

Resolución TP7:

Ejercicio 23 -a- modificado

Resolver la integral triple I con el recinto V.

$$V: \{(x, y, z) \in \mathbb{R}^3 / x^2 + y^2 + z^2 \leq 2 \wedge x^2 + y^2 \leq z\}$$

$$I = \iiint_V x^2 + y^2 dx dy dz$$

$$x^2 + y^2 + z^2 \leq 2 \wedge x^2 + y^2 \leq z$$

$$z^2 \leq 2 - (x^2 + y^2)$$

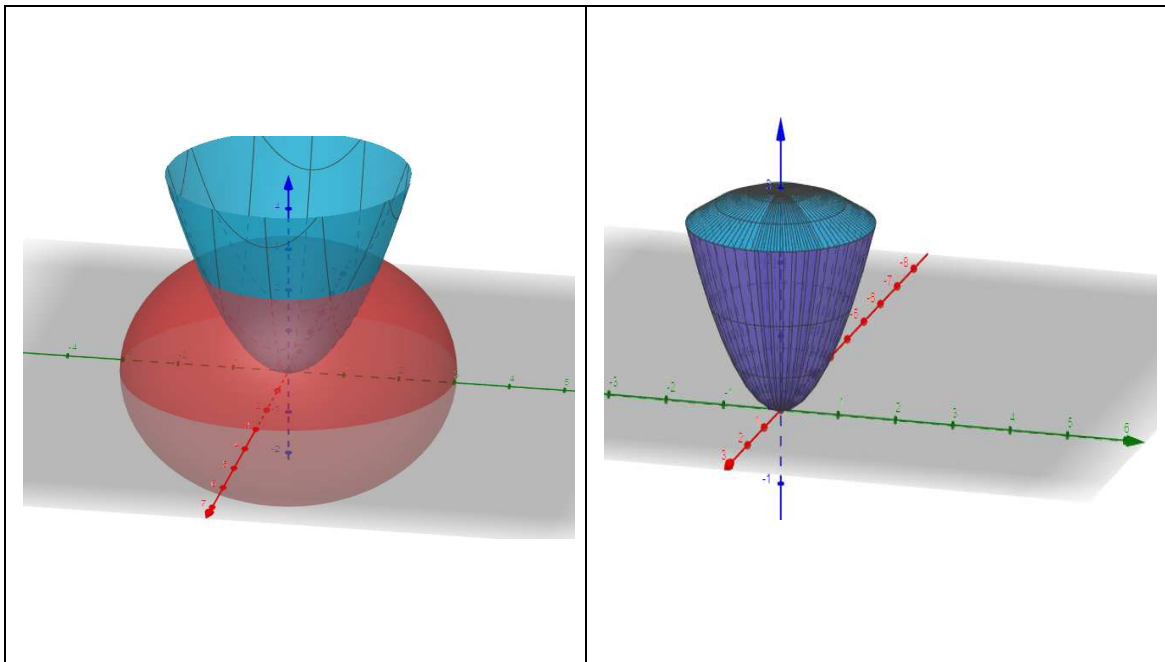
$$-\sqrt{2 - (x^2 + y^2)} \leq z \leq \sqrt{2 - (x^2 + y^2)}$$

$$x^2 + y^2 \leq z \text{ implica } 0 \leq z \text{ (que } z \text{ es positivo)}$$

$$\text{por lo que solo vale } z \leq \sqrt{2 - (x^2 + y^2)}$$

por transitividad

$$x^2 + y^2 \leq z \leq \sqrt{2 - (x^2 + y^2)}$$



Buscando Limites para x e y:

Tomando la interseccion:

$$x^2 + y^2 + z^2 = 2 \wedge x^2 + y^2 = z$$

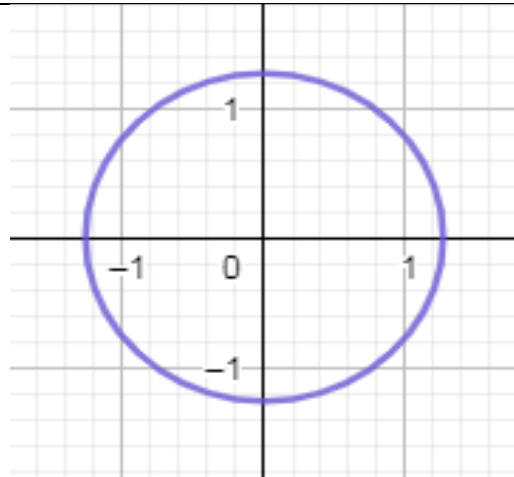
$$z + z^2 = 2 \rightarrow \begin{cases} z_1 = \frac{-1+3}{2} = 1 \\ z_2 = \frac{-1-3}{2} = -2 \end{cases}$$

