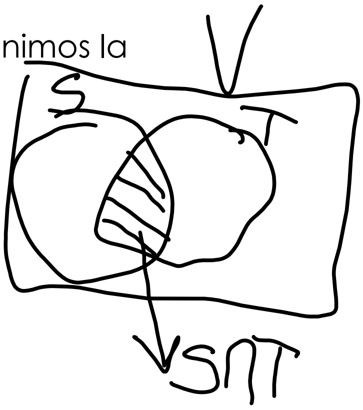
INTERSECCION DE SUBESPACIOS VECTORIALES

Sean S y T subespacios del mismo espacio vectorial V. Definimos la intersección como sigue:

$$S \cap T = \{v \in V, v \in S \wedge v \in T\}$$

Propiedad: $S \cap T$ es subespacio de V.



EJEMPLO 1

$S = \{(x, y, z, w) \in R^4 | x - 3w = 0\}$

$T = \{(x, y, z, w) \in R^4 | x + y - z = 0\}$

a) Hallar una base de S y una base de T.

S) $\{x - 3w = 0 \Rightarrow x = 3w$

$(x, y, z, w) = (3w, y, z, w) = w(3,0,0,1) + y(0,1,0,0) + z(0,0,1,0)$

$S = \text{gen}\{(3,0,0,1), (0,1,0,0), (0,0,1,0)\}$

M Corto $\begin{pmatrix} 3 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ LI

Base $S = \{(3,0,0,1), (0,1,0,0), (0,0,1,0)\}$

Dim $S = 3$

T) $\{x + y - z = 0$

$x = z - y$

$(x, y, z, w) = (z - y, y, z, w)$

$= z(1,0,1,0) + y(-1,1,0,0) + w(0,0,0,1)$

MC $\begin{pmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ LI $T = \{(1,0,1,0), (-1,1,0,0), (0,0,0,1)\}$

Dim $T = 3$

b) Hallar $S \cap T$. $v \in S \cap T \Leftrightarrow v \in S \wedge v \in T$

$$\boxed{\begin{matrix} S \\ T \end{matrix} \begin{cases} x - 3w = 0 \\ x + y - z = 0 \end{cases}} \quad \begin{pmatrix} 1 & 0 & 0 & -3 \\ 1 & 1 & -1 & 0 \end{pmatrix} \xrightarrow{f_2 - f_1} \begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & -1 & 3 \end{pmatrix}$$

$$\cancel{y} - z + 3w = 0 \qquad x = 3w \leftarrow x - 3w = 0$$

$$y = z - 3w$$

$$(x, y, z, w) = (3w, z - 3w, z, w) = w(3, -3, 0, 1) + z(0, 1, 1, 0)$$

$$S \cap T = \{(3, -3, 0, 1), (0, 1, 1, 0)\} \qquad \begin{array}{c} \\ \begin{pmatrix} 3 & -3 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array} \begin{array}{l} MC \\ LI \end{array}$$

$$\text{Dim}(S \cap T) = 2$$

SUMA

Dados S, T subespacios de V, se define la suma de los subespacios S y T

$$S + T = \{v \in V : v = v_1 + v_2,\ con\ v_1 \in S,\ v_2 \in T\}$$

Se puede demostrar que $S + T$ es un subespacio del espacio vectorial V.

base

Si conocemos conjuntos generadores de S y de T, podemos hallar generadores de la suma:

$$S = gen\{v_1, v_2, \dots, v_q\} \text{ y } T = gen\{w_1, w_2, \dots, w_r\} \Rightarrow S + T = gen\{v_1, v_2, \dots v_q, w_1, w_2, \dots, w_r\}$$

Para hallar la suma es usual buscar las bases de S y T. Como las bases son conjuntos generadores LI, si conocemos una base de cada subespacio podremos obtener un conjunto generador de la suma.

EJEMPLO 2

con generadores

S = <(1,0,-3) ; (0,1,2), (1,1,-1)>   T = {X ϵ R^3 / x + y=3x – z = 0}

Encontrar S+T y S∩T.

