8 Electrostatics

After completing this section, students should be able to do the following.

- distinguish between the distance vector and
- vector at a point whose length describes strength of the field
- Identify charge of electron, neutron and proton
- State and explain Coulombs law
- Apply Coulombs law using vector algebra to find electrostatic force at a point due to several point charges
- Explain definition of electric field
- Derive and calculate electric field due to several point charges using vector algebra
- Define conditions under which Gausss law can be used.
- Outline steps necessary to apply Gausss law
- Derive electric field due to line and sphere of charge
- Discuss analogy of circuits with water pipes
- Define potential as difference in potential energy per charge
- Derive potential of point charge
- Find electric field from potential and vice versa
- Define capacitance
- Derive capacitance of simple symetrical structures.
- Explain displacement current
- Describe how materials affect electric fields.

8.1 Electrostatic Force

Electic Charges

Electric charges observed in nature are multiples of a charge of an electron $e = -1.610^{-19}$ C. JJ Thomson discovered the electron in his cathode ray tube experiments in 1897. R. Millikan measured the mass to charge ratio of the electron in 1909 through his oil-drop experiment. In 1960, J. G. King proved experimentally that one proton carries a positive charge of $e = 1.610^{-19}$ C.

Electrostatic Force

The electrostatic force acts between electric charges in the following way:

- two positive charges repel each other.
- two negative charges repel each other.
- a positive and a negative charge attract each other.
- the force between two charges decreases inversely proportional to the square of the distance.
- the force acts along the line that connects the charges.
- in nature, positive and negative charges are balanced, and the net result is electrical neutrality! Balance is formed by tight fine mixtures of positive and negative charges.

A demonstration of the electric force by the MIT professor emeritus Walter Lewin.

YouTube link: https://www.youtube.com/watch?v=FKqWmnem3M4

Coulomb's Law

What we described is exactly the electrostatic force. All matter is a mixture of positive protons and negative electrons in a perfect balance. Coulomb described the strength and direction of the electrostatic force through his torsion-balance experiment in 1785. We can represent this electrostatic force visually. Figure 73 shows a stationary charge $+q_1$, repelling charge $+q_2$ with a force F_2. The figure shows the unit vector $\hat{r}_{21}$, and the distance vector $\overrightarrow{\mathbf{R}}_{21} = r_{21}\hat{r}_{21}$, where $r_{21} = |\overrightarrow{\mathbf{R}}_{21}|$ is the distance between the two point charges. The equation that describes the electrostatic (Coulomb) force $\overrightarrow{\mathbf{F}}_2$ is given in Equation 240.

$$\overrightarrow{\mathbf{F_2}} = \frac{q_1 q_2}{4\pi\varepsilon_0 r_{21}^2}\hat{r}_{21} \tag{240}$$

In the above equation, $\varepsilon_0 = 8.81\,10^{-12}\,\mathrm{F/m}$ is the electrical permittivity of air, r_{21} is the distance between charges, vector $\hat{r}_{21}$ is a unit vector oriented from charge 1 to charge 2. The unit vector is on the line that connects charges 1 and 2, and therefore the electrostatic force is also on the line that connects the two charges. The force will either point in the direction of the unit vector if the force is repulsive (charges have the same sign), or in the opposite direction when the force is attractive (charges have the opposite sign). Note that we need at least two charges to find the electrostatic force.

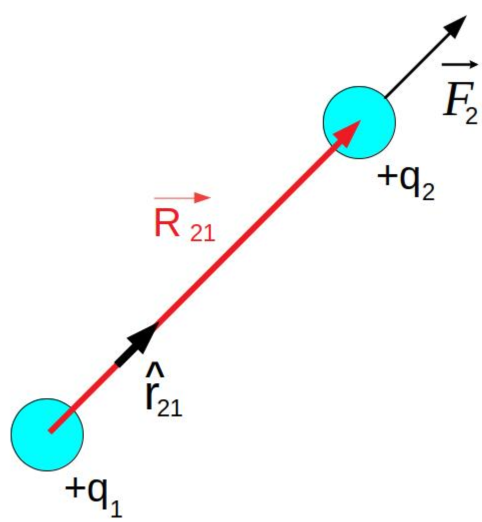

Figure 73: Vector representation of Coulomb's force between two static charges.

If the total net charge of an object is q, and if that object has n_e electrons and n_p protons, then the total charge is $q = n_p e - n_e e$.

Example 31. *Two positive unit charges $q_1 = 1\,nC$ and $q_2 = 1\,nC$ are fixed in air in Cartesian coordinate system at points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, see*

Figure 74. Find the electric force that a charge at point A exerts on charge at point B, and the force that charge at point B exerts on charge at point A. How would your answer change if one charge becomes negative?

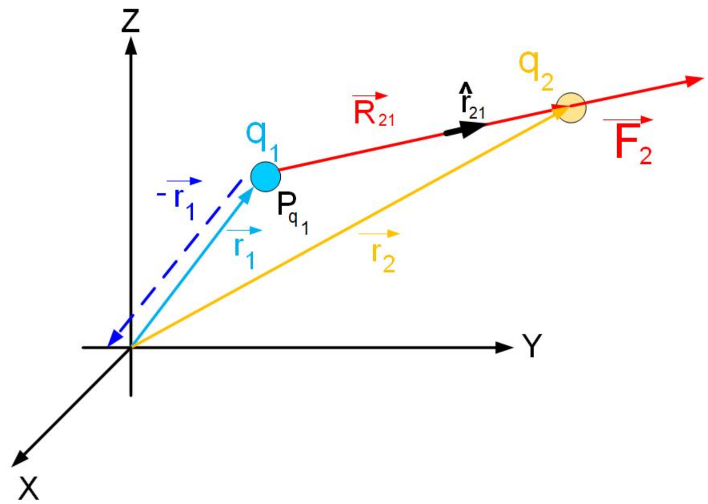

Figure 74: Electric Field due to a unit charge in Rectangular coordinate system.

Explanation. *The electrostatic force F_2 on charge q_2 is given by*

$$\overrightarrow{\mathbf{F_2}} = \frac{q_1 q_2}{4\pi\varepsilon_0 r_{21}^2}\hat{r}_{21} \tag{241}$$

To find the force, we have to calculate

(a) *The magnitude of the force $|\overrightarrow{\mathbf{F_2}}| = \frac{q_1 q_2}{4\pi\varepsilon_0 r_{21}^2}$. To find the magnitude, we need to find the distance between the two charges r_{21}. To find the distance, we need to find the distance vector $\overrightarrow{\mathbf{R}}_{21}$.*

(b) *To find the unit vector, we need to find the distance vector first. $\hat{r}_{21} = \frac{\overrightarrow{\mathbf{R}}_{21}}{r_{21}}$.*

The distance vector is a particular type of vector that starts at one point in the coordinate system and ends at another point. To find the distance vector $\overrightarrow{\mathbf{R}}_{21}$, we first have to know the position of charge q_1, where the distance vector starts, and the position of charge q_2 where the distance vector ends. In this problem, *the location of two charges is given by two points in the coordinate system A and B.*

The next step is finding the position vector r_1 *of point A, and the position vector* r_2 *of point B. Position vectors are special vectors that start in the coordinate system origin and point to various points in the coordinate system. Charge* q_1 *is at point* $A(x_1, y_1, z_1)$*, therefore the position vector* $\overrightarrow{\mathbf{r_1}}$ *of this point is shown in Equation 242.*

$$\overrightarrow{\mathbf{r_1}} = x_1 \overrightarrow{\mathbf{x}} + y_1 \overrightarrow{\mathbf{y}} + z_1 \overrightarrow{\mathbf{z}} \tag{242}$$

The position vector of charge q_2 *is*

$$\overrightarrow{\mathbf{r_2}} = x_2 \overrightarrow{\mathbf{x}} + y_2 \overrightarrow{\mathbf{y}} + z_2 \overrightarrow{\mathbf{z}} \tag{243}$$

The two vectors mark the beginning and the end of the distance vector $\overrightarrow{\mathbf{R}}_{21}$ *between charges* q_1 *and* q_2*. The vector* $\overrightarrow{\mathbf{R}}_{21}$ *is the sum of vectors* $-\overrightarrow{\mathbf{r_1}}$ *and* $\overrightarrow{\mathbf{r_2}}$*.*

$$\overrightarrow{\mathbf{R}}_{21} = \overrightarrow{\mathbf{r_2}} + (-\overrightarrow{\mathbf{r_1}}) \tag{244}$$

When we substitute position vectors r_1 *and* r_2*:*

$$\overrightarrow{\mathbf{R}}_{21} = (x_2 - x_1)\overrightarrow{\mathbf{x}} + (y_2 - y_1)\overrightarrow{\mathbf{y}} + (z_2 - z_1)\overrightarrow{\mathbf{z}} \tag{245}$$

The magnitude of vector $\overrightarrow{\mathbf{R}}_{21}$ *is*

$$|\overrightarrow{\mathbf{R}}_{21}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{246}$$

Unit vector in the direction of vector $\overrightarrow{\mathbf{R}}_{21}$ *is:*

$$\hat{r}_{21} = \frac{\overrightarrow{\mathbf{R}}_{21}}{|\overrightarrow{\mathbf{R}}_{21}|} \tag{247}$$

$$\hat{r}_{21} = \frac{\overrightarrow{\mathbf{R}}_{21}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}} \tag{248}$$

Substituting expressions for $\hat{r}_{21}$, and $|\overrightarrow{\mathbf{R}}_{21}|$ in equation for the electrostatic force

$$\overrightarrow{\mathbf{F_2}} = \frac{q_1 q_2}{4\pi\varepsilon_0 {r_a}^2}\hat{r_a} \tag{249}$$

We get

$$\overrightarrow{\mathbf{F_2}} = \frac{q_1 q_2}{4\pi\varepsilon_0 \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}^3}\overrightarrow{\mathbf{R}}_{21} \tag{250}$$

What if the charge is in an insulator (aka dielectric) other than air?

If the charge is within a dielectric material, then we need to account for that by changing this ε_0 somehow. If we place the charge inside a dielectric material, what do you think will happen with the atoms in the material? The atoms will get distorted and polarized. Such a polarized atom we call an electric dipole. The distortion process is called polarization. Because the material polarizes, the electric field around this point charge is different than if there was no material. To compensate for this new polarization, we multiply the dielectric permittivity of free space ε_0 with a unitless quantity of ε_r. ε_r is called a relative dielectric constant. ε_r values for different materials can found on the internet. Some examples of dielectric constants are ε_r: air ε_r=1, Teflon ε_r=2.2, glass ε_r=4.4, Silicon ε_r= 11, GaAs ε_r=12, distilled water ε_r= 80. Equation 251 is the definition of the electrostatic force between two charges. Sometimes, the product of $\varepsilon_0\varepsilon_r$ is written as ε.

$$\overrightarrow{\mathbf{F_e}} = \frac{q_1 q_2}{4\pi\varepsilon_0\varepsilon_r r^2}\hat{R_{12}} \tag{251}$$

$$\varepsilon = \varepsilon_0\varepsilon_r \tag{252}$$

Principle of Superposition

The principle of superposition states that in linear systems, we can calculate contributions of forces individually from different charges, then add them all up to get the total force on a charge.

If we have three or more charges, the total force from two charges to one charge is equal to the vector sum of the forces due to individual charges, see Figure 75.

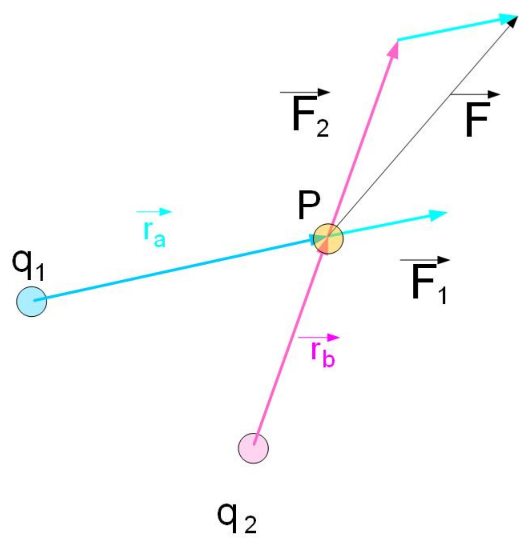

Figure 75: Electric Field due to two charges.

The force on the yellow charge below from charges q_1 and q_2 are:

$$\overrightarrow{\mathbf{F_1}} = \frac{q_1 q_y}{4\pi\varepsilon_0 {r_a}^2}\hat{r_a} \tag{253}$$

$$\overrightarrow{\mathbf{F_2}} = \frac{q_2 q_y}{4\pi\varepsilon_0 {r_b}^2}\hat{r_b} \tag{254}$$

Where $\hat{r_a}$ and $\hat{r_b}$ are unit vectors in the direction of r_a and r_b. The total field due to both charges is

$$\overrightarrow{\mathbf{F}} = \overrightarrow{\mathbf{F_1}} + \overrightarrow{\mathbf{F_2}} \tag{255}$$

Example 32. *Calculate the total force on a positive charge q_2 at a point (x_2, y_2, z_2) due to two other postive charges q_1 at a point (x_1, y_1, z_1) and charge q_3 at a point (x_3, y_3, z_3)*

Explanation. *Figure 76 shows the charges, distance vectors, position vectors and forces on charge q_2. The total force is equal to the sum of two individual forces from charges q_1 and q_2.*

$$\overrightarrow{\mathbf{F}} = \frac{q_1 q_2}{4\pi\varepsilon_0 \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}^3}\overrightarrow{\mathbf{r}}_{21} + \frac{q_2 q_3}{4\pi\varepsilon_0 \sqrt{(x_2-x_3)^2+(y_2-y_3)^2+(z_2-z_3)^2}^3}\overrightarrow{\mathbf{r}}_{23} \tag{256}$$

Where $\overrightarrow{\mathbf{r}}_{21}$ ist the distance vector from charge q_1 to q_2 and , $\overrightarrow{\mathbf{r}}_{23}$ is the distance vector from charge q_3 to q_2.

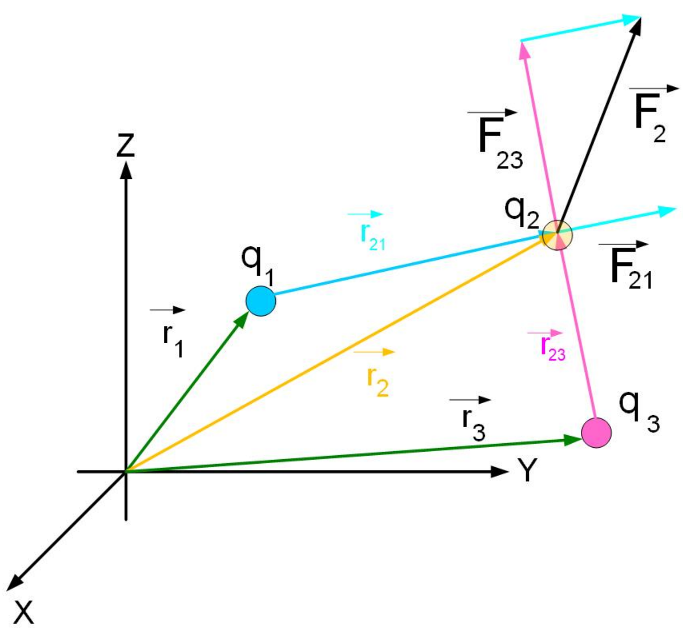

Figure 76: Electric field due to two charges in Rectangular coordinate system.

Problem 31 *Three positive charges, each $q = 100\mu C$ are placed at A(0,0), B(8,0), and C(4,4). Calculate the magnitude and direction of total force exerted on B due to charges A and C. Check your result with the calculator below. Explore with the calculator below how would the direction of the net force on B change if the charges A and C become negative.*

Geogebra link: `https://tube.geogebra.org/m/xqytpecf`

Question 32 *Four negative charges Q are distributed at (-1,0), (1,0), (0,1) and (0, -1). If we place the fifth charge Q at the origin (0,0), what will be the total force on this charge regardless of it's polarity?*

Multiple Choice:

(a) 0

(b) *Not enough information*

(c) $\dfrac{4Q^2}{4\pi\varepsilon_0}$

(d) $\dfrac{-4Q^2}{4\pi\varepsilon_0}$

Now, take a look at the simulation of the Baloon experiment. Charge the baloon by rubbing it on the sweater, then bring it to the wall. What happens? Observe how the neutral balloon is not attracted to the wall or sweater. When you rub it on the sweater, it will become attracted to the neutral wall. Why?

Geogebra link: `https://tube.geogebra.org/m/NcNVtwAQ`

A live demonstration of the electrostatic force between charged and charged, and charged and neutral body.Charging by induction. Observe the types of materials: metals and dielectrics (insulators). Towards the end of the video is a live demonstration of the balloon experiment you worked on in Geogebra app. The demonstration is by the MIT professor emeritus Walter Lewin.

YouTube link: `https://www.youtube.com/watch?v=3x0SuvuWe8Q`

8.2 Electrostatic Field

More about the Electrical force and field

We talked about the Electric Force in the previous section. To find the force on one charge, we had to know the position of all charges and what is their charge. To introduce Electric Field, we'll call charge q_1 the source of the force and charge q_2 the charge that feels the force. Charge one produces the force, and charge two feels the force.

By introducing the concept of the field, we can separate the cause of the field and the effect that the field has on other charges. We can find a field due to a charge or charge distribution, and once we know this field, we don't have to keep track of the source charge. We can just find the field's effect on other charges or charge distributions.

In the Equation 257 q_1 and q_2 are charges, r is the distance between the two charges, and $\hat{r}$ is the unit vector directed from charge one to charge two. The electric field of a source charge q_1 is defined as the force that a charge q_1 would impress on a positive charge q_2, divided by the amount of charge q_2, as shown in Equation 258. One thing to remember is that in the definition of the electric field, we always assume that the charge q_2 is positive! This way, we remove the ambiguity of the field direction. The direction of the electric field at a point P from charge q_1 is always in the direction of the force that would act on a positive charge q_2 placed at that point, as shown in Figure 77.

$$\overrightarrow{\mathbf{F_2}} = \frac{q_1 q_2}{4\pi\varepsilon r^2}\hat{r} \tag{257}$$

$$\overrightarrow{\mathbf{E}} = \frac{\overrightarrow{\mathbf{F_2}}}{q_2} \tag{258}$$

$$\overrightarrow{\mathbf{E}} = \frac{q_1}{4\pi\varepsilon_0 r^2}\hat{r} \tag{259}$$

If you carefully look at the Equation 259, you see that the electric field depends only on the source charge q_1.

