Electrical Engineering Reference — Problem Sets

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Contents

Chapter 1 — Section 1.1: Power Generation

Practice problems covering fossil fuel plants, hydroelectric plants, nuclear plants, solar PV, wind turbines, microgrids, GSU transformers, and auxiliary transformers.

Problem 1.1.1

Given: A combined-cycle gas turbine plant has a gas turbine (Brayton cycle) efficiency of 35% and a heat recovery steam generator (HRSG) that captures 50% of the waste heat for a steam turbine (Rankine cycle). The plant burns natural gas at a thermal input of 800 MW.

Find: (a) The gas turbine electrical output, (b) the steam turbine electrical output, (c) the total plant output, and (d) the overall plant efficiency.

Solution:

(a) Gas turbine output: $P_{GT} = \eta_{GT} \times Q_{in} = 0.35 \times 800 = 280$ MW

(b) Waste heat from gas turbine: $Q_{waste} = Q_{in} - P_{GT} = 800 - 280 = 520$ MW

Steam turbine output: $P_{ST} = \eta_{HRSG} \times Q_{waste} = 0.50 \times 520 = 260$ MW

(c) Total plant output: $P_{total} = P_{GT} + P_{ST} = 280 + 260 = 540$ MW

(d) Overall efficiency: $\eta_{overall} = P_{total} / Q_{in} = 540 / 800 = 0.675 = 67.5\%$

Problem 1.1.2

Given: A hydroelectric plant has a hydraulic head of 120 m and a water flow rate of 40 m^3/s. The turbine efficiency is 92% and the generator efficiency is 97%. Use $\rho = 1000$ kg/m^3 and $g = 9.81$ m/s^2.

Find: (a) The available hydraulic power, (b) the turbine mechanical output, and (c) the electrical power output.

Solution:

(a) Hydraulic power: $P_{hydraulic} = \rho \times g \times Q \times h = 1000 \times 9.81 \times 40 \times 120 = 47{,}088{,}000$ W = 47.09 MW

(b) Turbine mechanical output: $P_{mech} = \eta_{turbine} \times P_{hydraulic} = 0.92 \times 47.09 = 43.32$ MW

(c) Electrical power output: $P_{elec} = \eta_{gen} \times P_{mech} = 0.97 \times 43.32 = 42.02$ MW

Overall efficiency: $\eta = P_{elec} / P_{hydraulic} = 42.02 / 47.09 = 89.2\%$

Problem 1.1.3

Given: A nuclear power plant has a rated thermal output of 4,000 MW(th) and operates a Rankine cycle with a thermal efficiency of 34%. The plant operates at a capacity factor of 88% over one year (8,760 hours).

Find: (a) The electrical output at full power, (b) the annual electrical energy generated, and (c) the amount of waste heat that must be rejected (in MW).

Solution:

(a) Electrical output: $P_{elec} = \eta \times P_{thermal} = 0.34 \times 4{,}000 = 1{,}360$ MW

(b) Annual energy at capacity factor: $E = P_{elec} \times CF \times 8{,}760 = 1{,}360 \times 0.88 \times 8{,}760 = 10{,}479{,}168$ MWh $= 1.048 \times 10^7$ MWh

(c) Waste heat: $Q_{reject} = P_{thermal} - P_{elec} = 4{,}000 - 1{,}360 = 2{,}640$ MW

This waste heat is rejected through cooling towers or to a body of water.

Problem 1.1.4

Given: A utility-scale solar PV farm is rated at 75 MW_{DC}. The site receives an average of 4.8 peak sun hours (PSH) per day. The inverter efficiency is 97%, DC wiring losses are 2%, and soiling losses are 3%.

Find: (a) The effective daily DC energy after wiring and soiling losses, (b) the daily AC energy output, (c) the annual AC energy output, and (d) the capacity factor.

Solution:

(a) Effective DC energy: Loss factor $= (1 - 0.02)(1 - 0.03) = 0.98 \times 0.97 = 0.9506$ $E_{DC,eff} = 75 \times 4.8 \times 0.9506 = 342.2$ MWh/day

(b) Daily AC energy: $E_{AC} = \eta_{inv} \times E_{DC,eff} = 0.97 \times 342.2 = 331.9$ MWh/day

(c) Annual AC energy: $E_{annual} = 331.9 \times 365 = 121{,}143$ MWh/year

(d) Capacity factor: $CF = E_{annual} / (P_{rated} \times 8{,}760) = 121{,}143 / (75 \times 8{,}760) = 121{,}143 / 657{,}000 = 0.184 = 18.4\%$

Problem 1.1.5

Given: A wind turbine has a rotor diameter of 150 m and operates at a hub height where air density is $\rho = 1.18$ kg/m³. The wind speed is 10 m/s. The turbine power coefficient is $C_p = 0.45$ and the gearbox/generator efficiency is 94%.

Find: (a) The rotor swept area, (b) the total kinetic power in the wind, (c) the rotor mechanical power, (d) the electrical output, and (e) the percentage of the Betz limit achieved.

Solution:

(a) Rotor swept area: $A = \pi(D/2)^2 = \pi(75)^2 = 17{,}671.5\ \text{m}^2$

(b) Wind power: $P_{wind} = 0.5 \times \rho \times A \times v^3 = 0.5 \times 1.18 \times 17{,}671.5 \times (10)^3$ $P_{wind} = 0.5 \times 1.18 \times 17{,}671.5 \times 1{,}000 = 10{,}426{,}185\ \text{W} = 10.43\ \text{MW}$

(c) Rotor mechanical power: $P_{rotor} = C_p \times P_{wind} = 0.45 \times 10.43 = 4.69$ MW

(d) Electrical output: $P_{elec} = \eta_{drive} \times P_{rotor} = 0.94 \times 4.69 = 4.41$ MW

(e) Betz limit is $C_{p,max} = 16/27 = 0.5926$. Fraction achieved: 0.45 / 0.5926 = 0.759 = 75.9% of the Betz limit

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Problem 1.1.6

Given: A microgrid serves a data center with a peak load of 1,200 kW. Resources include a 600 kW solar array (currently producing 450 kW), a 1,000 kWh battery at 75% state of charge (minimum SOC = 15%), a 500 kW natural gas generator, and inverter efficiency of 95%.

Find: (a) The power deficit that must be supplied by the battery or generator, (b) whether the gas generator alone can meet the deficit, and (c) how long the battery alone could sustain the deficit.

Solution:

(a) Power deficit: Deficit = Peak load - Solar output = 1,200 - 450 = 750 kW

(b) The gas generator is rated at 500 kW, which is less than 750 kW. The generator alone cannot meet the deficit. It would need 750 - 500 = 250 kW from the battery simultaneously.

(c) Usable battery energy: $E_{usable} = (SOC_{current} - SOC_{min}) \times \text{Capacity} = (0.75 - 0.15) \times 1{,}000 = 600$ kWh AC energy delivered: $E_{AC} = \eta_{inv} \times E_{usable} = 0.95 \times 600 = 570$ kWh Duration at 750 kW: t = 570 / 750 = 0.76 hours (approximately 45.6 minutes)

Problem 1.1.7

Given: A 350 MW natural gas plant has a generator rated at 412 MVA, 18 kV, 0.85 power factor. The GSU transformer is rated 412 MVA, 18 kV delta / 230 kV wye-grounded, with an impedance of 9.0%.

Find: (a) The rated current on the 18 kV and 230 kV sides, (b) the maximum symmetrical fault current on the 230 kV side (assuming infinite bus on the generator side), and (c) the fault MVA.

Solution:

(a) Rated currents: $I_{LV} = S / (\sqrt{3} \times V_{LV}) = 412 \times 10^6 / (\sqrt{3} \times 18{,}000) = 412 \times 10^6 / 31{,}177 = 13{,}216$ A $I_{HV} = S / (\sqrt{3} \times V_{HV}) = 412 \times 10^6 / (\sqrt{3} \times 230{,}000) = 412 \times 10^6 / 398{,}372 = 1{,}034$ A

(b) Fault current at 230 kV: $I_{fault} = I_{HV,rated} / Z_{pu} = 1{,}034 / 0.09 = 11{,}489$ A = 11.5 kA

(c) Fault MVA: $S_{fault} = S_{rated} / Z_{pu} = 412 / 0.09 = 4{,}578$ MVA = 4.58 GVA

Problem 1.1.8

Given: A 700 MW coal-fired generating unit has a generator rated at 824 MVA, 24 kV. Station service loads total 52 MW at 0.88 power factor lagging. The unit auxiliary transformer (UAT) steps down from 24 kV to 6.9 kV.

Find: (a) Station service as a percentage of gross generation, (b) the net plant output, (c) the required UAT MVA rating with 20% margin, and (d) the UAT rated current on the 6.9 kV side.

Solution:

(a) Station service percentage: $SS\% = P_{aux} / P_{gross} \times 100 = 52 / 700 \times 100 = 7.43\%$

(b) Net plant output: $P_{net} = 700 - 52 = 648$ MW

(c) Required UAT apparent power: $S_{UAT} = P_{aux} / PF = 52 / 0.88 = 59.1$ MVA With 20% margin: $S_{rated} = 59.1 \times 1.20 = 70.9$ MVA (select a standard 75 MVA rating)

(d) UAT current at 6.9 kV: $I_{6.9kV} = S_{rated} / (\sqrt{3} \times V) = 75 \times 10^6 / (\sqrt{3} \times 6{,}900) = 75 \times 10^6 / 11{,}950 = 6{,}276$ A

Problem 1.1.9

Given: A Rankine cycle coal plant has the following operating parameters: boiler thermal input of 1,200 MW, boiler efficiency of 88%, turbine isentropic efficiency of 90%, generator efficiency of 98%, and the condenser rejects heat at 35 degrees C.

Find: (a) The steam energy entering the turbine, (b) the turbine shaft power, (c) the electrical output, and (d) the overall plant heat rate in BTU/kWh (1 MW = 3,412,141 BTU/hr).

Solution:

(a) Steam energy into turbine: $P_{steam} = \eta_{boiler} \times Q_{fuel} = 0.88 \times 1{,}200 = 1{,}056$ MW

(b) Turbine shaft power: $P_{shaft} = \eta_{turbine} \times P_{steam} = 0.90 \times 1{,}056 = 950.4$ MW

(c) Electrical output: $P_{elec} = \eta_{gen} \times P_{shaft} = 0.98 \times 950.4 = 931.4$ MW

(d) Overall efficiency: $\eta = P_{elec} / Q_{fuel} = 931.4 / 1{,}200 = 0.7762$ Heat rate $= 3{,}412{,}141 / \eta_{overall} = 3{,}412{,}141 / (931.4/1{,}200 \times 1{,}000) = (3{,}412{,}141 \times 1{,}200) / (931.4 \times 1{,}000)$ Simpler: $HR = 3{,}412{,}141 / (\eta \times 1{,}000)$ BTU/kWh $\eta = 0.7762$, so $HR = 3{,}412 / 0.7762 = 4{,}396$ BTU/kWh

Note: This is the net heat rate from fuel input to electrical output. Typical coal plants have heat rates of 9,000-10,500 BTU/kWh because the Carnot efficiency is not accounted for here. The turbine isentropic efficiency is applied to the actual steam energy, not ideal Carnot output. A more realistic calculation would use the Carnot-limited steam cycle efficiency (typically 40-45%) rather than the isentropic turbine efficiency alone.

Problem 1.1.10

Given: Compare three generation sources for a 200 MW load operating 6,000 hours per year: (a) Natural gas combined-cycle at 58% efficiency with gas cost of $4.50/MMBTU, (b) Solar PV with a capacity factor of 22% and installation cost of $1,200/kW amortized at $120/kW-yr, (c) Onshore wind with a capacity factor of 35% and installation cost of $1,500/kW amortized at $150/kW-yr. Assume 1 MWh = 3.412 MMBTU of thermal input.

Find: The annual fuel or levelized energy cost for each source to supply 200 MW × 6,000 hr = 1,200,000 MWh of energy.

Solution:

(a) Natural gas combined-cycle: Thermal input = E / η = 1,200,000 / 0.58 = 2,068,966 $\text{MWh}_{\text{thermal}}$ In MMBTU: 2,068,966 × 3.412 = 7,059,229 MMBTU Annual fuel cost = 7,059,229 × $4.50 = $31,766,531/year LCOE (fuel only) = $31,766,531 / 1,200,000 = $26.47/MWh

(b) Solar PV: Required installed capacity = 200 MW / 0.22 × (6,000/8,760) = effective capacity needed. Actually, annual energy from installed capacity P: E = P × CF × 8,760. For 1,200,000 MWh: P = 1,200,000 / (0.22 × 8,760) = 1,200,000 / 1,927.2 = 622.6 MW Annual cost = 622.6 × 1,000 × $120 = $74,712,000/year LCOE = $74,712,000 / 1,200,000 = $62.26/MWh

(c) Onshore wind: P = 1,200,000 / (0.35 × 8,760) = 1,200,000 / 3,066 = 391.4 MW Annual cost = 391.4 × 1,000 × $150 = $58,710,000/year LCOE = $58,710,000 / 1,200,000 = $48.93/MWh

Chapter 1 — Section 1.2: Power Transmission

Practice problems covering short, medium, and long transmission lines, underground transmission cables, and transmission line parameters.

Problem 1.2.1

Given: A 50 km, 115 kV, single-phase short transmission line has a series impedance of z = 0.20 + j0.50 Ω/km. The line delivers 15 MVA at 115 kV with a 0.90 lagging power factor at the receiving end.

Find: (a) The total line impedance, (b) the receiving-end current, (c) the sending-end voltage, and (d) the voltage regulation.

Solution:

(a) Total line impedance: $Z_{total} = 50 \times (0.20 + j0.50) = 10.0 + j25.0\ \Omega$ $|Z_{total}| = \sqrt{(10^2 + 25^2)} = \sqrt{(725)} = 26.93\ \Omega$ $\angle Z = \tan^{-1}(25/10) = 68.20°$ $Z_{total} = 26.93\angle 68.20°\ \Omega$

(b) Receiving-end current: $I_R = S / V_R = 15 \times 10^6 / 115{,}000 = 130.43$ A Power factor angle: $\theta = \cos^{-1}(0.90) = 25.84°$ $I_R = 130.43\angle\text{-}25.84°$ A

(c) Sending-end voltage: $V_S = V_R + I_R \times Z_{total}$ $I_R \times Z = 130.43\angle\text{-}25.84° \times 26.93\angle 68.20° = 3{,}512.5\angle 42.36° = 3{,}512.5(\cos 42.36° + j \sin 42.36°) = 2{,}596.0 + j2{,}365.8$ V $V_S = 115{,}000 + 2{,}596.0 + j2{,}365.8 = 117{,}596.0 + j2{,}365.8$ $|V_S| = \sqrt{(117{,}596^2 + 2{,}365.8^2)} = 117{,}620$ V = 117.62 kV

(d) Voltage regulation: $VR\% = (|V_S| - |V_R|) / |V_R| \times 100 = (117{,}620 - 115{,}000) / 115{,}000 \times 100 = 2.28\%$

Problem 1.2.2

Given: A 180 km, 345 kV, three-phase medium transmission line has per-phase parameters: z = 0.04 + j0.40 Ω/km and $y = j3.0 \times 10^{-6}$ S/km. Use the nominal π model.

Find: (a) The total series impedance Z, (b) the total shunt admittance Y, (c) the ABCD parameters, and (d) the sending-end voltage if the line delivers 400 MVA at 345 kV, 0.95 lagging power factor.

Solution:

(a) Total series impedance: $Z = z \times l = (0.04 + j0.40) \times 180 = 7.2 + j72.0\ \Omega$ $|Z| = \sqrt{(7.2^2 + 72^2)} = \sqrt{(51.84 + 5{,}184)} = \sqrt{5{,}235.84} = 72.36\ \Omega$ $\angle Z = \tan^{-1}(72/7.2) = 84.29°$

(b) Total shunt admittance: $Y = y \times l = j3.0 \times 10^{-6} \times 180 = j5.4 \times 10^{-4}$ S

(c) ABCD parameters: $YZ/2 = (j5.4 \times 10^{-4})(7.2 + j72)/2 = (j0.003888 + j^2 0.03888)/2 = (-0.03888 + j0.003888)/2$ $YZ/2 = -0.01944 + j0.001944$

$A = D = 1 + YZ/2 = 1 - 0.01944 + j0.001944 = 0.9806 + j0.0019 = 0.9806\angle 0.11°$

$B = Z = 7.2 + j72.0 = 72.36\angle 84.29°\ \Omega$

$YZ/4 = -0.00972 + j0.000972$ $C = Y(1 + YZ/4) = j5.4 \times 10^{-4} \times (0.99028 + j0.000972)$ $C = j5.347 \times 10^{-4} - 5.249 \times 10^{-7} = -5.25 \times 10^{-7} + j5.347 \times 10^{-4} \approx 5.347 \times 10^{-4}\angle 90.06°$ S

(d) Receiving-end quantities (per phase): $V_R = 345{,}000 / \sqrt{3} = 199{,}186$ V (phase voltage) $I_R = S / (\sqrt{3} \times V_{LL}) = 400 \times 10^6 / (\sqrt{3} \times 345{,}000) = 669.4$ A $\theta = \cos^{-1}(0.95) = 18.19°$, so $I_R = 669.4\angle{-18.19°}$ A

Sending-end voltage (per phase): $V_S = A \times V_R + B \times I_R$ $A \times V_R = 0.9806\angle 0.11° \times 199{,}186\angle 0° = 195{,}322\angle 0.11°$ $B \times I_R = 72.36\angle 84.29° \times 669.4\angle{-18.19°} = 48{,}438\angle 66.10° = 48{,}438(\cos 66.10° + j \sin 66.10°) = 19{,}612 + j44{,}295$

$V_S = 195{,}322 + 19{,}612 + j(339 + 44{,}295) = 214{,}934 + j44{,}634$ $|V_S| = \sqrt{(214{,}934^2 + 44{,}634^2)} = 219{,}521$ V per phase $V_{S,LL} = \sqrt{3} \times 219{,}521 = 380.2$ kV

Voltage regulation = (380.2 - 345) / 345 × 100 = 10.2%

Problem 1.2.3

Given: A 500 km, 500 kV, three-phase long transmission line has per-phase distributed parameters: $z = 0.02 + j0.30\ \Omega$/km and $y = j4.0 \times 10^{-6}$ S/km.

Find: (a) The characteristic impedance Z_c, (b) the propagation constant γ, (c) the surge impedance loading (SIL), and (d) the attenuation and phase shift over the full line length.

Solution:

(a) Characteristic impedance: $z/y = (0.02 + j0.30) / (j4.0 \times 10^{-6})$ Multiply by -j/-j: $= (0.02 + j0.30)(-j) / (4.0 \times 10^{-6}) = (-j0.02 + 0.30) / (4.0 \times 10^{-6}) = (0.30 - j0.02) / (4.0 \times 10^{-6}) = 75{,}000 - j5{,}000$ $|z/y| = \sqrt{(75{,}000^2 + 5{,}000^2)} = \sqrt{(5.625 \times 10^9 + 25 \times 10^6)} = \sqrt{(5.65 \times 10^9)} = 75{,}166.5$ $\angle(z/y) = \tan^{-1}(-5{,}000/75{,}000) = -3.81°$ $Z_c = \sqrt{75{,}166.5}\ \angle(-3.81°/2) = 274.2\angle{-1.91°}\ \Omega \approx 274.2\ \Omega$

(b) Propagation constant: $z \times y = (0.02 + j0.30)(j4.0 \times 10^{-6}) = j8.0 \times 10^{-8} - 1.2 \times 10^{-6} = -1.2 \times 10^{-6} + j8.0 \times 10^{-8} = 1.2027 \times 10^{-6}\angle 176.18°$ $\gamma = \sqrt{(1.2027 \times 10^{-6})}\angle(176.18°/2) = 1.097 \times 10^{-3}\angle 88.09°$ $\gamma = \alpha + j\beta = 3.66 \times 10^{-5} + j1.096 \times 10^{-3}$ per km

(c) Surge impedance loading: $SIL = V_{LL}^2 / Z_c = (500{,}000)^2 / 274.2 = 912$ MW

(d) Over 500 km: Attenuation: $\alpha l = 3.66 \times 10^{-5} \times 500 = 0.0183$ Np $= 0.0183 \times 8.686 = 0.159$ dB Phase shift: $\beta l = 1.096 \times 10^{-3} \times 500 = 0.548$ rad $= 31.4°$

Problem 1.2.4

Given: A 230 kV underground XLPE cable is 40 km long. Each phase has a capacitance of 0.25 μF/km and an ampacity of 700 A. System frequency is 60 Hz.

Find: (a) The charging current per phase, (b) the three-phase reactive power generated, (c) the remaining ampacity for load current, (d) the maximum load power transfer, and (e) the required shunt reactor compensation.

Solution:

(a) Total capacitance per phase: $C_{total} = 0.25 \times 40 = 10.0\ \mu F$ Line-to-neutral voltage: $V_{LN} = 230{,}000 / \sqrt{3} = 132{,}791$ V $I_{charging} = 2\pi f C V_{LN} = 2\pi \times 60 \times 10.0 \times 10^{-6} \times 132{,}791 = 500.7$ A per phase

(b) Three-phase reactive power: $Q = 3 \times V_{LN} \times I_{charging} = 3 \times 132{,}791 \times 500.7 = 199.4 \times 10^6$ VAR = 199.4 MVAR

(c) Remaining ampacity (charging and load currents are ~90° apart): $I_{load,max} = \sqrt{(I_{rated}{}^2 - I_{charging}{}^2)} = \sqrt{(700^2 - 500.7^2)} = \sqrt{(490{,}000 - 250{,}700)} = \sqrt{239{,}300} = 489.2$ A

(d) Maximum load power: $S_{max} = \sqrt{3} \times V_{LL} \times I_{load,max} = \sqrt{3} \times 230{,}000 \times 489.2 = 194.8 \times 10^6$ VA = 194.8 MVA

Without charging current: $S = \sqrt{3} \times 230{,}000 \times 700 = 278.8$ MVA. The charging current reduces capacity by (278.8 - 194.8)/278.8 = 30.1%.

(e) A shunt reactor of approximately 200 MVAR is needed to compensate the capacitive charging current, which would restore the full 700 A ampacity for load current.

Problem 1.2.5

Given: Two parallel 138 kV transmission lines each have a series impedance of $Z = 5 + j40\ \Omega$ and a thermal rating of 400 MVA. One line trips due to a fault. The total load is 600 MVA at 0.92 lagging power factor.

Find: (a) The current through the remaining line, (b) whether the remaining line exceeds its thermal rating, (c) the voltage drop across the remaining line, and (d) the voltage regulation.

Solution:

(a) Current through the remaining line: $I = S / (\sqrt{3} \times V) = 600 \times 10^6 / (\sqrt{3} \times 138{,}000) = 2{,}510$ A

(b) The thermal rating current: $I_{rated} = S_{rated} / (\sqrt{3} \times V) = 400 \times 10^6 / (\sqrt{3} \times 138{,}000) = 1{,}674$ A Since 2,510 A > 1,674 A, the line exceeds its thermal rating by 50%. This is an N-1 contingency violation.

(c) Voltage drop (magnitude): $|V_{drop}| = |I \times Z| = 2{,}510 \times |5 + j40| = 2{,}510 \times \sqrt{(25 + 1{,}600)} = 2{,}510 \times 40.31 = 101{,}178$ V

More precisely, with power factor angle $\theta = \cos^{-1}(0.92) = 23.07°$: $V_{drop} = I(R\cos\theta + X\sin\theta) = 2{,}510 \times (5 \times 0.92 + 40 \times 0.3919) = 2{,}510 \times (4.60 + 15.68) = 2{,}510 \times 20.28 = 50{,}903$ V

(d) Voltage regulation (using the approximate formula): $VR\% = V_{drop} / V_R \times 100 = 50{,}903 / (138{,}000/\sqrt{3}) \times 100 = 50{,}903 / 79{,}674 \times 100 = 63.9\%$

This extreme voltage regulation confirms the N-1 violation and the need for load shedding or alternate supply paths.

Problem 1.2.6

Given: A 345 kV three-phase transmission line is 300 km long with a characteristic impedance of $Z_c = 290\ \Omega$. The line currently transfers 500 MW.

Find: (a) The surge impedance loading (SIL), (b) whether the line is above or below SIL, (c) the reactive power behavior of the line (absorbing or generating VARs), and (d) the approximate receiving-end voltage if the sending-end voltage is 345 kV (use the simplified relationship for lines near SIL).

Solution:

(a) SIL: $SIL = V^2 / Z_c = (345{,}000)^2 / 290 = 119{,}025{,}000{,}000 / 290 = 410.4$ MW

(b) The line transfers 500 MW, which is above the SIL (500 > 410.4).

(c) When a line operates above SIL, the reactive power absorbed by the series inductance exceeds the reactive power generated by the shunt capacitance. The line absorbs reactive power (net inductive), causing the voltage to drop along the line.

(d) For a line operating above SIL, the receiving-end voltage is lower than the sending-end voltage. Using the approximate relationship: $V_R/V_S \approx \cos(\beta l)$ for lines at SIL, with correction for loading above SIL. At 300 km with $\beta \approx 1.1 \times 10^{-3}$ rad/km: $\beta l = 0.33$ rad $= 18.9°$ $V_R \approx V_S \times \cos(\beta l) \times (SIL/P)^{\wedge}(0.5) \approx 345 \times \cos(18.9°) \times \sqrt{(410.4/500)}$ $V_R \approx 345 \times 0.946 \times 0.906 = 295.7$ kV

The receiving-end voltage drops to approximately 296 kV, requiring reactive compensation (shunt capacitors) at the receiving end to maintain acceptable voltage.

Chapter 1 — Section 1.3: Power Distribution

Practice problems covering substations (transformers, autotransformers, circuit breakers, voltage regulators, CTs, VTs, switching), distribution poles, underground distribution, three-phase connections, AC analysis, and power factor correction.

Problem 1.3.1

Given: A three-phase, 75 MVA, 230 kV / 34.5 kV, wye-delta transformer has a nameplate impedance of 10%. The source (230 kV side) has a short-circuit capacity of 2,000 MVA.

Find: (a) The turns ratio, (b) the rated current on each side, (c) the total impedance seen at the 34.5 kV bus (source + transformer) in per-unit on the transformer base, and (d) the fault current at the 34.5 kV bus.

Solution:

(a) Turns ratio: $N_1/N_2 = V_1/V_2 = 230{,}000 / 34{,}500 = 6.667{:}1$

(b) Rated currents: $I_{HV} = S / (\sqrt{3} \times V_{HV}) = 75 \times 10^6 / (\sqrt{3} \times 230{,}000) = 188.3$ A $I_{LV} = S / (\sqrt{3} \times V_{LV}) = 75 \times 10^6 / (\sqrt{3} \times 34{,}500) = 1{,}255$ A

(c) Source impedance on transformer base: $Z_{source} = S_{base} / S_{SC} = 75 / 2{,}000 = 0.0375$ pu Total impedance: $Z_{total} = Z_{source} + Z_{xfmr} = 0.0375 + 0.10 = 0.1375$ pu

(d) Fault current at 34.5 kV bus: $I_{fault} = 1.0 / Z_{total} \times I_{base} = (1/0.1375) \times 1{,}255 = 7.273 \times 1{,}255 = 9{,}128$ A $= 9.13$ kA

Problem 1.3.2

Given: A single-phase, 200 MVA, 345/138 kV autotransformer supplies a 138 kV bus.

Find: (a) The turns ratio, (b) the rated load current at 138 kV, (c) the source current at 345 kV, (d) the current in the series winding, (e) the transformed (coupled) power, and (f) the power advantage ratio.

Solution:

(a) Turns ratio: $a = V_1/V_2 = 345/138 = 2.5{:}1$

(b) Load current: $I_{load} = S / V_2 = 200 \times 10^6 / 138{,}000 = 1{,}449$ A

(c) Source current: $I_{source} = S / V_1 = 200 \times 10^6 / 345{,}000 = 580$ A

(d) Series winding current: $I_{series} = I_{load} - I_{source} = 1{,}449 - 580 = 869$ A

(e) Transformed power: $S_{transformed} = S_{auto} \times (1 - 1/a) = 200 \times (1 - 1/2.5) = 200 \times 0.60 = 120$ MVA

(f) Power advantage: $PA = S_{auto} / S_{transformed} = 200 / 120 = 1.667$

A two-winding transformer rated at only 120 MVA can deliver 200 MVA as an autotransformer at this ratio.

Problem 1.3.3

Given: A 230 kV substation bus has a rated short-circuit current of 50 kA symmetrical. The circuit breaker must interrupt the fault within 3 cycles at 60 Hz.

Find: (a) The interrupting time in milliseconds, (b) the three-phase short-circuit MVA, and (c) the asymmetrical peak current assuming an X/R ratio of 20 (asymmetry factor = 2.6 × symmetrical RMS).

Solution:

(a) Interrupting time: $t = 3 \times (1/60) = 0.05$ s = 50 ms

(b) Short-circuit MVA: $S_{fault} = \sqrt{3} \times V_{LL} \times I_{fault} = \sqrt{3} \times 230{,}000 \times 50{,}000 = 19.92 \times 10^9$ VA = 19,919 MVA ≈ 19.9 GVA

(c) Asymmetrical peak current: $I_{peak} = 2.6 \times I_{sym} = 2.6 \times 50{,}000 = 130$ kA peak

The circuit breaker must have a close-and-latch rating of at least 130 kA.

Problem 1.3.4

Given: A step voltage regulator on a 14.4 kV (line-to-neutral) distribution feeder has R = 2.5 V and X = 7.5 V compensator settings (on a 120 V base). The CT ratio is 400:5 (80:1). The feeder current is 250 A at 0.88 lagging power factor.

Find: (a) The secondary current, (b) the compensator voltage drop, and (c) the required regulator output voltage (on 120 V base and converted to primary).

Solution:

(a) Secondary current: $I_{sec} = 250 / 80 = 3.125$ A

(b) Power factor angle: $\theta = \cos^{-1}(0.88) = 28.36°$ $I_{sec} = 3.125\angle{-28.36°} = 2.75 - j1.484$ A

$V_{drop} = I_{sec} \times (R + jX) = (2.75 - j1.484)(2.5 + j7.5) = 6.875 + j20.625 - j3.710 + 11.13 = 18.005 + j16.915$
$|V_{drop}| = \sqrt{(18.005^2 + 16.915^2)} = \sqrt{(324.2 + 286.1)} = \sqrt{610.3} = 24.7$ V

(c) Regulator output on 120 V base: $V_{reg} = 120 + 24.7 = 144.7$ V Convert to primary: $V_{out} = (144.7/120) \times 14{,}400 = 1.206 \times 14{,}400 = 17{,}362$ V

Tap position needed: (17,362 - 14,400) / 14,400 = 0.2057 = 20.6% Since regulator range is ±10%, this exceeds the single regulator range. The regulator would operate at its maximum boost (+10%), providing $14{,}400 \times 1.10 = 15{,}840$ V. A second regulator or capacitor bank is required.

Problem 1.3.5

Given: A 1200:5 CT with accuracy class C400 is connected to a relay with impedance of 1.5 Ω and lead wire resistance of 2.0 Ω (total loop). A primary fault current of 18,000 A flows.

Find: (a) The secondary current, (b) the voltage across the burden, and (c) whether the CT maintains accuracy.

Solution:

(a) CT ratio = 1200/5 = 240 Secondary current: $I_{sec} = 18{,}000 / 240 = 75$ A

(b) Total burden: $Z_{burden} = 1.5 + 2.0 = 3.5\ \Omega$ Voltage: $V_{sec} = I_{sec} \times Z_{burden} = 75 \times 3.5 = 262.5$ V

(c) C400 means the CT can deliver up to 400 V at 20 times rated secondary (100 A) without exceeding 10% ratio error. At 75 A (15× rated), the required 262.5 V is within the 400 V rating. The CT maintains accuracy with margin of 400 - 262.5 = 137.5 V.

Problem 1.3.6

Given: A 34,500:120 V voltage transformer is connected to two relays (each 30 VA burden) and two meters (each 40 VA burden). The VT has a thermal burden rating of 250 VA and an accuracy class of 0.3 at 100 VA.

Find: (a) The total connected burden, (b) the secondary current, (c) whether the VT thermal rating is exceeded, and (d) whether the metering accuracy is maintained.

Solution:

(a) Total burden: $S_{total} = 2 \times 30 + 2 \times 40 = 60 + 80 = 140$ VA

(b) Secondary current: $I_{sec} = S_{total} / V_{sec} = 140 / 120 = 1.167$ A

(c) 140 VA < 250 VA thermal rating: within thermal rating (56% loaded).

(d) The accuracy class 0.3 is specified at 100 VA burden. At 140 VA, the VT is loaded beyond its accuracy-rated burden. The metering accuracy may not meet the 0.3% specification at this burden. Revenue metering should be placed on a separate VT secondary winding with burden within 100 VA.

Problem 1.3.7

Given: A three-phase, 480Y/277 V wye-connected system supplies three loads: Phase A = 60 kW at unity PF, Phase B = 45 kW at 0.85 lagging PF, Phase C = 70 kW at 0.90 lagging PF.

Find: (a) The current in each phase, (b) the neutral current (magnitude), and (c) the total three-phase apparent power.

Solution:

(a) Phase currents: $I_A = P_A / (V_{LN} \times PF_A) = 60{,}000 / (277 \times 1.0) = 216.6$ A at 0° $I_B = P_B / (V_{LN} \times PF_B) = 45{,}000 / (277 \times 0.85) = 191.2$ A at $\theta = -\cos^{-1}(0.85) = -31.79°$ relative to Phase B $I_C = P_C / (V_{LN} \times PF_C) = 70{,}000 / (277 \times 0.90) = 280.8$ A at $\theta = -\cos^{-1}(0.90) = -25.84°$ relative to Phase C

(b) Phase angles referenced to Phase A: $I_A = 216.6\angle 0° = 216.6 + j0$ $I_B = 191.2\angle(-120° - 31.79°) = 191.2\angle{-151.79°} = -168.6 - j90.5$ $I_C = 280.8\angle(120° - 25.84°) = 280.8\angle 94.16° = -20.4 + j280.1$

$I_N = I_A + I_B + I_C = (216.6 - 168.6 - 20.4) + j(0 - 90.5 + 280.1) = 27.6 + j189.6$ $|I_N| = \sqrt{(27.6^2 + 189.6^2)} = \sqrt{(762 + 35{,}948)} = \sqrt{36{,}710} = 191.6$ A

The significant neutral current results from the unbalanced loading.

(c) Total apparent power: $S_A = 60/1.0 = 60$ kVA; $S_B = 45/0.85 = 52.9$ kVA; $S_C = 70/0.90 = 77.8$ kVA $S_{total} = 60 + 52.9 + 77.8 = 190.7$ kVA (arithmetic sum; vector sum would differ)

Problem 1.3.8

Given: A three-phase, 240 V delta-connected heater bank has three identical resistance elements of 12 Ω each.

Find: (a) The phase current, (b) the line current, (c) the total three-phase power, and (d) the equivalent wye impedance per phase.

Solution:

(a) Phase current (delta: phase voltage = line voltage): $I_{phase} = V_{LL} / R = 240 / 12 = 20.0$ A

(b) Line current: $I_L = \sqrt{3} \times I_{phase} = 1.732 \times 20.0 = 34.64$ A

(c) Total power (resistive, PF = 1): $P = \sqrt{3} \times V_{LL} \times I_L \times PF = \sqrt{3} \times 240 \times 34.64 \times 1.0 = 14{,}400$ W = 14.4 kW Alternatively: $P = 3 \times V_{phase}^2 / R = 3 \times 240^2 / 12 = 3 \times 4{,}800 = 14{,}400$ W. Check.

(d) Equivalent wye impedance: $Z_Y = Z_\Delta / 3 = 12 / 3 = 4\ \Omega$ per phase

Verify: $I_L = V_{LN} / Z_Y = (240/\sqrt{3}) / 4 = 138.56 / 4 = 34.64$ A. Check.

Problem 1.3.9

Given: A series RLC circuit has R = 15 Ω, L = 30 mH, and C = 200 μF, connected to a 240 V_{rms}, 50 Hz source.

Find: (a) The inductive and capacitive reactances, (b) the impedance in polar form, (c) the current magnitude and phase, (d) the power factor, and (e) the real, reactive, and apparent power.

Solution:

(a) $X_L = 2\pi fL = 2\pi \times 50 \times 0.030 = 9.42\ \Omega$ $X_C = 1/(2\pi fC) = 1/(2\pi \times 50 \times 200 \times 10^{-6}) = 15.92\ \Omega$

(b) Net reactance: $X = X_L - X_C = 9.42 - 15.92 = -6.50\ \Omega$ (capacitive) $Z = 15 - j6.50\ \Omega$ $|Z| = \sqrt{(15^2 + 6.50^2)} = \sqrt{(225 + 42.25)} = \sqrt{267.25} = 16.35\ \Omega$ $\theta = \tan^{-1}(-6.50/15) = -23.43°$ $Z = 16.35\angle -23.43°\ \Omega$

(c) Current: $I = V/|Z| = 240/16.35 = 14.68$ A at phase angle +23.43° (leading)

(d) Power factor: $PF = \cos(23.43°) = 0.918$ leading

(e) Real power: $P = V \times I \times PF = 240 \times 14.68 \times 0.918 = 3{,}233$ W Reactive power: $Q = V \times I \times \sin(23.43°) = 240 \times 14.68 \times 0.3975 = 1{,}400$ VAR (capacitive) Apparent power: $S = V \times I = 240 \times 14.68 = 3{,}523$ VA

Problem 1.3.10

Given: An industrial plant draws 800 kW at 0.68 lagging power factor from a 600 V, 60 Hz, three-phase supply. The utility requires correction to 0.92 lagging.

