Surface Acoustic Wave Driven Flows over Topography

Bhargav Samineni

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Abstract

Motivated by the acoustowetting phenomenon and its applications to the dynamic wetting of objects by a coating liquid, we explore the influence of surface acoustic waves and gravity on the motion of a thin film flowing over a surface. We consider surfaces that may be both inclined and include topographical features like trenches, mounds, bumps, etc. Using the lubrication approximation, we reduce the equations of motion for the film to a single nonlinear partial differential equation that describes the evolution of the film height relative to the surface topography in time and space.

1 Introduction

Let $s(x, y)$ describe the topography of the surface, $h(x, y, t)$ be the film thickness relative to $s(x, y)$ at any point in the (x, y) plane at a time t, and $\phi(x, y, t) = s(x, y) + h(x, y, t)$ be the height of the free surface at any point in the (x, y) plane at a time t.

2 Governing Equation

2.1 Lubrication Approximation

As shown in [Kon03], if the Reynold's number is sufficiently low, the inertial terms of the Navier-Stokes equations can be ignored. Thus, under the effects of gravity and surface acoustic wave streaming forces (see [SM94] for more information), the lubrication approximation gives

$$\nabla_2 p = \mu \frac{\partial^2 \boldsymbol{v}}{\partial z^2} + \rho g \sin\beta \boldsymbol{i} - \rho J e^{2k_i(x+\alpha_1 z)} \boldsymbol{i} \tag{1}$$

$$\frac{\partial p}{\partial z} = -\rho g \cos\beta - \rho J \alpha_1 e^{2k_i(x+\alpha_1 z)} \tag{2}$$

where $J = \left(1+\alpha_1^2\right) A^2 \omega^2 k_i$, $\nabla_2 = (\partial_x, \partial_y)$, $\boldsymbol{v} = (u, v)$, and x_0 is a function of time denoting the position of the contact line between the SAW and the fluid.

2.2 Boundary Conditions

The Laplace-Young boundary condition states that at the interface $z = \phi(x, y, t)$, the pressure is given by $p(\phi) = -\gamma\kappa + p_0$, where κ is the curvature of the boundary, γ is the surface tension, and p_0 is the atmospheric pressure. Thus, integrating Eq. (2) gives

$$\begin{aligned}
\int_\phi^z \frac{\partial p}{\partial z}\, dz &= \int_\phi^z -\rho g \cos\beta - \rho J \alpha_1 e^{2k_i(x+\alpha_1 z)}\, dz \\
p(z) - p(\phi) &= -\left(z-\phi\right)\rho g \cos\beta - \frac{\rho J}{2k_i}\left(e^{2k_i(x+\alpha_1 z)} - e^{2k_i(x+\alpha_1 \phi)}\right) \\
p(z) &= -\left(z-\phi\right)\rho g \cos\beta - \frac{\rho J}{2k_i}\left(e^{2k_i(x+\alpha_1 z)} - e^{2k_i(x+\alpha_1 \phi)}\right) - \gamma\kappa + p_0.
\end{aligned} \tag{3}$$

If we define $P(x,y,t) = \phi\rho g\cos\beta + \frac{\rho J}{2k_i}e^{2k_i(x+\alpha_1\phi)} - \gamma\kappa$, this simplifies Eq. (3) to

$$p(z) = P - z\rho g\cos\beta - \frac{\rho J}{2k_i}e^{2k_i(x+\alpha_1 z)} + p_0$$

which further gives

$$\nabla_2 p = \nabla_2\left(P - \frac{\rho J}{2k_i}e^{2k_i(x+\alpha_1 z)}\right) = \nabla P - \rho J e^{2k_i(x+\alpha_1 z)}\boldsymbol{i}. \tag{4}$$

Further boundary conditions include

$$\boldsymbol{v}\big|_{z=s(x,y)} = \boldsymbol{0} \tag{5}$$

$$\frac{\partial \boldsymbol{v}}{\partial z}\bigg|_{z=\phi(x,y,t)} = \boldsymbol{0} \tag{6}$$

where Eq. (5) is a no-slip boundary condition along the surface $z = s(x,y)$ and Eq. (6) enforces vanishing shear stresses along the fluid-air boundary $z = \phi(x,y,t)$.