If the electric charge is in a medium other than air, the electric field becomes

$$\overrightarrow{\mathbf{E}} = \frac{q_1}{4\pi\varepsilon_0\varepsilon_r r^2}\hat{R_{12}} \tag{260}$$

$$\varepsilon = \varepsilon_0\varepsilon_r \tag{261}$$

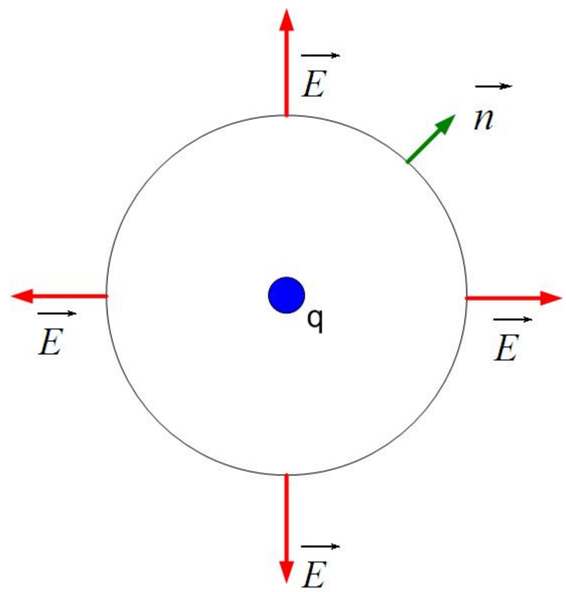

Figure 77: Electric field due to a unit charge q.

We added a unitless quantity, ε_r, called the relative dielectric constant, or relative permittivity, of a material. ε_r values for different materials can be found online. For example, you can see its values for different materials here `https://en.wikipedia.org/wiki/Relative_permittivity`.

Problem 33 *Find the electric field at a center of the Cartesian Coordinate system if positive charges q are placed at points (0,1) and (-1,0).*

positive charge that is in an electric field experiences a force that is

Problem 34 *A positive charge that is in an electric field E experiences a force that is*

Principle of Superposition

What is the electric field if we have more than one charge?

The total electric field at a point in space from the two charges is equal to the sum of the electric fields from the individual charges at that point.

If we have two charges, the total field due to both charges is equal to the vector sum of the fields due to individual charges, see Figure 78. The field at

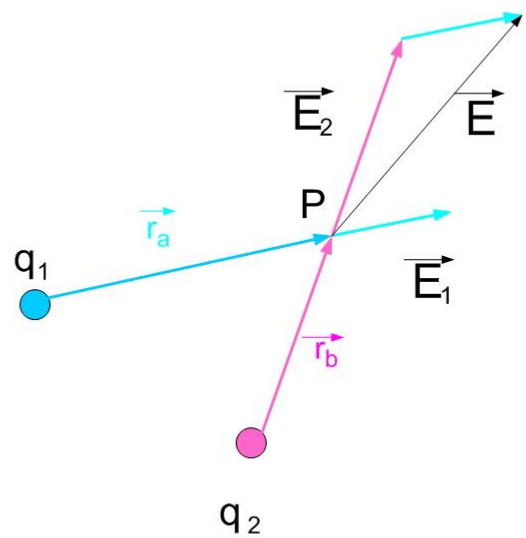

Figure 78: Electric Field due to two charges.

The fields or charges q_1 and q_2 are:

$$\overrightarrow{\mathbf{E_1}} = \frac{q_1}{4\pi\varepsilon_0 {r_a}^2}\hat{r_a} \tag{262}$$

$$\overrightarrow{\mathbf{E_2}} = \frac{q_1}{4\pi\varepsilon_0 {r_b}^2}\hat{r_b} \tag{263}$$

Where $\hat{r_a}$ and $\hat{r_b}$ are unit vectors in the direction of r_a and r_b. The total field due to both charges is

$$\overrightarrow{\mathbf{E}} = \overrightarrow{\mathbf{E_1}} + \overrightarrow{\mathbf{E_2}} \tag{264}$$

Electric Field in Rectangular Coordinates

The general equation for the electric field is given as

$$\overrightarrow{\mathbf{E}} = \frac{q_1}{4\pi\varepsilon_0 {r_a}^2}\hat{r_a} \tag{265}$$

The electric field at a point $P(x, y, z)$ due to a charge q_1 positioned at a point $P_{q_1}(x_1, y_1, z_1)$ in the rectangular coordinate system is shown in Figure 79. The position vector of the point P_{q_1} is

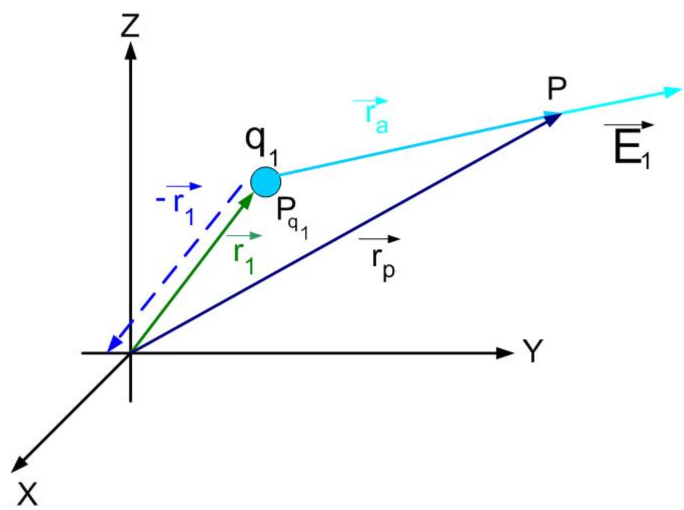

Figure 79: Electric Field due to a unit charge in Rectangular coordinate system.

$$\overrightarrow{\mathbf{r_1}} = x_1\overrightarrow{\mathbf{x}} + y_1\overrightarrow{\mathbf{y}} + z_1\overrightarrow{\mathbf{z}} \tag{266}$$

The position vector of point P is equal to

$$\overrightarrow{\mathbf{r_p}} = x\overrightarrow{\mathbf{x}} + y\overrightarrow{\mathbf{y}} + z\overrightarrow{\mathbf{z}} \tag{267}$$

The two vectors mark the beginning and the end of the distance vector $\overrightarrow{\mathbf{r_a}}$ between points P_{q_1} and P. The vector $\overrightarrow{\mathbf{r_a}}$ is the sum of vectors $-\overrightarrow{\mathbf{r_p}}$ and $\overrightarrow{\mathbf{r_1}}$

$$\overrightarrow{\mathbf{r_a}} = \overrightarrow{\mathbf{r_p}} + (-\overrightarrow{\mathbf{r_1}}) \tag{268}$$

When we substitute position vectors r_1 and r_p:

$$\overrightarrow{\mathbf{r_a}} = (x - x_1)\overrightarrow{\mathbf{x}} + (y - y_1)\overrightarrow{\mathbf{y}} + (z - z_1)\overrightarrow{\mathbf{z}} \tag{269}$$

Vector $\overrightarrow{\mathbf{r_a}}$ has the magnitude of:

$$|\overrightarrow{\mathbf{r_a}}| = \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2} \tag{270}$$

Unit vector in the direction of vector $\overrightarrow{\mathbf{r_a}}$ is:

$$\hat{r_a} = \frac{\overrightarrow{\mathbf{r_a}}}{|\overrightarrow{\mathbf{r_a}}|} \tag{271}$$

$$\hat{r_a} = \frac{\overrightarrow{\mathbf{r_a}}}{\sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}} \tag{272}$$

$$\overrightarrow{\mathbf{E_1}} = \frac{q_1}{4\pi\varepsilon_0 {r_a}^2}\hat{r_a} \tag{273}$$

Where r_a is the distance between the charge q_1 and the point P. Substituting expressions for $\hat{r_a}$, and $|\overrightarrow{\mathbf{r_a}}|$ in equation 265 we get

$$\overrightarrow{\mathbf{E_1}} = \frac{q_1}{4\pi\varepsilon_0 \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}^3}\overrightarrow{\mathbf{r_a}} \tag{274}$$

Substituting

For two charges, as shown in Figure 80 equation 274 becomes

$$\begin{aligned}\overrightarrow{\mathbf{E}} = &\frac{q_1}{4\pi\varepsilon_0 \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}^3}\overrightarrow{\mathbf{r_a}} + \\ &\frac{q_2}{4\pi\varepsilon_0 \sqrt{(x - x_2)^2 + (y - y_2)^2 + (z - z_2)^2}^3}\overrightarrow{\mathbf{r_b}}\end{aligned} \tag{275}$$

Example 33. *Find the point P where total electric field is zero inside of an equilateral triangle, if the three charges of magnitude $3\,nC$, $3\,nC$, and $3\,nC$ are placed in the corners of equilateral triangle of side $2\,m$. Use the app below to confirm your result.*

Geogebra link: `https://tube.geogebra.org/m/kupge9gc`

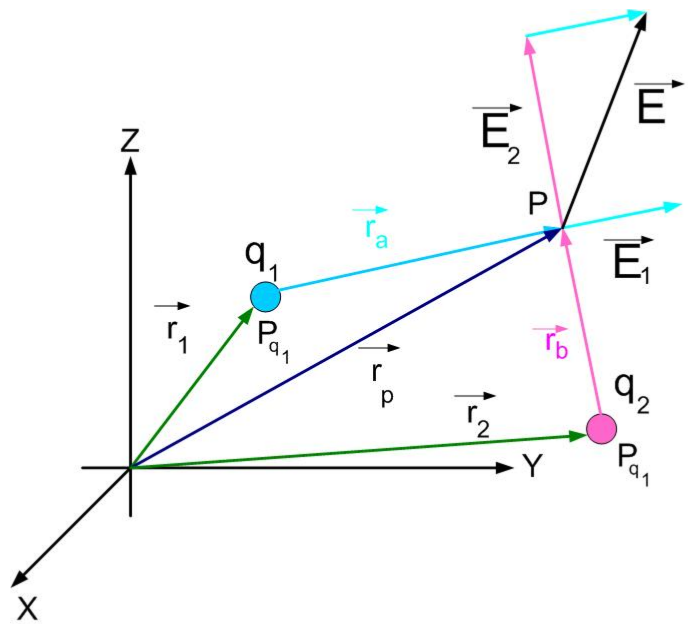

Figure 80: Electric field due to two charges in Rectangular coordinate system.

8.3 Electrostatic Potential

Electrostatic Potential Energy

When charges are pushed around in an electric field, the energy is not lost. Electrical forces are conservative.

How much energy do we have to bring into the system to bring two far-away positive charges at a distance r?

To answer that question, we can look at Figure 81. Bringing the first charge to a certain position in space would require no work since there are no other charged particles around, no electric field, and therefore no force. To bring the second charge at a distance r from the first charge, we need to overcome the repulsive force between the charges. How much work does this take? The work that we need to do to bring charges together is equal to the work that the first charge has to do to repel the second charge. The only difference is that we have to move the charges closer together, from infinity to some distance r, and the repulsive force does the work (pushes the charge q_2 away) from the distance r to infinity.

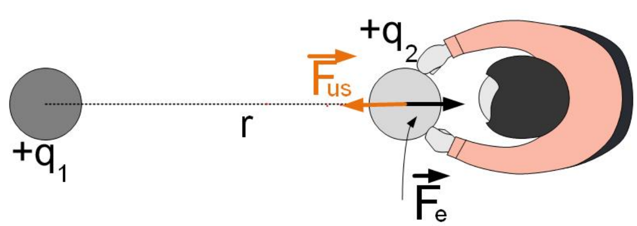

Figure 81: Moving the charge q_2 with force $\overrightarrow{\mathbf{F}}_{us}$ against the electrostatic force $\overrightarrow{\mathbf{F}}_{e}$ due to charge q_1 to find the electrostatic potential energy

$$W_{us} = We \tag{276}$$

$$W_{us} = \int_{\infty}^{r} \overrightarrow{\mathbf{F}}_{us} \cdot \overrightarrow{\mathbf{dr}} \tag{277}$$

$$W_e = \int_{r}^{\infty} \overrightarrow{\mathbf{F}}_{e} \cdot \overrightarrow{\mathbf{dr}} \tag{278}$$

The electric force is $\overrightarrow{\mathbf{F}}_e = \frac{q_1 q_2}{4\pi\varepsilon_o r^2}\overrightarrow{\mathbf{r}}$, so we can calculate the total work necessary to bring the two charges at distance r.

$$W_{us} = \int_r^{\infty} \overrightarrow{\mathbf{F}}_e \cdot \overrightarrow{\mathbf{dr}} \tag{279}$$

$$W_{us} = \int_r^{\infty} \frac{q_1 q_2}{4\pi\varepsilon_o r^2}\overrightarrow{\mathbf{r}}\,\overrightarrow{\mathbf{dr}} \tag{280}$$

Since the charge is moved in the direction of the force, we can drop vectors and calculate the work we have to do.

$$W_{us} = \frac{q_1 q_2}{4\pi\varepsilon_o}\int_r^{\infty} \frac{dr}{r^2} \tag{281}$$

$$W_{us} = \frac{q_1 q_2}{4\pi\varepsilon_o}(-\frac{1}{r}\bigg|_r^{\infty})W_{us} = \frac{q_1 q_2}{4\pi\varepsilon_o r} \tag{282}$$

This work is the same no matter what path we take to move charge q_2 from infinity to a distance r from charge q_1, see Figure 82. It depends only on the initial and end position of the charge.

Definition of Potential and Voltage

The work needed to move the two charges closer together depends on both charges q_1 and $q2$, just like the electrostatic force depends on both charges. It would be easier to define another variable that will separate the cause, charge q_1, and the effect, the work that we have to do to move charge q_2. This variable is Electric Potential. Electric potential is defined as the work we need to do to move the charge divided by the amount of charge q_2.

$$V = \frac{W_{us}}{q_2} \tag{283}$$

$$V = \frac{q_1}{4\pi\varepsilon_o r} \tag{284}$$

Observe that in Equation 284 the potential is a function of the "source" charge q_1. We again separated the source and the effect, but this time of the potential energy. The source is a charge q_1 that produces potential V. If we now want to see what is the potential energy or work that we need to do to move another charge, we don't have to know which charge produced it. We only need to know the potential in an area, from which we can find the potential energy change of charge q_2.

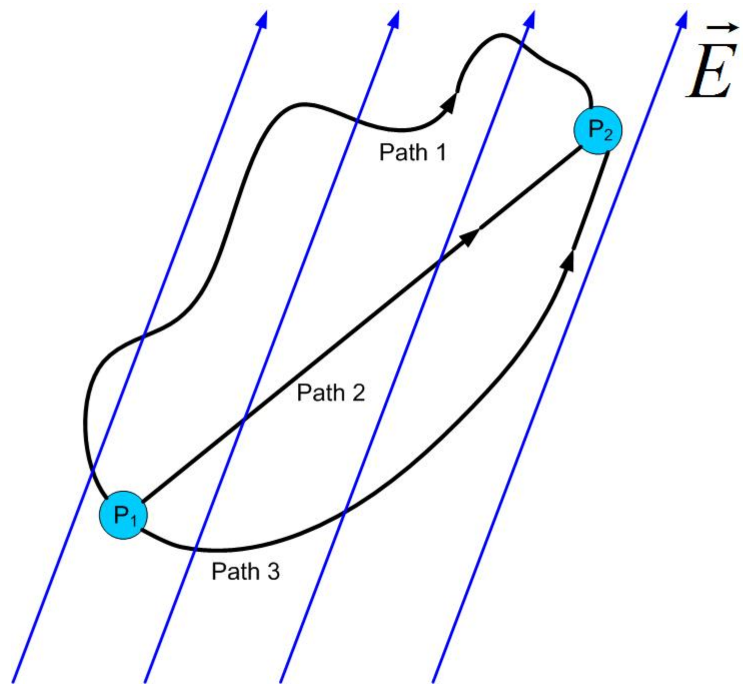

Figure 82: Potential is not dependent on the specific path.

We can find potential energy and potential of any number of charges using the above expression and the principle of superposition.

Question 35 *The difference in electric potential is most closely associated with*

Multiple Choice:

(a) *Work per unit charge*

(b) *Number of electrons in an atom*

(c) *Mechanical force on a charge*

The general relationship between the electric field and potential

In the first section, we defined the work necessary to bring two charges togeter at a distance r as

$$W_{us} = \int_r^\infty \overrightarrow{\mathbf{F}}_e \cdot \overrightarrow{\mathbf{dr}} \tag{285}$$

Electric force can be expressed in terms of electric field as

$$\overrightarrow{\mathbf{F}} = q\overrightarrow{\mathbf{F}} \tag{286}$$

If we substitute the electric force from the above equation into Equation 285, we get

$$V_R = \int_r^\infty q\overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dr}} \tag{287}$$

The potential of a point R with respect to infinity is then defined as

$$V_R = \frac{W_e}{q} = \int_r^\infty \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dr}} \tag{288}$$

The potential at a point due to a unit positive charge is found to be V. If the distance between the charge and the point is tripled, the potential becomes

Question 36 *The potential at a point due to a unit positive charge is V. If the distance between the charge and the point is doubled, the potential becomes*

Multiple Choice:

(a) V^2

(b) $2V$

(c) $V/2$

(d) *need more information*

Voltage - the potential difference

The potential difference between two points is defined as $V_{AB} = V_A - V_B$. We defined the potential at a point in the equation above as

$$V_A = \int_A^\infty \vec{\mathbf{E}} \cdot \vec{\mathbf{dr}} \tag{289}$$

$$V_B = \int_B^\infty \vec{\mathbf{E}} \cdot \vec{\mathbf{dr}} \tag{290}$$

The potential difference, or voltage is then defined as

$$\Delta V = V_A - V_B = \int_A^\infty \vec{\mathbf{E}} \cdot \vec{\mathbf{dr}} - \int_B^\infty \vec{\mathbf{E}} \cdot \vec{\mathbf{dr}} \tag{291}$$

$$\Delta V = \int_A^\infty \vec{\mathbf{E}} \cdot \vec{\mathbf{dr}} + \int_\infty^B \vec{\mathbf{E}} \cdot \vec{\mathbf{dr}} \tag{292}$$

$$\Delta V = \int_A^B \vec{\mathbf{E}} \cdot \vec{\mathbf{dr}} \tag{293}$$

Demonstration of potential difference in radial electric field of a VanDenGraaf generator by Prof. Emeritus at MIT, Walter Lewin.

