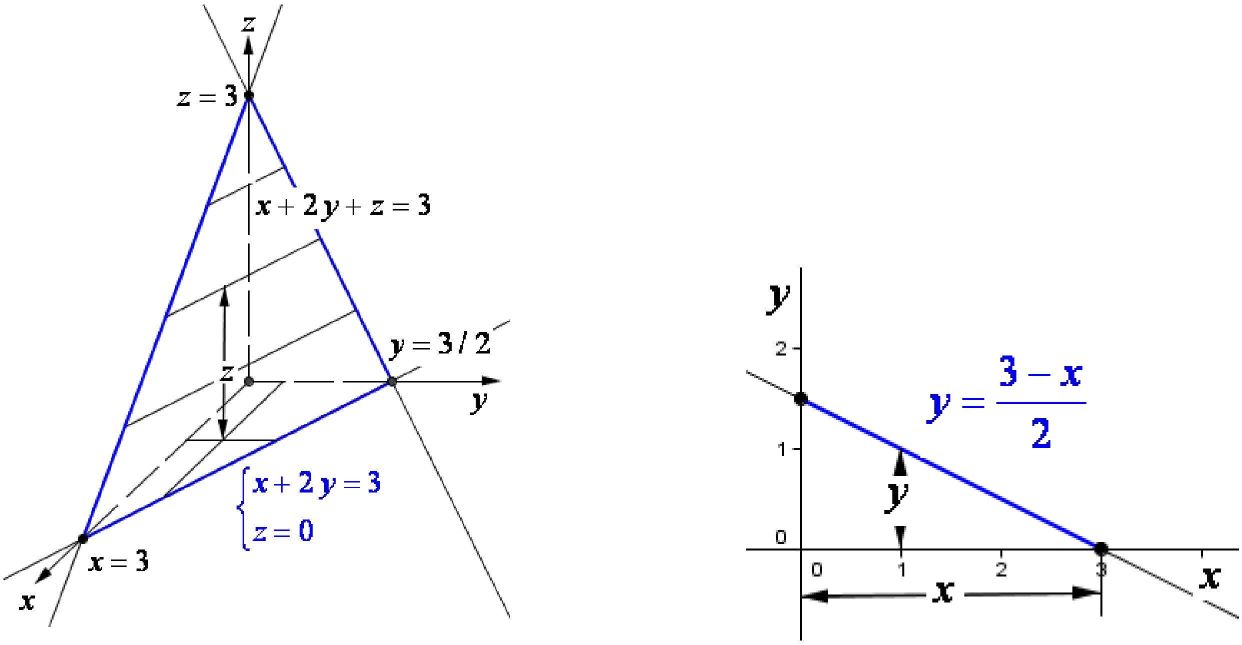
TP 7. Ejercicios adicionales integrales triples-03

1. Calcular: $\iiint_S (x+y+z)\,dxdydz$

Donde S es el volumen limitado por los planos coordenados y el plano de ecuación:

$$x+2y+z=3$$



De: $x+2y+z=3 \rightarrow z=3-x-2y$

De ambos gráficos, en la región S se verifica:

$$0 \le z \le 3-x-2y$$
$$0 \le y \le \frac{3-x}{2}$$
$$0 \le x \le 3$$

$$\therefore \iiint_S (x+y+z)\,dxdydz = \int_{x=0}^{3}\left\{\int_{y=0}^{\frac{3-x}{2}} \underbrace{\left[\int_{z=0}^{(3-x-2y)} (x+y+z)dz\right]}_{*} dy\right\} dx$$

$$*= \left.\left((x+y)z+\frac{z^2}{2}\right)\right|_{z=0}^{3-x-2y} = (x+y)(3-x-2y)+\frac{(3-x-2y)^2}{2}$$
$$=\frac{9}{2}-\frac{x^2}{2}-xy-3y$$

