Modelling Surface Acoustic Wave Driven Flows over Topography

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Want to numerically simulate the flow of a fluid driven by high frequency surface acoustic waves over a surface that may include topography and may be inclined.

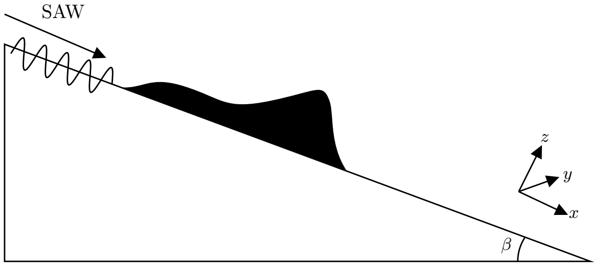


Figure: A simplified sketch of a fluid flowing down an inclined plane with no topography.

We let

- $s(x, y)$ describe the topography of the surface
- $h(x, y, t)$ describe the film thickness relative to $s(x, y)$
- $\phi(x, y, t) = s(x, y) + h(x, y, t)$ be the height of the free surface
- b describe the precursor film height (Using a precursor film accounts for any contact line singularities)

For completeness, we derive an equation to model fluids in a general 3-D system, but we perform simulations on the simplified non-inclined 2-D case.

We start with the Incompressible Navier-Stokes Equation

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho g \sin \beta \mathbf{i} - \rho g \cos \beta \mathbf{k} - \rho J e^{2k_i(x+\alpha_1 z)} \mathbf{i} - \rho J \alpha_1 e^{2k_i(x+\alpha_1 z)} \mathbf{k} \quad (1)$$

where

- $\mathbf{u} = (u, v, w) =$ Fluid velocity, $p =$ Fluid pressure, $\rho =$ Fluid density, $\mu =$ Fluid viscosity
- $k_i =$ Attenuation coefficient, $\alpha_1 =$ Geometric constant, and $J = \left(1 + \alpha_1^2\right) A^2 \omega^2 k_i$ is a constant we define to consolidate terms

The Lubrication Approximation assumes we are dealing with thin films and allows us to ignore the inertial terms of the Navier-Stokes equation (LHS) as well as the in plane derivatives and normal component of **u**. Hence, Eq. (1) reduces to

$$\begin{aligned} \nabla_2 p &= \mu \frac{\partial^2 \mathbf{v}}{\partial z^2} + \rho g \sin\beta \mathbf{i} - \rho J e^{2k_i(x+\alpha_1 z)} \mathbf{i} \\ \frac{\partial p}{\partial z} &= -\rho g \cos\beta - \rho J \alpha_1 e^{2k_i(x+\alpha_1 z)} \end{aligned} \tag{2}$$

where $\nabla_2 = (\partial_x, \partial_y)$ and $\mathbf{v} = (u, v)$.

We use the following boundary conditions

- Laplace-Young: At the interface $z = \phi(x, y, t)$ the pressure is given by $p(\phi) = -\gamma\kappa + p_0$ where κ is the curvature of the boundary, γ is the surface tension, and p_0 is the atmospheric pressure
- Vanishing shear stresses: $\frac{\partial \mathbf{v}}{\partial z} = \mathbf{0}$ along $z = \phi(x, y, t)$
- No slip: $\mathbf{v} = \mathbf{0}$ along the surface $z = s(x, y)$

Using these conditions and averaging over the height gives

$$\bar{\mathbf{v}} = -\frac{h^2}{3\mu}\left(\rho g \cos\beta \nabla\phi - \gamma\nabla\kappa - \rho g \sin\beta \mathbf{i} + \frac{\rho J}{2k_i}\nabla e^{2k_i(x+\alpha_1\phi)}\right).$$