YouTube link: `https://www.youtube.com/watch?v=cYOBSzh2SLE`

Now you can enjoy shocking John Travoltage in this PhET simulation.

Geogebra link: `https://tube.geogebra.org/m/zm6qzDwt`

Electric field calculation from potential difference

From the equation above, if we look at the potenital of two points that are very close on the x-axis, V_A, and $V_B = V_A + dV$, the voltage is equal to $V_A - V_B = V_A - (V_A + dV) = -dV = -E_x dx$. Therefore the electric field in the x-direction is

$$\overrightarrow{\mathbf{E_x}} = -\frac{\partial V}{\partial x} \tag{294}$$

The above equations states that the field is proportional to the change of potential over distance. The larger the change of potential over distance, the stronger the field.

To find the electric field in 3 dimensions, we use the gradient function, which represents a 3-dimensional derivative.

$$\overrightarrow{\mathbf{E}} = -\boldsymbol{\nabla} V = -(\frac{\partial V}{\partial x}\overrightarrow{\mathbf{x}} + \frac{\partial V}{\partial y}\overrightarrow{\mathbf{y}} + \frac{\partial V}{\partial z}\overrightarrow{\mathbf{z}}) \tag{295}$$

Example 34. *Figure 83 shows an electric field, and equipotential lines in an x-y plane. Equipotential lines are the lines of the same potential. We can observe that the electric field is always normal to the equipotential lines. The difference in potential between two points is* $dV = \overrightarrow{\mathbf{E}}\overrightarrow{\mathbf{dr}}$. *The lines, or surfaces, normal to the electric field vector are always equipotential surfaces, as the angle between the electric field vector and a perpendicular vector is* 90^0, *and the dot product between them is zero. In other words, the work that we have to do to move the charge in the direction perpendicular to the electric force is zero.*

Example 35. *Two charges, a positive and a negative one are placed on x-y plane. The potential around the charges is shown as a red surface. Observe the simulation below.*

(a) *Where is the potential positive, and where is it negative?*

(b) *What is the direction of the electric field?*

(c) *If we place a positive point charge closer to the positive charge Q1, which way will it go?*

(d) *What if we have a negative charge?*

Now click on the 2D view.

(a) *Where is the field the strongest?*

(b) *Can you tell the direction of the field from equipotential lines?*

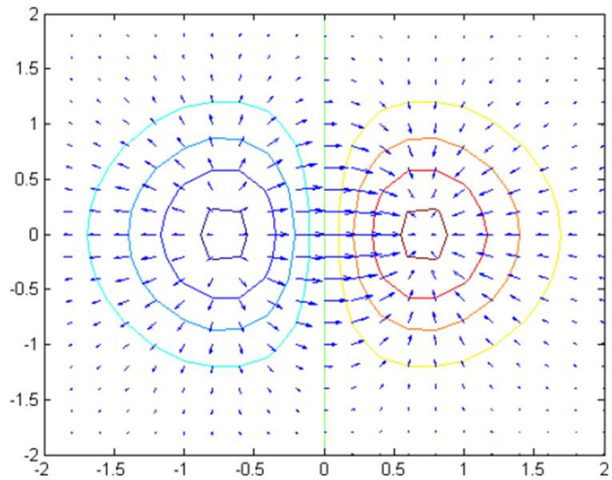

Figure 83: Electric field and equipotential lines.

Geogebra link: `https://tube.geogebra.org/m/tmkauhfc`

Problem 37 *Rougly adjust the direction and length of the vectors shown in the graph below. When done, click on "Grade Update " to check your answer.*

Geogebra link: `https://tube.geogebra.org/m/dqvffmrh`

8.4 Electrostatic fields from distributed charges

Electric Field due to a Charge Distribution

This is an optional section.

Derive Analytical Solution for the Electric field Due to a Loop of Charge

We will first find the electric field due to a loop of charge. The loop of charge is charged with the line charge density ρ_l and is in the X-Y plane, as shown in Figure 84. To solve this problem, we first divide the loop into small pieces. The small (blue) arc obtained in this way can be considered a point charge. The electric field due to a point charge is shown in Figure 84(the blue arrow). The position of the point charge is defined by a position vector $\overrightarrow{\mathbf{r_2}}$. The position of point P is defined with the position vector $\overrightarrow{\mathbf{r_1}}$. The vector $\overrightarrow{\mathbf{dE}}$ is defined in Equation 296.

$$\overrightarrow{\mathbf{dE}} = \frac{dQ}{4\pi\varepsilon_0 r^2}\hat{r} \tag{296}$$

The total electric field at a point P is then equal to the sum of all the fields due to the point charges, as shown in Figure 85. The equation for the total field is given in Equation 297.

$$\overrightarrow{\mathbf{E}} = \int\limits_{all\ point\ charges} \overrightarrow{\mathbf{dE}} \tag{297}$$

The problem now is to represent all the variables in the Equation 296 (dQ, dl, $\hat{r}$ and r) using appropriate coordinate system and given charge distribution. The total charge on the segment dl is equal to $dQ = \rho_l dl$. As seen in Figure 85, dl is an arc length in the direction of theta, $dl = a\,d\theta$, where a is the radius of the loop. The vector $\overrightarrow{\mathbf{r_2}}$ is the position vector of the arc length dl, and the vector $\overrightarrow{\mathbf{r_1}}$ is the position vector of the point be of the electric field calculation. Point P is an arbitrary point in the Cartesian coordinate system, P(x,y,z), therefore its vector is shown in Equation 298. The vector $\overrightarrow{\mathbf{r}}$ is the distance vector between the point charge (the source) and the point at which we are calculating the electric field.

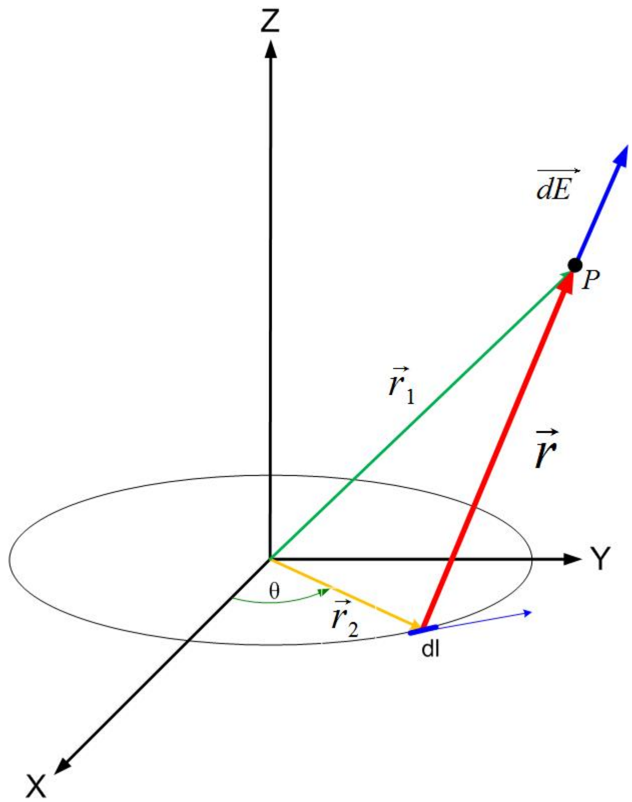

Figure 84: Loop of wire uniformly charged with line charge density ρ_l. Electric field is shown due to a very small section (arc length) of the loop dl.

$$\overrightarrow{\mathbf{r_1}} = x\overrightarrow{\mathbf{a_x}} + y\overrightarrow{\mathbf{a_y}} + z\overrightarrow{\mathbf{a_z}} \tag{298}$$

The vector $\overrightarrow{\mathbf{r_2}}$ can be written in Polar Coordinates as in Equation 299,where a is the radius of the loop. The equation 299 can be rewritten in Cartesian coordinate system as shown in Equation 301.

$$\overrightarrow{\mathbf{r_2}} = a\,\overrightarrow{\mathbf{a_r}} \tag{299}$$

$$\overrightarrow{\mathbf{a_r}} = \; cos\theta\,\overrightarrow{\mathbf{a_x}} + sin\theta\,\overrightarrow{\mathbf{a_y}} \tag{300}$$

$$\overrightarrow{\mathbf{r_2}} = a\,cos\theta\,\overrightarrow{\mathbf{a_x}} + a\,sin\theta\,\overrightarrow{\mathbf{a_y}} \tag{301}$$

The two vectors mark the beginning and the end of the distance vector $\overrightarrow{\mathbf{r}}$. The

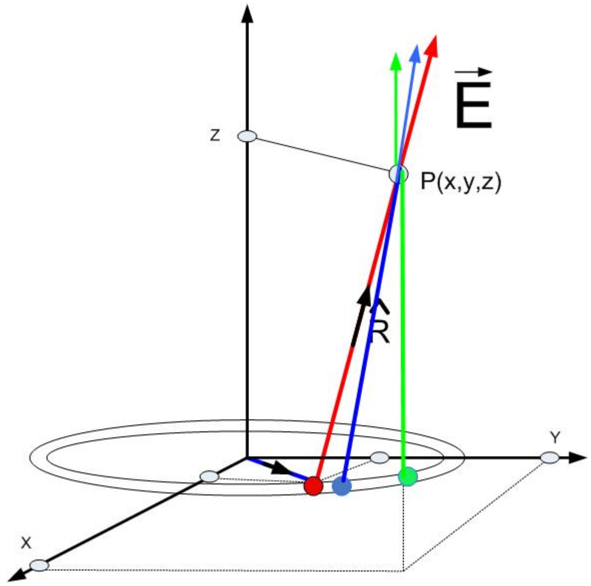

Figure 85: Loop of wire uniformly charged with line charge density ρ_l. Electric field is shown due to several very small sections (arc lengths) of the loop dl.

vector $\overrightarrow{\mathbf{r}}$ is the sum of vectors $-\overrightarrow{\mathbf{r_2}}$ and $\overrightarrow{\mathbf{r_1}}$.

$$\overrightarrow{\mathbf{r}} = \overrightarrow{\mathbf{r_1}} + (-\overrightarrow{\mathbf{r_2}}) \tag{302}$$

Or:

$$\overrightarrow{\mathbf{r}} = (x - a\,cos\theta)\overrightarrow{\mathbf{a_x}} + (y - a\,sin\theta)\overrightarrow{\mathbf{a_y}} + z\overrightarrow{\mathbf{a_z}} \tag{303}$$

Vector $\overrightarrow{\mathbf{r}}$ has the magnitude of:

$$|\overrightarrow{\mathbf{r}}| = \sqrt{(x - a\,cos\theta)^2 + (y - a\,sin\theta)^2 + z^2} \tag{304}$$

Unit vector in the direction of vector $\overrightarrow{\mathbf{r}}$ is:

$$\hat{r} = \frac{\overrightarrow{\mathbf{r}}}{|\overrightarrow{\mathbf{r}}|} \tag{305}$$

$$\hat{r} = \frac{\overrightarrow{\mathbf{r}}}{\sqrt{(x - a\,cos\theta)^2 + (y - a\,sin\theta)^2 + z^2}} \tag{306}$$

Replacing other variables in the Equations 298-306 to the Equation 296, we get the Equation 307 for the electric field $\overrightarrow{\mathbf{dE}}$ at a point P.

$$\overrightarrow{\mathbf{dE}} = \frac{\rho_l \, a \, d\theta}{4\pi\varepsilon_0 \sqrt{(x - a\,cos\theta)^2 + (y - a\,sin\theta)^2 + z^2}^3} \cdots$$
$$\cdots (x - a\,cos\theta)\overrightarrow{\mathbf{a_x}} + (y - a\,sin\theta)\overrightarrow{\mathbf{a_y}} + z\overrightarrow{\mathbf{a_z}} \tag{307}$$

Components of the electric field are given in Equations 308-310.

$$\overrightarrow{\mathbf{dE_x}} = \frac{\rho_l \, a \, d\theta}{4\pi\varepsilon_0 \sqrt{(x - a\,cos\theta)^2 + (y - a\,sin\theta)^2 + z^2}^3} (x - a\,cos\theta)\overrightarrow{\mathbf{a_x}} \tag{308}$$

$$\overrightarrow{\mathbf{dE_y}} = \frac{\rho_l \, a \, d\theta}{4\pi\varepsilon_0 \sqrt{(x - a\,cos\theta)^2 + (y - a\,sin\theta)^2 + z^2}^3} (y - a\,sin\theta)\overrightarrow{\mathbf{a_y}} \tag{309}$$

$$\overrightarrow{\mathbf{dE_z}} = \frac{\rho_l \, a \, d\theta}{4\pi\varepsilon_0 \sqrt{(x - a\,cos\theta)^2 + (y - a\,sin\theta)^2 + z^2}^3} z\overrightarrow{\mathbf{a_z}} \tag{310}$$

Each component of the field can be integrated separately, as shown in Equations 311-313.

$$\overrightarrow{\mathbf{E_x}} = \int_0^{2\pi} \frac{\rho_l \, a \, (x - a\,cos\theta) d\theta}{4\pi\varepsilon_0 \sqrt{(x - a\,cos\theta)^2 + (y - a\,sin\theta)^2 + z^2}^3} \overrightarrow{\mathbf{a_x}} \tag{311}$$

$$\overrightarrow{\mathbf{E_y}} = \int_0^{2\pi} \frac{\rho_l \, a \, (y - a\,sin\theta) d\theta}{4\pi\varepsilon_0 \sqrt{(x - a\,cos\theta)^2 + (y - a\,sin\theta)^2 + z^2}^3} \overrightarrow{\mathbf{a_y}} \tag{312}$$

$$\overrightarrow{\mathbf{E_z}} = \int_0^{2\pi} \frac{\rho_l \, a \, z d\theta}{4\pi\varepsilon_0 \sqrt{(x - a\,cos\theta)^2 + (y - a\,sin\theta)^2 + z^2}^3} \overrightarrow{\mathbf{a_z}} \tag{313}$$

The integrals in Equations 311-313 can be integrated analytically in some special cases only. In general, this is in an elliptical integral and cannot be solved analytically. However, this integral can be sold numerically.

Derive Numerical solution to the Electric Field due to a Loop of Charge

The integrals in equations 311-313 can be represented as infinite sums using simple numerical integration as shown in Equations 314-316. Here we see that the continuous function of θ was replaced with the discrete values of θ.

$$\overrightarrow{\mathbf{E_x}} = \sum_{i=0}^{n} \frac{\rho_l\, a\,(x - a\,cos\theta_i)\Delta\theta}{4\pi\varepsilon_0 \sqrt{(x - a\,cos\theta_i)^2 + (y - a\,sin\theta_i)^2 + z^2}^3} \overrightarrow{\mathbf{a_x}} \tag{314}$$

$$\overrightarrow{\mathbf{E_y}} = \sum_{i=0}^{n} \frac{\rho_l\, a\,(y - a\,sin\theta_i)\Delta\theta}{4\pi\varepsilon_0 \sqrt{(x - a\,cos\theta_i)^2 + (y - a\,sin\theta_i)^2 + z^2}^3} \overrightarrow{\mathbf{a_y}} \tag{315}$$

$$\overrightarrow{\mathbf{E_z}} = \sum_{i=0}^{n} \frac{\rho_l\, a\, z\Delta\theta}{4\pi\varepsilon_0 \sqrt{(x - a\,cos\theta_i)^2 + (y - a\,sin\theta_i)^2 + z^2}^3} \overrightarrow{\mathbf{a_z}} \tag{316}$$

$\Delta\theta$ represents the length of the interval that the line is divided into, θ_i represents the value of angle at a certain point, and i designates different points on the loop. It should be noted here that the error due to the trapezoidal rule can be improved by using more advanced numerical integration techniques.

Matlab Code to Find the Electric Field due to a Loop of Charge

Equations 314-316 can be implemented in Matlab as shown below. Cut-and-paste the program below in Matlab editor. Play with values for the size of the ring and meshgrid values. For some values of the step, the vectors on the graph will completely disappearcomment on why that may be.

```
clear all
%Specify the extents of x,y,z axes
rad=-2:1.9965:2;
%Make X,Y,Z matrices
[X Y Z]=meshgrid(rad);
%Define constants epsilon, Q. They are obviously not physical.
%They are set
%to small constants so that we get smaller numbers.
eps=1;
Q=1;
a=1;
const=Q/(2*pi*eps);
%Set the initial electric field components to zero.
Ex=0;
Ey=0;
Ez=0;
%Here we find the fieldl at all X,Y,Z points defined previously
% from each of the "unit" charges on the ring.
%th variable starts from 0.1
%to avoid the infinite value of V at the point r=1,theta=0.
for th=0.1:pi/20:2*pi
```

```
t=const./ (sqrt((X-cos(th)).^2 + (Y-sin(th)).^2 + (Z).^2)).^(3);
Ex = Ex+t.*(X-cos(th));
Ey = Ey+t.*(Y-sin(th));
Ez = Ez+t.*Z;
end
%Plot the field components Ex,Ey, Ez at points X,Y,Z.
%Scale the vectors by factor 3.
quiver3(X,Y,Z,Ex,Ey,Ez,3)
hold on;
%Plot the ring of charge where it is positioned in x-y plane.
t = linspace(0,2*pi,1000);
r=1;
x = r*cos(t);
y = r*sin(t);
plot(x,y,'b');
hold off
```

Short line of charge

Problem 38 *Derive electric field integral, numerical solution for integral, then implement the code in MATLAB, to find the electric field from a 1 m stick of charge, charged with a uniform line charge density of $\rho_l = 1nC/m$. The stick is positioned as in Figure 86.*

In the simulation below, observe how changing the section of the selected piece of charge (a red square) changes the field at a point P.

Geogebra link: `https://tube.geogebra.org/m/fmwgw66c`

Then click on the play button in the lower left corner of the simulation below to watch how each piece of charge contributes to the total electric field.

Geogebra link: `https://tube.geogebra.org/m/akfavjkx`

Potential

Derive Analytical Solution for the Potential Due to a Loop of Charge

Derive the potential due to a ring of charge, charged with a line charge density ρ_l.