Find: (a) The original and target reactive power, (b) the required capacitor bank in kVAR, (c) the capacitance per phase for a wye-connected bank, and (d) the line current reduction achieved.

Solution:

(a) Original: $\theta_1 = \cos^{-1}(0.68) = 47.16°$ $Q_1 = P \times \tan(\theta_1) = 800 \times \tan(47.16°) = 800 \times 1.0785 = 862.8$ kVAR

Target: $\theta_2 = \cos^{-1}(0.92) = 23.07°$ $Q_2 = P \times \tan(\theta_2) = 800 \times \tan(23.07°) = 800 \times 0.4259 = 340.7$ kVAR

(b) Required capacitor bank: $Q_{cap} = Q_1 - Q_2 = 862.8 - 340.7 = 522.1$ kVAR

(c) For a wye-connected bank, each capacitor sees line-to-neutral voltage: $V_{LN} = 600 / \sqrt{3} = 346.4$ V Q per phase: $Q_{phase} = 522.1 / 3 = 174.0$ kVAR $X_C = V_{LN}^2 / Q_{phase} = (346.4)^2 / 174{,}000 = 119{,}993 / 174{,}000 = 0.6896\ \Omega$ $C = 1/(2\pi fX_C) = 1/(2\pi \times 60 \times 0.6896) = 3{,}847\ \mu F$ per phase

(d) Original current: $I_1 = S_1/(\sqrt{3} \times V) = (800/0.68)/(\sqrt{3} \times 600) = 1{,}176.5/1{,}039.2 = 1{,}132$ A Corrected current: $I_2 = S_2/(\sqrt{3} \times V) = (800/0.92)/(\sqrt{3} \times 600) = 869.6/1{,}039.2 = 836.8$ A Reduction: $(1{,}132 - 836.8)/1{,}132 \times 100 = 26.1\%$ reduction in line current

Chapter 1 — Section 1.4: Power System Protection

Practice problems covering protective relays, fault analysis, protection coordination, and symmetrical components.

Problem 1.4.1

Given: A time-overcurrent relay (51) protects a 13.8 kV feeder with a pickup current of 800 A and uses the IEEE Extremely Inverse characteristic: $t = (28.2 / (M^2 - 1) + 0.1217) \times TDS$, where $M = I_{fault}/I_{pickup}$. The time dial setting is TDS = 3.0.

Find: The relay operating time for fault currents of (a) 2,400 A, (b) 4,800 A, and (c) 12,000 A.

Solution:

(a) M = 2,400 / 800 = 3.0 t = (28.2/(9 - 1) + 0.1217) × 3.0 = (28.2/8 + 0.1217) × 3.0 = (3.525 + 0.1217) × 3.0 = 3.647 × 3.0 = 10.94 s

(b) M = 4,800 / 800 = 6.0 t = (28.2/(36 - 1) + 0.1217) × 3.0 = (28.2/35 + 0.1217) × 3.0 = (0.806 + 0.1217) × 3.0 = 0.928 × 3.0 = 2.78 s

(c) M = 12,000 / 800 = 15.0 t = (28.2/(225 - 1) + 0.1217) × 3.0 = (28.2/224 + 0.1217) × 3.0 = (0.126 + 0.1217) × 3.0 = 0.248 × 3.0 = 0.74 s

The extremely inverse curve shows dramatic time reduction with increasing fault current: a 5x increase in fault current (from 2,400 to 12,000 A) reduces operating time by a factor of nearly 15.

Problem 1.4.2

Given: A 4.16 kV industrial bus is supplied through a transformer with Z = 6.5% on a 10 MVA base. The source impedance on the same base is 1.5%. The system X/R ratio is 10.

Find: (a) The base current, (b) the three-phase fault current in per-unit and amperes, (c) the fault MVA, and (d) the asymmetrical fault current (use asymmetry factor of $1.0 + e^{-\pi/(X/R)}$ for the peak).

Solution:

(a) Base current: $I_{base} = S_{base} / (\sqrt{3} \times V_{base}) = 10 \times 10^6 / (\sqrt{3} \times 4{,}160) = 1{,}388$ A

(b) Total impedance: $Z_{total} = 0.015 + 0.065 = 0.080$ pu $I_{fault} = 1.0 / 0.080 = 12.5$ pu $= 12.5 \times 1{,}388 = 17{,}350$ A $= 17.35$ kA

(c) Fault MVA: $S_{fault} = S_{base} / Z_{total} = 10 / 0.080 = 125$ MVA

(d) Peak asymmetrical factor: Peak multiplier $= \sqrt{2} \times (1 + e^{-\pi/10}) = 1.414 \times (1 + e^{-0.3142}) = 1.414 \times (1 + 0.7304) = 1.414 \times 1.7304 = 2.447$ $I_{peak} = 2.447 \times 17{,}350 = 42{,}455$ A $= 42.5$ kA peak

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Problem 1.4.3

Given: Three series protective devices on a radial feeder must be coordinated: - Fuse C (downstream): 100K fuse link, total clearing time = 0.03 s at 5,000 A - Relay B (midstream): pickup = 500 A, IEEE Very Inverse, TDS = 2.5 - Relay A (upstream): pickup = 800 A, IEEE Very Inverse, must coordinate with Relay B

The IEEE Very Inverse characteristic is: $t = (19.61/(M^2 - 1) + 0.491) \times TDS$. CTI between relays = 0.3 s.

Find: (a) Relay B operating time at 5,000 A, (b) whether Relay B coordinates with Fuse C (margin > 0.2 s), (c) required Relay A TDS to coordinate with Relay B at the maximum fault of 8,000 A.

Solution:

(a) Relay B at 5,000 A: $M_B = 5{,}000/500 = 10.0$ $t_B = (19.61/(100 - 1) + 0.491) \times 2.5 = (0.198 + 0.491) \times 2.5 = 0.689 \times 2.5 = 1.72$ s

(b) Coordination with Fuse C: Margin $= t_B - t_{Fuse\ C} = 1.72 - 0.03 = 1.69$ s 1.69 s > 0.2 s: Relay B coordinates with Fuse C with ample margin.

(c) Relay A at 8,000 A: First, Relay B at 8,000 A: $M_B = 8{,}000/500 = 16.0$ $t_B = (19.61/(256 - 1) + 0.491) \times 2.5 = (0.0769 + 0.491) \times 2.5 = 0.568 \times 2.5 = 1.42$ s

Required Relay A time: $t_A = t_B + CTI = 1.42 + 0.3 = 1.72$ s

Relay A at 8,000 A: $M_A = 8{,}000/800 = 10.0$ $1.72 = (19.61/(100 - 1) + 0.491) \times TDS_A = (0.198 + 0.491) \times TDS_A = 0.689 \times TDS_A$ $TDS_A = 1.72/0.689 = 2.50$

Problem 1.4.4

Given: A 138 kV generator has sequence impedances: $Z_1 = j0.20$ pu, $Z_2 = j0.18$ pu, $Z_0 = j0.06$ pu. The generator neutral is grounded through a reactor with $Z_n = j0.05$ pu. Pre-fault voltage is 1.0 pu. $S_{base} = 50$ MVA.

Find: (a) The single-line-to-ground (SLG) fault current, (b) the line-to-line (LL) fault current, and (c) the three-phase fault current. Express results in per-unit and amperes.

Solution:

Base current: $I_{base} = 50 \times 10^6 / (\sqrt{3} \times 138{,}000) = 209.2$ A

(a) SLG fault (all sequences in series): For SLG: $I_1 = V / (Z_1 + Z_2 + Z_0 + 3Z_n)$ $Z_{total} = j0.20 + j0.18 + j0.06 + j0.15 = j0.59$ $I_1 = I_2 = I_0 = 1.0/j0.59 = -j1.695$ pu $I_{fault} = 3 \times I_1 = 3 \times 1.695 = 5.085$ pu $I_{fault} = 5.085 \times 209.2 = 1{,}064$ A

(b) Line-to-line fault (positive and negative in series): $I_1 = V / (Z_1 + Z_2) = 1.0 / (j0.20 + j0.18) = 1.0/j0.38 = -j2.632$ pu $I_{fault} = \sqrt{3} \times |I_1| = \sqrt{3} \times 2.632 = 4.557$ pu $I_{fault} = 4.557 \times 209.2 = 953$ A

(c) Three-phase fault (positive sequence only): $I_{fault} = V / Z_1 = 1.0/j0.20 = -j5.0$ pu $= 5.0$ pu $I_{fault} = 5.0 \times 209.2 = 1{,}046$ A

Note: In this case the SLG fault (1,064 A) exceeds the three-phase fault (1,046 A) because of the relatively low zero-sequence impedance and grounding impedance. This is common for solidly or low-impedance grounded generators.

Problem 1.4.5

Given: A 13.8 kV distribution system has the following sequence impedance data on a 100 MVA base: generator $Z_1 = j0.12$, $Z_2 = j0.12$, $Z_0 = j0.04$; transformer $Z_1 = Z_2 = Z_0 = j0.08$ (delta-wye, with grounded wye). A single-line-to-ground fault occurs on the 13.8 kV bus.

Find: (a) The total sequence impedances, (b) the sequence currents, (c) the fault current, and (d) the fault current in amperes.

Solution:

Base current: $I_{base} = 100 \times 10^6 / (\sqrt{3} \times 13{,}800) = 4{,}184$ A

(a) Total sequence impedances: $Z_{1total} = Z_{1gen} + Z_{1xfmr} = j0.12 + j0.08 = j0.20$ pu $Z_{2total} = Z_{2gen} + Z_{2xfmr} = j0.12 + j0.08 = j0.20$ pu $Z_{0total} = Z_{0xfmr} = j0.08$ pu (delta winding blocks zero-sequence from generator)

(b) Sequence currents: $I_1 = I_2 = I_0 = V/(Z_1 + Z_2 + Z_0) = 1.0/(j0.20 + j0.20 + j0.08) = 1.0/j0.48 = -j2.083$ pu

(c) Phase A fault current: $I_a = 3 \times I_1 = 3 \times 2.083 = 6.25$ pu

(d) In amperes: $I_{fault} = 6.25 \times 4{,}184 = 26{,}150$ A $= 26.15$ kA

Problem 1.4.6

Given: A differential relay protects a 50 MVA, 138/13.8 kV, delta-wye transformer. The CT ratios are 300:5 on the 138 kV side and 2500:5 on the 13.8 kV side. The relay has a minimum pickup of 0.3 A and a slope setting of 25%.

Find: (a) The rated currents on each side of the transformer, (b) the CT secondary currents at rated load, (c) the relay operating and restraint currents at rated load (accounting for the delta-wye 30°

phase shift and $\sqrt{3}$ factor), and (d) the minimum internal fault current (as a percentage of rated) that will trip the relay.

Solution:

(a) Rated currents: $I_{HV} = 50 \times 10^6/(\sqrt{3} \times 138{,}000) = 209.2$ A $I_{LV} = 50 \times 10^6/(\sqrt{3} \times 13{,}800) = 2{,}091.8$ A

(b) CT secondary currents: HV side: $I_{CT,HV} = 209.2 \times (5/300) = 3.49$ A LV side: $I_{CT,LV} = 2{,}091.8 \times (5/2500) = 4.18$ A

The CTs on the delta (HV) side are connected in wye, and those on the wye (LV) side in delta to compensate for the transformer phase shift. The delta-connected CTs multiply the current by $\sqrt{3}$: $I_{CT,LV,delta} = 4.18 \times \sqrt{3} = 7.24$ A

There is a mismatch: 3.49 A vs. 7.24 A. Auxiliary CTs or relay tap settings are used to match them. Assuming the relay compensates (modern numerical relays do this internally): Ratio correction applied so both sides read approximately equal at rated load.

(c) At rated, balanced load with proper matching: Operating current: $I_{op} = |I_1 - I_2| \approx 0$ A (ideally) Restraint current: $I_{res} = (|I_1| + |I_2|)/2 \approx 3.49$ A (per relay scaling)

(d) Minimum fault current to trip: With 25% slope: $I_{op} = 0.25 \times I_{res}$ must exceed 0.3 A pickup. At minimum: $I_{op} = 0.3$ A This corresponds to a differential current of 0.3/3.49 = 8.6% of rated, so a fault producing approximately 8.6% of rated current on the HV side would trip the relay.

Chapter 1 — Section 1.5: Power Quality

Practice problems covering harmonics, voltage sags and swells, power quality monitoring, and arc flash analysis.

Problem 1.5.1

Given: A 12-pulse VFD produces the following harmonic current spectrum (as percentage of fundamental): 11th = 8.5%, 13th = 6.5%, 23rd = 3.2%, 25th = 2.8%. The VFD draws 200 A fundamental at 480 V on a system where the available short-circuit current is I_{SC} = 25,000 A.

Find: (a) The current THD, (b) the I_{SC}/I_L ratio, (c) the IEEE 519 TDD limit for that ratio, and (d) whether the installation is compliant.

Solution:

(a) THD = $\sqrt{}(8.5^2 + 6.5^2 + 3.2^2 + 2.8^2) = \sqrt{}(72.25 + 42.25 + 10.24 + 7.84) = \sqrt{}132.58 = 11.5\%$

(b) I_{SC}/I_L = 25,000 / 200 = 125

(c) For $I_{SC}/I_L > 100$: IEEE 519 TDD limit = 15.0%. Individual harmonic limits: 11th-16th ≤ 7.0%, 17th-22nd ≤ 2.5%, 23rd-34th ≤ 1.4%.

(d) TDD = 11.5% < 15.0%: TDD is compliant. Individual harmonics: 11th = 8.5% > 7.0%: Non-compliant on the 11th harmonic. The 23rd = 3.2% > 1.4%: Non-compliant on the 23rd harmonic. The 25th = 2.8% > 1.4%: Non-compliant on the 25th harmonic.

Overall: Does not comply due to individual harmonic violations. A passive tuned filter at the 11th harmonic or an active harmonic filter would be needed.

Problem 1.5.2

Given: A 208 V bus experiences a voltage sag to 72% of nominal lasting 150 ms. A sensitive semiconductor fabrication tool requires a minimum of 85% voltage. The tool draws 50 kVA during the sag.

Find: (a) The sag magnitude in volts, (b) whether the tool will malfunction, (c) the voltage that a dynamic voltage restorer (DVR) must inject, and (d) the energy the DVR must supply during the event.

Solution:

(a) Sag voltage: $0.72 \times 208 = 149.8$ V

(b) Required minimum: $0.85 \times 208 = 176.8$ V. Since 149.8 V < 176.8 V, the tool will malfunction.

(c) DVR injection voltage: $V_{DVR} = V_{nominal} - V_{sag} = 208 - 149.8 = 58.2$ V (28% of nominal)

(d) DVR power: $S_{DVR} = (V_{DVR}/V_{nominal}) \times S_{load} = 0.28 \times 50 = 14$ kVA Energy: $E = S_{DVR} \times t = 14{,}000 \times 0.150 = 2{,}100$ J = 2.1 kJ

Problem 1.5.3

Given: A power quality monitor at a pharmaceutical plant records the following over 90 days: 120 total voltage sags, of which 35 are between 80-90%, 55 between 70-80%, 20 between 50-70%, and 10 below 50%. Each sag below 70% causes a batch loss costing $25,000.

Find: (a) The SARFI-90 index (events below 90%) per 30 days, (b) the SARFI-70 index per 30 days, (c) the average sag frequency per week, and (d) the annualized cost of power quality events.

Solution:

(a) SARFI-90 = all events below 90% per period. Total below 90% = 35 + 55 + 20 + 10 = 120 in 90 days. Per 30 days: SARFI-90 = 120/3 = 40 events per 30 days

(b) SARFI-70 = events below 70%. Total below 70% = 20 + 10 = 30 in 90 days. Per 30 days: SARFI-70 = 30/3 = 10 events per 30 days

(c) Average sag frequency: 120 sags / (90/7 weeks) = 120/12.86 = 9.3 sags per week

(d) Events causing batch loss: below 70% = 30 in 90 days. Annualized: $30 \times (365/90) = 121.7$ events per year. Annual cost: $121.7 \times \$25{,}000 = \$3{,}041{,}667$/year

This cost would justify a UPS or DVR system costing up to several million dollars with a payback under two years.

Problem 1.5.4

Given: A 13.8 kV switchgear has a bolted fault current of 25 kA. The upstream relay clears faults in 0.5 seconds (30 cycles at 60 Hz). The working distance is 910 mm (36 inches). Using the IEEE 1584 simplified method with $E_n = 4.0$ J/cm², $C_f = 1.0$ (voltage > 1 kV), and distance exponent x = 2.0.

Find: (a) The incident energy at the working distance, (b) the required PPE category, and (c) the incident energy if an arc flash relay reduces clearing time to 0.05 s (3 cycles).

Solution:

(a) $E = 4.184 \times C_f \times E_n \times (t/0.2) \times (610^x/D^x)$ $E = 4.184 \times 1.0 \times 4.0 \times (0.5/0.2) \times (610^2/910^2)$ $E = 4.184 \times 4.0 \times 2.5 \times (372{,}100/828{,}100)$ $E = 41.84 \times 0.4493 = 18.80$ J/cm² = 4.49 cal/cm²

(b) PPE categories: Cat 1 ≤ 4 cal/cm², Cat 2 ≤ 8 cal/cm², Cat 3 ≤ 25 cal/cm², Cat 4 ≤ 40 cal/cm². At 4.49 cal/cm²: Category 2 (arc-rated clothing rated 8 cal/cm² minimum).

(c) With arc flash relay (t = 0.05 s): $E = 4.184 \times 1.0 \times 4.0 \times (0.05/0.2) \times (610^2/910^2)$ $E = 4.184 \times 4.0 \times 0.25 \times 0.4493 = 1.88$ J/cm² = 0.45 cal/cm²

This is below Category 1 (4 cal/cm²), meaning no arc-rated PPE is required beyond standard work clothing. The arc flash relay reduces incident energy by a factor of 10, from 4.49 to 0.45 cal/cm².

Problem 1.5.5

Given: A three-phase rectifier bridge draws a square-wave current from the AC source. The fundamental component is 500 A, and the harmonic amplitudes follow the ideal pattern of $I_h = I_1/h$ for odd harmonics (no even harmonics). The system has a transformer with 5% impedance on a 2,000 kVA base at 480 V.

Find: (a) The current THD, (b) the voltage THD at the PCC (using $V_h = I_h \times Z_h$, where $Z_h = h \times Z_1$), (c) whether the voltage THD meets IEEE 519 limits (5% for systems < 69 kV).

Solution:

(a) Harmonic currents: $I_3 = 500/3 = 166.7$ A, $I_5 = 100$ A, $I_7 = 71.4$ A, $I_9 = 55.6$ A, $I_{11} = 45.5$ A, $I_{13} = 38.5$ A. THD = $\sqrt{}(166.7^2 + 100^2 + 71.4^2 + 55.6^2 + 45.5^2 + 38.5^2)$ / 500 = $\sqrt{}(27{,}789 + 10{,}000 + 5{,}098 + 3{,}091 + 2{,}070 + 1{,}482)$ / 500 = $\sqrt{}49{,}530$ / 500 = 222.6 / 500 = 44.5%

(b) System impedance at fundamental: $Z_{base} = V^2/S = 480^2/(2{,}000 \times 10^3) = 0.1152\ \Omega$ $Z_1 = 0.05 \times 0.1152 = 0.00576\ \Omega$ (primarily reactive)

Harmonic voltage drops: $V_3 = I_3 \times 3 \times Z_1 = 166.7 \times 3 \times 0.00576 = 2.88$ V $V_5 = 100 \times 5 \times 0.00576 = 2.88$ V $V_7 = 71.4 \times 7 \times 0.00576 = 2.88$ V $V_9 = 55.6 \times 9 \times 0.00576 = 2.88$ V $V_{11} = 45.5 \times 11 \times 0.00576 = 2.88$ V $V_{13} = 38.5 \times 13 \times 0.00576 = 2.88$ V

$V_{1,LN} = 480/\sqrt{}3 = 277.1$ V $THD_V = \sqrt{}(6 \times 2.88^2)$ / 277.1 = $\sqrt{}49.79$ / 277.1 = 7.06 / 277.1 = 2.55%

(c) Voltage THD = 2.55% < 5.0% limit: Compliant with IEEE 519. The individual harmonic voltages are 2.88/277.1 = 1.04%, each well below the 3% individual limit.

Problem 1.5.6

Given: A 480 V motor control center (MCC) has a bolted fault current of 42 kA. Two scenarios are compared: (a) Standard relay with 0.3 s clearing time, and (b) Current-limiting fuse with 0.004 s (quarter-cycle) clearing time. Working distance = 610 mm, $C_f = 1.5$, $E_n = 3.2$ J/cm², x = 1.641.

Find: The incident energy and PPE category for each scenario.

Solution:

(a) Standard relay ($t = 0.3$ s): $E = 4.184 \times 1.5 \times 3.2 \times (0.3/0.2) \times (610^{1.641}/610^{1.641})$ The distance ratio $= 1.0$ since $D = 610$ mm (reference distance). $E = 4.184 \times 1.5 \times 3.2 \times 1.5 \times 1.0 = 30.13$ J/cm² $=$ 7.20 cal/cm² PPE: Category 2 (8 cal/cm² minimum)

(b) Current-limiting fuse ($t = 0.004$ s): $E = 4.184 \times 1.5 \times 3.2 \times (0.004/0.2) \times 1.0$ $E = 4.184 \times 1.5 \times 3.2 \times 0.02 = 0.40$ J/cm² $= 0.096$ cal/cm² PPE: No arc-rated PPE required (below Category 1)

The current-limiting fuse reduces incident energy from 7.20 to 0.096 cal/cm² – a factor of 75 reduction – by clearing in a quarter-cycle instead of 18 cycles.

Chapter 1 — Section 1.6: HVDC Transmission

Practice problems covering HVDC fundamentals and converter technologies.

Problem 1.6.1

Given: A bipolar HVDC link operates at ±400 kV and transmits 2,000 MW over a distance of 800 km. Each conductor has a resistance of 0.015 Ω/km.

Find: (a) The DC current per pole, (b) the total I^2R losses in both conductors, (c) the loss percentage, and (d) the equivalent AC losses for comparison using three conductors at 400 kV AC with the same resistance per conductor.

Solution:

(a) Each pole carries half the power: $P_{pole} = 2{,}000/2 = 1{,}000$ MW $I_{dc} = P_{pole}/V_{dc} = 1{,}000 \times 10^6/(400 \times 10^3) = 2{,}500$ A per pole

(b) Conductor resistance: $R = 0.015 \times 800 = 12\ \Omega$ per conductor Losses per pole: $P_{loss} = I^2R = 2{,}500^2 \times 12 = 75$ MW Total losses: $P_{total} = 2 \times 75 = 150$ MW

(c) Loss percentage: $150/2{,}000 \times 100 = 7.5\%$

(d) AC equivalent at 400 kV (three-phase): $I_{AC} = P/(\sqrt{3} \times V \times PF) = 2{,}000 \times 10^6/(\sqrt{3} \times 400 \times 10^3 \times 1.0) = 2{,}887$ A (assuming unity PF for comparison) AC losses $= 3 \times I^2 \times R = 3 \times 2{,}887^2 \times 12 = 3 \times 100.1 \times 10^6 = 300.2$ MW $= 15.0\%$

HVDC losses (7.5%) are half the AC losses (15.0%).

Problem 1.6.2

Given: A 12-pulse LCC-HVDC rectifier operates with firing angle $\alpha = 20°$ and commutation overlap $\mu = 25°$. The no-load DC voltage per 6-pulse bridge is $V_{d0} = 300$ kV. The DC link transmits 1,500 MW.

Find: (a) The actual DC voltage per bridge, (b) the total DC voltage (12-pulse), (c) the DC current, (d) the converter power factor, and (e) the reactive power consumed.

Solution:

(a) DC voltage per bridge: $V_d = V_{d0} \times (\cos\alpha + \cos(\alpha + \mu))/2 = 300 \times (\cos 20° + \cos 45°)/2 = 300 \times (0.9397 + 0.7071)/2 = 300 \times 0.8234 = 247.0$ kV per bridge

(b) Total DC voltage (two bridges in series): $V_{dc} = 2 \times 247.0 = 494.0$ kV

(c) DC current: $I_{dc} = P/V_{dc} = 1{,}500 \times 10^6/494{,}000 = 3{,}036$ A

(d) Converter power factor: $\cos\varphi \approx (\cos\alpha + \cos(\alpha + \mu))/2 = 0.8234$ $\varphi = \cos^{-1}(0.8234) = 34.6°$ PF $= 0.823$ lagging

(e) Reactive power: $Q = P \times \tan\varphi = 1{,}500 \times \tan(34.6°) = 1{,}500 \times 0.691 = 1{,}036$ MVAR

This large reactive power demand requires shunt capacitor banks and harmonic filters totaling approximately 1,000+ MVAR at the converter station.

Problem 1.6.3

Given: A VSC-HVDC link connects an offshore wind farm to the onshore grid. The link is rated at ±320 kV, 1,000 MW, with submarine cables 150 km long. Each cable has resistance of 0.010 Ω/km. The VSC converter has a power loss of 1.0% per conversion stage.

Find: (a) The DC current, (b) the cable I^2R losses, (c) the total converter losses (two converters), (d) the total system losses, and (e) the overall transmission efficiency.

Solution:

(a) DC current per pole: $P_{pole} = 1{,}000/2 = 500$ MW $I_{dc} = 500 \times 10^6/(320 \times 10^3) = 1{,}562.5$ A per pole

(b) Cable losses: $R = 0.010 \times 150 = 1.5\ \Omega$ per cable $P_{cable} = 2 \times I^2 \times R = 2 \times 1{,}562.5^2 \times 1.5 = 2 \times 3{,}662{,}109 = 7.32$ MW

(c) Converter losses (rectifier + inverter): $P_{conv} = 2 \times 0.01 \times 1{,}000 = 20$ MW

(d) Total losses: $P_{total\ loss} = P_{cable} + P_{conv} = 7.32 + 20 = 27.32$ MW

(e) Transmission efficiency: $\eta = (P_{delivered})/(P_{generated}) = (1{,}000 - 27.32)/1{,}000 = 972.68/1{,}000 = 97.3\%$

The VSC converter losses (2.0%) dominate over cable losses (0.73%), which is typical for shorter HVDC links.

Problem 1.6.4

Given: An HVDC submarine cable link is being evaluated as an alternative to an AC cable. The route is 60 km at 230 kV. The AC cable has capacitance of 0.22 μF/km per phase and ampacity of 600 A. The HVDC option operates at ±200 kV with the same conductors.

Find: (a) The AC cable charging current per phase, (b) the remaining AC ampacity for load, (c) the maximum AC power transfer, (d) the HVDC power transfer (no charging current at DC), and (e) the power transfer advantage of HVDC.

Solution:

(a) AC charging current: $C_{total} = 0.22 \times 60 = 13.2$ μF per phase $V_{LN} = 230{,}000/\sqrt{3} = 132{,}791$ V $I_{charging} = 2\pi fCV = 2\pi \times 60 \times 13.2 \times 10^{-6} \times 132{,}791 = 661$ A per phase

(b) $I_{charging} = 661$ A $> I_{rated} = 600$ A! The charging current exceeds the cable ampacity. The AC cable cannot carry any load current over 60 km at 230 kV. This is a practical impossibility for AC at this distance and voltage.

(c) Maximum AC power transfer = 0 MW (the cable is completely consumed by charging current).

Even with shunt reactors at each end: $I_{load} \approx \sqrt{(600^2 - (661/2)^2)} = \sqrt{(360{,}000 - 109{,}056)} = \sqrt{250{,}944}$ = 500.9 A (with mid-point compensation) $S_{AC} = \sqrt{3} \times 230{,}000 \times 500.9 = 199.5$ MVA

(d) HVDC power transfer (two poles): $P_{HVDC} = 2 \times V_{dc} \times I = 2 \times 200{,}000 \times 600 = 240$ MW

(e) Without reactive compensation, HVDC delivers 240 MW versus 0 MW for AC – an infinite advantage. Even with ideal compensation, HVDC delivers 240 MW versus ~190 MW for AC = 1.26x advantage. This illustrates why HVDC is required for long submarine cables.

Chapter 1 — Section 1.7: Load Flow Analysis

Practice problems covering power flow calculations using the bus admittance matrix, Gauss-Seidel iteration, and power balance equations.

Problem 1.7.1

Given: A 3-bus power system has the following configuration: - Bus 1: Slack bus, $V_1 = 1.05\angle 0°$ pu - Bus 2: PV bus, $P_2 = 0.5$ pu (generation), $|V_2| = 1.02$ pu - Bus 3: PQ bus, $P_3 = -0.8$ pu (load), $Q_3 = -0.3$ pu (load)

Line admittances: $y_{12} = 10 - j30$ pu, $y_{13} = 8 - j24$ pu, $y_{23} = 5 - j15$ pu.

Find: (a) The Y_{bus} matrix, and (b) the initial power mismatch at Bus 3 using a flat start ($V_3^{(0)} = 1.0\angle 0°$, $V_2^{(0)} = 1.02\angle 0°$).

Solution:

(a) Y_{bus} diagonal elements (sum of admittances connected to each bus): $Y_{11} = y_{12} + y_{13} = (10 - j30) + (8 - j24) = 18 - j54$ pu $Y_{22} = y_{12} + y_{23} = (10 - j30) + (5 - j15) = 15 - j45$ pu $Y_{33} = y_{13} + y_{23} = (8 - j24) + (5 - j15) = 13 - j39$ pu

Off-diagonal elements (negative of line admittance): $Y_{12} = Y_{21} = -(10 - j30) = -10 + j30$ pu $Y_{13} = Y_{31} = -(8 - j24) = -8 + j24$ pu $Y_{23} = Y_{32} = -(5 - j15) = -5 + j15$ pu

(b) Power injected at Bus 3 (calculated): $P_{3calc} = |V_3| \times \Sigma|V_k| \times (G_{3k} \cos(\delta_3 - \delta_k) + B_{3k} \sin(\delta_3 - \delta_k))$

With flat start (all angles = 0°), sin terms vanish: $P_{3calc} = |V_3| \times (|V_1|G_{31} + |V_2|G_{32} + |V_3|G_{33}) = 1.0 \times (1.05\times(-8) + 1.02\times(-5) + 1.0\times 13) = 1.0 \times (-8.4 - 5.1 + 13.0) = -0.5$ pu

Specified $P_3 = -0.8$ pu. Mismatch: $\Delta P_3 = P_{3spec} - P_{3calc} = -0.8 - (-0.5) = -0.3$ pu

$Q_{3calc} = |V_3| \times (|V_1|G_{31} \sin(0) - |V_1|B_{31} \cos(0) + \ldots) = |V_3| \times \Sigma|V_k|(G_{3k} \sin(\delta_3\text{-}\delta_k) - B_{3k} \cos(\delta_3\text{-}\delta_k)) = 1.0 \times (0 - 1.05\times 24 + 0 - 1.02\times 15 + 0 - 1.0\times(-39)) = 1.0 \times (-25.2 - 15.3 + 39.0) = -1.5$ pu

$\Delta Q_3 = Q_{3spec} - Q_{3calc} = -0.3 - (-1.5) = +1.2$ pu

The large mismatches confirm that the flat start is far from the solution. Newton-Raphson iterations would converge to the solution in 3-5 iterations.

Problem 1.7.2

Given: A simple 2-bus system has Bus 1 (slack, $V_1 = 1.0\angle 0°$) connected to Bus 2 (PQ, $P_2 = -2.0$ pu, $Q_2 = -0.8$ pu) through a line with admittance $y_{12} = 3 - j12$ pu. Initial guess: $V_2^{(0)} = 1.0\angle 0°$.

Find: Perform two Gauss-Seidel iterations to find V_2.

Solution:

$Y_{22} = y_{12} = 3 - j12$; $Y_{21} = -(3 - j12) = -3 + j12$.

GS update: $V_2^{(k+1)} = (1/Y_{22}) \times [(P_2 - jQ_2)/V_2^{(k)*} - Y_{21}V_1]$

Iteration 1 ($V_2^{(0)} = 1.0 + j0$): $(P_2 - jQ_2)/V_2^* = (-2.0 + j0.8)/1.0 = -2.0 + j0.8$ $-Y_{21}V_1 = (3 - j12)(1.0) = 3 - j12$ Sum $= (-2.0 + j0.8) + (3 - j12) = 1.0 - j11.2$ $V_2^{(1)} = (1.0 - j11.2)/(3 - j12) = (1.0 - j11.2)(3 + j12)/((3)^2 + (12)^2) = (3 + j12 - j33.6 + 134.4)/153 = (137.4 - j21.6)/153 = 0.8980 - j0.1412 = 0.9090\angle -8.94°$ pu

Iteration 2 ($V_2^{(1)} = 0.8980 - j0.1412$): $V_2^{(1)*} = 0.8980 + j0.1412$ $(P_2 - jQ_2)/V_2^* = (-2.0 + j0.8)/(0.8980 + j0.1412)$

Multiply by conjugate: $(-2.0 + j0.8)(0.8980 - j0.1412)/(0.8980^2 + 0.1412^2)$ Numerator: $-1.796 + j0.2824 + j0.7184 + 0.1130 = -1.683 + j1.0008$ Denominator: $0.8064 + 0.01994 = 0.8264 = -2.036 + j1.211$

Sum $= (-2.036 + j1.211) + (3 - j12) = 0.964 - j10.789$ $V_2^{(2)} = (0.964 - j10.789)/(3 - j12) = (0.964 - j10.789)(3 + j12)/153 = (2.892 + j11.568 - j32.367 + 129.468)/153 = (132.36 - j20.799)/153 = 0.8651 - j0.1359 = 0.8757\angle -8.93°$ pu

The voltage at Bus 2 is converging toward approximately $0.876\angle -8.9°$ pu, indicating a significant voltage drop due to the heavy load (2.0 pu on the line).

Problem 1.7.3

Given: A 345 kV transmission line connects Bus A (sending, $V_A = 1.02\angle 10°$ pu) to Bus B (receiving, $V_B = 0.98\angle 0°$ pu). The line impedance is $Z = 0.01 + j0.10$ pu, and the shunt admittance is negligible.

Find: (a) The complex power flow from A to B, (b) the real and reactive power delivered to Bus B, (c) the real and reactive power losses in the line, and (d) the line current in amperes ($S_{base} = 100$ MVA).

Solution:

(a) Line admittance: $y = 1/Z = 1/(0.01 + j0.10) = (0.01 - j0.10)/(0.01^2 + 0.10^2) = (0.01 - j0.10)/0.0101 = 0.990 - j9.901$ pu

Power from A to B: $S_{AB} = V_A \times (V_A - V_B)^* \times y^*$

$V_A - V_B = 1.02\angle 10° - 0.98\angle 0° = (1.02 \cos 10° + j1.02 \sin 10°) - 0.98 = (1.0045 - 0.98) + j0.1771 = 0.0245 + j0.1771$

$(V_A - V_B)^* = 0.0245 - j0.1771$

$y^* = 0.990 + j9.901$

$V_A(V_A\text{-}V_B)y = (1.0045 + j0.1771)(0.0245 - j0.1771)(0.990 + j9.901)$

First: $(0.0245 - j0.1771)(0.990 + j9.901) = 0.02426 + j0.24258 - j0.17533 + 1.75348 = 1.77774 + j0.06725$

Then: $(1.0045 + j0.1771)(1.77774 + j0.06725) = 1.7858 + j0.0675 + j0.3149 + (j^2)0.01191 = (1.7858 - 0.01191) + j(0.0675 + 0.3149) = 1.7739 + j0.3824$

$S_{AB} = 1.774 + j0.382$ pu = (177.4 + j38.2) MVA

(b) Power at Bus B: $S_{BA} = V_B(V_B - V_A)y$ (power into Bus B = $-S_{BA}$)

The real power delivered to B equals sending power minus losses: $P_{losses} = I^2 \times R$. We need I first.

$I = (V_A - V_B)/Z = (0.0245 + j0.1771)/(0.01 + j0.10) = (0.0245 + j0.1771)(0.01 - j0.10)/(0.0101) = (0.000245 - j0.00245 + j0.001771 + 0.01771)/0.0101 = (0.017955 - j0.000679)/0.0101 = 1.778 - j0.0672$

$|I| = \sqrt{(1.778^2 + 0.0672^2)} = 1.779$ pu

(c) Line losses: $P_{loss} = |I|^2 \times R = 1.779^2 \times 0.01 = 0.0316$ pu = 3.16 MW $Q_{loss} = |I|^2 \times X = 1.779^2 \times 0.10 = 0.3165$ pu = 31.65 MVAR

Real power to Bus B: $P_B = 177.4 - 3.16 = 174.2$ MW

(d) Base current: $I_{base} = 100 \times 10^6/(\sqrt{3} \times 345{,}000) = 167.3$ A $I = 1.779 \times 167.3 = 297.6$ A

Chapter 1 — Section 1.8: SCADA Systems

Practice problems covering SCADA architecture, RTU data acquisition, communication protocols (DNP3, IEC 61850), scan rates, data throughput, cybersecurity (NERC CIP, IDS performance), and system reliability.

Problem 1.8.1

Given: A regional utility operates a SCADA system monitoring 85 substations. Each RTU reports 120 analog measurements (4 bytes each) and 96 digital status points (1 bit each). The master station uses DNP3 over TCP/IP with a scan cycle of 6 seconds and a protocol overhead of 25%.

Find: (a) The total number of monitored data points, (b) the raw payload per scan cycle, (c) the total data per scan with overhead, and (d) the minimum communication link bandwidth required.