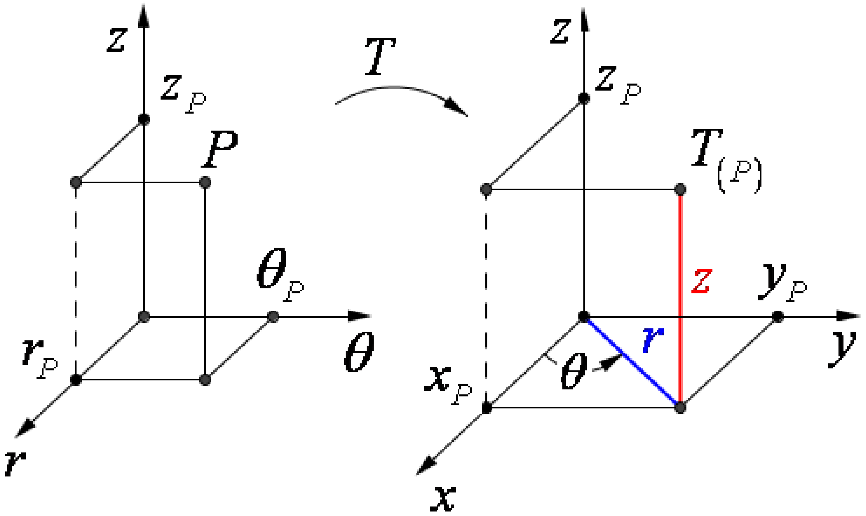
Resta calcular

$$\int_{x=0}^{3}\left[\int_{y=0}^{\left(\frac{3-x}{2}\right)}\left(\frac{9}{2}-\frac{x^2}{2}-xy-3y\right)dy\right]dx=\int_{x=0}^{3}\left(\frac{9}{2}y-\frac{-x^2}{2}y-x\frac{y^2}{2}-3\frac{y^2}{2}\right)\Bigg|_{y=0}^{\frac{3-x}{2}}dx$$

$$=\int_{x=0}^{3}\left[\frac{x^3-3x^2-9x+27}{8}\right]dx=\frac{135}{32}$$

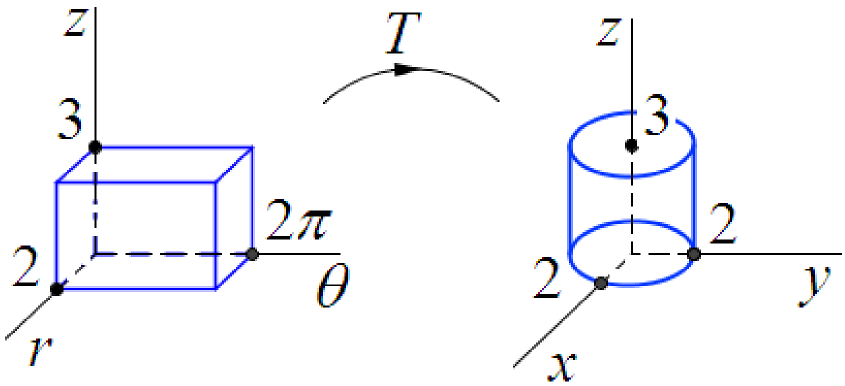
Transformación basada en coordenadas cilíndricas

$$T: \mathbb{R}^3 \to \mathbb{R}^3$$
$$T(r,\theta,z) = \big(x(r,\theta,z), y(r,\theta,z), z(r,\theta,z)\big) = (r\cos\theta\,, r sen\theta, z)$$



Ejemplos

A)



Región original

Región transformada

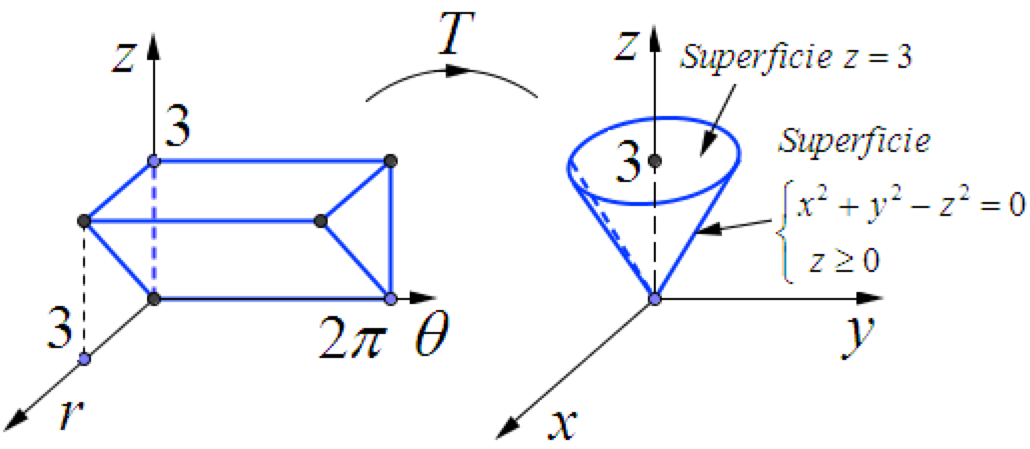
Paralelepípedo de lados $2, 2\pi$ y 3.