The conservation of mass, when depth-averaged, gives

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\bar{\mathbf{v}}) = 0.$$

Approximating $\kappa \approx \nabla^2 \phi$, this gives a final dimensional equation

$$\begin{aligned}\frac{\partial h}{\partial t} = \frac{1}{3\mu}\left[\nabla \cdot \left[\rho g \cos\beta h^3 \nabla \phi\right] - \nabla \cdot \left[\gamma h^3 \nabla \nabla^2 \phi\right]\right] \\ + \frac{1}{3\mu}\left[-\rho g \sin\beta \frac{\partial h^3}{\partial x} + \nabla \cdot \left[\frac{\rho J}{2k_i} h^3 \nabla e^{2k_i(x+\alpha_1 \phi)}\right]\right]. \quad (3)\end{aligned}$$

Scaling the coordinates and time by

$$\overline{x} = \frac{x}{x_c}, \quad \overline{y} = \frac{y}{x_c}, \quad \overline{z} = \frac{z}{h_c}, \quad \bar{t} = \frac{t}{t_c}$$

to get dimensionless quantities and using the following characteristics

$$t_c = \frac{3\mu x_c^3}{\gamma h_c^3}, \quad \varepsilon = \frac{h_c}{x_c}, \quad \mathrm{Bo} = \frac{x_c^2 \rho g}{\gamma}, \quad \mathrm{We}_{\mathrm{ac}} = \frac{\rho \omega^2 A^2 x_c}{\gamma}$$

gives the dimensionless equation

$$\begin{aligned} \frac{\partial h}{\partial t} = \mathrm{Bo} \cos\beta \nabla \cdot \left[h^3 \nabla \phi\right] - \nabla \cdot \left[h^3 \nabla \nabla^2 \phi\right] - \frac{\mathrm{Bo}}{\varepsilon} \sin\beta \frac{\partial h^3}{\partial x} \\ + \frac{\left(1+\alpha_1^2\right) \mathrm{We}_{\mathrm{ac}}}{2\varepsilon} \nabla \cdot \left[h^3 \nabla e^{2k_i(x+\alpha_1 \varepsilon \phi)}\right] \end{aligned} \tag{4}$$

after removing any overlines.

We make the further simplification that the free surface of the film does not change in the transverse direction (i.e. h and s are both y-independent). This simplifies Eq. (4) to

$$\frac{\partial h}{\partial t} = \mathrm{Bo}\cos\beta\left[h^3\phi_x\right]_x - \left[h^3\phi_{xxx}\right]_x - \frac{\mathrm{Bo}}{\varepsilon}\sin\beta\left[h^3\right]_x + \frac{k_i\left(1+\alpha_1^2\right)\mathrm{We}_{\mathrm{ac}}}{\varepsilon}\left[h^3 e^{2k_i(x+\alpha_1\varepsilon\phi)}\left(1+\alpha_1\varepsilon\phi_x\right)\right]_x. \quad (5)$$

To enforce that the SAW forcing occurs starting from the film front on the left, we redefine k_i (in dimensionless form) as

$$k_i(h) = x_c\left(\left(k_i^{\text{liquid}} - k_i^{\text{air}}\right)\left(1 - e^{-x_c(h-b)/\lambda}\right) + k_i^{\text{air}}\right).$$

If we define the domain of interest as $[0, L_x]$, we can discretize the domain into $N_x + 1$ points such that

$$x_j = j\Delta x \qquad j = 0, \ldots, N_x \qquad \Delta x = L_x / N_x.$$

We further define $h_j(t) = h(x_j, t)$ which allows us to discretize Eq. (5) into a system of ODEs of the form

$$\frac{dh_j}{dt} = \mathrm{Bo}\cos\beta f_j^{(1)} - f_j^{(2)} - \frac{\mathrm{Bo}}{\varepsilon}\sin\beta f_j^{(3)} + \frac{k_i\left(1+\alpha_1^2\right)\mathrm{We}_{\mathrm{ac}}}{\varepsilon} f_j^{(4)} \tag{6}$$

where $j = 1, \ldots, N_x - 1$ and $f_j^{(k)}$ is the discretization of the k-th term.

