Notes on Metric Bipartite Minimum Weight b-Factors

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1 Introduction

When constructing a b-matching, an edge can potentially be included in the matching multiple times. If an edge can be matched at most once, the resulting solution is called a simple b-matching. A b-factor is perfect simple b-matching in a graph. That is, it is a subset $F \subseteq E$ with $\deg_F(v) = b(v)$ for each $v \in V$. Hence, a 1-factor is equivalent to a perfect matching.

Theorem 1 ([8]). *Let $G = (V, E)$ be a bipartite graph with node capacity function $b\colon V \to \mathbb{Z}_+$. Then G has a b-factor iff each subset $X \subseteq V$ spans at least $b(X) - \frac{1}{2}b(V)$ edges.*

Theorem 1 implies that some necessary (but not sufficient) conditions for a bipartite graph to admit a b-factor is that for each vertex $v \in V$, $b(v) \le \frac{1}{2}b(V)$, and for each partition A, B in the bipartition, $b(A) = b(B) = \frac{1}{2}b(V)$. If a bipartite graph admits a b-factor for some capacity function b, a natural extension of the Min Weight Perfect Matching problem is to ask for a Min Weight b-Factor.

Problem 1 (Min Weight b-Factor)

Let $G = (V, E)$ be a bipartite graph. Given $b\colon V \to \mathbb{Z}_+$ and $w\colon E \to \mathbb{Q}_+$, find a minimum weight b-factor of G.

The LP-relaxation of the Min Weight b-Factor problem is

$$\begin{aligned} \text{minimize} \quad & \sum_{e \in E} w(e)\, x(e) \\ \text{subject to} \quad & \sum_{e \in \delta(v)} x(e) = b(v) \quad && \forall v \in V \\ & 0 \le x(e) \le 1 && \forall e \in E \end{aligned} \tag{1}$$

and its dual is

$$\begin{aligned} \text{maximize} \quad & \sum_{v \in V} b(v)\, y(v) - \sum_{e \in E} z(e) \\ \text{subject to} \quad & y(i) + y(j) - z(e) \le w(e) \quad && \forall e = (i, j) \in E \\ & z(e) \ge 0 && \forall e \in E. \end{aligned} \tag{2}$$

Lemma 1 (Complementary Slackness). *For an edge $e = (i, j)$, define $y(e) = y(i) + y(j)$. Then,*

1. $x(e) > 0 \to y(e) - z(e) = w(e) \to y(e) \ge w(e)$
2. $z(e) > 0 \to x(e) = 1$. *Alternatively, this gives us that* $x(e) < 1 \to z(e) = 0 \to y(e) \le w(e)$

By Lemma 1, we find that we do not have to maintain the edge dual $z\colon E \to \mathbb{R}_+$ as its optimum value can be given by explicitly by $z(e) = \max\{y(e) - w(e), 0\}$.

2 Related Work

The Min Weight b-Factor problem is a special case of Min Weight Perfect b-Matching where edges can have multiplicity at most 1. A closely related problem is b-Transportation, which is essentially the fractional version of Min Weight Perfect b-Matching (i.e. the capacity function $b\colon V \to \mathbb{R}_+$ and the matching function $x\colon E \to \mathbb{R}_+$ allow for fractional values).
[4] Gabow and Tarjan, "Faster Scaling Algorithms for Network Problems", 1989

- In a bipartite graph, a min weight perfect DCS can be found in $O\big(m \min\{m^{1/2}, n^{2/3}\} \log(nW)\big)$ time, where $W = \max_{e \in E} w(e)$.
- Finding a perfect DCS is equivalent to finding a b-factor when $u(v) = b(v), l(v) = 0$.
- For a complete bipartite graph, $m = O(n^2)$. Hence, the algorithm runs in $O\big(mn^{2/3} \log(nW)\big)$ time

[9] Sharathkumar and Agarwal, "Algorithms for the Transportation Problem in Geometric Settings", 2012

- Study the Min Weight Perfect b-Matching problem in bipartite graphs
- Given an ε-close approximation to the problem under any non-negative edge cost function $d\colon E \to R_+$.
- Use this ε-close approximation to get a $(1 - \varepsilon)$ approximation when d is a metric
- Instead of blowing up the graph using the standard reduction from b-matching to perfect matching, they maintain a compact representation of the graph such that the total number of vertices are bounded by $4n$ where n is the number of vertices in the input graph.

[6] Lahn et al., "A Graph Theoretic Additive Approximation of Optimal Transport", 2019
[7] Lahn et al., "A Push-Relabel Based Additive Approximation for Optimal Transport", 2022

- Study the b-Transportation problem. Do not assume metric spaces
- Both papers give ε-close approximations based on the scaling algorithm
- The second paper gives an analysis/algorithm that allows for parallelization

[2] Bertsekas and Castanon, "The auction algorithm for the transportation problem", 1989
[3] Bertsekas et al., "Reverse auction and the solution of inequality constrained assignment problems", 1993
[5] Khosla and Anand, "Revisiting the auction algorithm for weighted bipartite perfect matchings", 2021
[1] Alfaro et al., "The assignment problem revisited", 2022