Cilindro de radio 2 y altura 3.

$$0 \le r \le 2$$
$$0 \le \theta \le 2\pi$$
$$0 \le z \le 3$$

$$0 \le x^2 + y^2 \le 4$$
$$0 \le z \le 3$$

B)



Cuña

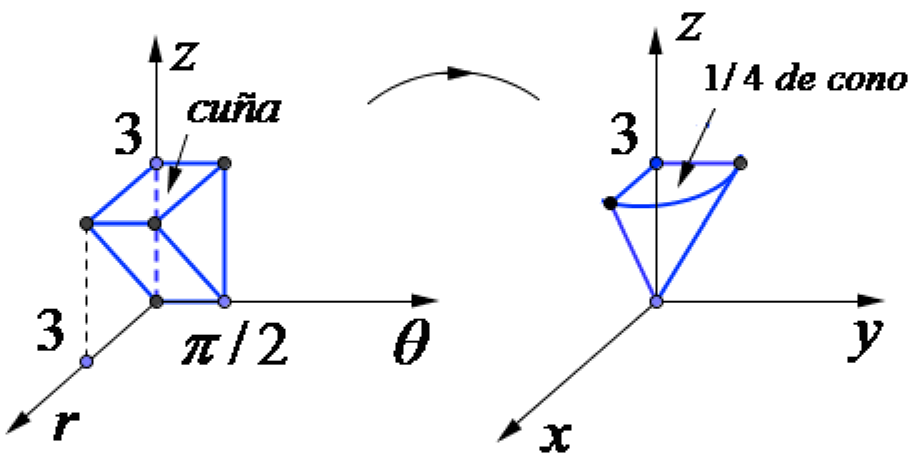
Cono

$$0 \le r \le 3$$
$$0 \le \theta \le 2\pi$$

$r \le z \le 3$

$$\sqrt{x^2+y^2} \le z \le 3$$
$$0 \le x^2+y^2 \le 9$$

C)



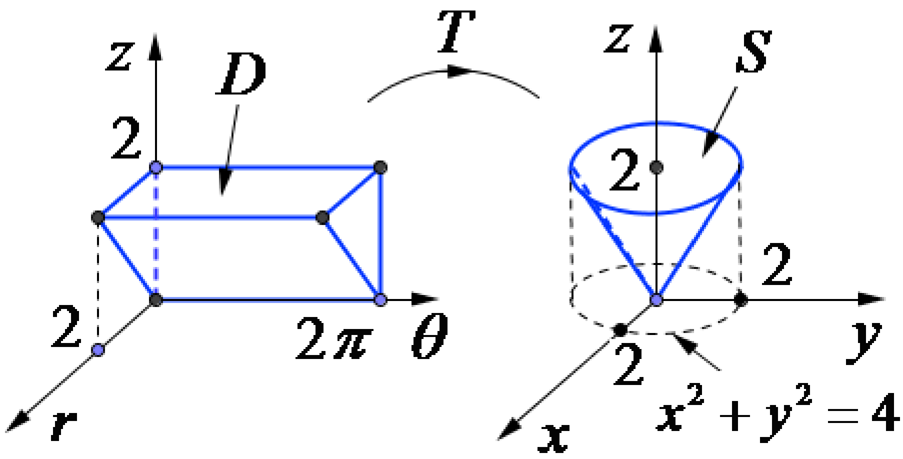
$$r \le z \le 3$$
$$0 \le r \le 3$$

$0 \le \theta \le \pi/2$

$$\sqrt{x^2+y^2} \le z \le 3$$
$$0 \le y \le \sqrt{9-x^2}$$
$$0 \le x \le 3$$

2. Calcular: $\iiint_S [1 + (x^2 + y^2)]\, dxdydz$

Donde S es el cono limitado por el plano de ecuación: $z = 2$
y la superficie de ecuación: $x^2 + y^2 - z^2 = 0, z \geq 0$



Con la transformación ……..$T = \begin{cases} x = r\cos\theta \\ y = r sen\theta \\ z = z \end{cases}$ $\left\| \frac{\partial(x,y,z)}{\partial(r,\theta,z)} \right\| = r$

Como $x^2 + y^2 \leq z^2 \leq 4 \to \sqrt{x^2 + y^2} \leq z \leq 2$

Resulta:

$$D = \begin{cases} r \leq z \leq 2 \\ 0 \leq r \leq 2 \\ 0 \leq \theta \leq 2\pi \end{cases}$$

La región *D* se aplica sobre la región *S*, *T(D)=S*.

$$\iiint_S \left(1 + (x^2 + y^2)\right) dxdydz = \iiint_D (1 + r^2)\, r dr d\theta dz =$$

