Notes on Metric Online Maximum Weighted b-Matching

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1 Prior Work

1.1 Unweighted Problems

Problem 1.1 (Online Maximum Matching)

Given an unweighted bipartite graph $G = (S \cup R, E)$, the server vertices S are known beforehand and the request vertices R and their incident edges arrive online one by one. When a request $r_i \in R$ arrives, it must either irrevocably be matched to one of its unmatched neighbors $s_k \in S$ or be unmatched forever. The objective is to maximize the number of requests matched.

[6] Karp et al., "An Optimal Algorithm for On-line Bipartite Matching", 1990

- Show that for Online Maximum Matching (OMM), no deterministic algorithm has a competitive ratio better than $\frac{1}{2}$
  - Greedy achieves this ratio
- Show that a randomized algorithm Ranking achieves a ratio of $1 - \frac{1}{e}$ which is optimal
  - [2] give a simplified Primal-Dual analysis of this algorithm
  - [3] give an economics based analysis of this algorithm

[5] Kalyanasundaram and Pruhs, "An Optimal Deterministic Algorithm for Online b-Matching", 2000

- Study the Online Maximum b-Matching (OM b-M) problem which is equivalent to OMM with the addition that the server vertices now have a capacity b (i.e. they can be matched b times)
- Show that no deterministic algorithm has a competitive ratio better than $1 - \frac{1}{\left(1+\frac{1}{b}\right)^b}$
  - They give a deterministic algorithm Balance that achieves this ratio

[1] Albers and Schubert, "Optimal Algorithms for Online b-Matching with Variable Vertex Capacities", 2021

- Study the OM b-M problem where each vertex $s \in S$ can have a variable b_s capacity
- Give a deterministic algorithm RelativeBalance with competitive ratio $1 - \frac{1}{\left(1+\frac{1}{b_{\min}}\right)^{b_{\min}}}$
- Extend the analysis of the original Ranking algorithm to show that it is still $1 - \frac{1}{e}$ competitive

1.2 Weighted Problems

In the weighted version of OMM and its b-Matching variants, there exists no algorithm that can achieve a non-trivial competitive ratio (see **??**). This necessitates the need for some kind of additional assumption on the model.

[4] Kalyanasundaram and Pruhs, "Online Weighted Matching", 1993

- Study the Online Maximum Weight Matching (OMWM) problem under the assumption that the weights of edges represent distances in some metric space.
  - Denote this problem as Metric Online Maximum Weight Matching (Metric OMWM)
- Show that Greedy for Metric OMWM gives competitive ratio $\frac{1}{3}$
- Show that no deterministic online algorithm can achieve a ratio better than $\frac{1}{3}$

2 Metric Online Maximum Weight Matching

2.1 Hardness without Metric Assumption

2.2 The Greedy Algorithm

2.3 Extension to Metric Online Maximum Weight b-Matching

References

[1] S. Albers and S. Schubert. “Optimal Algorithms for Online b-Matching with Variable Vertex Capacities”. In: *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2021)*. 2021.

[2] N. R. Devanur, K. Jain, and R. D. Kleinberg. “Randomized Primal-Dual Analysis of RANKING for Online Bipartite Matching”. In: *Proceedings of the twenty-fourth annual ACM-SIAM symposium on Discrete algorithms*. SIAM. 2013, pp. 101–107.

[3] A. Eden, M. Feldman, A. Fiat, and K. Segal. “An Economics-Based Analysis of RANKING for Online Bipartite Matching”. In: *Symposium on Simplicity in Algorithms (SOSA)*. SIAM. 2021, pp. 107–110.

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[5] B. Kalyanasundaram and K. R. Pruhs. “An Optimal Deterministic Algorithm for Online b-Matching”. In: *Theoretical Computer Science* 233.1-2 (2000), pp. 319–325.

[6] R. M. Karp, U. V. Vazirani, and V. V. Vazirani. “An Optimal Algorithm for On-line Bipartite Matching”. In: *Proceedings of the twenty-second annual ACM symposium on Theory of computing*. 1990, pp. 352–358.