Modelling Surface Acoustic Wave Driven Flows over Topography

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Want to numerically simulate the flow of a fluid driven by high frequency surface acoustic waves over a surface that may include topography and may be inclined.

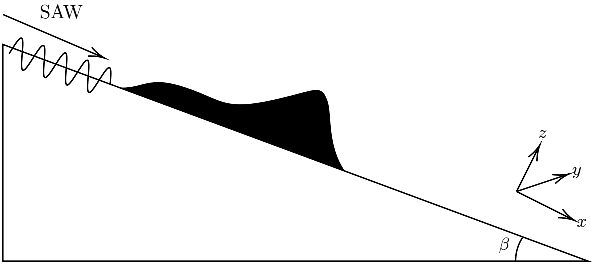

Figure: A simplified sketch of a fluid flowing down an inclined plane with no topography.

We let

- $s(x, y)$ describe the topography of the surface
- $h(x, y, t)$ describe the film thickness relative to $s(x, y)$
- $\phi(x, y, t) = s(x, y) + h(x, y, t)$ be the height of the free surface
- b describe the precursor film height

For completeness, we derive an equation to model fluids in a 3-D system, but we perform simulations on the simplified 2-D case.

We start with the Incompressible Navier-Stokes Equation

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\,\mathbf{u}\right)=-\nabla p+\mu\nabla^2\mathbf{u}+\rho g\sin\beta\mathbf{i}-\rho g\cos\beta\mathbf{k} \\ -\rho J e^{2k_i(x+\alpha_1 z)}\mathbf{i}-\rho J\alpha_1 e^{2k_i(x+\alpha_1 z)}\mathbf{k} \quad (1)$$

where

- $\mathbf{u}=(u,v,w)=$ Fluid velocity, $p=$ Fluid pressure, $\rho=$ Fluid density, $\mu=$ Fluid viscosity
- $k_i=$ Attenuation coefficient, $\alpha_1=$ Geometric constant, and $J=\left(1+\alpha_1^2\right)A^2\omega^2k_i$ is a constant we define to consolidate terms

The Lubrication Approximation assumes we are dealing with thin films and allows us to ignore the inertial terms of the Navier-Stokes equation (LHS) as well as the in plane derivatives and normal component of **u**. Hence, Eq. (1) reduces to

$$\begin{aligned} \nabla_2 p &= \mu \frac{\partial^2 \mathbf{v}}{\partial z^2} + \rho g \sin\beta \mathbf{i} - \rho J e^{2k_i(x+\alpha_1 z)} \mathbf{i} \\ \frac{\partial p}{\partial z} &= -\rho g \cos\beta - \rho J \alpha_1 e^{2k_i(x+\alpha_1 z)} \end{aligned} \tag{2}$$

where $\nabla_2 = (\partial_x, \partial_y)$ and $\mathbf{v} = (u, v)$.

We use the following boundary conditions

- Laplace-Young: At the interface $z = \phi(x, y, t)$ the pressure is given by $p(\phi) = -\gamma\kappa + p_0$ where κ is the curvature of the boundary, γ is the surface tension, and p_0 is the atmospheric pressure
- Vanishing shear stresses: $\frac{\partial \mathbf{v}}{\partial z} = \mathbf{0}$ along $z = \phi(x, y, t)$
- No slip: $\mathbf{v} = \mathbf{0}$ along the surface $z = s(x, y)$

Using these conditions and averaging over the height gives

$$\bar{\mathbf{v}} = -\frac{h^2}{3\mu}\left(\rho g \cos\beta \nabla\phi - \gamma\nabla\kappa - \rho g \sin\beta \mathbf{i} + \frac{\rho J}{2k_i}\nabla e^{2k_i(x+\alpha_1\phi)}\right).$$

The conservation of mass, when depth-averaged, gives

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\bar{\mathbf{v}}) = 0.$$

Approximating $\kappa \approx \nabla^2\phi$, this gives a final dimensional equation

$$\begin{aligned}\frac{\partial h}{\partial t} = \frac{1}{3\mu}\left[\nabla \cdot \left[\rho g \cos\beta h^3 \nabla\phi\right] - \nabla \cdot \left[\gamma h^3 \nabla\nabla^2\phi\right]\right] \\ + \frac{1}{3\mu}\left[-\rho g \sin\beta \frac{\partial h^3}{\partial x} + \nabla \cdot \left[\frac{\rho J}{2k_i} h^3 \nabla e^{2k_i(x+\alpha_1\phi)}\right]\right]. \quad (3)\end{aligned}$$