Solution:

(a) Total data points: Analog: $85 \times 120 = 10{,}200$ Digital: $85 \times 96 = 8{,}160$ Total $= 10{,}200 + 8{,}160 = 18{,}360$ points

(b) Raw payload per scan: Analog data: $10{,}200 \times 4 = 40{,}800$ bytes Digital data: 8,160 bits / 8 = 1,020 bytes Raw payload $= 40{,}800 + 1{,}020 = 41{,}820$ bytes

(c) With 25% DNP3 overhead: Total $= 41{,}820 \times 1.25 = 52{,}275$ bytes per scan

(d) Minimum bandwidth: Throughput = 52,275 bytes / 6 s = 8,712.5 bytes/s Bandwidth $= 8{,}712.5 \times 8 = 69{,}700$ bps = 69.7 kbps

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Problem 1.8.2

Given: A SCADA system uses DNP3 polling with a master station that communicates with 200 RTUs. Each RTU poll-response transaction takes 45 ms (including request, response, and processing). The utility requires a maximum scan cycle time of 10 seconds.

Find: (a) The time to poll all RTUs sequentially, (b) whether the scan cycle requirement is met, (c) the minimum number of parallel communication channels needed to meet the requirement, and (d) the RTU polling rate per channel.

Solution:

(a) Sequential polling time: $T_{seq} = 200 \times 45$ ms = 9,000 ms = 9.0 s

(b) 9.0 s < 10.0 s, so the requirement is met with sequential polling (with 1.0 s margin).

(c) Since 9.0 s < 10.0 s, the minimum is 1 channel. However, for redundancy and margin, consider that if RTU count grows to 250: $T_{seq} = 250 \times 45$ ms = 11.25 s > 10 s Minimum channels = $\lceil 11.25 / 10 \rceil = 2$ channels

(d) With 2 channels and 200 RTUs: RTUs per channel = 200 / 2 = 100 Polling rate per channel = 100 / 10 s = 10 RTUs/s

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Problem 1.8.3

Given: A substation RTU samples 64 analog channels using a 16-bit ADC with a full-scale range of ±10 V. The transducer scaling is configured so that one channel monitors a 345 kV bus voltage through a VT with ratio 345,000:115 V and a transducer output of 0–10 V for 0–120 V secondary.

Find: (a) The ADC resolution in volts, (b) the transducer output voltage when the bus voltage is 342 kV, (c) the corresponding ADC digital count (assuming unipolar 0–10 V maps to 0–65535), and (d) the measurement resolution in kV at the primary bus.

Solution:

(a) ADC resolution (full bipolar range): Resolution = 20 V / 2^{16} = 20 / 65,536 = 305.2 μV/count

(b) VT secondary voltage at 342 kV primary: V_{sec} = 342,000 × (115 / 345,000) = 342,000 / 3,000 = 114.0 V

Transducer output (0–10 V for 0–120 V): $V_{transducer}$ = (114.0 / 120) × 10 = 9.500 V

(c) ADC count (unipolar 0–10 V → 0–65535): Count = (9.500 / 10) × 65,535 = 62,258

(d) Primary bus resolution: One ADC count in transducer volts = 10 / 65,536 = 152.6 μV In secondary volts = 152.6 μV × (120 / 10) = 1.831 mV In primary kV = $1.831 \times 10^{-3} \times 3{,}000 / 1{,}000$ = 0.00549 kV = 5.49 V

Problem 1.8.4

Given: An IEC 61850 substation automation system has 12 IEDs communicating over a redundant Ethernet LAN (PRP — Parallel Redundancy Protocol). Each IED publishes GOOSE messages at a normal rate of 1 message per second, but during a protection event the rate increases to 1 message every 4 ms for 2 seconds (exponential retransmission). Each GOOSE frame is 150 bytes including Ethernet overhead.

Find: (a) The normal aggregate GOOSE traffic rate, (b) the peak traffic rate during a protection event from a single IED, (c) the peak aggregate rate if 3 IEDs simultaneously detect a fault, and (d) the percentage of a 100 Mbps Ethernet link consumed during the peak event.

Solution:

(a) Normal aggregate traffic: Rate = 12 IEDs × 1 msg/s × 150 bytes = 1,800 bytes/s = 1,800 × 8 = 14,400 bps = 14.4 kbps

(b) Peak rate from one IED: Messages per second = 1000 ms / 4 ms = 250 msg/s Data rate = 250 × 150 = 37,500 bytes/s = 37,500 × 8 = 300,000 bps = 300 kbps

(c) Peak aggregate (3 event IEDs + 9 normal IEDs): Event traffic = 3 × 300 kbps = 900 kbps Normal traffic = 9 × 1.2 kbps = 10.8 kbps Total = 900 + 10.8 = 910.8 kbps

(d) Link utilization: Utilization = 910.8 kbps / 100,000 kbps × 100% = 0.91%

The low utilization confirms that GOOSE messaging has minimal impact on network capacity, even during fault events.

Problem 1.8.5

Given: A SCADA master station performs state estimation using telemetered measurements from RTUs. A particular 5-bus system has 14 measurements available (voltages, power flows, power injections) but only 9 state variables (voltage magnitudes and angles; slack bus angle is reference). The measurement residual vector after state estimation has a weighted sum of squared residuals (objective function) J = 8.73.

Find: (a) The measurement redundancy ratio, (b) the degrees of freedom for the chi-squared (χ^2) bad data test, (c) whether the measurement set passes the χ^2 test at the 95% confidence level ($\chi^2_{0.05,5}$ = 11.07), and (d) the minimum number of measurements that could be lost while still maintaining observability (redundancy > 1.0).

Solution:

(a) Measurement redundancy ratio: Redundancy = m / n = 14 / 9 = 1.556

(b) Degrees of freedom: ν = m − n = 14 − 9 = 5

(c) Chi-squared test: J = 8.73 < $\chi^2_{0.05,5}$ = 11.07 Since J < threshold, the measurement set passes the bad data test — no gross errors detected at 95% confidence.

(d) Minimum measurements for observability: For observability, m ≥ n, so m_{min} = 9 Maximum measurements that can be lost = 14 − 9 = 5 measurements

(After losing 5, redundancy = 9/9 = 1.0 — the system is barely observable with zero redundancy for bad data detection.)

Problem 1.8.6

Given: A utility's SCADA cybersecurity monitoring system processes an average of 150,000 network packets per hour on the OT network. The IDS has a true positive rate (sensitivity) of 99.2% and a false positive rate of 0.05%. During a 12-hour shift, 8 actual malicious packets are injected by a penetration tester.

Find: (a) The total packets processed in 12 hours, (b) the expected true positives, (c) the expected false positives, (d) the precision (positive predictive value), and (e) the F1 score.

Solution:

(a) Total packets in 12 hours: N = 150,000 × 12 = 1,800,000 packets

(b) True positives: TP = 0.992 × 8 = 7.94 ≈ 8

(c) False positives: Legitimate packets = 1,800,000 − 8 = 1,799,992 FP = 0.0005 × 1,799,992 = 900.0

(d) Precision: Precision = TP / (TP + FP) = 8 / (8 + 900) = 8 / 908 = 0.00881 = 0.88%

(e) F1 score: Recall = TP / (TP + FN) = 8 / (8 + 0) = 1.0 (assuming all 8 detected) F1 = 2 × (Precision × Recall) / (Precision + Recall) F1 = 2 × (0.00881 × 1.0) / (0.00881 + 1.0) = 0.01762 / 1.00881 = 0.01747 ≈ 1.75%

The extremely low F1 score highlights the base-rate paradox: the overwhelming number of legitimate packets makes even a very low false positive rate generate far more false alarms than true detections.

Problem 1.8.7

Given: A SCADA communication network uses three independent paths between a control center and a critical substation: a primary fiber link (availability 99.95%), a backup microwave link (availability 99.80%), and an emergency cellular link (availability 99.50%). The system uses automatic failover — it fails only if all three paths are simultaneously unavailable.

Find: (a) The unavailability of each individual link, (b) the combined unavailability (probability all three are down simultaneously), (c) the combined availability, and (d) the expected downtime per year for each link and for the combined system.

Solution:

(a) Individual unavailabilities: $U_{fiber} = 1 - 0.9995 = 0.0005$ $U_{microwave} = 1 - 0.9980 = 0.0020$ $U_{cellular} = 1 - 0.9950 = 0.0050$

(b) Combined unavailability (independent failures): $U_{combined} = U_1 \times U_2 \times U_3 = 0.0005 \times 0.0020 \times 0.0050 = 5.0 \times 10^{-9}$

(c) Combined availability: $A_{combined} = 1 - 5.0 \times 10^{-9} = 0.999999995$ (nine 9's)

(d) Expected downtime per year (8,760 hours): Fiber: 0.0005 × 8,760 = 4.38 hours/year Microwave: 0.0020 × 8,760 = 17.52 hours/year Cellular: 0.0050 × 8,760 = 43.80 hours/year Combined: $5.0 \times 10^{-9} \times 8{,}760 = 4.38 \times 10^{-5}$ hours = 0.158 seconds/year

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Problem 1.8.8

Given: A DNP3 Secure Authentication implementation adds a 32-byte HMAC (Hash-based Message Authentication Code) challenge-response to each critical control command. The control center sends an average of 80 supervisory control commands per hour. Each authenticated command exchange consists of: (1) command request (60 bytes), (2) authentication challenge from RTU (48 bytes), (3) HMAC response from master (92 bytes), and (4) command execution confirmation (40 bytes). The link operates at 9,600 bps (serial).

Find: (a) The total bytes per authenticated command exchange, (b) the time to complete one authenticated exchange, (c) the total authentication traffic per hour, and (d) the percentage of link capacity consumed by authenticated control traffic.

Solution:

(a) Total bytes per exchange: $60 + 48 + 92 + 40 = 240$ bytes

(b) Time per exchange: Bits = $240 \times 8 = 1{,}920$ bits Transmission time = $1{,}920 / 9{,}600 = 0.200$ s = 200 ms

(This is transmission time only; actual round-trip includes propagation and processing delays.)

(c) Traffic per hour: Bytes/hour = $80 \times 240 = 19{,}200$ bytes/hour Bits/hour = $19{,}200 \times 8 = 153{,}600$ bits/hour

(d) Link utilization: Link capacity per hour = $9{,}600 \times 3{,}600 = 34{,}560{,}000$ bits/hour Utilization = $153{,}600 / 34{,}560{,}000 \times 100\% = 0.44\%$

Problem 1.8.9

Given: A SCADA system uses report-by-exception (RBE) to reduce communication traffic. An RTU monitors 200 analog points with a deadband of 0.5% of full scale. Under normal steady-state conditions, 5% of analog points exceed their deadband per scan cycle. During a system disturbance, 60% of points change per scan. The scan cycle is 4 seconds, each analog report is 12 bytes (including timestamp and quality flags), and the link operates at 56 kbps.

Find: (a) The steady-state report rate (reports per scan), (b) the steady-state data rate, (c) the disturbance report rate, (d) the disturbance data rate, and (e) the bandwidth reduction factor of RBE versus polling all points every scan.

Solution:

(a) Steady-state reports per scan: Reports = $0.05 \times 200 = 10$ reports/scan

(b) Steady-state data rate: Data per scan = $10 \times 12 = 120$ bytes Rate = $120 / 4 = 30$ bytes/s = $30 \times 8 = 240$ bps

(c) Disturbance reports per scan: Reports = $0.60 \times 200 = 120$ reports/scan

(d) Disturbance data rate: Data per scan = $120 \times 12 = 1{,}440$ bytes Rate = $1{,}440 / 4 = 360$ bytes/s = $360 \times 8 = 2{,}880$ bps

(e) Full polling data rate (all 200 points every scan): Data per scan = $200 \times 12 = 2{,}400$ bytes Rate = $2{,}400 / 4 = 600$ bytes/s = 4,800 bps

RBE reduction factor: Steady-state: $4{,}800 / 240 = 20\times$ reduction Disturbance: $4{,}800 / 2{,}880 = 1.67\times$ reduction

Even during disturbances, RBE reduces traffic by 40%. Under normal conditions, RBE achieves a 95% bandwidth savings.

Problem 1.8.10

Given: A NERC CIP compliance audit requires a utility to demonstrate that all Electronic Security Perimeter (ESP) access points have logging enabled. The utility has 3 control centers and 45 substations classified as medium-impact BES Cyber Systems. Each control center has 4 ESP access points (firewalls), and each substation has 2 ESP access points. Each access point generates an average of 500 log entries per day. Log entries are retained for 90 days as required by CIP-007. Each log entry averages 250 bytes.

Find: (a) The total number of ESP access points, (b) the total log entries generated per day, (c) the total storage required for 90-day retention, and (d) the storage required if the utility compresses logs at a 10:1 ratio.

Solution:

(a) Total ESP access points: Control centers: $3 \times 4 = 12$ Substations: $45 \times 2 = 90$ Total $= 12 + 90 = 102$ access points

(b) Log entries per day: Entries $= 102 \times 500 = 51{,}000$ entries/day

(c) Storage for 90-day retention: Daily storage $= 51{,}000 \times 250$ bytes $= 12{,}750{,}000$ bytes $= 12.75$ MB/day 90-day storage $= 12.75 \times 90 = 1{,}147.5$ MB ≈ 1.15 GB

(d) With 10:1 compression: Compressed $= 1{,}147.5 / 10 = 114.75$ MB ≈ 115 MB

Chapter 2 — Section 2.1: Analog Signal Processing

Practice problems covering AM radio and FM radio fundamentals.

Problem 2.1.1

Given: An AM transmitter has a carrier power of 25 kW and is modulated by a single-tone audio signal with a modulation index of m = 0.85.

Find: (a) The total transmitted power, (b) the power in each sideband, (c) the efficiency (fraction of power carrying information), and (d) the total power if the modulation index is increased to m = 1.0 (100% modulation).

Solution:

(a) Total power: $P_{total} = P_c(1 + m^2/2) = 25(1 + 0.85^2/2) = 25(1 + 0.3613) = 25 \times 1.3613 = 34.03$ kW

(b) Power in both sidebands: $P_{sb} = P_c \times m^2/2 = 25 \times 0.7225/2 = 25 \times 0.3613 = 9.03$ kW Each sideband: $P_{each} = 9.03/2 = 4.52$ kW

(c) Efficiency: $\eta = P_{sb}/P_{total} = 9.03/34.03 = 0.2654 = 26.5\%$

(d) At m = 1.0: $P_{total} = 25(1 + 1.0/2) = 25 \times 1.5 = 37.5$ kW Efficiency = $(25 \times 0.5)/37.5 = 12.5/37.5 = 33.3\%$

Even at 100% modulation, two-thirds of the power is wasted in the carrier.

Problem 2.1.2

Given: A DSB-SC (Double Sideband Suppressed Carrier) transmitter produces a total power of 10 kW with a 5 kHz audio signal. The carrier frequency is 1 MHz.

Find: (a) The bandwidth of the transmitted signal, (b) the power in each sideband, (c) the efficiency compared to standard AM with m = 1.0 transmitting the same sideband power, and (d) the required receiver complexity difference.

Solution:

(a) Bandwidth: $B = 2 \times f_m = 2 \times 5{,}000 = 10$ kHz

(b) Since the carrier is suppressed, all power is in the sidebands: $P_{\text{each sideband}} = 10/2 = 5$ kW each

(c) For standard AM at m = 1.0 to deliver 10 kW of sideband power: $P_{sb} = P_c \times m^2/2 = P_c \times 0.5$ So $P_c = 10/0.5 = 20$ kW $P_{\text{total,AM}} = 20 + 10 = 30$ kW

DSB-SC needs 10 kW vs. AM needing 30 kW for the same information power. Efficiency improvement = 30/10 = 3x (or 4.77 dB)

(d) DSB-SC requires coherent detection (a local oscillator synchronized to the carrier frequency and phase), unlike AM which uses a simple envelope detector. This is why AM broadcasting uses standard AM despite its inefficiency – receiver simplicity was critical for mass adoption.

Problem 2.1.3

Given: An FM signal has a carrier frequency of 100 MHz and modulates with an audio signal of bandwidth 20 kHz. The maximum frequency deviation is $\Delta f = 50$ kHz.

Find: (a) The modulation index, (b) Carson's rule bandwidth, (c) the number of significant sideband pairs (for $\beta = 2.5$, approximately 5 pairs), and (d) the exact bandwidth from significant sidebands.

Solution:

(a) Modulation index: $\beta = \Delta f/f_m = 50{,}000/20{,}000 = 2.5$

(b) Carson's rule: $B = 2(\Delta f + f_m) = 2(50{,}000 + 20{,}000) = 140$ kHz

(c) For $\beta = 2.5$, the Bessel function table shows approximately 5 significant sideband pairs (J_0 through J_5 with magnitude > 0.01).

(d) Exact bandwidth: $B_{exact} = 2 \times 5 \times f_m = 2 \times 5 \times 20{,}000 = 200$ kHz

This is a wideband FM signal ($\beta > 1$). Carson's rule captures about 98% of the power in 140 kHz, while the full sideband extent is 200 kHz.

Problem 2.1.4

Given: An FM broadcast system uses pre-emphasis with a time constant of $\tau = 75\ \mu$s (North American standard) and de-emphasis at the receiver. The audio bandwidth extends to 15 kHz.

Find: (a) The pre-emphasis corner frequency, (b) the pre-emphasis boost at 15 kHz relative to 1 kHz, (c) the SNR improvement provided by the pre-emphasis/de-emphasis system (approximate), and (d) the effective modulation index if $\Delta f = 75$ kHz at the maximum pre-emphasized frequency.

Solution:

(a) Corner frequency: $f_c = 1/(2\pi\tau) = 1/(2\pi \times 75 \times 10^{-6}) = 2{,}122$ Hz

(b) Pre-emphasis is a first-order high-pass characteristic: Boost at frequency $f = 20 \log_{10}(\sqrt{(1 + (f/f_c)^2)})$ At 15 kHz: boost $= 20 \log_{10}(\sqrt{(1 + (15{,}000/2{,}122)^2)}) = 20 \log_{10}(\sqrt{(1 + 50.0)}) = 20$

$\log_{10}(7.14) = 17.1$ dB At 1 kHz: boost $= 20 \log_{10}(\sqrt{}(1 + (1{,}000/2{,}122)^2)) = 20 \log_{10}(\sqrt{}(1 + 0.222))$ $= 20 \log_{10}(1.105) = 0.87$ dB Relative boost at 15 kHz vs 1 kHz: 17.1 - 0.87 = 16.2 dB

(c) The SNR improvement from pre-emphasis/de-emphasis is approximately: SNR improvement $\approx (f_m/f_c)^2 / 3$ (for high-frequency emphasis) $= (15{,}000/2{,}122)^2 / 3 = 50/3 = 16.7 = 12.2$ dB

(d) $\beta = \Delta f/f_m = 75{,}000/15{,}000 = 5$ (same as standard FM broadcast)

Problem 2.1.5

Given: An SSB (Single Sideband) transmitter operates with a carrier frequency of 14.2 MHz and transmits only the upper sideband. The audio bandwidth is 300 Hz to 3,000 Hz.

Find: (a) The transmitted bandwidth, (b) the frequency range of the transmitted signal, (c) the power savings compared to standard AM (m = 1.0) for a peak envelope power (PEP) of 100 W, and (d) the bandwidth savings compared to standard AM.

Solution:

(a) Bandwidth: $B = f_{max} - f_{min} = 3{,}000 - 300 = 2{,}700$ Hz

(b) USB frequency range: $f_{lower} = f_c + 300 = 14{,}200{,}300$ Hz $f_{upper} = f_c + 3{,}000 = 14{,}203{,}000$ Hz Range: 14.2003 to 14.203 MHz

(c) SSB PEP = 100 W. All power carries information. For AM at m = 1.0 with equivalent sideband power: The SSB signal has power in one sideband = 100 W PEP. AM total power for same PEP sideband: $P_c = 2 \times P_{ssb}/m^2 \times 2$ = needs clarification. More directly: AM at m = 1.0 has $P_{total} = 1.5 \times P_c$, with $P_{sb} = 0.5 \times P_c$ total in both sidebands. For the same information as SSB at 100 W: $P_c = 2 \times 100/0.5 = 400$ W, $P_{total} = 600$ W. Power savings: (600 - 100)/600 = 83.3% power savings

(d) AM bandwidth = $2 \times 3{,}000 = 6{,}000$ Hz. SSB bandwidth = 2,700 Hz. Savings: (6,000 - 2,700)/6,000 = 55% bandwidth savings

Chapter 2 — Section 2.2: Digital Signal Processing

Practice problems covering Fourier analysis, Nyquist sampling, quantization, encoding, DFT/FFT, SQNR, and analog-to-digital conversion.

Problem 2.2.1

Given: A periodic triangular wave has a fundamental frequency of 500 Hz and a peak amplitude of 3 V. The Fourier series of a triangular wave contains only odd harmonics with coefficients $a_n = 8A/(n^2\pi^2)$ for n = 1, 3, 5, …

Find: (a) The amplitude of the 1st, 3rd, and 5th harmonic components, (b) the frequency of each component, and (c) the RMS voltage of the signal truncated to these three harmonics.

Solution:

(a) Harmonic amplitudes (peak): 1st harmonic: $a_1 = 8 \times 3 / (1^2 \times \pi^2) = 24 / 9.8696 = 2.431$ V 3rd harmonic: $a_3 = 8 \times 3 / (3^2 \times \pi^2) = 24 / 88.826 = 0.270$ V 5th harmonic: $a_5 = 8 \times 3 / (5^2 \times \pi^2) = 24 / 246.740 = 0.0973$ V

(b) Frequencies: 1st harmonic: $f_1 = 500$ Hz 3rd harmonic: $f_3 = 3 \times 500 = 1{,}500$ Hz 5th harmonic: $f_5 = 5 \times 500 = 2{,}500$ Hz

(c) RMS voltage of truncated signal: $V_{rms} = \sqrt{}(a_1^2/2 + a_3^2/2 + a_5^2/2) = \sqrt{}(2.431^2/2 + 0.270^2/2 + 0.0973^2/2)$ $V_{rms} = \sqrt{}(2.955 + 0.0365 + 0.00473) = \sqrt{}2.996 = 1.731$ V

The true RMS of a triangular wave is $A/\sqrt{}3 = 3/\sqrt{}3 = 1.732$ V, so the three-harmonic approximation captures $(1.731/1.732)^2 = 99.9\%$ of the total power — triangular waves converge very rapidly due to the $1/n^2$ coefficient rolloff.

Problem 2.2.2

Given: A signal contains three frequency components: 2 kHz, 5 kHz, and 11 kHz. It is sampled at a rate of 20 kHz.

Find: (a) The Nyquist rate for this signal, (b) the Nyquist frequency (folding frequency) of the sampling system, (c) whether aliasing occurs, and (d) the aliased frequency of any component that aliases.

Solution:

(a) Nyquist rate: $f_{\text{Nyquist rate}} = 2 \times f_{max} = 2 \times 11{,}000 = 22$ kHz

(b) Nyquist frequency (folding frequency): $f_N = f_s / 2 = 20{,}000 / 2 = 10$ kHz

(c) The 2 kHz and 5 kHz components are below the Nyquist frequency of 10 kHz, so they are sampled correctly. The 11 kHz component exceeds 10 kHz. Aliasing occurs for the 11 kHz component.

(d) The aliased frequency of the 11 kHz component: $f_{alias} = f_s - f_{signal} = 20{,}000 - 11{,}000 = 9$ kHz

The 11 kHz component appears as a spurious 9 kHz tone in the sampled data, corrupting any legitimate signal content near 9 kHz. An anti-aliasing filter with a cutoff below 10 kHz would prevent this.

Problem 2.2.3

Given: A 12-bit ADC has an input range of 0 to 5 V (unipolar). A full-scale sinusoidal signal is applied.

Find: (a) The number of quantization levels, (b) the quantization step size, (c) the maximum quantization error, (d) the SQNR for a full-scale sinusoid, and (e) the dynamic range in dB.

Solution:

(a) Number of levels: $2^{12} = 4{,}096$ levels

(b) Step size: $\Delta = V_{range} / 2^N = 5 / 4{,}096 = 1.221$ mV

(c) Maximum quantization error: $\pm\Delta/2 = \pm 0.610$ mV $= \pm 0.610$ mV

(d) SQNR $= 6.02N + 1.76 = 6.02 \times 12 + 1.76 = 72.24 + 1.76 = 74.0$ dB

(e) Dynamic range $= 20\log_{10}(2^N) = 20\log_{10}(4{,}096) = 20 \times 3.6124 = 72.2$ dB

The 74 dB SQNR is adequate for high-quality audio recording but falls short of the 96 dB achieved by 16-bit CD-quality systems.

Problem 2.2.4

Given: An 8-bit μ-law companded PCM system ($\mu = 255$) digitizes voice at 8 kHz sampling rate. The input signal range is ±1 V.

Find: (a) The bit rate, (b) the equivalent dynamic range of the companded system (μ-law provides approximately 13-bit equivalent dynamic range with 8 bits), (c) the step size at the smallest signal level (near zero), and (d) the step size at the largest signal level (near ±1 V).

Solution:

(a) Bit rate: $R_b = N \times f_s = 8 \times 8{,}000 = 64$ kbps

(b) The dynamic range of μ-law with $\mu = 255$: $DR = 6.02 \times 8 + 1.76 + 10 \log_{10}(3(\mu)^2 / (\ln(1 + \mu))^2)$ The compression gain: $10 \log_{10}(3 \times 255^2 / (\ln(256))^2) = 10 \log_{10}(195{,}075 / 30.68) = 10 \log_{10}(6{,}358)$ $= 38.0$ dB Total DR $\approx 49.9 + 38.0 - 38.0 \approx 72$ dB (equivalent to approximately 12 uniform bits)

More directly, μ-law companding with 8 bits achieves an SQNR that is approximately constant at 38 dB over a 40 dB input dynamic range, equivalent to roughly 12–13 uniform bits for small signals.

(c) Near zero, the compression curve slope is steepest. The effective step size is: $\Delta_{min} \approx (2V_{max}) / (2^N) \times 1/(\ln(1 + \mu)) = 2 / 256 \times 1/\ln(256) = 0.00781 \times 0.1803 = 1.41$ mV

(d) Near full scale, the compression curve slope is flattest. The effective step size is: $\Delta_{max} \approx (2V_{max}) / (2^N) \times (1 + \mu)/(\ln(1 + \mu)) = 2 / 256 \times 256/5.55 = 0.00781 \times 46.1 = 360$ mV

The ratio $\Delta_{max}/\Delta_{min} = 360/1.41 = 255 = \mu$, confirming that the step size varies by a factor of μ across the input range.

Problem 2.2.5

Given: A 1,024-point FFT is applied to a signal sampled at $f_s = 48$ kHz. The signal contains tones at 3 kHz and 7.5 kHz.

Find: (a) The frequency resolution (bin spacing), (b) the FFT bin indices corresponding to the two tones, (c) the total number of unique frequency bins (considering the Nyquist symmetry), and (d) the maximum unambiguous frequency.

Solution:

(a) Frequency resolution: $\Delta f = f_s / N = 48{,}000 / 1{,}024 = 46.875$ Hz

(b) Bin indices: For 3 kHz: $k_1 = f_1 / \Delta f = 3{,}000 / 46.875 = 64$ For 7.5 kHz: $k_2 = f_2 / \Delta f = 7{,}500 / 46.875 = 160$

(c) Due to the conjugate symmetry of the FFT for real signals, only the first $N/2 + 1$ bins contain unique information: Unique bins $= 1{,}024/2 + 1 = 513$ bins (from DC through Nyquist)

(d) Maximum unambiguous frequency: $f_{max} = f_s / 2 = 48{,}000 / 2 = 24$ kHz (bin 512)

The FFT computation requires $N \log_2(N) = 1{,}024 \times 10 = 10{,}240$ complex multiply-accumulate operations, compared to $N^2 = 1{,}048{,}576$ for a direct DFT — a 102× speedup.

Problem 2.2.6

Given: A biomedical signal (ECG) has a bandwidth of 150 Hz. It is digitized using a 10-bit ADC with a ±5 mV input range. The sampling rate is set to 500 Hz.

Find: (a) Whether the sampling rate meets the Nyquist criterion, (b) the quantization step size in μV, (c) the SQNR, (d) the data rate in bps, and (e) the data volume for a 24-hour Holter monitor recording.

Solution:

(a) Nyquist rate $= 2 \times 150 = 300$ Hz. The sampling rate of 500 Hz exceeds 300 Hz, so the Nyquist criterion is met with a guard band of $500/2 - 150 = 100$ Hz.

(b) Step size: $\Delta = (5 - (-5))$ mV / $2^{10} = 10$ mV / 1,024 = 9.766 μV

(c) SQNR $= 6.02 \times 10 + 1.76 = 60.2 + 1.76 = 61.96$ dB

(d) Data rate: $R = N \times f_s = 10 \times 500 = 5{,}000$ bps = 5 kbps

(e) Data volume for 24 hours: Volume $= 5{,}000 \times 24 \times 3{,}600 = 432{,}000{,}000$ bits = 432 Mbits In bytes: 432,000,000 / 8 = 54,000,000 bytes = 54 MB

With lossless compression (typically 2:1 for ECG), this reduces to approximately 27 MB — easily stored on modern portable devices.

Problem 2.2.7

Given: A real-valued signal is sampled at 10 kHz and processed with a 256-point DFT. The magnitude spectrum shows a peak at bin $k = 40$.

Find: (a) The frequency corresponding to bin 40, (b) the minimum frequency separation between two tones that can be resolved by this DFT, (c) the total observation time (window length), and (d) the bin index where a 2,800 Hz tone would appear.

Solution:

(a) Frequency at bin k: $f = k \times f_s / N = 40 \times 10{,}000 / 256 = 400{,}000 / 256 = 1{,}562.5$ Hz

(b) Minimum resolvable frequency separation equals the bin spacing: $\Delta f = f_s / N = 10{,}000 / 256 = 39.0625$ Hz

Two tones must be separated by at least 39.06 Hz to appear in distinct bins. With windowing (e.g., Hanning window), the effective resolution is approximately $2\Delta f = 78.1$ Hz due to main lobe widening.

(c) Observation time: $T = N / f_s = 256 / 10{,}000 = 25.6$ ms

(d) Bin index for 2,800 Hz: $k = f \times N / f_s = 2{,}800 \times 256 / 10{,}000 = 71.68$

Since this is not an integer, the energy of the 2,800 Hz tone will be spread across adjacent bins (spectral leakage), with the peak at bin 72 and significant energy in bins 71 and 73. Windowing reduces the sidelobe leakage at the cost of wider main lobe width.

Problem 2.2.8

Given: A 14-bit ADC samples a signal at 100 MSPS (mega-samples per second). The ADC has an effective number of bits (ENOB) of 11.5 due to nonlinearities and clock jitter.

Find: (a) The ideal SQNR for 14 bits, (b) the actual SQNR based on the ENOB, (c) the SQNR degradation due to imperfections, (d) the Nyquist bandwidth, and (e) the raw data throughput.

Solution:

(a) Ideal SQNR = $6.02 \times 14 + 1.76 = 84.28 + 1.76 = 86.04$ dB

(b) Actual SQNR = $6.02 \times \text{ENOB} + 1.76 = 6.02 \times 11.5 + 1.76 = 69.23 + 1.76 = 70.99$ dB

(c) Degradation = $86.04 - 70.99 = 15.05$ dB

This 15 dB degradation (equivalent to 2.5 lost bits) is caused by differential nonlinearity (DNL), integral nonlinearity (INL), and aperture jitter of the sampling clock.

(d) Nyquist bandwidth = $f_s / 2 = 100 \times 10^6 / 2 = 50$ MHz

(e) Raw data throughput = $N \times f_s = 14 \times 100 \times 10^6 = 1{,}400$ Mbps = 1.4 Gbps

This high data rate requires careful digital interface design — typically using JESD204B or LVDS serial interfaces.

Problem 2.2.9

Given: A speech signal is band-limited to 3.4 kHz and sampled at 8 kHz. The signal is quantized using a uniform quantizer with 256 levels (8 bits). The signal has a probability density that approximates a Laplacian distribution with a peak-to-RMS ratio (crest factor) of 4:1, meaning the signal uses only 1/4 of the full ADC range on average.

Find: (a) The Nyquist rate, (b) the SQNR for a full-scale sinusoid, (c) the effective SQNR penalty due to the high crest factor, and (d) the actual SQNR for the speech signal.

Solution:

(a) Nyquist rate = $2 \times 3{,}400 = 6{,}800$ Hz. The 8 kHz sampling rate exceeds this by 1,200 Hz.

(b) SQNR for a full-scale sinusoid: SQNR = $6.02 \times 8 + 1.76 = 48.16 + 1.76 = 49.92$ dB

(c) The crest factor penalty occurs because the signal's RMS value is 1/4 of the peak value (-12 dB below full scale). This means the signal power is reduced while the quantization noise power stays the same: Crest factor penalty = $20 \log_{10}(4) = 12.04$ dB

(d) Actual SQNR for speech = $49.92 - 12.04 = 37.88$ dB

This 38 dB SQNR is considered marginal for toll-quality speech (which requires approximately 34–38 dB). This is exactly why telephone systems use μ-law or A-law companding: companding matches the quantization step size to the signal amplitude, recovering most of the 12 dB crest factor penalty and achieving consistent SQNR across the speech dynamic range.

Problem 2.2.10

Given: A radar system samples return pulses using a dual-channel (I and Q) ADC at 50 MSPS per channel with 16-bit resolution. The pulse repetition interval (PRI) is 1 ms, and each pulse return window is 200 μs. The system processes 128 pulses per coherent processing interval (CPI).

Find: (a) The number of samples per pulse return window per channel, (b) the total samples per CPI (both channels), (c) the total data per CPI in bytes, (d) the frequency resolution after a 128-point FFT across pulses (Doppler processing), and (e) the sustained data rate during operation.

Solution:

(a) Samples per pulse return window per channel: $N_{samples} = f_s \times T_{window} = 50 \times 10^6 \times 200 \times 10^{-6}$ = 10,000 samples

(b) Total samples per CPI (both I and Q channels): Total = 2 channels × 10,000 samples × 128 pulses = 2,560,000 samples

(c) Data per CPI: Data = 2,560,000 × 16 bits / 8 bits/byte = 2,560,000 × 2 = 5,120,000 bytes = 5.12 MB

(d) Doppler frequency resolution: The FFT is performed across 128 pulses with PRI = 1 ms, so the coherent integration time is 128 × 1 ms = 128 ms. $\Delta f_{Doppler} = 1 / (N_{pulses} \times PRI) = 1 / (128 \times 0.001) = 7.8125$ Hz

(e) CPI duration = 128 × 1 ms = 128 ms. Sustained data rate = 5,120,000 × 8 / 0.128 = 320 Mbps (combining both channels)

This high data rate requires dedicated FPGA-based processing to perform the range-Doppler FFT in real time.

Chapter 2 — Section 2.3: Digital Modulation

Practice problems covering ASK, FSK, PSK, QAM, constellation diagrams, BER performance, and MIMO spatial multiplexing.

Problem 2.3.1

Given: A 4-ASK (4-level amplitude shift keying) system transmits at a symbol rate of 4,800 symbols/s using raised-cosine pulse shaping with roll-off factor $\alpha = 0.5$. The four amplitude levels are 1, 3, 5, and 7 volts.

Find: (a) The bit rate, (b) the required bandwidth, (c) the bandwidth efficiency, and (d) the average transmitted power (assuming equal probability of each level and a 1 Ω load).

Solution:

(a) Bits per symbol: $\log_2(4) = 2$ bits/symbol Bit rate: $R_b = 2 \times 4{,}800 = 9{,}600$ bps

(b) Bandwidth: $B = R_s(1 + \alpha) = 4{,}800 \times (1 + 0.5) = 4{,}800 \times 1.5 = 7{,}200$ Hz

(c) Bandwidth efficiency: $\eta = R_b / B = 9{,}600 / 7{,}200 = 1.33$ bits/s/Hz

(d) Average power with equal probability (p = 1/4 each): $P_{avg} = (1/4)(1^2 + 3^2 + 5^2 + 7^2) / 1 = (1 + 9 + 25 + 49) / 4 = 84 / 4 = 21$ W

The peak-to-average power ratio is $7^2/21 = 49/21 = 2.33 = 3.68$ dB, which is significantly worse than constant-envelope modulations like FSK.

Problem 2.3.2

Given: A Gaussian Minimum Shift Keying (GMSK) system (as used in GSM) has a bit rate of 270.833 kbps and a bandwidth-time product BT = 0.3. The 99% power containment bandwidth for GMSK is approximately $1.22 \times R_b \times BT/0.3$.

Find: (a) The 99% power containment bandwidth, (b) the spectral efficiency, (c) the number of GSM channels in a 200 kHz allocation, and (d) the comparison to standard MSK bandwidth ($1.5 \times R_b$).

Solution:

(a) 99% power bandwidth for GMSK with BT = 0.3: $B_{99\%} = 1.22 \times R_b = 1.22 \times 270{,}833 = 330{,}416$ Hz $\approx$ 330.4 kHz

However, GSM allocates 200 kHz per channel, which contains approximately 99.8% of the power due to the Gaussian filtering. The occupied bandwidth within the 200 kHz channel is effectively: B_{GSM} = 200 kHz (allocated channel bandwidth)

(b) Spectral efficiency: $\eta = R_b / B = 270{,}833 / 200{,}000 = 1.354$ bits/s/Hz

(c) Each GSM carrier occupies one 200 kHz channel. Using TDMA with 8 time slots per carrier: Users per carrier = 8 (each user transmits during one time slot per frame) Total spectral efficiency = $8 \times 270.833/8 / 200 = 270.833 / 200 = 1.354$ bits/s/Hz per user slot

(d) Standard MSK bandwidth: $B_{MSK} = 1.5 \times R_b = 1.5 \times 270{,}833 = 406{,}250$ Hz GMSK with BT = 0.3 reduces the bandwidth by: $330.4/406.25 = 0.813$, a 18.7% reduction over standard MSK, at the cost of slight intersymbol interference introduced by the Gaussian filter.

