Resolución TP10:

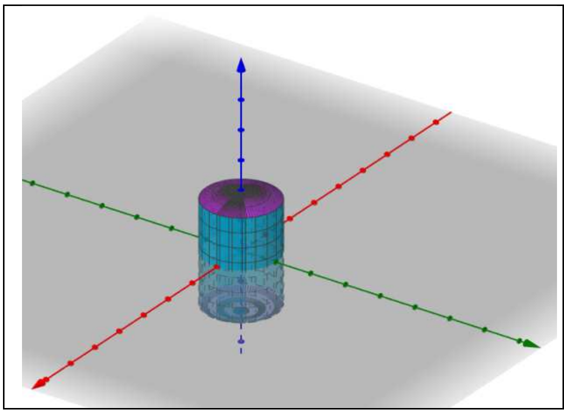
Ejercicio 6 - c

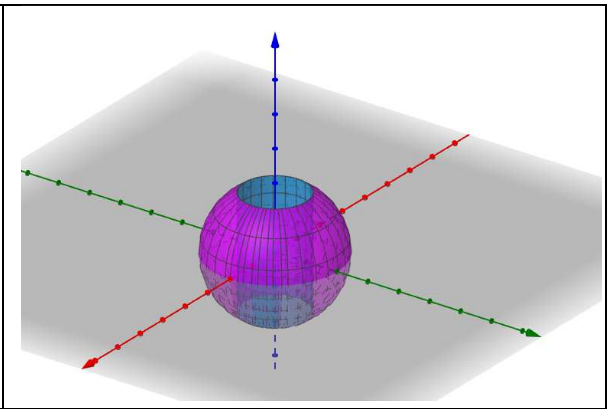
Dado el campo vectorial F y la superficie S, calcular el flujo entrante.

$$F(x, y, z) = (x, y, z)$$

S: la superficie limitada por $x^2 + y^2 = 1$ e $x^2 + y^2 + z^2 = 4$

Advertencia: El enunciado puede ser considerado de las siguientes formas.

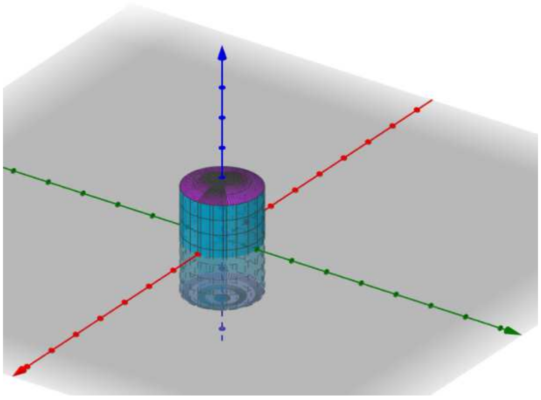




Vamos a utilizar la primera.

Resolviendo:

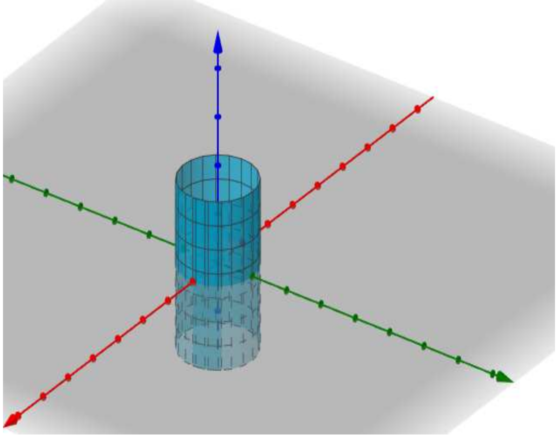
$$I = \iint_S F \cdot dS = \sum \iint_{R_\Phi} F(\Phi) \cdot (\Phi_u X \Phi_v) du dv$$



Límite entre el cilindro y las dos etapas:

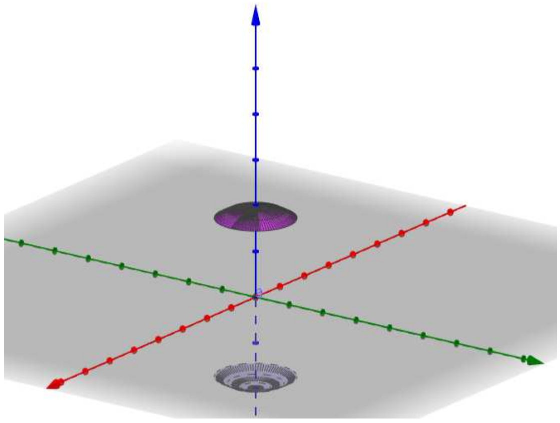
$$\begin{cases} x^2 + y^2 = 1 \\ x^2 + y^2 + z^2 = 4 \end{cases} \rightarrow 1 + z^2 = 4 \rightarrow z = \pm\sqrt{3}$$

Método 1: Considerando que se trata de resolver con cartesianas, la superficie se dividirá en 4:



$$S_1:\begin{cases}\Phi(x,z) = \left(x, \sqrt{1-x^2}, z\right) \\ R_\Phi:\begin{cases} -1 \le x \le 1 \\ -\sqrt{3} \le z \le \sqrt{3}\end{cases}\end{cases}$$

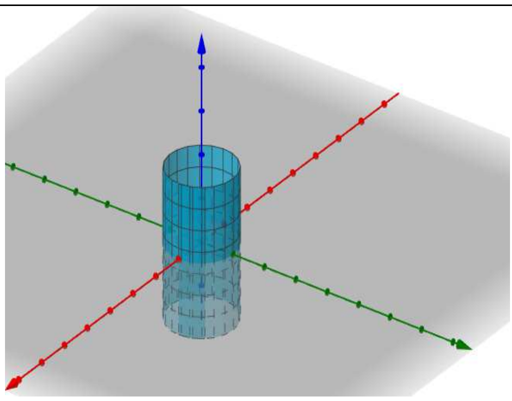
$$S_2:\begin{cases}\Phi(x,z) = \left(x, -\sqrt{1-x^2}, z\right) \\ R_\Phi:\begin{cases} -1 \le x \le 1 \\ -\sqrt{3} \le z \le \sqrt{3}\end{cases}\end{cases}$$



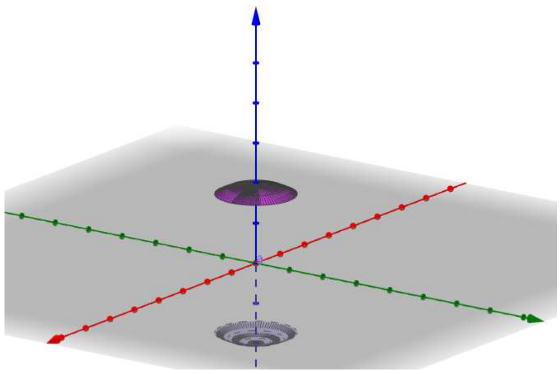
$$S_3:\begin{cases}\Phi(x,y) = \left(x, y, \sqrt{1-x^2-y^2}\right) \\ R_\Phi:\{x^2+y^2 \le 1\end{cases}$$

$$S_4:\begin{cases}\Phi(x,y) = \left(x, y, -\sqrt{1-x^2-y^2}\right) \\ R_\Phi:\{x^2+y^2 \le 1\end{cases}$$

Método 2: Considerando que se puede resolver con derivadas menos complejas con coordenadas cilíndricas y solo 3 particiones:



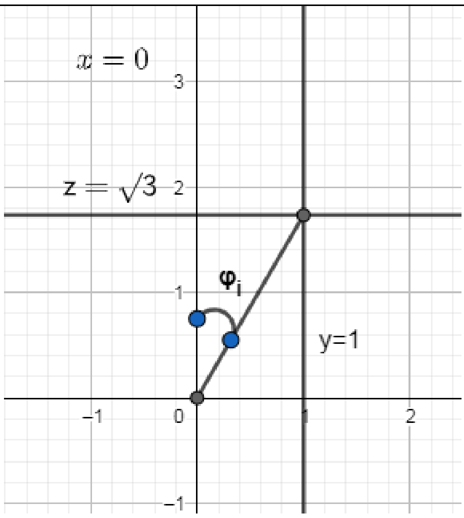
$$S_1:\begin{cases}\Phi(\theta,z) = (\cos(\theta), sen(\theta), z) \\ R_\Phi:\begin{cases} 0 \le \theta \le 2\pi \\ -\sqrt{3} \le z \le \sqrt{3}\end{cases}\end{cases}$$



