Notes on Metric Bipartite Minimum Weight b-Factors

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1 Introduction

When constructing a b-matching, an edge can potentially be included in the matching multiple times. If an edge can be matched at most once, the resulting solution is called a simple b-matching. A b-factor is perfect simple b-matching in a graph. That is, it is a subset $F \subseteq E$ with $\deg_F(v) = b(v)$ for each $v \in V$. Hence, a 1-factor is equivalent to a perfect matching.

Theorem 1 ([8])**.** *Let $G = (V, E)$ be a bipartite graph with node capacity function $b\colon V \to \mathbb{Z}_+$. Then G has a b-factor iff each subset $X \subseteq V$ spans at least $b(X) - \frac{1}{2}b(V)$ edges.*

Theorem 1 implies that some necessary (but not sufficient) conditions for a bipartite graph to admit a b-factor is that for each vertex $v \in V$, $b(v) \leq \frac{1}{2}b(V)$, and for each partition A, B in the bipartition, $b(A) = b(B) = \frac{1}{2}b(V)$. If a bipartite graph admits a b-factor for some capacity function b, a natural extension of the Min Weight Perfect Matching problem is to ask for a Min Weight b-Factor.

Problem 1 (Min Weight b-Factor)

Let $G = (V, E)$ be a bipartite graph. Given $b\colon V \to \mathbb{Z}_+$ and $w\colon E \to \mathbb{Q}_+$, find a minimum weight b-factor of G.

The LP-relaxation of the Min Weight b-Factor problem is

$$\begin{aligned} \text{minimize} \quad & \sum_{e \in E} w(e)\, x(e) & \\ \text{subject to} \quad & \sum_{e \in \delta(v)} x(e) = b(v) & \forall v \in V \\ & 0 \leq x(e) \leq 1 & \forall e \in E \end{aligned} \tag{1}$$

and its dual is

$$\begin{aligned} \text{maximize} \quad & \sum_{v \in V} b(v)\, y(v) - \sum_{e \in E} z(e) & \\ \text{subject to} \quad & y(i) + y(j) - z(e) \leq w(e) & \forall e = (i, j) \in E \\ & z(e) \geq 0 & \forall e \in E. \end{aligned} \tag{2}$$

Lemma 1 (Complementary Slackness)**.** *For an edge $e = (i, j)$, define $y(e) = y(i) + y(j)$. Then,*

1. $x(e) > 0 \to y(e) - z(e) = w(e) \to y(e) \geq w(e)$
2. $z(e) > 0 \to x(e) = 1$. *Alternatively, this gives us that* $x(e) < 1 \to z(e) = 0 \to y(e) \leq w(e)$

By Lemma 1, we find that we do not have to maintain the edge dual $z\colon E \to \mathbb{R}_+$ as its optimum value can be given by explicitly by $z(e) = \max\{y(e) - w(e), 0\}$.

2 Related Work

The Min Weight b-Factor problem is a special case of Min Weight Perfect b-Matching where edges can have multiplicity at most 1. A closely related problem is b-Transportation, which is essentially the fractional version of Min Weight Perfect b-Matching (i.e. the capacity function $b\colon V \to \mathbb{R}_+$ and the matching function $x\colon E \to \mathbb{R}_+$ allow for fractional values).
[4] Gabow and Tarjan, "Faster Scaling Algorithms for Network Problems", 1989

- In a bipartite graph, a min weight perfect DCS can be found in $O\big(m \min\{m^{1/2}, n^{2/3}\} \log(nW)\big)$ time, where $W = \max_{e \in E} w(e)$.
- Finding a perfect DCS is equivalent to finding a b-factor when $u(v) = b(v), l(v) = 0$.
- For a complete bipartite graph, $m = O(n^2)$. Hence, the algorithm runs in $O\big(mn^{2/3} \log(nW)\big)$ time

[9] Sharathkumar and Agarwal, "Algorithms for the Transportation Problem in Geometric Settings", 2012

- Study the Min Weight Perfect b-Matching problem in bipartite graphs
- Given an ε-close approximation to the problem under any non-negative edge cost function $d\colon E \to R_+$.
- Use this ε-close approximation to get a $(1-\varepsilon)$ approximation when d is a metric
- Instead of blowing up the graph using the standard reduction from b-matching to perfect matching, they maintain a compact representation of the graph such that the total number of vertices are bounded by $4n$ where n is the number of vertices in the input graph.