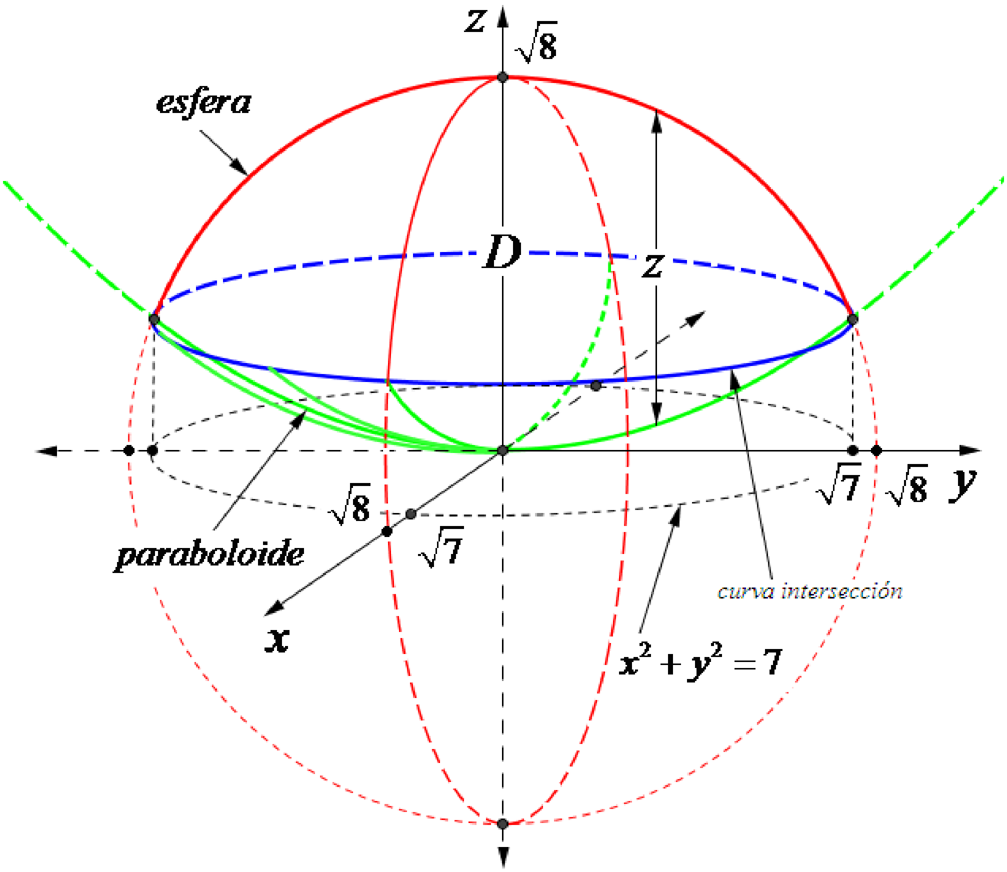
$$= \int_{\theta=0}^{2\pi} \left[\int_{r=0}^{2} \left\{ \int_{z=r}^{2} (r + r^3) dz \right\} dr \right] d\theta$$

$$= 2\pi \int_{r=0}^{2} (r + r^3)\, (z|_{z=r}^{2}) dr = 2\pi \int_{r=0}^{2} (r + r^3)\, (2 - r) dr =$$

$$= 2\pi \int_{r=0}^{2} (2r + 2r^3 - r^2 - r^4)\, dr = \frac{88}{15}\pi$$

3) Calcular el volumen del sólido D, limitado por las superficies:
S_1: $7z = x^2 + y^2$, y S_2: $x^2 + y^2 + z^2 = 8$,
y ubicado en el semiespacio definido por la inecuación $z \geq 0$.

S_1 es un paraboloide elíptico.
S_2 es la superficie de una esfera de radio $\sqrt{8}$.



La intersección de S_1 y S_2, se determina con el sistema de ecuaciones:

$$\begin{cases} x^2 + y^2 + z^2 = 8 \\ 7z = x^2 + y^2 \end{cases}$$

Queda $7z + z^2 = 8$, $z^2 + 7z - 8 = 0 \rightarrow z = \frac{7 \pm \sqrt{(-7)^2 - 4.1.(-8)}}{2}$

$$z = \frac{-7 \pm \sqrt{81}}{9} = \begin{cases} 1 \\ -8 \end{cases}$$

Como $z \geq 0$, queda $z = 1$, luego $x^2 + y^2 = 7$
$\therefore Si(x, y, z) \in D$, debe cumplirse:

De S_1, $z = \frac{x^2+y^2}{7}$

De S_2: $z = \sqrt{8 - (x^2 + y^2)}$

Entonces $\frac{x^2+y^2}{7} \leq z \leq \sqrt{8-(x^2+y^2)}(1)$

Con $0 \leq x^2 + y^2 \leq 7 \rightarrow 0 \leq \sqrt{x^2+y^2} \leq \sqrt{7}(2)$

Usando la transformación basada en coordenadas cilíndricas

$$T = \begin{cases} x = r\cos\theta \\ y = rsen\theta \\ z = z \end{cases} \qquad \left\| \frac{\partial(r,\theta,z)}{\partial(x,y,z)} \right\| = r$$

