Ejercicio 18.a: Verificar el Teorema de Green para el campo vectorial especificado en camino dado.

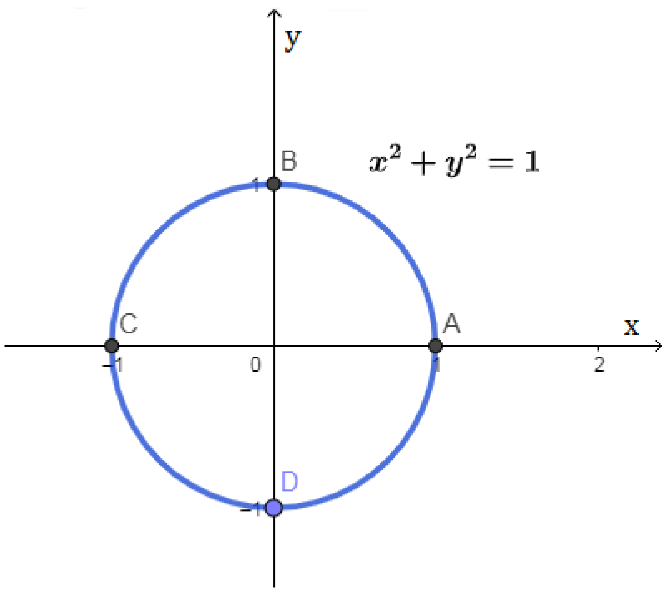
$\bar{F}(x, y) = (3x + 2y\ ;\ x - y)$

$\mathbb{C}: \bar{r}(t) = (\cos t\ ;\ \sin t) \ \rightarrow\ \bar{r}(t): [0\,, 2\pi] \rightarrow R^2 \ \rightarrow\ 0 \leq t \leq 2\pi$

Teorema de Green:

$$\oint_{\mathbb{C}^+} \overline{\boldsymbol{F}} \bullet \overline{\boldsymbol{ds}} = \iint_D \left(\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{x}} - \frac{\partial \boldsymbol{P}}{\partial \boldsymbol{y}}\right).\boldsymbol{dA}$$

Resolvemos en primer lugar la integral de línea, entonces:



Recorremos la curva en sentido positivo, esto es siguiendo la secuencia de puntos $A \rightarrow B \rightarrow C \rightarrow D$

$$\oint_{\mathbb{C}^+} \overline{\boldsymbol{F}} \bullet \overline{\boldsymbol{ds}} = \oint_{\mathbb{C}^+} \overline{\boldsymbol{F}}[\overline{\boldsymbol{r}}(\boldsymbol{t})] \bullet \dot{\boldsymbol{r}}(\boldsymbol{t}).\boldsymbol{dt} \quad \rightarrow$$

$\bar{F}(x, y) = \big(P(x, y)\ ;\ Q(x, y)\big) = (3x + 2y\ ;\ x - y)$

$$\bar{r}(t) = (cos\, t\ ;\ sin\, t)\quad ;\quad 0 \leq t \leq 2\pi$$

$$\bar{F}[\bar{r}(t)] = (3.cos\, t + 2.sin\, t\ ;\ cos\, t - sin\, t)$$

$$\dot{r}(t) = (-\,sin\, t\ ;\ cos\, t)$$

$$\oint_{\mathbb{C}^+} \bar{F} \bullet \overline{ds} = \int_0^{2\pi} (3.cos\, t + 2.sin\, t\ ;\ cos\, t - sin\, t) \bullet (-\,sin\, t\ ;\ cos\, t)\, .dt$$

$$\oint_{\mathbb{C}^+} \bar{F} \bullet \overline{ds} = \int_0^{2\pi} (-3.sin(t).cos(t) - 2.sin^2(t) + cos^2(t) - sin(t).cos(t))\, .dt$$

$$\oint_{\mathbb{C}^+} \bar{F} \bullet \overline{ds} = \int_0^{2\pi} (-4.sin(t).cos(t) - 2.sin^2(t) + cos^2(t)).dt$$

$$\oint_{\mathbb{C}^+} \bar{F} \bullet \overline{ds} = -4\int_0^{2\pi} (sin(t).cos\, t).dt - 2\int_0^{2\pi} sin^2(t).dt + \int_0^{2\pi} cos^2(t).dt$$

