STA 243 Spring 2018 **Assignment 3** Due date: May 3 (5:02pm), 2018

For this assignment you can do it in a group of maximum three people. Objectives of this assignment:

- To gain experience in using the EM algorithm.
- To learn about the inverse transform method and the rejection sampling method.

Altogether there are six questions.

1. Consider the multinomial distribution with four outcomes, that is, the multinomial with pdf

$$p(x_1, x_2, x_3, x_4) = \frac{n!}{x_1!x_2!x_3!x_4!} p_1^{x_1} p_2^{x_2} p_3^{x_3} p_4^{x_4}, \quad \sum_{i=1}^{4} x_i = n, \quad \sum_{i=1}^{4} p_i = 1.$$

Suppose the probabilities are related by a single parameter $0 \leq \theta \leq 1$:

$$\begin{aligned} p_1 &= \frac{1}{2} + \frac{1}{4}\theta \\ p_2 &= \frac{1}{4} - \frac{1}{4}\theta \\ p_3 &= \frac{1}{4} - \frac{1}{4}\theta \\ p_4 &= \frac{1}{4}\theta. \end{aligned}$$

Given an observation $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$, the log-likelihood is

$$l(\theta) = x_1 \log(2 + \theta) + (x_2 + x_3) \log(1 - \theta) + x_4 \log \theta + c. \tag{1}$$

To use the EM algorithm on this problem, consider a multinomial with five classes formed from the original multinomial by splitting the first class into two with probabilities $1/2$ and $\theta/4$. The original variable x_1 is now split into $x_1 = x_{11} + x_{12}$. Under this reformulation, we now have a MLE of θ by considering $x_{12} + x_4$ to be a realization of a binomial with $n = x_{12} + x_4 + x_2 + x_3$ and $p = \theta$. However, we do not know x_{12}, and the complete data log-likelihood is

$$l_c(\theta) = (x_{12} + x_4) \log \theta + (x_2 + x_3) \log(1 - \theta). \tag{2}$$

   (a) Suppose $\boldsymbol{x} = (125, 21, 20, 33)$. Find the MLE of θ by maximizing (1).

   (b) Using (2), develop an EM algorithm for estimating θ. Note: you should be able to combine the E-Step and the M-Step together; i.e., $\hat{\theta}^{(t+1)}$ can be expressed in terms of $\hat{\theta}^{(t)}$.

   (c) Compare your answers obtained in (a) and (b).

2. Consider an iid sample drawn from a bivariate normal distribution with mean vector $\mu = (\mu_1, \mu_2)$ and covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}.$$

Suppose through some random accident that the first p observations are missing their first component, the next q observations are missing their second component, and the last r observations are complete. Design an EM algorithm for estimating the five mean and variance parameters, taking the original data before the accidental loss as complete data.

3. Use the inverse transform method to sample from the density

$$f(x) \propto e^{-x}, \quad 0 < x < 2.$$

Note that x is less than 2. Detail your algorithm. Draw a sample of 5000 observations and plot the estimated density of the sample.

4. Consider the following probability density function:

$$f(x) \propto q(x) = \frac{e^{-x}}{1+x^2}, \quad x > 0.$$

Use rejection sampling to sample from $f(x)$ with the following envelope density functions:

$$g_1(x) = e^{-x}, \quad g_2(x) = \frac{2}{\pi(1+x^2)}, \quad x > 0.$$

   (a) For each density function ($f(x)$, $g_1(x)$ and $g_2(x)$), draw a sample of 5000 random observations and plot the estimated density function for $0 < x < 5$.

   (b) Comments on the speeds of sampling and the results using $g_1(x)$ and $g_2(x)$.

5. Consider the following two density functions:

$$f(x) \propto \sqrt{4+x}\,x^{\theta-1}e^{-x}, \quad g(x) \propto (2x^{\theta-1} + x^{\theta-1/2})e^{-x}, \quad x > 0.$$

   (a) Find the value of the normalizing constant for $g(x)$.

   (b) Show that $g(x)$ is a mixture of Gamma distributions. Identify the component distributions and their weights in the mixture.

   (c) Design a procedure to sample from $g(x)$.

   (d) Design a procedure using rejection sampling to sample from $f(x)$ using $g(x)$ as the auxiliary distribution. Graph the estimated density of a random sample generated by your procedure. Use $\theta = 0.5, 1, 1.5$.

6. Design a rejection algorithm to sample from the following density on the upper right quarter of the unit disc:

$$f(x,y) \propto x^{\alpha}y, \quad x > 0, \quad y > 0, \quad x^2 + y^2 \le 1.$$

You don't need to run simulations for this problem. Just describe your algorithm in detail.

—— End of Assignment 3 ——