Resultan de (1) y (2) las siguientes inecuaciones del recinto H en (r,θ,z):

$$H = \begin{cases} \frac{r^2}{7} \leq z \leq \sqrt{8-r^2} \\ 0 \leq r \leq \sqrt{7} \\ 0 \leq \theta \leq 2\pi \end{cases}$$

∴ el volumen del sólido *D* resulta igual a:

$$Vol(D) = \iiint_D dxdydz = \iiint_H rdrd\theta dz = \int_{\theta=0}^{2\pi} \left\{ \int_{r=0}^{\sqrt{7}} r. \left[\int_{z=\frac{r^2}{7}}^{\sqrt{8-r^2}} dz \right] dr \right\} d\theta =$$

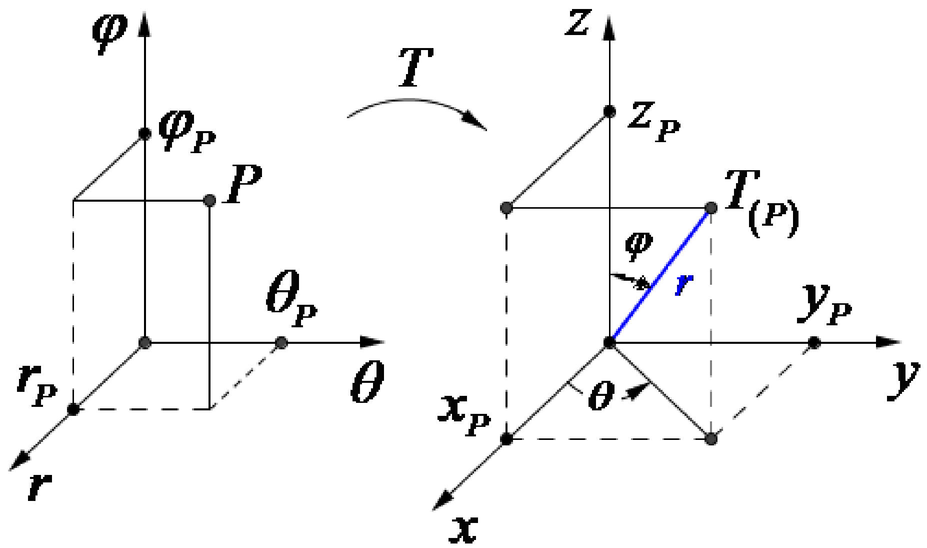
$$= 2\pi \int_{r=0}^{\sqrt{7}} r. \left(z|_{\frac{r^2}{7}}^{\sqrt{8-r^2}} \right) dr = 2\pi \int_{r=0}^{\sqrt{7}} r \left(\sqrt{8-r^2} - \frac{r^2}{7} \right) dr = \left(\frac{32\sqrt{2}}{3} - \frac{25}{6} \right) \pi$$

Transformación basada en coordenadas esféricas

$$T: \mathbb{R}^3 \to \mathbb{R}^3$$

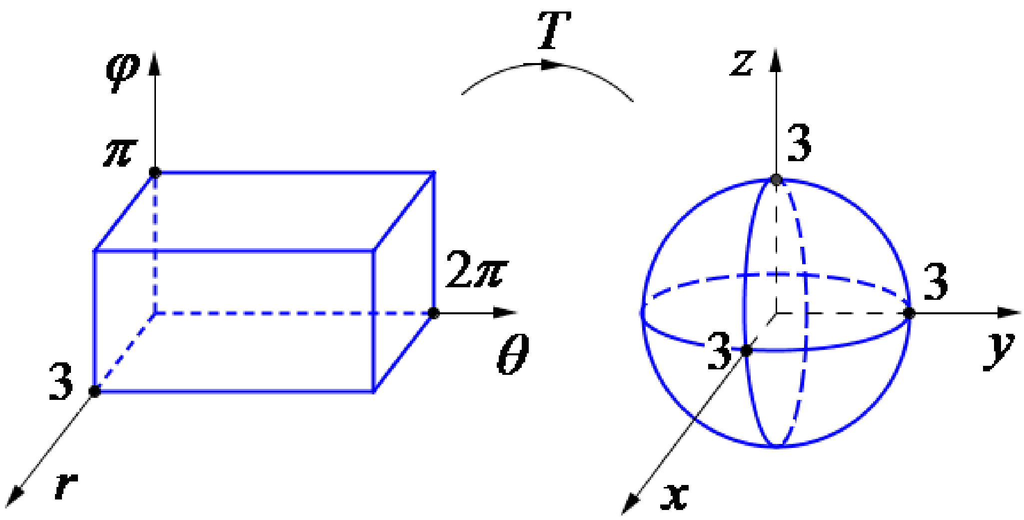
$$T_{(r,\theta,\phi)} = \big(x(r,\theta,\phi), y(r,\theta,\phi), z(r,\theta,\phi)\big)$$