[6] Lahn et al., "A Graph Theoretic Additive Approximation of Optimal Transport", 2019
[7] Lahn et al., "A Push-Relabel Based Additive Approximation for Optimal Transport", 2022

- Study the b-Transportation problem. Do not assume metric spaces
- Both papers give ε-close approximations based on the scaling algorithm
- The second paper gives an analysis/algorithm that allows for parallelization

[2] Bertsekas and Castanon, "The auction algorithm for the transportation problem", 1989
[3] Bertsekas et al., "Reverse auction and the solution of inequality constrained assignment problems", 1993
[5] Khosla and Anand, "Revisiting the auction algorithm for weighted bipartite perfect matchings", 2021
[1] Alfaro et al., "The assignment problem revisited", 2022

- Study a different algorithm for weighted perfect matchings called the Auction algorithm
- Algorithms runs in $O\big(n^2 \frac{W}{e}\big)$ time and gives an $\mathrm{OPT} + \varepsilon n$ solution
- Benefits: Can be easily parallelized
- [1] shows that it often outperforms Hungarian and Scaling even in sequential
- [2] show that it can be extended to Min Weight Perfect b-Matching

3 An Auction Based Approach

Recall that the LP-Relaxation of the Min Weight b-Factor problem in a bipartite graph $G = (L \cup R, E)$ is given by

$$
\begin{aligned}
\text{minimize} \quad & \sum_{(i,j)\in E} w(i,j)\, x(i,j) \\
\text{subject to} \quad & \sum_{j \in N(i)} x(i,j) = b(i) \quad \forall i \in L \\
& \sum_{i \in N(j)} x(i,j) = b(j) \quad \forall j \in R \\
& 0 \le x(i,j) \le 1 \qquad\quad \forall\,(i,j) \in E
\end{aligned}
\tag{3}
$$

and its dual by

$$
\begin{aligned}
\text{maximize} \quad & \sum_{i\in L} b(i)\, r(i) + \sum_{j \in R} b(j)\, p(j) - \sum_{(i,j)\in E} z(i,j) \\
\text{subject to} \quad & r(i) + p(j) - z(i,j) \le w(i,j) \quad \forall\,(i,j) \in E \\
& z(i,j) \ge 0 \qquad\qquad\qquad\qquad \forall\,(i,j) \in E.
\end{aligned}
\tag{4}
$$

For exposition purposes, we refer to vertices in L as *bidders*, and vertices in R as *objects*.

By complementary slackness conditions (Lemma 1), we find that we do not have to maintain the edge duals z as their optimal value can be given explicitly by $z(i,j) = \max\{r(i) + p(j) - w(i,j)\,, 0\}$. This further implies that for all $(i,j) \in E$, $r(i) + p(j) \le w(i,j)$. To condense notation, we define $w'(i,j) = w(i,j) - p(j)$ as the *augmented cost* of object j for bidder i. This gives us that the dual can be equivalently described by

$$
\begin{aligned}
\text{maximize} \quad & \sum_{i\in L} b(i)\, r(i) + \sum_{j \in R} b(j)\, p(j) - \sum_{(i,j)\in E} \max\left\{r(i) - w'(i,j)\,, 0\right\} \\
\text{subject to} \quad & r(i) + p(j) \le w(i,j) \quad \forall\,(i,j) \in E.
\end{aligned}
\tag{5}
$$

If we are given a price function p, then the objective value of Eq. (5) is maximized when $r(i) = \min_{k \in N(i)} w'(i,k)$ while still respecting the constraints of the problem. Since the edge duals z depend on r, they can be given explicitly by $z(i,j) = \max\left\{\min_{k\in N(i)} w'(i,k) - w'(i,j)\,, 0\right\}$. However, since clearly $\min_{k \in N(i)} w'(i,k) \le w'(i,j)$, all the edge duals are given by $z(i,j) = 0$. Hence, the dual problem can be further equivalently described by

$$
\underset{p}{\text{maximize}} \quad \sum_{j\in R} b(j)\, p(j) + \sum_{i \in L} b(i) \min_{k \in N(i)} w'(i,k)
\tag{6}
$$

3.1 ε-Happiness

Let F^* be an optimal b-factor and p^* an optimal price function. Consider the incidence function x induced by F^*. By complementary slackness conditions in Eq. (6), this implies that if $x(i,j) > 0$,

$$
\begin{aligned}
w(i,j) - p^*(j) &= \min_{k \in N(i)} w'(i,k) \\
w'(i,j) &= \min_{k \in N(i)} w'(i,k)\,.
\end{aligned}
\tag{7}
$$

