Resolución TP6:

Ejercicio 11

Se tiene $F(x;y;z) = z^3 + 3x^2y - y^3z + y^2 - 3x - 1 = 0$ $y\ pide\ que\ se\ evalue\ en\ el\ P = (1;1;1)$

$Dom_F = \{\mathbb{R}\}$

Derivadas de primer orden (implícitos - TFI1)

Es un polinomio de grado 3 y por lo tanto es continua todo su dominio sin embargo se define lo siguiente:

Teorema de Cauchy: si f es de clase C^1($es\ de\ clase\ C^1\ si\ y\ solo\ si\ las\ derivadas\ parciales\ de\ f\ son\ continuas\ en\ un\ conjunto\ abierto\ no\ vacio\ U$) en U($f \in C^1$), entonces F es diferenciable en U.

$$\begin{cases} \dfrac{\partial F}{\partial x}(x;y;z) = 6yx - 3 \\ \dfrac{\partial F}{\partial y}(x;y;z) = 3x^2 - 3zy^2 + 2y \\ \dfrac{\partial F}{\partial z}(x;y;z) = 3z^2 - y^3 \end{cases} \Rightarrow \begin{cases} \dfrac{\partial F}{\partial x}(1;1;1) = 3 \\ \dfrac{\partial F}{\partial y}(1;1;1) = 2 \\ \dfrac{\partial F}{\partial z}(1;1;1) = 2 \end{cases}$$

Por lo tanto, al ser continuas en su dominio ($\mathbb{R}$) son diferenciables ahí mismos.

Entonces puede definirse lo siguiente:

$$f: E_2 \Rightarrow E_1 /\ z = \varphi(x;y)$$

Donde E_1 es un entorno del $z_0 = 1$ y E_2 de $P'(x_0;y_0) = (1;1)$. Entonces

$$\frac{\partial F(x;y;\varphi(x;y))}{\partial x} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial \varphi}{\partial x} = 0$$

$$\frac{\partial F}{\partial x}1 + \frac{\partial F}{\partial z}\frac{\partial \varphi}{\partial x} = 0$$

$$\frac{\partial \varphi}{\partial x}(1;1) = -\frac{\frac{\partial F}{\partial x}(1;1;1)}{\frac{\partial F}{\partial z}(1;1;1)} = -\frac{3}{2}$$

$$\frac{\partial F(x;y;\varphi(x;y))}{\partial y} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial \varphi}{\partial y} = 0$$

$$\frac{\partial F}{\partial y}1 + \frac{\partial F}{\partial z}\frac{\partial \varphi}{\partial y} = 0$$

$$\frac{\partial \varphi}{\partial y}(1;1) = -\frac{\frac{\partial F}{\partial y}(1;1;1)}{\frac{\partial F}{\partial z}(1;1;1)} = -1$$

Derivadas de segundo orden (implícitos - TFI1)

Entonces para aplicar la derivada de segundo orden se procede a lo siguiente

Usando $\left(\frac{u}{v}\right)' = \frac{u'v-uv'}{v^2}$

$$\textit{primera derivada x segunda en ...}\begin{cases} \dfrac{\partial^2\varphi}{\partial x^2}(x_0;y_0) = \dfrac{\partial}{\partial x}\left(-\dfrac{\frac{\partial F}{\partial x}(x;y;\varphi(x;y))}{\frac{\partial F}{\partial z}(x;y;\varphi(x;y))}\right) = -\dfrac{\left(\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial z\partial x}\frac{\partial\varphi}{\partial x}\right)*\frac{\partial F}{\partial z}-\frac{\partial F}{\partial x}*\left(\frac{\partial^2 F}{\partial x\partial z}+\frac{\partial^2 F}{\partial z^2}\frac{\partial\varphi}{\partial x}\right)}{\left(\frac{\partial F}{\partial z}\right)^2} \\ \dfrac{\partial\varphi}{\partial x\partial y}(x_0;y_0) = \dfrac{\partial}{\partial y}\left(-\dfrac{\frac{\partial F}{\partial x}(x;y;\varphi(x;y))}{\frac{\partial F}{\partial z}(x;y;\varphi(x;y))}\right) = -\dfrac{\left(\frac{\partial^2 F}{\partial y\partial x}+\frac{\partial^2 F}{\partial z\partial x}\frac{\partial\varphi}{\partial y}\right)*\frac{\partial F}{\partial z}-\frac{\partial F}{\partial x}*\left(\frac{\partial^2 F}{\partial y\partial z}+\frac{\partial^2 F}{\partial z^2}\frac{\partial\varphi}{\partial y}\right)}{\left(\frac{\partial F}{\partial z}\right)^2} \end{cases}$$

