PSO

1. (a) $f(x) = \frac{1}{2} x^T A x + b^T x$

$$\nabla f(x) = \nabla\left(\frac{1}{2} x^T A x\right) + \nabla\left(b^T x\right)$$

$$= \frac{1}{2} \cdot 2Ax + b = \boxed{Ax + b}$$

(b) $f(x) = g(h(x))$

$$\nabla f(x) = \nabla g(h(x)) = \cancel{\nabla \begin{bmatrix} g(h(x_1)) \\ g(h(x_2)) \\ \vdots \\ g(h(x_n)) \end{bmatrix}}$$

$$= \begin{bmatrix} \frac{\partial}{\partial x_1} g(h(x)) \\ \frac{\partial}{\partial x_2} g(h(x)) \\ \vdots \\ \frac{\partial}{\partial x_n} g(h(x)) \end{bmatrix}$$

$$= \begin{bmatrix} g'(h(x)) \frac{\partial}{\partial x_1} h(x) \\ \vdots \\ g'(h(x)) \frac{\partial}{\partial x_n} h(x) \end{bmatrix} = \boxed{g'(h(x)) \nabla h(x)}$$

(c) $f(x) = \frac{1}{2} x^T A x + b^T x$

$$\nabla^2 f(x) = \nabla(\nabla f(x))$$

$$= \nabla(Ax + b)$$

$$= \nabla Ax + \nabla b = \boxed{A}$$

(d) $f(x) = g(a^T x)$

$$\nabla f(x) = g'(a^T x)\, \nabla a^T x = g'(a^T x)\, a$$

$$\nabla^2 f(x) = \nabla(\nabla f(x)) = \nabla(g'(a^T x)\, a)$$

$$= \left[g''(a^T x) a . a_1 \quad g''(a^T x) a . a_2 \cdots g''(a^T x) a a_n \right.$$

$$= \boxed{g''(a^T x)\, a a^T}$$

2. (a) $zz^T = A$, $z \in \mathbb{R}^n$

(i) $A^T = (zz^T)^T = (z^T)^T z^T = zz^T = A$

(ii) $x^T A x = x^T z z^T x = (x \cdot z) \cdot (x \cdot z) = (x \cdot z)^2 \geq 0$

(b) $A = zz^T$, $N(A) = \{x \in \mathbb{R}^n : Ax = \vec{0}\}$

$= \{x \in \mathbb{R}^n : zz^T x = \vec{0}\}$

$= \{x \in \mathbb{R}^n : z^T x = \vec{0}\}$ $(\because z \neq \vec{0})$.

$R(A) = R(zz^T) = 1$.

(c) $A \to PSD$. $M = BAB^T$

(i) $(BAB^T)^T = (B^T)^T A^T B^T = BAB^T = M$

(ii) $x^T M x = x^T B A B^T x = y^T A y \geq 0$ $(\because A$ is PSD$)$

3. (a) $A = T \Lambda T^{-1} \Rightarrow AT = T\Lambda$

$$\Rightarrow A[t^{(1)} \; t^{(2)} \cdots t^{(n)}] = [t^{(1)} \; t^{(2)} \cdots t^{(n)}] \begin{bmatrix} \lambda_1 & 0 & \cdots \\ \vdots & \lambda_2 & \\ 0 & \vdots & \\ & 0 & \end{bmatrix}$$

$$\Rightarrow [At^{(1)} \; At^{(2)} \cdots At^{(n)}] = [\lambda_1 t^{(1)} \; \lambda_2 t^{(2)} \cdots \lambda_n t^{(n)}]$$

$$\Rightarrow \boxed{At^{(i)} = \lambda_i t^{(i)}}, \; i \in [1, \ldots n]$$

(b) $A = U \Lambda U^T$

$\Rightarrow AU = U \Lambda U^T U$

$\Rightarrow AU = U \Lambda I = U \Lambda$

$\Rightarrow [Au^{(1)} \; Au^{(2)} \; \cdots \; Au^{(n)}] = [u^{(1)}\lambda_1 \; u^{(2)}\lambda_2 \; \cdots \; u^{(n)}\lambda_n]$

$\Rightarrow \boxed{Au^{(i)} = \lambda_i u^{(i)}}, \; i \in [1, n]$

(c) A is PSD

$\cancel{\Rightarrow x^T U \Lambda U^T x \geq 0}$ $\quad At^{(i)} = \lambda_i t^{(i)}$

$\cancel{\Rightarrow y^T \Lambda y \geq 0}$ $\Rightarrow t^{(i)T} A t^{(i)} = t^{(i)T} \lambda_i t^{(i)} \geq 0$

$\Rightarrow t^{(i)T} A t^{(i)} = \lambda_i t^{(i)T} t^{(i)} \geq 0$

$\Rightarrow \lambda_i \|t^{(i)}\|^2 \geq 0$

$\Rightarrow \lambda_i \geq 0 \quad (\because \|k\|^2 \geq 0)$