Resolución TP10:

Ejercicio 3 - a

Calcular el área de la superficie de la grafica del paraboloide de ecuación $z = x^2 + y^2$, limitado superiormente por $z = 1$ usando coordenadas cartesianas y polares.

Resolviendo:

Sabemos que

$$Area(S) = \iint_R ||\Phi_u X \Phi_v|| dudv$$

En el caso de coordenadas cartesianas:

$$S: \begin{cases} \Phi(x,y) = (x, y, x^2 + y^2) \\ Dom\Phi = \{(x,y)\epsilon\mathbb{R}^2 / \ \ x^2 + y^2 \leq 1\} \end{cases} \rightarrow Area(S) = \iint_{x^2+y^2 \leq 1} \left\|\Phi_x X \Phi_y\right\| dxdy$$

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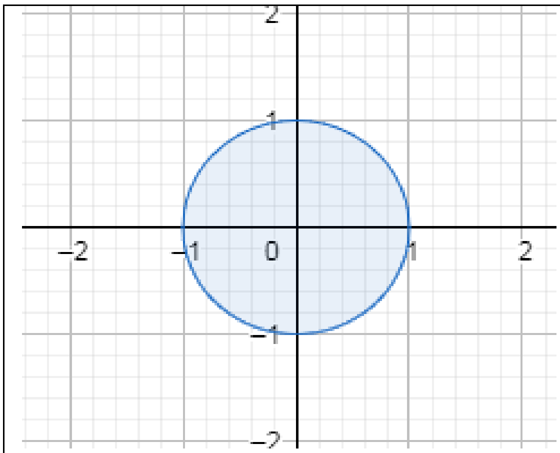
<td>
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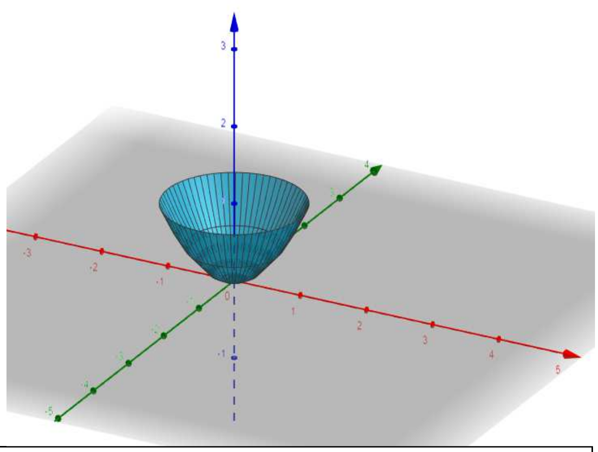
<td></td>
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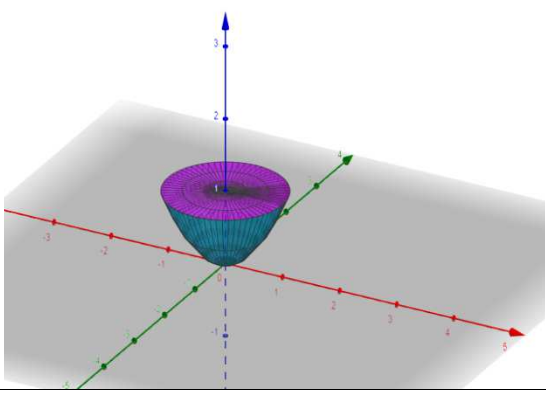
<tr>
<td>No se debe confundir con la grafica
</td>
<td>En dicho caso, se habria pedido

"Calcular el área de la superficie del volumen encerrado por paraboloide de ecuación $z = x^2 + y^2$, y el plano $z = 1$"

"Calcular el área de la superficie del volumen $x^2 + y^2 \leq \ z \leq 1$ "</td>
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$$\Phi(x,y) = (x, y, x^2 + y^2)$$

$$\Phi_x(x,y) = (1,0,2x)$$

$$\Phi_y(x,y) = (0,1,2y)$$

$$\left|\Phi_x X \Phi_y\right| = \begin{bmatrix} i & j & k \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{bmatrix} = \left(\begin{bmatrix} 0 & 2x \\ 1 & 2y \end{bmatrix}, -\begin{bmatrix} 1 & 2x \\ 0 & 2y \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = (-2x, -2y, 1)$$

$$\left|\left|\Phi_x X \Phi_y\right|\right| = \sqrt{(-2x)^2 + (-2y)^2 + (1)^2} = \sqrt{4x^2 + 4y^2 + 1}$$

$$Area(S) = \iint_{x^2+y^2\leq 1} \left|\left|\Phi_x X \Phi_y\right|\right| dxdy$$

$$Area(S) = \iint_{x^2+y^2\leq 1} \sqrt{4x^2 + 4y^2 + 1}\, dxdy$$

$$Area(S) \overset{\substack{Transformacion \\ Polar}}{\triangleq} \int_0^{2\pi} \int_0^1 r\sqrt{4r^2 + 1}\, dr\, d\alpha$$

$$Area(S) = \int_0^{2\pi} \frac{1}{12}\left[\sqrt{(4r^2+1)^3}\right]_0^1 d\alpha$$