2.3 Film Equation

Using the Laplace-Young boundary condition and substituting Eq. (4) into Eq. (1) yields

$$\nabla P = \mu\frac{\partial^2 \boldsymbol{v}}{\partial z^2} + \rho g\sin\beta\boldsymbol{i}. \tag{7}$$

Integrating Eq. (7) twice with respect to z and utilizing the boundary conditions in Eq. (5) and Eq. (6) gives

$$\begin{aligned}
\int_s^z\int_z^\phi \frac{\partial^2 \boldsymbol{v}}{\partial z^2}\,dzdz &= \frac{1}{\mu}\int_s^z\int_z^\phi \left(\nabla P - \rho g\sin\beta\boldsymbol{i}\right)\,dzdz \\
\int_s^z \frac{\partial \boldsymbol{v}}{\partial z}\,dz &= \frac{1}{\mu}\left(\nabla P - \rho g\sin\beta\boldsymbol{i}\right)\int_s^z (z-\phi)\,dz \\
\boldsymbol{v} &= \frac{1}{\mu}\left(\nabla P - \rho g\sin\beta\boldsymbol{i}\right)\left(\frac{z^2}{2} - \phi z - \frac{s^2}{2} + \phi s\right).
\end{aligned} \tag{8}$$

Averaging over the height removes the z dependence of $\boldsymbol{v} = (u,v)$ and gives the equation $\overline{\boldsymbol{v}} = \frac{1}{h}\int_s^\phi \boldsymbol{v}\,dz$. Plugging in Eq. (8) and solving this integral then gives

$$\begin{aligned}
\overline{\boldsymbol{v}} &= \frac{1}{h}\int_s^\phi \frac{1}{\mu}\left(\nabla P - \rho g\sin\beta\boldsymbol{i}\right)\left(\frac{z^2}{2} - \phi z - \frac{s^2}{2} + \phi s\right)\,dz \\
&= \frac{1}{\mu h}\left(\nabla P - \rho g\sin\beta\boldsymbol{i}\right)\left(-\frac{\phi^3}{3} + \phi^2 s - \phi s^2 + \frac{s^3}{3}\right) \\
&= -\frac{h^2}{3\mu}\left(\nabla P - \rho g\sin\beta\boldsymbol{i}\right).
\end{aligned} \tag{9}$$

The conservation of mass, when depth-averaged, gives $\frac{\partial h}{\partial t} + \nabla\cdot(h\overline{\boldsymbol{v}}) = 0$ which results in

$$\begin{aligned}
\frac{\partial h}{\partial t} &= \frac{1}{3\mu}\nabla\cdot\left[h^3\left(\nabla P - \rho g\sin\beta\boldsymbol{i}\right)\right] \\
&= \frac{1}{3\mu}\nabla\cdot\left[h^3\left(\rho g\cos\beta\nabla\phi - \gamma\nabla\kappa - \rho g\sin\beta\boldsymbol{i} + \frac{\rho J}{2k_i}\nabla e^{2k_i(x+\alpha_1\phi)}\right)\right].
\end{aligned} \tag{10}$$

when plugging in Eq. (9). Approximating the curvature $\kappa \approx \nabla^2\phi$ then gives

$$\begin{aligned}
\frac{\partial h}{\partial t} &= \frac{1}{3\mu}\nabla\cdot\left[h^3\left(\rho g\cos\beta\nabla\phi - \gamma\nabla\nabla^2\phi - \rho g\sin\beta\boldsymbol{i} + \frac{\rho J}{2k_i}\nabla e^{2k_i(x+\alpha_1\phi)}\right)\right] \\
&= \frac{1}{3\mu}\left[\nabla\cdot\left[\rho g\cos\beta h^3\nabla\phi\right] - \nabla\cdot\left[\gamma h^3\nabla\nabla^2\phi\right] - \rho g\sin\beta\frac{\partial h^3}{\partial x} + \nabla\cdot\left[\frac{\rho J}{2k_i}h^3\nabla e^{2k_i(x+\alpha_1\phi)}\right]\right].
\end{aligned} \tag{11}$$

2.4 Dimensionless Form

Scale the in-plane coordinates and time by

$$\overline{x} = \frac{x}{x_c}, \quad \overline{y} = \frac{y}{x_c}, \quad \overline{z} = \frac{z}{h_c}, \quad \overline{t} = \frac{t}{t_c}$$

where an overline denotes a non-dimensional quantity. Substituting these scales into Eq. (11) and removing any overlines gives

$$\begin{aligned}\frac{h_c}{t_c}\frac{\partial h}{\partial t} =& \frac{1}{3\mu}\left[\frac{h_c^4}{x_c^2}\rho g\cos\beta\nabla\cdot\left[h^3\nabla\phi\right] - \frac{\gamma h_c^4}{x_c^4}\nabla\cdot\left[h^3\nabla\nabla^2\phi\right] - \frac{h_c^3}{x_c}\rho g\sin\beta\frac{\partial h^3}{\partial x}\right] \\ &+ \frac{1}{3\mu}\left[\frac{\rho J^* h_c^3}{2x_c^2}\nabla\cdot\left[h^3\nabla e^{2k_i(x+\alpha_1\phi(h_c/x_c))}\right]\right]\end{aligned} \tag{12}$$

where $J^* = \left(1+\alpha_1^2\right)A^2\omega^2$. By virtue of the fact that we are looking at thin films, we define $\varepsilon = h_c/x_c$ where $h_c \ll x_c$. This gives

