Modelling Surface Acoustic Wave Driven Flows over Topography

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Want to numerically simulate the flow of a fluid driven primarily by high frequency surface acoustic waves over a surface that may include topography and may be inclined.

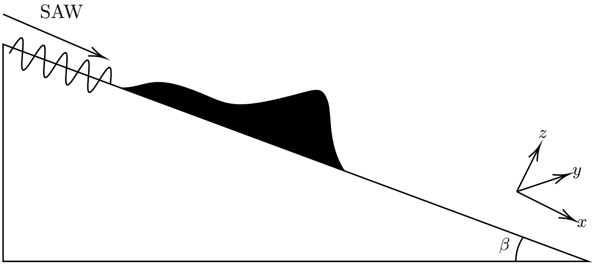
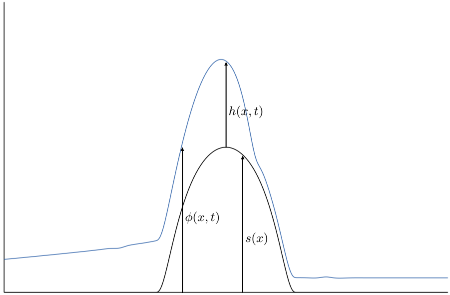


Figure: A simplified sketch of a fluid flowing down an inclined plane with no topography.

We let

- $s(x, y)$ describe the topography of the surface
- $h(x, y, t)$ describe the film thickness relative to $s(x, y)$
- $\phi(x, y, t) = s(x, y) + h(x, y, t)$ be the height of the free surface

For completeness, we derive an equation to model fluids in a 3-D system, but we perform simulations on the simplified 2-D case.



We start with the Incompressible Navier-Stokes Equation

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\,\mathbf{u}\right)=-\nabla p+\mu\nabla^2\mathbf{u}+\rho g\sin\beta\mathbf{i}-\rho g\cos\beta\mathbf{k} \\ -\rho J e^{2k_i(x+\alpha_1 z)}\mathbf{i}-\rho J\alpha_1 e^{2k_i(x+\alpha_1 z)}\mathbf{k} \quad (1)$$

where $\mathbf{u}=$ Fluid velocity, $p=$ Fluid pressure, $\rho=$ Fluid density, $\mu=$ Fluid viscosity, $k_i=$ Attenuation coefficient, $\alpha_1=$ Geometric constant, and $J=\left(1+\alpha_1^2\right)A^2\omega^2 k_i$ is a constant we define to consolidate terms.

The first two vector terms in the x and z direction represent the in plane and out of plane components of gravity, while the last two vector terms represent the in plane and out of plane components of the SAW forcing.

The Lubrication Approximation assumes we are dealing with thin films and allows us to ignore the inertial terms of the Navier-Stokes equation (LHS) as well as the in plane derivatives and normal component of **u**. Hence, Eq. (1) reduces to

$$\begin{aligned} \nabla_2 p &= \mu \frac{\partial^2 \mathbf{v}}{\partial z^2} + \rho g \sin\beta \mathbf{i} - \rho J e^{2k_i(x+\alpha_1 z)} \mathbf{i} \\ \frac{\partial p}{\partial z} &= -\rho g \cos\beta - \rho J \alpha_1 e^{2k_i(x+\alpha_1 z)} \end{aligned} \tag{2}$$

where $\nabla_2 = (\partial_x, \partial_y)$ and $\mathbf{v} = (u, v)$.

Boundary Conditions

If we define the domain of interest as $[0, L_x]$, we can discretize the domain into $N_x + 1$ points such that

$$x_j = j\Delta x \qquad j = 0, \ldots, N_x \qquad \Delta x = L_x / N_x.$$

We further define $h_j(t) = h(x_j, t)$ which allows us to discretize (temp eq holder) into a system of $N_x + 1$ ODEs of the form

$$\frac{dh_j}{dt} = \mathrm{Bo}\cos\beta f_j^{(1)} - f_j^{(2)} - \frac{\mathrm{Bo}}{\varepsilon}\sin\beta f_j^{(3)} + \frac{k_i\left(1+\alpha_1^2\right)\mathrm{We}_{\mathrm{ac}}}{\varepsilon} f_j^{(4)} \tag{3}$$

where $f_j^{(k)}$ is the discretization of the k-th term.

Discretizing $f_j^{(2)}$ (i.e. the fourth order term) requires the most care in order to not lead to a large computational stencil. Using a combination of forward and backward differences

$$h_{x,j} = \frac{h_{j+1} - h_j}{\Delta x} \text{ (Forward)} \qquad h_{\overline{x},j} = \frac{h_j - h_{j-1}}{\Delta x} \text{ (Backward)}$$

then a possible discretization is

$$\left[h^3 \phi_{xxx}\right]_x \approx f_j^{(2)} = \left(\frac{1}{2}\left(h_{i-1}^3 + h_i^3\right)\phi_{\overline{x}x\overline{x},j}\right)_x$$

which is a second order approximation that gives a five point stencil.

Because the other terms are lower order, we can apply simpler differencing methods and not worry about increasing the size of our stencil.

$$\left[h^3\phi_x\right]_x \approx f_j^{(1)} = \frac{1}{2}\left(\left(h_{i-1}^3 + h_i^3\right)\phi_{x,j}\right)_{\overline{x}}$$

$$\left[h^3\right]_x \approx f_j^{(3)} = \left(h_j^3\right)_{x^*}$$

$$\left[h^3 e^{2k_i(x+\alpha_1\varepsilon\phi)}\left(1+\alpha_1\varepsilon\phi_x\right)\right]_x \approx f_j^{(4)} = \left(h_j^3 e^{2k_i(x_j+\alpha_1\varepsilon\phi_j)}\left(1+\alpha_1\varepsilon\phi_{x^*,j}\right)\right)_{x^*}$$