Resolución TP6:

Ejercicio 17 - iv

Hallar los puntos extremos para $f(x,y) = x^2y^2$ dado la siguiente restricción $x^2 + y^2 - 1 = 0$, y clasificar como máximo o mínimo.

Para empezar:

- El dominio de ambas funciones es todo $\mathbb{R}^2$ por lo que no tenemos restricción alguna para los puntos que hallaremos

Primeras Derivadas:

$$f_x = 2xy^2$$
$$f_y = 2x^2y$$
$$g_x = 2x$$
$$g_y = 2y$$

Sistema de ecuaciones:

$$\begin{cases} g(x,y) = 0 \\ \nabla f(x,y) = \ell \nabla g(x,y) \end{cases} \rightarrow \begin{cases} x^2 + y^2 - 1 = 0 \\ 2xy^2 = \ell 2x \\ 2x^2y = \ell 2y \end{cases}$$

Despejamos

Esquema de decisiones: $x = 0$ o $x \neq 0$

$$x \neq 0, y \neq 0 \begin{cases} x^2 + y^2 - 1 = 0 \\ \dfrac{2xy^2}{2x} = \ell \\ \dfrac{2x^2y}{2y} = \ell \end{cases}$$

$$\begin{cases} x^2 + y^2 - 1 = 0 \\ y^2 = \ell \\ x^2 = \ell \end{cases} \quad para\ x \neq 0\ e\ y \neq 0$$

Sustitución en $g(x,y) = 0$

$$\ell + \ell - 1 = 0$$

$$2\ell = 1$$

$$\ell = \frac{1}{2}$$

$$y^2 = \frac{1}{2} \to \mathrm{y} = \pm\frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$$

Entonces

$$\begin{cases} \ell = \frac{1}{2} \\ y^2 = \ell \\ x^2 = \ell \end{cases} \Rightarrow \begin{cases} y = \frac{\sqrt{2}}{2} \vee y = -\frac{\sqrt{2}}{2} \\ x = \frac{\sqrt{2}}{2} \vee x = -\frac{\sqrt{2}}{2} \end{cases} \Rightarrow \begin{matrix} Pc_1 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) \\ Pc_2 = (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) \\ Pc_3 = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) \\ Pc_4 = (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) \end{matrix}$$

Falta determinar qué pasa si $x = 0$ o $y = 0$

$$\begin{cases} x^2 + y^2 - 1 = 0 \\ 2xy^2 = \ell 2x \\ 2x^2 y = \ell 2y \end{cases}$$

$$si\ x = 0 \Rightarrow \begin{cases} y^2 - 1 = 0 \\ 2 \cdot 0 \cdot y^2 = \ell \cdot 2 \cdot 0 \\ 2 \cdot 0^2 \cdot y = \ell 2y \end{cases} \Rightarrow \begin{cases} y^2 - 1 = 0 \\ 0 = \ell 2y \end{cases}$$

$si\ y = 0;\ \ y^2 - 1 = 0\ es\ absurdo\ lo\ unico\ que\ puede\ valer\ es\ \ \ell = 0$

$$\begin{cases} y = 1 \vee y = -1 \\ 0 = \ell \end{cases} \Rightarrow \begin{matrix} Pc_5 = (0,1) \\ Pc_6 = (0,-1) \end{matrix}$$

$$\begin{cases} x^2 + y^2 - 1 = 0 \\ 2xy^2 = \ell 2x \\ 2x^2 y = \ell 2y \end{cases}$$

$$si\ y = 0 \Rightarrow \begin{cases} x^2 - 1 = 0 \\ 0 = \ell \cdot 2x \\ x^2 \cdot 0 = \ell \cdot 2 \cdot 0 \end{cases} \Rightarrow \begin{cases} x^2 - 1 = 0 \\ 0 = \ell \cdot 2x \end{cases}$$

$si\ x = 0;\ \ x^2 - 1 = 0\ es\ absurdo\ lo\ unico\ que\ puede\ valer\ es\ \ \ell = 0$

$$\begin{cases} x = 1 \vee x = -1 \\ 0 = \ell \end{cases} \Rightarrow \begin{matrix} Pc_7 = (1,0) \\ Pc_8 = (-1,0) \end{matrix}$$

Clasificación:

Método 1: Ya sabemos que ambos puntos cumplen la condición, debemos compáralos entre sí para saber si son máximo o mínimo.