$$\frac{h_c}{t_c}\frac{\partial h}{\partial t} = \frac{1}{3\mu}\left[\frac{h_c^4}{x_c^2}\rho g\cos\beta\nabla\cdot\left[h^3\nabla\phi\right] - \frac{\gamma h_c^4}{x_c^4}\nabla\cdot\left[h^3\nabla\nabla^2\phi\right] - \frac{h_c^3}{x_c}\rho g\sin\beta\frac{\partial h^3}{\partial x} + \frac{\rho J^* h_c^3}{2x_c^2}\nabla\cdot\left[h^3\nabla e^{2k_i(x+\alpha_1\varepsilon\phi)}\right]\right],$$

which can be further manipulated to the form

$$\frac{\partial h}{\partial t} = \frac{\gamma h_c^3 t_c}{3\mu x_c^4}\left[\frac{x_c^2\rho g}{\gamma}\cos\beta\nabla\cdot\left[h^3\nabla\phi\right] - \nabla\cdot\left[h^3\nabla\nabla^2\phi\right] - \frac{x_c^3\rho g}{\gamma h_c}\sin\beta\frac{\partial h^3}{\partial x} + \frac{\rho J^* x_c^2}{2\gamma h_c}\nabla\cdot\left[h^3\nabla e^{2k_i(x+\alpha_1\varepsilon\phi)}\right]\right].$$

Choose t_c such that

$$t_c = \frac{3\mu x_c^4}{\gamma h_c^3}.$$

Additionally, we define the Bond number $\mathrm{Bo} = x_c^2\rho g/\gamma$ and acoustic Weber number $\mathrm{We}_{\mathrm{ac}} = \rho\omega^2 A^2 x_c/\gamma$. Substituting these constants and expanding J^* yields a final equation

$$\frac{\partial h}{\partial t} = \mathrm{Bo}\cos\beta\nabla\cdot\left[h^3\nabla\phi\right] - \nabla\cdot\left[h^3\nabla\nabla^2\phi\right] - \frac{\mathrm{Bo}}{\varepsilon}\sin\beta\frac{\partial h^3}{\partial x} + \frac{\left(1+\alpha_1^2\right)\mathrm{We}_{\mathrm{ac}}}{2\varepsilon}\nabla\cdot\left[h^3\nabla e^{2k_i(x+\alpha_1\varepsilon\phi)}\right]. \tag{13}$$

The first term and third terms represent the out of plane and in plane influences of gravity, respectively, on the film while the second term represents the influence of capillary forces and the fourth term represents the contribution of SAW driving.

2.5 Two-Dimensional Equation and Enforcing a Film Front

Although Eq. (13) is a simplified partial differential equation, it is still strongly nonlinear. To gain a basic understanding of some of its solutions, we make the further simplification that the free surface of the thin film does not change in the transverse direction (i.e. h and s are both y-independent). This assumption further reduces our problem to only one variable in space and simplifies Eq. (13) to

$$\frac{\partial h}{\partial t} = \mathrm{Bo}\cos\beta\left[h^3\phi_x\right]_x - \left[h^3\phi_{xxx}\right]_x - \frac{\mathrm{Bo}}{\varepsilon}\sin\beta\left[h^3\right]_x + \frac{k_i\left(1+\alpha_1^2\right)\mathrm{We}_{\mathrm{ac}}}{\varepsilon}\left[h^3 e^{2k_i(x+\alpha_1\varepsilon\phi)}\left(1+\alpha_1\varepsilon\phi_x\right)\right]_x \tag{14}$$

where h and ϕ are now functions of x and t.

Additionally, to enforce that the SAW forcing occurs starting from the film front, we redefine k_i (in dimensionless form) as

$$k_i(h) = x_c\left(\left(k_i^{\mathrm{oil}} - k_i^{\mathrm{air}}\right)\left(1 - e^{-x_c(h-b)/\lambda}\right) + k_i^{\mathrm{air}}\right) \tag{15}$$

where k_i^{oil} denotes the attenuation in the film and k_i^{air} denotes the attenuation outside the film. The λ term is a dimensional constant that controls the steepness of the change from k_i^{air} to k_i^{oil}, while b denotes the precursor film height. In essence, when $h = b$, $k_i = x_c k_i^{\mathrm{air}}$ and decays to $x_c k_i^{\mathrm{oil}}$ as h increases. See Fig. 1 for an example of such a graph.