Buscar S y T

$$MC \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix} \xrightarrow{f_3 - f_1} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow{f_3 - f_2} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Base $S = \{(1,0,-3)\ (0,1,2)\}$

S∩T

$S = \{(1,0,-3), (0,1,2)\}$ $T = \{X \in R^3 \ \ x+y=0,\ 3x-z=0\}$

$(x,y,z) = a(1,0,-3) + b(0,1,2)$

$(x,y,z) = (a, b, -3a+2b) \to S$

$x+y=0$

$a+b=0$

$3x-z=0$

$3a-(-3a+2b)=0$

$6a-2b=0$

$$\begin{cases} a+b=0 \\ 6a-2b=0 \end{cases}$$

$$\begin{pmatrix} 1 & 1 \\ 6 & -2 \end{pmatrix} \underset{f_2-6f_1}{} \begin{pmatrix} 1 & 1 \\ 0 & -8 \end{pmatrix} \quad rg(A) = rg(M) = n = 2 \quad SCD$$

$$a = b = 0$$

$$(a, b, -3a+2b) = (0,0,0) \quad S \cap T = \{(0,0,0)\}$$

$$Dim(S \cap T) = 0$$

$S+T$

Buscar base de T
$$\begin{cases} x+y=0 \\ 3x-z=0 \end{cases}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & -1 \end{pmatrix} \underset{f_2-3f_1}{} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -3 & -1 \end{pmatrix} \quad \underbrace{-3y = z}_{x=-y}$$

$$(x,y,z) = (-y, y, -3y) = y(-1,1,-3)$$

$$T = \{(-1,1,-3)\} \quad Dim\, T = 1 \quad \Big/ \quad S = \{(1,0,-3)(0,1,2)\}$$

$$(x,y,z) = a(1,0,-3) + b(0,1,2) + c(-1,1,-3) \rightarrow S+T$$

$$S+T = gen\{(1,0,-3),(0,1,2),(-1,1,-3)\}$$

MC
$$\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ -1 & 1 & -3 \end{pmatrix} \underset{f_3+f_1}{} \begin{pmatrix} 1 & 0 & -3 \\ 0 & \boxed{1} & 2 \\ 0 & 1 & -6 \end{pmatrix} \underset{f_3-f_2}{} \underbrace{\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & -8 \end{pmatrix}}_{} \; LI$$

$S+T=\{(1,0,-3)\ (0,1,2)\ (-1,1,-3)\}$ $Dim\, S+T=3$

$S+T=\mathbb{R}^3$

$Dim(S+T) = Dim\, S + Dim\, T - Dim(S\cap T)$

$3 \stackrel{?}{=} 2 + 1 - 0$ ✓

Suma $S+T$ es suma directa $\Rightarrow Dim(S\cap T)=0$

$S\cap T=\{\vec{0}\}$

1. EJEMPLO 3

S=gen{($S=\left\{A\in R^{2x2}, \begin{pmatrix} a & b \\ b+a & a\end{pmatrix}\right\}$ $T=gen\{\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}2&1\\1&2\end{pmatrix}\}$

Hallar base de S∩T y S+T.

a) Hallar bases de S y T

$\begin{pmatrix} a & b \\ b+a & a\end{pmatrix} = a\begin{pmatrix}1&0\\1&1\end{pmatrix}+b\begin{pmatrix}0&1\\1&0\end{pmatrix}$

MC $\begin{pmatrix}1&0&1&1\\0&1&1&0\end{pmatrix}$ LI

$S=\left\{\begin{pmatrix}1&0\\1&1\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix}\right\}$ · $Dim\, S=2$.

base T MC $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 1 & 2 \end{pmatrix} \underset{f_3-2f_1}{} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \underset{f_3-f_2}{} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$$T=\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\} \quad \text{Dim } T = 2$$

$$S+T = \text{gen}\left\{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$$

MC $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \underset{f_3-f_1}{} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \underset{f_4-f_2}{} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$

TEOREMA DE LA DIMENSION PARA S Y T.

DIM (S+T)= DIM(S) + DIM(T) – DIM(S∩T)

$$S+T=\left\{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\} \quad \text{Dim}(S+T)=3$$

$$\text{Dim}(S+T) = \text{Dim } S + \text{Dim } T - \boxed{\text{Dim}(S\cap T)}$$
$$3 = 2 + 2 - \boxed{1} \quad ✓$$

$$S=\left\{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\} \qquad T=\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$$

S $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a & b \\ a+b & a \end{pmatrix}$$

$$T) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = p\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + q\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} p & q \\ q & p \end{pmatrix}$$

$$S \cap T \Rightarrow \begin{pmatrix} a & b \\ a+b & a \end{pmatrix} = \begin{pmatrix} p & q \\ q & p \end{pmatrix}$$

$$\left.\begin{matrix} a = p \\ b = q \\ a+b = q \end{matrix}\right\} \qquad p+q = q \Rightarrow \boxed{p=0} \Rightarrow \boxed{a=0}$$

$$a = p$$

$$\begin{pmatrix} a & b \\ a+b & a \end{pmatrix} = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} = b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} p & q \\ q & p \end{pmatrix} = \begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix} = q\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S \cap T = \left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\} \qquad \mathrm{Dim}(S \cap T) = 1$$