For any feasible simple b-matching (not necessarily a b-factor) F and price function p, we call an edge in F that satisfies Eq. (7) *happy* with respect to p. We can relax this condition to consider bidders assigned to objects within ε of attaining the minimum augmented cost, where $\varepsilon \geq 0$. That is,

$$w'(i,j) \leq \min_{k \in N(i)} w'(i,k) + \varepsilon. \tag{8}$$

If an edge in F satisfies Eq. (8), then it is called *ε-happy* with respect to p. If F consists of only ε-happy (resp. happy) edges, then it is also ε-happy (resp. happy) with respect to p. If F is a b-factor and is happy, then it must be optimal by Eq. (7). However, if it is a b-factor and ε-happy, then its weight falls within an additive loss of the optimal min weight b-factor.

Lemma 2. *Let F, p be some feasible b-factor and price function, respectively, such that F is ε-happy with respect to p. Then*

$$w(F^*) \leq w(F) \leq w(F^*) + \varepsilon U$$

where F^* is an optimal b-factor and $U = \frac{1}{2} b(V) = |F|$.

Proof. Since F is ε-happy, each edge $(i,j) \in F$ satisfies Eq. (8), which implies that

$$\min_{k \in N(i)} w'(i,k) \geq w(i,j) - p(j) - \varepsilon.$$

Additionally, by definition of a b-factor,

$$\sum_{j \in R} b(j)\, p(j) + \sum_{i \in L} b(i) \min_{k \in N(i)} w'(i,k) = \sum_{(i,j) \in F} \left(p(j) + \min_{k \in N(i)} w'(i,k) \right)$$

since each vertex $v \in L \cup R$ is matched exactly $b(v)$ times in F. Hence, by weak duality,

$$\begin{aligned} w(F^*) &\geq \sum_{j \in R} b(j)\, p(j) + \sum_{i \in L} b(i) \min_{k \in N(i)} w'(i,k) \\ &= \sum_{(i,j) \in F} \left(p(j) + \min_{k \in N(i)} w'(i,k) \right) \\ w(F^*) &\geq \sum_{(i,j) \in F} (w(i,j) - \varepsilon) = w(F) - \varepsilon U \geq w(F^*) - \varepsilon U. \end{aligned}$$

Adding εU to the last line gives $w(F^*) + \varepsilon U \geq w(F) \geq w(F^*)$. ■

3.2 The Auction Algorithm

The auction algorithm uses the alternative expression of the dual problem Eq. (6) to find a b-factor by iteratively constructing a feasible price function p. In each iteration, an unsaturated bidder u finds the objects v_1, v_2 that give it the lowest and second-lowest augmented cost, respectively, with the intention of bidding on the lowest cost object. The bid is calculated by simply taking the difference in the augmented costs of the two objects, plus some additional ε value. The price of object v_1 is then lowered by the bid, and u is matched to v_1. If v_1 is already saturated in F before adding the edge (u, v_1), then a random edge in F containing v_1 is removed and replaced with (u, v_1). This process continues as long as there are unsaturated bidders, and if it terminates, it must necessarily return a feasible b-factor.

Algorithm 1 Sequential Auction

Input: A bipartite graph $G = (V = L \cup R, E)$, edge weight function $w \colon E \to \mathbb{R}_{\geq 0}$, node capacity function $b \colon V \to \mathbb{Z}_+$, $\varepsilon \geq 0$, and prices $p \colon V \to \mathbb{R}$

Output: A b-factor F and price function p such that F is ε-happy with respect to p

$U \leftarrow L$ ▷ *Unsaturated nodes*
$F \leftarrow \emptyset$
while $U \neq \emptyset$ **do**
  Choose any $u \in U$
  $\textsc{Bid}(u, p, U, F)$
return (F, p)

procedure $\textsc{Bid}(u,\ p,\ U,\ F)$
  $v_1 \leftarrow \arg\min_{v \in N(u)} w'(u, v)$ ▷ *Lowest augmented cost object*
  $v_2 \leftarrow \arg\min_{v \in N(u) \setminus \{v_1\}} w'(u, v)$ ▷ *Second lowest augmented cost object*
  $\gamma \leftarrow w'(u, v_2) - w'(u, v_1) + \varepsilon$ ▷ *Bid*
  $p(v_1) \leftarrow p(v_1) - \gamma$
  if $|\{e \in F \colon v_1 \in e\}| = b(v_1)$ **then** ▷ *v_1 is saturated*
    Choose y randomly from the set of bidders v_1 is matched to
    $F \leftarrow F \setminus \{(y, v_1)\} \cup \{(u, v_1)\}$ ▷ *Remove previous edge and insert new one*
    $U \leftarrow U \cup \{y\}$
  else
    $F \leftarrow F \cup \{(u, v_1)\}$
  if $|\{e \in F \colon u \in e\}| = b(u)$ **then** ▷ *u is saturated*
    $U \leftarrow U \setminus \{u\}$