- Study a different algorithm for weighted perfect matchings called the Auction algorithm
- Algorithms runs in $O\big(n^2 \frac{W}{e}\big)$ time and gives an $\mathrm{OPT} + \varepsilon n$ solution
- Benefits: Can be easily parallelized
- [1] shows that it often outperforms Hungarian and Scaling even in sequential
- [2] show that it can be extended to Min Weight Perfect b-Matching

3 The Auction Algorithm

Recall that the LP-Relaxation of the Min Weight b-Factor problem in a bipartite graph $G = (L \cup R, E)$ is given by

$$\begin{aligned}
\text{minimize} \quad & \sum_{(i,j)\in E} w(i,j)\, x(i,j) \\
\text{subject to} \quad & \sum_{j\in N(i)} x(i,j) = b(i) \quad \forall i \in L \\
& \sum_{i\in N(j)} x(i,j) = b(j) \quad \forall j \in R \\
& 0 \le x(i,j) \le 1 \qquad \forall\, (i,j) \in E
\end{aligned} \tag{3}$$

and its dual by

$$\begin{aligned}
\text{maximize} \quad & \sum_{i\in L} b(i)\, r(i) + \sum_{j\in R} b(j)\, p(j) - \sum_{(i,j)\in E} z(i,j) \\
\text{subject to} \quad & r(i) + p(j) - z(i,j) \le w(i,j) \quad \forall\, (i,j) \in E \\
& z(i,j) \ge 0 \qquad \forall\, (i,j) \in E.
\end{aligned} \tag{4}$$

By complementary slackness conditions (Lemma 1), we find that we do not have to maintain the edge duals z as their optimal value can be given explicitly by $z(i,j) = \max\{r(i) + p(j) - w(i,j)\,, 0\}$. This further implies that for all $(i,j) \in E$, $r(i) + p(j) \le w(i,j)$. To condense notation, we define $w'(i,j) = w(i,j) - p(j)$ as the reduced cost of item j for person i. This gives us that the dual can be equivalently described by

$$\begin{aligned}
\text{maximize} \quad & \sum_{i\in L} b(i)\, r(i) + \sum_{j\in R} b(j)\, p(j) - \sum_{(i,j)\in E} \max\left\{r(i) - w'(i,j)\,, 0\right\} \\
\text{subject to} \quad & r(i) + p(j) \le w(i,j) \quad \forall\, (i,j) \in E
\end{aligned} \tag{5}$$

If we are given a price function p, then the objective value of Eq. (5) is maximized when $r(i) = \min_{k\in N(i)} w(i,k) - p(k)$ while still respecting the constraints of the problem. Since the edge duals z depend on r, they can be given explicitly by $z(i,j) = \max\left\{\min_{k\in N(i)} w'(i,k) - w'(i,j)\,, 0\right\}$. However, since clearly $w'(i,k) \le w'(i,j)$, the edge duals are given by $z(i,j) = 0$. Hence, the dual problem can be further equivalently restated as

$$\underset{p}{\text{maximize}} \quad \sum_{j\in R} b(j)\, p(j) + \sum_{i\in L} b(i) \min_{k\in N(i)} w'(i,k) \tag{6}$$

3.1 ε-Happy Edges

Let F be a b-factor and p a price function. By the Strong Duality theorem, if F and p are both primal and dual optimal, respectively, then

$$\begin{aligned}
\sum_{(i,j)\in F} w(i,j) &= \sum_{j\in R} b(j)\, p(j) + \sum_{i\in L} b(i) \min_{k\in N(i)} w'(i,k) \\
\sum_{(i,j)\in F} w(i,j) - p(j) &= \sum_{i\in L} b(i) \min_{k\in N(i)} w'(i,k) \\
\sum_{(i,j)\in F} w'(i,j) &= \sum_{i\in L} b(i) \min_{k\in N(i)} w'(i,k)
\end{aligned}$$

which implies that

$$w'(i,j) = \min_{k \in N(i)} w'(i,k) \qquad \forall\, (i,j) \in F. \tag{7}$$

For any feasible b-factor F and price function p, we call an edge in F that satisfies Eq. (7) *happy*.

A relaxation of the concept of happy edges are ε-happy edges which allow persons to be assigned to objects within ε of attaining the minimum reduced cost. That is,

$$w'(i,j) \leq \min_{k \in N(i)} w'(i,k) + \varepsilon \qquad \forall\, (i,j) \in F \tag{8}$$

with ε being some non-negative constant. We can use the concept of ε-happy edges to get within an additive loss of the optimal min cost b-factor.

Theorem 2. *Let F be some feasible b-factor and p a price function such that F consists of ε-happy edges. Then*

$$w(F^*) \leq w(F) \leq w(F^*) + \varepsilon U$$

where F^* is an optimal b-factor and $U = \frac{1}{2} b(V)$.

Proof. By weak duality,

$$\begin{aligned}
w(F^*) &\geq \sum_{j \in R} b(j)\, p(j) + \sum_{i \in L} b(i) \min_{k \in N(i)} w'(i,k) \\
&= \sum_{(i,j) \in F} \left(p(j) + \min_{k \in N(i)} w'(i,k) \right) \\
&\geq \sum_{(i,j) \in F} (w(i,j) - \varepsilon) = w(F) - \varepsilon U \geq w(F^*) - \varepsilon U
\end{aligned}$$

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