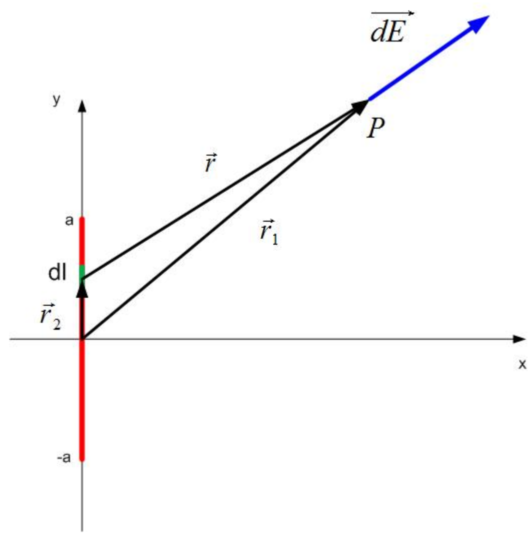

Figure 86: Stick of wire uniformly charged with line charge density ρ_l. Electric field is shown due to a very small section of the line dl.

Potential due to a point charge is given in Equation 317. We will first find the potential due to a loop of charge. Assume that the loop of charge is charged with the line charge density ρ_l. The loop of charge is in the X-Y plane, as shown in Figure 87. First, we will divide the loop into small pieces. We will assume that the small arc obtained in this way can be considered a point charge. The potential due to a point charge at a point P is labeled in Figure 87 as dV. The position of the point charge is defined by a position vector $\overrightarrow{\mathbf{r_2}}$. The position of point P is defined with the position vector $\overrightarrow{\mathbf{r_1}}$. The electric scalar potential dV is defined in Equation 317.

$$dV = \frac{dQ}{4\pi\varepsilon_0 r} \tag{317}$$

The total potential at a point P is then equal to the sum of all the potentials due to the point charges, as shown in Figure 88. The equation for the total potential is given in 318.

$$V = \int_{all\ point\ charges} dV \tag{318}$$

The problem now is to represent all the variables in the Equation 317 (dQ, dl, $\hat{r}$ and r) using appropriate coordinate system and given charge distribution. The

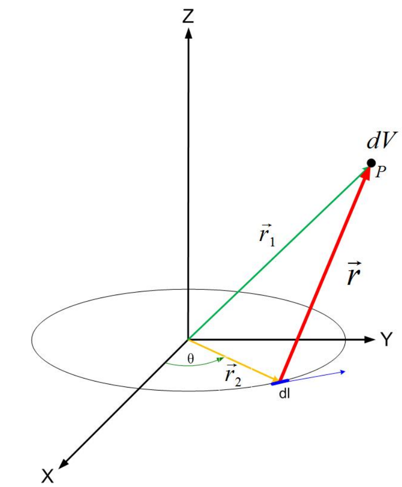

Figure 87: Loop of wire uniformly charged with line charge density ρ_l. Electric field is shown due to a very small section (arc length) of the loop dl.

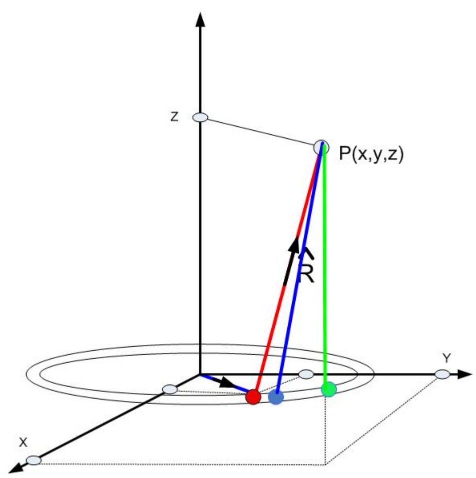

Figure 88: Loop of wire uniformly charged with line charge density ρ_l. Electric field is shown due to several very small sections (arc lengths) of the loop dl. Each section is modeled by a point charge dQ.

total charge on the segment dl is equal to $dQ = \rho_l dl$. As seen in Figure 87, dl is an arc length in the direction of theta (blue arrow next to dl) $dl = a\, d\theta$, where a is the radius of the loop. The vector $\overrightarrow{\mathbf{r_2}}$ is the position vector of the arc length dl, and the vector $\overrightarrow{\mathbf{r_1}}$ is the position vector of the point be of the electric field calculation. Point P is an arbitrary point in the Cartesian coordinate system, P(x,y,z), therefore its vector is shown in Equation 319. The vector $\overrightarrow{\mathbf{r}}$ is the distance vector between the point charge (the source) and the point at which we are calculating the electric field.

$$\overrightarrow{\mathbf{r_1}} = x\overrightarrow{\mathbf{a_x}} + y\overrightarrow{\mathbf{a_y}} + z\overrightarrow{\mathbf{a_z}} \tag{319}$$

The vector $\overrightarrow{\mathbf{r_2}}$ can be written in Polar Coordinates as in equation 320,where a is the radius of the loop. The Equation 320 can be rewritten in Cartesian coordinate system as in Equation 322.

$$\overrightarrow{\mathbf{r_2}} = a\,\overrightarrow{\mathbf{a_r}} \tag{320}$$

$$\overrightarrow{\mathbf{a_r}} = \ cos\theta\,\overrightarrow{\mathbf{a_x}} + sin\theta\,\overrightarrow{\mathbf{a_y}} \tag{321}$$

$$\overrightarrow{\mathbf{r_2}} = a\,cos\theta\,\overrightarrow{\mathbf{a_x}} + a\,sin\theta\,\overrightarrow{\mathbf{a_y}} \tag{322}$$

The two vectors mark the beginning and the end of the distance vector $\overrightarrow{\mathbf{r}}$. The vector $\overrightarrow{\mathbf{r}}$ is the sum of vectors $-\overrightarrow{\mathbf{r_2}}$ and $\overrightarrow{\mathbf{r_1}}$.

$$\overrightarrow{\mathbf{r}} = \overrightarrow{\mathbf{r_1}} + (-\overrightarrow{\mathbf{r_2}}) \tag{323}$$

Therefore the vector's $\overrightarrow{\mathbf{r}}$ magnitude is shown in Equations 325.

$$\overrightarrow{\mathbf{r}} = (x - a\,cos\theta)\overrightarrow{\mathbf{a_x}} + (y - a\,sin\theta)\overrightarrow{\mathbf{a_y}} + z\overrightarrow{\mathbf{a_z}} \tag{324}$$

Vector $\overrightarrow{\mathbf{r}}$ has the magnitude of:

$$|\overrightarrow{\mathbf{r}}| = \sqrt{(x - a\,cos\theta)^2 + (y - a\,sin\theta)^2 + z^2} \tag{325}$$

Replacing other variables in the Equations 319-325 to the Equation 317, we get the Equation 326 for the potential dV at a point P.

$$dV = \frac{\rho_l\,a\,d\theta}{4\pi\varepsilon_0\sqrt{(x - a\,cos\theta)^2 + (y - a\,sin\theta)^2 + z^2}} \tag{326}$$

$$V = \int_0^{2\pi} \frac{\rho_l\,a\,d\theta}{4\pi\varepsilon_0\sqrt{(x - a\,cos\theta)^2 + (y - a\,sin\theta)^2 + z^2}} \tag{327}$$

The integral in Equation 327 can be integrated analytically in some special cases; however, this equation can be solved numerically, as shown in the next section.

Derive Numerical solution to the Electric Field due to a Loop of Charge

The integrals in equations 327 can be represented with infinite sum using simple numerical integration (trapezoidal rule), as shown in Equations 328. Here we see that the continuous function of θ was replaced with the discrete values of θ.

$$V = \sum_{i=0}^{n} \frac{\rho_l \, a\Delta\theta}{4\pi\varepsilon_0 \sqrt{(x - a\,cos\theta_i)^2 + (y - a\,sin\theta_i)^2 + z^2}} \tag{328}$$

Where $\Delta\theta$ represents the length of the interval that the line is divided into, θ_i represents the value of angle at a certain point, and i designates different points on the loop. It should be noted here that the error due to the trapezoidal rule can be improved by using more advanced numerical integration techniques.

Matlab Code to Find the Electric Potential due to a Loop of Charge

Plot the cross-section of the potential and the equipotential lines on the planes x=0, y=0, z=0.

```
clear all
%Specify the extents of x,y,z axes
rad=-1.5:0.1:1.5

%Make X,Y,Z matrices
[X Y Z]=meshgrid(rad,rad,rad)
%Define constants epsilon, Q. They are obviously not physical.
%They are set
%to small constants so that we get smaller numbers.
eps=1
Q=1
const=Q/(2*pi*eps)
%Set the initial potential value to zero. Initialize V.
V=0
%Here we find the potential at all X, Y, Z points defined
% previously from
%each of the "unit" charges on the ring.
```

```
%The th variable starts from 0.1
%to avoid the infinite value of V at the point r=1,theta=0.

 for th=0.1:pi/20:2*pi
     t=const./ sqrt((X-cos(th)).^2 + (Y-sin(th)).^2 + (Z).^2)
V= V+t;
 end
%Plot the volume distribution of potential at planes
% x=0, y=0, z=0
slice(X,Y,Z,V,[0],[0],[0])
%Keep the same figure
hold on;
%Plot contours of potential at planes x=0, y=0, z=0. See
%more on contours in appendix.
h=contourslice(X,Y,Z,V,[0],[0],[0])
set(h,'EdgeColor','k','LineWidth',1.5)
```

Plot several equipotential surfaces.

```
clear all
%Specify the extents of x,y,z axes
rad=-1.5:0.1:1.5

%Make X,Y,Z matrices
[X Y Z]=meshgrid(rad,rad,rad)
%Define constants epsilon, Q. They are obviously not physical.
%They are set
%to small constants so that we get smaller numbers.
eps=1
Q=1
const=Q/(2*pi*eps)
%Set the initial potential value to zero. Initialize V.
V=0
%Here we find the potential at all X, Y, Z points defined
%previously from
%each of the "unit" charges on the ring. The th variable
% starts from 0.1
%to avoid the infinite value of V at the point r=1,theta=0.

 for th=0.1:pi/20:2*pi
     t=const./ sqrt((X-cos(th)).^2 + (Y-sin(th)).^2 + (Z).^2)
V= V+t;
 end
%Plot the volume distribution of potential at planes
```

```
% x=0, y=0, z=0
p = patch(isosurface(X,Y,Z,V,7));
isonormals(X,Y,Z,V,p)
set(p,'FaceColor','red','EdgeColor','none');
daspect([1 1 1])
view(3); axis tight
camlight
lighting gouraud
```

Observe the equipotential surfaces. Explain why the surfaces doughnut are shaped? What is the electric field direction on the surface? Change the isosurface from 7 to 10 or some larger number. How does the isosurface look now? How does the potential of a ring of charge looks at the distances far away from the ring? What about the field?

Visualizing Scalar Fields in Matlab

To visualize scalar fields in Matlab, we can use the following functions: slice, contourslice, patch, isonormals, camlight, and lightning. Please note that a more detailed explanation about these functions shown here can be found in Matlab's help.

slice

Slice is a command that shows the magnitude of a scalar field on a plane that slices the volume where the potential field is visualized. The format of this command is as shown below.

```
slice(x,y,z,v,xslice,yslice,zslice)
```

Where X, Y, and Z are coordinates of points where the scaler function is calculated, V is v the scalar function at those points, and the last three vectors xslice, yslice, and zslice are showing where will the volume will be sliced.

An example of slice command is given below. In the example below, there is an additional command colormap that colors the volume with a specific palette. To see more about different color maps, see Matlab's help. xslice has three points at which the x-axis will be slice. They are -1.2, .8, 2. This means that the volume will be slice with a plane that is perpendicular to the x-axis, and it crosses the x-axis at points -1.2, .8, and 2.

```
clc
clear all
```

```
[x,y,z] = meshgrid(-2:.2:2,-2:.25:2,-2:.16:2);
v = x.*exp(-x.^2-y.^2-z.^2);
xslice = [-1.2,.8,2]; yslice = 1; zslice = [-2,0];
slice(x,y,z,v,xslice,yslice,zslice)
colormap hsv
```

contourslice

Contourslice command will display equipotential lines on a plane being the volume where the potential field is visualized. An example of a contourslice function is shown below.

```
[x,y,z] = meshgrid(-2:.2:2,-2:.25:2,-2:.16:2);
v = x.*exp(-x.^2-y.^2-z.^2); % Create volume data
[xi,yi,zi] = sphere; % Plane to contour
contourslice(x,y,z,v,xi,yi,zi)
view(3)
```

patch

Patch command creates a patch of color.

isonormals

Command isonormals create equipotential surfaces.

camlight

```
camlight('headlight') creates a light at the camera position.

camlight('right') creates a light
right and up from camera.

camlight('left') creates a light
left and up from camera.camlight with no arguments is the
same as camlight('right').

camlight(az,el) creates a light
at the specified azimuth (az) and elevation (el)
with respect to the camera position.
```

```
The camera target is the center of rotation,
and az and el are in degrees.
```

lighting

```
Lighting flat selects flat lighting.

Lighting gouraud selects gouraud lighting.

Lighting phong selects phong lighting.

lighting none turns off lighting.
```

8.5 Calculation of electric field using Gauss's Law

Field Visualization

There are several ways of visualizing fields:

(a) vectors of different lengths represent the strength and direction of the field at different points.

(b) streamlines show the field flow. The field vector direction is tangential to a flow line. Streamlines are lines that a particle would follow in a field.

(c) Streamlines above can be drawn to show that the vector's strength is proportional to the density of field lines (number of field lines per unit area). Vector lines are usually represented as a fixed number of streamlines, not individual vectors. The lines bunch up where the field is stronger and diverge from each other where the field is weak.

Flux

Let's assume that in this example, we'll visualize a field with a density of field lines per area, as we have shown in 3 above. Rivers are symbols of flux. For example, if we imagine a river flow and a small rectangular frame submerged in it, we can qualitatively explain the "amount" of the field going through a surface. A simple way to understand flux is to count how many field lines poke a surface. If many field lines poke the surface, the flux is strong; otherwise, the flux is weak. The flux is the rate of flow of water through the frame.

Mathematically, the flux of any vector $\overrightarrow{\mathbf{A}}$ through a surface $\overrightarrow{\mathbf{S}}$ is defined as

$$\Phi = \int_S \overrightarrow{\mathbf{A}} \cdot \overrightarrow{\mathbf{dS}} \tag{329}$$

In the equation above, the surface is a vector so that we can define the direction of the flow of the vector. The surface vector $\overrightarrow{\mathbf{dS}}$ is defined as a surface of the frame dS multiplied with a vector perpendicular to the surface $dS\overrightarrow{\mathbf{n}}$. The flux is the highest if the normal to the surface and the vector field point in the same direction.

Example 36. *Observe the simulation below, change the angle between the frame and the field, then change the strenght of the field. How does the flux change?*

Geogebra link: *https://tube.geogebra.org/m/CdSZEQcW*

Gauss's Law

Gauss's Law states that the flux of electric field through a **closed** surface is equal to the charge enclosed divided by a constant.

$$\oint_S \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} = \frac{Q_{inS}}{\varepsilon} \tag{330}$$

It can be shown that no matter the shape of the closed surface, the flux will always be equal to the charge enclosed. This proof is beyond the scope of these lectures.

Gauss's Law is used to find the electric field when a charge distribution is given. We can apply Gauss's Law using analytical expressions only to a specific set of symmetric charge distributions.

The key to finding the Electric field from Gauss's Law is selecting the simplest surface to perform the integration in Equation 330. If the simplest surface is found, the above equation simplifies to $E\,S = \frac{Q_{inS}}{\varepsilon}$, where S is the surface (or sum of surfaces) where the flux exists, E is the electric field on that surface, and Q is the charge enclosed.

Example 37. *Observe the 2D flux from electric charge through a square. Set all charges to zero, except for one. Move the charge inside and outside of the square. What can you conclude about the flux when the charge is inside the square, and what when it is outside?*

Geogebra link: *https://tube.geogebra.org/m/r7Ue9Nac*

Question 39 *An positive Q and a negative charge -Q separated by a distance d (a dipole) are enclosed in a cube. What is the flux through the cube?*

Multiple Choice:

(a) Q/ε

(b) *zero*

(c) $2Q/\varepsilon$

(d) $Q/2\varepsilon$

Applying Gauss's Law to special, symmetric charge distributions

The key to applying Gauss's Law to find the field from a symmetric charge distribution is to find the surface S so that the normal to the surface is either perpendicular or parallel to the electric field. Practically speaking, this means that if the charge distribution has spherical symmetry, we'll choose a sphere for the surface. If the charge density has cylindrical symmetry, we'll choose a cylinder for the surface. If the charge density is an infinite plane, we'll choose a box (or, as we'll see later, a cylinder again). As you will see, before we apply Gauss's Law to find the electric field, we have to know how the electric field looks from a particular charge distribution so that we can carry out the integration.

(a) The parts of the surface where the electric field $\overrightarrow{\mathbf{E}}$ is perpendicular to the normal on the surface $dS\overrightarrow{\mathbf{n}}$ have zero flux through it, as the field does not poke the surface. Mathematically this can be explained through the dot product. Since the angle between the electric field and the normal to the surface is $\angle(\overrightarrow{\mathbf{E}}, \overrightarrow{\mathbf{dS}}) = 90^o$, the dot product becomes zero $\overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} = E\,dS\,\cos(\angle(\overrightarrow{\mathbf{E}}, \overrightarrow{\mathbf{dS}}))$ and so the integral $\int_S \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}}$ through that surface is zero.

(b) The parts of the surface where the electric field $\overrightarrow{\mathbf{E}}$ is parallel with the normal on the surface $dS\overrightarrow{\mathbf{n}}$ the dot product between the two quantities $\overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}}$ becomes $EdS\cos(\angle(\overrightarrow{\mathbf{E}}, \overrightarrow{\mathbf{dS}})) = EdS$, because $\angle(\overrightarrow{\mathbf{E}}, \overrightarrow{\mathbf{dS}}) = 0$, and $\cos(\angle(\overrightarrow{\mathbf{E}}, \overrightarrow{\mathbf{dS}})) = 1$. If we select a surface in such a way that all points of this surface are the same distance from the charge, the electric field is constant on the surface, so it can be taken out of the integral, and Gauss's law then simplifies to:

$$E \oint_S dS = \frac{Q_{inS}}{\varepsilon} \tag{331}$$

In the equation above, you can see that the integral $\int_S dS$ is just the surface area of the chosen surface, so the equation further simplifies to $E\,S = \dfrac{Q_{inS}}{\varepsilon}$

Electric Field of a Point Charge

Our first example is to find the electric field of a point charge +Q.

To start this problem, we have to know the direction of the field. The electric field lines from a point charge are pointed radially outward from the charge

(Figure 89). Mathematically we can write that the field direction is $\overrightarrow{\mathbf{E}} = E\,\hat{r}$. We have to know the direction and distribution of the field if we want to apply Gauss's Law to find the electric field.