$$T = \begin{cases} x(r,\theta,\phi) = r\cos\theta\,.sen\phi \\ y(r,\theta,\phi) = rsen\theta.sen\phi \\ z(r,\theta,\phi) = r.\cos\phi \end{cases}$$



Ejemplos

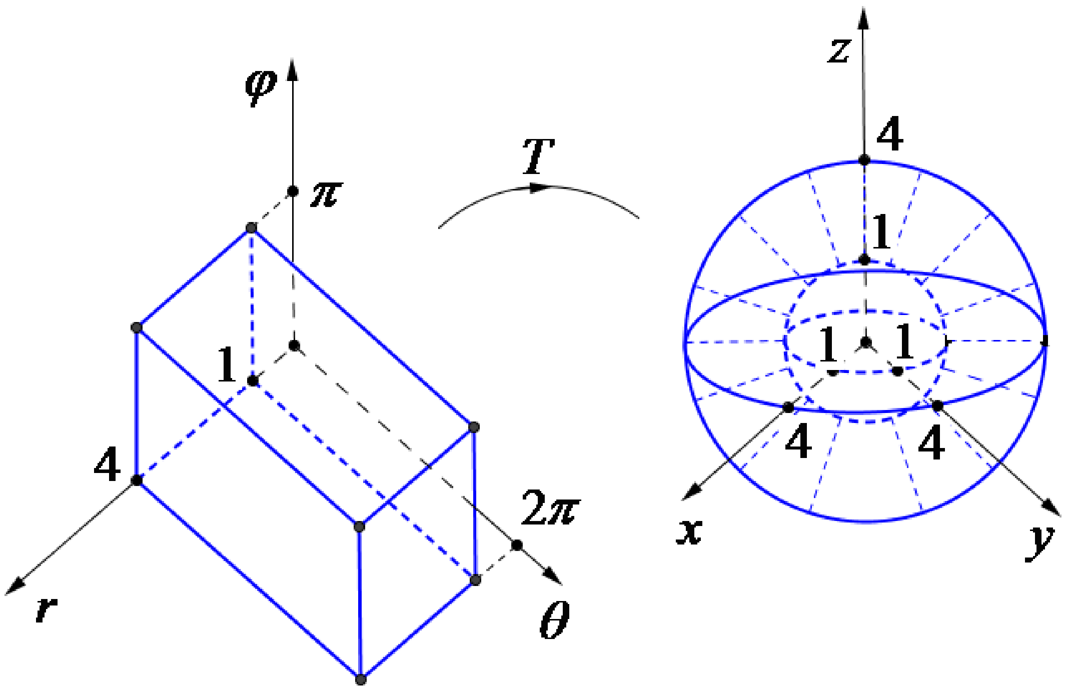
A)