Se evalúan en $f(x, y) = x^2y^2$

- $f(Pc_1) = \left(\frac{\sqrt{2}}{2}\right)^2 \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{4}$
- $f(Pc_2) = \left(\frac{\sqrt{2}}{2}\right)^2 \left(-\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{4}$
- $f(Pc_3) = \left(-\frac{\sqrt{2}}{2}\right)^2 \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{4}$
- $f(Pc_4) = \left(-\frac{\sqrt{2}}{2}\right)^2 \left(-\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{4}$
- $f(Pc_5) = (0)^2(1)^2 = 0$
- $f(Pc_6) = (0)^2(-1)^2 = 0$
- $f(Pc_7) = (-1)^2(0)^2 = 0$
- $f(Pc_8) = (1)^2(0)^2 = 0$

$Pc_1, Pc_2, Pc_3\ y\ Pc_4$ son máximo condicionados de $g(x, y) = 0$

$Pc_5, Pc_6, Pc_7\ y\ Pc_8$ son mínimos condicionados de $g(x, y) = 0$

Método 2: Matriz Hessiana Reducida

Si $L(x, y, \ell) = f(x, y) - \ell g(x, y)$

$$f_x(x, y) - \ell g_X(x, y) = 0 \rightarrow f_x(x, y) = \ell g_X(x, y)$$

$$f_y(x, y) - \ell g_y(x, y) = 0 \rightarrow f_y(x, y) = \ell g_y(x, y)$$

Se toma $-\ell$ para que se cumpla $\nabla f(x, y) = \ell \nabla g(x, y)$

$$f_x = 2xy^2 \rightarrow f_{xx} = 2y^2\,, f_{xy} = 4xy$$
$$f_y = 2x^2y \rightarrow f_{yy} = 2x^2, f_{yx} = 4xy$$
$$g_x = 2x \rightarrow g_{xx} = 2\,, g_{xy} = 0$$
$$g_y = 2y \rightarrow g_{yy} = 2\,, g_{yx} = 0$$

$$L_{xx}(x, y, \ell) = f_{xx}(x, y) - \ell g_{xx}(x, y) = 2y^2 - \ell 2$$

$$L_{xy}(x, y, \ell) = f_{xy}(x, y) - \ell g_{xy}(x, y)$$

$$L_{yy}(x, y, \ell) = f_{yy}(x, y) - \ell g_{yy}(x, y)$$

$$L_{yx}(x,y,\ell) = f_{yx}(x,y) - \ell g_{yx}(x,y)$$

$$H(f,g) = \begin{pmatrix} 0 & -g_x & -g_y \\ -g_x & L_{xx} & L_{xy} \\ -g_y & L_{yx} & L_{yy} \end{pmatrix} = \begin{pmatrix} 0 & -2x & -2y \\ -2x & 2y^2 - \ell 2 & 4xy - \ell 0 \\ -2y & 4xy - \ell 0 & 2x^2 - \ell 2 \end{pmatrix}$$

$$H(f,g) = \begin{pmatrix} 0 & -2x & -2y \\ -2x & 2y^2 - \ell 2 & 4xy \\ -2y & 4xy & 2x^2 - \ell 2 \end{pmatrix}$$

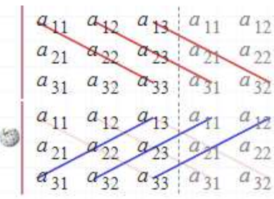
1. Si |**H**|>0 entonces P es un máximo condicionado en *g(x,y)=0*
2. Si |**H**|<0 entonces P es un mínimo condicionado en *g(x,y)=0*
3. Si |**H**|=0 entonces el criterio no concluye nada