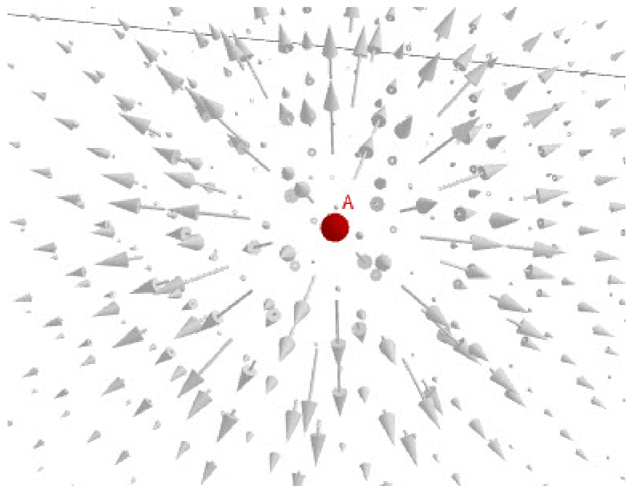

Figure 89: Electric field of a point charge

We want to find the magnitude of the electric field as a function of distance r from the charge. To do this, we will enclose the charge with an imaginary surface. We first have to decide on what kind of imaginary surface we are going to use in this case. Since the field has spherical symmetry, we will use a sphere, as shown in Figure 90. There are multiple reasons why we should use a sphere:

(a) Symmetry of the charge dictates using a sphere.

(b) The field on the sphere is in the same direction as the outward normal to the sphere.

(c) Electric field is constant everywhere on the sphere.

$$\oint_S \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} = \frac{Q_{inS}}{\varepsilon} \tag{332}$$

Because the electric field is in the same direction with the outward normal to the sphere as shown in Figure 90, the dot product becomes just the product of E and dS $\overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} = E\,dS\cos(\angle(\overrightarrow{\mathbf{E}}, \overrightarrow{\mathbf{dS}})) = E\,dS\cos(\angle(0^o)) = EdS$, and the Gauss's law equation becomes

$$\oint_S E\,dS = \frac{Q_{inS}}{\varepsilon} \tag{333}$$

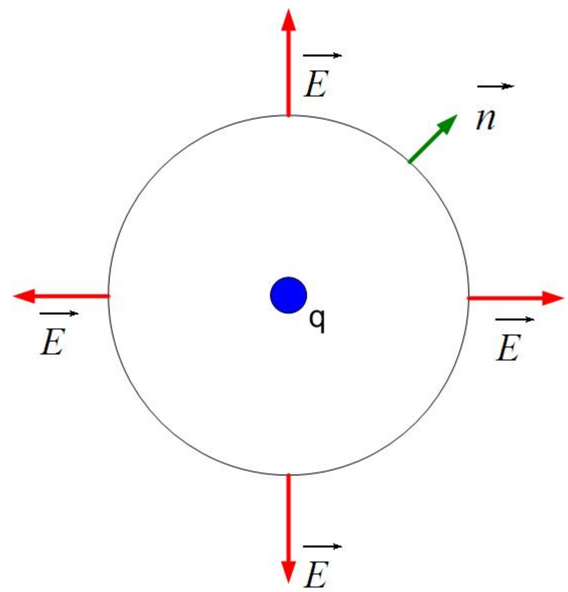

Figure 90: Application of Gauss's Law to find Electric Field of a point charge.

Since the electric field is constant everywhere on the surface, we can take it out of the integral.

$$E \oint_S dS = \frac{Q_{inS}}{\varepsilon} \tag{334}$$

In the above equation, the $\oint_S dS$ is just the surface area of the sphere, $4\pi r^2$.

$$E\, 4\pi r^2 = \frac{Q_{inS}}{\varepsilon} \tag{335}$$

The final expression for the magnitude of the field is:

$$E = \frac{Q_{inS}}{4\pi\varepsilon\, r^2} \tag{336}$$

Note that we only found the magnitude of the field in the above equation. We started this problem with the knowledge of field radial direction. So the final expression for the field is

$$\overrightarrow{\mathbf{E}} = \frac{Q_{inS}}{4\pi\varepsilon\, r^2}\, \hat{r} \tag{337}$$

Note that this equation can also be obtained from Coulomb's Law. One last step in finding the electric field is to find the total charge enclosed by the imaginary surface S. Depending on whether we're looking for the field inside or outside of a charge distribution, and whether the charge distribution is a volume charge distribution ρ, surface charge distribution σ or line charge distribution ρ_l, the whole or fraction of volume V, surface S or line l of the charge distribution will be enclosed.

$$Q_{inS} = \int_V \rho \, dV \tag{338}$$

$$Q_{inS} = \int_S \sigma \, dS \tag{339}$$

$$Q_{inS} = \int_l \rho_l \, dl \tag{340}$$

Electric Field of a Spherical Charge Distribution

A spherical region of radius a is charged with uniform volume charge density ρ=const. Find the field inside the spherical region of charge at a distance r from the center of the charge density and the field outside of the spherical region of charge at (another) distance r away from the center of the charge.

Electric Field outside of the sphere

We will first look at the field outside of the spherical charge distribution. This process is the same as in the previous problem, where we found the field from a point charge. Following the reasoning in the previous problem, we select a sphere for the integration surface. It is important to mention that we pick a point outside of the distributed charge at the distance r from the center, and that will be one point on the sphere's surface. We picked a point at random distance r, not at r=a because we want to find out electric field anywhere outside of the charge distribution E(r), and r could be any point $r > a$.

$$\oint_S \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} = \frac{Q_{inS}}{\varepsilon} \tag{341}$$

Because the electric field is in the same direction with the outward normal to the sphere as shown in Figure 91, the dot product becomes just the product of scalars E and dS $\overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} = E\,dS\cos(\angle(\overrightarrow{\mathbf{E}}, \overrightarrow{\mathbf{dS}})) = E\,dS\cos(\angle(0^o)) = EdS$, and the Gauss's law equation becomes

$$\oint_S E\,dS = \frac{Q_{inS}}{\varepsilon} \tag{342}$$

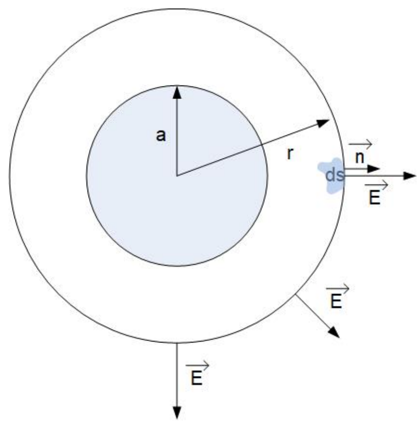

Figure 91: Application of Gauss's Law to find Electric Field of a spherical volume charge density.

Since the electric field is constant everywhere on the surface, we can take it out of the integral.

$$E \oint_S dS = \frac{Q_{inS}}{\varepsilon} \tag{343}$$

In the above equation, the $\oint_S dS$ is just the surface area of the sphere, $4\pi r^2$.

$$E\, 4\pi r^2 = \frac{Q_{inS}}{\varepsilon} \tag{344}$$

The final expression for the magnitude of the field is:

$$E = \frac{Q_{inS}}{4\pi\varepsilon\, r^2} \tag{345}$$

Note that we only found the magnitude of the field in the above equation. We started this problem with the knowledge of field radial direction. So the final expression for the field is

$$\overrightarrow{\mathbf{E}} = \frac{Q_{inS}}{4\pi\varepsilon\, r^2}\, \hat{r} \tag{346}$$

Note that this equation can also be obtained from Coulomb's Law. The charge Q_{inS} is the total charge in the spherical volume where the charge is located. If the total charge in the volume was known, then this is the solution. However, in this problem, the charge density is known, so we have to find the total charge.

$$Q_{inS} = \int_V \rho \, dV \tag{347}$$

Since the charge density ρ is constant, it can be taken outside of the integral.

$$Q_{inS} = \rho \int_V dV \tag{348}$$

The integral above is then just the entire volume of the charge distribution, which is $V = \dfrac{4\pi a^3}{3}$. The final expression for the electric field is

$$\vec{\mathbf{E}} = \frac{\rho \, 4\pi \, a^3}{12 \, \pi\varepsilon \, r^2} \, \hat{r} \tag{349}$$

$$\vec{\mathbf{E}} = \frac{\rho \, a^3}{3 \, \varepsilon \, r^2} \, \hat{r} \tag{350}$$

Note that now the electric field is inversely proportional to the square of the distance from the center of the sphere, $E \sim 1/r^2$. The electric field decreases as we move away from the sphere.

Electric field inside the sphere

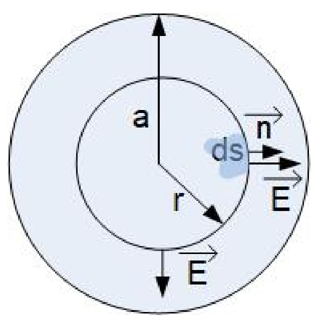

Figure 92: Application of Gauss's Law to find Electric Field of a .

Inside the spherical charge distribution, we'll again use a spherical imaginary surface S to enclose charge because of the spherical symmetry of the problem. The normal to the imaginary surface S is in the same direction with the electric field inside the spherical charge distribution as shown in Figure 92, and therefore the same analysis can be applied as above to get to the conclusion that:

$$E\,4\pi r^2 = \frac{Q_{inS}}{\varepsilon} \tag{351}$$

The difference in the analysis here is if we look at the right side of Gauss's law equation because we now have to determine the amount of charge enclosed by the imaginary surface we created. The charge enclosed in the imaginary surface is not the total charge Q. It is a fraction of the total charge. What fraction of the charge is enclosed? It is the fraction of the charge in the volume enclosed by the surface S.

$$Q_{inS} = \int_V \rho\, dV \tag{352}$$

Since the charge density ρ is constant, it can be taken outside of the integral.

$$Q_{inS} = \rho \int_V dV \tag{353}$$

The integral above is then just the fraction of the volume of the charge distribution enclosed by the surface S, which is $V = \dfrac{4\pi r^3}{3}$. The final expression for the electric field is

$$\overrightarrow{\mathbf{E}} = \frac{\rho\, 4\pi\, r^3}{12\,\pi\varepsilon\, r^2}\,\hat{r} \tag{354}$$

$$\overrightarrow{\mathbf{E}} = \frac{\rho\, r}{3\,\varepsilon}\,\hat{r} \tag{355}$$

Note that now the electric field is proportional to the distance from the center of the sphere, $E \sim r$, the field increases from the center of the spere out until $r = a$. At $r = a$, the entire charge has been enclosed, and the field is maximum at that point.

Electric field due to an infinite line of charge

A cylindrical region of radius a and infinite length is charged with uniform volume charge density ρ=const and centered on the z-axis. Find the field inside the cylindrical region of charge at a distance r from the axis of the charge density and the field outside of the spherical region of charge at (another) distance r away from the z-axis.

We will use Gauss's Law to solve this problem.

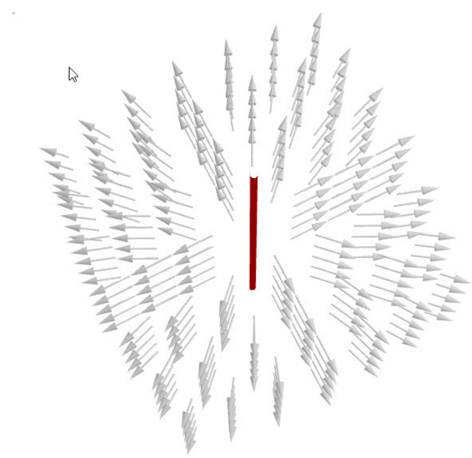

Figure 93: 3-dimensional electric field of a wire.

Electric field outside the line of charge (a wire)

We first look at the field outside of the cylindrical charge distribution. The wire is shown as a blue line in the direction of the z-axis, as shown in Figure 94. Because of the symmetry of the problem, we select a cylinder for the closed surface S. It is important to mention that we pick a point outside of the distributed charge at the distance r from the z-axis. The point that we picked is one point on the sphere's surface. We picked a point from the z-axis at a distance r, not at r=a, because we want to find out electric field anywhere outside of the charge distribution E(r), and r could be any point $r > a$. Gauss's Law states that

$$\oint_S \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} = \frac{Q_{inS}}{\varepsilon} \tag{356}$$

We will now apply Gauss's law to the three surfaces shown in Figure 94. The cylindrical surface consists of three surfaces: two bases S1 and S3 and the side surface S2. The normals to the bases are S1 and S3 are $\hat{n}_1 = -\hat{z}$, $\hat{n}_3 = \hat{z}$ and the normal to the side surface is $\hat{n}_2 = \hat{r}$. We can now split the flux of the electric field vector (the left-side of Gauss's law) through this closed cylindrical surface intro three integrals:

$$\oint_S \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} = \int_{S1} \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} + \int_{S2} \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} + \int_{S3} \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} \tag{357}$$

The electric field shown in Figure 94 is in the radial direction. The electric field lines only poke the side surface S2 and not surfaces S1 and S3. We, therefore, see that the flux through surfaces S1 and S3 is zero. This can also be shown

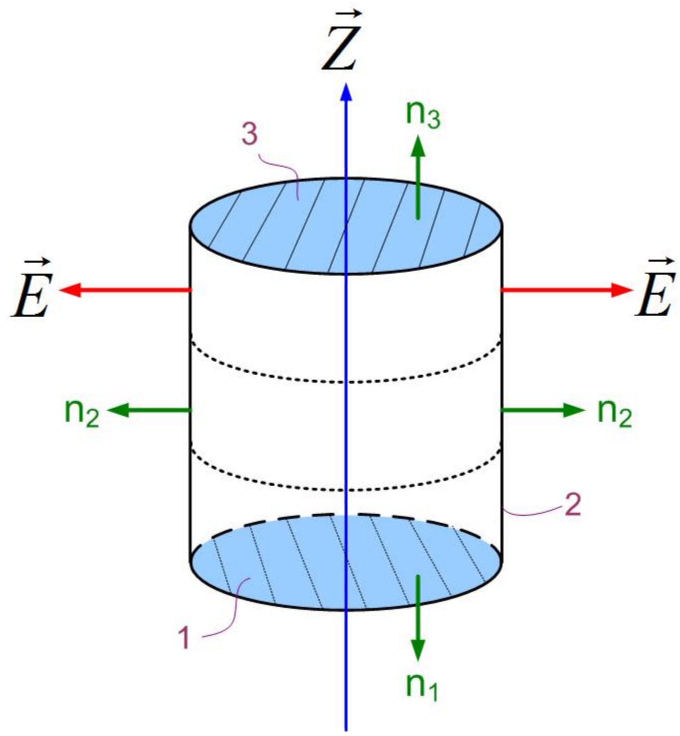

Figure 94: Application of Gauss's Law to find electric field of a wire.

mathematically. For the bases of the cylinder, the dot product between the top surface vector and the electric field vector in Gauss's law becomes $\overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} = E\,dS\cos(\angle(\overrightarrow{\mathbf{E}}, \overrightarrow{\mathbf{dS}})) = E\,dS\cos(\angle(90^o)) = 0$, and the Gauss's law equation becomes

$$\int_{S2} E\,dS = \frac{Q_{inS}}{\varepsilon} \tag{358}$$

Since the electric field is constant everywhere on the side surface, we can take it out of the integral.

$$E\int_{S2} dS = \frac{Q_{inS}}{\varepsilon} \tag{359}$$

In the above equation, the $\int_{S2} dS$ is just the rectangular surface area of the side of the cylinder, $2\pi rl$. Where r is the radius of the cylinder, and l is the length of the cylinder.

$$E\,2\pi rl = \frac{Q_{inS}}{\varepsilon} \tag{360}$$

From here we can find the magnitude of the field:

$$E = \frac{Q_{inS}}{2\pi\varepsilon\,l\,r} \tag{361}$$

Note that we only found the magnitude of the field in the above equation. We started this problem with the knowledge of the field's radial direction.

$$\overrightarrow{\mathbf{E}} = \frac{Q_{inS}}{2\pi\varepsilon\,l\,r}\,\hat{r} \tag{362}$$

The charge Q_{inS} is the total charge in the spherical volume where the charge is located. If the total charge in the volume was known, then this is the solution. However, in this problem, the charge density is known, so we have to find the total charge.

$$Q_{inS} = \int_V \rho\,dV \tag{363}$$

Since the charge density ρ is constant, it can be taken outside of the integral.

$$Q_{inS} = \rho\int_V dV \tag{364}$$

The integral above is then just the entire volume of the charge distribution, which is $V = a^2\pi l$. The final expression for the electric field is

$$\overrightarrow{\mathbf{E}} = \frac{\rho\, a^2\pi l}{2\,\pi\varepsilon\, l\, r}\,\hat{r} \tag{365}$$

$$\overrightarrow{\mathbf{E}} = \frac{\rho\, a^2}{2\,\varepsilon\, r}\,\hat{r} \tag{366}$$

Note that now the electric field is inversely proportional to the distance from the z-axis, $E \sim 1/r$. The electric field decreases as we move away from the sphere, but slower than in the case of the sphere of charge.

Electric field inside the line of charge

To find the electric field inside the cylindrical charge distribution, we zoom in on the wire in the previous figure and select a cylindrical imaginary surface S inside the wire, as shown in Figure 95. Since the electric field is in the same direction inside the wire, and the flux of the electric field is the same, we conclude that the left side of Gauss's Law is the same as in the previous case.

$$E\,S = \frac{Q_{inS}}{\varepsilon} \tag{367}$$

$$E\,2\pi r l = \frac{Q_{inS}}{\varepsilon} \tag{368}$$

The difference here is the amount of charge enclosed by the surface S.

$$Q_{inS} = \int_V \rho\, dV \tag{369}$$

Since the charge density ρ is constant, it can be taken outside of the integral.

$$Q_{inS} = \rho \int_V dV \tag{370}$$

The integral above is then just the fraction of the volume of the charge distribution enclosed by the surface S, which is $V = r^2\pi\, l$. The final expression for the electric field is

$$\overrightarrow{\mathbf{E}} = \frac{\rho\, r^2\,\pi l}{2\pi l\varepsilon\, r}\,\hat{r} \tag{371}$$

$$\overrightarrow{\mathbf{E}} = \frac{\rho\, r}{2\,\varepsilon}\,\hat{r} \tag{372}$$

The electric field is proportional to the distance from the center of the sphere, $E \sim r$, the field increases from the center of the spere out until $r = a$. At $r = a$, the entire charge has been enclosed, and the field is maximum at that point.