$$S_3:\begin{cases}\Phi(r,\theta) = \left(\mathrm{r}\cos(\theta), rsen(\theta), \sqrt{1-r^2}\right) \\ R_\Phi:\begin{cases} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi\end{cases}\end{cases}$$

$$S_4:\begin{cases}\Phi(r,\theta) = \left(\mathrm{r}\cos(\theta), rsen(\theta), -\sqrt{1-r^2}\right) \\ R_\Phi:\begin{cases} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi\end{cases}\end{cases}$$

Método 3:Considerando que se puede resolver las tapas del cilindro con derivadas menos complejas con coordenadas esféricas de radio fijo, solo hace falta conocer el límite de φ:

<table>
<tr>
<td>


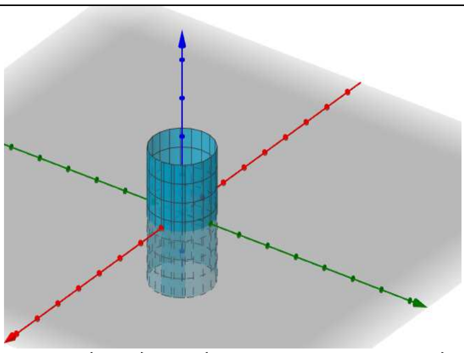
</td>
<td>
En la tapa superior

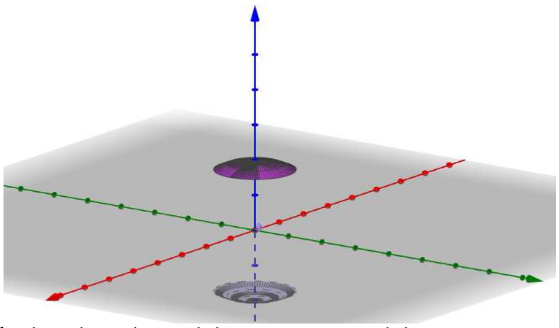
$$tan(\varphi_{\mathrm{final}}) = \frac{\mathrm{Opuesto}}{\mathrm{Adyacente}}$$

$$tan(\varphi_{\mathrm{final}}) = \frac{\mathrm{y}}{\mathrm{z}} = \frac{1}{\sqrt{3}}$$

$$\varphi_{\mathrm{final}} = \frac{\pi}{6}$$

En la tapa inferior

$$\varphi_{\mathrm{inicial}} = \pi - \frac{\pi}{6} = \frac{5}{6}\pi$$
</td>
</tr>
<tr>
<td>


$$S_1\colon\begin{cases}\Phi(\theta, z) = (\cos(\theta), sen(\theta), z)\\ R_\Phi\colon\begin{cases}0 \le \theta \le 2\pi\\ -\sqrt{3} \le z \le \sqrt{3}\end{cases}\end{cases}$$
</td>
<td>


$$S_3\colon\begin{cases}\Phi(\theta, \varphi) = (2\cos(\theta)\, sen(\varphi), 2\,\mathrm{sen}(\theta)\, sen(\varphi), 2\, cos(\varphi))\\ R_\Phi\colon\begin{cases}0 \le \theta \le 2\pi\\ 0 \le \varphi \le \frac{\pi}{6}\end{cases}\end{cases}$$

$$S_4\colon\begin{cases}\Phi(\theta, \varphi) = (2\cos(\theta)\, sen(\varphi), 2\,\mathrm{sen}(\theta)\, sen(\varphi), 2\, cos(\varphi))\\ R_\Phi\colon\begin{cases}0 \le \theta \le 2\pi\\ \frac{5}{6}\pi \le \varphi \le \pi\end{cases}\end{cases}$$
</td>
</tr>
</table>