Problem 2.3.3

Given: An 8-PSK satellite modem operates at a symbol rate of 5 Msymbols/s with raised-cosine shaping ($\alpha = 0.2$). The required E_b/N_0 for BER = 10^{-5} with 8-PSK is approximately 14.0 dB.

Find: (a) The bit rate, (b) the occupied bandwidth, (c) the bandwidth efficiency, (d) the minimum symbol spacing in the constellation (normalized to unit average power), and (e) the E_b/N_0 penalty compared to QPSK at the same BER.

Solution:

(a) 8-PSK: $\log_2(8) = 3$ bits/symbol $R_b = 3 \times 5 \times 10^6 = 15$ Mbps

(b) Bandwidth: $B = R_s(1 + \alpha) = 5 \times 10^6 \times 1.2 = 6$ MHz

(c) Bandwidth efficiency: $\eta = 15 / 6 = 2.5$ bits/s/Hz

(d) For unit-power 8-PSK, the constellation points lie on a unit circle at angles $2\pi k/8$ for $k = 0, 1, \ldots, 7$. Minimum distance: $d_{min} = 2\sin(\pi/8) = 2 \times 0.3827 = 0.7654$ For comparison, QPSK has $d_{min} = 2\sin(\pi/4) = \sqrt{2} = 1.414$ for unit power.

(e) QPSK requires $E_b/N_0 \approx 9.6$ dB for BER = 10^{-5}. 8-PSK penalty = $14.0 - 9.6 = 4.4$ dB

This 4.4 dB penalty is the price paid for the 50% increase in bandwidth efficiency (2.5 vs. 2.0 bits/s/Hz). For this reason, 16-QAM (which achieves 4 bits/s/Hz at ~13.4 dB) is generally preferred over 8-PSK when amplitude variations are acceptable.

Problem 2.3.4

Given: A 256-QAM cable modem (DOCSIS 3.0) operates in a 6 MHz channel with a symbol rate of 5.361 Msymbols/s. The channel SNR is measured at 34 dB.

Find: (a) The bits per symbol, (b) the raw bit rate, (c) the net data rate after 7% FEC and framing overhead, (d) the required E_b/N_0 for BER = 10^{-8} with 256-QAM, and (e) whether the measured SNR is sufficient.

Solution:

(a) Bits per symbol: $\log_2(256) = 8$ bits/symbol

(b) Raw bit rate: $R_b = 8 \times 5.361 \times 10^6 = 42.88$ Mbps

(c) Net data rate: $R_{net} = 42.88 \times (1 - 0.07) = 42.88 \times 0.93 = 39.88$ Mbps

(d) For 256-QAM, approximate BER: BER $\approx (4/8)(1 - 1/16) \times Q(\sqrt{}(3 \times 8 \times E_b / (255 \times N_0)))$ Setting BER = 10^{-8}: $Q(\sqrt{}(24 E_b / (255 N_0))) \approx 10^{-8} / 0.469 = 2.13 \times 10^{-8}$ $Q^{-1}(2.13 \times 10^{-8}) \approx 5.50$ 24 E_b / (255 N_0) = 30.25 $E_b/N_0 = 30.25 \times 255 / 24 = 321.4 = 10 \log_{10}(321.4) = 25.07$ dB

Converting to SNR: SNR = $E_b/N_0 + 10 \log_{10}(R_b/B) = 25.07 + 10 \log_{10}(42.88/6) = 25.07 + 8.54 = 33.61$ dB

(e) The measured SNR of 34 dB exceeds the required 33.61 dB by 0.39 dB. The link marginally meets the requirement, but the margin is slim. In practice, DOCSIS systems require approximately 33 dB SNR for reliable 256-QAM operation, so 34 dB provides about 1 dB of margin.

Problem 2.3.5

Given: A wireless system operates at E_b/N_0 = 20 dB. Three modulation schemes are available: QPSK (BER = $Q(\sqrt{}(2E_b/N_0))$), 16-QAM (BER $\approx (3/8)Q(\sqrt{}(4E_b/(5N_0)))$), and 64-QAM (BER $\approx (7/24)Q(\sqrt{}(2E_b/(7N_0)))$).

Find: (a) The BER for each scheme, (b) the bandwidth efficiency of each, (c) the throughput at symbol rate R_s = 1 Msymbol/s in a 1.25 MHz channel ($\alpha = 0.25$), and (d) the highest-order modulation meeting BER < 10^{-6}.

Solution:

$E_b/N_0 = 20$ dB $= 10^2 = 100$ (linear).

(a) QPSK: BER = $Q(\sqrt{}(2 \times 100)) = Q(\sqrt{}200) = Q(14.14)$ BER $\approx < 10^{-44}$ (essentially error-free)

16-QAM: BER = $(3/8) \times Q(\sqrt{}(4 \times 100/5)) = 0.375 \times Q(\sqrt{}80) = 0.375 \times Q(8.944)$ BER $\approx 0.375 \times 1.8 \times 10^{-19} = 6.75 \times 10^{-20}$ (essentially error-free)

64-QAM: BER = $(7/24) \times Q(\sqrt{}(2 \times 100/7)) = 0.2917 \times Q(\sqrt{}28.57) = 0.2917 \times Q(5.345)$ BER $\approx 0.2917 \times 4.5 \times 10^{-8} = 1.31 \times 10^{-8}$

(b) Bandwidth efficiencies: QPSK = 2 bits/s/Hz, 16-QAM = 4 bits/s/Hz, 64-QAM = 6 bits/s/Hz

(c) With R_s = 1 Msymbol/s: QPSK: 2×10^6 = 2 Mbps 16-QAM: 4×10^6 = 4 Mbps 64-QAM: 6×10^6 = 6 Mbps

(d) All three schemes meet BER < 10^{-6}. The highest-order modulation meeting the target is 64-QAM with BER = 1.31×10^{-8}, providing 6 Mbps throughput. At 20 dB E_b/N_0, 64-QAM has comfortable margin.

Problem 2.3.6

Given: A BPSK coherent demodulator has a carrier phase error of $\theta = 15°$ due to imperfect carrier recovery. The channel $E_b/N_0 = 10$ dB. For BPSK with phase error, BER = $Q(\sqrt{(2E_b/N_0)} \times \cos(\theta))$.

Find: (a) The ideal BER with no phase error, (b) the BER with the 15° phase error, (c) the effective E_b/N_0 loss due to the phase error, and (d) the maximum tolerable phase error if BER must remain below 10^{-5}.

Solution:

$E_b/N_0 = 10$ dB $= 10$ (linear).

(a) Ideal BER: $Q(\sqrt{(2 \times 10)}) = Q(\sqrt{20}) = Q(4.472) = 3.87 \times 10^{-6}$

(b) BER with $\theta = 15°$: $Q(\sqrt{20} \times \cos(15°)) = Q(4.472 \times 0.9659) = Q(4.320)$ BER $= 7.8 \times 10^{-6}$

(c) The effective E_b/N_0 with phase error $= (E_b/N_0) \times \cos^2(\theta) = 10 \times \cos^2(15°) = 10 \times 0.9330 = 9.330$ In dB: $10 \log_{10}(9.330) = 9.698$ dB Loss $= 10.0 - 9.698 = 0.30$ dB

(d) For BER $= 10^{-5}$, we need $Q(\sqrt{20} \times \cos(\theta)) = 10^{-5}$. $Q^{-1}(10^{-5}) = 4.265$ $\sqrt{20} \times \cos(\theta) = 4.265$ $\cos(\theta) = 4.265/4.472 = 0.9537$ $\theta = \arccos(0.9537) = 17.5°$

Phase errors beyond 17.5° at this E_b/N_0 will violate the BER requirement.

Problem 2.3.7

Given: A 16-QAM transmitter has a constellation with minimum distance $d_{min} = 2$ (arbitrary units) and average symbol energy E_{avg}. The constellation is a 4×4 grid with points at (±1, ±3) and (±3, ±1) and (±1, ±1) and (±3, ±3).

Find: (a) The average symbol energy, (b) the peak symbol energy, (c) the peak-to-average power ratio (PAPR), (d) the number of nearest neighbors for a corner point versus an inner point, and (e) the approximate symbol error rate (SER) at $E_s/N_0 = 20$ dB.

Solution:

(a) The 16 constellation points have coordinates (±1, ±1), (±1, ±3), (±3, ±1), (±3, ±3). Energies: $|\pm1, \pm1|^2 = 2$ (4 points); $|\pm1, \pm3|^2 = 10$ (4 points); $|\pm3, \pm1|^2 = 10$ (4 points); $|\pm3, \pm3|^2 = 18$ (4 points) $E_{avg} = (4{\times}2 + 4{\times}10 + 4{\times}10 + 4{\times}18) / 16 = (8 + 40 + 40 + 72) / 16 = 160/16 = 10$

(b) Peak energy (corner points): $E_{peak} = 3^2 + 3^2 = 18$

(c) PAPR $= E_{peak}/E_{avg} = 18/10 = 1.8 = 2.55$ dB

(d) Corner point (3, 3): 2 nearest neighbors at distance 2 (points (1,3) and (3,1)). Inner point (1, 1): 4 nearest neighbors at distance 2 (points (−1,1), (3,1), (1,−1), (1,3)). Corner points have 2 neighbors; inner points have 4 neighbors.

(e) SER $\approx 4(1 - 1/\sqrt{M}) \times Q(\sqrt{(3E_s/((M-1)N_0))})$ $E_s/N_0 = 20$ dB $= 100$. SER $= 4(1 - 1/4) \times Q(\sqrt{(3 \times 100/15)}) = 3 \times Q(\sqrt{20}) = 3 \times Q(4.472)$ SER $= 3 \times 3.87 \times 10^{-6} = 1.16 \times 10^{-5}$

Problem 2.3.8

Given: A differential QPSK (DQPSK) system encodes data in the phase change between consecutive symbols. The phase changes are 0°, 90°, 180°, and 270° corresponding to bit pairs 00, 01, 11, and 10. The symbol rate is 2 Msymbols/s and the channel E_b/N_0 is 12 dB.

Find: (a) The bit rate, (b) the BER for coherent DQPSK detection (BER $\approx Q(\sqrt{(2E_b/N_0)}) \times 2$), (c) the BER for noncoherent DQPSK detection (BER $\approx e^{-Eb/N_0}$), and (d) the E_b/N_0 penalty relative to coherent QPSK.

Solution:

(a) Bit rate: $R_b = 2 \times R_s = 2 \times 2 \times 10^6 = 4$ Mbps

$E_b/N_0 = 12$ dB $= 15.85$ (linear).

(b) Coherent DQPSK BER: BER $\approx 2 \times Q(\sqrt{(2 \times 15.85)}) = 2 \times Q(\sqrt{31.70}) = 2 \times Q(5.630)$ BER $= 2 \times 9.0 \times 10^{-9} = 1.80 \times 10^{-8}$

(c) Noncoherent DQPSK BER: BER $\approx e^{-Eb/N_0} = e^{-15.85} = 1.32 \times 10^{-7}$

(d) Coherent QPSK at the same E_b/N_0: BER $= Q(\sqrt{(2 \times 15.85)}) = Q(5.630) = 9.0 \times 10^{-9}$

Comparing at BER $= 10^{-5}$: Coherent QPSK requires $E_b/N_0 = 9.6$ dB. Coherent DQPSK requires approximately $9.6 + 0.5 = 10.1$ dB (about 0.5 dB penalty). Noncoherent DQPSK requires approximately $9.6 + 1.8 = 11.4$ dB (about 1.8 dB penalty).

The penalty is the cost of not requiring a coherent carrier reference, which simplifies the receiver significantly.

Problem 2.3.9

Given: A 1024-QAM system (used in Wi-Fi 6E) transmits at a symbol rate of 234 data subcarriers per OFDM symbol in an 80 MHz channel. The OFDM symbol duration is 12.8 μs (useful part) plus 0.8 μs guard interval. The coding rate is 5/6.

Find: (a) The bits per symbol (per subcarrier), (b) the raw data rate per spatial stream, (c) the net data rate after coding, (d) the required SNR per subcarrier for BER $= 10^{-5}$ with 1024-QAM, and (e) the bandwidth efficiency.

Solution:

(a) Bits per symbol: $\log_2(1024) = 10$ bits/subcarrier

(b) Total OFDM symbol time $= 12.8 + 0.8 = 13.6$ μs Raw rate $= 234 \times 10 / 13.6 \times 10^{-6} = 2{,}340 / 13.6 \times 10^{-6} = 172.06$ Mbps

(c) Net data rate = $172.06 \times 5/6 = 143.38$ Mbps

The Wi-Fi 6 (802.11ax) standard specifies MCS 11 (1024-QAM, 5/6) at approximately 143.4 Mbps per spatial stream for 80 MHz with 0.8 μs GI, confirming this calculation.

(d) For 1024-QAM at BER = 10^{-5}: $E_b/N_0 \approx 25.1$ dB (from tables). SNR per subcarrier = $E_b/N_0 + 10 \log_{10}(\text{bits/symbol}) - 10 \log_{10}(\text{code rate})$ SNR = $25.1 + 10 \log_{10}(10) - 10 \log_{10}(5/6) = 25.1 + 10.0 - (-0.79) = 35.9$ dB

(e) Bandwidth efficiency = 143.38 / 80 = 1.79 bits/s/Hz per stream With 2 spatial streams: $2 \times 143.38 / 80 = 3.58$ bits/s/Hz. This demonstrates why 1024-QAM requires extremely clean channels.

Problem 2.3.10

Given: A 2×2 MIMO system uses spatial multiplexing with QPSK modulation on each stream. The channel bandwidth is 20 MHz, the symbol rate is 18 Msymbols/s per stream, and the coding rate is 3/4. The channel matrix H has singular values $\sigma_1 = 1.8$ and $\sigma_2 = 0.6$.

Find: (a) The aggregate data rate, (b) the effective SNR on each spatial stream if the total transmit SNR is 20 dB (equally split between antennas), (c) the BER on each stream, and (d) the average BER across both streams.

Solution:

(a) QPSK: 2 bits/symbol per stream. Rate per stream = $2 \times 18 \times 10^6 \times 3/4 = 27$ Mbps Aggregate rate = $2 \times 27 = 54$ Mbps

(b) Total transmit SNR = 20 dB = 100 (linear). With equal power allocation (SNR = 50 per antenna): SNR on stream 1 = $\sigma_1^2 \times \text{SNR}_{\text{per antenna}} = 1.8^2 \times 50 = 3.24 \times 50 = 162 = 22.1$ dB SNR on stream 2 = $\sigma_2^2 \times \text{SNR}_{\text{per antenna}} = 0.6^2 \times 50 = 0.36 \times 50 = 18 = 12.6$ dB

(c) For QPSK with coding rate 3/4, E_b/N_0 = SNR / (bits per symbol × code rate) = SNR / 1.5: Stream 1: $E_b/N_0 = 162/1.5 = 108 \rightarrow \text{BER} = Q(\sqrt{(2 \times 108)}) = Q(14.7) \approx < 10^{-40}$ (error-free) Stream 2: $E_b/N_0 = 18/1.5 = 12 \rightarrow \text{BER} = Q(\sqrt{(2 \times 12)}) = Q(4.899) = 4.8 \times 10^{-7}$

(d) Average BER = $(\text{BER}_1 + \text{BER}_2) / 2 \approx (0 + 4.8 \times 10^{-7}) / 2 = 2.4 \times 10^{-7}$

The weaker stream ($\sigma_2 = 0.6$) dominates the error performance. Water-filling power allocation would shift more power to stream 1 and less to stream 2, but at this SNR both streams are usable. If σ_2 were too small (rank-deficient channel), the system would fall back to single-stream transmission with transmit diversity.

Chapter 2 — Section 2.4: Channel Coding and Error Correction

Practice problems covering error detection (parity, CRC), forward error correction (Hamming, Reed-Solomon, convolutional codes, LDPC), interleaving, and burst error protection.

Problem 2.4.1

Given: A data link transmits 512-byte frames protected by a CRC-16 checksum. The channel has a random bit error rate of 10^{-6}. The CRC-16 has a Hamming distance of 4, guaranteeing detection of all 1, 2, and 3-bit errors, and its undetected error probability for random patterns longer than 16 bits is approximately 2^{-16}.

Find: (a) The total number of bits per frame, (b) the probability that a frame contains at least one bit error, (c) the probability of an undetected error per frame, and (d) the expected number of frames transmitted between undetected errors at a data rate of 10 Mbps.

Solution:

(a) Bits per frame: $512 \times 8 = 4{,}096$ data bits + 16 CRC bits = 4,112 bits

(b) P(one or more errors) $= 1 - (1 - 10^{-6})^{4112} \approx 4{,}112 \times 10^{-6} = 4.112 \times 10^{-3}$ (about 1 in 243 frames)

(c) The CRC-16 fails to detect with probability $\approx 2^{-16} = 1.526 \times 10^{-5}$ for random error patterns of more than 16 bits. For single, double, and triple bit errors (which dominate at BER $= 10^{-6}$), all are detected (Hamming distance 4).

Probability of 4+ errors in a frame is very small: P(≥4 errors) $\approx C(4112, 4) \times (10^{-6})^4 \approx 1.19 \times 10^{13} \times 10^{-24} = 1.19 \times 10^{-11}$ P(undetected) $\approx$ P(≥4 errors) $\times 2^{-16} = 1.19 \times 10^{-11} \times 1.526 \times 10^{-5} = 1.82 \times 10^{-16}$

(d) Frames per second: $10 \times 10^6 / 4{,}112 = 2{,}432$ frames/s Mean time between undetected errors: $1 / (2{,}432 \times 1.82 \times 10^{-16}) = 2.26 \times 10^{12}$ s $\approx$ 71,610 years

This extraordinary reliability is why CRC combined with retransmission (ARQ) is used in Ethernet and most data links rather than FEC alone.

Problem 2.4.2

Given: A (7, 4) Hamming code is used to protect data on a memory bus. The 4 data bits are $d_3d_2d_1d_0$ = 1011, and the 3 parity bits are computed as: $p_1 = d_3 \oplus d_2 \oplus d_0$, $p_2 = d_3 \oplus d_1 \oplus d_0$, $p_3 = d_2 \oplus d_1 \oplus d_0$.

Find: (a) The encoded 7-bit codeword, (b) the syndrome if bit position 5 (d_1) is flipped during transmission, (c) the corrected codeword, and (d) the code rate and overhead percentage.

Solution:

(a) Computing parity bits for $d_3d_2d_1d_0 = 1011$: $p_1 = 1 \oplus 0 \oplus 1 = 0$ $p_2 = 1 \oplus 1 \oplus 1 = 1$ $p_3 = 0 \oplus 1 \oplus 1 = 0$

Codeword (positions 1-7): $p_1p_2d_3p_3d_2d_1d_0 = 0\ 1\ 1\ 0\ 0\ 1\ 1$

(b) If bit 5 (d_1 at position 6 in standard Hamming) is flipped: received = 0 1 1 0 0 0 1 Syndrome bits: $s_1 = p_1 \oplus d_3 \oplus d_2 \oplus d_0 = 0 \oplus 1 \oplus 0 \oplus 1 = 0$ $s_2 = p_2 \oplus d_3 \oplus d_1 \oplus d_0 = 1 \oplus 1 \oplus 0 \oplus 1 = 1$ $s_3 = p_3 \oplus d_2 \oplus d_1 \oplus d_0 = 0 \oplus 0 \oplus 0 \oplus 1 = 1$ Syndrome = $s_3s_2s_1 = 110_2 = 6$ (decimal), indicating error at position 6

(c) Flip bit 6: corrected = 0 1 1 0 0 1 1 → data = 1011 (matches original)

(d) Code rate: $R_c = k/n = 4/7 = 0.571$ Overhead: $(n - k)/k \times 100 = 3/4 \times 100 = 75\%$

Problem 2.4.3

Given: A Reed-Solomon RS(255, 239) code is used in a fiber optic communication system. The code operates over $GF(2^8)$ with 8-bit symbols. The pre-FEC bit error rate is 2×10^{-4}.

Find: (a) The number of parity symbols, (b) the maximum number of correctable symbol errors, (c) the code rate, (d) the probability of a symbol error at the input, and (e) the post-FEC BER (using the approximation for RS code output BER).

Solution:

(a) Parity symbols: $n - k = 255 - 239 = 16$ symbols (128 bits of overhead per block)

(b) Correctable symbol errors: $t = (n - k)/2 = 16/2 = 8$ symbol errors per block

(c) Code rate: $R_c = 239/255 = 0.937$ (6.3% overhead)

(d) Symbol error probability (each symbol is 8 bits): $P_s = 1 - (1 - \text{BER})^8 = 1 - (1 - 2 \times 10^{-4})^8 \approx 8 \times 2 \times 10^{-4} = 1.6 \times 10^{-3}$

(e) Expected symbol errors per block: $\mu = 255 \times 1.6 \times 10^{-3} = 0.408$ Since $\mu = 0.408$ is far less than $t = 8$, the probability of decoder failure (more than 8 errors) is extremely small.

Using the binomial approximation: $P(\text{block failure}) = \Sigma(j{=}9 \text{ to } 255)\ C(255,j) \times P_s^j \times (1{-}P_s)^{255-j} \approx C(255,9) \times (1.6 \times 10^{-3})^9$

The post-FEC BER is approximately $< 10^{-15}$, well below the 10^{-12} target for optical transport systems.

Problem 2.4.4

Given: A rate-1/2 convolutional code with constraint length K = 3 has generator polynomials $g_1 = [1, 1, 1]$ and $g_2 = [1, 0, 1]$. The input data sequence is 1 0 1 1.

Find: (a) The output coded sequence (encoding each bit through both generators), (b) the total number of output bits, (c) the code rate, and (d) the free distance d_{free} of this code (which is 5 for K=3, rate-1/2).

Solution:

(a) The encoder uses a 3-stage shift register (initialized to 000). Process each input bit:

Input 1: register = 1 0 0 $\text{Output}_1 = 1 \oplus 0 \oplus 0 = 1$, $\text{Output}_2 = 1 \oplus 0 = 1 \to 11$

Input 0: register = 0 1 0 $\text{Output}_1 = 0 \oplus 1 \oplus 0 = 1$, $\text{Output}_2 = 0 \oplus 0 = 0 \to 10$

Input 1: register = 1 0 1 $\text{Output}_1 = 1 \oplus 0 \oplus 1 = 0$, $\text{Output}_2 = 1 \oplus 1 = 0 \to 00$

Input 1: register = 1 1 0 $\text{Output}_1 = 1 \oplus 1 \oplus 0 = 0$, $\text{Output}_2 = 1 \oplus 0 = 1 \to 01$

Flush (input 0): register = 0 1 1 $\text{Output}_1 = 0 \oplus 1 \oplus 1 = 0$, $\text{Output}_2 = 0 \oplus 1 = 1 \to 01$

Flush (input 0): register = 0 0 1 $\text{Output}_1 = 0 \oplus 0 \oplus 1 = 1$, $\text{Output}_2 = 0 \oplus 1 = 1 \to 11$

Complete output sequence: 11 10 00 01 01 11

(b) Total output bits: 4 data bits + 2 flush bits = 6 input bits × 2 = 12 output bits

(c) Code rate = 4 data bits / 12 output bits = 1/3 for finite block. For continuous operation: $R_c = 1/2$ (each input bit produces 2 output bits)

(d) The free distance $d_{free} = 5$ for this code. This means the Viterbi decoder can correct up to $\lfloor (5-1)/2 \rfloor = 2$ errors in any window of decoded bits, and can detect up to 4 errors.

Problem 2.4.5

Given: A block interleaver is used with an RS(15, 11) code that corrects t = 2 symbol errors per codeword. The channel produces burst errors up to 8 symbols long.

Find: (a) The required interleaving depth, (b) the interleaver matrix dimensions, (c) the total interleaver memory in symbols, (d) the latency introduced by the interleaver/de-interleaver pair, and (e) the effective burst error correction capability after de-interleaving.

Solution:

(a) Required depth: $D = \lceil L/t \rceil = \lceil 8/2 \rceil = 4$ rows

With D = 4, a burst of 8 consecutive symbols distributes as 2 per codeword across 4 codewords — exactly at the correction limit. For safety margin, use D = 5 rows (handles bursts up to 10 symbols).

(b) Matrix dimensions: $D \times n = 5 \times 15 = 5$ rows × 15 columns

(c) Memory: $5 \times 15 = 75$ symbols (at the interleaver; another 75 at the de-interleaver)

(d) Latency: The interleaver must fill the complete matrix before reading. With D = 5: One-way latency = $D \times n = 75$ symbols Total round-trip latency = $2 \times 75 = 150$ symbols

If each symbol is 4 bits (GF(2^4) for RS(15,11)) and the symbol rate is 10,000 symbols/s: Latency = 150 / 10,000 = 15 ms

(e) Without interleaving: corrects bursts up to 2 symbols. With D = 5 interleaving: corrects bursts up to $D \times t = 5 \times 2 = 10$ symbols — a 5× improvement.

Problem 2.4.6

Given: A turbo code consists of two rate-1/2 recursive systematic convolutional (RSC) encoders connected in parallel through a pseudo-random interleaver of length 1,024 bits. The overall code rate after puncturing is 1/3 (no puncturing) or 1/2 (with puncturing). The code operates at $E_b/N_0 = 1.0$ dB with 8 iterations of turbo decoding.

Find: (a) The number of parity bits produced by each encoder (for rate 1/3), (b) the overhead percentage for rate 1/2, (c) the Shannon limit for a rate-1/2 code on an AWGN channel, (d) the gap to the Shannon limit at $E_b/N_0 = 1.0$ dB, and (e) the decoding complexity in terms of trellis states visited.

Solution:

(a) For rate 1/3: each RSC encoder produces 1,024 parity bits from 1,024 data bits. Total output = 1,024 (systematic) + 1,024 (parity 1) + 1,024 (parity 2) = 3,072 bits. Each encoder produces 1,024 parity bits.

(b) For rate 1/2 (puncturing every other parity bit from each encoder): Total output = 2,048 bits for 1,024 data bits. Overhead = (2,048 − 1,024)/1,024 × 100 = 100% (doubles the transmitted bits)

(c) Shannon limit for rate-1/2 AWGN: $C = R_c \times \log_2(1 + SNR)$, where $SNR = 2 \times R_c \times E_b/N_0$ At capacity: $1/2 = 1/2 \times \log_2(1 + E_b/N_0)$ $\log_2(1 + E_b/N_0) = 1 \rightarrow E_b/N_0 = 1 = 0$ dB

(d) Gap to Shannon limit = 1.0 − 0 = 1.0 dB At this operating point, turbo codes achieve BER ≈ 10^{-5}, operating just 1 dB from the theoretical limit.

(e) Each RSC encoder has K = 4 (constraint length), giving $2^{K-1} = 8$ trellis states. Per iteration: 2 decoders × 1,024 trellis stages × 8 states = 16,384 state computations. For 8 iterations: 8 × 16,384 = 131,072 state computations per block of 1,024 bits.

Problem 2.4.7

Given: An LDPC code used in DVB-S2 satellite broadcasting has a block length of n = 64,800 bits and a code rate of 3/4 (k = 48,600 data bits). The LDPC code achieves BER = 10^{-7} at $E_b/N_0 = 3.10$ dB on an AWGN channel.

Find: (a) The number of parity bits, (b) the Shannon limit for rate 3/4, (c) the gap to the Shannon limit, (d) the coding gain compared to uncoded QPSK, and (e) the net data rate for a 36 MHz transponder using QPSK at 27.5 Msymbols/s.

Solution:

(a) Parity bits: n − k = 64,800 − 48,600 = 16,200 parity bits per block

(b) Shannon limit for rate 3/4: $3/4 = 3/4 \times \log_2(1 + SNR) \rightarrow \log_2(1 + SNR) = 1 \rightarrow SNR = 1$ $E_b/N_0 = SNR / (2R_c) = 1 / (2 \times 3/4) = 2/3$ But more precisely: $C/B = \log_2(1 + 2R_c \times E_b/N_0) = R_c \log_2(1 + 1.5 \times E_b/N_0) = 0.75$ $1.5 \times E_b/N_0 = 2^{0.75} - 1 = 1.6818 - 1 = 0.6818$ $E_b/N_0 = 0.4545 = -3.42$ dB...

Correction: $E_b/N_{0min} = (2^R - 1) / R = (2^{0.75} - 1) / 0.75 = 0.6818 / 0.75 = 0.909 = -0.41$ dB

(c) Gap to Shannon limit = $3.10 - (-0.41) = 3.51$ dB

More commonly referenced: for rate 3/4 on AWGN, the Shannon limit for coded BER = 10^{-7} is approximately 1.89 dB, giving a gap of $3.10 - 1.89 = 1.21$ dB — remarkably close to the theoretical limit.

(d) Uncoded QPSK requires $E_b/N_0 \approx 11.3$ dB for BER = 10^{-7}. Coding gain = $11.3 - 3.10 = 8.2$ dB

(e) QPSK: 2 bits/symbol. Raw rate = $2 \times 27.5 = 55$ Mbps. Net rate = $55 \times 3/4 = 41.25$ Mbps

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Problem 2.4.8

Given: A wireless system uses a convolutional interleaver with parameters (I, M) = (12, 17), where I is the number of delay branches and M is the delay increment per branch (in symbols). The FEC code is a (204, 188) Reed-Solomon code over GF(2^8).

Find: (a) The maximum delay in the interleaver in symbols, (b) the total interleaver memory, (c) the maximum burst length (in symbols) that can be corrected after de-interleaving, (d) the total system latency, and (e) the effective burst correction capability in bytes.

Solution:

(a) Maximum delay: The deepest branch has delay = $(I - 1) \times M = 11 \times 17 = 187$ symbols

(b) Total memory = $I \times (I - 1) \times M / 2 = 12 \times 11 \times 17 / 2 = 1{,}122$ symbols

(c) The RS(204, 188) code corrects $t = (204 - 188)/2 = 8$ symbol errors per codeword. After de-interleaving, a burst of B consecutive symbols maps to at most $\lceil B/I \rceil$ errors per codeword. Maximum correctable burst: $B_{max} = I \times t = 12 \times 8 = 96$ symbols

(d) The convolutional interleaver introduces a one-way delay equal to the maximum branch delay: Interleaver delay = 187 symbols; De-interleaver delay = 187 symbols. Total latency = $2 \times 187 = 374$ symbols

At a symbol rate of 6.75 Msymbols/s (DVB-T standard), this is $374 / 6.75 \times 10^6 = 55.4$ µs.

(e) Since each RS symbol is 8 bits = 1 byte: Burst correction = $96 \times 1 = 96$ bytes (768 bits)

This is the burst correction capability used in DVB-T digital television, protecting against impulse noise and multipath fading in the terrestrial channel.

Problem 2.4.9

Given: A deep-space probe communicates with Earth using a rate-1/6 convolutional code with K = 15 and Viterbi decoding, achieving a coding gain of 9.5 dB at BER = 10^{-5}. The raw channel capacity is C = 500 bps due to the extreme distance (2 AU). The transmitter power is 20 W, antenna gain is 40 dBi, and the link operates at 8.4 GHz (X-band).

Find: (a) The effective data rate after coding, (b) the uncoded E_b/N_0 required for BER = 10^{-5} with BPSK, (c) the coded E_b/N_0 required, (d) the margin gained by using the code, and (e) the minimum link SNR.

Solution:

(a) Effective data rate: $R_{data} = R_{channel} \times R_c = 500 \times 1/6 = 83.3$ bps

Wait — the code rate reduces the data rate. The 500 bps is the coded symbol rate on the channel. Actual data rate = $500 \times (1/6) = 83.3$ bps. However, if 500 bps is the channel's information capacity: $R_{data} = 500 / 6 \times 6 = 500$ bps at the channel rate → 83.3 bps of actual user data.

(b) Uncoded BPSK at BER = 10^{-5}: $E_b/N_0 = 9.6$ dB

(c) Coded $E_b/N_0 = 9.6 - 9.5 = 0.1$ dB (just barely above 0 dB)

(d) The code allows the system to operate at $E_b/N_0 = 0.1$ dB instead of 9.6 dB, reducing the required transmitter power by a factor of $10^{9.5/10} = 8.91$, or equivalently: Power savings = 9.5 dB → the 20 W transmitter with coding is equivalent to 178 W without coding.

(e) Minimum link SNR for the coded system: $\text{SNR} = E_b/N_0 + 10\log_{10}(R_{data}/B)$ With $B \approx 500$ Hz (matched to channel rate): $\text{SNR} = 0.1 + 10\log_{10}(83.3/500) = 0.1 + (-7.78) = -7.68$ dB

The system operates at a negative SNR — the signal is buried below the noise floor, but the powerful code extracts it reliably. This is the hallmark of deep-space communication.

Problem 2.4.10

Given: A 5G NR system uses an LDPC code for the data channel (PDSCH) and a Polar code for the control channel (PDCCH). The data channel has a transport block size of 8,424 bits encoded to 16,848 bits (rate 1/2). The control channel carries 48 bits of DCI encoded to 432 bits. Both codes use successive cancellation list (SCL) decoding with list size L = 8 for Polar and min-sum decoding with 10 iterations for LDPC.

Find: (a) The LDPC code rate, (b) the Polar code rate, (c) the overhead percentage for each, (d) the approximate E_b/N_0 performance advantage of the Polar code for short blocks compared to LDPC, and (e) why different codes are used for data versus control.

Solution:

(a) LDPC code rate: $R_{LDPC} = 8{,}424 / 16{,}848 = 1/2 = 0.500$

(b) Polar code rate: $R_{Polar} = 48 / 432 = 1/9 = 0.111$

(c) LDPC overhead: $(16{,}848 - 8{,}424) / 8{,}424 \times 100 = 100\%$ Polar overhead: $(432 - 48) / 48 \times 100 = 800\%$

(d) For short block lengths (< 200 bits), Polar codes with SCL decoding outperform LDPC by approximately 0.5–1.5 dB at BER = 10^{-3}. At long block lengths (> 1,000 bits), LDPC codes perform as well or better. The crossover point is typically around 200–400 bits.

(e) Different codes are optimal for different block sizes:

- The control channel carries small payloads (20–140 bits) where Polar codes excel due to their provable capacity-achieving property for binary-input memoryless channels and superior short-block performance.
- The data channel carries large payloads (hundreds to thousands of bits) where LDPC codes achieve near-Shannon-limit performance with efficient parallelizable hardware decoders.
- LDPC decoding is highly parallelizable (message-passing on a sparse graph), enabling high-throughput implementation for multi-Gbps data rates.
- Polar decoding (SCL) is inherently sequential but excellent for the low-rate, latency-sensitive control channel.

Chapter 2 — Section 2.5: Multiplexing

Practice problems covering frequency division multiplexing (FDM), OFDM, time division multiplexing (TDM), SONET/SDH, code division multiple access (CDMA), OFDMA, and SC-FDMA.

Problem 2.5.1

Given: A traditional analog FDM telephone system combines 12 voice channels into a Group. Each voice channel has a 4 kHz bandwidth (300 Hz to 3,400 Hz of voice plus guard bands), and channels are stacked starting at 60 kHz using SSB modulation with carrier suppression.

Find: (a) The total bandwidth of one Group, (b) the frequency range occupied, (c) the number of Groups in a Supergroup (5 Groups), (d) the Supergroup bandwidth, and (e) the total voice channels in a Mastergroup (10 Supergroups).

Solution:

(a) Group bandwidth: 12 channels $\times$ 4 kHz = 48 kHz

(b) Frequency range: The first channel occupies 60–64 kHz, the last channel occupies 104–108 kHz. Range = 60 kHz to 108 kHz

(c) Groups per Supergroup: 5 Groups = $5 \times 12 = 60$ voice channels

(d) Supergroup bandwidth: 5×48 kHz = 240 kHz, plus guard bands between groups. Standard Supergroup occupies 312–552 kHz = 240 kHz (60 channels)

(e) Mastergroup = 10 Supergroups: Total channels = $10 \times 60 = 600$ voice channels Bandwidth $\approx$ $10 \times 240 = 2{,}400$ kHz = 2.4 MHz

This hierarchical FDM structure was the backbone of the analog telephone network for decades, carrying thousands of simultaneous calls over coaxial cables and microwave radio links.

Problem 2.5.2

Given: An OFDM system for a broadband wireless access network uses 2,048 subcarriers (2K FFT) in an 8 MHz channel. Of the 2,048 subcarriers, 1,705 carry data, 193 are pilot/reference subcarriers, and 150 are null guard subcarriers. The guard interval (cyclic prefix) is 1/4 of the useful symbol duration. Each data subcarrier uses 16-QAM with a 2/3 FEC code rate.

Find: (a) The subcarrier spacing, (b) the useful symbol duration, (c) the total symbol duration (including guard interval), (d) the net data rate, and (e) the spectral efficiency.

Solution:

(a) Subcarrier spacing: $\Delta f = B / N = 8 \times 10^6 / 2{,}048 = 3{,}906.25$ Hz ≈ 3.906 kHz

Actually, for DVB-T style systems, $\Delta f = 8$ MHz $\times (7/8) / 2{,}048 = 7 \times 10^6 / 2{,}048 = 3{,}418$ Hz (using 7 MHz usable bandwidth in 8 MHz channel). Let us use the simpler relationship: $\Delta f = 1/T_u$ where T_u is the useful symbol duration.

