Notes on Metric Online Maximum Weighted b-Matching

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1 Prior Work

1.1 Unweighted Problems

Problem 1 (Online Maximum Matching)

Given an unweighted bipartite graph $G = (S \cup R, E)$, the server vertices S are known beforehand and the request vertices R and their incident edges arrive online one by one. When a request $r_i \in R$ arrives, it must either irreversibly be matched to one of its unmatched neighbors $s_k \in S$ or be unmatched forever. The objective is to maximize the number of requests matched.

[6] Karp et al., "An Optimal Algorithm for On-line Bipartite Matching", 1990

- Show that for Online Maximum Matching (OMM), no deterministic algorithm has a competitive ratio better than $\frac{1}{2}$
  - Greedy achieves this ratio
- Show that a randomized algorithm Ranking achieves a ratio of $1 - \frac{1}{e}$ which is optimal
  - [2] give a simplified Primal-Dual analysis of this algorithm
  - [3] give an economics based analysis of this algorithm

[5] Kalyanasundaram and Pruhs, "An Optimal Deterministic Algorithm for Online b-Matching", 2000

- Study the Online Maximum b-Matching (OM b-M) problem which is equivalent to OMM with the addition that the server vertices now have a capacity b (i.e. they can be matched up to b times)
- Show that no deterministic algorithm has a competitive ratio better than $1 - \frac{1}{\left(1+\frac{1}{b}\right)^b}$
  - They give a deterministic algorithm Balance that achieves this ratio
  - Approaches $1 - \frac{1}{e}$ as $b \to \infty$

[1] Albers and Schubert, "Optimal Algorithms for Online b-Matching with Variable Vertex Capacities", 2021

- Study the OM b-M problem where each vertex $s \in S$ can have a variable b_s capacity
- Give a deterministic algorithm RelativeBalance with competitive ratio $1 - \frac{1}{\left(1+\frac{1}{b_{\min}}\right)^{b_{\min}}}$
- Extend the analysis of the original Ranking algorithm to show that it is still $1 - \frac{1}{e}$ competitive

1.2 Weighted Problems

In the weighted version of OMM and its b-Matching variants, there exists no algorithm that can achieve a non-trivial competitive ratio (see Section 2.1). This necessitates the need for some kind of additional assumption on the model.

[4] Kalyanasundaram and Pruhs, "Online Weighted Matching", 1993

- Study the Online Maximum Weight Matching (OMWM) problem under the assumption that the weights of edges represent distances in some metric space.
  - Denote this problem as Metric OMWM
- Show that Greedy for Metric OMWM gives competitive ratio $\frac{1}{3}$
- Show that no deterministic online algorithm can achieve a ratio better than $\frac{1}{3}$

Most research on this metric case has been in finding minimum weight perfect matchings but there is essentially nothing on Metric OMWM. Possible further directions for Metric OMWM:

- Can randomization improve the bounds on the competitive ratio?
- Can allowing for b-Matching improve the bounds as it does in OMM?

2 Metric Online Maximum Weight Matching

2.1 Hardness without Metric Assumption

2.2 The Greedy Algorithm

Algorithm 1 Greedy

Input: Offline vertices S and capacities b_s for all $s \in S$
Output: A matching M

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while a new request r ∈ R arrives do
    N(r) ← the set of neighbors s ∈ S of r with remaining capacity
    if N(r) = ∅ then
        leave r unmatched
    else
        s_max ← arg max_{s∈N(r)} d(s, r)
        M ← M ∪ {(s_max, r)}, b_{s_max} ← b_{s_max} − 1
return M
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Theorem 1. *Greedy is $\frac{1}{3}$ competitive for Metric OMWM*

Proof. Let s_i be the server the algorithm matches request r_i to. By definition of Greedy it must be that

$$d(s_j, r_i) \leq d(s_i, r_i) \text{ for all } j > i. \tag{1}$$

Let $s_{\pi(i)}$ be the server the optimal solution matches r_i to.

Define $D(r_i) = d\big(s_{\pi(i)}, r_i\big) - d(s_i, r_i)$. If $i \leq \pi(i)$ then clearly $D(r_i) \leq 0$ by Eq. (1). If $i > \pi(i)$

consider that $d(s_i, r_{\pi(i)}) \leq d(s_{\pi(i)}, r_{\pi(i)})$ again by Eq. (1). By the triangle inequality,

$$d(s_{\pi(i)}, s_i) \leq d(s_{\pi(i)}, r_{\pi(i)}) + d(s_i, r_{\pi(i)}) \leq 2d(s_{\pi(i)}, r_{\pi(i)}) \,. \tag{2}$$

Then

$$\begin{aligned} D(r_i) = d(s_{\pi(i)}, r_i) - d(s_i, r_i) &\leq d(s_{\pi(i)}, s_i) + d(s_i, r_i) - d(s_i, r_i) \\ &= d(s_{\pi(i)}, s_i) \\ &\leq 2d(s_{\pi(i)}, r_{\pi(i)}) \,. \end{aligned}$$

Therefore, $D(r_i)$ is upper bounded by $2d(s_{\pi(i)}, r_{\pi(i)})$ for all i. Hence,

$$\begin{aligned} \sum_{i=1}^{n} D(r_i) &\leq \sum_{i=1}^{n} 2d(s_{\pi(i)}, r_{\pi(i)}) \\ w(M^*) - w(M) &\leq 2w(M) \\ w(M^*) &\leq 3w(M) \end{aligned}$$

and Greedy is $\frac{1}{3}$ competitive. ■

References

[1] S. Albers and S. Schubert. “Optimal Algorithms for Online b-Matching with Variable Vertex Capacities”. In: *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2021)*. 2021.

[2] N. R. Devanur, K. Jain, and R. D. Kleinberg. “Randomized Primal-Dual Analysis of RANKING for Online Bipartite Matching”. In: *Proceedings of the twenty-fourth annual ACM-SIAM symposium on Discrete algorithms*. SIAM. 2013, pp. 101–107.

[3] A. Eden, M. Feldman, A. Fiat, and K. Segal. “An Economics-Based Analysis of RANKING for Online Bipartite Matching”. In: *Symposium on Simplicity in Algorithms (SOSA)*. SIAM. 2021, pp. 107–110.

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[5] B. Kalyanasundaram and K. R. Pruhs. “An Optimal Deterministic Algorithm for Online b-Matching”. In: *Theoretical Computer Science* 233.1-2 (2000), pp. 319–325.

[6] R. M. Karp, U. V. Vazirani, and V. V. Vazirani. “An Optimal Algorithm for On-line Bipartite Matching”. In: *Proceedings of the twenty-second annual ACM symposium on Theory of computing*. 1990, pp. 352–358.