Resolviendo con Método 3:

$$FlujoEntrante = I_{Cilindro} + I_{Tapa\, superior} + I_{Tapa\, inferior}$$

Cilindro:

$$\Phi(\theta, z) = (cos(\theta), sen(\theta), z)$$

$$\Phi_\theta(\theta, z) = (-sen(\theta), \cos(\theta), 0)$$

$$\Phi_z(\theta, z) = (0,0,1)$$

$$\Phi_\theta X \Phi_z = \begin{bmatrix} i & j & k \\ -\operatorname{sen}(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Phi_\theta X \Phi_z = \left(\begin{bmatrix} \cos(\theta) & 0 \\ 0 & 1 \end{bmatrix}, - \begin{bmatrix} -\operatorname{sen}(\theta) & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -\operatorname{sen}(\theta) & 2\cos(\theta) \\ 0 & 0 \end{bmatrix} \right)$$

$$\Phi_\theta X \Phi_z = (\cos(\theta), \operatorname{sen}(\theta), 0)$$

$$¿\, Que\ direccion\ posee?$$

$$si: \begin{cases} \theta = 0 \\ z = 0 \end{cases} \rightarrow \Phi_\theta X \Phi_z = (1,0,0)$$

$$N = -\Phi_\theta X \Phi_z = (-\cos(\theta), -\operatorname{sen}(\theta), 0)$$

N es la dirección buscada

$$F(\Phi) = \Phi(\theta, z) = (cos(\theta), sen(\theta), z)$$

$$F(\Phi) \cdot N = -\cos^2(\theta) - sen^2(\theta) + 0 = -1$$

$$I_{Cilindro} = \int_0^{2\pi} \int_{-\sqrt{3}}^{\sqrt{3}} -1 \mathrm{dzd\theta}$$

$$I_{Cilindro} = \int_0^{2\pi} \int_{-\sqrt{3}}^{\sqrt{3}} -1 \mathrm{dzd\theta} = -4\sqrt{3}\pi$$

Tapas:

$$\Phi(\theta,\varphi) = (2\cos(\theta)\, sen(\varphi), 2\,\mathrm{sen}(\theta)\, sen(\varphi), 2\, cos(\varphi))$$

$$\Phi_\theta = (-2\mathrm{sen}(\theta)\, sen(\varphi), 2\cos(\theta)\, sen(\varphi), 0)$$

$$\Phi_\varphi = (2\cos(\theta)\, cos(\varphi), 2\,\mathrm{sen}(\theta)\, cos(\varphi), -2sen(\varphi))$$

$$\Phi_\theta X \Phi_\varphi = \begin{bmatrix} i & j & k \\ -2\mathrm{sen}(\theta)\, sen(\varphi) & 2\cos(\theta)\, sen(\varphi) & 0 \\ 2\cos(\theta)\, cos(\varphi) & 2\,\mathrm{sen}(\theta)\, cos(\varphi) & -2sen(\varphi) \end{bmatrix}$$

$$\Phi_\theta X \Phi_\varphi = \left(-4\cos(\theta)\, sen^2(\varphi), -4\,\mathrm{sen}(\theta)\, sen^2(\varphi), -4\,\mathrm{sen}(\varphi)\, cos(\varphi)\right)$$

$$¿\, Que\ direccion\ posee?$$

$$si: \begin{cases} \theta = 0 \\ \varphi = 0 \end{cases} \to \Phi_\theta X \Phi_\varphi = (0,0,0) no\ sirve\ para\ determinar$$

Pero

$$\Phi_\theta X \Phi_\varphi = 2\,\mathrm{sen}(\varphi) \underbrace{\left(-\cos(\theta)\,\mathrm{sen}(\varphi), -\mathrm{sen}(\theta)\,\mathrm{sen}(\varphi), -cos(\varphi)\right)}_{v}$$

$$si: \begin{cases} \theta = 0 \\ \varphi = 0 \end{cases} \to v = (0,0,-1)\ direccion\ entrante$$

es la dirección buscada

$$F(\Phi) = \Phi(\theta,\varphi) = (2\cos(\theta)\, sen(\varphi), 2\,\mathrm{sen}(\theta)\, sen(\varphi), 2cos(\varphi))$$

$$F(\Phi)\cdot \Phi_\theta X \Phi_\varphi = -8\cos^2(\theta)\, sen^3(\varphi) - 8\,\mathrm{sen}^2(\theta)\, sen^3(\varphi) - 8\,\mathrm{sen}(\varphi)\cos^2(\varphi)$$

$$F(\Phi)\cdot \Phi_\theta X \Phi_\varphi = -8sen^3(\varphi) - 8\,\mathrm{sen}(\varphi)\cos^2(\varphi)$$