///////////////////C.A//Regla de la Cadena//

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}(x;y;\varphi(x;y))\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right)\frac{\partial x}{\partial x} + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)\frac{\partial y}{\partial x} + \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial x}\right)\frac{\partial\varphi}{\partial x} = \frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial z\partial x}\frac{\partial\varphi}{\partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z}(x;y;\varphi(x;y))\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z}\right)\frac{\partial x}{\partial x} + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z}\right)\frac{\partial y}{\partial x} + \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial z}\right)\frac{\partial\varphi}{\partial x} = \frac{\partial^2 F}{\partial x\partial z}+\frac{\partial^2 F}{\partial z^2}\frac{\partial\varphi}{\partial x}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}(x;y;\varphi(x;y))\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right)\frac{\partial x}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)\frac{\partial y}{\partial y} + \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial x}\right)\frac{\partial\varphi}{\partial y} = \frac{\partial^2 F}{\partial y\partial x}+\frac{\partial^2 F}{\partial z\partial x}\frac{\partial\varphi}{\partial y}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z}(x;y;\varphi(x;y))\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z}\right)\frac{\partial x}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z}\right)\frac{\partial y}{\partial y} + \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial z}\right)\frac{\partial\varphi}{\partial y} = \frac{\partial^2 F}{\partial y\partial z}+\frac{\partial^2 F}{\partial z^2}\frac{\partial\varphi}{\partial y}$$

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Usando $\left(\frac{u}{v}\right)' = \frac{u'v-uv'}{v^2}$

$$\textit{primera derivada y segunda en ...}\begin{cases} \dfrac{\partial^2\varphi}{\partial y\partial x}(x_0;y_0) = \dfrac{\partial}{\partial x}\left(-\dfrac{\frac{\partial F}{\partial y}(x;y;\varphi(x;y))}{\frac{\partial F}{\partial z}(x;y;\varphi(x;y))}\right) = -\dfrac{\left(\frac{\partial^2 F}{\partial x\partial y}+\frac{\partial^2 F}{\partial z\partial y}\frac{\partial\varphi}{\partial x}\right)*\frac{\partial F}{\partial z}-\frac{\partial F}{\partial y}*\left(\frac{\partial^2 F}{\partial x\partial z}+\frac{\partial^2 F}{\partial z^2}\frac{\partial\varphi}{\partial x}\right)}{\left(\frac{\partial F}{\partial z}\right)^2} \\ \dfrac{\partial\varphi}{\partial y^2}(x_0;y_0) = \dfrac{\partial}{\partial y}\left(-\dfrac{\frac{\partial F}{\partial y}(x;y;\varphi(x;y))}{\frac{\partial F}{\partial z}(x;y;\varphi(x;y))}\right) = -\dfrac{\left(\frac{\partial^2 F}{\partial y^2}+\frac{\partial^2 F}{\partial z\partial y}\frac{\partial\varphi}{\partial y}\right)*\frac{\partial F}{\partial z}-\frac{\partial F}{\partial y}*\left(\frac{\partial^2 F}{\partial y\partial z}+\frac{\partial^2 F}{\partial z^2}\frac{\partial\varphi}{\partial y}\right)}{\left(\frac{\partial F}{\partial z}\right)^2} \end{cases}$$

///////////////////C.A//Regla de la Cadena//

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}(x;y;\varphi(x;y))\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)\frac{\partial x}{\partial x} + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y}\right)\frac{\partial y}{\partial x} + \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial y}\right)\frac{\partial\varphi}{\partial x} = \frac{\partial^2 F}{\partial x\partial y}+\frac{\partial^2 F}{\partial z\partial y}\frac{\partial\varphi}{\partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z}(x;y;\varphi(x;y))\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z}\right)\frac{\partial x}{\partial x} + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z}\right)\frac{\partial y}{\partial x} + \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial z}\right)\frac{\partial\varphi}{\partial x} = \frac{\partial^2 F}{\partial x\partial z}+\frac{\partial^2 F}{\partial z^2}\frac{\partial\varphi}{\partial x}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y}(x;y;\varphi(x;y))\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)\frac{\partial x}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y}\right)\frac{\partial y}{\partial y} + \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial y}\right)\frac{\partial\varphi}{\partial y} = \frac{\partial^2 F}{\partial y^2}+\frac{\partial^2 F}{\partial z\partial y}\frac{\partial\varphi}{\partial y}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z}(x;y;\varphi(x;y))\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z}\right)\frac{\partial x}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z}\right)\frac{\partial y}{\partial y} + \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial z}\right)\frac{\partial\varphi}{\partial y} = \frac{\partial^2 F}{\partial y\partial z}+\frac{\partial^2 F}{\partial z^2}\frac{\partial\varphi}{\partial y}$$