Entonces:

$$\boldsymbol{\int_0^{2\pi} (sin(t).cos\, t).dt \qquad (I) \rightarrow}$$

Realizando el siguiente cambio de variables, tenemos:

$$u = sin(t)$$

$$du = cos(t).dt$$

$$\int u\, .du\ = \frac{u^2}{2} + C$$

Volviendo a la variable original, tenemos:

$$\int_0^{2\pi} (sin(t).cos\, t).dt\ = \frac{sin^2(t)}{2}\Bigg|_0^{2\pi} = 0 \quad \rightarrow$$

$$\boldsymbol{\int_0^{2\pi} (sin(t).cos\, t).dt = 0 \qquad (I)}$$

$$\int_{0}^{2\pi} \boldsymbol{sin^2(t).dt} \qquad (\boldsymbol{II})$$

Sabiendo que:

a) $cos^2(t) + sin^2(t) = 1$
b) $cos^2(t) - sin^2(t) = cos(2t)$

Restando la expresión a) a la b), tenemos:

$$2.sin^2(t) = 1 - cos(2t) \quad \rightarrow$$

$$sin^2(t) = \frac{1}{2} - \frac{1}{2}.cos(2t)$$

Reemplazando en (II)

$$\int_{0}^{2\pi} sin^2(t).dt \ = \ \int_{0}^{2\pi} \left(\frac{1}{2} - \frac{cos(2t)}{2}\right).dt \quad \rightarrow$$

$$\int_{0}^{2\pi} sin^2(t).dt \ = \ \frac{1}{2}\int_{0}^{2\pi} dt \quad - \frac{1}{2}\int_{0}^{2\pi} cos(2t).dt \rightarrow$$

$$\int_{0}^{2\pi} sin^2(t).dt \ = \ \frac{1}{2}.t\Big|_{0}^{2\pi} \ - \frac{1}{4} sin(2t)\Big|_{0}^{2\pi} \quad \rightarrow$$

$$\int_{0}^{2\pi} sin^2(t).dt \ = \ \frac{1}{2}.2\pi - 0 \quad \rightarrow$$

$$\int_{0}^{2\pi} \boldsymbol{sin^2(t).dt} \ = \ \boldsymbol{\pi} \qquad (\boldsymbol{II})$$

$$\int_{0}^{2\pi} \boldsymbol{cos^2(t).dt} \qquad (\boldsymbol{III})$$

Sabiendo que:

c) $cos^2(t) + sin^2(t) = 1$
d) $cos^2(t) - sin^2(t) = cos(2t)$

Sumando la expresión a) a la b), tenemos:

$2. cos^2(t) = 1 + cos(2t) \quad \rightarrow$

$$cos^2(t) = \frac{1}{2} + \frac{1}{2}. cos(2t)$$

Reemplazando en (III)

$$\int_0^{2\pi} cos^2(t). dt \; = \int_0^{2\pi} \left(\frac{1}{2} + \frac{cos(2t)}{2}\right). dt \quad \rightarrow$$

$$\int_0^{2\pi} cos^2(t). dt \; = \frac{1}{2}\int_0^{2\pi} dt + \frac{1}{2}\int_0^{2\pi} cos(2t). dt \quad \rightarrow$$

$$\int_0^{2\pi} cos^2(t). dt \; = \frac{1}{2}. t\Big|_0^{2\pi} + \frac{1}{4} sin(2t)\Big|_0^{2\pi} \quad \rightarrow$$

$$\int_0^{2\pi} cos^2(t). dt \; = \; \frac{1}{2}. 2\pi + 0 \quad \rightarrow$$

$$\boldsymbol{\int_0^{2\pi} cos^2(t). dt \; = \; \pi \qquad (III)}$$

Finalmente:

$$\oint_{\mathbb{C}^+} \bar{F} \bullet \overline{ds} = -4\int_0^{2\pi} (sin(t). cos\, t). dt - 2\int_0^{2\pi} sin^2(t). dt + \int_0^{2\pi} cos^2(t). dt$$

$$\oint_{\mathbb{C}^+} \bar{F} \bullet \overline{ds} = -4. (I) \; - 2. (II) \; + \; (III) \quad \rightarrow$$

$$\oint_{\mathbb{C}^+} \bar{F} \bullet \overline{ds} = -4. 0 \; - 2. \pi \; + \; \pi \quad \rightarrow$$

