T P 7 Ej 23 b

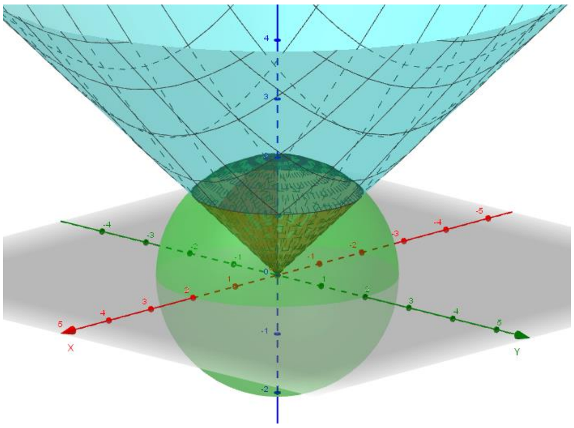
Calcular la integral:

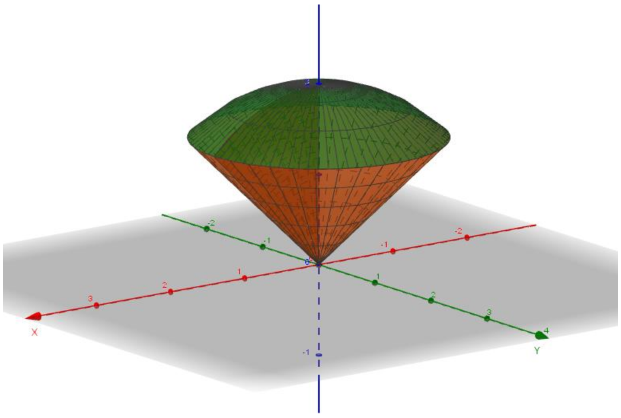
$$\iiint_V \frac{1}{\sqrt{x^2+y^2+z^2}}\,dx\,dy\,dz$$

Donde

$$V=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2\le 4\}\cup\left\{(x,y,z)\in\mathbb{R}^3 : z\ge\sqrt{x^2+y^2}\right\}$$

En principio representamos el sólido en el cual calcularemos la integral.





Utilizando coordenadas esféricas.

$$\begin{cases} x=\rho\cdot\cos(\theta)\cdot sen(\phi)\\ y=\rho\cdot sen(\theta)\cdot sen(\phi)\\ z=\rho\cdot\cos(\phi)\end{cases}$$

Siendo

$$J=\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}=\rho^2\cdot sen(\phi)$$

Con

$$0\le\theta\le 2\pi$$

$$0\le\phi\le\frac{\pi}{4}$$

$$0\le\rho\le 2$$

Aplicando el cambio de variable en la integral nos queda.

$$\int_{\rho=0}^{2}\int_{\phi=0}^{\frac{\pi}{4}}\int_{\theta=0}^{2\pi}\left(\frac{1}{\sqrt{\left(\rho\cdot\cos(\theta)\cdot sen(\phi)\right)^2+\left(\rho\cdot sen(\theta)\cdot sen(\phi)\right)^2+(\rho\cdot\cos(\phi))^2}}\right)\cdot\rho^2\cdot sen(\phi)\;d\rho d\theta d\phi$$

$$\int_{\rho=0}^{2}\int_{\phi=0}^{\frac{\pi}{4}}\int_{\theta=0}^{2\pi}\left(\frac{\rho^2\cdot sen(\phi)}{\sqrt{\rho^2(\cos^2(\theta)sen^2(\phi)+\ \mathrm{sen}^2(\theta)sen^2(\phi)+\cos^2(\phi))}}\right)d\rho d\theta d\phi$$

$$\int_{\rho=0}^{2}\int_{\phi=0}^{\frac{\pi}{4}}\int_{\theta=0}^{2\pi}\left(\frac{\rho^2\cdot sen(\phi)}{\sqrt{\rho^2\left(sen^2(\phi)\cdot\left[\underbrace{\cos^2(\theta)+sen^2(\theta)}_{1}\right]+\cos^2(\theta)\right)}}\right)d\rho d\theta d\phi$$

$$\int_{\rho=0}^{2}\int_{\phi=0}^{\frac{\pi}{4}}\int_{\theta=0}^{2\pi}\left(\frac{\rho^2\cdot sen(\phi)}{\sqrt{\rho^2\left(sen^2(\phi)\cdot\left[\underbrace{\cos^2(\theta)+sen^2(\theta)}_{1}\right]+\cos^2(\theta)\right)}}\right)d\rho d\theta d\phi$$

$$\int_{\rho=0}^{2}\int_{\phi=0}^{\frac{\pi}{4}}\int_{\theta=0}^{2\pi}\left(\frac{\rho^2\cdot sen(\phi)}{\sqrt{\rho^2\left(\underbrace{sen^2(\phi)+\cos^2(\theta)}_{1}\right)}}\right)d\rho d\theta d\phi$$

$$\int_{\rho=0}^{2}\int_{\phi=0}^{\frac{\pi}{4}}\int_{\theta=0}^{2\pi}\left(\frac{\rho^2\cdot sen(\phi)}{\sqrt{\rho^2}}\right)d\rho d\theta d\phi$$

$$\int_{\rho=0}^{2}\int_{\phi=0}^{\frac{\pi}{4}}\int_{\theta=0}^{2\pi}\left(\frac{\rho^2\cdot sen(\phi)}{\rho}\right)d\rho d\theta d\phi$$

$$\int_{\rho=0}^{2}\int_{\phi=0}^{\frac{\pi}{4}}\int_{\theta=0}^{2\pi}\left(\rho\cdot sen(\phi)\right)\ d\rho d\theta d\phi$$

Estamos en condiciones de resolver la integral triple reescribiéndola se la siguiente manera:

Con lo cual, la integral triple queda definida como:

$$\int_0^{2\pi}\left[\int_0^{2}\left[\int_0^{\frac{\pi}{4}}(\rho\cdot sen(\phi))\ d\phi\right]d\rho\right]d\theta$$

Resolvemos la primera integral respecto de θ

$$\int_0^{\frac{\pi}{4}}(\rho\cdot sen(\phi))\ d\phi=-\rho\cdot\cos(\phi)\Big|_0^{\pi/4}=-\frac{\sqrt{2}}{2}\cdot\rho+\rho$$

Reemplazando en la integral original nos queda

$$\int_0^{2\pi}\left[\int_0^{2}\left(-\frac{\sqrt{2}}{2}\cdot\rho+\rho\right)\ d\rho\right]d\theta$$

Ahora se resuelve la integral respecto de la variable ρ

$$\int_0^{2}\left(-\frac{\sqrt{2}}{2}\cdot\rho+\rho\right)\ d\rho=-\frac{\rho^2}{4}\sqrt{2}+\frac{\rho^2}{2}\Big|_0^2=-\sqrt{2}+2$$

Reemplazando en la integral nos queda simplemente calcularla respecto de la variable θ

$$\int_0^{2\pi}(-\sqrt{2}+2)d\theta=(-\sqrt{2}+2)\cdot\theta\Big|_0^{2\pi}=2\pi(-\sqrt{2}+2)$$