Resolución TP7:

Ejercicio 23 -a- modificado

Resolver la integral triple I con él recinto V.

$V: \{(x, y, z) \epsilon R^3 / \ x^2 + y^2 + z^2 \leq 2 \ \wedge \ x^2 + y^2 \leq z\}$

$$I = \iiint\limits_V x^2 + y^2 dxdydz$$

$$x^2 + y^2 + z^2 \leq 2 \ \wedge \ x^2 + y^2 \leq z$$

$$z^2 \leq 2 - (x^2 + y^2)$$

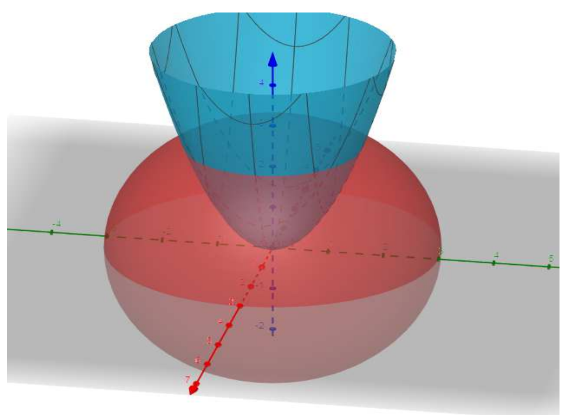
$$-\sqrt{2 - (x^2 + y^2)} \leq z \leq \sqrt{2 - (x^2 + y^2)}$$

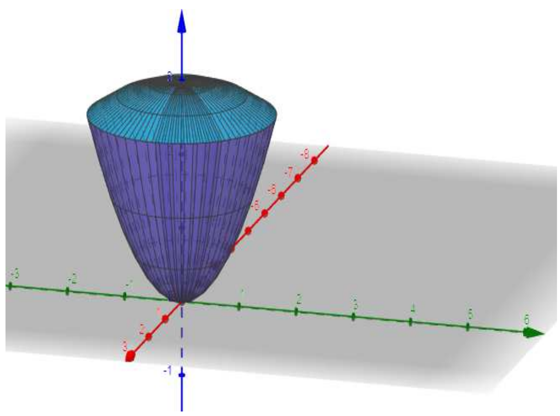
$x^2 + y^2 \leq z$ implica $0 \leq z$ (que z es positivo)

por lo que solo vale $z \leq \sqrt{2 - (x^2 + y^2)}$

por transitividad

$$x^2 + y^2 \leq z \leq \sqrt{2 - (x^2 + y^2)}$$



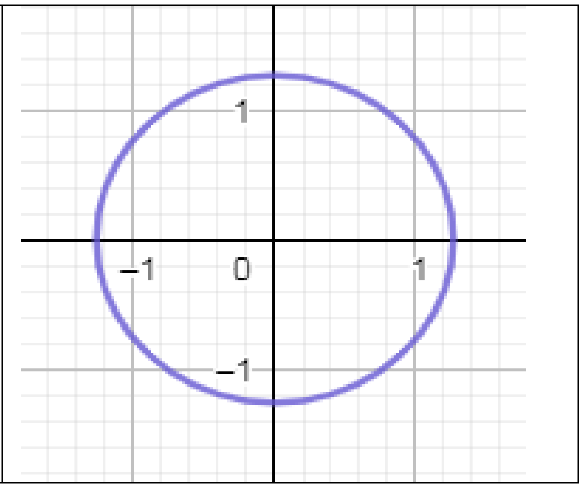


Buscando Limites para x e y:

Tomando la interseccion:
$\underbrace{x^2+y^2}_{z}+z^2=2 \wedge x^2+y^2=z$

$$z+z^2=2 \rightarrow \begin{cases} z_1=\dfrac{-1+3}{2}=1 \\ z_2=\dfrac{-1-3}{2}=-2 \end{cases}$$

$x^2+y^2=z$ indica que z es positivo por lo que vale solo
$z_1=1$
por transitibidad tomamos la proyeccion $x^2+y^2\leq 1$



