T P 05 Ej 8

Supongamos que la expresión $F(x,y,z)=0$ determina impícitamente funciones diferenciables

$x=x(y,z)$, $y=y(x,z)$, $z=z(x,y)$. Pruebe que:

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x}=-1$$

Resolución.

Del enunciado se desprende lo siguiente

$$F(x,y,z)=0\begin{cases}x=x(y,z) & diferenciable \quad (1)\\ y=y(x,z) & diferenciable \quad (2)\\ z=z(x,y) & diferenciable \quad (3)\end{cases}$$

De (1), considerando $P(y,z)=F(x(y,z),y,z)=0$, se deriva respecto de y, aplicando la regla de la cadena.

$$\frac{\partial P(y,z)}{\partial y}=\frac{\partial F(x,y,z)}{\partial x}\frac{\partial x}{\partial y}+\frac{\partial F(x,y,z)}{\partial y}\frac{\partial y}{\partial y}+\frac{\partial F(x,y,z)}{\partial z}\frac{\partial z}{\partial y}=0$$

De dónde resulta

$$\frac{\partial x}{\partial y}=-\frac{\frac{\partial F(x,y,z)}{\partial y}}{\frac{\partial F(x,y,z)}{\partial x}}=-\frac{F_y}{F_x}$$

De (2), considerando $Q(x,z)=F(x,y(x,z),z)=0$, se deriva respecto de z, aplicando la regla de la cadena.

$$\frac{\partial Q(x,z)}{\partial z}=\frac{\partial F(x,y,z)}{\partial x}\frac{\partial x}{\partial z}+\frac{\partial F(x,y,z)}{\partial y}\frac{\partial y}{\partial z}+\frac{\partial F(x,y,z)}{\partial z}\frac{\partial z}{\partial z}=0$$

De dónde resulta

$$\frac{\partial y}{\partial z}=-\frac{\frac{\partial F(x,y,z)}{\partial z}}{\frac{\partial F(x,y,z)}{\partial y}}=-\frac{F_z}{F_y}$$

De (3), considerando $\mathrm{R}(x,y)=F(x,y,z(x,y))=0$, se deriva respecto de x.

$$\frac{\partial R(x,y)}{\partial x}=\frac{\partial F(x,y,z)}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial F(x,y,z)}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial F(x,y,z)}{\partial z}\frac{\partial z}{\partial x}=0$$

De dónde resulta

$$\frac{\partial z}{\partial x}=-\frac{\frac{\partial F(x,y,z)}{\partial x}}{\frac{\partial F(x,y,z)}{\partial z}}=-\frac{F_x}{F_z}$$

Finalmente:

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x}=\left(-\frac{F_y}{F_x}\right)\left(-\frac{F_z}{F_y}\right)\left(-\frac{F_x}{F_z}\right)=-1$$