(b) Useful symbol duration: $T_u = N / B_{usable}$ For 2K mode in 8 MHz: $T_u = 2{,}048 / (8 \times 10^6 \times 7/8) = 2{,}048 / 7{,}000{,}000 = 224\ \mu s$ (This gives $\Delta f = 1/224\ \mu s = 4{,}464$ Hz)

More standard: $T_u = 1/\Delta f$. With $\Delta f = 8 \times 10^6 / 2{,}048$: $T_u = 2{,}048 / (8 \times 10^6) = 256\ \mu s$

(c) Total symbol time with 1/4 guard interval: $T_{total} = T_u + T_u/4 = 256 \times (1 + 1/4) = 256 \times 1.25 = 320\ \mu s$

(d) Net data rate: Bits per OFDM symbol = 1,705 data subcarriers $\times$ 4 bits (16-QAM) $\times$ 2/3 (FEC) $= 1{,}705 \times 2.667 = 4{,}547$ bits $R_{net} = 4{,}547 / 320 \times 10^{-6} = 14.21$ Mbps

(e) Spectral efficiency: $\eta = 14.21 / 8 = 1.78$ bits/s/Hz

Problem 2.5.3

Given: A DS-1 (T1) frame consists of 24 DS-0 channels multiplexed using TDM. Each DS-0 carries one 8-bit voice sample per frame. A framing bit is added to each frame. The frame rate is 8,000 frames per second. Four T1 circuits are multiplexed into a DS-2.

Find: (a) The number of bits per T1 frame, (b) the T1 line rate, (c) the payload capacity (voice data only), (d) the DS-2 line rate (4 $\times$ T1 plus overhead), and (e) the number of T1s in a DS-3 (28 T1s) and the DS-3 line rate.

Solution:

(a) Bits per T1 frame: 24 channels $\times$ 8 bits + 1 framing bit $= 192 + 1 = 193$ bits

(b) T1 line rate: $193 \times 8{,}000 = 1{,}544{,}000$ bps = 1.544 Mbps

(c) Payload capacity: $24 \times 8 \times 8{,}000 = 1{,}536{,}000$ bps = 1.536 Mbps Overhead $= 1{,}544 - 1{,}536 = 8$ kbps (the framing channel)

(d) DS-2 rate: 4×1.544 Mbps = 6.176 Mbps (payload) + stuffing and overhead bits. Actual DS-2 line rate = 6.312 Mbps Overhead $= 6.312 - 6.176 = 0.136$ Mbps (bit stuffing for clock synchronization)

(e) DS-3: $28 \times$ T1 $= 28 \times 1.544 = 43.232$ Mbps (payload) + overhead. DS-3 line rate = 44.736 Mbps This carries $28 \times 24 = 672$ voice channels

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Problem 2.5.4

Given: An OC-192 SONET link operates at 9.953 Gbps. The link carries concatenated STS-192c payload for a high-bandwidth Ethernet connection. The SONET overhead is approximately 3.3% (27 columns out of 810 per STS-1 row, but for concatenated payloads, path overhead is shared).

Find: (a) The number of STS-1 signals multiplexed, (b) the STS-1 line rate (for verification), (c) the approximate payload bandwidth, (d) the maximum Ethernet payload that can be carried (after mapping overhead), and (e) the number of DS-3 signals that could alternatively be carried.

Solution:

(a) STS-1 count: OC-192 = 192 STS-1 signals

(b) STS-1 line rate: 9,953.28 / 192 = 51.84 Mbps (confirmed)

(c) Approximate payload: $9{,}953.28 \times (1 - 0.033) = 9{,}953.28 \times 0.967 = 9{,}624.8$ Mbps

For STS-192c: payload = 192×50.112 Mbps (SPE per STS-1) = 9,621.5 Mbps Actual usable payload ≈ 9.58 Gbps (after path overhead and pointer adjustments)

(d) An OC-192 carries one 10 Gigabit Ethernet (10GbE) WAN PHY signal. The 10GbE WAN PHY rate is 9.953 Gbps (matching OC-192 exactly). Maximum Ethernet payload = $9{,}953 \times (1 -$ overhead) ≈ 9.29 Gbps after 64B/66B encoding and SONET mapping overhead.

(e) DS-3 capacity: Each STS-1 carries 1 DS-3 (or 28 DS-1s). OC-192 carries: $192 \times 1 = 192$ DS-3 signals (or $192 \times 28 = 5{,}376$ DS-1 signals = 129,024 voice channels)

Problem 2.5.5

Given: A CDMA system (IS-95) operates in a 1.25 MHz band. Each user transmits at 9.6 kbps, and the Walsh code length is 64 chips per bit (processing gain = 64). The system uses a 3-sector cell with voice activity factor v = 0.4 and frequency reuse factor f = 0.6 (indicating 60% of interference comes from other cells). The required E_b/N_0 is 6 dB.

Find: (a) The chip rate, (b) the processing gain in dB, (c) the number of users per sector, (d) the total users per cell, and (e) the sector capacity improvement if soft handoff and power control reduce the required E_b/N_0 by 2 dB.

Solution:

(a) Chip rate: R_c = processing gain × R_b = $64 \times 9{,}600 = 614{,}400$ chips/s However, IS-95 uses R_c = 1.2288 Mchips/s (128 chips/bit for rate 9.6 kbps, or the actual spread bandwidth is 1.25 MHz) Processing gain: $G_p = W/R = 1{,}228{,}800 / 9{,}600 = 128$

(b) Processing gain in dB: $10 \log_{10}(128) = 21.07$ dB

(c) Users per sector: $N = G_p / ((E_b/N_0) \times (1 + f) \times v) + 1$ $E_b/N_0 = 10^{6/10} = 3.981$ $N = 128 / (3.981 \times (1 + 0.6) \times 0.4) + 1 = 128 / (3.981 \times 1.6 \times 0.4) + 1$ $N = 128 / 2.548 + 1 = 50.2 + 1 \approx 51$ users per sector

Wait, the formula accounts for other-cell interference: $N = G_p / ((E_b/N_0) \times v \times (1 + f))$ $N = 128 / (3.981 \times 0.4 \times 1.6) = 128 / 2.548 = 50$ users per sector

(d) Total per cell (3 sectors): $3 \times 50 = 150$ users per cell

(e) With 2 dB reduction: $E_b/N_0 = 4$ dB $= 2.512$ $N_{new} = 128 / (2.512 \times 0.4 \times 1.6) = 128 / 1.608 = 80$ users per sector Improvement: $80/50 = 1.6\times$ (60% more users)

Problem 2.5.6

Given: A 5G NR OFDMA system uses a 100 MHz channel at 3.5 GHz (sub-6 GHz band) with 30 kHz subcarrier spacing. The channel has 273 resource blocks (RBs) of 12 subcarriers each. A slot consists of 14 OFDM symbols at duration 0.5 ms. Two users are allocated: User A gets 200 RBs with 256-QAM (8 bits) and 0.93 code rate; User B gets 73 RBs with QPSK (2 bits) and 0.5 code rate.

Find: (a) The total number of data subcarriers, (b) the OFDM symbol duration (useful + CP), (c) the data rate for each user, and (d) the total cell throughput per slot.

Solution:

(a) Total subcarriers: $273 \times 12 = 3{,}276$ subcarriers User A: $200 \times 12 = 2{,}400$ subcarriers User B: $73 \times 12 = 876$ subcarriers

(b) Slot = 14 symbols in 0.5 ms: Symbol duration = 500 μs / 14 = 35.71 μs With 30 kHz spacing: $T_u = 1/30{,}000 = 33.33$ μs useful + 2.38 μs CP = 35.71 μs ✓

(c) Data rates (assuming 12 data symbols per slot, 2 for control/reference): User A: $2{,}400 \times 8 \times 0.93 \times 12 / 0.5 \times 10^{-3} = 2{,}400 \times 7.44 \times 12 / 5 \times 10^{-4} = 214{,}272 / 5 \times 10^{-4} = 428.5$ Mbps

User B: $876 \times 2 \times 0.5 \times 12 / 0.5 \times 10^{-3} = 876 \times 1.0 \times 12 / 5 \times 10^{-4} = 10{,}512 / 5 \times 10^{-4} = 21.0$ Mbps

(d) Total throughput: $428.5 + 21.0 = 449.5$ Mbps

This represents a single-layer (single antenna) throughput. With 4×4 MIMO, the peak would approach $4 \times 449.5 \approx 1.8$ Gbps, which aligns with typical 5G sub-6 GHz peak rates.

Problem 2.5.7

Given: An LTE uplink uses SC-FDMA with 15 kHz subcarrier spacing. A user is allocated 25 resource blocks (300 subcarriers) for a 5 MHz channel allocation. The modulation is 16-QAM with 1/2 coding rate. The subframe duration is 1 ms with 14 OFDM symbols (2 used for reference signals, 12 for data).

Find: (a) The occupied bandwidth, (b) the DFT size used for SC-FDMA precoding, (c) the data rate, (d) the PAPR advantage compared to OFDMA, and (e) the power amplifier efficiency improvement.

Solution:

(a) Occupied bandwidth: 25 RBs $\times$ 12 subcarriers $\times$ 15 kHz $= 300 \times 15{,}000 = 4.5$ MHz (within the 5 MHz allocation)

(b) The DFT precoding block size equals the number of allocated subcarriers: DFT size = 300 points (one DFT per OFDM symbol)

The 300-point DFT converts the time-domain signal to frequency-domain, which is then mapped to the allocated subcarriers before the standard IFFT. This converts the multicarrier OFDM signal into a single-carrier-like waveform.

(c) Data rate: Bits per subframe = 300 subcarriers × 4 bits (16-QAM) × 1/2 (code rate) × 12 data symbols = 300 × 2 × 12 = 7,200 bits R = 7,200 / 1 × 10^{-3} = 7.2 Mbps

(d) OFDMA PAPR for 300 subcarriers: approximately 10–12 dB (for 99.9% CCDF) SC-FDMA PAPR: approximately 6–8 dB (for 99.9% CCDF) PAPR reduction: 3–4 dB

(e) A 3 dB PAPR reduction allows the power amplifier to operate 3 dB closer to saturation: PA efficiency at 6 dB backoff ≈ 15–20% PA efficiency at 3 dB backoff ≈ 30–35% Improvement: approximately 2× better PA efficiency, which directly translates to longer battery life for mobile devices — a critical advantage for the uplink.

Problem 2.5.8

Given: A Wavelength Division Multiplexing (WDM) fiber optic system uses Dense WDM (DWDM) in the C-band (1530–1565 nm) with 50 GHz channel spacing. Each channel carries a 100 Gbps signal using DP-QPSK (dual-polarization QPSK) modulation.

Find: (a) The number of channels that fit in the C-band, (b) the total system capacity, (c) the channel bandwidth in nm at 1550 nm center wavelength, (d) the spectral efficiency per channel, and (e) the spectral efficiency of the full system.

Solution:

(a) C-band range in frequency: $f_{start} = c/\lambda_{end} = 3 \times 10^8$ / 1,565 × 10^{-9} = 191.69 THz $f_{end} = 3 \times 10^8$ / 1,530 × 10^{-9} = 196.08 THz Total bandwidth = 196.08 − 191.69 = 4.39 THz Channels = 4,390 GHz / 50 GHz = 87 channels (88 with edge channels)

Using the ITU grid: typically 80 channels in the C-band with 50 GHz spacing.

(b) Total capacity: 80 × 100 = 8,000 Gbps = 8 Tbps

(c) Channel spacing in wavelength at 1,550 nm: $\Delta\lambda = \lambda^2 \times \Delta f / c = (1{,}550 \times 10^{-9})^2 \times 50 \times 10^9 / (3 \times 10^8) = 2.4025 \times 10^{-12} \times 50 \times 10^9 / 3 \times 10^8$ $\Delta\lambda = 1.2013 \times 10^{-1} \times 10^{-9}$ = 0.40 nm

(d) Per channel: DP-QPSK carries 4 bits/symbol (2 polarizations × 2 bits/symbol). Symbol rate = 100 Gbps / 4 = 25 Gbaud. In 50 GHz spacing: η = 100 / 50 = 2 bits/s/Hz

(e) Full system spectral efficiency: Total capacity / Total bandwidth = 8,000 / 4,000 = 2 bits/s/Hz

Modern systems using DP-16QAM at 32 Gbaud achieve 400 Gbps per channel, pushing spectral efficiency to 8 bits/s/Hz per channel.

Problem 2.5.9

Given: An OFDMA system (similar to Wi-Fi 6, 802.11ax) operates in a 20 MHz channel with 256 subcarriers (9.765625 kHz spacing). The resource allocation divides the channel into resource units (RUs):

9 RUs of 26 subcarriers each (234 data subcarriers total). Three stations are scheduled simultaneously: STA1 gets 4 RUs (106 subcarriers) with 256-QAM, 3/4 coding; STA2 gets 3 RUs (78 subcarriers) with 64-QAM, 2/3 coding; STA3 gets 2 RUs (52 subcarriers) with QPSK, 1/2 coding.

Find: (a) The OFDM symbol duration (with 0.8 μs GI), (b) the data rate per station, (c) the total aggregate throughput, and (d) the per-station spectral efficiency.

Solution:

(a) Useful symbol duration: $T_u = 1/\Delta f = 1/9{,}765.625 = 102.4$ μs Wait — 802.11ax uses 78.125 kHz spacing for 20 MHz with 256 subcarriers: $\Delta f = 78.125$ kHz, $T_u = 1/78{,}125 = 12.8$ μs Total: $T_{total} = 12.8 + 0.8 = 13.6$ μs

(b) Data rates: STA1: 106 subcarriers × 8 bits × 3/4 / 13.6 μs = $106 \times 6 / 13.6 \times 10^{-6} = 636 / 13.6 \times 10^{-6} = 46.76$ Mbps STA2: $78 \times 6 \times 2/3 / 13.6 \times 10^{-6} = 78 \times 4 / 13.6 \times 10^{-6} = 312 / 13.6 \times 10^{-6} = 22.94$ Mbps STA3: $52 \times 2 \times 1/2 / 13.6 \times 10^{-6} = 52 \times 1 / 13.6 \times 10^{-6} = 52 / 13.6 \times 10^{-6} = 3.82$ Mbps

(c) Total throughput: $46.76 + 22.94 + 3.82 = 73.52$ Mbps

(d) Per-station spectral efficiency: STA1: $46.76 / (106 \times 78.125 \times 10^{-3}) = 46.76 / 8.28 = 5.65$ bits/s/Hz STA2: $22.94 / (78 \times 78.125 \times 10^{-3}) = 22.94 / 6.09 = 3.77$ bits/s/Hz STA3: $3.82 / (52 \times 78.125 \times 10^{-3}) = 3.82 / 4.06 = 0.94$ bits/s/Hz System: $73.52 / 20 = 3.68$ bits/s/Hz

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Problem 2.5.10

Given: A statistical TDM (stat-mux) system aggregates 20 bursty data sources, each with a peak rate of 2 Mbps and an average utilization of 30%. The shared output link has a capacity of 15 Mbps. Overflow traffic is buffered in a 500 KB queue.

Find: (a) The total peak aggregate input rate, (b) the average aggregate input rate, (c) the oversubscription ratio (peak input / link capacity), (d) the probability that instantaneous demand exceeds link capacity (using the Gaussian approximation), and (e) the maximum burst duration before the buffer overflows.

Solution:

(a) Total peak rate: $20 \times 2 = 40$ Mbps

(b) Average aggregate rate: $20 \times 2 \times 0.30 = 12$ Mbps

(c) Oversubscription ratio: 40 / 15 = 2.67:1

(d) Using the Gaussian approximation for the sum of 20 independent sources: Each source: mean $= 0.3 \times 2 = 0.6$ Mbps, variance $= 0.3 \times 0.7 \times 2^2 = 0.84$ Mbps^2 Aggregate: $\mu = 20 \times 0.6 = 12$ Mbps, $\sigma^2 = 20 \times 0.84 = 16.8$, $\sigma = 4.099$ Mbps

P(demand > 15) = $Q((15 - 12) / 4.099) = Q(0.732) = 0.232$ (23.2%)

This is the probability that instantaneous demand exceeds the link rate, requiring buffering.

(e) Maximum burst scenario: all 20 sources transmit at peak simultaneously (40 Mbps). Buffer drain rate during overflow: input − output = 40 − 15 = 25 Mbps. Buffer capacity: 500 KB = 500

$\times$ 8 = 4,000 kbits = 4 Mbits. Maximum burst before overflow: t = 4,000 / 25,000 = 0.16 s = 160 ms

In practice, the statistical multiplexing gain is substantial: 20 sources at 2 Mbps peak are served by a 15 Mbps link (62.5% savings) because the probability that all sources transmit simultaneously is negligibly small ($(0.3)^{20} = 3.5 \times 10^{-11}$).

Chapter 2 — Section 2.6: Information Theory

Practice problems covering Shannon's channel capacity, entropy, source coding, Huffman coding, rate-distortion theory, and capacity limits of various channels.

Problem 2.6.1

Given: A digital subscriber line (DSL) uses a frequency band from 25 kHz to 1.1 MHz. The average SNR across the band is 40 dB, but the actual SNR varies: 50 dB from 25–300 kHz, 40 dB from 300–600 kHz, 30 dB from 600–900 kHz, and 20 dB from 900 kHz–1.1 MHz.

Find: (a) The Shannon capacity assuming a flat 40 dB SNR across the entire band, (b) the capacity of each sub-band, (c) the total capacity using the sub-band calculation, and (d) the percentage difference between the flat and sub-band estimates.

Solution:

(a) Flat SNR model: $B_{total} = 1{,}100 - 25 = 1{,}075$ kHz SNR = 40 dB = 10,000 $C_{flat} = 1{,}075{,}000 \times \log_2(1 + 10{,}000) = 1{,}075{,}000 \times 13.29 = 14.29$ Mbps

(b) Sub-band capacities: Band 1 (25–300 kHz, B = 275 kHz, SNR = 50 dB = 100,000): $C_1 = 275{,}000 \times \log_2(100{,}001) = 275{,}000 \times 16.61 = 4.568$ Mbps

Band 2 (300–600 kHz, B = 300 kHz, SNR = 40 dB = 10,000): $C_2 = 300{,}000 \times \log_2(10{,}001) = 300{,}000 \times 13.29 = 3.987$ Mbps

Band 3 (600–900 kHz, B = 300 kHz, SNR = 30 dB = 1,000): $C_3 = 300{,}000 \times \log_2(1{,}001) = 300{,}000 \times 9.968 = 2.990$ Mbps

Band 4 (900–1,100 kHz, B = 200 kHz, SNR = 20 dB = 100): $C_4 = 200{,}000 \times \log_2(101) = 200{,}000 \times 6.658 = 1.332$ Mbps

(c) Total sub-band capacity: $4.568 + 3.987 + 2.990 + 1.332 = 12.877$ Mbps

(d) Difference: $(14.29 - 12.88) / 14.29 \times 100 = 9.9\%$

The flat model overestimates capacity by about 10% because it ignores the degraded SNR at higher frequencies. Real DSL systems use DMT (Discrete Multi-Tone, a form of OFDM) to allocate bits per subcarrier based on the actual SNR at each frequency — a practical implementation of water-filling that approaches the sub-band capacity.

Problem 2.6.2

Given: A discrete memoryless source emits four symbols A, B, C, D with probabilities P(A) = 0.5, P(B) = 0.25, P(C) = 0.125, P(D) = 0.125.

Find: (a) The entropy of the source, (b) the Huffman code for these symbols, (c) the average code length, (d) the coding efficiency ($\eta = H/L_{avg}$), and (e) the maximum compression ratio relative to a fixed 2-bit code.

Solution:

(a) Entropy: $H(X) = -0.5\log_2(0.5) - 0.25\log_2(0.25) - 0.125\log_2(0.125) - 0.125\log_2(0.125)$ $H(X) = 0.5 \times 1 + 0.25 \times 2 + 0.125 \times 3 + 0.125 \times 3$ $H(X) = 0.5 + 0.5 + 0.375 + 0.375 = 1.75$ bits/symbol

(b) Huffman code construction: Combine D(0.125) and C(0.125) → CD(0.25) Combine CD(0.25) and B(0.25) → BCD(0.5) Combine BCD(0.5) and A(0.5) → root(1.0)

Assignments: A = 0, B = 10, C = 110, D = 111

Symbol	Probability	Code	Length
A	0.500	0	1
B	0.250	10	2
C	0.125	110	3
D	0.125	111	3

(c) Average code length: $L_{avg} = 0.5 \times 1 + 0.25 \times 2 + 0.125 \times 3 + 0.125 \times 3 = 0.5 + 0.5 + 0.375 + 0.375 = 1.75$ bits/symbol

(d) Efficiency: $\eta = H(X) / L_{avg} = 1.75 / 1.75 = 100\%$

This is an optimal Huffman code because the probabilities are all negative powers of 2 (dyadic distribution).

(e) Compression ratio: 2.0 / 1.75 = 1.143:1 (14.3% compression)

Problem 2.6.3

Given: A binary symmetric channel (BSC) has a crossover probability p = 0.01 (1% error rate). The channel is used with a transmission rate of 10 Mbps.

Find: (a) The channel capacity in bits per channel use, (b) the capacity in Mbps, (c) the maximum achievable reliable data rate, (d) the minimum code rate required to achieve arbitrarily low error probability, and (e) the capacity at p = 0.1 for comparison.

Solution:

(a) BSC capacity: $C = 1 - H(p) = 1 - [-p \log_2(p) - (1-p) \log_2(1-p)]$ $H(0.01) = -0.01 \times \log_2(0.01) - 0.99 \times \log_2(0.99)$ $H(0.01) = -0.01 \times (-6.644) - 0.99 \times (-0.01449)$ $H(0.01) = 0.06644 + 0.01434 = 0.08078$ $C = 1 - 0.08078 = 0.919$ bits/channel use

(b) Capacity: $C_{rate} = 0.919 \times 10 = 9.19$ Mbps

(c) Maximum reliable data rate $= C_{rate} = 9.19$ Mbps

By Shannon's theorem, any rate below 9.19 Mbps can be achieved with arbitrarily low error probability using sufficiently long codes. Rates above 9.19 Mbps will inevitably have a non-zero error floor.

(d) Minimum code rate: $R_{c,min} = C = 0.919$ Codes must have rate ≤ 0.919, meaning at least $(1 - 0.919) \times 100 = 8.1\%$ redundancy.

(e) At $p = 0.1$: $H(0.1) = -0.1 \times \log_2(0.1) - 0.9 \times \log_2(0.9) = 0.3322 + 0.1368 = 0.4690$ $C = 1 - 0.469 = 0.531$ bits/channel use $= 5.31$ Mbps

A 10× increase in error rate (0.01 to 0.1) reduces capacity by only 42% (9.19 to 5.31 Mbps), demonstrating the logarithmic sensitivity of capacity to channel quality.

Problem 2.6.4

Given: An English text source has 26 letters plus a space character (27 symbols). In natural English, the letter frequencies are approximately: space (18.3%), E (10.2%), T (7.7%), A (6.5%), O (6.2%), I (5.7%), N (5.7%), and the remaining 20 characters share the remaining 39.7%.

Find: (a) The maximum entropy if all 27 symbols were equally likely, (b) the approximate first-order entropy using the given frequencies (and assuming the 20 remaining characters have equal probability of 39.7/20 = 1.985% each), (c) the redundancy, (d) the theoretical compression limit, and (e) comparison to practical compression (gzip achieves ~2.5 bits/character for English).

Solution:

(a) Maximum entropy: $H_{max} = \log_2(27) = 4.755$ bits/symbol

(b) First-order entropy (using the given frequencies): $H_1 = -[0.183 \log_2(0.183) + 0.102 \log_2(0.102) + 0.077 \log_2(0.077)$

- $0.065 \log_2(0.065) + 0.062 \log_2(0.062) + 0.057 \log_2(0.057)$
- $0.057 \log_2(0.057) + 20 \times 0.01985 \log_2(0.01985)]$

Computing each term: Space: $0.183 \times 2.450 = 0.4484$ E: $0.102 \times 3.293 = 0.3359$ T: $0.077 \times 3.698 = 0.2847$ A: $0.065 \times 3.943 = 0.2563$ O: $0.062 \times 4.012 = 0.2487$ I: $0.057 \times 4.133 = 0.2356$ N: $0.057 \times 4.133 = 0.2356$ 20 others: $20 \times 0.01985 \times 5.654 = 20 \times 0.1122 = 2.2448$

$H_1 = 0.4484 + 0.3359 + 0.2847 + 0.2563 + 0.2487 + 0.2356 + 0.2356 + 2.2448 = 4.290$ bits/symbol

(c) Redundancy: $R = 1 - H_1/H_{max} = 1 - 4.290/4.755 = 1 - 0.902 = 0.098 = 9.8\%$

This is only the first-order redundancy. When letter sequences (digrams, trigrams) are considered, the true entropy of English drops to approximately 1.0–1.5 bits/character (Shannon's estimate), giving redundancy of 68–79%.

(d) Theoretical compression limit: $4.290/4.755 = 0.902$ of original size using first-order statistics, or $4.290/8.0 = 0.536$ of ASCII encoding (46.4% reduction).

(e) Practical gzip: ~2.5 bits/character = $2.5/8.0 = 31.25\%$ of ASCII. This exceeds the first-order limit (4.29 bits) because gzip exploits higher-order statistical patterns (repeated words and phrases). The Shannon limit ($\approx$1.3 bits/char) suggests further 1.9$\times$ improvement is theoretically possible.

Problem 2.6.5

Given: A 4G LTE downlink channel has 10 MHz bandwidth and operates with various SNR levels depending on the user's distance from the tower. Consider three scenarios: (a) cell center with SNR = 25 dB, (b) mid-cell with SNR = 15 dB, (c) cell edge with SNR = 3 dB.

Find: For each scenario: (a) the Shannon capacity, (b) the spectral efficiency, (c) the closest standard LTE modulation and coding scheme (MCS), and (d) the practical achievable rate (typically 60–75% of Shannon capacity due to overhead and imperfect coding).

Solution:

(a) Cell center (SNR = 25 dB = 316.2): $C = 10 \times 10^6 \times \log_2(1 + 316.2) = 10 \times 10^6 \times 8.31 = 83.1$ Mbps Spectral efficiency = 8.31 bits/s/Hz Closest MCS: 64-QAM with 0.93 code rate $\rightarrow 6 \times 0.93 = 5.58$ bits/s/Hz $\rightarrow$ ~55.8 Mbps Practical rate: $0.70 \times 83.1 = 58.2$ Mbps (matches 64-QAM, high code rate)

(b) Mid-cell (SNR = 15 dB = 31.62): $C = 10 \times 10^6 \times \log_2(1 + 31.62) = 10 \times 10^6 \times 5.03 = 50.3$ Mbps Spectral efficiency = 5.03 bits/s/Hz Closest MCS: 16-QAM with 0.60 code rate $\rightarrow 4 \times 0.60 = 2.40$ bits/s/Hz $\rightarrow$ ~24 Mbps Practical rate: $0.65 \times 50.3 = 32.7$ Mbps (between 16-QAM and 64-QAM MCS levels)

(c) Cell edge (SNR = 3 dB = 2.0): $C = 10 \times 10^6 \times \log_2(1 + 2.0) = 10 \times 10^6 \times 1.585 = 15.85$ Mbps Spectral efficiency = 1.585 bits/s/Hz Closest MCS: QPSK with 0.44 code rate $\rightarrow 2 \times 0.44 = 0.88$ bits/s/Hz $\rightarrow$ ~8.8 Mbps Practical rate: $0.60 \times 15.85 = 9.5$ Mbps (matches QPSK, low code rate)

The gap between Shannon capacity and achievable rate widens at lower SNR because fixed block-length codes are less efficient and higher overhead fractions consume more of the limited throughput.

Problem 2.6.6

Given: A source emits symbols from an alphabet of 8 characters with probabilities: {0.30, 0.20, 0.15, 0.12, 0.10, 0.06, 0.04, 0.03}.

Find: (a) The entropy, (b) the Huffman code, (c) the average code length, (d) the coding efficiency, and (e) the average code length if two symbols are coded jointly (2nd-order Huffman).

Solution:

(a) Entropy: $H = -[0.30 \log_2(0.30) + 0.20 \log_2(0.20) + 0.15 \log_2(0.15) + 0.12 \log_2(0.12)$

- $0.10 \log_2(0.10) + 0.06 \log_2(0.06) + 0.04 \log_2(0.04) + 0.03 \log_2(0.03)]$

$H = 0.30(1.737) + 0.20(2.322) + 0.15(2.737) + 0.12(3.059) + 0.10(3.322) + 0.06(4.059) + 0.04(4.644) + 0.03(5.059)$ $H = 0.521 + 0.464 + 0.411 + 0.367 + 0.332 + 0.244 + 0.186 + 0.152$ $H = 2.677$ bits/symbol

(b) Huffman code (building the tree by combining lowest probabilities):

Symbol	Prob	Code	Length
S_1	0.30	00	2
S_2	0.20	10	2
S_3	0.15	010	3
S_4	0.12	110	3
S_5	0.10	111	3
S_6	0.06	0110	4
S_7	0.04	01110	5
S_8	0.03	01111	5

(c) Average code length: $L = 0.30(2) + 0.20(2) + 0.15(3) + 0.12(3) + 0.10(3) + 0.06(4) + 0.04(5) + 0.03(5)$ $L = 0.60 + 0.40 + 0.45 + 0.36 + 0.30 + 0.24 + 0.20 + 0.15 = 2.70$ bits/symbol

(d) Efficiency: $\eta = H/L = 2.677/2.70 = 99.1\%$

(e) Second-order Huffman codes blocks of 2 symbols (64 pairs). The average code length per original symbol approaches the entropy: $L_2 \approx H + 1/(2 \times 64) \approx 2.677 + 0.008 \approx 2.685$ bits/symbol per original symbol

The improvement from 2.70 to 2.685 bits/symbol is small but meaningful at high data rates. Arithmetic coding would achieve even closer to 2.677 without the combinatorial explosion of higher-order Huffman trees.

Problem 2.6.7

Given: A MIMO channel has $N_t = 4$ transmit and $N_r = 4$ receive antennas. The channel is rich scattering (full rank). The total transmit SNR is 20 dB, equally distributed among transmit antennas. The singular values of the channel matrix H are $\sigma_1 = 3.2$, $\sigma_2 = 2.1$, $\sigma_3 = 1.0$, $\sigma_4 = 0.3$.

Find: (a) The SNR per transmit antenna, (b) the SNR on each spatial sub-channel, (c) the capacity of each sub-channel, (d) the total MIMO capacity, and (e) the comparison to a SISO system at the same total power.

Solution:

(a) Total SNR = 20 dB = 100. Per antenna: $SNR_{ant} = 100/4 = 25$ per antenna

(b) SNR per spatial sub-channel: $SNR_i = \sigma_i^2 \times SNR_{ant}$ Channel 1: $3.2^2 \times 25 = 10.24 \times 25 = 256.0$ (24.08 dB) Channel 2: $2.1^2 \times 25 = 4.41 \times 25 = 110.25$ (20.42 dB) Channel 3: $1.0^2 \times 25 = 1.0 \times 25 = 25.0$ (13.98 dB) Channel 4: $0.3^2 \times 25 = 0.09 \times 25 = 2.25$ (3.52 dB)

(c) Sub-channel capacities: $C_1 = \log_2(1 + 256.0) = \log_2(257) = 8.01$ bits/s/Hz $C_2 = \log_2(1 + 110.25) = \log_2(111.25) = 6.80$ bits/s/Hz $C_3 = \log_2(1 + 25.0) = \log_2(26) = 4.70$ bits/s/Hz $C_4 = \log_2(1 + 2.25) = \log_2(3.25) = 1.70$ bits/s/Hz

(d) Total MIMO capacity: $C_{MIMO} = 8.01 + 6.80 + 4.70 + 1.70 = 21.21$ bits/s/Hz

(e) SISO capacity at same total power (SNR = 100): $C_{SISO} = \log_2(1 + 100) = \log_2(101) = 6.66$ bits/s/Hz

MIMO gain = 21.21 / 6.66 = 3.18×

The MIMO system achieves 3.18× the SISO capacity. The gain is less than 4× (the number of spatial channels) because the 4th sub-channel has low SNR ($\sigma_4 = 0.3$). Water-filling power allocation would shift power from the weak 4th channel to the stronger channels, potentially increasing total capacity slightly.

Problem 2.6.8

Given: A rate-distortion problem: a Gaussian source with variance $\sigma^2 = 1$ is to be compressed with a maximum mean-squared error distortion of $D = 0.01$. The rate-distortion function for a Gaussian source is $R(D) = (1/2) \log_2(\sigma^2/D)$ bits/sample.

Find: (a) The minimum bit rate required, (b) the equivalent number of quantization bits, (c) the rate for $D = 0.1$, (d) the rate for $D = 0.001$, and (e) the rate saving from allowing 10× more distortion ($D = 0.01$ vs $D = 0.1$).

Solution:

(a) $R(D) = (1/2) \log_2(1/0.01) = (1/2) \log_2(100) = (1/2) \times 6.644 = 3.322$ bits/sample

(b) Equivalent bits: A uniform quantizer with N bits achieves $D \approx \sigma^2 \times 2^{-2N}$. For $D = 0.01$: $2^{-2N} = 0.01 \rightarrow 2N = \log_2(100) = 6.644 \rightarrow N = 3.32$ bits. Equivalent to approximately 3.3 bits per sample (between 3 and 4 bit uniform quantization).

(c) $R(0.1) = (1/2) \log_2(1/0.1) = (1/2) \times 3.322 = 1.661$ bits/sample

(d) $R(0.001) = (1/2) \log_2(1/0.001) = (1/2) \times 9.966 = 4.983$ bits/sample

(e) Rate saving from $D = 0.01$ to $D = 0.1$: $\Delta R = 3.322 - 1.661 = 1.661$ bits/sample saved

This is exactly $(1/2) \log_2(10) = 1.661$ bits — allowing 10× more distortion saves 1.66 bits per sample. This is the rate-distortion trade-off: each halving of the distortion costs exactly 0.5 bits/sample for Gaussian sources.

Problem 2.6.9

Given: A wireless channel undergoes flat Rayleigh fading with an average SNR of $\bar{\gamma} = 20$ dB. The ergodic capacity of a Rayleigh fading channel is $C = E[\log_2(1 + \gamma)] = e^{1/\bar{\gamma}} \times E_1(1/\bar{\gamma}) / \ln(2)$, where E_1 is the exponential integral. For high average SNR ($\bar{\gamma} \gg 1$), this simplifies to $C \approx \log_2(\bar{\gamma}) - 0.833$ bits/s/Hz.

Find: (a) The AWGN capacity at SNR = 20 dB (for comparison), (b) the ergodic Rayleigh fading capacity using the high-SNR approximation, (c) the capacity loss due to fading, (d) the outage capacity at 10% outage probability (the rate achievable 90% of the time), and (e) the benefit of 2-branch receive diversity (which doubles the average SNR).

Solution:

(a) AWGN capacity: $C_{AWGN} = \log_2(1 + 100) = \log_2(101) = 6.66$ bits/s/Hz

(b) Ergodic Rayleigh capacity (high-SNR approximation): $\bar{\gamma} = 100$ (linear) $C_{Rayleigh} \approx \log_2(100) - 0.833 = 6.644 - 0.833 = 5.81$ bits/s/Hz

(c) Capacity loss due to fading: $6.66 - 5.81 = 0.85$ bits/s/Hz (12.8% reduction)

The ergodic capacity is the average over all fading states. Fading reduces capacity because the channel spends some time at very low SNR, where the capacity is near zero, and these deep fades are not fully compensated by the high-SNR peaks (due to the logarithm's concavity).

(d) Outage capacity at 10% outage: For Rayleigh fading, $P(\gamma < \gamma_{th}) = 1 - e^{-\gamma th/\bar{\gamma}} = 0.10$ $\gamma_{th} = -\bar{\gamma} \times \ln(0.90) = -100 \times (-0.1054) = 10.54$ (10.23 dB) $C_{out,10\%} = \log_2(1 + 10.54) = \log_2(11.54) = 3.53$ bits/s/Hz

This is much lower than the ergodic capacity because 10% of the time the channel is in a deep fade.

(e) With 2-branch MRC diversity, average SNR doubles to $2\bar{\gamma} = 200$: $C_{diversity} \approx \log_2(200) - 0.833 = 7.644 - 0.833 = 6.81$ bits/s/Hz Gain from diversity: $6.81 - 5.81 = 1.0$ bits/s/Hz — the diversity gain recovers more than the fading loss.

Problem 2.6.10

Given: A joint source-channel coding problem: a 720p video stream requires 5 Mbps after source coding (H.264). The wireless channel has bandwidth B = 2 MHz and SNR = 18 dB.

Find: (a) The Shannon capacity of the channel, (b) whether the video can be transmitted reliably, (c) the minimum channel coding rate needed, (d) the maximum source coding rate (video quality reduction) if the channel cannot support the full rate, and (e) the bandwidth required to reliably transmit the 5 Mbps stream.

Solution:

(a) Channel capacity: SNR = 18 dB = 63.10 $C = 2 \times 10^6 \times \log_2(1 + 63.10) = 2 \times 10^6 \times 6.00 = 12.0$ Mbps

(b) The required rate of 5 Mbps is below the channel capacity of 12 Mbps. Yes, the video can be transmitted reliably. There is $12.0 - 5.0 = 7.0$ Mbps of margin for channel coding overhead.

(c) The channel coding rate must satisfy: $R_{source} / R_{channel\ code} \leq C$ $R_{channel\ code} \geq R_{source} / C = 5.0 / 12.0 = 0.417$

So a code rate of at least 0.417 (e.g., rate 1/2) provides reliable transmission. The remaining bandwidth supports the coding redundancy.

(d) If the channel had lower capacity (say, C = 4 Mbps, insufficient for 5 Mbps): Maximum video rate = $C \times R_{\text{channel code}} = 4 \times 0.75 = 3$ Mbps (with rate 3/4 FEC) This would require reducing video quality from 5 Mbps to 3 Mbps — dropping resolution from 720p to approximately 480p, or reducing frame rate from 30 fps to 18 fps.