$0 \leq r \leq 3$
$0 \leq \theta \leq 2\pi$
$0 \leq \phi \leq \pi$

$0 \leq x^2 + y^2 + z^2 \leq 9$

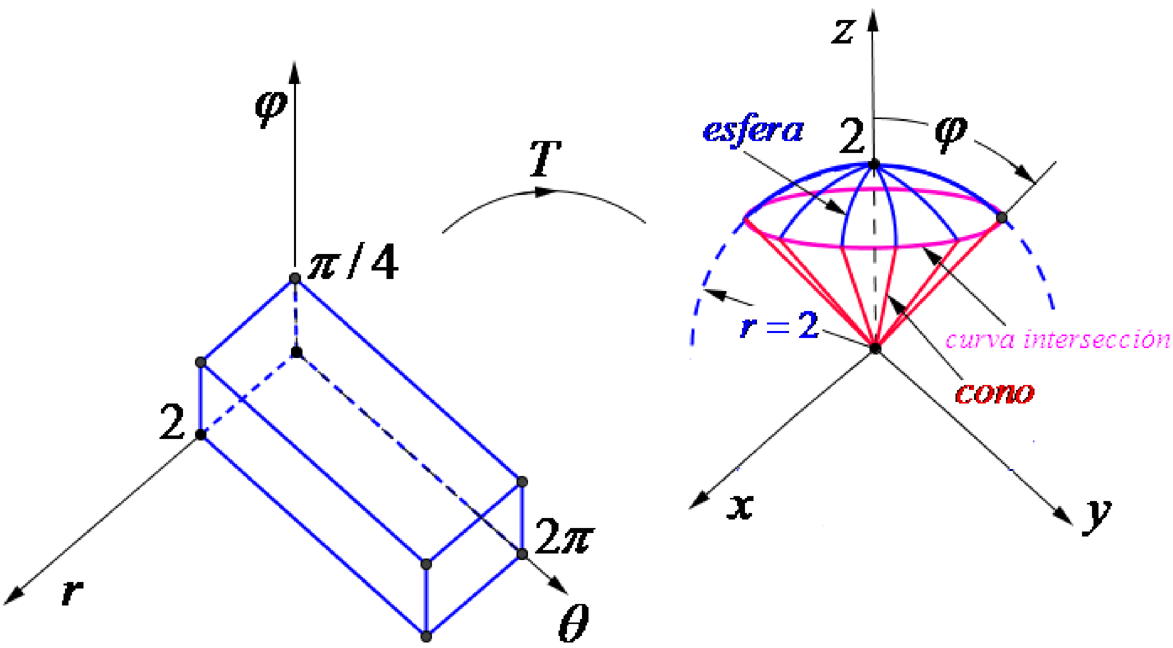
B)



$1 \leq r \leq 4$
$0 \leq \theta \leq 2\pi$
$0 \leq \phi \leq \pi$

$1 \leq x^2 + y^2 + z^2 \leq 16$

C)



$0 \leq r \leq 2$
$0 \leq \theta \leq 2\pi$
$0 \leq \phi \leq \frac{\pi}{4}$

$Esfera: x^2 + y^2 + z^2 = 4$
$Cono: z = \sqrt{x^2 + y^2}$
$\sqrt{x^2 + y^2} \leq z \leq \sqrt{4 - (x^2 + y^2)}$

4) Calcular: $\iiint_V \sqrt{x^2+y^2}\, dxdydz$

Donde: $V=\{(x,y,z)/x^2+y^2+z^2\leq 9; z\geq 0\}$

Usando la transformación T basada en coordenadas esféricas, resulta: $V=T(H)$

$$T=\begin{cases} x(r,\theta,\phi)=r\cos\theta\,.sen\phi \\ y(r,\theta,\phi)=rsen\theta.sen\phi \\ z(r,\theta,\phi)=r.\cos\phi \end{cases} \qquad \left\|\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\right\|=|-r^2sen\phi|=r^2sen\phi$$

Donde H, en el espacio de coordenadas (r,θ,ϕ) , es el sólido definido por:

$$H=\begin{cases} 0\leq r\leq 3 \\ 0\leq\theta\leq 2\pi \\ 0\leq\phi\leq\frac{\pi}{2} \end{cases}$$

Además: $\sqrt{x^2+y^2}=r\ sen\phi$

$$\therefore \iiint_V \sqrt{x^2+y^2}dxdydz=\iiint_H r.sen\phi.r^2sen\phi drd\varphi d\theta=\int_{\theta=0}^{2\pi}\left\{\int_{\phi=0}^{\frac{\pi}{2}}\left[\int_{r=0}^{3} r^3sen^2\phi dr\right]d\phi\right\}d\theta$$

$$=\left[\int_0^{2\pi}d\theta\right].\left[\int_0^{\frac{\pi}{2}}sen^2\phi d\phi\right].\left[\int_0^{3}r^3dr\right]=2\pi.\left[\frac{1}{2}(\phi-sen\phi.\cos\phi)\Big|_{\theta=0}^{\frac{\pi}{2}}\right].\left[\frac{r^4}{4}\Big|_{r=0}^{3}\right]$$

$$=2\pi.\frac{1}{2}\frac{\pi}{2}.\frac{3^4}{4}=\frac{81}{8}\pi^2$$

5) Calcular: $\iiint_V \frac{e^z}{\sqrt{x^2+y^2+z^2}}dxdydz *$

Donde $V = \{(x,y,z)/4 \leq x^2+y^2+z^2 \leq 9, z \geq 0\}$

Usando coordenadas esféricas $V = T(H)$

$$T = \begin{cases} x(r,\theta,\phi) = r\cos\theta \,.\, sen\phi \\ y(r,\theta,\phi) = rsen\theta . sen\phi \\ z(r,\theta,\phi) = r.\cos\phi \end{cases} \qquad \left\|\frac{\partial(x,y,z)}{\partial(r,\theta\phi)}\right\| = r^2 sen\phi$$

Donde H está definido por :

$$H = \begin{cases} 2 \leq r \leq 3 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \frac{\pi}{2} \end{cases}$$

