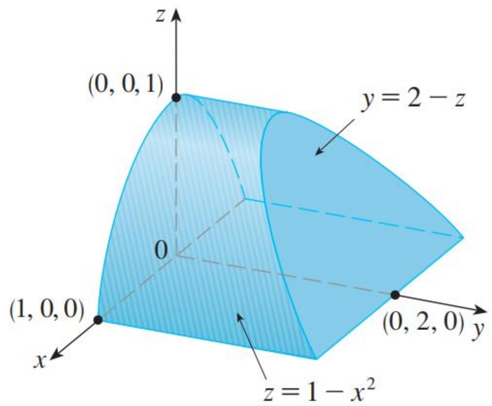
Teorema de Gauss.

Ejemplo 1

Calcular

$$\iint_S \vec{F} \bullet \vec{n}_{ext} dS$$

Donde $\vec{F}(x,y,z) = \left(xy\,, y^2 + e^x z^2\,, sen(xy)\right)$ donde S es la superficie es la región acotada por el cilindro parabólico $z = 1 - x^2$ y los planos $z = 0$, $y = 0$ y $y + z = 2$.



Teorema de Gauss (divergencia).

Sea el campo vectorial

$$F: U \subseteq \mathbb{R}^3 \to \mathbb{R}^3 : F(x,y,z) = \left(P_{(x,y,z)}, Q_{(x,y,x)}, R_{(x,y,z)}\right)$$

De clase C^1 en el conjunto abierto U de $\mathbb{R}^3$. Y sea la superficie cerrada

$$S = \Omega \subset U$$

Frontera del solido $\Omega \subset U$. entonces

$$\iint_S \vec{F} \bullet \vec{n} dS = \iiint_\Omega Div\,(F) dxdydz$$

Siendo $\vec{n}$ la normal exterior a la superficie S.

Entonces

$$F(x,y,z)=\big(P(x,y,z),Q(x,y,z),R(x,y,z)\big)$$

$$Div(F)=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}$$

$$\vec{F}(x,y,z)=\left(\underbrace{xy}_{P},\underbrace{y^2+e^xz^2}_{Q},\underbrace{sen(xy)}_{R}\right)$$

$$Div(F)=y+2y+0=3y$$

Por lo tanto,

$$\iint_S \vec{F}\bullet\vec{n}_{ext}dS=\iiint_\Omega Div\,(F)dxdydz=\iiint_\Omega 3y\,dxdydz$$

Donde

$$\Omega=\{(x,y,z)\in\mathbb{R}^3:\ -1\le x\le 1\,,0\le y\le 2-z\,,0\le z\le 1-x^2\}$$

$$3\int_{-1}^{1}\int_{0}^{1-x^2}\int_{0}^{2-z} y\ dydzdx$$

Resolviendo respecto de y

$$\int_0^{2-z} y\,dy=\frac{y^2}{2}\bigg|_0^{2-z}=\frac{(2-z)^2}{2}$$

$$3\int_{-1}^{1}\int_{0}^{1-x^2}\frac{(2-z)^2}{2}\ dzdx$$

Resolviendo respecto de z

$$\int_0^{1-x^2}\frac{(2-z)^2}{2}dz=-\frac{1}{6}(2-z)^3\bigg|_0^{1-x^2}=-\frac{1}{6}\big((x^2+1)-8\big)$$

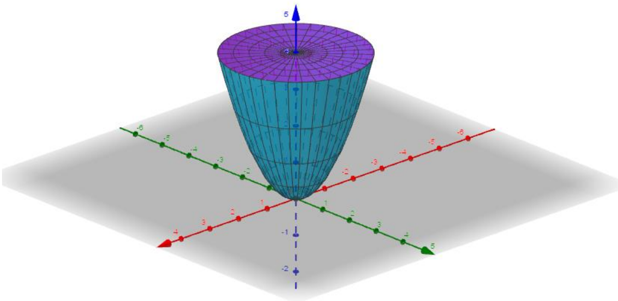
$$-\frac{1}{2}\int_0^1 x^6+3x^4+3x^2-7\ dx=\frac{184}{35}$$

Ejemplo 2

Calcular

$$\iint_S \vec{F} \bullet \vec{n}_{ext} dS$$

Donde $\vec{F}(x, y, z) = (\cos(z) + xy^2 , xe^{-z} , sen(y) + x^2z)$ donde S es la superficie del solido acotado por el paraboloide $z = x^2 + y^2$ y el plano $z = 4$.