(e) Minimum bandwidth for 5 Mbps with 18 dB SNR: $5 \times 10^6 = B \times \log_2(1 + 63.10) = B \times 6.00$ $B = 5 \times 10^6 / 6.00 = 833$ kHz

This is the theoretical minimum. In practice, with a rate-1/2 code: Required channel rate = 5 / 0.5 = 10 Mbps. With 16-QAM (4 bits/s/Hz): B = 10 / 4 = 2.5 MHz (practical minimum)

Chapter 2 — Section 2.7: Noise in Communication Systems

Practice problems covering thermal noise, noise power, noise voltage, noise figure, noise temperature, cascaded system noise, Friis formula, satellite link budgets, and G/T calculations.

Problem 2.7.1

Given: A microwave receiver front-end operates at a physical temperature of T = 300 K. The receiver has a bandwidth of 25 MHz and an input impedance of 50 Ω.

Find: (a) The thermal noise power in watts and dBm, (b) the noise power spectral density (N_0) in dBm/Hz, (c) the RMS noise voltage across the 50 Ω input, (d) the RMS noise voltage if the impedance were 75 Ω (common in cable TV), and (e) the noise power if the system were cooled to 77 K (liquid nitrogen temperature).

Solution:

(a) Noise power: $N = kTB = 1.381 \times 10^{-23} \times 300 \times 25 \times 10^6 = 1.036 \times 10^{-13}$ W $N(\text{dBm}) = 10 \log_{10}(1.036 \times 10^{-13} / 10^{-3}) = 10 \log_{10}(1.036 \times 10^{-10}) = -99.8$ dBm

Quick check: $-174 + 10 \log_{10}(25 \times 10^6) = -174 + 74.0 = -100.0$ dBm (at 290 K) At 300 K: $-100.0 + 10 \log_{10}(300/290) = -100.0 + 0.15 = -99.8$ dBm ✓

(b) $N_0 = kT = 1.381 \times 10^{-23} \times 300 = 4.143 \times 10^{-21}$ W/Hz $N_0(\text{dBm/Hz}) = 10 \log_{10}(4.143 \times 10^{-21} / 10^{-3}) = -173.8$ dBm/Hz

(c) RMS noise voltage (50 Ω): $V_n = \sqrt{(4kTRB)} = \sqrt{(4 \times 1.381 \times 10^{-23} \times 300 \times 50 \times 25 \times 10^6)}$ $V_n = \sqrt{(2.071 \times 10^{-11})} = 4.55\ \mu V$

(d) For 75 Ω: $V_n = \sqrt{(4kT \times 75 \times B)} = \sqrt{(4 \times 1.381 \times 10^{-23} \times 300 \times 75 \times 25 \times 10^6)}$ $V_n = \sqrt{(3.107 \times 10^{-11})} = 5.57\ \mu V$

Note: despite the higher voltage, the noise power delivered to a matched load remains kTB = −99.8 dBm regardless of impedance.

(e) At 77 K: $N = kTB = 1.381 \times 10^{-23} \times 77 \times 25 \times 10^6 = 2.659 \times 10^{-14}$ W $N(\text{dBm}) = -105.8$ dBm Improvement: $-99.8 - (-105.8) = 5.9$ dB reduction in noise floor by cryogenic cooling.

Problem 2.7.2

Given: A receiver chain consists of three stages: (1) a bandpass filter with 1.5 dB insertion loss, (2) an LNA with noise figure 1.2 dB and gain 25 dB, (3) a mixer with noise figure 10 dB and conversion gain 0 dB (unity).

Find: (a) The linear noise figures and gains, (b) the system noise figure using the Friis formula, (c) the system noise temperature, and (d) the improvement if the filter is moved after the LNA.

Solution:

(a) Converting to linear: Filter: $F_1 = 10^{1.5/10} = 1.413$, $G_1 = 10^{-1.5/10} = 0.708$ (passive loss) LNA: $F_2 = 10^{1.2/10} = 1.318$, $G_2 = 10^{25/10} = 316.2$ Mixer: $F_3 = 10^{10/10} = 10.0$, $G_3 = 10^{0/10} = 1.0$

(b) Friis formula: $F_{sys} = F_1 + (F_2 - 1)/G_1 + (F_3 - 1)/(G_1 \times G_2)$ $F_{sys} = 1.413 + (1.318 - 1)/0.708 + (10.0 - 1)/(0.708 \times 316.2)$ $F_{sys} = 1.413 + 0.449 + 0.0402 = 1.902$ $NF_{sys} = 10 \log_{10}(1.902) = 2.79$ dB

(c) Noise temperature: $T_{sys} = T_0(F_{sys} - 1) = 290 \times (1.902 - 1) = 290 \times 0.902 = 261.6$ K

(d) With LNA first, then filter, then mixer: $F_{sys} = F_{LNA} + (F_{filter} - 1)/G_{LNA} + (F_{mixer} - 1)/(G_{LNA} \times G_{filter})$ $F_{sys} = 1.318 + (1.413 - 1)/316.2 + (10.0 - 1)/(316.2 \times 0.708)$ $F_{sys} = 1.318 + 0.00131 + 0.0402 = 1.359$ $NF_{sys} = 10 \log_{10}(1.359) = 1.33$ dB

Improvement: $2.79 - 1.33 = 1.46$ dB by placing the LNA before the filter. The system noise figure drops from 2.79 dB to 1.33 dB — nearly matching the LNA's own noise figure.

Problem 2.7.3

Given: A Ka-band satellite earth station operates at 20 GHz (downlink). The antenna has a diameter of 1.2 m with 60% aperture efficiency. The system noise temperature breakdown: antenna noise T_a = 30 K (clear sky, elevation 30°), LNA noise temperature T_{LNA} = 65 K, and subsequent receiver stages contribute 15 K (referred to input).

Find: (a) The antenna gain, (b) the system noise temperature, (c) the G/T figure of merit, (d) the G/T degradation during rain (T_a increases to 120 K due to rain emission), and (e) the effective noise figure of the LNA.

Solution:

(a) Antenna gain at 20 GHz: $\lambda = c/f = 3 \times 10^8 / 20 \times 10^9 = 0.015$ m $G = \eta \times (\pi D/\lambda)^2 = 0.60 \times (\pi \times 1.2/0.015)^2 = 0.60 \times (251.3)^2 = 0.60 \times 63{,}155$ $G = 37{,}893 = 10 \log_{10}(37{,}893) = 45.8$ dBi

(b) System noise temperature: $T_{sys} = T_a + T_{LNA} + T_{rest} = 30 + 65 + 15 = 110$ K

(c) $G/T = G(\text{dBi}) - 10 \log_{10}(T_{sys}) = 45.8 - 10 \log_{10}(110) = 45.8 - 20.41 = 25.4$ dB/K

(d) During rain, $T_a = 120$ K: $T_{sys,rain} = 120 + 65 + 15 = 200$ K $G/T_{rain} = 45.8 - 10 \log_{10}(200) = 45.8 - 23.01 = 22.8$ dB/K G/T degradation $= 25.4 - 22.8 = 2.6$ dB

This 2.6 dB degradation adds to the direct rain attenuation loss, making Ka-band particularly sensitive to rain.

(e) LNA noise figure: $NF = 10 \log_{10}(1 + T_{LNA}/T_0) = 10 \log_{10}(1 + 65/290) = 10 \log_{10}(1.224) = 0.88$ dB

Problem 2.7.4

Given: A radio astronomy receiver operates at 1,420 MHz (hydrogen line) with a system noise temperature of 25 K (cryogenically cooled). The antenna has an effective collecting area of 500 m² and the observation bandwidth is 1 MHz.

Find: (a) The system noise power, (b) the antenna gain, (c) the minimum detectable flux density for SNR = 5 after 1 hour of integration (radiometer equation: $SNR = S_{signal}/(S_{noise}/\sqrt{(Bt)})$), (d) the noise power spectral density, and (e) the equivalent noise voltage across a 50 Ω feed.

Solution:

(a) Noise power: $N = kT_{sys}B = 1.381 \times 10^{-23} \times 25 \times 10^6 = 3.453 \times 10^{-16}$ W $N(dBW) = 10 \log_{10}(3.453 \times 10^{-16}) = -154.6$ dBW

(b) Antenna gain: $G = 4\pi A_{eff}/\lambda^2$ $\lambda = c/f = 3 \times 10^8 / 1.42 \times 10^9 = 0.2113$ m $G = 4\pi \times 500 / (0.2113)^2 = 6{,}283.2 / 0.04465 = 140{,}730$ $G(dBi) = 10 \log_{10}(140{,}730) = 51.5$ dBi

(c) Radiometer equation: $SNR_{out} = (T_{source}/T_{sys}) \times \sqrt{(Bt)}$ For SNR = 5: $T_{source} = 5 \times T_{sys}/\sqrt{(Bt)}$ $\sqrt{(Bt)} = \sqrt{(10^6 \times 3{,}600)} = \sqrt{(3.6 \times 10^9)} = 60{,}000$ $T_{source,min} = 5 \times 25 / 60{,}000 = 0.00208$ K = 2.08 mK

Minimum flux density: $S = 2kT_{source}/A_{eff} = 2 \times 1.381 \times 10^{-23} \times 0.00208 / 500$ $S = 1.149 \times 10^{-28}$ W/m²/Hz = 11.5 mJy (1 Jy = 10^{-26} W/m²/Hz)

(d) $N_0 = kT_{sys} = 1.381 \times 10^{-23} \times 25 = 3.453 \times 10^{-22}$ W/Hz = −214.6 dBW/Hz

(e) RMS noise voltage: $V_n = \sqrt{(4kT_{sys}RB)} = \sqrt{(4 \times 1.381 \times 10^{-23} \times 25 \times 50 \times 10^6)}$ $V_n = \sqrt{(6.905 \times 10^{-14})} = 0.263\ \mu V = 263$ nV

Problem 2.7.5

Given: A cellular base station receiver chain: Stage 1: Duplexer/filter (loss = 0.8 dB) Stage 2: Tower-mounted LNA (NF = 0.5 dB, gain = 15 dB) Stage 3: Coaxial cable run (loss = 6 dB, length = 50 m) Stage 4: Base station amplifier (NF = 4 dB, gain = 30 dB) Stage 5: Mixer/IF chain (NF = 12 dB)

Find: (a) The overall system noise figure, (b) the system noise temperature, (c) the receiver sensitivity for a 200 kHz channel bandwidth at 10 dB required SNR, and (d) the noise figure if the cable loss increases to 10 dB (longer cable run).

Solution:

(a) Converting to linear: $F_1 = 10^{0.8/10} = 1.202$, $G_1 = 10^{-0.8/10} = 0.832$ $F_2 = 10^{0.5/10} = 1.122$, $G_2 = 10^{15/10} = 31.62$ $F_3 = 10^{6/10} = 3.981$, $G_3 = 10^{-6/10} = 0.251$ $F_4 = 10^{4/10} = 2.512$, $G_4 = 10^{30/10} = 1000$ $F_5 = 10^{12/10} = 15.85$

Friis formula: $F_{sys} = F_1 + (F_2-1)/G_1 + (F_3-1)/(G_1G_2) + (F_4-1)/(G_1G_2G_3) + (F_5-1)/(G_1G_2G_3G_4)$ $F_{sys} = 1.202 + 0.122/0.832 + 2.981/(0.832 \times 31.62) + 1.512/(0.832 \times 31.62 \times 0.251) + 14.85/(0.832 \times 31.62 \times 0.251 \times 1000)$ $F_{sys} = 1.202 + 0.147 + 0.1133 + 0.229 + 0.00225$ $F_{sys} = 1.693$ $NF_{sys} = 10 \log_{10}(1.693) = 2.29$ dB

(b) $T_{sys} = 290 \times (1.693 - 1) = 290 \times 0.693 = 200.9$ K

(c) Sensitivity $= -174 + 10 \log_{10}(B) + NF + SNR_{req} = -174 + 10 \log_{10}(200{,}000) + 2.29 + 10 = -174 + 53.01 + 2.29 + 10 = -108.7$ dBm

(d) With 10 dB cable loss: $F_3 = 10$, $G_3 = 0.1$ $F_{sys} = 1.202 + 0.147 + 9.0/(0.832 \times 31.62) + 1.512/(0.832 \times 31.62 \times 0.1) + 14.85/(0.832 \times 31.62 \times 0.1 \times 1000)$ $F_{sys} = 1.202 + 0.147 + 0.342 + 0.575 + 0.00565$ $F_{sys} = 2.272$ $NF_{sys} = 10 \log_{10}(2.272) = 3.56$ dB

The 4 dB additional cable loss degrades the noise figure by $3.56 - 2.29 = 1.27$ dB, reducing sensitivity by the same amount. This is why tower-mounted LNAs are critical.

Problem 2.7.6

Given: A geostationary satellite transponder at 36,000 km has an EIRP of 48 dBW at 12 GHz (Ku-band). The earth station uses a 0.75 m dish with 62% aperture efficiency. The LNB (Low-Noise Block) has a noise figure of 0.7 dB. Atmospheric losses are 0.3 dB and miscellaneous losses are 0.5 dB. Antenna noise temperature is 35 K.

Find: (a) The free-space path loss, (b) the receive antenna gain, (c) the received carrier power, (d) the system noise temperature, and (e) the C/N in a 27 MHz transponder bandwidth.

Solution:

(a) FSPL at 12 GHz, 36,000 km: $\lambda = 3 \times 10^8 / 12 \times 10^9 = 0.025$ m $\text{FSPL} = 20 \log_{10}(4\pi d/\lambda) = 20 \log_{10}(4\pi \times 3.6 \times 10^7 / 0.025) = 20 \log_{10}(1.810 \times 10^{10}) = 20 \times 10.258 = 205.2$ dB

(b) Receive gain: $G = \eta(\pi D/\lambda)^2 = 0.62 \times (\pi \times 0.75/0.025)^2 = 0.62 \times (94.25)^2 = 0.62 \times 8{,}883 = 5{,}507$ $G(\text{dBi}) = 10 \log_{10}(5{,}507) = 37.4$ dBi

(c) Received power: $P_{rx} = \text{EIRP} - \text{FSPL} + G_{rx} - L_{atm} - L_{misc}$ $P_{rx} = 48 - 205.2 + 37.4 - 0.3 - 0.5 = -120.6$ dBW $= -90.6$ dBm

(d) LNB noise temperature: $T_{LNB} = T_0(F - 1) = 290 \times (10^{0.7/10} - 1) = 290 \times (1.175 - 1) = 290 \times 0.175 = 50.7$ K System noise temperature: $T_{sys} = T_a + T_{LNB} = 35 + 50.7 = 85.7$ K

(e) Noise power: $N = kT_{sys}B = 1.381 \times 10^{-23} \times 85.7 \times 27 \times 10^6 = 3.197 \times 10^{-14}$ W $N(\text{dBW}) = 10 \log_{10}(3.197 \times 10^{-14}) = -134.95$ dBW

Or: $N = -228.6 + 10 \log_{10}(85.7) + 10 \log_{10}(27 \times 10^6) = -228.6 + 19.33 + 74.31 = -134.96$ dBW

$C/N = P_{rx} - N = -120.6 - (-135.0) = 14.4$ dB

This provides comfortable margin for DVB-S2 QPSK (requires ~5.5 dB C/N) with approximately 9 dB of rain margin.

Problem 2.7.7

Given: A two-antenna diversity receiver combines signals using maximal ratio combining (MRC). Each antenna path has: antenna gain = 5 dBi, cable loss = 2 dB, LNA noise figure = 2 dB, LNA gain = 20 dB. The signal arrives at SNR = 8 dB per branch (before combining). The noise is independent between branches.

Find: (a) The combined SNR after MRC (ideal: $SNR_{MRC} = SNR_1 + SNR_2$ for equal-SNR branches), (b) the diversity gain in dB, (c) the effective system noise figure per branch, (d) the combined noise figure, and (e) the BER improvement for BPSK (BER with diversity vs. without).

Solution:

(a) Each branch: SNR = 8 dB = 6.31 (linear) MRC combined SNR = $SNR_1 + SNR_2 = 6.31 + 6.31 = 12.62$ $SNR_{MRC}(dB) = 10 \log_{10}(12.62) = 11.0$ dB

(b) Diversity gain = 11.0 − 8.0 = 3.0 dB This is the ideal 2-branch MRC gain for equal-SNR branches with independent noise.

(c) Per branch: cable (loss) then LNA: $F_{cable} = 10^{2/10} = 1.585$, $G_{cable} = 10^{-2/10} = 0.631$ $F_{LNA} = 10^{2/10} = 1.585$, $G_{LNA} = 100$

$F_{branch} = F_{cable} + (F_{LNA} - 1)/G_{cable} = 1.585 + 0.585/0.631 = 1.585 + 0.927 = 2.512$ $NF_{branch} = 10 \log_{10}(2.512) = 4.0$ dB

(d) For MRC with independent noise, the combined noise figure is the same as each branch: $NF_{combined} = 4.0$ dB (the diversity gain comes from coherent signal addition, not noise reduction per se)

(e) BPSK BER comparison at SNR per bit = 8 dB = 6.31: Without diversity: BER = $Q(\sqrt{(2 \times 6.31)}) = Q(3.553) = 1.90 \times 10^{-4}$ With 2-branch MRC (SNR = 12.62): BER = $Q(\sqrt{(2 \times 12.62)}) = Q(5.024) = 2.53 \times 10^{-7}$ Improvement: $1.90 \times 10^{-4} / 2.53 \times 10^{-7} = 751\times$ reduction in BER

Problem 2.7.8

Given: A 5G millimeter-wave link operates at 28 GHz over a distance of 200 m. The transmitter has $P_{tx} = 30$ dBm and uses a 16-element phased array ($G_{tx} = 18$ dBi). The receiver has a 16-element array ($G_{rx} = 18$ dBi) with overall noise figure NF = 6 dB. Additional losses include atmospheric absorption (0.2 dB) and foliage loss (10 dB for a tree in the path). The channel bandwidth is 400 MHz.

Find: (a) The free-space path loss, (b) the total path loss (including atmospheric and foliage), (c) the received power, (d) the noise floor, and (e) the achievable SNR and corresponding maximum modulation order.

Solution:

(a) FSPL at 28 GHz, 200 m: $\lambda = 3 \times 10^8 / 28 \times 10^9 = 0.01071$ m FSPL = $20 \log_{10}(4\pi \times 200/0.01071) = 20 \log_{10}(234{,}512) = 20 \times 5.370 = 107.4$ dB

(b) Total path loss: $L_{total} = 107.4 + 0.2 + 10 = 117.6$ dB

(c) Received power: $P_{rx} = P_{tx} + G_{tx} - L_{total} + G_{rx} = 30 + 18 - 117.6 + 18 = -51.6$ dBm

(d) Noise floor: $N = -174 + 10\log_{10}(400 \times 10^6) + NF = -174 + 86.0 + 6 = -82.0$ dBm

(e) $SNR = P_{rx} - N = -51.6 - (-82.0) = 30.4$ dB

At 30.4 dB SNR: - 256-QAM requires ~27 dB SNR → margin = 3.4 dB → feasible but tight - 64-QAM requires ~22 dB SNR → margin = 8.4 dB → comfortable - 64-QAM is recommended for reliability with the foliage loss

Without the foliage loss: SNR = 40.4 dB, supporting up to 1024-QAM. This demonstrates the severe impact of obstructions at mmWave frequencies.

Problem 2.7.9

Given: A bent-pipe satellite transponder has the following link parameters: Uplink: EIRP = 72 dBW, path loss = 207 dB, satellite G/T = −2 dB/K, bandwidth = 36 MHz Downlink: satellite EIRP = 44 dBW, path loss = 196 dB, earth station G/T = 20 dB/K, bandwidth = 36 MHz Intermodulation noise: C/I = 22 dB

Find: (a) The uplink C/N, (b) the downlink C/N, (c) the overall C/N using the reciprocal addition formula, and (d) the impact on overall C/N if the uplink power is increased by 3 dB.

Solution:

(a) Uplink C/N: $(C/N)_{up} = EIRP + G/T - L_{path} - k - 10\log_{10}(B) = 72 + (-2) - 207 - (-228.6) - 10\log_{10}(36 \times 10^6) = 72 - 2 - 207 + 228.6 - 75.56 = 16.04$ dB

(b) Downlink C/N: $(C/N)_{down} = 44 + 20 - 196 - (-228.6) - 75.56 = 44 + 20 - 196 + 228.6 - 75.56 = 21.04$ dB

(c) Overall C/N (reciprocal addition in linear): $(C/N)_{up} = 10^{16.04/10} = 40.18$ $(C/N)_{down} = 10^{21.04/10} = 127.1$ $(C/I) = 10^{22/10} = 158.5$

$1/(C/N)_{total} = 1/40.18 + 1/127.1 + 1/158.5 = 0.02489 + 0.00787 + 0.00631 = 0.03907$ $(C/N)_{total} = 1/0.03907 = 25.60 = 14.08$ dB

The uplink (16.04 dB) is the bottleneck, pulling the overall C/N down to 14.08 dB.

(d) Increasing uplink power by 3 dB: $(C/N)_{up,new} = 16.04 + 3 = 19.04$ dB $= 80.22$ $1/(C/N)_{total} = 1/80.22 + 1/127.1 + 1/158.5 = 0.01247 + 0.00787 + 0.00631 = 0.02665$ $(C/N)_{total} = 37.52 = 15.74$ dB

Improvement: 15.74 − 14.08 = 1.66 dB — less than the 3 dB uplink increase because the downlink and intermodulation now become comparable contributors.

Problem 2.7.10

Given: An Earth-to-Mars communication link during opposition (closest approach, distance = 0.52 AU = 7.78×10^{10} m). The Deep Space Network (DSN) 70 m antenna operates at 8.4 GHz (X-band) with $G_{tx} = 74$ dBi and transmit power = 400 kW. The Mars orbiter has a 1.5 m HGA with $G_{rx} = 39$ dBi and system noise temperature of 200 K. The data is coded with a rate-1/6 turbo code requiring E_b/N_0 = 0.5 dB for BER = 10^{-5}.

Find: (a) The EIRP, (b) the free-space path loss, (c) the received power, (d) the noise spectral density, and (e) the maximum achievable data rate.

Solution:

(a) EIRP = P_{tx}(dBW) + G_{tx} = 10 $\log_{10}$(400,000) + 74 = 56.02 + 74 = 130.0 dBW

(b) FSPL at 8.4 GHz, 7.78×10^{10} m: $\lambda = 3 \times 10^8 / 8.4 \times 10^9 = 0.03571$ m FSPL = 20 $\log_{10}(4\pi \times 7.78 \times 10^{10} / 0.03571) = 20 \log_{10}(2.741 \times 10^{13}) = 20 \times 13.438 = 268.8$ dB

(c) P_{rx} = EIRP − FSPL + G_{rx} = 130.0 − 268.8 + 39 = −99.8 dBW (= −69.8 dBm)

Wait — this is the uplink from Earth to Mars. Actually: P_{rx} = EIRP − FSPL + G_{rx} = 130.0 − 268.8 + 39 = −99.8 dBW

(d) Noise spectral density: $N_0 = kT_{sys} = -228.6 + 10 \log_{10}(200) = -228.6 + 23.0 = -205.6$ dBW/Hz

(e) Maximum data rate: $P_{rx}/N_0 = -99.8 - (-205.6) = 105.8$ dB·Hz

For coded system: E_b/N_0 = 0.5 dB R_b(dB·Hz) = $P_{rx}/N_0 - E_b/N_0$ = 105.8 − 0.5 = 105.3 dB·Hz $R_b = 10^{105.3/10} = 10^{10.53} = 3.39 \times 10^{10}$ bps

This seems too high. Let me reconsider — 400 kW transmit from the DSN is the uplink (Earth to Mars), and the spacecraft has limited transmit power. Let me recalculate for the Mars-to-Earth downlink instead:

For the downlink (Mars orbiter transmitter): P_{tx} = 25 W (typical), G_{tx} = 39 dBi. EIRP = 10 $\log_{10}$(25) + 39 = 13.98 + 39 = 52.98 dBW DSN 70 m: G_{rx} = 74 dBi, T_{sys} = 20 K

P_{rx} = 52.98 − 268.8 + 74 = −141.8 dBW $N_0 = -228.6 + 10 \log_{10}(20) = -228.6 + 13.0 = -215.6$ dBW/Hz $P_{rx}/N_0 = -141.8 - (-215.6) = 73.8$ dB·Hz $R_b = 10^{(73.8-0.5)/10} = 10^{7.33}$ = 21.4 Mbps (at closest approach)

For the original calculation (uplink from Earth): the 400 kW + 74 dBi DSN can deliver: $R_b = 10^{10.53}$ = 33.9 Gbps theoretical maximum at closest approach.

In practice, the uplink capacity is limited by the spacecraft receiver bandwidth and processing capability, not the RF link. The downlink at 21.4 Mbps is the practical bottleneck and matches NASA's reported Mars Reconnaissance Orbiter data rates at opposition.

Chapter 2 — Section 2.8: Link Budget Analysis

Practice problems covering free-space path loss, link margin, receiver sensitivity, Wi-Fi range estimation, cellular link budgets, microwave point-to-point links, Fresnel zone clearance, and fade margin calculations.

Problem 2.8.1

Given: A point-to-point microwave link operates at 6 GHz over a distance of 30 km. The transmit power is 1 W (30 dBm), and both antennas are 1.2 m parabolic dishes with 55% aperture efficiency. Miscellaneous losses (waveguide, connectors, radome) total 3 dB at each end. The receiver requires a minimum input power of −75 dBm.

Find: (a) The antenna gain at each end, (b) the free-space path loss, (c) the received power, (d) the link margin, and (e) the fade margin available for rain and multipath fading.

Solution:

(a) Antenna gain: $\lambda = c/f = 3 \times 10^8 / 6 \times 10^9 = 0.05$ m $G = \eta(\pi D/\lambda)^2 = 0.55 \times (\pi \times 1.2/0.05)^2 = 0.55 \times (75.40)^2 = 0.55 \times 5{,}685 = 3{,}127$ $G(\text{dBi}) = 10 \log_{10}(3{,}127) = 34.95$ dBi ≈ 35.0 dBi

(b) FSPL at 6 GHz, 30 km: $\text{FSPL} = 20 \log_{10}(4\pi d/\lambda) = 20 \log_{10}(4\pi \times 30{,}000/0.05) = 20 \log_{10}(7.540 \times 10^6) = 20 \times 6.877 = 137.5$ dB

(c) Received power: $P_{rx} = P_{tx} + G_{tx} - L_{tx,misc} - \text{FSPL} + G_{rx} - L_{rx,misc}$ $P_{rx} = 30 + 35.0 - 3 - 137.5 + 35.0 - 3 = -43.5$ dBm

(d) Link margin: $\text{Margin} = P_{rx} - P_{sensitivity} = -43.5 - (-75) = 31.5$ dB

(e) Typical allocations from the 31.5 dB margin:

- Rain attenuation (99.99% availability at 6 GHz): ~5 dB
- Multipath fading (99.99% availability, Vigants model): ~20 dB
- Equipment aging margin: ~3 dB
- Total allocated: 28 dB
- Remaining unallocated margin: 3.5 dB

The link is well-designed with adequate margin for carrier-grade (99.99%) availability.

Problem 2.8.2

Given: A 5 GHz Wi-Fi 6 access point (802.11ax) operates at P_{tx} = 23 dBm (200 mW, the EIRP regulatory limit) with G_{tx} = 5 dBi. The client device has G_{rx} = 0 dBi. The receiver sensitivity at various MCS rates is: MCS 11 (1024-QAM, 143 Mbps) = −55 dBm, MCS 7 (64-QAM, 86 Mbps) = −65 dBm, MCS 3 (16-QAM, 43 Mbps) = −73 dBm, MCS 0 (BPSK, 8.6 Mbps) = −82 dBm.

Find: (a) The FSPL at 5 GHz and 20 m, (b) the received power at 20 m (free space), (c) the maximum data rate achievable at 20 m, (d) the maximum range for 64-QAM with a 10 dB fade margin (for indoor multipath), and (e) the maximum range for MCS 0 with a 15 dB wall penetration loss.

Solution:

(a) FSPL at 5 GHz, 20 m: $\lambda = 3 \times 10^8 / 5 \times 10^9 = 0.06$ m FSPL = 20 $\log_{10}(4\pi \times 20/0.06)$ = 20 $\log_{10}(4{,}189) = 20 \times 3.622 = 72.4$ dB

(b) $P_{rx} = 23 + 5 - 72.4 + 0 = -44.4$ dBm

(c) At −44.4 dBm, the signal exceeds the sensitivity for all MCS rates. Maximum rate = MCS 11, 143 Mbps (margin = −44.4 − (−55) = 10.6 dB).

(d) Maximum range for MCS 7 (−65 dBm) with 10 dB fade margin: Required $P_{rx} = -65 + 10 = -55$ dBm (need 10 dB above sensitivity) Allowable FSPL = $P_{tx} + G_{tx} + G_{rx} - P_{rx,req} = 23 + 5 + 0 - (-55) = 83.0$ dB FSPL = 20 $\log_{10}(4\pi d/\lambda) = 83.0$ $4\pi d/\lambda = 10^{83/20} = 10^{4.15} = 14{,}125$ $d = 14{,}125 \times \lambda / (4\pi) = 14{,}125 \times 0.06 / 12.566 = 67.4$ m

(e) Maximum range for MCS 0 (−82 dBm) with 15 dB wall loss: Allowable FSPL = $23 + 5 + 0 - (-82) - 15 = 95.0$ dB $4\pi d/\lambda = 10^{95/20} = 10^{4.75} = 56{,}234$ $d = 56{,}234 \times 0.06 / 12.566 = 268$ m

In practice, indoor propagation at 5 GHz experiences 3–8 dB loss per wall and path loss exponents of 3.0–4.5 (vs. 2.0 for free space), significantly reducing these theoretical ranges.

Problem 2.8.3

Given: A cellular LTE uplink operates at 700 MHz (Band 12). The mobile device transmits at P_{tx} = 23 dBm with G_{tx} = 0 dBi. The base station has G_{rx} = 18 dBi and receiver sensitivity of −103 dBm (for QPSK, 1/3 code rate in 10 MHz). Body loss = 3 dB, building penetration loss = 15 dB. Use the Hata model for urban propagation: $L_{path} = 69.55 + 26.16 \log_{10}(f_{MHz}) - 13.82 \log_{10}(h_b) - a(h_m) + (44.9 - 6.55 \log_{10}(h_b)) \times \log_{10}(d_{km})$, with h_b = 30 m (base station height), h_m = 1.5 m (mobile height), and $a(h_m)$ = 0 dB for small city correction.

Find: (a) The maximum allowable path loss, (b) the Hata model path loss coefficients, (c) the maximum range, (d) the link margin at 5 km, and (e) the range improvement by moving to 450 MHz (Band 31).

Solution:

(a) Maximum allowable path loss (MAPL): MAPL = $P_{tx} + G_{tx} + G_{rx}$ − body loss − building loss − $P_{sensitivity}$ MAPL = 23 + 0 + 18 − 3 − 15 − (−103) = 126 dB

(b) Hata model coefficients at 700 MHz, $h_b = 30$ m: $A = 69.55 + 26.16 \log_{10}(700) - 13.82 \log_{10}(30) - 0$ $A = 69.55 + 26.16 \times 2.845 - 13.82 \times 1.477 = 69.55 + 74.44 - 20.42 = 123.57$ dB $B = 44.9 - 6.55 \log_{10}(30) = 44.9 - 6.55 \times 1.477 = 44.9 - 9.67 = 35.23$ dB/decade

$L_{Hata} = 123.57 + 35.23 \times \log_{10}(d_{km})$

(c) Maximum range: Set L_{Hata} = MAPL: $123.57 + 35.23 \times \log_{10}(d) = 126$ $\log_{10}(d) = (126 - 123.57) / 35.23 = 2.43 / 35.23 = 0.0690$ $d = 10^{0.0690} = 1.17$ km

(d) Link margin at 5 km: $L_{Hata} = 123.57 + 35.23 \times \log_{10}(5) = 123.57 + 35.23 \times 0.699 = 123.57 + 24.63 = 148.2$ dB Margin = MAPL $- L_{Hata} = 126 - 148.2 = -22.2$ dB (link fails)

The link cannot penetrate buildings at 5 km. Without building penetration loss: MAPL = 141 dB, giving range $= 10^{(141-123.57)/35.23} = 10^{0.495} = 3.12$ km (outdoor coverage only).

(e) At 450 MHz: $A_{450} = 69.55 + 26.16 \times \log_{10}(450) - 13.82 \times 1.477 = 69.55 + 69.44 - 20.42 = 118.57$ dB Range at 450 MHz: $\log_{10}(d) = (126 - 118.57) / 35.23 = 7.43 / 35.23 = 0.211$ $d = 10^{0.211} = 1.63$ km

Improvement: 1.63 / 1.17 = 1.39× (39% greater range). Lower frequencies provide better building penetration and reduced path loss.

Problem 2.8.4

Given: A 60 GHz wireless backhaul link (IEEE 802.11ad/ay) operates over 500 m between two buildings. $P_{tx} = 10$ dBm, $G_{tx} = G_{rx} = 38$ dBi (narrow-beam horn antennas). Oxygen absorption at 60 GHz adds 15 dB/km. Rain attenuation at 60 GHz during heavy rain (50 mm/hr) is approximately 20 dB/km. The receiver noise figure is 8 dB and the channel bandwidth is 2.16 GHz.

Find: (a) The FSPL, (b) the total atmospheric loss (clear air), (c) the received power (clear air), (d) the noise floor, (e) the SNR and achievable data rate in clear air and during heavy rain.

Solution:

(a) FSPL at 60 GHz, 500 m: $\lambda = 3 \times 10^8 / 60 \times 10^9 = 0.005$ m FSPL $= 20 \log_{10}(4\pi \times 500/0.005) = 20 \log_{10}(1.257 \times 10^6) = 20 \times 6.099 = 122.0$ dB

(b) Oxygen absorption: $L_{O_2} = 15 \times 0.5 = 7.5$ dB

(c) Clear-air received power: $P_{rx} = 10 + 38 - 122.0 + 38 - 7.5 = -43.5$ dBm

(d) Noise floor: $N = -174 + 10 \log_{10}(2.16 \times 10^9) + 8 = -174 + 93.35 + 8 = -72.65$ dBm

(e) Clear-air SNR: SNR $= -43.5 - (-72.65) = 29.2$ dB

At 29.2 dB SNR with 2.16 GHz bandwidth: Spectral efficiency $\approx \log_2(1 + 10^{29.2/10}) = \log_2(1 + 832) = \log_2(833) = 9.70$ bits/s/Hz Maximum rate $= 9.70 \times 2.16 = 20.95$ Gbps (theoretical) Practical (802.11ay, 64-QAM OFDM): approximately 8–10 Gbps

During heavy rain: Rain attenuation $= 20 \times 0.5 = 10$ dB $P_{rx,rain} = -43.5 - 10 = -53.5$ dBm $SNR_{rain} = -53.5 - (-72.65) = 19.2$ dB Rate drops to approximately 64-QAM level: 4–5 Gbps

Heavy rain reduces throughput by roughly 50% but the link remains operational.

Problem 2.8.5

Given: A Fresnel zone clearance calculation for a 15 GHz microwave link spanning 10 km between two towers. Tower heights are equal. The midpoint terrain elevation is 5 m above the direct line of sight.

Find: (a) The first Fresnel zone radius at the midpoint, (b) the required clearance (60% of the first Fresnel zone), (c) the minimum tower height above terrain for adequate clearance, (d) the Fresnel zone radius at a point 3 km from one end, and (e) the effect of Earth curvature over the 10 km path (assuming K = 4/3 effective Earth radius factor).

Solution:

(a) First Fresnel zone radius at midpoint: $r_1 = \sqrt{(n\lambda d_1 d_2 / (d_1 + d_2))}$ At midpoint: $d_1 = d_2 = 5{,}000$ m, $n = 1$ $\lambda = c/f = 3 \times 10^8 / 15 \times 10^9 = 0.02$ m $r_1 = \sqrt{(1 \times 0.02 \times 5{,}000 \times 5{,}000 / 10{,}000)} = \sqrt{(50)} = 7.07$ m

(b) Required clearance (60% rule): $0.6 \times r_1 = 0.6 \times 7.07 = 4.24$ m

(c) The terrain at midpoint is 5 m above the line of sight. Required additional height to achieve 4.24 m clearance above the obstruction: $h_{clearance} = 5 + 4.24 = 9.24$ m above the direct LOS

Since both towers are equal height and the obstruction is at midpoint, each tower must be raised by 9.24 m above the current LOS. Total tower height = terrain base height + 9.24 m above the LOS datum.

If the current tower heights place the LOS at terrain level at midpoint: each tower needs to be at least 9.24 m above the surrounding terrain.

(d) At 3 km from one end ($d_1 = 3$ km, $d_2 = 7$ km): $r_1 = \sqrt{(0.02 \times 3{,}000 \times 7{,}000 / 10{,}000)} = \sqrt{(42)} = 6.48$ m

The Fresnel zone is slightly smaller than at midpoint, as expected for an asymmetric position.

(e) Earth curvature bulge at midpoint: $h_{bulge} = d_1 \times d_2 / (2 \times K \times R_e)$ $R_e = 6{,}371$ km, $K = 4/3$ (standard atmosphere) $h_{bulge} = 5{,}000 \times 5{,}000 / (2 \times 4/3 \times 6{,}371{,}000) = 25 \times 10^6 / 16{,}989{,}333 = 1.47$ m

The Earth's curvature adds 1.47 m of effective obstruction at midpoint. This must be added to the clearance requirement: Total required tower height adjustment = 5 + 4.24 + 1.47 = 10.71 m above the LOS baseline.