$Pc_1 = (x_0, y_0, \ell_0) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}\right)$ $Pc_2 = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$ $Pc_3 = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}\right)$ $Pc_4 = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$	$Pc_5 = (0,1,0)$ $Pc_6 = (0,-1,0)$ $Pc_7 = (1,0,0)$ $Pc_8 = (-1,0,0)$

$$H(f,g) = \begin{pmatrix} 0 & -2x & -2y \\ -2x & 2y^2 - \ell 2 & 4xy \\ -2y & 4xy & 2x^2 - \ell 2 \end{pmatrix}$$

$$H(PC1) = \begin{pmatrix} 0 & -\frac{2\sqrt{2}}{2} & -\frac{2\sqrt{2}}{2} \\ -2\frac{\sqrt{2}}{2} & 2\left(\frac{\sqrt{2}}{2}\right)^2 - \frac{1}{2}2 & 4\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} \\ -\frac{2\sqrt{2}}{2} & 4\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} & 2\left(\frac{\sqrt{2}}{2}\right)^2 - \frac{1}{2}2 \end{pmatrix} = \begin{pmatrix} 0 & -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & 0 & 2 \\ -\sqrt{2} & 2 & 0 \end{pmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = +a_{11}\times a_{22}\times a_{33} + a_{12}\times a_{23}\times a_{31} + a_{13}\times a_{21}\times a_{32} - a_{13}\times a_{22}\times a_{31} - a_{11}\times a_{23}\times a_{32} - a_{12}\times a_{21}\times a_{33}\ (?)$$



$$\begin{vmatrix} 0 & -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & 0 & 2 \\ -\sqrt{2} & 2 & 0 \end{vmatrix} = 0\times 0\times 0 + (-\sqrt{2})\times 2\times(-\sqrt{2}) + (-\sqrt{2})\times(-\sqrt{2})\times 2 - (-\sqrt{2})\times 0\times(-\sqrt{2}) - 2\times 2\times 0 - 0\times(-\sqrt{2})\times(-\sqrt{2}) = 8$$

$$H(PC2) = \begin{pmatrix} 0 & -\frac{2\sqrt{2}}{2} & \frac{2\sqrt{2}}{2} \\ -2\frac{\sqrt{2}}{2} & 2\left(-\frac{\sqrt{2}}{2}\right)^2 - \frac{1}{2}2 & 4*\frac{\sqrt{2}}{2}*-\frac{\sqrt{2}}{2} \\ \frac{2\sqrt{2}}{2} & 4*\frac{\sqrt{2}}{2}*-\frac{\sqrt{2}}{2} & 2\left(\frac{\sqrt{2}}{2}\right)^2 - \frac{1}{2}2 \end{pmatrix} = \begin{pmatrix} 0 & -\sqrt{2} & \sqrt{2} \\ -\sqrt{2} & 0 & -2 \\ \sqrt{2} & -2 & 0 \end{pmatrix}$$

$$\begin{vmatrix} 0 & -\sqrt{2} & \sqrt{2} \\ -\sqrt{2} & 0 & -2 \\ \sqrt{2} & -2 & 0 \end{vmatrix} = 0\times 0\times 0+(-\sqrt{2})\times(-2)\times\sqrt{2}+\sqrt{2}\times(-\sqrt{2})\times(-2)-\sqrt{2}\times 0\times\sqrt{2}-(-2)\times(-2)\times 0-0\times(-\sqrt{2})\times(-\sqrt{2})=8$$

$$H(PC3)=\begin{pmatrix} 0 & \frac{2\sqrt{2}}{2} & -\frac{2\sqrt{2}}{2} \\ 2\frac{\sqrt{2}}{2} & 2\left(\frac{\sqrt{2}}{2}\right)^2-\frac{1}{2}2 & 4*-\frac{\sqrt{2}}{2}*\frac{\sqrt{2}}{2} \\ -\frac{2\sqrt{2}}{2} & 4*-\frac{\sqrt{2}}{2}*\frac{\sqrt{2}}{2} & 2\left(-\frac{\sqrt{2}}{2}\right)^2-\frac{1}{2}2 \end{pmatrix}=\begin{pmatrix} 0 & \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & 0 & -2 \\ -\sqrt{2} & -2 & 0 \end{pmatrix}$$