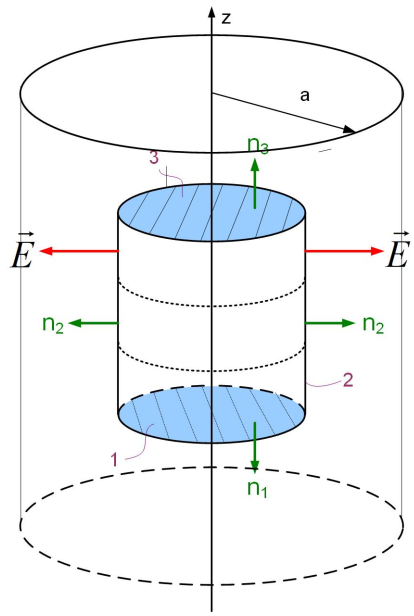

Figure 95: Infinite plane charged with positive surface charge density ρ_S.

Electric Field due to an infinite plane of Charge

An infinite plane of charge has an electric field in the direction away from it, as shown in Figure 96. In this figure, the light blue plane represents the charged plane, and only electric field vectors are represented above the plane. Below the plane, the vectors will be in the opposite direction, away from the plane.

The flux of the electric field vector is zero for any frame that is perpendicular to the plane of charge. For our imaginary surface, we can then use a box, where the flux will only exist through the top and bottom surfaces that are perpendicular to the field, or we can use a cylinder whose bases are parallel to the plane, as shown in Figure 97.

We can now separate the integral around the closed cylindrical surface to three surfaces, two that are parallel to the plane of charge, where the flux is not zero, and one side surface where the flux is zero.

$$\oint_S \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} = \int_{S1} \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} + \int_{S2} \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} + \int_{S3} \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} \tag{373}$$

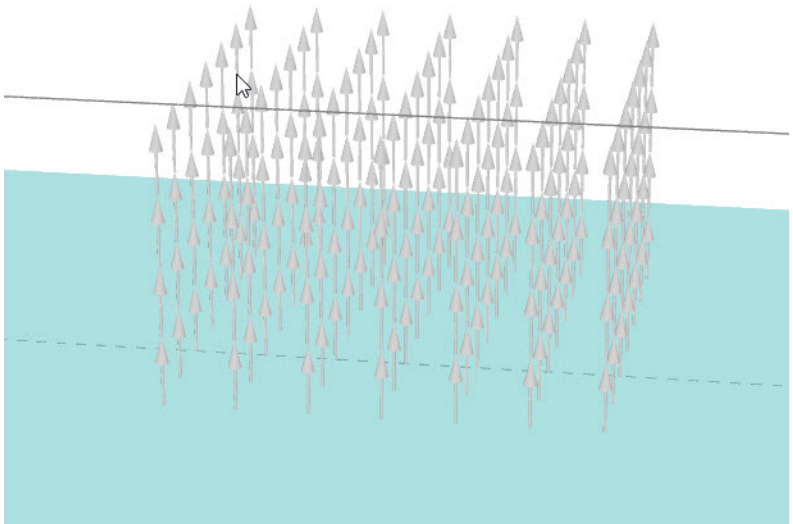

Figure 96: Electric field above an infinite plane charged with positive surface charge density ρ_S.

Mathematically the dot product between the electric field and the normal to the cylindrical top (S3) and bottom surface (S1) is just the product of the magnitudes of E and dS $\overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} = E\,dS\cos(\angle(\overrightarrow{\mathbf{E}}, \overrightarrow{\mathbf{dS}})) = E\,dS\cos(\angle(0^o)) = EdS$. The dot product on the side surface (S2) is zero, because the angle between the normal to the surface and the electric field is 90^o, therefore the $\overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} = E\,dS\cos(\angle(\overrightarrow{\mathbf{E}}, \overrightarrow{\mathbf{dS}})) = E\,dS\cos(\angle(90^o)) = 0$. We can subsequently take the electric field outside of the integral, and the integral around the closed surface then becomes

$$E\int_{S1} dS + E\int_{S2} dS = \frac{Q_{inS}}{\varepsilon} \tag{374}$$

The integral of surface S1 and S1 is just the surface area of two surfaces.

$$E\,S + E\,S = \frac{Q_{inS}}{\varepsilon} \tag{375}$$

$$2E\,S = \frac{Q_{inS}}{\varepsilon} \tag{376}$$

$$E = \frac{Q_{inS}}{2\varepsilon S} \tag{377}$$

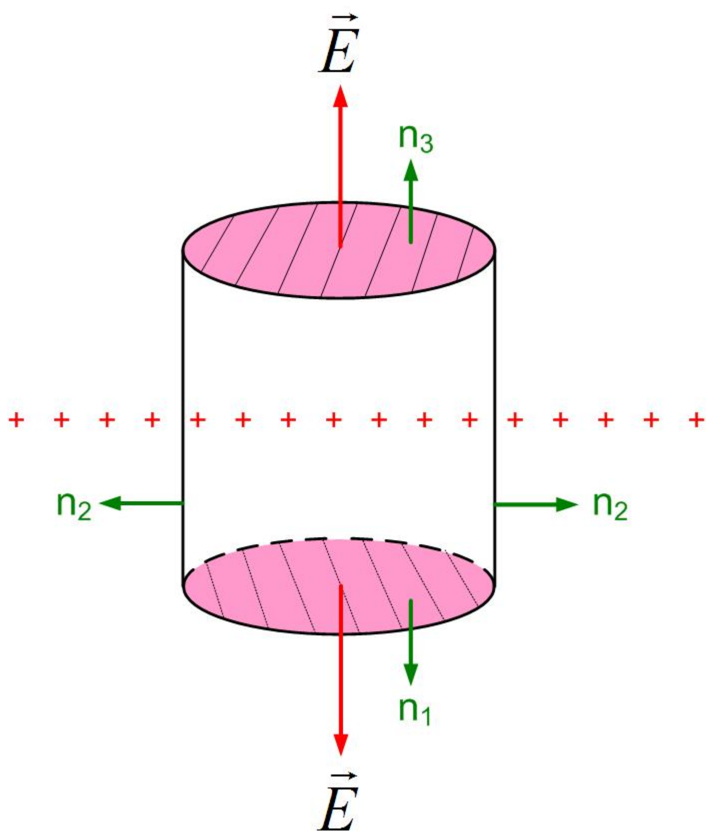

Figure 97: Infinite plane charged with positive surface charge density ρ_S.

In the above equation, the ration $\dfrac{Q_{inS}}{S}$ is just the surface charge density σ. The final electric field expression for the infinite sheet of charge becomes

$$\overrightarrow{\mathbf{E}} = \pm\frac{\sigma}{2\varepsilon}\,\hat{z} \tag{378}$$

Note that in the above expression, we added $\pm$ because we knew ahead of time the direction of the electric field. Below the z-axis, it is in the negative z-direction $-\hat{z}$, and above the z-axis, it is in the positive z-direction $\hat{z}$. Further, the electric field does not depend on the distance from the infinite plane. It is a constant that only depends on the surface charge density σ and the dielectric permittivity of surrounding space ε.

Two Infinite Planes

A parallel-plate capacitor can be modeled with two infinite parallel plates of opposite charge densities. To find the total field, we can use the principle of superposition, as shown in Figure 98. The Figure shows first just the negative sheet of charge (the first region to the left), then only the positive sheet of charge (the region between two red vertical lines), and finally both sheets of charge in the region to the right. We first find the field separately for the negative sheet of charge, then the positive sheet of charge, finally, we sum them up to get the total field.

In the previous section, we found that the electric field from an infinite sheet of charge is constant. The electric field anywhere around the negative sheet of charge is

$$\overrightarrow{\mathbf{E}_-} = \mp\frac{\sigma}{2\varepsilon}\,\hat{z} \tag{379}$$

The electric field, due to the positive sheet of charge, is

$$\overrightarrow{\mathbf{E}_+} = \pm\frac{\sigma}{2\varepsilon}\,\hat{z} \tag{380}$$

If you look at the direction of the field, the fields from individual plates between the plates of the capacitor add up, and the fields above and below the capacitor subtract. The final field is only between the plates of the capacitor, and it is equal to double the value of the one sheet of charge.

$$\overrightarrow{\mathbf{E}} = \frac{\sigma}{\varepsilon}\,\hat{z} \tag{381}$$

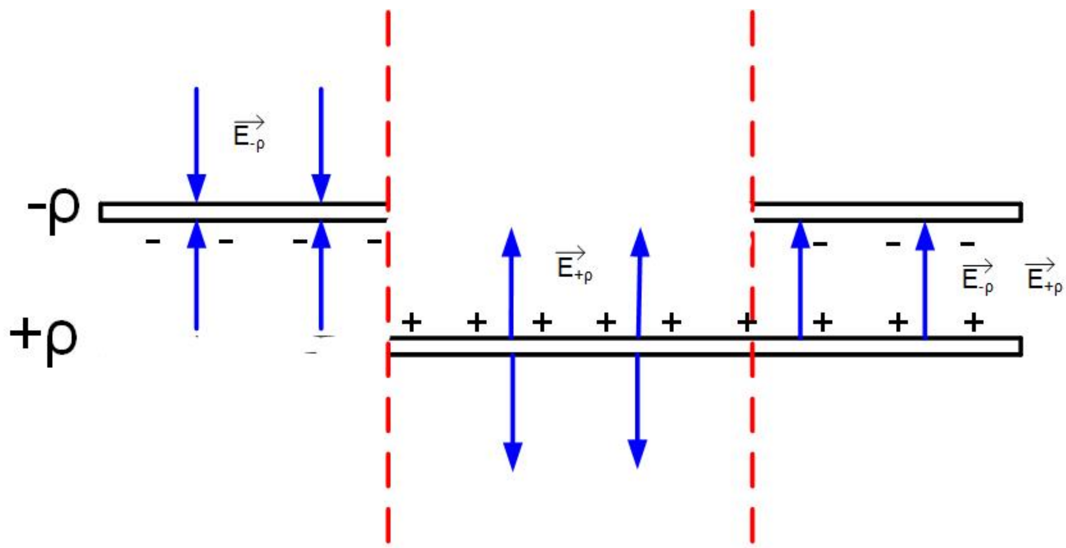

Figure 98: Two infinite planes charged with positive surface charge density ρ_S and $-\rho_S$.

8.6 Electrostatic Boundary Conditions

Conductors in the electrostatic field

Conductors conduct current well because the atoms of good conductors have many loosely bound electrons that can leave the atoms in the presence of an external electric field. In the absence of an external electric field, a piece of metal is shown in Figure 99 on the left. When there is no electric field, the electrons are close to the nucleus. On the right, the same metal piece is placed in an electric field of the battery. Under the influence of the external field, E_v, electrons can freely move away from the atoms in the direction opposite to the direction of the external field. This type of current is called conduction current. The point form of Ohm's law states that $E = J/\sigma$, where J is the current density, E is the electric field, and σ is the conductivity of the material. Conductors have very high conductivity, and so the electric field inside the conductors is zero.

In electrostatics, we assume that the charges are not moving, so there is no conduction current. The electric field inside the conductors is zero. As we will see later in this section, the charges on the conductor in electrostatic fields can exist only on its surface, and the vector of the electric field must be perpendicular to the surface of the metal. The tangential electric field is zero. All points on a conductor in electrostatic fields have the same potential, and so the conductor is an equipotential surface.

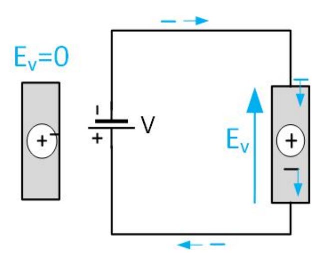

Figure 99: Conductor in Electric Field.

Dielectrics in the electrostatic field

As shown in Figure 100, in dielectrics, in the absence of an electric field, the electrons are close to the nucleus. The difference here is that the electrons are tightly bound to the nucleus, and they cannot escape in the presence of an electric field. When a battery establishes electric field E_v inside the dielectric, the atoms of the dielectric stretch because the nucleus is pulled in the direction of the field, and electrons in the opposite direction and the atom can be represented by a dipole. On the other hand, the free electrons in the wire connected to the dielectric start bunching up on top of the dielectric piece, and the dipole's positive charge is attracted to electrons. The negative dipole's bound charge pushes electrons away from the bottom conductor. Looking from the outside, the current flows, but the electrons are not flowing through the dielectric. This type of current is called a displacement current. If the battery is removed, the free negative and positive charges are trapped on the top and bottom of the dielectric piece.

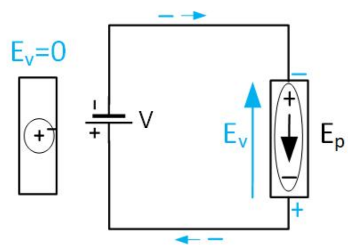

Figure 100: Dielectric in Electric Field.

The electrons in the metal on top of the dielectric establish an electric field across it, as shown in Figure 101. This field, in turn, produces electric dipoles in the dielectric, as explained above. The internal positive and negative charges cancel each other, and the positive bound charge from the dielectric on top and negative on the bottom produce their own field, which is in the opposite direction from the external field, as shown in Figure 102.

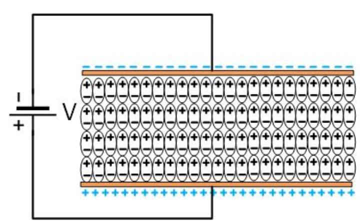

Figure 101: Polarization of a dielectric in external electric Field. Each oval represents one atom.

The total field in the dielectric is the sum of the electric fields from free charges on top and bottom metal pieces E_v, and the electric field from the separated polarization charges of the dielectric E_p, as shown in Equation 382. The induced field E_v is a fraction of the external field, and we can represent it in terms of the

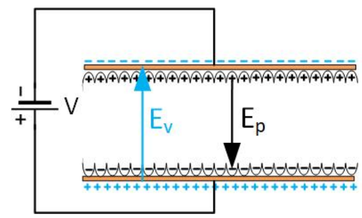

Figure 102: Two fields acting inside the dielectric. The external field E_v from the free charges in the metal on top and bottom, and the polarized dielectric field E_p. The inner part of the dielectric is removed to show clearly the fields.

external field as $E_p = mE_v$, where m is some constant. We can then express the total field as a fraction of the external field in Equation 384.

$$E_{total} = E_v - E_p \tag{382}$$

$$E_{total} = E_v - mE_v \tag{383}$$

$$E_{total} = E_v(1 - m) \tag{384}$$

Relative dielectric permittivity of the material ε is defined as $1 - m = \frac{1}{\varepsilon_r}$. Therefore the total field inside the dielectric is lower than if no dielectric is present.

$$E_{total} = \frac{E_v}{\varepsilon_r} \tag{385}$$

Dielectric permittivity of a material is defined as the relative permittivity multiplied by the permittivity of free space $\varepsilon_0 = 8.85\ 10^{-12}$ F/m.

Relative dielectric constant

Relative dielectric constant is in general a complex number $\varepsilon_r = \varepsilon_r{}' + j\varepsilon_r{}''$.

$\varepsilon_r{}'$ in data sheets is called a dielectric constant, or design dielectric constant, and it varies from 1 in the air to 13 in GaAs. An outlier is a dielectric constant of distilled water, $\varepsilon_r{}' = 80$.

We can sketch the complex relative dielectric constant on a complex plane. The angle between the magnitude of the dielectric constant and the x-axis is called $\tan\delta$, and it is used to describe the losses in the dielectric material. In datasheets for PC boards, you can see that $\tan\delta$ is from about 0.001 for microwave substrates, such as Rogers Duroid, to 0.02 for low-frequency FR4 substrates.

Boundary conditions at a dielectric-dielectric boundary

In many electrical structures, more than one dielectric is used so that the electric field exists in different dielectrics. In such cases, we are interested in how will the electric field change from one dielectric to the other. Figure 103 shows the boundary between the two dielectrics with permittivities ε_1 and ε_2, and the electric fields E_1 in material 1 and E_2 in material 2. At the boundary between two materials, we may have surface charge density ρ_s.

At the boundary between any two dielectrics, the tangential components of the electric field E_{1t}, E_{2t} are continuous, and the normal components E_{1n}, E_{2n} are discontinuous and equal to the surface charge density.

$$E_{1t} = E_{2t} \tag{386}$$
$$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s \tag{387}$$

If the free surface charge density at the boundary is zero, then the normal components of the electric field at the boundary are

$$E_{1t} = E_{2t} \tag{388}$$
$$\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n} \tag{389}$$

We can also write electric flux density vectors at the boundary. Since $D_1 = \varepsilon_1\, E_1$ and $D_2 = \varepsilon_2\, E_2$, the above equations can be re-written as

$$\varepsilon_2 D_{1t} = \varepsilon_1 D_{2t} \tag{390}$$
$$D_{1n} = D_{2n} \tag{391}$$

Question 40 *The four equations below show the tangential and normal electric field at the boundary of two dielectrics. Dielectric 1 is a Teflon with a relative dielectric constant of 2.2, and dielectric 2 is Silicon with a relative dielectric constant of 11.2. Which set of equations represents a possible electric field?*

Multiple Choice:

(a) $2.2\, E_{1t} = 11.2\, E_{2t}$ and $E_{1n} = E_{2n}$

(b) $E_{1t} = E_{2t}$ and $2.2\, E_{1n} = 11.2\, E_{2n}$

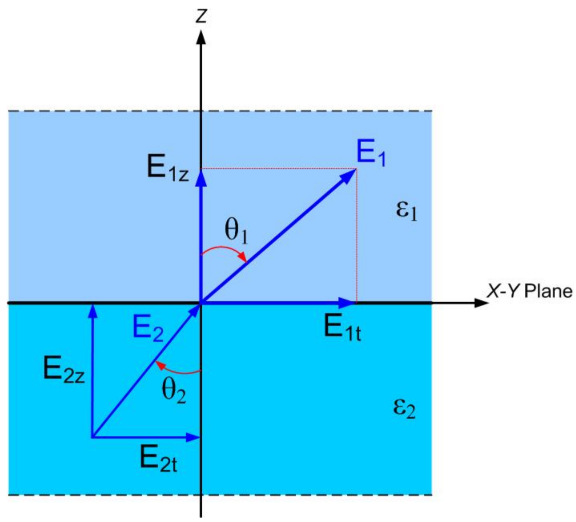

Figure 103: Boundary Conditions for Electric Field.

