Adicionales:

Ejercicio 16-Adicional

Tomando $F(x, y, z) = 0$ que define $z = f(x, y)$ implicitamente. Hallar las derivadas de segundo orden.

Resolución:

Herramientas:

- Se utilizar regla de la cadena.

Para empezar:

En este caso podemos componer $H(x, y) = F(x, y, z = f(x, y)))$

Derivadas de H:

$H(x, y)$ se puede derivar en x y en y.

Para H_x:

$$H_x = \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}$$

Sabemos que $\frac{\partial x}{\partial x} = 1$, $\frac{\partial y}{\partial x} = 0$, $\frac{\partial z}{\partial x} = f_x$

$$H_x = F_x + F_z f_x$$

Si $F(P) = 0$ entonces $H(x_0) = 0$ entonces derivando lado a lado $H_x(x_0) = 0$

$$f_x(x_0) = -\frac{F_x(P)}{F_z(P)}$$

Para H_y:

$$H_y = \frac{\partial F}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y}$$

Sabemos que $\frac{\partial x}{\partial y} = 0$, $\frac{\partial y}{\partial y} = 1$, $\frac{\partial z}{\partial y} = f_y$

$$H_y = F_y + F_z f_y$$

Si $F(P) = 0$ entonces $H(x_0) = 0$ entonces derivando lado a lado $H_y(x_0) = 0$

$$f_y(x_0) = -\frac{F_y(P)}{F_z(P)}$$

$H_x(x)$ se puede derivar en x y en y.

$H_y(x)$ se puede derivar en x y en y.

$F_x(x, y, z)$ se puede derivar en x y en y por lo que se le aplica el mismo proceso que a H(x)

$F_y(x, y, z)$ se puede derivar en x y en y por lo que se le aplica el mismo proceso que a H(x)

$F_z(x, y, z)$ se puede derivar en x y en y por lo que se le aplica el mismo proceso que a H(x)

A los productos $F_z f_x$ y $F_y f_x$ se le aplica $uv = u'v + uv'$

Para H_{xx}:

$$H_{xx} = \left(\frac{\partial^2 \mathrm{F}}{\partial^2 \mathrm{x}}\frac{\partial \mathrm{x}}{\partial \mathrm{x}} + \frac{\partial^2 \mathrm{F}}{\partial \mathrm{x}\,\partial \mathrm{y}}\frac{\partial \mathrm{y}}{\partial \mathrm{x}} + \frac{\partial^2 \mathrm{F}}{\partial \mathrm{x}\,\partial \mathrm{z}}\frac{\partial \mathrm{z}}{\partial \mathrm{x}}\right) + \left(\frac{\partial^2 \mathrm{F}}{\partial \mathrm{z}\,\partial \mathrm{x}}\frac{\partial \mathrm{x}}{\partial \mathrm{x}} + \frac{\partial^2 \mathrm{F}}{\partial \mathrm{z}\,\partial \mathrm{y}}\frac{\partial \mathrm{y}}{\partial \mathrm{x}} + \frac{\partial^2 \mathrm{F}}{\partial^2 \mathrm{z}}\frac{\partial \mathrm{z}}{\partial \mathrm{x}}\right)\frac{\partial \mathrm{z}}{\partial \mathrm{x}} + \frac{\partial \mathrm{F}}{\partial \mathrm{z}}\frac{\partial^2 \mathrm{z}}{\partial^2 \mathrm{x}}$$

Sabemos que $\frac{\partial \mathrm{x}}{\partial \mathrm{x}} = 1$, $\frac{\partial \mathrm{y}}{\partial \mathrm{x}} = 0$, $\frac{\partial \mathrm{z}}{\partial \mathrm{x}} = f_x$ $\frac{\partial^2 \mathrm{z}}{\partial^2 \mathrm{x}} = f_{xx}$ $F_{xz} = F_{zx}$

$$H_{xx} = (F_{xx} + F_{xz}f_x) + (F_{zx}f_x + F_{zz}f_x^2) + F_z f_{xx} = F_{xx} + 2F_{xz}f_x + F_{zz}f_x^2 + F_z f_{xx}$$

Si $H_x(x_0) = 0$ entonces $H_{xx}(x_0) = 0$

$$[F_{xx} + 2F_{xz}f_x + F_{zz}f_x^2 + F_z f_{xx}]_{(P)} = 0$$

$$[F_z f_{xx}]_{(P)} = -[F_{xx} + 2F_{xz}f_x + F_{zz}f_x^2]_{(P)}$$

$$f_{xx}(x_0) = -\left[\frac{F_{xx} + 2F_{xz}f_x + F_{zz}f_x^2}{F_z}\right]_{(P)}$$

Sabemos que $f_x(x_0) = -\frac{F_x(P)}{F_z(P)}$

$$f_{xx}(x_0) = -\left[\frac{F_{xx} + 2F_{xz}(-\frac{F_x}{F_z}) + F_{zz}\left(-\frac{F_x}{F_z}\right)^2}{F_y}\right]_{(P)}$$

$$f_{xx}(x_0) = -\left[\frac{F_{xx} - 2F_{xy}\frac{F_x}{F_z} + F_{zz}\frac{F_x^2}{F_z^2}}{F_z}\right]_{(P)}$$

$$f_{xx}(x_0) = -\left[\frac{F_{xx}F_z^2 - 2F_{xz}F_z + F_{zz}F_x^2}{F_z^3}\right]_{(P)}$$