En resumen

$$V:\begin{cases} x^2+y^2\leq z\leq \sqrt{2-(x^2+y^2)} \\ x^2+y^2\leq 1 \end{cases}$$

$$V:\begin{cases} x=rcos\theta \\ y=rsen\theta \\ z=z \\ |J|=r \\ V'=\begin{cases} r^2\leq z\leq \sqrt{2-r^2} \\ r^2\leq 1 \end{cases} \end{cases}$$

$$V:\begin{cases} x=rcos\theta \\ y=rsen\theta \\ z=z \\ |J|=r \\ V'=\begin{cases} r^2\leq z\leq \sqrt{2-r^2} \\ 0\leq r\leq 1 \\ 0\leq \theta\leq 2\pi \end{cases} \end{cases}$$

$$I=\iiint_V x^2+y^2dxdydz=\iiint_{V'} r^2\overbrace{r}^{|J|}\,drd\theta dz$$

$$I=\int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} r^3\,dz\,drd\theta$$

$$I=\int_0^{2\pi}\int_0^1 r^3\left(\sqrt{2-r^2}-r^2\right)drd\theta$$

$$I=\int_0^{2\pi}d\theta\int_0^1\left(r^3\sqrt{2-r^2}-r^5\right)dr$$

$$I = 2\pi \int_0^1 \left(r^3\sqrt{2-r^2} - r^5\right) dr$$

$$I = 2\pi \left(\int_0^1 r^3\sqrt{2-r^2}dr - \int_0^1 r^5 dr\right)$$

$$\int r^3\sqrt{2-r^2}dr = ?$$

$$sustitucion\ 2-r^2 = t \to -2rdr = dt \to rdr = -\frac{dt}{2}$$

$$\int r^2 r\sqrt{2-r^2}dr = -\int \frac{r^2\sqrt{t}dt}{2} = -\frac{1}{2}\int r^2\sqrt{t}dt$$

$$2-r^2 = t \to r^2 = 2-t$$

$$\int r^2\sqrt{t}dt = \int (2-t)\sqrt{t}dt$$

$$\int r^3\sqrt{2-r^2}dr \overset{2-r^2=t}{\triangleq} -\frac{1}{2}\int (2-t)\sqrt{t}dt$$

$$\int 2\sqrt{t}dt - \int t\sqrt{t}dt$$

$$\int 2\sqrt{t}dt - \int \sqrt{t^3}dt$$

$$2\frac{2}{3}\sqrt{t^3} - \frac{2}{5}\sqrt{t^5}$$

$$\frac{4}{3}\sqrt{t^3} - \frac{2}{5}\sqrt{t^5}$$

$$\int r^3\sqrt{2-r^2}dr \overset{2-r^2=t}{\triangleq} -\frac{1}{2}\int (2-t)\sqrt{t}dt = -\frac{1}{2}\left(\frac{4}{3}\sqrt{t^3} - \frac{2}{5}\sqrt{t^5}\right)$$

$$\int r^3\sqrt{2-r^2}dr \overset{2-r^2=t}{\triangleq} -\frac{2}{3}\sqrt{t^3} + \frac{1}{5}\sqrt{t^5}$$

$$\overset{2-r^2=t}{\hat{=}} \quad -\frac{2}{3}\sqrt{(2-r^2)^3}+\frac{1}{5}\sqrt{(2-r^2)^5}$$

$$\int r^3\sqrt{2-r^2}dr = -\frac{2}{3}\sqrt{(2-r^2)^3}+\frac{1}{5}\sqrt{(2-r^2)^5}$$

$$I \,=\, 2\pi\left(\int_0^1 r^3\sqrt{2-r^2}dr \,-\, \int_0^1 r^5 dr\right)$$

$$I \,=\, 2\pi\left[-\frac{2}{3}\sqrt{(2-r^2)^3}+\frac{1}{5}\sqrt{(2-r^2)^5}-\frac{r^6}{6}\right]_0^1$$

$$P(r) = -\frac{2}{3}\sqrt{(2-r^2)^3}+\frac{1}{5}\sqrt{(2-r^2)^5}-\frac{r^6}{6}$$

$$P(1) = -\frac{2}{3}\sqrt{(2-1)^3}+\frac{1}{5}\sqrt{(2-1)^5}-\frac{1}{6} = -\frac{2}{3}+\frac{1}{5}-\frac{1}{6} = -\frac{19}{30}$$

$$P(0) = -\frac{2}{3}\sqrt{(2-0)^3}+\frac{1}{5}\sqrt{(2-0)^5}-\frac{0}{6} = -\frac{4}{3}\sqrt{2}+\frac{2}{5}\sqrt{2} = -\frac{14}{15}\sqrt{2}$$

$$I \,=\, 2\pi\left(\underbrace{\left(-\frac{19}{30}\right)}_{P(1)}-\underbrace{\left(-\frac{14}{15}\sqrt{2}\right)}_{P(0)}\right)$$

$$I \,=\, 2\pi\left(\frac{14\sqrt{2}\cdot 2-19}{30}\right)$$

$$I \,=\, 2\pi\left(\frac{28\sqrt{2}-19}{30}\right)$$

$$I = \frac{\pi}{15}\left(28\sqrt{2}-19\right)$$

$$\iiint_V \underbrace{x^2+y^2}_{\geq 0 \to Rdo+} dxdydz$$