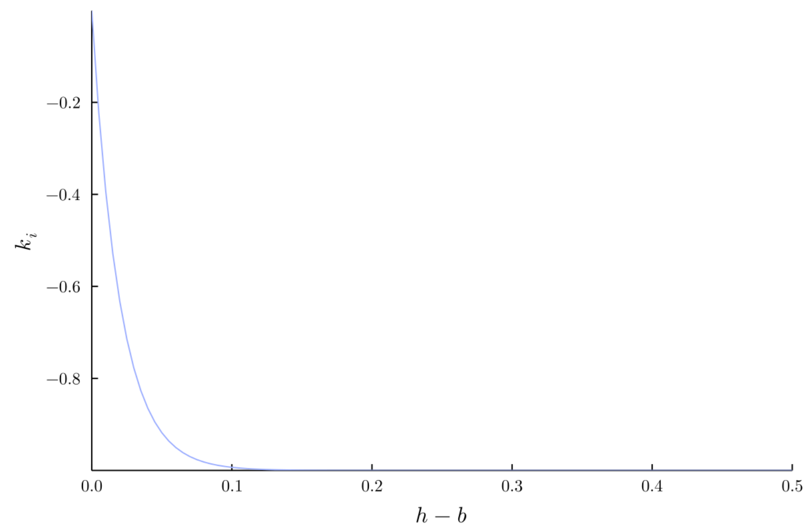

Figure 1: Attenuation as a function of h for $k_i^{\text{oil}} = -1000\left[m^{-1}\right], k_i^{\text{air}} = -1\left[m^{-1}\right]$, $\lambda = 20 * 10^{-3}\,[m]$, and $x_c = 10^{-3}$.

3 Method of Solution

3.1 Spatial Discretization

If we define the domain of interest as $[0, L_x]$, we can discretize the domain into points $x_j = j\Delta x$ for $j = 0, \ldots, N_x$ where $\Delta x = L_x/N_x$ and N_x is the number of grid points excluding the origin. If we further define $h_j(t) = h(x_j, t)$, we can discretize Eq. (14) into a system of ordinary differential equations of the form

$$\frac{dh_j}{dt} = f_j = \mathrm{Bo}\cos\beta f_j^{(1)} - f_j^{(2)} - \frac{\mathrm{Bo}}{\varepsilon}\sin\beta f_j^{(3)} + \frac{k_i\left(1+\alpha_1^2\right)\mathrm{We}_{\mathrm{ac}}}{\varepsilon} f_j^{(4)}, \quad j = 0, \ldots, N_x \tag{16}$$

where $f_j^{(k)}$ is the discretization of the k-th term in the right-hand side of Eq. (14). Because the governing equation contains high order derivatives, the discretization used for certain components needs special attention in order to not lead to a large computational stencil.

3.1.1 Fourth-Order Term

Following the method outlined in [Kon03], discretizing the fourth order term (i.e. $f_j^{(2)}$) can be done by a combination of forward and backward differences. If we define

$$h_{x,j} = \frac{h_{j+1} - h_j}{\Delta x} \qquad h_{\overline{x},j} = \frac{h_j - h_{j-1}}{\Delta x}$$

as the forward and backward differences, respectively, then a possible discretization is

$$f_j^{(2)} = \left(a(h_{i-1}, h_i)\,\phi_{\overline{x}x\overline{x}}\right)_x \tag{17}$$

where $a(j_1, j_2) = \frac{1}{2}\left({j_1}^3 + {j_2}^3\right)$. This discretization leads to a second order approximation that has a five point stencil, which is better than the seven point stencil that would result from using a central differencing scheme.

3.1.2 Lower Order Terms

Appendices

A Topography Definitions

B Spatial Discretization Expanded

$$f_j^{(1)} = \frac{1}{2\Delta x^2}\left(\left(h_{j-1}^3 + h_j^3\right)(\phi_{j-1} - \phi_j)) + \left(h_j^3 + h_{j+1}^3\right)(\phi_{j+1} - \phi_j)\right)$$

$$f_j^{(2)} = \frac{1}{2\Delta x^4}\left(\left(h_{j-1}^3 + h_j^3\right)(\phi_{j-2} - 3\phi_{j-1} + 3\phi_j - \phi_{j+1}) + \left(h_j^3 + h_{j+1}^3\right)(-\phi_{j-1} + 3\phi_j - 3\phi_{j+1} + \phi_{j+2})\right)$$

$$f_j^{(3)} = \frac{1}{2\Delta x}\left(h_{j+1}^3 - h_{j-1}^3\right)$$

$$f_j^{(4)} = \frac{1}{2\Delta x}\left(h_{j+1}^3 e^{2k_i x_{j+1}} - h_{j-1}^3 e^{2k_i x_{j-1}}\right)$$

References

[Kon03] L. Kondic. "Instabilities in gravity driven flow of thin fluid films". In: *Siam review* 45.1 (2003), pp. 95–115.

[SM94] S. Shiokawa and Y. Matsui. "The Dynamics of SAW Streaming and its Application to Fluid Devices". In: *MRS Proceedings* 360 (1994), p. 53.