Para H_{xy}:

$$H_{xy} = \left(\frac{\partial^2 \mathrm{F}}{\partial^2 \mathrm{x}}\frac{\partial \mathrm{x}}{\partial \mathrm{y}} + \frac{\partial^2 \mathrm{F}}{\partial \mathrm{x}\,\partial \mathrm{y}}\frac{\partial \mathrm{y}}{\partial \mathrm{y}} + \frac{\partial^2 \mathrm{F}}{\partial \mathrm{x}\,\partial \mathrm{z}}\frac{\partial \mathrm{z}}{\partial \mathrm{y}}\right) + \left(\frac{\partial^2 \mathrm{F}}{\partial \mathrm{z}\,\partial \mathrm{x}}\frac{\partial \mathrm{x}}{\partial \mathrm{y}} + \frac{\partial^2 \mathrm{F}}{\partial \mathrm{z}\,\partial \mathrm{y}}\frac{\partial \mathrm{y}}{\partial \mathrm{y}} + \frac{\partial^2 \mathrm{F}}{\partial^2 \mathrm{z}}\frac{\partial \mathrm{z}}{\partial \mathrm{y}}\right)\frac{\partial \mathrm{z}}{\partial \mathrm{x}} + \frac{\partial \mathrm{F}}{\partial \mathrm{z}}\frac{\partial^2 \mathrm{z}}{\partial \mathrm{x}\,\partial \mathrm{y}}$$

Sabemos que $\frac{\partial \mathrm{x}}{\partial \mathrm{x}} = 1$, $\frac{\partial \mathrm{y}}{\partial \mathrm{x}} = 0$, $\frac{\partial \mathrm{z}}{\partial \mathrm{x}} = f_x$ $\frac{\partial^2 \mathrm{z}}{\partial \mathrm{x}\,\partial \mathrm{y}} = f_{xy}$

$$H_{xx} = (F_{xy} + F_{xz}f_y) + \left(F_{zy}f_x + F_{zz}f_yf_x\right) + F_zf_{xy}$$

Si $H_x(x_0) = 0$ entonces $H_{xy}(x_0) = 0$

$$\left[F_{xy} + F_{xz}f_y + F_{zy}f_x + F_{zz}f_yf_x + F_zf_{xy}\right]_{(P)} = 0$$

$$\left[F_zf_{xy}\right]_{(P)} = -\left[F_{xy} + F_{xz}f_y + F_{zy}f_x + F_{zz}f_yf_x\right]_{(P)}$$

$$f_{xy}(x_0) = -\left[\frac{F_{xy} + F_{xz}f_y + F_{zy}f_x + F_{zz}f_yf_x}{F_z}\right]_{(P)}$$

Sabemos que $f_x(x_0) = -\frac{F_x(P)}{F_z(P)}$ y $f_y(x_0) = -\frac{F_y(P)}{F_z(P)}$

$$f_{xy}(x_0) = -\left[\frac{F_{xy} - \frac{F_{xz}F_y}{F_z} - \frac{F_{zy}F_x}{F_z} + \frac{F_{zz}F_yF_x}{F_z^2}}{F_y}\right]_{(P)}$$

$$f_{xy}(x_0) = -\left[\frac{F_{xy}F_z^2 - F_{xz}F_yF_z + F_{zy}F_xF_z + F_{zz}F_yF_x}{F_z^3}\right]_{(P)}$$

Para H_{yx}:

$$H_{yx}=\left(\frac{\partial^2 \mathrm{F}}{\partial \mathrm{y}\,\partial \mathrm{x}}\frac{\partial \mathrm{x}}{\partial \mathrm{x}}+\frac{\partial^2 \mathrm{F}}{\partial^2 \mathrm{y}}\frac{\partial \mathrm{y}}{\partial \mathrm{x}}+\frac{\partial^2 \mathrm{F}}{\partial \mathrm{y}\,\partial \mathrm{z}}\frac{\partial \mathrm{z}}{\partial \mathrm{x}}\right)+\left(\frac{\partial^2 \mathrm{F}}{\partial \mathrm{z}\,\partial \mathrm{x}}\frac{\partial \mathrm{x}}{\partial \mathrm{x}}+\frac{\partial^2 \mathrm{F}}{\partial \mathrm{z}\,\partial \mathrm{y}}\frac{\partial \mathrm{y}}{\partial \mathrm{x}}+\frac{\partial^2 \mathrm{F}}{\partial^2 \mathrm{z}}\frac{\partial \mathrm{z}}{\partial \mathrm{x}}\right)\frac{\partial \mathrm{z}}{\partial \mathrm{y}}+\frac{\partial \mathrm{F}}{\partial \mathrm{z}}\frac{\partial^2 \mathrm{z}}{\partial \mathrm{y}\,\partial \mathrm{x}}$$