Teorema de Schwarz: si f es de clase C^2($es\ de\ clase\ C^2\ si\ y\ solo\ si\ las\ derivadas\ segundas\ parciales\ de\ f\ son\ continuas\ en\ un\ conjunto\ abierto\ no\ vacio\ U$) en U, entonces f posee derivadas mixtas que son iguales en U. Por lo tanto, se podría definir lo siguiente:

$$\begin{cases} \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x} \\ \frac{\partial^2 F}{\partial x \partial z} = \frac{\partial^2 F}{\partial z \partial x} \\ \frac{\partial^2 F}{\partial y \partial z} = \frac{\partial^2 F}{\partial z \partial y} \end{cases}$$

Y como ya demostrado anteriormente, al ser polinómicas y continuas se cumple el teorema ya antes dicho.

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Explicitas de F

$$\begin{cases} \frac{\partial F}{\partial x}(1;1;1) = 3 \\ \frac{\partial F}{\partial y}(1;1;1) = 2 \\ \frac{\partial F}{\partial z}(1;1;1) = 2 \end{cases}$$

Implícitas de φ

$$\frac{\partial \varphi}{\partial x}(1;1) = -\frac{\frac{\partial F}{\partial x}(1;1;1)}{\frac{\partial F}{\partial z}(1;1;1)} = -\frac{3}{2}$$

$$\frac{\partial \varphi}{\partial y}(1;1) = -\frac{\frac{\partial F}{\partial y}(1;1;1)}{\frac{\partial F}{\partial z}(1;1;1)} = -1$$
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Explicitas de F - 2do orden

$$\frac{\partial^2 F}{\partial x^2}(x;y;z) = 6y \Rightarrow \frac{\partial^2 F}{\partial x^2}(1;1;1) = 6$$

$$\frac{\partial^2 F}{\partial y^2}(x;y;z) = -6zy + 2 \Rightarrow \frac{\partial^2 F}{\partial y^2}(1;1;1) = -4$$

$$\frac{\partial^2 F}{\partial z^2}(x;y;z) = 6z \Rightarrow \frac{\partial^2 F}{\partial z^2}(1;1;1) = 6$$

$$\frac{\partial^2 F}{\partial x \partial y}(x;y;z) = 6x \Rightarrow \frac{\partial^2 F}{\partial x \partial y}(1;1;1) = 6$$

$$\frac{\partial^2 F}{\partial x \partial z}(x;y;z) = 0 \Rightarrow \frac{\partial^2 F}{\partial x \partial z}(1;1;1) = 0$$

$$\frac{\partial^2 F}{\partial y \partial z}(x;y;z) = -3y^2 \Rightarrow \frac{\partial^2 F}{\partial y \partial z}(1;1;1) = -3$$
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$$\frac{\partial^2 \varphi}{\partial x^2}(1{,}1) = -\frac{\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial z \partial x}\frac{\partial \varphi}{\partial x}\right) * \frac{\partial F}{\partial z} - \frac{\partial F}{\partial x} * \left(\frac{\partial^2 F}{\partial x \partial z} + \frac{\partial^2 F}{\partial z^2}\frac{\partial \varphi}{\partial x}\right)}{\left(\frac{\partial F}{\partial z}\right)^2} = -\frac{\left(6 + 0 \cdot -\frac{3}{2}\right) \cdot 2\ - 3 \cdot \left(0 + 6 \cdot -\frac{3}{2}\right)}{2^2} = -\frac{17}{2}$$

$$\frac{\partial \varphi}{\partial x \partial y}(1{,}1) = -\frac{\left(\frac{\partial^2 F}{\partial y \partial x} + \frac{\partial^2 F}{\partial z \partial x}\frac{\partial \varphi}{\partial y}\right) * \frac{\partial F}{\partial z} - \frac{\partial F}{\partial x} * \left(\frac{\partial^2 F}{\partial y \partial z} + \frac{\partial^2 F}{\partial z^2}\frac{\partial \varphi}{\partial y}\right)}{\left(\frac{\partial F}{\partial z}\right)^2} = -\frac{(6 + 0 \cdot -1) \cdot 2 - 3 \cdot (-3 + 6 \cdot -1)}{(2)^2} = -\frac{39}{4}$$