Observation 1. *In each iteration of Algorithm 1, only one object has a price update. The price of this object decreases by at least ε.*

The choice of the bidding value $\gamma = w'(u, v_2) - w'(u, v_1) + \varepsilon$ for an unsaturated bidder u is made to maintain a simple b-matching that is ε-happy with respect to the price function p in each iteration.

Lemma 3. *If at the start of an iteration of Algorithm 1 F is ε-happy with respect to the price function p, the updated matching is still ε-happy with respect to the updated price function at the end of the iteration.*

Proof. Suppose that in this iteration, bidder i finds objects j and $\bar{j}$ that have the lowest and second-lowest augmented cost, respectively, and makes a bid on object j. Let p and p' be the price function before and after the bid, respectively. Since the bid $\gamma = w'\big(i, \bar{j}\big) - w'(i, j) + \varepsilon$, we get that

$$p'(j) = p(j) - \big(w'\big(i, \bar{j}\big) - w'(i, j) + \varepsilon\big) = w(i, j) - w'\big(i, \bar{j}\big) - \varepsilon$$

and hence

$$w(i, j) - p'(j) = w'\big(i, \bar{j}\big) + \varepsilon = \min_{k \in N(i) \setminus \{j\}} (w(i, k) - p(k)) + \varepsilon.$$

By Observation 1, we find that for all $r \in R$, $p'(r) \leq p(r)$ with equality for all objects besides j. Hence,

$$w(i,j) - p'(j) = \min_{k \in N(i) \setminus \{j\}} (w(i,k) - p(k)) + \varepsilon \leq \min_{k \in N(i)} \big(w(i,k) + p'(k)\big) + \varepsilon$$

so the edge (i,j) is ε-happy with respect to p'. Additionally, for each edge (a,b) in F where $b \neq j$, we find that

$$w(a,b) - p'(b) = w(a,b) - p(b) \leq \min_{c \in N(a)} (w(a,c) - p(c)) + \varepsilon \leq \min_{c \in N(a)} \big(w(a,c) - p'(c)\big) + \varepsilon.$$

Thus, F is also ε-happy with respect to p'. Therefore, the updated matching $F' = F \cup \{(i,j)\}$ (plus any additional removals if j is already saturated) is ε-happy with respect to p'. ■

References

[1] C. A. Alfaro, S. L. Perez, C. E. Valencia, and M. C. Vargas. "The assignment problem revisited". In: *Optimization Letters* 16.5 (2022), pp. 1531–1548.

[2] D. P. Bertsekas and D. A. Castanon. "The auction algorithm for the transportation problem". In: *Annals of Operations Research* 20.1 (1989), pp. 67–96.

[3] D. P. Bertsekas, D. A. Castanon, and H. Tsaknakis. "Reverse auction and the solution of inequality constrained assignment problems". In: *SIAM Journal on Optimization* 3.2 (1993), pp. 268–297.

[4] H. N. Gabow and R. E. Tarjan. "Faster Scaling Algorithms for Network Problems". In: *SIAM Journal on Computing* 18.5 (1989), pp. 1013–1036.

[5] M. Khosla and A. Anand. "Revisiting the auction algorithm for weighted bipartite perfect matchings". In: *arXiv preprint arXiv:2101.07155* (2021).

[6] N. Lahn, D. Mulchandani, and S. Raghvendra. "A Graph Theoretic Additive Approximation of Optimal Transport". In: *Advances in Neural Information Processing Systems* 32 (2019).

[7] N. Lahn, S. Raghvendra, and K. Zhang. "A Push-Relabel Based Additive Approximation for Optimal Transport". In: *arXiv preprint arXiv:2203.03732* (2022).

[8] A. Schrijver. *Combinatorial Optimization: Polyhedra and Efficiency*. Vol. 24. Springer, 2003.

[9] R. Sharathkumar and P. K. Agarwal. "Algorithms for the Transportation Problem in Geometric Settings". In: *Proceedings of the twenty-third annual ACM-SIAM Symposium on Discrete Algorithms*. SIAM. 2012, pp. 306–317.