Sabemos que $\frac{\partial \mathrm{x}}{\partial \mathrm{x}}=1$, $\frac{\partial \mathrm{y}}{\partial \mathrm{x}}=0,$ $\frac{\partial \mathrm{z}}{\partial \mathrm{x}}=f_x$ $\frac{\partial^2 \mathrm{z}}{\partial^2 \mathrm{x}}=f_{xx}$ $F_{xz}=F_{zx}$

$$H_{xx}=(F_{yx}+F_{yz}f_x)+\left(F_{zx}f_y+F_{zz}f_xf_y\right)+F_zf_{yx}$$

Si $H_x(x_0)=0$ entonces $H_{xy}(x_0)=0$

$$\left[F_{yx}+F_{yz}f_x+F_{zz}f_y+F_{zz}f_xf_y+F_zf_{yx}\right]_{(P)}=0$$

$$\left[F_zf_{yx}\right]_{(P)}=-\left[F_{yx}+F_{yz}f_x+F_{zz}f_y+F_{zz}f_xf_y\right]_{(P)}$$

$$f_{yx}(x_0)=-\left[\frac{F_{yx}+F_{yz}f_x+F_{zz}f_y+F_{zz}f_xf_y}{F_z}\right]_{(P)}$$

Sabemos que $f_x(x_0)=-\frac{F_x(P)}{F_z(P)}$ y $f_y(x_0)=-\frac{F_y(P)}{F_z(P)}$

$$f_{yx}(x_0)=-\left[\frac{F_{yx}+F_{yz}\left(-\frac{F_x}{F_z}\right)+F_{zz}\left(-\frac{F_y}{F_z}\right)+F_{zz}\left(-\frac{F_x}{F_z}\right)\left(-\frac{F_y}{F_z}\right)}{F_z}\right]_{(P)}$$

$$f_{yx}(x_0)=-\left[\frac{F_{yx}F_z^2+F_{yz}F_xF_z+F_{zz}F_yF_z+F_{zz}F_xF_y}{F_z^3}\right]_{(P)}$$

Para H_{yy}:

$$H_{xy}=\left(\frac{\partial^2 \mathrm{F}}{\partial \mathrm{y}\,\partial \mathrm{x}}\frac{\partial \mathrm{x}}{\partial \mathrm{y}}+\frac{\partial^2 \mathrm{F}}{\partial^2 \mathrm{y}}\frac{\partial \mathrm{y}}{\partial \mathrm{y}}+\frac{\partial^2 \mathrm{F}}{\partial \mathrm{y}\,\partial \mathrm{z}}\frac{\partial \mathrm{z}}{\partial \mathrm{y}}\right)+\left(\frac{\partial^2 \mathrm{F}}{\partial \mathrm{z}\,\partial \mathrm{x}}\frac{\partial \mathrm{x}}{\partial \mathrm{y}}+\frac{\partial^2 \mathrm{F}}{\partial \mathrm{z}\,\partial \mathrm{y}}\frac{\partial \mathrm{y}}{\partial \mathrm{y}}+\frac{\partial^2 \mathrm{F}}{\partial^2 \mathrm{z}}\frac{\partial \mathrm{z}}{\partial \mathrm{y}}\right)\frac{\partial \mathrm{z}}{\partial \mathrm{y}}+\frac{\partial \mathrm{F}}{\partial \mathrm{z}}\frac{\partial^2 \mathrm{z}}{\partial^2 \mathrm{y}}$$

Sabemos que $\frac{\partial \mathrm{x}}{\partial \mathrm{x}}=1$, $\frac{\partial \mathrm{y}}{\partial \mathrm{x}}=0$, $\frac{\partial \mathrm{z}}{\partial \mathrm{x}}=f_x$ $\frac{\partial^2 \mathrm{z}}{\partial \mathrm{x}\,\partial \mathrm{y}}=f_{xy}$

$$H_{xx}=(F_{yy}+F_{yz}f_y)+\left(F_{zy}f_y+F_{zz}f_y^2\right)+F_zf_{yy}$$

Si $H_x(x_0)=0$ entonces $H_{xy}(x_0)=0$

$$\left[F_{yy}+2F_{yz}f_y+F_{zz}f_y^2+F_zf_{yy}\right]_{(P)}=0$$

$$\left[F_zf_{yy}\right]_{(P)}=-\left[F_{yy}+2F_{yz}f_y+F_{zz}f_y^2\right]_{(P)}$$

$$f_{yy}(x_0)=-\left|\frac{F_{yy}+2F_{yz}f_y+F_{zz}f_y^2}{F_z}\right|_{(P)}$$

Sabemos que $f_y(x_0)=-\frac{F_y(P)}{F_z(P)}$

$$f_{yy}(x_0)=-\left|\frac{F_{yy}+2F_{yz}\left(-\frac{F_y}{F_z}\right)+F_{zz}\left(-\frac{F_y}{F_z}\right)^2}{F_y}\right|_{(P)}$$

$$f_{yy}(x_0)=-\left|\frac{F_{yy}F_z^2-2F_{yz}F_yF_z+F_{zz}F_y^2}{F_z^3}\right|_{(P)}$$