Problem 2.8.6

Given: A LoRa (Long Range) IoT sensor link operates at 915 MHz with the following parameters: $P_{tx} = 20$ dBm, $G_{tx} = 2$ dBi (omnidirectional), $G_{rx} = 6$ dBi (gateway antenna). LoRa spreading factor SF = 10, bandwidth = 125 kHz, coding rate 4/5. The receiver sensitivity at SF10 is −134 dBm. The environment is suburban with log-distance path loss model: $L = L_0 + 10n \times \log_{10}(d/d_0)$, where $L_0 = 31.5$ dB at $d_0 = 1$ m and $n = 3.3$.

Find: (a) The LoRa data rate at SF10, (b) the maximum allowable path loss, (c) the maximum range in suburban conditions, (d) the link margin at 5 km, and (e) the range if SF is increased to 12 (sensitivity improves to −140 dBm).

Solution:

(a) LoRa data rate: $R_b = SF \times BW \times CR / 2^{SF}$ $R_b = 10 \times 125{,}000 \times (4/5) / 2^{10} = 10 \times 125{,}000 \times 0.8 / 1{,}024$ $R_b = 1{,}000{,}000 / 1{,}024 = 976.6$ bps ≈ 977 bps

(b) $MAPL = P_{tx} + G_{tx} + G_{rx} - P_{sensitivity}$ $MAPL = 20 + 2 + 6 - (-134) = 162$ dB

(c) Maximum range: $L = L_0 + 10n \times \log_{10}(d)$ $162 = 31.5 + 10 \times 3.3 \times \log_{10}(d)$ $33 \times \log_{10}(d) = 130.5$ $\log_{10}(d) = 3.955$ $d = 10^{3.955} = 9{,}016$ m ≈ 9.0 km

(d) Path loss at 5 km: $L = 31.5 + 33 \times \log_{10}(5{,}000) = 31.5 + 33 \times 3.699 = 31.5 + 122.1 = 153.6$ dB Margin $= 162 - 153.6 = 8.4$ dB

(e) At SF12 (sensitivity = −140 dBm): $MAPL = 20 + 2 + 6 - (-140) = 168$ dB $\log_{10}(d) = (168 - 31.5) / 33 = 136.5 / 33 = 4.136$ $d = 10^{4.136} = 13{,}690$ m ≈ 13.7 km

Data rate at SF12: $R_b = 12 \times 125{,}000 \times 0.8 / 2^{12} = 1{,}200{,}000 / 4{,}096 = 293$ bps

The 6 dB sensitivity improvement extends range by 13.7/9.0 = 1.52× (52%), but the data rate drops from 977 to 293 bps (3.3× slower). This is the fundamental LoRa trade-off: spreading factor increases range at the cost of data rate.

Problem 2.8.7

Given: A fiber optic link uses single-mode fiber (SMF) at 1,310 nm over 40 km. The transmitter output is 0 dBm (1 mW) from an SFP+ module. The fiber attenuation is 0.35 dB/km at 1,310 nm. There are 4 fusion splices (0.1 dB each) and 2 connectors (0.3 dB each). The receiver sensitivity is −24 dBm for 10 Gbps with $BER = 10^{-12}$.

Find: (a) The total fiber loss, (b) the total connector and splice loss, (c) the total link loss, (d) the power margin, and (e) the maximum achievable distance with 3 dB margin.

Solution:

(a) Fiber loss: $L_{fiber} = 0.35 \times 40 = 14.0$ dB

(b) Connector and splice loss: $L_{splice} = 4 \times 0.1 = 0.4$ dB $L_{connector} = 2 \times 0.3 = 0.6$ dB Total: 1.0 dB

(c) Total link loss: $L_{total} = 14.0 + 1.0 = 15.0$ dB

(d) Received power: $P_{rx} = 0 - 15.0 = -15.0$ dBm Power margin: $M = P_{rx} - P_{sensitivity} = -15.0 - (-24) = 9.0$ dB

(e) Maximum distance with 3 dB margin: Available loss budget $= P_{tx} - P_{sensitivity} -$ margin $- L_{conn/splice} = 0 - (-24) - 3 - 1.0 = 20.0$ dB for fiber Maximum distance $= 20.0 / 0.35 = 57.1$ km

At 1,550 nm with 0.20 dB/km attenuation: $d_{max} = 20.0/0.20 = 100$ km. The lower attenuation at 1,550 nm nearly doubles the reach, which is why long-haul systems use the C-band (1,550 nm).

Problem 2.8.8

Given: A Bluetooth Low Energy (BLE) link at 2.4 GHz is used for an indoor asset tracking system. P_{tx} = 0 dBm, $G_{tx} = G_{rx}$ = 0 dBi. The receiver sensitivity is −97 dBm at 1 Mbps data rate (BLE 1M PHY). The indoor propagation model uses a log-distance model with n = 2.8 and reference loss L_0 = 40 dB at 1 m. Wall penetration loss is 5 dB per wall.

Find: (a) The MAPL, (b) the maximum range in open indoor space (no walls), (c) the range through 2 interior walls, (d) the received signal strength at 10 m (for RSSI-based ranging accuracy estimation), and (e) the ranging error if RSSI varies by ±6 dB due to multipath.

Solution:

(a) MAPL = 0 + 0 + 0 − (−97) = 97 dB

(b) Maximum range (no walls): $97 = 40 + 10 \times 2.8 \times \log_{10}(d)$ $28 \times \log_{10}(d) = 57$ $\log_{10}(d) = 2.036$ $d = 10^{2.036} = 108.6$ m

(c) Through 2 walls: available loss = $97 - 2 \times 5 = 87$ dB $28 \times \log_{10}(d) = 87 - 40 = 47$ $\log_{10}(d) = 1.679$ $d = 10^{1.679} = 47.7$ m

(d) RSSI at 10 m: $L = 40 + 28 \times \log_{10}(10) = 40 + 28 = 68$ dB $P_{rx} = 0 - 68 = -68$ dBm

(e) RSSI-based distance estimation at −68 dBm: $d_{est} = 10^{(Prx - P_0 + L_0)/(10n)}$

With ±6 dB variation: At −62 dBm: $L = 62 \rightarrow d = 10^{(62-40)/28} = 10^{0.786} = 6.11$ m At −74 dBm: $L = 74 \rightarrow d = 10^{(74-40)/28} = 10^{1.214} = 16.37$ m

True distance = 10 m, but RSSI estimates range from 6.1 to 16.4 m. Ranging error: −3.9 m to +6.4 m (−39% to +64%)

This illustrates why RSSI-based indoor positioning is limited to room-level accuracy (3–5 m). BLE 5.1 direction-finding (Angle of Arrival) achieves sub-meter accuracy by using phase information instead of amplitude.

Problem 2.8.9

Given: A 5G NR mmWave outdoor urban small cell operates at 39 GHz. The base station transmits at 36 dBm using a 256-element phased array with 30 dBi beamforming gain. The UE (user equipment) has G_{rx} = 5 dBi (small phone array). NF_{rx} = 7 dB, channel bandwidth = 100 MHz. The propagation model is the 3GPP UMi street canyon model: $L = 32.4 + 21.0 \log_{10}(d_{3D}) + 20 \log_{10}(f_{GHz})$. Minimum required SNR is 5 dB for QPSK 1/2.

Find: (a) The noise floor, (b) the minimum required received power, (c) the allowable path loss, (d) the maximum LOS range, and (e) the link margin at 100 m.

Solution:

(a) Noise floor: $N = -174 + 10 \log_{10}(100 \times 10^6) + 7 = -174 + 80 + 7 = -87$ dBm

(b) Minimum received power: $P_{rx,min} = N + SNR_{req} = -87 + 5 = -82$ dBm

(c) Allowable path loss: $MAPL = P_{tx} + G_{tx} + G_{rx} - P_{rx,min} = 36 + 30 + 5 - (-82) = 153$ dB

(d) Maximum LOS range: $L = 32.4 + 21.0 \log_{10}(d) + 20 \log_{10}(39) = 32.4 + 21.0 \log_{10}(d) + 31.82$ $L = 64.22 + 21.0 \log_{10}(d)$

Setting L = MAPL: $153 = 64.22 + 21.0 \log_{10}(d)$ $\log_{10}(d) = 88.78 / 21.0 = 4.228$ $d = 10^{4.228} = 16{,}896$ m ≈ 16.9 km (theoretical LOS)

This seems very large — this is because the 30 dBi array gain is substantial. However, mmWave is limited by blockage and NLOS conditions, not LOS range.

(e) Link margin at 100 m: $L = 64.22 + 21.0 \times \log_{10}(100) = 64.22 + 42.0 = 106.22$ dB $P_{rx} = 36 + 30 + 5 - 106.22 = -35.22$ dBm $SNR = -35.22 - (-87) = 51.78$ dB Margin over QPSK 1/2 requirement: $51.78 - 5 = 46.8$ dB

At 100 m with 51.8 dB SNR, the system can use 256-QAM (~27 dB SNR) with massive margin, achieving peak rates of ~2 Gbps in 100 MHz. The real challenge at mmWave is blockage: a human body attenuates 20–35 dB, and building materials cause 40+ dB loss, which is why 5G mmWave requires dense small cell deployment.

Problem 2.8.10

Given: A maritime VHF radio link operates at 156 MHz (Channel 16, distress frequency). The ship transmitter has $P_{tx} = 25$ W (43.98 dBm) and the antenna height is 5 m above sea level with $G_{tx} = 3$ dBi. The coast station antenna is at 50 m height with $G_{rx} = 6$ dBi. Receiver sensitivity is −107 dBm. The radio horizon distance is given by $d(\text{km}) \approx 4.12 \times (\sqrt{h_1} + \sqrt{h_2})$ where h is in meters.

Find: (a) The radio horizon distance, (b) the FSPL at the horizon distance, (c) the received power at the horizon, (d) the link margin at the horizon, and (e) the maximum range for reliable communication (accounting for a 10 dB fade margin for sea-state fading).

Solution:

(a) Radio horizon: $d = 4.12 \times (\sqrt{5} + \sqrt{50}) = 4.12 \times (2.236 + 7.071) = 4.12 \times 9.307 = 38.3$ km

(b) FSPL at 156 MHz, 38.3 km: $\lambda = 3 \times 10^8 / 156 \times 10^6 = 1.923$ m $FSPL = 20 \log_{10}(4\pi \times 38{,}300 / 1.923) = 20 \log_{10}(249{,}870) = 20 \times 5.398 = 107.96$ dB

(c) Received power at horizon: $P_{rx} = 43.98 + 3 - 107.96 + 6 = -54.98$ dBm

(d) Link margin: $M = -54.98 - (-107) = 52.0$ dB

(e) With 10 dB fade margin, maximum allowable path loss: $MAPL = 43.98 + 3 + 6 - (-107) - 10 = 149.98$ dB

FSPL = 149.98: $20 \log_{10}(4\pi d/1.923) = 149.98$ $4\pi d/1.923 = 10^{149.98/20} = 10^{7.499} = 3.157 \times 10^7$ $d = 3.157 \times 10^7 \times 1.923 / (4\pi) = 4.833 \times 10^6$ m = 4,833 km

This far exceeds the radio horizon (38.3 km), so the limiting factor is not RF power but the line of sight. The maximum reliable range is the radio horizon: 38.3 km — beyond this, the signal is blocked by Earth's curvature.

This is why maritime VHF is considered a short-range service (typically 20–40 nautical miles = 37–74 km depending on antenna heights), and long-range maritime communication uses HF (sky-wave) or satellite systems.

Chapter 3 — Section 3.1: Semiconductor Fundamentals

Practice problems covering energy bands, doping, carrier concentrations, drift, and diffusion in semiconductor materials.

Problem 3.1.1

Given: A germanium sample ($E_g = 0.66$ eV) has an intrinsic carrier concentration $n_i = 2.4 \times 10^{13}$ cm^{-3} at 300 K. The material constant B is the same for both temperatures in the approximation $n_i \sim B \times T^{3/2} \times e^{-Eg/(2kT)}$.

Find: The intrinsic carrier concentration at 350 K. Use $k = 8.617 \times 10^{-5}$ eV/K.

Solution: Taking the ratio at two temperatures (B cancels):

$n_i(T_2)/n_i(T_1) = (T_2/T_1)^{3/2} \times e^{-Eg/(2k) \times (1/T_2 - 1/T_1)}$

Temperature ratio: $(350/300)^{3/2} = (1.1667)^{3/2} = 1.260$

Exponent: $-E_g/(2k) \times (1/T_2 - 1/T_1) = -0.66/(2 \times 8.617 \times 10^{-5}) \times (1/350 - 1/300) = -3{,}830 \times (0.002857 - 0.003333) = -3{,}830 \times (-4.762 \times 10^{-4}) = 1.824$

$e^{1.824} = 6.197$

$n_i(350 \text{ K}) = 2.4 \times 10^{13} \times 1.260 \times 6.197 = 1.87 \times 10^{14}$ cm^{-3}

The carrier concentration increases by a factor of ~7.8 for a 50 K rise, illustrating the strong temperature dependence of germanium (which has a smaller bandgap than silicon).

Problem 3.1.2

Given: A silicon sample is doped with boron (acceptor) at a concentration of $N_A = 8 \times 10^{15}$ cm^{-3}. The hole mobility is $mu_p = 450$ cm^2/(V*s). Use $n_i = 1.5 \times 10^{10}$ cm^{-3} and $q = 1.602 \times 10^{-19}$ C.

Find: (a) The majority and minority carrier concentrations, (b) the conductivity, and (c) the resistivity.

Solution:

(a) This is P-type silicon (boron is a Group III acceptor).

Majority carriers (holes): $p \sim N_A = 8 \times 10^{15}$ cm^{-3}

Minority carriers (electrons): $n = n_i^2 / p = (1.5 \times 10^{10})^2 / (8 \times 10^{15}) = 2.25 \times 10^{20} / 8 \times 10^{15} = 2.81 \times 10^4$ cm^{-3}

(b) Conductivity (dominated by majority carriers): sigma = q x p x mu_p = $1.602 \times 10^{-19} \times 8 \times 10^{15} \times 450$ = **0.577 (ohm*cm)$^{-1}$**

(c) Resistivity: rho = 1/sigma = 1/0.577 = **1.73 ohm*cm**

Problem 3.1.3

Given: A silicon wafer is doped with both phosphorus ($N_D = 3 \times 10^{16}$ cm^{-3}) and boron ($N_A = 5 \times 10^{15}$ cm^{-3}) — a compensated semiconductor. Use $n_i = 1.5 \times 10^{10}$ cm^{-3}, mu_n = 1,200 cm^2/(V*s), and q = 1.602×10^{-19} C.

Find: (a) The net doping type, (b) the majority and minority carrier concentrations, and (c) the resistivity.

Solution:

(a) Since $N_D > N_A$, the net doping is N-type.

(b) Net donor concentration: $N_D - N_A = 3 \times 10^{16} - 5 \times 10^{15} = 2.5 \times 10^{16}$ cm^{-3}

Majority carriers (electrons): $n \sim N_D - N_A = 2.5 \times 10^{16}$ cm^{-3}

Minority carriers (holes): $p = n_i^2 / n = (1.5 \times 10^{10})^2 / (2.5 \times 10^{16}) = 2.25 \times 10^{20} / 2.5 \times 10^{16} = 9.0 \times 10^3$ cm^{-3}

(c) sigma = q x n x mu_n = $1.602 \times 10^{-19} \times 2.5 \times 10^{16}$ x 1,200 = 4.806 (ohm*cm)$^{-1}$

rho = 1/sigma = 1/4.806 = **0.208 ohm*cm**

Problem 3.1.4

Given: An N-type silicon bar ($N_D = 5 \times 10^{16}$ cm^{-3}) is 1 cm long with a cross-sectional area of 0.01 cm^2. A voltage of 10 V is applied across its length. The electron mobility is mu_n = 1,100 cm^2/(V*s). Use q = 1.602×10^{-19} C.

Find: (a) The electric field, (b) the electron drift velocity, (c) the current density, (d) the total current, and (e) the resistance of the bar.

Solution:

(a) Electric field: E = V/L = 10/1 = 10 V/cm

(b) Drift velocity: v_d = mu_n x E = 1,100 x 10 = 11,000 cm/s = 110 m/s

(c) Current density: $J = q \times n \times mu_n \times E = 1.602 \times 10^{-19} \times 5 \times 10^{16} \times 1{,}100 \times 10 = 1.602 \times 10^{-19} \times 5.5 \times 10^{20} = 88.1\ A/cm^2$

(d) Total current: $I = J \times A = 88.1 \times 0.01 = 0.881$ A

(e) Resistance: $R = V/I = 10/0.881 = 11.35$ ohm

Alternatively: rho = $1/(q \times n \times mu_n) = 1/(1.602 \times 10^{-19} \times 5 \times 10^{16} \times 1{,}100) = 0.1135$ ohm*cm R = rho x L/A = 0.1135 x 1/0.01 = 11.35 ohm (confirms).

Problem 3.1.5

Given: In a P-type silicon region, the hole concentration decreases linearly from $p_1 = 10^{17}\ cm^{-3}$ to $p_2 = 10^{15}\ cm^{-3}$ over a distance of 2 um. The hole mobility is $mu_p = 400\ cm^2/(V*s)$ and T = 300 K. Use $q = 1.602 \times 10^{-19}$ C and $V_T = kT/q = 0.02585$ V.

Find: (a) The hole diffusion coefficient, (b) the concentration gradient, and (c) the hole diffusion current density.

Solution:

(a) Diffusion coefficient (Einstein relation): $D_p = V_T \times mu_p = 0.02585 \times 400 = 10.34\ cm^2/s$

(b) Concentration gradient: $dp/dx = (p_2 - p_1)$ / delta_x $= (10^{15} - 10^{17}) / (2 \times 10^{-4}) = -9.9 \times 10^{16} / (2 \times 10^{-4}) = -4.95 \times 10^{20}\ cm^{-4}$

(c) Hole diffusion current density: $J_{diff,p} = -q \times D_p \times (dp/dx) = -1.602 \times 10^{-19} \times 10.34 \times (-4.95 \times 10^{20}) = 819.7\ A/cm^2$

The positive value indicates current flows in the direction of decreasing hole concentration (from high to low), which is physically correct for hole diffusion.

Problem 3.1.6

Given: An N-type silicon sample ($N_D = 10^{16}\ cm^{-3}$) is illuminated, generating excess carriers at a uniform rate of $G = 10^{20}\ cm^{-3}/s$. The minority carrier (hole) lifetime is $tau_p = 10$ us.

Find: (a) The steady-state excess minority carrier concentration, (b) the total hole concentration under illumination, and (c) the percentage change in majority carrier concentration. Use $n_i = 1.5 \times 10^{10}\ cm^{-3}$.

Solution:

(a) Steady-state excess carrier concentration: delta_p = G x $tau_p = 10^{20} \times 10 \times 10^{-6} = 10^{15}\ cm^{-3}$

(b) Equilibrium hole concentration: $p_0 = n_i^2/N_D = (1.5 \times 10^{10})^2 / 10^{16} = 2.25 \times 10^4\ cm^{-3}$

Total hole concentration: $p = p_0$ + delta_p $= 2.25 \times 10^4 + 10^{15} \sim 10^{15}\ cm^{-3}$ (delta_p dominates over p_0 by 11 orders of magnitude)

(c) Majority carrier concentration also increases by delta_p to maintain charge neutrality: $n = N_D$ + delta_p $= 10^{16} + 10^{15} = 1.1 \times 10^{16}\ cm^{-3}$

Percentage change = delta_p/N_D x 100 = $10^{15}/10^{16}$ x 100 = 10%

While the minority carrier concentration changed by 11 orders of magnitude, the majority carrier concentration changed by only 10%.

Chapter 3 — Section 3.2: Semiconductor Materials

Practice problems covering silicon, gallium arsenide, and gallium nitride material properties and device comparisons.

Problem 3.2.1

Given: Intrinsic silicon at T = 300 K has $n_i = 1.5 \times 10^{10}$ cm^{-3}. A silicon sample is doped with arsenic (donor) at $N_D = 2 \times 10^{17}$ cm^{-3}. Use kT = 0.02585 eV at 300 K.

Find: (a) The electron and hole concentrations, (b) the Fermi level position relative to the intrinsic Fermi level, and (c) the Fermi level position relative to the conduction band edge if $E_g = 1.12$ eV.

Solution:

(a) Electron concentration: $n \sim N_D = 2 \times 10^{17}$ cm^{-3}

Hole concentration: $p = n_i^2/n = (1.5 \times 10^{10})^2 / (2 \times 10^{17}) = 2.25 \times 10^{20} / 2 \times 10^{17} = 1.125 \times 10^3$ cm^{-3}

(b) Fermi level position: $E_F - E_i = kT \times \ln(n/n_i) = 0.02585 \times \ln(2 \times 10^{17} / 1.5 \times 10^{10}) = 0.02585 \times \ln(1.333 \times 10^7) = 0.02585 \times 16.41 = 0.424$ eV above E_i

(c) The intrinsic Fermi level is approximately at mid-gap: $E_i \sim E_C - E_g/2 = E_C - 0.56$ eV.

$E_C - E_F = E_g/2 - (E_F - E_i) = 0.56 - 0.424 = 0.136$ eV below E_C

The Fermi level is close to the conduction band, confirming strong N-type behavior.

Problem 3.2.2

Given: A GaAs laser diode emits light at a wavelength corresponding to its bandgap of $E_g = 1.42$ eV. A silicon photodetector has a bandgap of $E_g = 1.12$ eV. Use $h = 6.626 \times 10^{-34}$ J*s, $c = 3 \times 10^8$ m/s, and 1 eV = 1.602×10^{-19} J.

Find: (a) The emission wavelength of the GaAs laser, (b) whether the silicon detector can absorb this wavelength, and (c) the maximum (cutoff) wavelength the silicon detector can absorb.

Solution:

(a) Emission wavelength: lambda = hc/E_g = ($6.626 \times 10^{-34} \times 3 \times 10^8$) / ($1.42 \times 1.602 \times 10^{-19}$) = 1.988×10^{-25} / 2.275×10^{-19} = 8.74×10^{-7} m = 874 nm (near infrared)

(b) For the silicon detector to absorb the photon, the photon energy must exceed silicon's bandgap: Photon energy = 1.42 eV > $E_{g,Si}$ = 1.12 eV. Yes, silicon can absorb the 874 nm GaAs emission.

(c) Silicon cutoff wavelength: $lambda_{cutoff} = hc/E_{g,Si}$ = 1.988×10^{-25} / ($1.12 \times 1.602 \times 10^{-19}$) = 1.988×10^{-25} / 1.794×10^{-19} = 1.108×10^{-6} m = 1,108 nm

Silicon can detect wavelengths shorter than 1,108 nm, so the 874 nm GaAs emission falls well within the detector's range.

Problem 3.2.3

Given: A GaAs HEMT has electron mobility mu_n = 8,000 $cm^2/(Vs)$ *and a silicon MOSFET has* mu_n = *1,200* $cm^2/(Vs)$. Both devices have a channel length of L = 0.25 um. The GaAs device operates at V_{DD} = 3.3 V and the silicon device at V_{DD} = 1.8 V.

Find: (a) The electron transit time for each device at an average field of E = 10 kV/cm, (b) the estimated f_T for each, and (c) the ratio of f_T values.

Solution:

(a) Drift velocity: GaAs: v_d = 8,000 x 10,000 = 8.0×10^7 cm/s Silicon: v_d = 1,200 x 10,000 = 1.2×10^7 cm/s

Transit time tau = L/v_d: GaAs: tau = 0.25×10^{-4} / 8.0×10^7 = 3.13×10^{-13} s = 0.313 ps Silicon: tau = 0.25×10^{-4} / 1.2×10^7 = 2.08×10^{-12} s = 2.08 ps

(b) f_T ~ 1/(2 x pi x tau): GaAs: f_T = 1/(2 x pi x 3.13×10^{-13}) = 509 GHz Silicon: f_T = 1/(2 x pi x 2.08×10^{-12}) = 76.5 GHz

(c) Ratio: 509/76.5 = 6.7x advantage for GaAs, consistent with its use in mmWave and 5G applications.

Problem 3.2.4

Given: A GaN HEMT power amplifier for a 5G base station operates at 3.5 GHz with V_{DS} = 28 V and I_D = 3.5 A. The device achieves a drain efficiency of 65%. An equivalent GaAs device at the same frequency achieves 45% efficiency at V_{DS} = 12 V and I_D = 7 A.

Find: (a) The DC input power and RF output power for each device, (b) the heat dissipated in each, and (c) the power density advantage of GaN (assuming the GaN die is 4 mm x 1 mm and the GaAs die is 8 mm x 1.5 mm).

Solution:

(a) GaN: $P_{DC} = V_{DS} \times I_D$ = 28 x 3.5 = 98 W P_{RF} = eta x P_{DC} = 0.65 x 98 = 63.7 W

GaAs: P_{DC} = 12 x 7 = 84 W P_{RF} = 0.45 x 84 = 37.8 W

(b) Heat dissipated: GaN: $P_{heat} = P_{DC} - P_{RF} = 98 - 63.7 = 34.3$ W GaAs: $P_{heat} = 84 - 37.8 = 46.2$ W

(c) Power density (RF output per die area): GaN: 63.7 / (4 x 1) = 15.9 W/mm^2 GaAs: 37.8 / (8 x 1.5) = 3.15 W/mm^2

Power density advantage: 15.9 / 3.15 = 5.0x, demonstrating why GaN enables smaller, higher-power amplifiers.

Problem 3.2.5

Given: A GaN-on-SiC power switching transistor operates at $V_{DS} = 650$ V with $R_{DS(on)} = 25$ mohm at 25 degrees C. The $R_{DS(on)}$ temperature coefficient is +0.8%/degree C. An equivalent SiC MOSFET has $R_{DS(on)} = 40$ mohm at 25 degrees C with a temperature coefficient of +1.2%/degree C. Both devices carry 30 A at a junction temperature of 150 degrees C.

Find: (a) $R_{DS(on)}$ at 150 degrees C for each device, and (b) the conduction losses at 150 degrees C for each.

Solution:

(a) Temperature rise: delta_T = 150 - 25 = 125 degrees C.

GaN: $R_{DS(on)}(150) = 25 \times (1 + 0.008 \times 125) = 25 \times 2.0 = 50$ mohm SiC: $R_{DS(on)}(150) = 40 \times (1 + 0.012 \times 125) = 40 \times 2.5 = 100$ mohm

(b) Conduction losses at 150 degrees C: GaN: $P_{cond} = I^2 \times R_{DS(on)} = 30^2 \times 0.050 = 900 \times 0.050 = 45.0$ W SiC: $P_{cond} = 30^2 \times 0.100 = 900 \times 0.100 = 90.0$ W

The GaN device has 50% lower conduction losses at operating temperature, though both devices show significant $R_{DS(on)}$ increases (2x and 2.5x) from the cold-start values — thermal derating must be included in the design.

Chapter 3 — Section 3.3: PN Junction

Practice problems covering depletion regions, built-in potential, forward/reverse bias characteristics, and junction capacitance.

Problem 3.3.1

Given: A silicon PN junction has $N_A = 10^{18}$ cm^{-3} and $N_D = 5 \times 10^{15}$ cm^{-3}. T = 300 K. Use $n_i = 1.5 \times 10^{10}$ cm^{-3}, $V_T = 0.02585$ V, $epsilon_{Si} = 1.036 \times 10^{-12}$ F/cm, and $q = 1.602 \times 10^{-19}$ C.

Find: (a) The built-in potential, (b) the depletion width at zero bias, (c) the depletion width at a reverse bias of 10 V, and (d) the ratio of depletion widths.

Solution:

(a) Built-in potential: $V_{bi} = V_T \times \ln(N_A \times N_D / n_i^2) = 0.02585 \times \ln(10^{18} \times 5 \times 10^{15} / (1.5 \times 10^{10})^2) = 0.02585 \times \ln(5 \times 10^{33} / 2.25 \times 10^{20}) = 0.02585 \times \ln(2.222 \times 10^{13}) = 0.02585 \times 30.73 = 0.795$ V

(b) Since $N_A \gg N_D$, the depletion extends mostly into the N-side: $1/N_A + 1/N_D \sim 1/N_D = 2 \times 10^{-16}$ cm^3

$W = \mathrm{sqrt}(2 \times epsilon_{Si} \times (V_{bi} - V_A) / (q \times N_D))$ for one-sided junction approximation: $W = \mathrm{sqrt}(2 \times 1.036 \times 10^{-12} \times 0.795 / (1.602 \times 10^{-19} \times 5 \times 10^{15})) = \mathrm{sqrt}(1.647 \times 10^{-12} / 8.01 \times 10^{-4}) = \mathrm{sqrt}(2.056 \times 10^{-9}) = 4.54 \times 10^{-5}$ cm = 0.454 um

(c) At $V_R = 10$ V ($V_A = -10$ V): $W = \mathrm{sqrt}(2 \times 1.036 \times 10^{-12} \times (0.795 + 10) / (1.602 \times 10^{-19} \times 5 \times 10^{15})) = \mathrm{sqrt}(2 \times 1.036 \times 10^{-12} \times 10.795 / 8.01 \times 10^{-4}) = \mathrm{sqrt}(2.238 \times 10^{-11} / 8.01 \times 10^{-4}) = \mathrm{sqrt}(2.794 \times 10^{-8}) = 1.672 \times 10^{-4}$ cm = 1.672 um

(d) Ratio: 1.672/0.454 = 3.68x

The depletion width increases by a factor of $\mathrm{sqrt}((V_{bi} + V_R)/V_{bi}) = \mathrm{sqrt}(10.795/0.795) = 3.68$, confirming the square-root voltage dependence.

Problem 3.3.2

Given: A silicon diode has $I_S = 5 \times 10^{-15}$ A and ideality factor n = 1.05 at T = 300 K. The diode carries a forward current of $I_F = 1.5$ mA.

Find: (a) The forward voltage across the diode, (b) the dynamic (small-signal) resistance at this operating point, and (c) the forward voltage if the temperature increases to 350 K (assume I_S doubles for every 10 degrees C increase).

Solution:

(a) From the Shockley equation, $I = I_S(e^{V/(nVT)} - 1) \sim I_S \times e^{V/(nVT)}$ (since $I \gg I_S$):

$V = n \times V_T \times \ln(I/I_S) = 1.05 \times 0.02585 \times \ln(1.5 \times 10^{-3} / 5 \times 10^{-15}) = 0.02714 \times \ln(3 \times 10^{11}) = 0.02714 \times 26.43 = 0.717$ V

(b) Dynamic resistance: $r_d = n \times V_T / I = 1.05 \times 0.02585 / 0.0015 = 0.02714 / 0.0015 = 18.1$ ohm

(c) At 350 K: $V_T(350) = kT/q = 8.617 \times 10^{-5} \times 350 = 0.03016$ V

I_S doubles per 10 degrees C, so over 50 degrees C: $I_S(350) = 5 \times 10^{-15} \times 2^5 = 5 \times 10^{-15} \times 32 = 1.6 \times 10^{-13}$ A

$V = n \times V_T(350) \times \ln(I/I_S(350)) = 1.05 \times 0.03016 \times \ln(1.5 \times 10^{-3} / 1.6 \times 10^{-13}) = 0.03167 \times \ln(9.375 \times 10^9) = 0.03167 \times 22.96 = 0.727$ V

The voltage change is only +10 mV despite a 50 K temperature rise — the increase in V_T is largely offset by the increase in I_S. In practice, the forward voltage typically decreases at approximately -2 mV/degree C at constant current.

Problem 3.3.3

Given: A silicon solar cell PN junction has $N_A = 10^{17}$ cm^{-3}, $N_D = 10^{16}$ cm^{-3}, and a junction area of A = 100 cm^2. The built-in potential is $V_{bi} = 0.757$ V and epsilon$_{Si}$ = 1.036×10^{-12} F/cm. The junction operates at a reverse bias of $V_R = 0.3$ V (the solar cell's maximum power point voltage creates a slight forward bias reducing the depletion capacitance, but here we consider the dark capacitance at reverse bias).

Find: (a) The depletion width, (b) the zero-bias junction capacitance per unit area, and (c) the total junction capacitance at $V_R = 0.3$ V.

Solution:

(a) Using $1/N_A + 1/N_D = 10^{-17} + 10^{-16} = 1.1 \times 10^{-16}$ cm^3:

W = sqrt($2 \times 1.036 \times 10^{-12} \times 1.1 \times 10^{-16} \times (0.757 + 0.3) / 1.602 \times 10^{-19}$) = sqrt($2.279 \times 10^{-28} \times 1.057 / 1.602 \times 10^{-19}$) = sqrt($2.409 \times 10^{-28} / 1.602 \times 10^{-19}$) = sqrt($1.504 \times 10^{-9}$) = 3.88×10^{-5} cm = 0.388 um

(b) Zero-bias junction capacitance per unit area: C_{j0}/A = epsilon$_{Si}$/W_0

W_0 = sqrt($2 \times 1.036 \times 10^{-12} \times 1.1 \times 10^{-16} \times 0.757 / 1.602 \times 10^{-19}$) = sqrt($1.724 \times 10^{-28} / 1.602 \times 10^{-19}$) = sqrt($1.076 \times 10^{-9}$) = 3.28×10^{-5} cm

$C_{j0}/A = 1.036 \times 10^{-12} / 3.28 \times 10^{-5} = 3.16 \times 10^{-8}$ F/cm^2 = 31.6 nF/cm^2

(c) At $V_R = 0.3$ V: $C_j = C_{j0}$ / sqrt($1 + V_R/V_{bi}$)

$C_{j0} = 31.6 \times 10^{-9} \times 100 = 3.16$ uF (total zero-bias capacitance) $C_j = 3.16 \times 10^{-6}$ / sqrt(1 + 0.3/0.757) = 3.16×10^{-6} / sqrt(1.396) = $3.16 \times 10^{-6} / 1.182 = 2.67$ uF

Problem 3.3.4

Given: A varactor diode has $C_{j0} = 50$ pF, $V_{bi} = 0.7$ V, and m = 0.45 (hyperabrupt junction). It is used in a VCO tank circuit with L = 10 nH.

Find: (a) The junction capacitance at reverse bias voltages of 0.5 V, 2 V, 5 V, and 12 V, and (b) the corresponding resonant frequencies and the tuning range.

Solution:

(a) $C_j = C_{j0} / (1 + V_R/V_{bi})^m$:

At $V_R = 0.5$ V: C = 50 / $(1.714)^{0.45}$ = 50 / 1.274 = 39.2 pF At $V_R = 2$ V: C = 50 / $(3.857)^{0.45}$ = 50 / 1.836 = 27.2 pF At $V_R = 5$ V: C = 50 / $(8.143)^{0.45}$ = 50 / 2.570 = 19.5 pF At $V_R = 12$ V: C = 50 / $(18.14)^{0.45}$ = 50 / 3.684 = 13.6 pF

(b) Resonant frequency $f_0 = 1/(2\pi\sqrt{(LC)})$:

At 0.5 V: $f_0 = 1/(2\pi\sqrt{(10 \times 10^{-9} \times 39.2 \times 10^{-12})}) = 1/(2\pi \times 6.261 \times 10^{-10})$ = 254 MHz At 2 V: $f_0 = 1/(2\pi\sqrt{(10 \times 10^{-9} \times 27.2 \times 10^{-12})}) = 1/(2\pi \times 5.215 \times 10^{-10})$ = 305 MHz At 5 V: $f_0 = 1/(2\pi\sqrt{(10 \times 10^{-9} \times 19.5 \times 10^{-12})}) = 1/(2\pi \times 4.416 \times 10^{-10})$ = 361 MHz At 12 V: $f_0 = 1/(2\pi\sqrt{(10 \times 10^{-9} \times 13.6 \times 10^{-12})}) = 1/(2\pi \times 3.688 \times 10^{-10})$ = 432 MHz

Tuning range: 432/254 = 1.70:1 (254 MHz to 432 MHz), suitable for a wideband VCO.

Problem 3.3.5

Given: Two silicon diodes are connected in series (same polarity) and driven by a constant current source of I = 2 mA. Diode 1 has $I_{S1} = 10^{-14}$ A and Diode 2 has $I_{S2} = 5 \times 10^{-14}$ A. Both have n = 1.0 and T = 300 K.

Find: (a) The forward voltage across each diode, (b) the total voltage, and (c) the voltage across each diode if the temperature rises to 85 degrees C (358 K), assuming I_S doubles every 10 degrees C.

Solution:

(a) At 300 K, $V_T = 0.02585$ V:

$V_1 = V_T \times \ln(I/I_{S1}) = 0.02585 \times \ln(2 \times 10^{-3} / 10^{-14}) = 0.02585 \times \ln(2 \times 10^{11}) = 0.02585 \times 26.02 = 0.673$ V

$V_2 = V_T \times \ln(I/I_{S2}) = 0.02585 \times \ln(2 \times 10^{-3} / 5 \times 10^{-14}) = 0.02585 \times \ln(4 \times 10^{10}) = 0.02585 \times 24.41 = 0.631$ V

(b) Total voltage: $V_{total} = 0.673 + 0.631 = 1.304$ V

(c) At 358 K (85 degrees C): $V_T(358) = 8.617 \times 10^{-5} \times 358 = 0.03085$ V Temperature rise = 58 degrees C, so I_S multiplier = $2^{5.8} = 55.7$

$I_{S1}(358) = 10^{-14} \times 55.7 = 5.57 \times 10^{-13}$ A $I_{S2}(358) = 5 \times 10^{-14} \times 55.7 = 2.785 \times 10^{-12}$ A

$V_1 = 0.03085 \times \ln(2 \times 10^{-3} / 5.57 \times 10^{-13}) = 0.03085 \times \ln(3.59 \times 10^{9}) = 0.03085 \times 22.00 = 0.679$ V $V_2 = 0.03085 \times \ln(2 \times 10^{-3} / 2.785 \times 10^{-12}) = 0.03085 \times \ln(7.18 \times 10^{8}) = 0.03085 \times 20.39 = 0.629$ V

Total: 0.679 + 0.629 = 1.308 V — remarkably stable despite a 58 degrees C temperature increase, demonstrating why series diode strings are used as voltage references.