Vale: $\sqrt{x^2+y^2+z^2} = r$

$$\therefore * = \iiint_H \frac{e^{r\cos\phi}}{r}.r^2 sen\phi dr d\theta d\phi = \int_{\theta=0}^{2\pi}\left\{\int_{r=2}^{3}\left[\int_{\phi=0}^{\frac{\pi}{2}} e^{r\cos\phi} rsen\phi d\phi\right]dr\right\}d\theta (**)$$

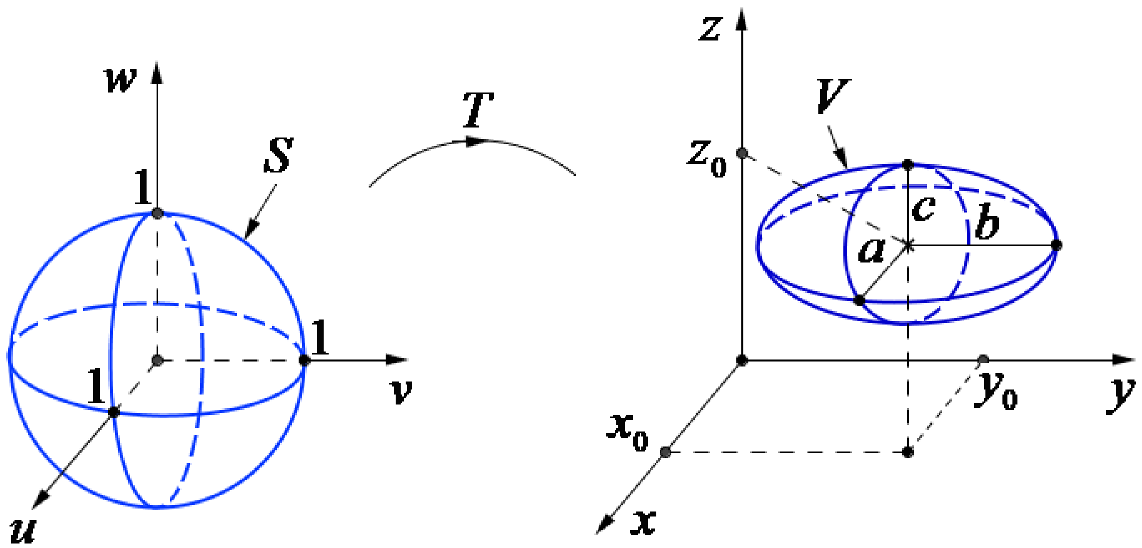
Haciendo $u = r\cos\phi \rightarrow du = -rsen\phi d\phi \rightarrow -du = rsen\phi d\phi$

$$\int e^{r\cos\phi} rsen\phi d\phi = \int e^u(-du) = -e^u = -e^{r\cos\phi}$$

$$** = 2\pi\int_{r=2}^{3}\left(\left(-e^{r\cos\phi}\right)\Big|_{\phi=0}^{\frac{\pi}{2}}\right)dr = 2\pi\int_{r=2}^{3}\left(-e^0-(-e^r)\right)dr = 2\pi\int_{r=2}^{3}(e^r - i)\,dr$$
$$= 2\pi(e^r - r)\big|_{r=2}^{3} = [e^3 - 3 - (e^2 - 2)] = 2\pi(e^3 - e^2 - 1)$$

6) Si V es el sólido cuyos puntos satisfacen: $\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}+\frac{(z-z_0)}{c^2}\leq 1$

Hallar su volumen



Haciendo $(x,y,z)=(au+x_0,bv+y_0,cw+z_0)$

La esfera S, de radio 1 y centro en $(0,0,0)$, se transforma en V

Vale: $\left\|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right\|=a.b.c \quad (verificar)$

Empleado además coordenadas esféricas, S es la imagen de H, definido por:

$H: 0\leq r\leq 1; 0\leq\theta\leq 2\pi; 0\leq\phi\leq\pi$

Vale entonces:

$$\iiint_V dxdydz=\iiint_S abcdudvdw=\int_{\theta=0}^{2\pi}\left\{\int_{\phi=0}^{\pi}\left[\int_{r=0}^{1}abcr^2sen\phi dr\right]d\phi\right\}d\theta$$

$$=a.b.c.2\pi\left[\int_{\phi=0}^{\pi}sen\phi d\phi\right].\left[\int_{r=0}^{1}r^2\,dr\right]=abc2\pi\left(-\cos\phi|_{\phi=0}^{\pi}\right).\frac{r^3}{3}\Big|_{r=0}^{1}=\frac{4}{3}\pi.a.b.c$$