$$F(\Phi)\cdot \Phi_\theta X \Phi_\varphi = -8\mathrm{sen}(\varphi)\, sen^2(\varphi) - 8\,\mathrm{sen}(\varphi)\cos^2(\varphi)$$

$$F(\Phi)\cdot \Phi_\theta X \Phi_\varphi = -8\mathrm{sen}(\varphi)$$

$$I_{Tapa\ superior} = \int_0^{2\pi}\int_0^{\frac{\pi}{6}} -8\mathrm{sen}(\varphi)\, d\varphi d\theta$$

$$I_{Tapa\ inferior} = \int_0^{2\pi}\int_{\frac{5}{6}\pi}^{\pi} -8\mathrm{sen}(\varphi)\, d\varphi d\theta$$

$$I_{Tapa\,superior} = \int_{0}^{2\pi} \int_{0}^{\frac{\pi}{6}} -8\mathrm{sen}(\varphi)\, \mathrm{d}\varphi \mathrm{d}\theta$$

$$I_{Tapa\,superior} = -8 \int_{0}^{2\pi} \mathrm{d}\theta \int_{0}^{\frac{\pi}{6}} \mathrm{sen}(\varphi)\, \mathrm{d}\varphi$$

$$I_{Tapa\,superior} = -8(2\pi)[-\cos\varphi]_{0}^{\frac{\pi}{6}}$$

$$I_{Tapa\,superior} = 8(2\pi)[\cos\varphi]_{0}^{\frac{\pi}{6}}$$

$$I_{Tapa\,superior} = 8(2\pi)\left(\cos\frac{\pi}{6} - \cos 0\right)$$

$$I_{Tapa\,superior} = 16\pi\left(\frac{\sqrt{3}}{2} - 1\right)$$

$$I_{Tapa\,superior} = \int_{0}^{2\pi} \int_{0}^{\frac{\pi}{6}} -8\mathrm{sen}(\varphi)\, \mathrm{d}\varphi \mathrm{d}\theta$$

$$I_{Tapa\,inferior} = -8 \int_{0}^{2\pi} \mathrm{d}\theta \int_{\frac{5}{6}\pi}^{\pi} \mathrm{sen}(\varphi)\, \mathrm{d}\varphi$$

$$I_{Tapa\,inferior} = -8(2\pi)[-\cos\varphi]_{\frac{5}{6}\pi}^{\pi}$$

$$I_{Tapa\,inferior} = 8(2\pi)[-\cos\varphi]_{\frac{5}{6}\pi}^{\pi}$$

$$I_{Tapa\,inferior} = 8(2\pi)\left(\cos\pi - \cos\frac{5}{6}\pi\right)$$

$$I_{Tapa\,inferior} = 16\pi\left(-1 - \left(-\frac{\sqrt{3}}{2}\right)\right)$$

$$I_{Tapa\,inferior} = 16\pi\left(\frac{\sqrt{3}}{2} - 1\right)$$

$$FlujoEntrante = I_{Cilindro} + I_{Tapa\ superior} + I_{Tapa\ inferior}$$

$$FlujoEntrante = -4\sqrt{3}\pi + 16\pi\left(\frac{\sqrt{3}}{2} - 1\right) + 16\pi\left(\frac{\sqrt{3}}{2} - 1\right)$$

$$FlujoEntrante = -4\sqrt{3}\pi + 32\pi\left(\frac{\sqrt{3}}{2} - 1\right) \simeq -35.23$$

$$FlujoEntrante = -\frac{8\sqrt{3}\pi}{2} + 32\pi\frac{\sqrt{3}}{2} - 32\pi \simeq -35.23$$

$$FlujoEntrante = -8\pi\frac{\sqrt{3}}{2} + 32\pi\frac{\sqrt{3}}{2} - 32\pi \simeq -35.23$$

$$FlujoEntrante = 24\pi\frac{\sqrt{3}}{2} - 32\pi \simeq -35.23$$

$$FlujoEntrante = 12\sqrt{3}\pi - 32\pi \simeq -35.23$$

Corolario.

Que el flujo entrante resulte negativo significa que en realidad el flujo sobre la superficie es saliente