$$\begin{vmatrix} 0 & \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & 0 & -2 \\ -\sqrt{2} & -2 & 0 \end{vmatrix} = 0\times 0\times 0+\sqrt{2}\times(-2)\times(-\sqrt{2})+(-\sqrt{2})\times\sqrt{2}\times(-2)-(-\sqrt{2})\times 0\times(-\sqrt{2})-(-2)\times(-2)\times 0-0\times\sqrt{2}\times\sqrt{2}=8$$

$$H(PC4)=\begin{pmatrix} 0 & \frac{2\sqrt{2}}{2} & \frac{2\sqrt{2}}{2} \\ 2\frac{\sqrt{2}}{2} & 2\left(-\frac{\sqrt{2}}{2}\right)^2-\frac{1}{2}2 & 4*-\frac{\sqrt{2}}{2}*-\frac{\sqrt{2}}{2} \\ \frac{2\sqrt{2}}{2} & 4*-\frac{\sqrt{2}}{2}*-\frac{\sqrt{2}}{2} & 2\left(-\frac{\sqrt{2}}{2}\right)^2-\frac{1}{2}2 \end{pmatrix}=\begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 0 & 2 \\ \sqrt{2} & 2 & 0 \end{pmatrix}$$

$$\begin{vmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 0 & 2 \\ \sqrt{2} & 2 & 0 \end{vmatrix} = 0\times 0\times 0+\sqrt{2}\times 2\times\sqrt{2}+\sqrt{2}\times\sqrt{2}\times 2-\sqrt{2}\times 0\times\sqrt{2}-2\times 2\times 0-0\times\sqrt{2}\times\sqrt{2}=8$$

$$H(PC5)=\begin{pmatrix} 0 & -2*0 & -2*1 \\ -2*0 & 2*1^2-\frac{1}{2}2 & 4*0*1 \\ -2*1 & 4*0*1 & 2*0^2-\frac{1}{2}2 \end{pmatrix}=\begin{pmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix}$$

$$\begin{vmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & -1 \end{vmatrix} = 0\times 1\times(-1)+0\times 0\times(-2)+(-2)\times 0\times 0-(-2)\times 1\times(-2)-0\times 0\times 0-(-1)\times 0\times 0=-4$$

$$H(PC6)=\begin{pmatrix} 0 & -2*0 & -2*(-1) \\ -2*0 & 2*(-1)^2-\frac{1}{2}2 & 4*0*(-1) \\ -2*(-1) & 4*0*(-1) & 2*0^2-\frac{1}{2}2 \end{pmatrix}=\begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$$

$$\begin{vmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{vmatrix} = 0\times 1\times(-1)+0\times 0\times 2+2\times 0\times 0-2\times 1\times 2-0\times 0\times 0-(-1)\times 0\times 0=-4$$

$$H(PC7)=\begin{pmatrix} 0 & -2*1 & -2*0 \\ -2*1 & 2*0^2-\frac{1}{2}2 & 4*1*0 \\ -2*0 & 4*1*0 & 2*1^2-\frac{1}{2}2 \end{pmatrix}=\begin{pmatrix} 0 & -2 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 0 & -2 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0\times(-1)\times 1+(-2)\times 0\times 0+0\times(-2)\times 0-0\times(-1)\times 0-0\times 0\times 0-1\times(-2)\times(-2)=-4$$

$$H(PC8) = \begin{pmatrix} 0 & -2*(-1) & -2*0 \\ -2*(-1) & 2*0^2 - \frac{1}{2}2 & 4*(-1)*0 \\ -2*0 & 4*(-1)*0 & 2*(-1)^2 - \frac{1}{2}2 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 0 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0\times(-1)\times 1 + 2\times 0\times 0 + 0\times 2\times 0 - 0\times(-1)\times 0 - 0\times 0\times 0 - 1\times 2\times 2 = -4$$

$Pc_1, Pc_2, Pc_3\ y\ Pc_4$ son máximo condicionados de $g(x,y)=0$

$Pc_5, Pc_6, Pc_7\ y\ Pc_8$ son mínimos condicionados de $g(x,y)=0$