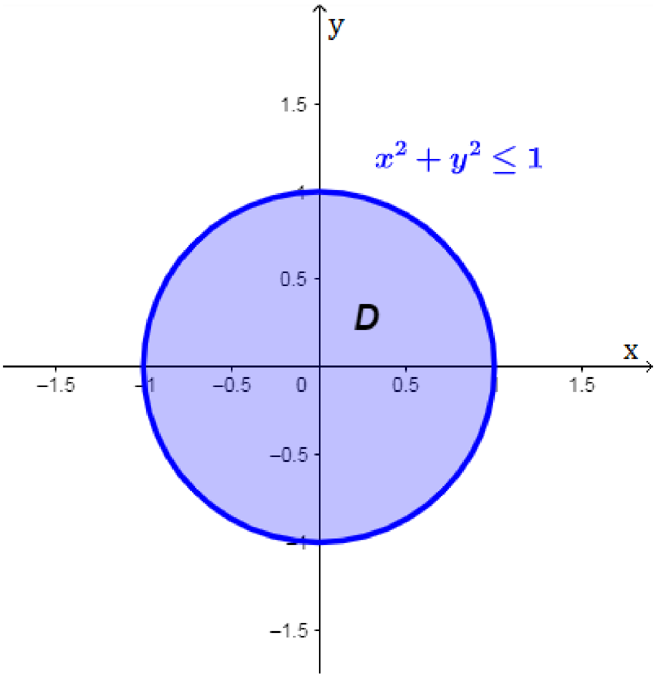
$$\boldsymbol{\oint_{\mathbb{C}^+} \bar{F} \bullet \overline{ds} = -\pi}$$

Ahora resolvemos la integral doble sobre la región D.

$$\iint_D \left(\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{x}} - \frac{\partial \boldsymbol{P}}{\partial \boldsymbol{y}}\right).\boldsymbol{dA} \quad \rightarrow$$

$$\bar{F}(x,y) = \big(P(x,y)\,;\,Q(x,y)\big) = (3x + 2y\,;\,x - y)$$

$$\frac{\partial Q}{\partial x} = 1 \qquad ; \qquad \frac{\partial P}{\partial y} = 2$$



Trabajando con coordenadas polares, tenemos:

$$x = \rho.\cos\varphi$$

$$y = \rho.\sin\varphi$$

$$|J| = \rho$$

$$0 \leq \varphi \leq 2\pi$$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).dA = \int_0^{2\pi}\int_0^1 \rho.(1-2).d\rho.d\varphi \quad \rightarrow$$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).dA = \int_0^{2\pi} d\varphi \int_0^1 -\rho.d\rho \quad \rightarrow$$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).dA = \int_0^{2\pi} d\varphi \left(-\frac{\rho^2}{2}\right)\Bigg|\begin{matrix}1\\0\end{matrix} \quad \rightarrow$$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).dA = -\frac{1}{2}\int_0^{2\pi} d\varphi \quad \rightarrow$$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).dA = -\frac{1}{2}.(\varphi)\Big|\begin{matrix}2\pi\\0\end{matrix} \quad \rightarrow$$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).dA = -\frac{1}{2}.2\pi \quad \rightarrow$$

$$\boldsymbol{\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).dA = -\pi}$$

Con lo cual queda demostrado el Teorema de Green.

$$\oint_{\mathbb{C}^+} \boldsymbol{\overline{F} \bullet \overline{ds}} = \boldsymbol{\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).dA = -\pi}$$