Scaling the coordinates and time by

$$\overline{x} = \frac{x}{x_c}, \quad \overline{y} = \frac{y}{x_c}, \quad \overline{z} = \frac{z}{h_c}, \quad \bar{t} = \frac{t}{t_c}$$

to get dimensionless quantities and using the following characteristics

$$t_c = \frac{3\mu x_c^3}{\gamma h_c^3}, \quad \varepsilon = \frac{h_c}{x_c}, \quad \mathrm{Bo} = \frac{x_c^2 \rho g}{\gamma}, \quad \mathrm{We}_{\mathrm{ac}} = \frac{\rho \omega^2 A^2 x_c}{\gamma}$$

gives the dimensionless equation

$$\frac{\partial h}{\partial t} = \mathrm{Bo}\cos\beta \nabla \cdot \left[h^3 \nabla \phi\right] - \nabla \cdot \left[h^3 \nabla \nabla^2 \phi\right] - \frac{\mathrm{Bo}}{\varepsilon} \sin\beta \frac{\partial h^3}{\partial x} + \frac{\left(1 + \alpha_1^2\right) \mathrm{We}_{\mathrm{ac}}}{2\varepsilon} \nabla \cdot \left[h^3 \nabla e^{2k_i(x + \alpha_1 \varepsilon \phi)}\right] \quad (4)$$

after removing any overlines.

We make the further simplification that the free surface of the film does not change in the transverse direction (i.e. h and s are both y-independent). This simplifies Eq. (4) to

$$\frac{\partial h}{\partial t} = \mathrm{Bo}\cos\beta\left[h^3\phi_x\right]_x - \left[h^3\phi_{xxx}\right]_x - \frac{\mathrm{Bo}}{\varepsilon}\sin\beta\left[h^3\right]_x + \frac{k_i\left(1+\alpha_1^2\right)\mathrm{We}_{\mathrm{ac}}}{\varepsilon}\left[h^3 e^{2k_i(x+\alpha_1\varepsilon\phi)}\left(1+\alpha_1\varepsilon\phi_x\right)\right]_x. \quad (5)$$

To enforce that the SAW forcing occurs starting from the film front on the left, we redefine k_i (in dimensionless form) as

$$k_i(h) = x_c\left(\left(k_i^{\text{liquid}} - k_i^{\text{air}}\right)\left(1 - e^{-x_c(h-b)/\lambda}\right) + k_i^{\text{air}}\right).$$

If we define the domain of interest as $[0, L_x]$, we can discretize the domain into $N_x + 1$ points such that

$$x_j = j\Delta x \qquad j = 0, \ldots, N_x \qquad \Delta x = L_x/N_x.$$

We further define $h_j(t) = h(x_j, t)$ which allows us to discretize Eq. (5) into a system of ODEs of the form

$$\frac{dh_j}{dt} = \mathrm{Bo}\cos\beta f_j^{(1)} - f_j^{(2)} - \frac{\mathrm{Bo}}{\varepsilon}\sin\beta f_j^{(3)} + \frac{k_i\left(1+\alpha_1^2\right)\mathrm{We_{ac}}}{\varepsilon} f_j^{(4)} \tag{6}$$

where $j = 1, \ldots, N_x - 1$ and $f_j^{(k)}$ is the discretization of the k-th term.

Discretizing $f_j^{(2)}$ (i.e. the fourth order term) requires the most care in order to not lead to a large computational stencil. Using a combination of forward and backward differences

$$h_{x,j} = \frac{h_{j+1} - h_j}{\Delta x} \text{ (Forward)} \qquad h_{\overline{x},j} = \frac{h_j - h_{j-1}}{\Delta x} \text{ (Backward)}$$

then a possible discretization is

$$\left[h^3 \phi_{xxx}\right]_x \to f_j^{(2)} = \left(\frac{1}{2}\left(h_{i-1}^3 + h_i^3\right) \phi_{\overline{x}x\overline{x},j}\right)_x$$

which is a second order approximation that gives a five point stencil.

Because the other terms are lower order, we can apply simpler differencing methods and not worry about increasing the size of our stencil

$$\left[h^3\phi_x\right]_x \to f_j^{(1)} = \frac{1}{2}\left(\left(h_{i-1}^3 + h_i^3\right)\phi_{x,j}\right)_{\overline{x}}$$

$$\left[h^3\right]_x \to f_j^{(3)} = \left(h_j^3\right)_{x^*}$$

$$\left[h^3 e^{2k_i(x+\alpha_1\varepsilon\phi)}\left(1+\alpha_1\varepsilon\phi_x\right)\right]_x \to f_j^{(4)} = \left(h_j^3 e^{2k_i(x_j+\alpha_1\varepsilon\phi_j)}\left(1+\alpha_1\varepsilon\phi_{x^*,j}\right)\right)_{x^*}$$

where x^* denotes a central difference.

We are mainly interested in situations where the SAW is the primary driving force, so we perform simulations with $\beta = 0$.

We consider the initial state of our system to be described by an equation $g(x)$. This allows us to define our boundary conditions

$$h(0,t) = g(0)\,, \quad h(L_x,t) = g(L_x)$$
$$h_x(0,t) = 0 = h_x(L_x,t)$$