$x^2 + y^2 = z$ indica que z es positivo por lo que vale solo

$$z_1 = 1$$

por transitividad tomamos la

$$\text{proyeccion } x^2 + y^2 \leq 1$$



En resumen

$$V: \begin{cases} x^2 + y^2 \leq z \leq \sqrt{2 - (x^2 + y^2)} \\ x^2 + y^2 \leq 1 \end{cases}$$

$$V: \begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \\ |J| = r \\ V' = \begin{cases} r^2 \leq z \leq \sqrt{2 - r^2} \\ r^2 \leq 1 \end{cases} \end{cases}$$

$$V: \begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \\ |J| = r \\ V' = \begin{cases} r^2 \leq z \leq \sqrt{2 - r^2} \\ 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{cases} \end{cases}$$

$$I = \iiint_V x^2 + y^2 dx dy dz = \iiint_{V'} r^2 r dr d\theta dz$$

$$I = \int_0^{2\pi} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} r^3 dz dr d\theta$$

$$I = \int_0^{2\pi} \int_0^1 r^3 (\sqrt{2-r^2} - r^2) dr d\theta$$

$$I = \int_0^{2\pi} \int_0^1 (r^3 \sqrt{2-r^2} - r^5) dr d\theta$$

$$I = 2\pi \int_0^1 (r^3 \sqrt{2-r^2} - r^5) dr$$

$$I = 2\pi \left(\int_0^1 r^3 \sqrt{2-r^2} dr - \int_0^1 r^5 dr \right)$$

$$\int r^3 \sqrt{2-r^2} dr = ?$$

sustitucion $2 - r^2 = t \rightarrow -2rdr = dt \rightarrow rdr = -\frac{dt}{2}$

$$\int r^2 r \sqrt{2-r^2} dr = - \int \frac{r^2 \sqrt{t} dt}{2} = -\frac{1}{2} \int r^2 \sqrt{t} dt$$

$$2 - r^2 = t \rightarrow r^2 = 2 - t$$

$$\int r^2 \sqrt{t} dt = \int (2-t) \sqrt{t} dt$$

$$\int r^3 \sqrt{2-r^2} dr \stackrel{2-r^2=t}{\cong} -\frac{1}{2} \int (2-t) \sqrt{t} dt$$

$$\int 2\sqrt{t} dt - \int \sqrt{t^3} dt$$

$$2\frac{2}{3}\sqrt{t^3} - \frac{2}{5}\sqrt{t^5}$$

$$\frac{4}{3}\sqrt{t^3} - \frac{2}{5}\sqrt{t^5}$$

$$\int r^3 \sqrt{2-r^2} dr \stackrel{2-r^2=t}{\cong} -\frac{1}{2} \int (2-t) \sqrt{t} dt = -\frac{1}{2} \left(\frac{4}{3}\sqrt{t^3} - \frac{2}{5}\sqrt{t^5} \right)$$

$$\int r^3 \sqrt{2-r^2} dr \stackrel{2-r^2=t}{\cong} -\frac{2}{3}\sqrt{t^3} + \frac{1}{5}\sqrt{t^5}$$

$$\int r^3 \sqrt{2-r^2} dr = -\frac{2}{3}\sqrt{(2-r^2)^3} + \frac{1}{5}\sqrt{(2-r^2)^5}$$

$$I = 2\pi \left(\int_0^1 r^3 \sqrt{2-r^2} dr - \int_0^1 r^5 dr \right)$$

$$I = 2\pi \left[-\frac{2}{3} \sqrt{(2-r^2)^3} + \frac{1}{5} \sqrt{(2-r^2)^5} - \frac{r^6}{6} \right]_0^1$$

$$P(r) = -\frac{2}{3} \sqrt{(2-r^2)^3} + \frac{1}{5} \sqrt{(2-r^2)^5} - \frac{r^6}{6}$$

$$P(1) = -\frac{2}{3} \sqrt{(2-1)^3} + \frac{1}{5} \sqrt{(2-1)^5} - \frac{1}{6} = -\frac{2}{3} + \frac{1}{5} - \frac{1}{6} = -\frac{19}{30} \sim -0.6\hat{3}$$

$$P(0) = -\frac{2}{3} \sqrt{(2-0)^3} + \frac{1}{5} \sqrt{(2-0)^5} - \frac{0}{6} = -\frac{4}{3} \sqrt{2} + \frac{4}{5} \sqrt{2} = -\frac{8}{15} \sqrt{2} \sim -0.7542472\hat{3}$$

$$I = 2\pi \left(\underbrace{-\frac{19}{30}}_{-0.6\hat{3}} + \underbrace{\frac{8}{15} \sqrt{2}}_{\sim 0.75} \right)$$

$$I > 0 \text{ observe } f(x, y, z) = x^2 + y^2$$

$$I = \pi \left(-\frac{19}{15} + \frac{16}{15} \sqrt{2} \right)$$

$$I = \frac{\pi}{15} (-19 + 16\sqrt{2})$$

$$I = \frac{\pi}{15} (16\sqrt{2} - 19)$$