(c) $2.2\, E_{1t} = 11.2\, E_{2t}$ *and* $E_{1n} = E_{2n}$

(d) $E_{1t} = E_{2t}$ *and* $11.2\, E_{1n} = 2.2\, E_{2n}$

Boundary conditions at a conductor-dielectric boundary

The electric field inside perfect conductors $\sigma \to \infty$ is zero. Ohm's law states that

$$E = \frac{J}{\sigma} \tag{392}$$

When $\sigma \to \infty$, from the above equation, we see that the electric field is zero. This means that at the boundary of the dielectric and metal, the tangential field in the dielectric must be zero as well, and the only field at the boundary of a metal is the normal electric field D_n, and it is equal to the induced charge at the surface of the conductor.

$$D_n = \rho_s \tag{393}$$

Figure 104 shows the field at the boundary of the metallic sphere. Watch this demonstration of separation of charges on a metallic sphere in the electric field of the VanDenGraaff generator.

YouTube link: `https://www.youtube.com/watch?v=hOD2TOfYuM8`

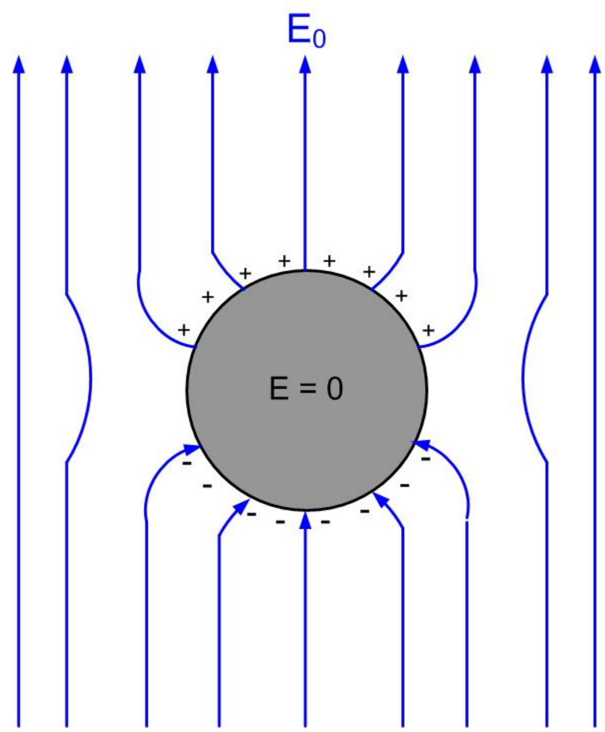

Figure 104: Metallic sphere in an external electric field.

Shielding with Faraday's Cage

The electric field is zero inside the closed metallic conductor, even if the conductor is hollow, as shown in Figure 105, and no charge is induced inside a metallic shield. This is Faraday's cage.

Watch a demonstration of zero electric fields, and no charge, inside a hollow conductor by Prof Emeritus of MIT Walter Lewin.

YouTube link: `https://www.youtube.com/watch?v=W_NelWgJ3LY`

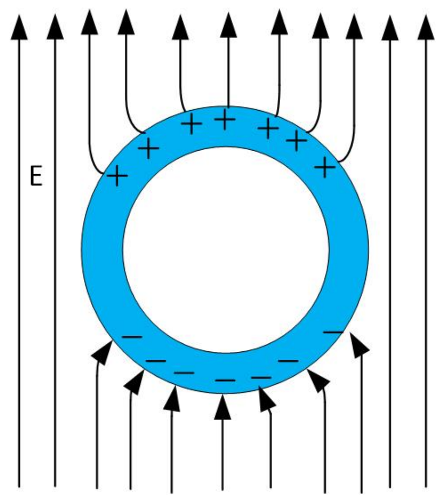

Figure 105: Electric field inside hollow metallic conductor (Faraday's Cage).

Grounding

In Figure 106, we introduce a charge inside a hollow conductor, and the electric field forms inside the conductor. The charge in the metallic shell will redistribute so that the field is zero inside the metal. The charge on the surface of the conductor will be uniformly distributed, regardless of the position of the charge inside the hollow part.

Figure 107 shows a grounded hollow conductor with a charge inside it. In this case, the positive charge on the outside of the conductor will attract negative charges from the ground that neutralize the positive charge inside the shell, so there will be no field outside the shell.

Watch a demonstration of Faraday's Cage by Prof Emeritus of MIT Walter Lewin. He will enter the Faraday's Cage with a tinsel, a transmitter (his wireless microphone that likely works at a frequency of a few GHz), and a receiver (a radio that works at a couple of hundred Megahertz frequency). The Faraday's cage is likely not grounded. He cannot receive the radio signal as the outside radio waves cannot enter the Faraday's cage, but the waves his microphone transmitter generates inside the cage still reach the receiver that is placed somewhere in the classroom since the cage is not grounded.

YouTube link: https://www.youtube.com/watch?v=t5uHzCfiXp4

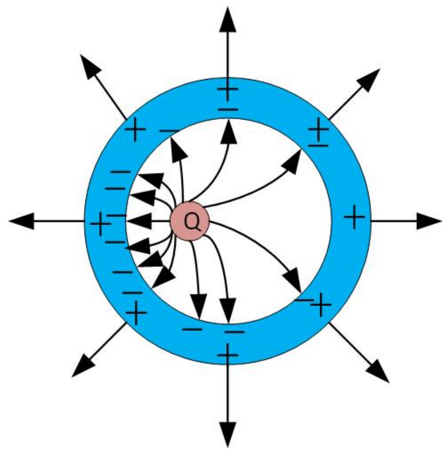

Figure 106: Hollow conductor with a charge inside.

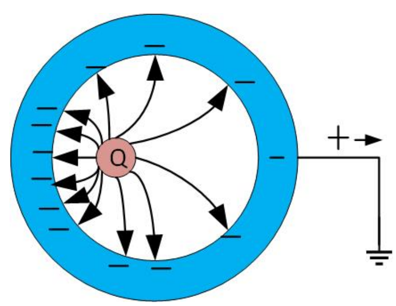

Figure 107: Grounded hollow conductor with a charge inside.

Proof of boundary conditions

We will now use Maxwell's equations to derive the electrostatic boundary conditions.

First, we will use Gauss's law to find the normal component of the fields at the boundary between two dielectrics, as shown in Figure 108. As we can see from the figure, the flux of the electric field exists through both bases and the side of the cylinder. We can find the components of the fields in both dielectrics, one parallel to the boundary x and one perpendicular to the boundary, in the direction of y.

$$\overrightarrow{\mathbf{E_1}} = \overrightarrow{\mathbf{E_{1x}}} + \overrightarrow{\mathbf{E_{1y}}} = E_{1x}\overrightarrow{\mathbf{x}} + E_{1y}\overrightarrow{\mathbf{y}} \tag{394}$$

$$\overrightarrow{\mathbf{E_2}} = \overrightarrow{\mathbf{E_{2x}}} + \overrightarrow{\mathbf{E_{2y}}} = E_{2x}\overrightarrow{\mathbf{x}} + E_{2y}\overrightarrow{\mathbf{y}} \tag{395}$$

The tangential components of the fields produce flux through the sides, and normal components produce flux through the bases. Since we are interested in what happens at the boundary, we will let the height of the cylinder be infinitesimally small $h \to 0$. Because the height of the cylinder is zero, and therefore the surface area is zero, the flux through the side surface S3 is zero. The flux through the top and bottom surfaces will only exist due to the normal components of the field.

$$\oint_S \overrightarrow{\mathbf{D}} \cdot \overrightarrow{\mathbf{dS}} = Q_{inS} \tag{396}$$

$$\int_{S1} \overrightarrow{\mathbf{D}} \cdot \overrightarrow{\mathbf{dS}} + \int_{S2} \overrightarrow{\mathbf{D}} \cdot \overrightarrow{\mathbf{dS}} + \int_{S3} \overrightarrow{\mathbf{D}} \cdot \overrightarrow{\mathbf{dS}} = Q_{inS} \tag{397}$$

$$\int_{S1} \overrightarrow{\mathbf{D}} \cdot \overrightarrow{\mathbf{dS}} + \int_{S2} \overrightarrow{\mathbf{D}} \cdot \overrightarrow{\mathbf{dS}} + 0 = Q_{inS} \tag{398}$$

$$\int_{S1} (\varepsilon_1 E_{1n}\overrightarrow{\mathbf{y}} + \varepsilon_1 E_{1t}\overrightarrow{\mathbf{x}}) \cdot dS\overrightarrow{\mathbf{y}} + (\varepsilon_2 E_{2n}\overrightarrow{\mathbf{y}} + \varepsilon_2 E_{2t}\overrightarrow{\mathbf{x}}) \cdot dS\overrightarrow{\mathbf{y}} = Q_{inS} \tag{399}$$

$$-\varepsilon_1 E_{1n} S + \varepsilon_2 E_{2n} S = Q_{inS} \tag{400}$$

$$\varepsilon_2 E_{2n} - \varepsilon_1 E_{1n} = \frac{Q_{inS}}{S} \tag{401}$$

The tangential components of the field can be obtained from the equation for Faraday's law for static fields, as shown in Figure 109. We choose a rectangular contour, as shown in the figure with length l and width w. Since again we are interested in the boundary, we will let the width of the contour go to zero. The integral along the w-pieces will then be zero. The integral along the l-pieces of contour will depend on the orientation of contour, and we will pick a counter-clockwise path. Because of the counter-clockwise path, the x-component of the

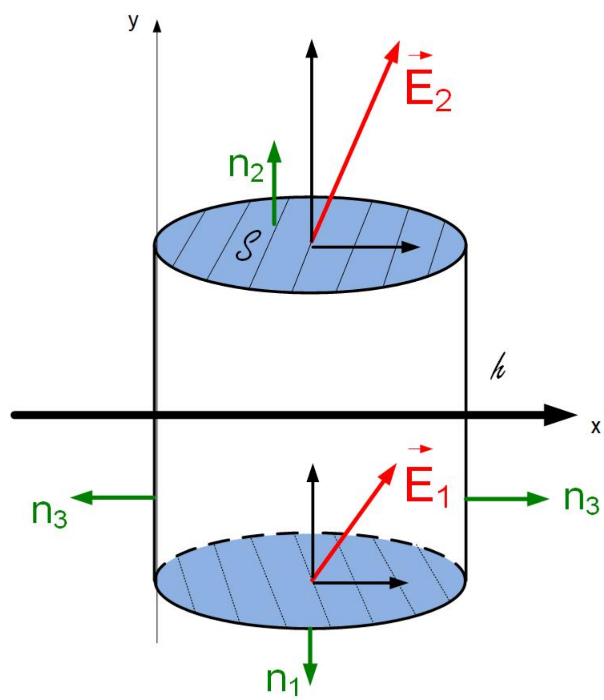

Figure 108: Derivation of equation for normal components of electric field on the boundary of two dielectrics.

E_1 field will be negative, and we have that the x-components of the fields in two dielectrics have to be the same.

$$\oint_C \vec{\mathbf{E}} \cdot \vec{\mathbf{dl}} = 0 \tag{402}$$

$$\int_{l1} (E_{1x}\vec{\mathbf{x}} + E_{1y}\vec{\mathbf{y}}) \cdot dy\vec{\mathbf{y}} + \int_{l2} (E_{2x}\vec{\mathbf{x}} + E_{2y}\vec{\mathbf{y}}) \cdot dy\vec{\mathbf{y}} = 0 \tag{403}$$

$$-E_{1x}l + E_{2x}l = 0 \tag{404}$$

$$E_{1x} = E_{2x} \tag{405}$$

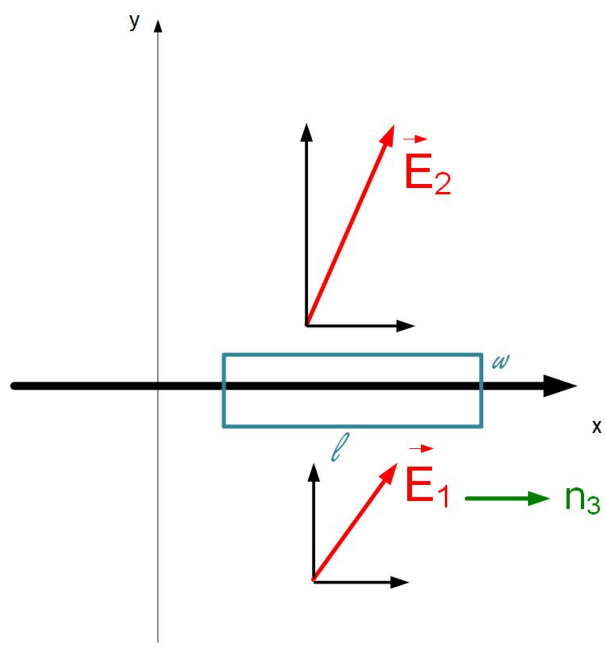

Figure 109: Derivation of equation for tangential components of electric field on the boundary of two dielectrics.

Charge distribution around sharp edges

Theshape of the conductive material impacts the chargde distribution and charge density, as shown in Figure 110. We can see that the charge distribution and electric field on round objects are uniform. The highest charge density and strongest electric fields are produced on sharp edges of condutive bodies.

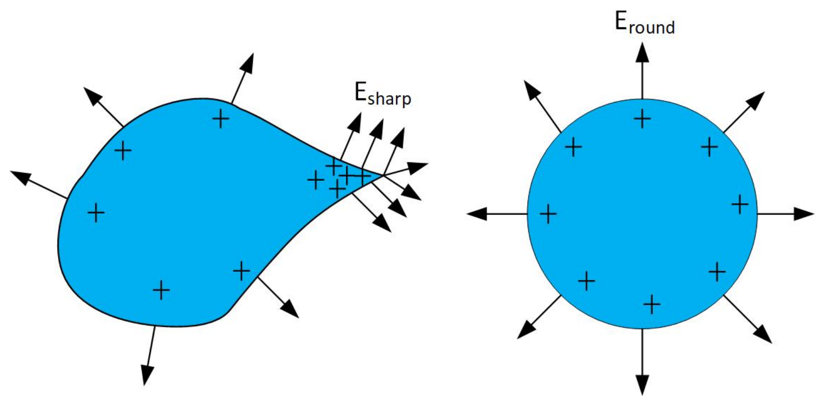

Figure 110: Electric field and charge distribution close to sharp edges.

Demonstration of higher charge density near sharp edges by Prof. Emeritus at MIT, Walter Lewin.

YouTube link: https://www.youtube.com/watch?v=cYOBSzh2SLE

8.7 Capacitance

What is Capacitance?

Capacitance is a constant that relates the amount of charge on a conductor with a potential difference between conductors. Whenever we have two conductors at different potentials, separated by an insulator (dielectric), there will be electric field between them, and we can define the ability for those two conductors to store charge. This is called a capacitance.

$$C = \frac{Q}{V} \tag{406}$$

From Gauss's law, we found that the electric field is proportional to the charge enclosed $Q \sim E$. The capacitance therefore represents the strenght of the electric field around a conductor for a fixed voltage between the conductors. If we for example have two sets of conductors, for the same voltage between them the electric field will be stronger around the conductor set with higher capacitance, and this conductor set will have higher ability to store charge. It will store more charge for the same voltage difference than the other conductor set.

Capacitance relates the current with the change in voltage. The current is defined as the change of charge in time $i = \frac{dq}{dt}$. Since charge is from the above equation equal to the product of capacitance and voltage, Q=CV, then the current is proportional to the change in voltage. If the voltage changes twice as fast, the current doubles. The voltage V is the potential difference between the positive and negative plate of the capacitor.

$$i = C\frac{dv}{dt} \tag{407}$$

Computing Capacitance

Capacitance can be computed:

(a) from its definition $C = \frac{Q}{V}$. You can first apply Gauss's law to find the electric field, then find the potential difference between the conductors. The potential difference depends on charge, and so, the charges will cancel.

(b) from the expression for the electrostatic energy $W_e = \dfrac{1}{2}\displaystyle\int_V \varepsilon_o \varepsilon_r E^2 \, dV$. First apply Gauss's law to find the electric field E, square it, and integrate throughout the volume enclosed by the two conductors.

Capacitors

Capacitors are components that provide capacitance in electronic circuits. They are used in filters, PCB power-distribution networks, matching circuits, delay lines etc. Typical capacitors are shown in Figure 111. Earth is a conductor, so any conductor either on it's own or in pair with another conductor can be modeled as a capacitor. For a single conductor, for example a metallic conductor in air, the other "conductor" is earth, or another nearby metallic object. Transmission lines have two conductors, a "signal" and "ground" conductor, so they have capacitance per unit lenght, and we can model them with a distributed capacitors. In this section, using Maxwell's equations, we will derive equations for capacitance for typical capacitors and transmission lines.

Figure 111: Various commercially available capacitors. Reference: Wikipedia

Analysis of a Parallel-Plate Capacitor with a homogeneous dielectric

A parallel plate capacitor consists of two conductive plates, as shown in Figure 112. The surface area of the plates is S and the charge on each plate is Q. The plates are separated by a distance d ($d << S$). Homogeneous dielectric with dielectric constant of ε_r fills the space between electrodes.

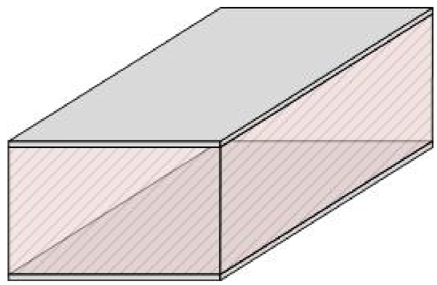

Figure 112: Two infinite planes charged with positive surface charge density ρ_S and $-\rho_S$.

The electric field in an actual parallel-plate capacitor is shown in Figure

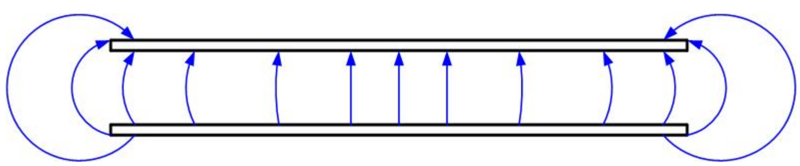

Figure 113: Fringing fields in a parallel-plate capacitor. The bottom plate is positively charged, and the top plate is negatively charged.

