Notes on Approximation Algorithms

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Abstract

A collection of some notes on the design and analysis of approximation algorithms and approximation techniques. Based mainly off of [1, 2].

1 LP-Duality

For a linear program in canonical form, the goal is to find a non-negative, rational vector $x \in \mathbb{Q}^n$ that minimizes a given linear objective function in x subject to some linear constraints on x. The coefficients of the linear objective function can be represented by a vector $c \in \mathbb{Q}^n$, the coefficients of the linear constraints can be represented by a matrix $A = (a_{ij}) \in \mathbb{Q}^{m \times n}$, and the values of the linear constraints can be represented by a vector $b \in \mathbb{Q}^m$.

Definition 1 (Primal and Dual)

Given a linear programming problem in canonical form denoted as (P), we can induce a problem denoted by (D) with the following form:

$$
\begin{array}{llll}
(P) & \text{minimize} & \displaystyle\sum_{j=1}^{n} c_j x_j & \\
& \text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j \geq b_i & i = 1, \ldots, m \\
& & x_j \geq 0 & j = 1, \ldots, n
\end{array}
\qquad
\begin{array}{llll}
(D) & \text{maximize} & \displaystyle\sum_{i=1}^{m} b_i y_i & \\
& \text{subject to} & \displaystyle\sum_{j=1}^{m} a_{ij} y_i \leq c_j & j = 1, \ldots, n \\
& & y_i \geq 0 & i = 1, \ldots, m
\end{array}
$$

where a_{ij}, b_i, and c_i are given rational numbers and y_i corresponds to the ith inequality of (P).

Definition 2 (Primal)

The problem (P) is referred to as the *primal.*

Definition 3 (Dual)

The *dual* of the primal is problem (D).

Note that the dual of a dual program is the primal program. Every feasible solution for the dual serves as a lower bound on the optimal objective function value of the primal. The reverse also holds in that every feasible solution to the primal serves as an upper bound on the optimal objective function value of the dual.

Theorem 1 (Weak Duality)**.** *If* $\boldsymbol{x} = (x_1, \ldots, x_n)$ *is a feasible solution to the LP* (P) *and* $\boldsymbol{y} = (y_1, \ldots, y_m)$ *a feasible solution to the LP* (D)*, then* $\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i$.

Proof.

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i \geq \sum_{i=1}^{m} b_i y_i$$

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As a consequence of Theorem 1, if we find that there exist some feasible solutions to (P) and (D) that have matching objection function values, then these solutions must be optimal.

Theorem 2 (Strong Duality/LP-Duality)**.** *If the LPs (P) and (D) are both feasible, and* $\boldsymbol{x}^* = (x_1^*, \ldots, x_n^*)$ *and* $\boldsymbol{y}^* = (y_1^*, \ldots, y_m^*)$ *are optimal solutions to (P) and (D), respectively, then* $\sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} b_i y_i$.

References

[1] V. V. Vazirani. *Approximation Algorithms*. Springer, 2001.

[2] D. P. Williamson and D. B. Shmoys. *The Design of Approximation Algorithms*. Cambridge University Press, 2011.