$$Div(F) = y^2 + 0 + x^2$$

$$\iint_S \vec{F} \bullet \vec{n}_{ext} dS = \iiint_\Omega Div\,(F) dxdydz = \iiint_\Omega x^2 + y^2\, dxdydz$$

Usamos cambio de coordenadas cilíndricas

$$\begin{cases} x = r \cdot cos(\theta) \\ y = r \cdot sen(\theta) \\ z = z \end{cases}$$

$$J = r$$

Sabiendo que $x^2 + y^2 \leq z \leq 4$

Los límites de integración nos quedan:

$$0 \leq r \leq 2$$

$$0 \leq \theta \leq 2\pi$$

$$r^2 \leq z \leq 4$$

$$\iiint_{\Omega} x^2 + y^2 \, dxdydz = \int_0^2 \int_0^{2\pi} \int_{r^2}^{4} r^2 \cdot r \; dzd\theta dr$$

Resolviendo la integral respecto de z

$$\int_{r^2}^{4} r^3 \, dz = zr^3\big|_{r^2}^{4} = 4r^3 - r^5$$

Reemplazando en la integral original

$$\int_0^2 \int_0^{2\pi} 4r^3 - r^5 \, d\theta dr$$

$$\left(\int_0^2 4r^3 - r^5 \, dr\right) \cdot \left(\int_0^{2\pi} 1 d\theta\right)$$

$$\left(r^4 - \frac{1}{6} r^6 \bigg|_0^2\right) \cdot \left(\theta\big|_0^{2\pi}\right)$$

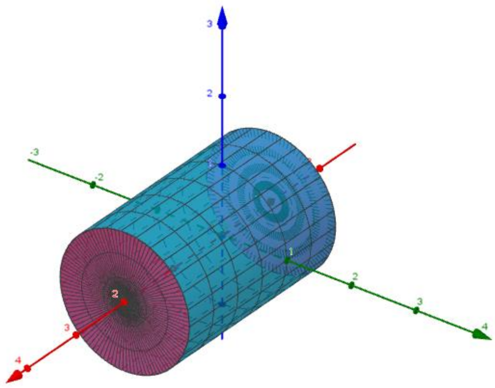
$$\frac{16}{3} \cdot 2\pi = \frac{32}{3}\pi$$

Ejemplo 3

Verificar el Teorema de Gauss para $F(x, y, z) = (3xy^2, xe^z, z^3)$ y la superficie S acotada por $y^2 + z^2 = 1$ y los planos $x = -1$ y $x = 2$.

Debo mostrar que

$$\iint_S \vec{F} \bullet \vec{n}_{ext} dS = \iiint_\Omega Div(F) dxdydz$$



Tenemos 3 superficies:

$$\Phi_1: disco\ de\ radio\ 1\ en\ x = -1$$

$$\Phi_2: disco\ de\ radio\ 1\ en\ x = 2$$

$$\Phi_3: al\ cilindro\ y^2 + z^2 = 1 \quad con\ -1 \leq x \leq 2$$

$$I = I_1 + I_2 + I_3$$

Parametrización de $\Phi_1: disco\ de\ radio\ 1\ en\ x = -1$

$$y^2 + z^2 = 1 \quad x = -1$$

$$\Phi_1 = (-1, y, z)$$

$$x = x$$
$$y = rcos(\theta)$$
$$z = rsen(\theta)$$

$$y^2 + z^2 = 1 \qquad x = -1$$

$$\Phi_1(r,\theta) = \left(-1\,, rcos(\theta)\,, rsen(\theta)\right)$$

$$0 \leq r \leq 1$$

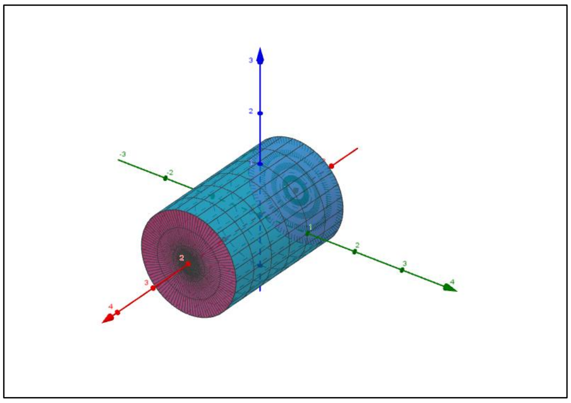
$$0 \leq \theta \leq 2\pi$$

Para la primera integral tenemos entonces:

$\vec{F}(x,y,z) = (3xy^2\,, xe^z\,, z^3)$

$\Phi_1(r,\theta) = \left(-1, r \cdot \cos(\theta)\,, r \cdot sen(\theta)\right)$

Buscamos los elementos para el armado de la primera integral



$F_{(\Phi_1(r,\theta))} = \left(-3r^2\cos^2(\theta)\,, -e^{rsen(\theta)}, r^3 sen^3(\theta)\right)$

$\Phi_r = \left(0, \cos(\theta)\ , sen(\theta)\right)$

$\Phi_\theta = \left(0, -r \cdot sen(\theta), r \cdot cos(\theta)\right)$

$$\Phi_r \times \Phi_\theta = \begin{vmatrix} i & j & k \\ 0 & \cos(\theta) & sen(\theta) \\ 0 & -r \cdot sen(\theta) & r \cdot cos(\theta) \end{vmatrix} = (r, 0{,}0)\ no\ es\ la\ direccion\ pedida$$

$F_{(\Phi(r,\theta))} \bullet (\Phi_r \times \Phi_\theta) = \left(-3r^2\cos^2(\theta)\,, -e^{rsen(\theta)}, r^3 sen^3(\theta)\right) \bullet (r, 0{,}0)$

$F_{(\Phi(r,\theta))} \bullet (\Phi_r \times \Phi_\theta) = -3r^3\cos^2(\theta)$

$$I_1 = -\int_0^1 \int_0^{2\pi} -3r^3\cos^2(\theta)\, drd\theta = \frac{3}{4}\pi$$

Parametrización de Φ_2: $disco\ de\ radio\ 1\ en\ x = 2$

$$\Phi_2(r,\theta) = \left(2\,, rcos(\theta)\,, rsen(\theta)\right)$$

$$0 \leq r \leq 1$$

$$0 \leq \theta \leq 2\pi$$

Para la primera integral tenemos entonces:

$\vec{F}(x,y,z) = (3xy^2\,, xe^z\,, z^3)$

$\Phi_2(r,\theta) = \big(2, r\cdot\cos(\theta)\,, r\cdot sen(\theta)\big)$

Buscamos los elementos para el armado de la primera integral:

$F_{(\Phi_2(r,\theta))} = \big(6r^2\cos^2(\theta)\,, 2e^{rsen(\theta)}, r^3sen^3(\theta)\big)$

$\Phi_r = \big(0, \cos(\theta)\ \ , sen(\theta)\big)$

$\Phi_\theta = \big(0, -r\cdot sen(\theta), r\cdot cos(\theta)\big)$

$\Phi_r \times \Phi_\theta = \begin{vmatrix} i & j & k \\ 0 & \cos(\theta) & sen(\theta) \\ 0 & -r\cdot sen(\theta) & r\cdot cos(\theta) \end{vmatrix} = (r, 0,0)$

$F_{(\Phi(r,\theta))} \bullet (\Phi_r \times \Phi_\theta) = \big(6r^2\cos^2(\theta)\,, 2e^{rsen(\theta)}, r^3sen^3(\theta)\big) \bullet (r, 0,0)$

$F_{(\Phi(r,\theta))} \bullet (\Phi_r \times \Phi_\theta) = 6r^3\cos^2(\theta)$

$$I_2 = \int_0^1 \int_0^{2\pi} 6r^3\cos^2(\theta)\, drd\theta = \frac{3}{2}\pi$$

Parametrización de Φ_3: $al\ cilindro\ y^2 + z^2 = 1 \quad con\ -1 \le x \le 2$

$$x = x$$
$$y = 1cos(\theta)$$
$$z = 1sen(\theta)$$

$$\Phi_3(x,\theta) = \big(x\,, cos(\theta)\,, sen(\theta)\big)$$

$$-1 \le x \le 2$$

$$0 \le \theta \le 2\pi$$

Para la primera integral tenemos entonces:

$\vec{F}(x,y,z) = (3xy^2\,, xe^z\,, z^3)$

$\Phi_3(x,\theta) = \big(x, \cos(\theta)\,, sen(\theta)\big)$

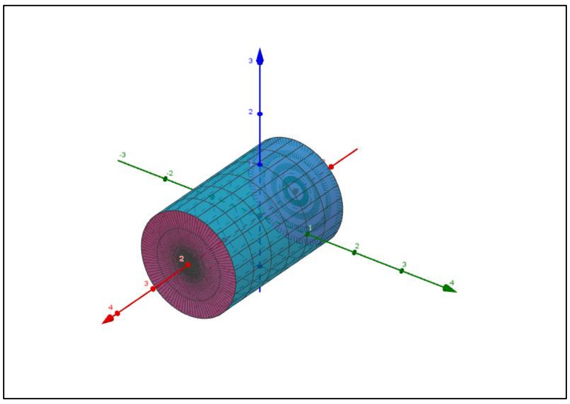
Buscamos los elementos para el armado de la integral:

$$F_{(\Phi_3(x,\theta))} = \left(3x\cos^2(\theta)\,, xe^{sen(\theta)}, sen^3(\theta)\right)$$

$$\Phi_x = (1,0\,,0)$$

$$\Phi_\theta = \left(0, -sen(\theta), cos(\theta)\right)$$

$$\Phi_x \times \Phi_\theta = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & -sen(\theta) & cos(\theta) \end{vmatrix} = \left(0, -\cos(\theta)\,, -sen(\theta)\right)\ \text{¿}es\ la\ direccion\ pedida?\quad NO$$



$$P = (0,0,1) \qquad \Phi_3(x,\theta) = \left(x, \cos(\theta)\,, sen(\theta)\right)$$

$$\Phi(x,\theta) = (0,0,1) = \Phi\left(0, \frac{\pi}{2}\right)$$

$$\Phi_x \times \Phi_\theta\left(0, \frac{\pi}{2}\right) = (0,0,-1)$$

$$F_{(\Phi(x,\theta))} \bullet (\Phi_x \times \Phi_\theta) = \left(3x\cos^2(\theta)\,, xe^{sen(\theta)}, sen^3(\theta)\right) \bullet \left(0, -\cos(\theta)\,, -sen(\theta)\right)$$

$$F_{(\Phi(x,\theta))} \bullet (\Phi_x \times \Phi_\theta) = -x \cdot cos(\theta) \cdot e^{sen(\theta)} - sen^4(\theta)$$

$$I_3 = - \int_{-1}^{2}\int_{0}^{2\pi} -x \cdot cos(\theta) \cdot e^{sen(\theta)} - sen^4(\theta)\, dxd\theta =$$

$$I_3 = \int_{-1}^{2}\int_{0}^{2\pi} x \cdot cos(\theta) \cdot e^{sen(\theta)} dxd\theta + \int_{-1}^{2}\int_{0}^{2\pi} sen^4(\theta)\, dxd\theta = \frac{9}{4}\pi$$

$$I = I_1 + I_2 + I_3 = \frac{3}{4}\pi + \frac{3}{2}\pi + \frac{9}{4}\pi = \frac{9}{2}\pi$$

Cálculo de la integral de superficie usando el Teorema de Gauss

$$F(x,y,z) = (3xy^2 \,, xe^z \,, z^3)$$

$$Div(F) = 3y^2 + 0 + 3z^2$$

$$3\iiint_{\Omega} y^2 + z^2 \, dxdydz$$

Usando coordenadas cilíndricas.

$$\begin{cases} x = x \\ y = r \cdot cos(\theta) \\ z = r \cdot sen(\theta) \end{cases}$$

Los límites de integración nos quedan:

$$-1 \leq x \leq 2$$

$$0 \leq r \leq 1$$

$$0 \leq \theta \leq 2\pi$$

$$J = r$$

$$3\iiint_{\Omega} y^2 + z^2 \, dxdydz = 3\int_{-1}^{2}\int_{0}^{1}\int_{0}^{2\pi} r^2 \cdot r \, d\theta drdx$$

$$3\int_{-1}^{2}\int_{0}^{1}\int_{0}^{2\pi} r^3 \, d\theta drdx$$

$$3 \cdot \left(\int_{0}^{2\pi} d\theta\right)\left(\int_{0}^{1} r^3 \, dr\right) \cdot \left(\int_{-1}^{2} dx\right)$$

$$3 \cdot (\theta|_0^{2\pi}) \cdot \left(\frac{1}{4} r^3\Big|_0^1\right) \cdot (x|_{-1}^2)$$

$$3 \cdot (2\pi) \cdot \left(\frac{1}{4}\right) \cdot (3) = \frac{9}{2}\pi$$