Notes on Metric Bipartite Minimum Weight b-Factors

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1 Introduction

When constructing a b-matching, an edge can potentially be included in the matching multiple times. If an edge can be matched at most once, the resulting solution is called a simple b-matching. A b-factor is perfect simple b-matching in a graph. That is, it is a subset $F \subseteq E$ with $\deg_F(v) = b(v)$ for each $v \in V$. Hence, a 1-factor is equivalent to a perfect matching.

Theorem 1 ([2])**.** *Let $G = (V, E)$ be a bipartite graph with node capacity function $b\colon V \to \mathbb{Z}_+$. Then G has a b-factor iff each subset $X \subseteq V$ spans at least $b(X) - \frac{1}{2}b(V)$ edges.*

Theorem 1 implies that some necessary (but not sufficient) conditions for a bipartite graph to admit a b-factor is that for each vertex $v \in V$, $b(v) \le \frac{1}{2}b(V)$, and for each partition A, B in the bipartition, $b(A) = b(B) = \frac{1}{2}b(V)$. If a bipartite graph admits a b-factor for some capacity function b, a natural extension of the Min Weight Perfect Matching problem is to ask for a Min Weight b-Factor.

Problem 1 (Min Weight b-Factor)

Let $G = (V, E)$ be a bipartite graph. Given $b\colon V \to \mathbb{Z}_+$ and $w\colon E \to \mathbb{Q}_+$, find a minimum weight b-factor of G.

The LP-relaxation of the Min Weight b-Factor problem is

$$\begin{aligned} \text{minimize} \quad & \sum_{e \in E} w(e)\, x(e) & \\ \text{subject to} \quad & \sum_{e \in \delta(v)} x(e) = b(v) & \forall v \in V \\ & 0 \le x(e) \le 1 & \forall e \in E \end{aligned} \tag{1}$$

and its dual is

$$\begin{aligned} \text{maximize} \quad & \sum_{v \in V} b(v)\, y(v) - \sum_{e \in E} z(e) & \\ \text{subject to} \quad & y(i) + y(j) - z(e) \le w(e) & \forall e = (i,j) \in E \\ & z(e) \ge 0 & \forall e \in E. \end{aligned} \tag{2}$$

Lemma 1 (Complementary Slackness)**.** *For an edge $e = (i, j)$, define $y(e) = y(i) + y(j)$. Then,*

1. $x(e) > 0 \to y(e) - z(e) = w(e) \to y(e) \ge w(e)$
2. $z(e) > 0 \to x(e) = 1$*. Alternatively, this gives us that* $x(e) < 1 \to z(e) = 0 \to y(e) \le w(e)$

By Lemma 1, we find that we do not have to maintain the edge dual $z\colon E \to \mathbb{R}_+$ as its optimum value can be given by explicitly by $z(e) = \max\{y(e) - w(e), 0\}$. This is, for some b-factor F, z can de defined as

$$z(e) = \begin{cases} y(e) - w(e) & \text{if } e \in F \\ 0 & \text{otherwise .} \end{cases}$$

2 Related Work

The Min Weight b-Factor problem is a special case of Min Weight Perfect b-Matching where edges can have multiplicity at most 1. A closely related problem is b-Transportation, which is essentially the fractional version of Min Weight Perfect b-Matching (i.e. the capacity function $b\colon V \to \mathbb{R}_+$ and the matching function $x\colon E \to \mathbb{R}_+$ allow for fractional values).

[1] Gabow and Tarjan, "Faster Scaling Algorithms for Network Problems", 1989

- In a bipartite graph, a min weight perfect DCS can be found in $O\left(m \min\left\{m^{1/2}, n^{2/3}\right\} \log(nW)\right)$ time, where $W = \max_{e \in E} w(e)$.
- Finding a perfect DCS is equivalent to finding a b-factor when $u(v) = b(v), l(v) = 0$.
- For a complete bipartite graph, $m = O\left(n^2\right)$. Hence, the algorithm runs in $O\left(mn^{2/3} \log(nW)\right)$ time

[3] Sharathkumar and Agarwal, "Algorithms for the Transportation Problem in Geometric Settings", 2012

- Study the Min Weight Perfect b-Matching problem in bipartite graphs
- Given an ε-close approximation to the problem under any non-negative edge cost function $d\colon E \to R_+$.
- Use this ε-close approximation to get a $(1 - \varepsilon)$ approximation when d is a metric
- Instead of blowing up the graph using the standard reduction from b-matching to perfect matching, they maintain a compact representation of the graph such that the total number of vertices are bounded by $4n$ where n is the number of vertices in the input graph.

References

[1] H. N. Gabow and R. E. Tarjan. “Faster Scaling Algorithms for Network Problems”. In: *SIAM Journal on Computing* 18.5 (1989), pp. 1013–1036.

[2] A. Schrijver. *Combinatorial Optimization: Polyhedra and Efficiency*. Vol. 24. Springer, 2003.

[3] R. Sharathkumar and P. K. Agarwal. “Algorithms for the Transportation Problem in Geometric Settings”. In: *Proceedings of the twenty-third annual ACM-SIAM Symposium on Discrete Algorithms*. SIAM. 2012, pp. 306–317.