The condition that d¡¡S allows us to ignore fringing fields at the edge of the capacitor, and to assume that the field is equal to the field of two infinite parallel plates of charge. This means that the field is constant in between the plates, oriented from positive to negative plate and zero outside of the plates, as shown in Figure 114.

To find the electric field inside the capacitor, we apply Gauss's law to a cylinder whose bottom half is in the dielectric, with the bottom base at the point where we want to find the field, and the top half is in the air above one of the charged plates as shown in Figure 115. We see that the flux through the capacitor exists only through the bottom base. The flux through the top base is zero because the field outside the capacitor is zero, and the flux through the side of the cylinder is zero because the electric field does not not go through that surface

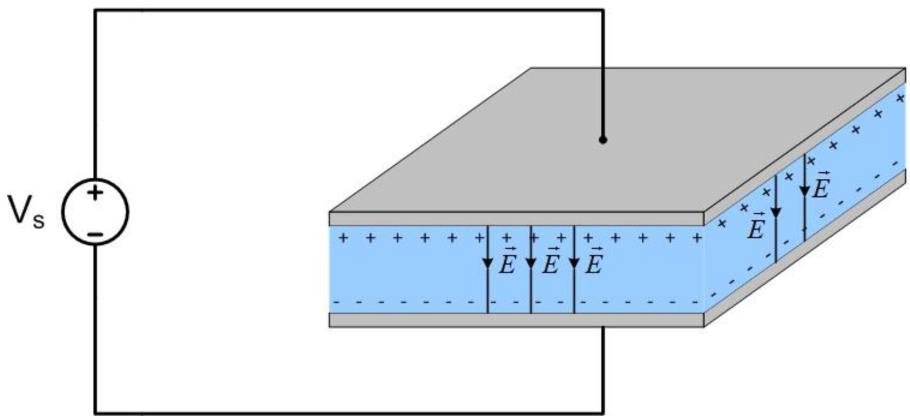

Figure 114: Two infinite planes charged with positive surface charge density ρ_S and $-\rho_S$.

(in other words the angle between the field and the normal to the surface is 90^o, and therefore the dot product is zero.) Mathematically, we start from the Gauss's law, assuming that the dielectric permeability is $\varepsilon = \varepsilon_o \varepsilon_r$, where ε_r is the dielectric constant of the material between capacitor's plates.

$$\oint_S \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} = \frac{Q_{inS}}{\varepsilon} \tag{408}$$

We split the cylindrical surface S into two base surfaces and a side surface.

$$\oint_S \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} = \int_{S1} \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} + \int_{S2} \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} + \int_{S3} \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} \tag{409}$$

The dot product between the electric field and the normal to the cylindrical top (S3) surface is zero, because electric field is zero outside of the capacitor. The dot product is zero on the side surface because there is no flux through it, the normal to the surface and the electric field are perpendicular $\overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} = E\,dS\cos(\angle(\overrightarrow{\mathbf{E}}, \overrightarrow{\mathbf{dS}})) = E\,dS\cos(\angle(90^o)) = 0$. The flux through the bottom surface (S1) is just the product of the magnitudes of E and dS $\overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} = E\,dS\cos(\angle(\overrightarrow{\mathbf{E}}, \overrightarrow{\mathbf{dS}})) = E\,dS\cos(\angle(0^o)) = EdS$, because the normal to the surface and the electric field are in the same direction. We can now take the electric field outside of the integral, and the integral around the closed surface then becomes

$$E \int_{S1} dS = \frac{Q_{inS}}{\varepsilon} \tag{410}$$

The integral of surface S1 is just the surface area of the bottom surface.

$$E\,S = \frac{Q_{inS}}{\varepsilon} \tag{411}$$
$$E = \frac{Q_{inS}}{\varepsilon S} \tag{412}$$

In the above equation, the ration $\frac{Q_{inS}}{S}$ is just the surface charge density σ. The final electric field expression for the infinite sheet of charge should include the unit vector of the direction of the field. We will assume that the z-axis is in the up direction from the bottom to the top plate. The field is then

$$\overrightarrow{\mathbf{E}} = \frac{\sigma}{\varepsilon}(-\overrightarrow{\mathbf{z}}) \tag{413}$$

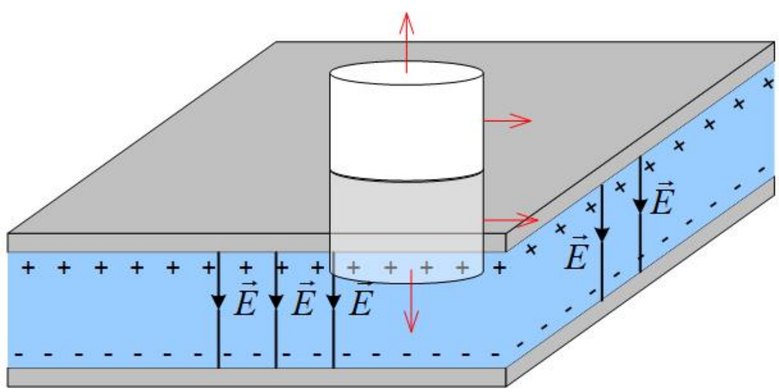

Figure 115: Two infinite planes charged with positive surface charge density ρ_S and $-\rho_S$.

Now that we found the electric field of the paralle plate capacitor, we need to find the potential difference between the two plates, from positive to negative plate. The positive plate in this case is the top plate, so we integrate from d to 0. The potential is defined as

$$V_{ab} = \int_d^0 \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}} \tag{414}$$

We will find the potential from the bottom plate at z=0 to the top plate at z=d. In the equation above $\overrightarrow{\mathbf{dl}}$ will always

$$V_{ab} = \int_d^0 \frac{\sigma}{\varepsilon}(\text{-}\overrightarrow{\mathbf{z}}) \cdot dz\overrightarrow{\mathbf{z}} \tag{415}$$

$$V_{ab} = \text{-}\frac{\sigma}{\varepsilon}\int_d^0 dz \tag{416}$$

$$V_{ab} = \frac{\sigma}{\varepsilon}d \tag{417}$$

This potential should be positive, because we integrated from the higher to the lower potential. The potential at a lower plate is lower than the potential at the higher plate. If you make a mistake and get a negative potential, it just means you should have started from the positive plate and integrated to the negative plate to get the positive potential difference. However, the $V_{ba} = -Vab$, so we can just take the magnitude of the negative potential difference. *If the potential difference is a* $\log x$ *function, make sure that the* $x > 1$. The total charge on one plate is $Q = \sigma S$, where S is the area of the plates. The capacitance is then

$$C = \frac{Q}{V} \tag{418}$$

$$C = \frac{\sigma S}{\frac{\sigma}{\varepsilon}d} \tag{419}$$

$$C = \varepsilon\frac{S}{d} \tag{420}$$

Question 41 *The potential difference between the plates of a capacitor is V. The distance between the plates is d. Ignore fringe effects. The magnitude of the electric field between the plates is:*

Multiple Choice:

(a) *not enough information*

(b) V^2/d

(c) d/V

(d) V/d

(e) d/V^2

Question 42 *The area of the plates of a parallel-plate capacitor is doubled, the capacitance is*

Multiple Choice:

(a) *not enough information*

(b) *stays the same*

(c) *quartered*

(d) *halved*

(e) *doubled*

If the area of the plates of a parallel-plate capacitor is doubled, the capacitance is

Coaxial-Cable Capacitance

Coaxial cable consists of a solid inner conductor and a conductive outer shell, Figure 118.

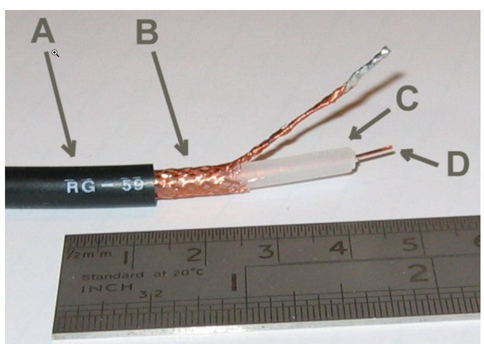

Figure 116: A coaxial cable. A - insulator, B - outer conductor, C - dielectric, D - inner conductor. Click to see the Wikipedia page on coaxial cables.

If we apply Gauss's law to a cylinder outside of the outer conductor, as shown in Figure 117, we see that the field is zero, because the total charge enclosed is zero.

To find the capacitance we need to find the electric field between the inner and outer conductor of a coax. We apply Gauss's law to the green closed cylindrical surface shown in Figure 118. Following the steps in the section on finding the field around the infinite line of charge, we find the field inside the capacitor.

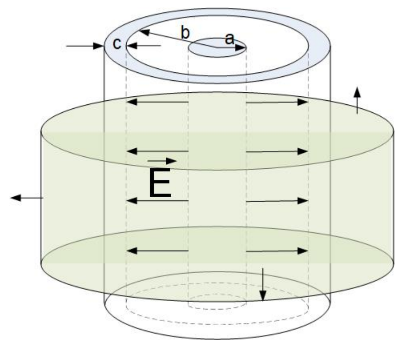

Figure 117: A coaxial cable, with inner conductor charged with positive scharge Q and outer conductor charged with $-Q$.

$$E\,S = \frac{Q_{inS}}{\varepsilon} \tag{421}$$

$$E = \frac{Q_{inS}}{2\pi h\varepsilon r} \tag{422}$$

To find the potential, we integrate from the inner to the outer conductor.

$$V = \int_a^b \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dr}} \tag{423}$$

$$V = \int_a^b \frac{Q_{inS}}{2\pi h\varepsilon r}\hat{r}\; dr\hat{r} \tag{424}$$

$$V = \frac{Q_{inS}}{2\pi h\varepsilon}\log\frac{b}{a} \tag{425}$$

The capacitance is then

$$C = \frac{Q}{V} \tag{426}$$

$$C = \frac{Q}{\frac{Q_{inS}\log\frac{b}{a}}{2\pi h\varepsilon}} \tag{427}$$

$$C = \frac{2\pi h\varepsilon}{\log\frac{b}{a}} \tag{428}$$

$$C' = \frac{C}{h} \tag{429}$$

$$C' = \frac{2\pi\varepsilon}{\log\frac{b}{a}} \tag{430}$$

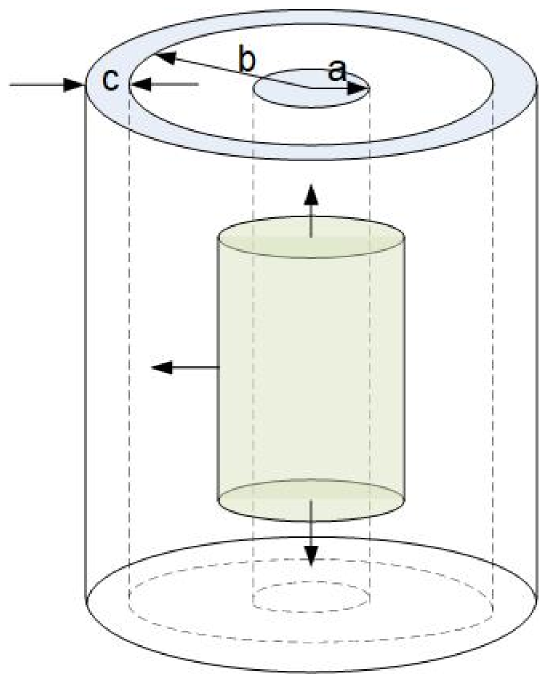

Figure 118: A coaxial cable, with inner conductor charged with positive scharge Q and outer conductor charged with $-Q$.

Use the calculator below to find the capacitance of RG6 Coaxial Cable, made by Mouser, part number 40001. See data sheet here. Assume PE dielectric constant is 2.25. Compare with the value in the data sheet.

Geogebra link: `https://tube.geogebra.org/m/whkrg2pu`

Spherical Capacitance

Our earth is a giant spherical capacitor, with ground as one electrode and the ionosphere as another. Spherical capacitor in Figure 119 consists of two concentric shells with radii a and b. The thickness of the outer shell is c-b. The inner shell is charged with charge Q and outer shell charged with charge -Q. If we apply Gauss's Law to a sphere larger than c, we see that the total charge enclosed is zero, and the field must be zero as well. Between the plates, the field is oriented radially from the positive to the negative charge. Application of Gauss's law to a sphere of a radius r between radii a and b leads to the following equation

$$E\,S = \frac{Q}{\varepsilon} \tag{431}$$

The surface area of the sphere is $4\pi r^2$.

$$E 4\pi r^2 = \frac{Q}{\varepsilon} \tag{432}$$

$$\overrightarrow{\mathbf{E}} = \frac{Q}{4\pi\varepsilon r^2}\hat{r} \tag{433}$$

The potential difference between the inner and outer shell is

$$V = \int_a^b \overrightarrow{\mathbf{E}} \cdot \hat{dr} \tag{434}$$

$$V = \int_a^b \frac{Q}{4\pi\varepsilon r^2}\hat{r}\ dr\hat{r} \tag{435}$$

$$V = -\frac{Q}{4\pi\varepsilon}\frac{1}{r}|_a^b \tag{436}$$

$$V = \frac{Q}{4\pi\varepsilon}\frac{b-a}{ab} \tag{437}$$

The capacitance is then

$$C = \frac{Q}{V} \tag{438}$$

$$C = 4\pi\varepsilon\frac{ab}{b-a} \tag{439}$$

Electrostatic energy of a charged capacitor

Electrostatic energy is defined as

$$W_e = \frac{1}{2}\int_V \varepsilon E^2 dv \tag{440}$$

Energy of a charged capacitor can be expressed by any of the following equations

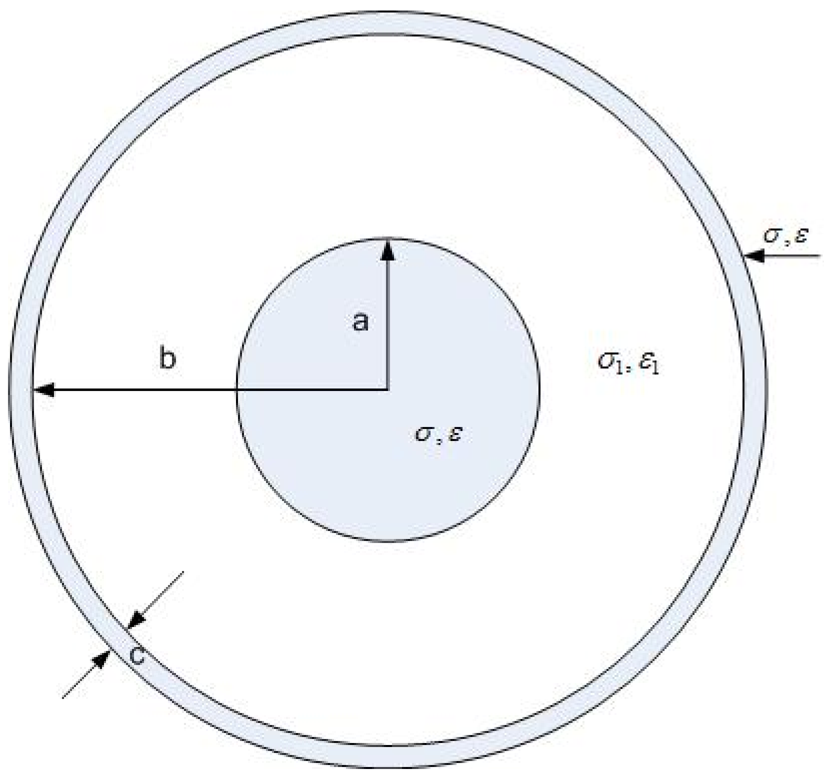

Figure 119: A spherical capacitor, with inner conductor charged with positive scharge Q and outer conductor charged with $-Q$.

$$W_e = \frac{1}{2}CV^2 \tag{441}$$

$$W_e = \frac{1}{2}QV \tag{442}$$

$$W_e = \frac{1}{2}\frac{Q^2}{C} \tag{443}$$

From the definition of electrostatic energy, and the expression for energy of a charged capacitor, we can find the capacitance:

$$C = \frac{Q^2}{\int_V \varepsilon E^2 dv} \tag{444}$$

You can try and solve the capacitance for the above problems again, by using the Equation 444.

Now, watch the videos below to experience the energy stored in a charged capacitor, and see how a fuse and camera flash work by discharging a capacitor.

Do not attempt these demonstrations at home, touching the capacitor's electrodes can be fatal, as humans are good conductors of electricity. Demonstrations were performed at MIT by Prof. Emeritus Walter Lewin.

YouTube link: `https://www.youtube.com/watch?v=ED-Aelxm13A`

YouTube link: `https://www.youtube.com/watch?v=_cx-PUh2Q7A`

8.8 Method of images

Method of images states that we can replace a charge above an infinite conducting plane with an equivalent configuration of the charge and its image. For example, if we have a positive charge Q, the field above the conducting plane will be the same if we replace the plane with an equal and opposite charge -Q as shown in Figure 120.

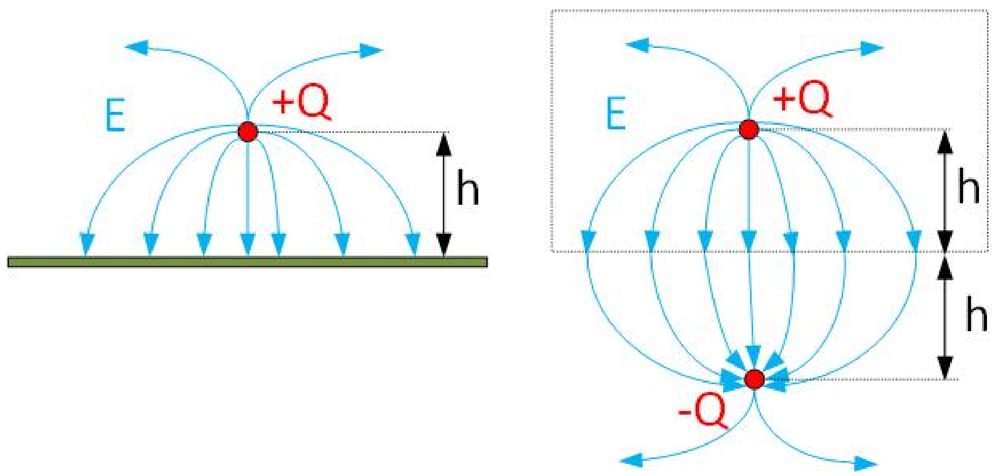

Figure 120: Electric Field due to a unit charge in Rectangular coordinate system.