{y=0}^{y=\sqrt{4-x^2}} dx$$

$$I_2 = \int_{x=1}^{x=2} \left[\left(x\sqrt{4-x^2} + \frac{(\sqrt{4-x^2})^2}{2}\right) - \left(x0 + \frac{(0)^2}{2}\right)\right] dx$$

$$I_2 = \int_{x=1}^{x=2} x\sqrt{4-x^2} + 2 - \frac{x^2}{2} \quad dx$$

$$I_2 = \int_{x=1}^{x=2} x\sqrt{4-x^2}\,dx + \int_{x=1}^{x=2} 2 - \frac{x^2}{2} \quad dx$$

Ver cálculos auxiliares

$$I_2 = +\sqrt{3} + \frac{5}{6}$$

Finalmente:

$$I = \iint_{R} (x+y)dxdy = \iint_{R1} (x+y)dydx + \iint_{R2} (x+y)dydx$$

$$I = I_1 + I_2 = \frac{23}{6} - \sqrt{3} + \sqrt{3} + \frac{5}{6} = \frac{14}{3}$$

$$I = \frac{14}{3}$$

C/A 1

$$\int_{x=0}^{x=1} x\sqrt{4-x^2}\,dx$$

$sustitucion\ en\ x\sqrt{4-x^2}$

$$x = 2sen\ t$$

$$dx = 2cos\ tdt$$

$$4-x^2 = 4-4sen^2\ t = 4\cos^2 t$$

$$\sqrt{4-x^2} = 2cost$$

$$x\sqrt{4-x^2}dx = 2sent2cost2costdt = 8sentcos^2tdt$$

$$\int_{x=0}^{x=1} x\sqrt{4-x^2}\,dx = \int_{x=0}^{x=1} 8sentcos^2tdt$$

$sustitucion\ en\ sentcos^2tdt$

$$u = cos\ t$$

$$du = -sen\ tdt$$

$$\int_{x=0}^{x=1} 8sentcos^2tdt\ = \int_{x=0}^{x=1} -8u^2du$$

$$-\frac{8}{3}\left[u^3\right]_{x=0}^{x=1}$$

$$-\frac{8}{3}\left[\cos^3\left(arcsen\left(\frac{x}{2}\right)\right)\right]_{x=0}^{x=1}$$

$$-\frac{8}{3}\left[\cos^3\left(arcsen\left(\frac{1}{2}\right)\right)\ \ -\cos^3(arcsen(0))\ \ \right]$$

$$-\frac{8}{3}\left[\cos^3\left(\frac{\pi}{6}\right)-\cos^3(0)\right]$$

$$-\frac{8}{3}\left[\frac{3}{8}\sqrt{3}-1\right] = \frac{8}{3}-\sqrt{3}$$

C/A 2

$$\int_{x=0}^{x=1} x\sqrt{1-x^2}\ \ dx$$

$sustitucion\ en\ x\sqrt{1-x^2}$

$$x = sen\ t$$

$$dx = cos\ tdt$$

$$1 - x^2 = 1 - 1sen^2\ t = 1\cos^2 t$$

$$\sqrt{1-x^2} = cost$$

$$x\sqrt{1-x^2}dx = sentcostcostdt = sentcos^2tdt$$

$$\int_{x=0}^{x=1} x\sqrt{1-x^2}\ dx = \int_{x=0}^{x=1} sentcos^2tdt$$

$sustitucion\ en\ sentcos^2tdt$

$$u = cos\ t$$

$$du = -sen\ tdt$$

$$\int_{x=0}^{x=1} sentcos^2tdt =$$

$$\int_{x=0}^{x=1} -u^2du =$$

$$-\frac{1}{3}\left[u^3\right]_{x=0}^{x=1} =$$

$$-\frac{1}{3}\left[\cos^3\left(arcsen(x)\right)\ \ \right]_{x=0}^{x=1}$$

$$-\frac{1}{3}\left[\cos^3\left(arcsen(1)\right)\ \ -\cos^3\left(arcsen(0)\right)\ \ \right] =$$

$$-\frac{1}{3}\left[\cos^3\left(\frac{\pi}{2}\right) - \cos^3(0)\right] =$$

$$-\frac{1}{3}[0-1] = +\frac{1}{3}$$

C/A 3

$$\int_{x=1}^{x=2} x\sqrt{4-x^2}\,dx$$

$sustitucion\ en\ x\sqrt{4-x^2}$

$$\int_{x=1}^{x=2} x\sqrt{4-x^2}\,dx\ = -2\left[\cos^3\left(arcsen\left(\frac{x}{2}\right)\right)\ \right]_{x=1}^{x=2}$$

$$-\frac{8}{3}\left[\cos^3(arcsen(1))\ \ -\cos^3\left(arcsen\left(\frac{1}{2}\right)\right)\ \right]=$$

$$-2\left[\cos^3\left(\frac{\pi}{2}\right)-\cos^3\left(\frac{\pi}{6}\right)\right]=$$

$$-\frac{8}{3}\left[0-\frac{3}{8}\sqrt{3}\right]=+\sqrt{3}$$

C/A 4

$$\int_{x=1}^{x=2} 2-\frac{x^2}{2}\,dx\ = \left[2x-\frac{x^3}{6}\right]_{x=1}^{x=2} = \left(4-\frac{8}{6}\right)-\left(2-\frac{1}{6}\right)=\frac{5}{6}$$