Discretizing $f_j^{(2)}$ (i.e. the fourth order term) requires the most care in order to not lead to a large computational stencil. Using a combination of forward and backward differences

$$h_{x,j} = \frac{h_{j+1} - h_j}{\Delta x} \text{ (Forward)} \qquad h_{\overline{x},j} = \frac{h_j - h_{j-1}}{\Delta x} \text{ (Backward)}$$

then a possible discretization is

$$\left[h^3 \phi_{xxx}\right]_x \to f_j^{(2)} = \left(\frac{1}{2}\left(h_{i-1}^3 + h_i^3\right)\phi_{\overline{x}x\overline{x},j}\right)_x$$

which is a second order approximation that gives a five point stencil.

Because the other terms are lower order, we can apply simpler differencing methods as long as they are also second order and don't increase the size of the stencil.

$$\left[h^3\phi_x\right]_x \to f_j^{(1)} = \frac{1}{2}\left(\left(h_{i-1}^3 + h_i^3\right)\phi_{x,j}\right)_{\overline{x}}$$

$$\left[h^3\right]_x \to f_j^{(3)} = \left(h_j^3\right)_{x^*}$$

$$\left[h^3 e^{2k_i(x+\alpha_1\varepsilon\phi)}\left(1+\alpha_1\varepsilon\phi_x\right)\right]_x \to f_j^{(4)} = \left(h_j^3 e^{2k_i(x_j+\alpha_1\varepsilon\phi_j)}\left(1+\alpha_1\varepsilon\phi_{x^*,j}\right)\right)_{x^*}$$

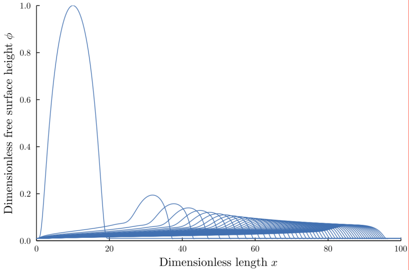
where x^* denotes a central difference.

We are mainly interested in situations where the SAW is the primary driving force, so we perform simulations with $\beta = 0$.

We let the initial state of our system to be described by an equation $g(x; b)$ where b is the precursor film thickness. This allows us to define our boundary conditions

$$h(0,t) = g(0) \quad h(L_x,t) = g(L_x)$$
$$h_x(0,t) = 0 = h_x(L_x,t)\,.$$

We use an implicit time stepping scheme (Rodas4) to simulate the fluid movement in time.

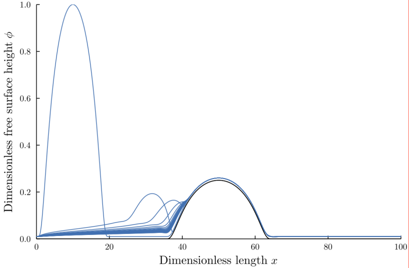


(a) Profile plot

(b) Profile animation

Figure: Fluid profile for a drop initial condition with a flat topography

Drop Initial Condition

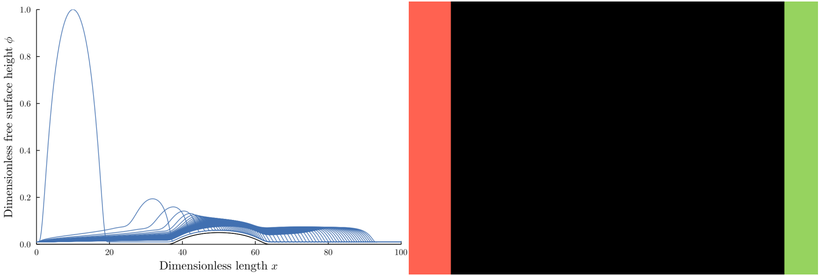


(a) Profile plot

(b) Profile animation

Figure: Fluid profile for a drop initial condition with a bump topography of height 0.25

Drop Initial Condition



(a) Profile plot

(b) Profile animation

Figure: Fluid profile for a drop initial condition with a bump topography of height 0.10