$$\frac{\partial^2 \varphi}{\partial y \partial x}(1{,}1) = -\frac{\left(\frac{\partial^2 F}{\partial x \partial y} + \frac{\partial^2 F}{\partial z \partial y}\frac{\partial \varphi}{\partial x}\right) * \frac{\partial F}{\partial z} - \frac{\partial F}{\partial y} * \left(\frac{\partial^2 F}{\partial x \partial z} + \frac{\partial^2 F}{\partial z^2}\frac{\partial \varphi}{\partial x}\right)}{\left(\frac{\partial F}{\partial z}\right)^2} = -\frac{\left(6 + (-3)\left(-\frac{3}{2}\right)\right) \cdot 2 - 2 \cdot \left(0 + 6\left(-\frac{3}{2}\right)\right)}{2^2} = -\frac{39}{4}$$

$$\frac{\partial \varphi}{\partial y^2}(x_0;y_0) = -\frac{\left(\frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z \partial y}\frac{\partial \varphi}{\partial y}\right) * \frac{\partial F}{\partial z} - \frac{\partial F}{\partial y} * \left(\frac{\partial^2 F}{\partial y \partial z} + \frac{\partial^2 F}{\partial z^2}\frac{\partial \varphi}{\partial y}\right)}{\left(\frac{\partial F}{\partial z}\right)^2} = -\frac{\left(-4 + (-3)(-1)\right) \cdot 2 - 2 \cdot (-3 + 6 \cdot (-1))}{2^2} = -4$$

Según Schwarz, al tener derivadas cruzadas iguales φ se puede asegurar que sus derivadas son continuas. Por lo tanto se permite fabricar el polinomio de Taylor

De esta manera se verifica además el procedimiento correctamente aplicado.

Matriz Hessiana con derivadas de segundo orden (implícitos - TFI1)

$$H(\varphi(x_0,y_0)) = \begin{pmatrix} \varphi_{xx} & \varphi_{xy} \\ \varphi_{yx} & \varphi_{yy} \end{pmatrix}_p = \begin{pmatrix} -\frac{17}{2} & -\frac{39}{4} \\ -\frac{39}{4} & -4 \end{pmatrix}$$

Polinomio de Taylor de segundo orden (implícitos - TFI1)

$$P_2(x;y) = \varphi(x_0,y_0) + \nabla\varphi(x_0,y_0)\cdot(x-x_0;y-y_0) + \frac{1}{2}\,[\,x-x_0\;\; y-y_0\,]\,H\left(\varphi(x_0,y_0)\right)\begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}$$

$$P_2(x;y) = \varphi(1,1) + \nabla\varphi(1,1)\cdot(x-1;y-1) + \frac{1}{2}\,[\,x-1\;\; y-1\,]\,H\left(\varphi(x_0,y_0)\right)\begin{bmatrix} x-1 \\ y-1 \end{bmatrix}$$

$$P_2(x;y) = 1 + \left(-\frac{3}{2},-1\right)\cdot(x-1;y-1) + \frac{1}{2}\,[\,x-1\;\; y-1\,]\begin{bmatrix} -\frac{17}{2} & -\frac{39}{4} \\ -\frac{39}{4} & -4 \end{bmatrix}\begin{bmatrix} x-1 \\ y-1 \end{bmatrix}$$

$$P_2(x;y) = 1 + -\frac{3}{2}(x-1) - 1(y-1) + \frac{1}{2}\begin{bmatrix} -\frac{17}{2}(x-1) - \frac{39}{4}(y-1) \\ -\frac{39}{4}(x-1) - 4(y-1) \end{bmatrix}\begin{bmatrix} x-1 \\ y-1 \end{bmatrix}$$

$$P_2(x;y) = 1 + -\frac{3}{2}(x-1) - 1(y-1) + \frac{1}{2}\left(-\frac{17}{2}(x-1)^2 - \frac{39}{4}(y-1)(x-1) - \frac{39}{4}(x-1)(y-1) - 4(y-1)^2\right)$$

$$P_2(x;y) = 1 + -\frac{3}{2}(x-1) - 1(y-1) + -\frac{17}{4}(x-1)^2 - \frac{39}{4}(y-1)(x-1) - 2(y-1)^2$$

Al ser continua ya asegura el teorema del resto, por lo tanto, no es necesario comprobar por limite.