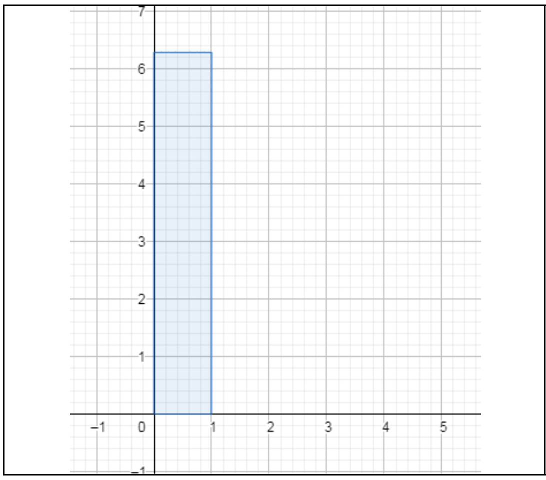
$$Area(S) = \int_0^{2\pi} \frac{1}{12}\left[\sqrt{(5)^3} - \sqrt{(1)^3}\right] d\alpha$$

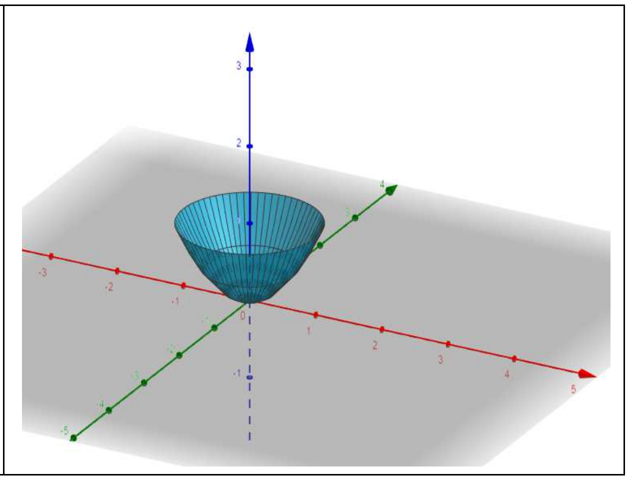
$$Area(S) = \int_0^{2\pi} \frac{1}{12}\left[5\sqrt{5} - 1\right] d\alpha$$

$$Area(S) = \frac{5\sqrt{5} - 1}{6}\pi$$

En el caso de coordenadas polares:

$$S:\begin{cases}\Phi(r,\alpha)=(rcos(\alpha),rsen(\alpha),r^2)\\ Dom\Phi=[0,2\pi]X[0,1]\end{cases}\rightarrow Area(S)=\iint_{[0,2\pi]X[0,1]}\left||\Phi_r X\Phi_\alpha|\right|drd\alpha$$





$$\Phi(r,\alpha)=(rcos(\alpha),rsen(\alpha),r^2)$$

$$\Phi_r(r,\alpha)=(\cos(\alpha)\,,\operatorname{sen}(\alpha)\,,2r)$$

$$\Phi_\alpha(r,\alpha)=(-r\operatorname{sen}(\alpha)\,,r\cos(\alpha)\,,0)$$

$$|\Phi_r X\Phi_\alpha|=\begin{bmatrix} i & j & k\\ \cos(\alpha) & \operatorname{sen}(\alpha) & 2r\\ -r\operatorname{sen}(\alpha) & r\cos(\alpha) & 0\end{bmatrix}$$

$$|\Phi_r X\Phi_\alpha|=\left(\begin{bmatrix}\operatorname{sen}(\alpha) & 2r\\ r\cos(\alpha) & 0\end{bmatrix},-\begin{bmatrix}\cos(\alpha) & 2r\\ -r\operatorname{sen}(\alpha) & 0\end{bmatrix},\begin{bmatrix}\cos(\alpha) & \operatorname{sen}(\alpha)\\ -r\operatorname{sen}(\alpha) & r\cos(\alpha)\end{bmatrix}\right)$$

$$|\Phi_r X\Phi_\alpha|=\left(0-2r^2\cos(\alpha)\,,-\left(0-(-2r^2\operatorname{sen}(\alpha))\right),r\cos(\alpha)-\ (-rsen(\alpha))\right)$$

$$|\Phi_r X\Phi_\alpha|=(-2r^2\cos(\alpha)\,,-2r^2\operatorname{sen}(\alpha)\,,r)$$

$$\left||\Phi_r X\Phi_\alpha|\right|=\sqrt{(-2r^2\cos(\alpha))^2+(-2r^2\operatorname{sen}(\alpha))^2+(r)^2}=r\sqrt{4r^2+1}$$

$$\iint_{[0,2\pi]X[0,1]}\left||\Phi_r X\Phi_\alpha|\right|drd\alpha\ =\int_0^{2\pi}\int_0^1 r\sqrt{4r^2+1}\,dr\,d\alpha=\frac{5\sqrt{5}-1}{6}\pi$$

Observación: No se necesito hacer transformaciones al resolver el área en este ultimo método.