Chapter 3 — Section 3.4: Diodes

Practice problems covering rectifier diodes, Zener diodes, Schottky diodes, and LEDs.

Problem 3.4.1

Given: A half-wave rectifier uses a single silicon diode ($V_f = 0.7$ V) powered by a 24 V_{rms} transformer secondary. The load is 50 ohm with a 2,200 uF filter capacitor. Frequency is 60 Hz.

Find: (a) The peak output voltage, (b) the DC load current, (c) the peak-to-peak ripple voltage, and (d) the ripple as a percentage of the DC output.

Solution:

(a) Peak transformer voltage: V_{peak} = 24 x sqrt(2) = 33.94 V One diode conducts: V_{out_peak} = 33.94 - 0.7 = 33.24 V

(b) DC load current: $I_{DC} \sim V_{out\_peak}/R_L$ = 33.24/50 = 664.8 mA

(c) For a half-wave rectifier, f_{ripple} = 60 Hz: $V_{ripple} = I_{DC}/(f_{ripple}$ x C) = 0.6648/(60 x 0.0022) = 0.6648/0.132 = 5.04 V peak-to-peak

(d) Ripple percentage: 5.04/33.24 x 100 = 15.2%

This poor ripple performance illustrates why full-wave rectifiers are preferred for most applications.

Problem 3.4.2

Given: A 12 V Zener diode has dynamic resistance r_z = 15 ohm and is rated for P_{max} = 5 W. The input voltage is 20 V and the series resistor R_S = 47 ohm. The load can vary from R_L = 100 ohm (heavy load) to R_L = open circuit (no load).

Find: (a) The Zener current at full load, (b) the Zener current at no load, (c) the maximum Zener power dissipation, and (d) the output voltage regulation (change in V_{out}) from full load to no load, accounting for r_z.

Solution:

(a) At full load (R_L = 100 ohm): $I_L = V_Z/R_L$ = 12/100 = 120 mA $I_{total} = (V_{in} - V_Z)/R_S$ = (20 - 12)/47 = 8/47 = 170.2 mA $I_Z = I_{total} - I_L$ = 170.2 - 120 = 50.2 mA

(b) At no load (R_L = open, $I_L = 0$): $I_Z = I_{total} = (20 - 12)/47 = 170.2$ mA

(c) Maximum power: $P_Z = V_Z \times I_{Z\_max} = 12 \times 0.1702 = 2.04$ W (within 5 W rating)

(d) Voltage change due to Zener current variation: delta_I_Z = 170.2 - 50.2 = 120 mA delta_V_{out} = delta_I_Z x r_z = 0.120 x 15 = 1.8 V

The output varies from approximately 12.0 V at full load to 12.0 + 1.8 = 13.8 V at no load. Load regulation = 1.8/12 x 100 = 15% — mediocre, suggesting a lower r_z Zener or an active regulator would be preferable.

Problem 3.4.3

Given: A synchronous buck converter switching at 500 kHz uses a Schottky body diode with V_f = 0.45 V during dead time. The dead time is t_{dead} = 30 ns per transition (two transitions per cycle). The inductor current (and diode current during dead time) is I_L = 15 A. The high-side MOSFET body diode would have V_f = 1.0 V during the same dead time.

Find: (a) The energy lost in the Schottky diode per dead-time event, (b) the total dead-time power loss with the Schottky diode, and (c) the power savings compared to using the MOSFET body diode.

Solution:

(a) Energy per dead-time event (Schottky): $E_{dead} = V_f \times I_L \times t_{dead} = 0.45 \times 15 \times 30 \times 10^{-9} = 202.5$ nJ

(b) Two dead-time events per switching cycle: $P_{dead} = 2 \times E_{dead} \times f_{sw} = 2 \times 202.5 \times 10^{-9} \times 500{,}000 = 0.203$ W

(c) With MOSFET body diode: $E_{dead} = 1.0 \times 15 \times 30 \times 10^{-9} = 450$ nJ $P_{dead} = 2 \times 450 \times 10^{-9} \times 500{,}000 = 0.450$ W

Power savings: 0.450 - 0.203 = 0.247 W (55% reduction in dead-time losses).

Problem 3.4.4

Given: An LED streetlight module contains 60 white LEDs arranged as 10 parallel strings of 6 LEDs in series. Each LED has V_f = 3.2 V at I_f = 350 mA and a luminous efficacy of 140 lm/W. The LED driver provides a constant current of 350 mA per string from a 24 V DC bus using a buck converter at 92% efficiency.

Find: (a) The forward voltage of each string, (b) the total luminous output, (c) the electrical power consumed by the LEDs, (d) the total input power from the 24 V bus, and (e) the system luminous efficacy (lumens per watt from the bus).

Solution:

(a) Each string has 6 LEDs in series: $V_{string} = 6 \times 3.2 = 19.2$ V

(b) Power per LED: $P_{LED} = V_f \times I_f = 3.2 \times 0.350 = 1.12$ W Lumens per LED: 140 x 1.12 = 156.8 lm Total: 60 x 156.8 = 9,408 lumens

(c) Total LED power: 60 x 1.12 = 67.2 W

(d) Input power from bus: $P_{in} = P_{LED}$/eta = 67.2/0.92 = 73.0 W

(e) System luminous efficacy: eta_{system} = 9,408/73.0 = 128.9 lm/W

This is approximately 8.6x more efficient than a 15 lm/W incandescent source producing the same light output.

Problem 3.4.5

Given: A full-wave bridge rectifier with four silicon diodes (V_f = 0.7 V each) is connected to a 120 V_{rms} / 18 V_{rms} transformer. The secondary winding resistance is R_w = 0.5 ohm. The load draws 2 A DC through a 4,700 uF filter capacitor at 60 Hz.

Find: (a) The no-load peak output voltage, (b) the approximate loaded DC output voltage accounting for diode drops and winding resistance, (c) the ripple voltage, and (d) the peak repetitive diode current (during the short capacitor charging pulse).

Solution:

(a) V_{peak} = 18 x sqrt(2) = 25.46 V Two diodes conduct: V_{out_peak} = 25.46 - 2 x 0.7 = 24.06 V

(b) With winding resistance loss: $V_{DC} \sim V_{out\_peak} - I_{DC} \times R_w - V_{ripple}/2$ Ripple first: $V_{ripple} = I_{DC}/(f_{ripple}$ x C) = 2/(120 x 0.0047) = 2/0.564 = 3.55 V V_{DC} ~ 24.06 - 2 x 0.5 - 3.55/2 = 24.06 - 1.0 - 1.78 = 21.28 V

(c) Ripple voltage: V_{ripple} = 3.55 V peak-to-peak (calculated above)

(d) The capacitor charging pulse occurs during a fraction of each half cycle. The conduction angle is approximately: $theta_c \sim$ sqrt(2 x V_{ripple}/V_{peak}) = sqrt(2 x 3.55/25.46) = sqrt(0.279) = 0.528 rad = 30.3 degrees

Peak diode current: $I_{peak} \sim I_{DC}$ x (2 x pi / $theta_c$) = 2 x (6.283/0.528) = 2 x 11.9 = 23.8 A

This large peak current is why rectifier diodes must have high surge current ratings and why transformer current ratings exceed the DC load current.

Problem 3.4.6

Given: An engineer needs to design a Zener-based voltage clamp to protect a 3.3 V microcontroller ADC input. The ADC input voltage range is 0 to 3.3 V. A 3.3 V Zener diode (r_z = 30 ohm) is placed from the ADC input to ground, with a 1 kohm series resistor from the signal source. The signal source can produce 0 to 12 V.

Find: (a) The ADC input voltage when the source is at 3.0 V (below Zener threshold), (b) the ADC input voltage when the source is at 5.0 V, (c) the ADC input voltage when the source is at 12 V, and (d) the maximum Zener power dissipation.

Solution:

(a) At V_{source} = 3.0 V: The Zener is not conducting ($V < V_Z$). The ADC has very high input impedance, so virtually no current flows through R_S: V_{ADC} ~ 3.0 V

(b) At V_{source} = 5.0 V: The Zener conducts. Using a simplified model: $I_Z = (V_{source} - V_Z)/(R_S + r_z)$ = (5.0 - 3.3)/(1,000 + 30) = 1.7/1,030 = 1.65 mA $V_{ADC} = V_Z + I_Z$ x r_z = 3.3 + 0.00165 x 30 = 3.3 + 0.050 = 3.35 V

(c) At V_{source} = 12 V: I_Z = (12 - 3.3)/1,030 = 8.7/1,030 = 8.45 mA V_{ADC} = 3.3 + 0.00845 x 30 = 3.3 + 0.254 = 3.55 V

This exceeds 3.3 V by 0.25 V, which may damage the ADC. The dynamic resistance r_z limits the clamping accuracy.

(d) Maximum Zener power: $P_Z = V_Z$ x I_Z ~ 3.3 x 0.00845 = 27.9 mW (negligible for most Zener ratings)

A better protection approach would use a Schottky diode clamp to V_{DD} (3.3 V) with a lower voltage drop, or use a precision TVS diode with lower r_z.

Chapter 3 — Section 3.5: Transistors

Practice problems covering BJTs, MOSFETs, and power MOSFETs in amplification and switching applications.

Problem 3.5.1

Given: An NPN BJT has beta = 200 and V_{BE} = 0.7 V. It is used in a common-emitter configuration with V_{CC} = 15 V, R_C = 3.3 kohm, R_E = 680 ohm, and a voltage divider bias network (R_1 = 56 kohm, R_2 = 12 kohm).

Find: (a) The base voltage V_B, (b) the emitter current I_E, (c) the collector current I_C, (d) V_{CE}, and (e) whether the transistor is in the active region.

Solution:

(a) Base voltage (voltage divider, assuming I_B is small): $V_B = V_{CC} \text{ x } R_2/(R_1 + R_2) = 15 \text{ x } 12{,}000/(56{,}000 + 12{,}000) = 15 \text{ x } 0.1765 = 2.65$ V

(b) Emitter voltage: $V_E = V_B - V_{BE} = 2.65 - 0.7 = 1.95$ V Emitter current: $I_E = V_E/R_E = 1.95/680 = 2.87$ mA

(c) $I_C \sim I_E$ x beta/(beta + 1) = 2.87 x 200/201 = 2.85 mA

(d) $V_{CE} = V_{CC} - I_C \text{ x } R_C - I_E \text{ x } R_E = 15 - 0.00285 \text{ x } 3{,}300 - 0.00287 \text{ x } 680 = 15 - 9.41 - 1.95 = 3.64$ V

(e) $V_{CE} = 3.64 \text{ V} > V_{CE(sat)} \sim 0.2$ V and $V_{CB} = V_{CE} - V_{BE} = 3.64 - 0.7 = 2.94 \text{ V} > 0$. The transistor is in the active region (suitable for linear amplification).

Problem 3.5.2

Given: An NPN BJT switching circuit has V_{CC} = 5 V, R_C = 1 kohm, and a base resistor R_B = 10 kohm driven by a 5 V logic signal. The transistor has $beta_{min}$ = 80 and $V_{CE(sat)}$ = 0.2 V, $V_{BE(sat)}$ = 0.8 V.

Find: (a) The base current, (b) the collector current in saturation, (c) the forced beta, (d) whether the transistor is indeed saturated, and (e) the power dissipated in the transistor when on.

Solution:

(a) Base current: $I_B = (V_{in} - V_{BE(sat)})/R_B = (5 - 0.8)/10{,}000 = 4.2/10{,}000 = 0.42$ mA

(b) Collector current in saturation: $I_{C(sat)} = (V_{CC} - V_{CE(sat)})/R_C = (5 - 0.2)/1{,}000 = 4.8$ mA

(c) Forced beta: $beta_{forced} = I_{C(sat)}/I_B = 4.8/0.42 = 11.4$

(d) Since $beta_{forced} = 11.4 < beta_{min} = 80$, the transistor has excess base drive and is confirmed saturated with an overdrive factor of 80/11.4 = 7.0x.

(e) Power dissipation when on: $P = V_{CE(sat)} \times I_C + V_{BE(sat)} \times I_B = 0.2 \times 0.0048 + 0.8 \times 0.00042 = 0.96$ mW + 0.34 mW = 1.30 mW

Problem 3.5.3

Given: An N-channel enhancement MOSFET has $V_{th} = 1.5$ V and $k_n = 2.0$ mA/V^2. It is biased at $V_{GS} = 4$ V and used with $R_D = 2$ kohm and $V_{DD} = 12$ V.

Find: (a) The drain current, (b) V_{DS}, (c) the operating region, (d) the small-signal transconductance g_m, and (e) the voltage gain with R_D as the load.

Solution:

(a) Assume saturation first: $I_D = (k_n/2) \times (V_{GS} - V_{th})^2 = (2.0 \times 10^{-3}/2) \times (4 - 1.5)^2 = 10^{-3} \times 6.25 = 6.25$ mA

(b) $V_{DS} = V_{DD} - I_D \times R_D = 12 - 0.00625 \times 2{,}000 = 12 - 12.5 = -0.5$ V

Negative V_{DS} is impossible. The MOSFET is in the triode (linear) region, not saturation.

(c) In triode: $I_D = k_n \times [(V_{GS} - V_{th}) \times V_{DS} - V_{DS}^2/2]$ KVL: $V_{DS} = V_{DD} - I_D \times R_D$

Substituting: $V_{DS} = 12 - 2{,}000 \times 2 \times 10^{-3} \times [(2.5) \times V_{DS} - V_{DS}^2/2]$ $V_{DS} = 12 - 4 \times [2.5 \times V_{DS} - V_{DS}^2/2]$ $V_{DS} = 12 - 10 \times V_{DS} + 2 \times V_{DS}^2$ $2V_{DS}^2 - 11V_{DS} + 12 = 0$ $V_{DS} = (11 +/- \text{sqrt}(121 - 96))/4 = (11 +/- 5)/4$

$V_{DS} = 4.0$ V or 1.5 V. Since $V_{DS} < V_{GS} - V_{th} = 2.5$ V for triode: $V_{DS} = 1.5$ V (the other root gives $V_{DS} = 4$ V > 2.5 V, which would be saturation)

Recalculate I_D: $I_D = (12 - 1.5)/2{,}000 = 5.25$ mA

(d) In saturation, g_m would be: $g_m = k_n \times (V_{GS} - V_{th}) = 2 \times 10^{-3} \times 2.5 = 5.0$ mA/V

(e) In saturation, voltage gain: $A_v = -g_m \times R_D = -5.0 \times 10^{-3} \times 2{,}000 = -10$ But since the device is in triode, it acts more like a resistor. To use this MOSFET as an amplifier with gain = -10, R_D must be reduced or V_{DD} increased to keep $V_{DS} > 2.5$ V.

Problem 3.5.4

Given: A power MOSFET drives a 24 V brushless DC motor at $f_{sw} = 20$ kHz using PWM. The MOSFET has $R_{DS(on)} = 12$ mohm at 25 degrees C (increasing to 18 mohm at $T_J = 100$ degrees C), $Q_g = 80$ nC, $t_{rise} = 25$ ns, $t_{fall} = 35$ ns. The motor draws 30 A at a PWM duty cycle of 75%.

Find: (a) The RMS current through the MOSFET, (b) the conduction loss at $T_J = 100$ degrees C, (c) the switching loss, (d) the gate drive loss at $V_{GS} = 10$ V, and (e) the total loss and junction temperature if $theta_{JA} = 40$ degrees C/W and $T_A = 40$ degrees C.

Solution:

(a) RMS current: $I_{RMS} = I_{load}$ x sqrt(D) = 30 x sqrt(0.75) = 30 x 0.866 = 25.98 A

(b) Conduction loss at 100 degrees C: $P_{cond} = I_{RMS}^2$ x $R_{DS(on)} = 25.98^2$ x 0.018 = 675 x 0.018 = 12.15 W

(c) Switching loss: $P_{sw} = 0.5$ x V_{DS} x I_{load} x ($t_{rise} + t_{fall}$) x f_{sw} = 0.5 x 24 x 30 x (25 + 35) x 10^{-9} x 20,000 = 360 x 60 x 10^{-9} x 20,000 = 0.432 W

(d) Gate drive loss: $P_{gate} = Q_g$ x V_{GS} x f_{sw} = 80 x 10^{-9} x 10 x 20,000 = 0.016 W

(e) Total loss: $P_{total} = 12.15 + 0.432 + 0.016 = 12.60$ W

Junction temperature: $T_J = T_A + P_{total}$ x $theta_{JA}$ = 40 + 12.60 x 40 = 40 + 504 = 544 degrees C

This far exceeds the 175 degrees C maximum junction temperature. The thermal resistance $theta_{JA}$ = 40 degrees C/W (no heatsink, free-standing package) is wholly inadequate. A heatsink with $theta_{JA}$ < (100 - 40)/12.60 = 4.76 degrees C/W is required.

Problem 3.5.5

Given: Two identical power MOSFETs ($R_{DS(on)} = 5.0$ mohm each at 25 degrees C, temperature coefficient +0.4%/degree C) are connected in parallel to share a 50 A load current. Due to layout asymmetry, MOSFET A runs 10 degrees C hotter than MOSFET B.

Find: (a) The $R_{DS(on)}$ of each MOSFET at their respective operating temperatures if $T_A = 60$ degrees C and $T_B = 50$ degrees C, (b) the current sharing ratio, and (c) whether the positive temperature coefficient helps or hurts current sharing.

Solution:

(a) $R_{DS(on)}$ at operating temperature: MOSFET A: R_A = 5.0 x (1 + 0.004 x (60 - 25)) = 5.0 x (1 + 0.14) = 5.0 x 1.14 = 5.70 mohm MOSFET B: R_B = 5.0 x (1 + 0.004 x (50 - 25)) = 5.0 x (1 + 0.10) = 5.0 x 1.10 = 5.50 mohm

(b) In parallel with common V_{DS}, current divides inversely with resistance: $I_A = I_{total}$ x $R_B/(R_A + R_B)$ = 50 x 5.50/(5.70 + 5.50) = 50 x 0.491 = 24.55 A I_B = 50 x 5.70/11.20 = 50 x 0.509 = 25.45 A

(c) The hotter MOSFET (A) carries less current (24.55 A vs 25.45 A) because its resistance increased. This is a self-balancing mechanism — the positive temperature coefficient of $R_{DS(on)}$ naturally promotes current sharing in parallel MOSFETs, unlike BJTs where thermal runaway causes current hogging.

Problem 3.5.6

Given: An N-channel MOSFET with V_{th} = 1.0 V and k_n = 4 mA/V^2 is used in a common-source amplifier with a drain resistor R_D = 1 kohm and V_{DD} = 5 V. The gate is biased at V_{GS} = 2.0 V. Channel-length modulation parameter lambda = 0.02 V^{-1}.

Find: (a) The DC operating point (I_D, V_{DS}), (b) g_m, (c) the output resistance r_o due to channel-length modulation, (d) the small-signal voltage gain, and (e) the output voltage swing (limited by saturation constraint V_{DS} >= V_{GS} - V_{th}).

Solution:

(a) I_D = (k_n/2)(V_{GS} - V_{th})^2(1 + lambda x V_{DS}) First approximation (ignore lambda): I_D = (4 x 10^{-3}/2)$(1.0)^2$ = 2.0 mA V_{DS} = V_{DD} - I_D x R_D = 5 - 2.0 x 1 = 3.0 V Check: V_{DS} = 3.0 V > V_{GS} - V_{th} = 1.0 V, so saturation confirmed.

Refine with lambda: I_D = 2.0 x (1 + 0.02 x 3.0) = 2.0 x 1.06 = 2.12 mA V_{DS} = 5 - 2.12 x 1 = 2.88 V

(b) g_m = k_n(V_{GS} - V_{th}) = 4 x 10^{-3} x 1.0 = 4.0 mA/V

(c) r_o = 1/(lambda x I_D) = 1/(0.02 x 0.00212) = 1/4.24 x 10^{-5} = 23.6 kohm

(d) Voltage gain: A_v = -g_m x (R_D || r_o) = -4 x 10^{-3} x (1,000 x 23,600)/(1,000 + 23,600) = -4 x 10^{-3} x 23,600,000/24,600 = -4 x 10^{-3} x 959 = -3.84

(e) Minimum V_{DS} for saturation = V_{GS} - V_{th} = 1.0 V. Maximum I_D = (V_{DD} - $V_{DS,min}$)/R_D = (5 - 1.0)/1,000 = 4.0 mA Maximum output swing downward: 2.88 - 1.0 = 1.88 V below Q-point. Maximum output swing upward (I_D -> 0): 5.0 - 2.88 = 2.12 V above Q-point.

Chapter 3 — Section 3.6: Voltage Regulators

Practice problems covering linear regulators, LDOs, and the TL431 programmable reference.

Problem 3.6.1

Given: An LM7812 fixed +12 V linear regulator is powered from an unregulated 18 V DC supply. It provides 800 mA to a load. The regulator has a dropout voltage of 2 V, a quiescent current of 6 mA, and is mounted in a TO-220 package with thermal resistance $theta_{JC}$ = 5 degrees C/W. A heatsink with $theta_{CS}$ = 0.5 degrees C/W and $theta_{SA}$ = 8 degrees C/W is attached. Ambient temperature is 35 degrees C.

Find: (a) The power dissipated in the regulator, (b) the efficiency, (c) the junction temperature, and (d) the maximum ambient temperature for safe operation ($T_{J,max}$ = 125 degrees C).

Solution:

(a) Power dissipation: $P_D = (V_{in} - V_{out}) \times (I_{out} + I_Q) = (18 - 12) \times (0.800 + 0.006) = 6.0 \times 0.806 = 4.84$ W

(b) Efficiency: $eta = P_{out}/P_{in} = (V_{out} \times I_{out}) / (V_{in} \times (I_{out} + I_Q)) = (12 \times 0.800) / (18 \times 0.806) = 9.6 / 14.51 = 66.2\%$

(c) Total thermal resistance: $theta_{JA} = theta_{JC} + theta_{CS} + theta_{SA} = 5 + 0.5 + 8 = 13.5$ degrees C/W $T_J = T_A + P_D \times theta_{JA} = 35 + 4.84 \times 13.5 = 35 + 65.3 = 100.3$ degrees C

(d) Maximum ambient temperature: $T_{A,max} = T_{J,max} - P_D \times theta_{JA} = 125 - 4.84 \times 13.5 = 125 - 65.3 = 59.7$ degrees C

Problem 3.6.2

Given: An LDO regulator converts 3.6 V (single Li-ion cell, ranging from 4.2 V fully charged to 3.0 V discharged) to 2.5 V for an FPGA I/O bank drawing 200 mA. The LDO has a dropout voltage of 200 mV, quiescent current I_Q = 50 uA, and PSRR = 60 dB at 1 kHz.

Find: (a) Whether the LDO can regulate over the full battery range, (b) the power dissipation at full charge and at near-discharge, (c) the battery efficiency at each point, and (d) the input ripple attenuation if the battery charger generates 10 mV_{rms} ripple at 1 kHz.

Solution:

(a) Minimum input for regulation: $V_{in,min} = V_{out} + V_{dropout} = 2.5 + 0.2 = 2.7$ V. Battery range is 3.0 to 4.2 V, so 3.0 V > 2.7 V. Yes, the LDO regulates over the full range.

(b) Power dissipation: At full charge (4.2 V): P_D = (4.2 - 2.5) x 0.200 = 1.7 x 0.200 = 340 mW At near-discharge (3.0 V): P_D = (3.0 - 2.5) x 0.200 = 0.5 x 0.200 = 100 mW

(c) Efficiency: At 4.2 V: eta = 2.5/4.2 = 59.5% At 3.0 V: eta = 2.5/3.0 = 83.3%

The LDO is most efficient when the battery is nearly discharged, which is fortunate since that is when energy conservation matters most.

(d) PSRR = 60 dB means 1,000:1 attenuation: Output ripple = 10 mV / 1,000 = 10 uV_{rms} at the FPGA supply

This excellent noise rejection is a key advantage of LDOs over switching regulators for sensitive analog/digital circuits.

Problem 3.6.3

Given: A TL431 is used to regulate a 5.0 V output rail. The internal reference voltage is $V_{ref} = 2.495$ V. The feedback resistor divider uses $R_2 = 4.7$ kohm (lower resistor). The TL431 cathode is connected through $R_S = 33$ ohm to an unregulated 8.0 V supply. The load draws 50 mA.

Find: (a) The required R_1 (upper resistor) for 5.0 V output, (b) the divider current, (c) the total cathode current, (d) the TL431 power dissipation, and (e) the output voltage shift if R_1 has a +1% tolerance and R_2 has a -1% tolerance (worst case).

Solution:

(a) $R_1 = R_2$ x (V_{out}/V_{ref} - 1) = 4,700 x (5.0/2.495 - 1) = 4,700 x (2.004 - 1) = 4,700 x 1.004 R_1 = 4,719 ohm (use standard 4.7 kohm, giving V_{out} = 2.495 x (4,700 + 4,700)/4,700 = 4.99 V)

With R_1 = 4.7 kohm: V_{out} = 2.495 x (4,700 + 4,700)/4,700 = 2.495 x 2 = 4.99 V

(b) Divider current: $I_{div} = V_{out}/(R_1 + R_2)$ = 4.99/(4,700 + 4,700) = 4.99/9,400 = 0.531 mA

(c) Total current through R_S: $I_{total} = (V_{supply} - V_{out})/R_S$ = (8.0 - 4.99)/33 = 3.01/33 = 91.2 mA I_{load} = 50 mA, I_{div} = 0.531 mA $I_{cathode} = I_{total} - I_{load} - I_{div}$ = 91.2 - 50 - 0.531 = 40.7 mA

(d) TL431 power dissipation: $P = V_{KA}$ x I_K = 4.99 x 0.0407 = 203 mW

(e) Worst-case tolerance analysis: R_1(+1%) = 4,700 x 1.01 = 4,747 ohm R_2(-1%) = 4,700 x 0.99 = 4,653 ohm $V_{out,max}$ = 2.495 x (4,747 + 4,653)/4,653 = 2.495 x 9,400/4,653 = 2.495 x 2.020 = 5.040 V

Voltage shift: 5.040 - 4.99 = +50 mV (1% resistor tolerances produce a 1% output shift)

Problem 3.6.4

Given: An engineer must select between an LDO and a buck converter for a 12 V to 3.3 V, 500 mA power supply in a battery-powered portable instrument. The LDO has I_Q = 30 uA. The buck converter has 88% efficiency and I_Q = 2 mA. Operating time from a 12 V, 5 Ah battery must be maximized.

Find: (a) The LDO power dissipation and efficiency, (b) the buck converter input current, (c) the total input current for each solution, (d) the battery runtime for each, and (e) the crossover load current where both solutions have equal efficiency.

Solution:

(a) LDO: P_D = (12 - 3.3) x 0.500 = 8.7 x 0.500 = 4.35 W eta = 3.3/12 = 27.5% $I_{in} = I_{out} + I_Q$ = 500 + 0.030 = 500.03 mA

(b) Buck converter input current: P_{out} = 3.3 x 0.500 = 1.65 W $P_{in} = P_{out}$/eta = 1.65/0.88 = 1.875 W $I_{in,load} = P_{in}/V_{in}$ = 1.875/12 = 156.3 mA

(c) Total input current: LDO: 500.0 mA (essentially all flows through) Buck: 156.3 + 2.0 = 158.3 mA

(d) Battery runtime (capacity / current): LDO: 5,000 mAh / 500.0 = 10.0 hours Buck: 5,000 / 158.3 = 31.6 hours

(e) Crossover: LDO efficiency = V_{out}/V_{in} = 27.5% regardless of load. Buck efficiency depends on quiescent overhead. At load current I_L: Buck total input power = 3.3 x I_L/0.88 + 12 x 0.002 LDO total input power = 12 x I_L

Setting equal: 3.75 x I_L + 0.024 = 12 x I_L 0.024 = 8.25 x I_L I_L = 0.024/8.25 = 2.9 mA

Below 2.9 mA load, the LDO is more efficient due to the buck converter's higher quiescent current. Above 2.9 mA, the buck converter wins decisively.

Chapter 3 — Section 3.7: Semiconductor Fabrication

Practice problems covering crystal growth, lithography, ion implantation, etching, CMOS integration, and advanced nodes.

Problem 3.7.1

Given: A 200 mm diameter silicon wafer is used to fabricate power management ICs with die dimensions of 3 mm x 4 mm. The edge exclusion is 5 mm.

Find: (a) The usable wafer area, (b) the maximum number of dies (before edge correction), (c) the estimated dies per wafer after edge loss correction, and (d) the number of good dies at 85% yield.

Solution:

(a) Usable wafer area: A = pi x (D/2 - edge)^2 = pi x (100 - 5)^2 = pi x 95^2 = pi x 9,025 = 28,353 mm^2

(b) Die area: 3 x 4 = 12 mm^2 Maximum dies: 28,353/12 = 2,363

(c) Edge loss correction: Die diagonal = sqrt(3^2 + 4^2) = sqrt(25) = 5.0 mm Edge dies lost ~ pi x (D - 2 x edge)/die_diagonal = pi x 190/5.0 = 119 Estimated dies: 2,363 - 119 = ~2,244 dies per wafer

(d) Good dies at 85% yield: 2,244 x 0.85 = ~1,907 good dies

Problem 3.7.2

Given: An advanced lithography system uses EUV at lambda = 13.5 nm with NA = 0.33 and k_1 = 0.32. A next-generation high-NA EUV tool uses NA = 0.55 with the same wavelength and k_1 = 0.28.

Find: (a) The minimum feature size (CD_{min}) for each tool, (b) the minimum half-pitch (feature + space), and (c) the transistor density improvement assuming density scales as 1/pitch^2.

Solution:

(a) Rayleigh criterion: CD_{min} = k_1 x lambda/NA

Standard EUV: CD_{min} = 0.32 x 13.5/0.33 = 13.1 nm High-NA EUV: CD_{min} = 0.28 x 13.5/0.55 = 6.87 nm

(b) Minimum half-pitch (assuming pitch = 2 x CD): Standard EUV: pitch = 2 x 13.1 = 26.2 nm High-NA EUV: pitch = 2 x 6.87 = 13.7 nm

(c) Density improvement: Density ratio = $(pitch_{old}/pitch_{new})^2 = (26.2/13.7)^2 = (1.912)^2 = 3.65x$

High-NA EUV enables approximately 3.65x higher transistor density, supporting the transition from 3 nm to sub-2 nm nodes.

Problem 3.7.3

Given: Boron is implanted into an N-type silicon substrate ($N_D = 5 \times 10^{15}$ cm^{-3}) at an energy of 50 keV. At this energy, the projected range is R_p = 170 nm and the standard deviation is delta_R_p = 60 nm. The implant dose is phi = 5×10^{13} cm^{-2}.

Find: (a) The peak boron concentration, (b) the junction depth where the boron concentration equals the background N_D, and (c) the surface concentration.

Solution:

(a) Peak concentration (at $x = R_p$) for a Gaussian profile: N_{peak} = phi / (sqrt(2 x pi) x delta_R_p) = 5 x 10^{13} / (sqrt(2 x pi) x 60 x 10^{-7}) = 5 x 10^{13} / (2.507 x 6 x 10^{-6}) = 5 x 10^{13} / 1.504 x 10^{-5} = 3.32 x 10^{18} cm^{-3}

(b) Junction depth where $N(x_j) = N_D$ = 5 x 10^{15}: N(x) = N_{peak} x exp(-(x - R_p)^2 / (2 x delta_R_p^{2})) 5 x 10^{15} = 3.32 x 10^{18} x exp(-(x_j - 170)^2 / (2 x 60^2))

exp(-(x_j - 170)^2/7,200) = 5 x 10^{15} / 3.32 x 10^{18} = 1.506 x 10^{-3}

-(x_j - 170)^2/7,200 = ln(1.506 x 10^{-3}) = -6.499 (x_j - 170)^2 = 7,200 x 6.499 = 46,793 x_j - 170 = +/- 216.3 nm

Taking the deeper junction: x_j = 170 + 216 = 386 nm (into the substrate) The shallow junction would be at 170 - 216 = -46 nm (surface side, not physically meaningful as it is above the surface).

(c) Surface concentration (x = 0): N(0) = N_{peak} x exp(-R_p^{2} / (2 x delta_R_p^{2})) = 3.32 x 10^{18} x exp(-170^2 / (2 x 60^2)) = 3.32 x 10^{18} x exp(-28,900/7,200) = 3.32 x 10^{18} x exp(-4.014) = 3.32 x 10^{18} x 0.01810 = 6.01 x 10^{16} cm^{-3}

Problem 3.7.4

Given: A PECVD process deposits SiO_2 at a rate of 80 nm/min. The target film thickness is 400 nm +/- 5%. The subsequent RIE etch process has an etch rate of 150 nm/min for SiO_2 and 8 nm/min for the underlying silicon nitride (Si_3N_4). A 15% overetch is standard.

Find: (a) The deposition time, (b) the selectivity of SiO_2 to Si_3N_4, (c) the etch time with overetch, and (d) the Si_3N_4 loss during overetch.

Solution:

(a) Deposition time: t_{dep} = 400/80 = 5.0 minutes

(b) Selectivity: S = Rate(SiO_2)/Rate(Si_3N_4) = 150/8 = 18.75:1

(c) Main etch time: t_{etch} = 400/150 = 2.667 min Overetch time: 0.15 x 2.667 = 0.400 min Total: 2.667 + 0.400 = 3.07 minutes

(d) Si_3N_4 loss during overetch: Loss = 8 x 0.400 = 3.2 nm

If the SiO_2 film is 5% thicker (420 nm), the main etch takes 420/150 = 2.80 min, and the overetch time becomes 0.15 x 2.80 = 0.42 min, with Si_3N_4 loss of 8 x 0.42 = 3.4 nm. The high selectivity ensures minimal damage to the underlayer.

Problem 3.7.5

Given: A 5 nm GAA nanosheet CMOS process has a gate pitch of 48 nm and a minimum metal pitch of 28 nm. The SRAM bit cell area is 0.021 um^2. A processor requires 8 MB of L2 cache (8 x 8 x 10^6 x 8 = 64 x 10^6 bits, plus ~30% overhead for control logic), and the logic portion requires 15 billion transistors at a density of 310 MTr/mm^2.

Find: (a) The SRAM area for L2 cache, (b) the logic area, (c) the total die area, and (d) the estimated die cost if the wafer cost is $18,000 for a 300 mm wafer (assume 90% yield and use the die count from a simplified calculation).

Solution:

(a) Total SRAM bits with overhead: 64 x 10^6 x 1.30 = 83.2 x 10^6 cells Each SRAM cell = 6 transistors, but area is per bit cell: A_{SRAM} = 83.2 x 10^6 x 0.021 x 10^{-6} mm^2 = 83.2 x 10^6 x 2.1 x 10^{-8} = 1.75 mm^2

(b) Logic area: A_{logic} = 15 x 10^9 / (310 x 10^6) = 48.4 mm^2

(c) Total die area (logic + SRAM + I/O and misc ~20% overhead): A_{total} = (48.4 + 1.75) x 1.20 = 50.15 x 1.20 = 60.2 mm^2

(d) Dies per 300 mm wafer: Usable area: pi x (150 - 3)^2 = 67,929 mm^2 Gross dies: 67,929/60.2 = 1,128 Edge loss: pi x 294/sqrt(2 x 60.2) ~ pi x 294/10.96 ~ 84 Net dies: 1,128 - 84 = 1,044 Good dies at 90%: 1,044 x 0.90 = 940

Die cost: $18,000/940 = $19.15 per die

Problem 3.7.6

Given: A chiplet-based processor uses a 3 nm compute chiplet (80 mm^2, $35/die), two 5 nm I/O chiplets (40 mm^2 each, $12/die each), and an HBM memory stack ($25/stack, 4 stacks). These are assembled on a silicon interposer (500 mm^2, $15). The assembly and test cost is $20 per package. The equivalent monolithic design would be a single 250 mm^2 die at 3 nm ($95/die) with assembly cost of $8.

Find: (a) The total cost of the chiplet solution, (b) the total cost of the monolithic solution, (c) the yield advantage of the chiplet approach (assuming 3 nm yield = 75% for 80 mm^2 and 55% for 250 mm^2), and (d) the effective cost per good package for each approach.

Solution:

(a) Chiplet solution cost per package: Compute chiplet: \$35 I/O chiplets: 2 x \$12 = \$24 HBM stacks: 4 x \$25 = \$100 Interposer: \$15 Assembly + test: \$20 Total: \$194 per package

(b) Monolithic solution: Die: \$95 Assembly + test: \$8 Total: \$103 per package

(c) The die costs above already assume testing, but not yield on final assembly. If the chiplet assembly yield is 95% and monolithic assembly yield is 98%:

Chiplet effective cost: \$194/0.95 = \$204 per good package Monolithic effective cost: \$103/0.98 = \$105 per good package

(d) However, the given die costs may not include yield. If we factor in raw die costs:

Chiplet: The 3 nm die at 75% yield has a raw cost already factored in at \$35. The monolithic 3 nm die at 55% yield costs more per good die.

If raw wafer cost is \$18,000 and the monolithic die gets 67,929/250 ~ 230 gross dies (minus ~30 edge = 200 net), then at 55% yield = 110 good dies, raw cost = \$18,000/110 = \$163/die (not \$95). This demonstrates the chiplet yield advantage: smaller dies have exponentially better yield, making the chiplet approach more cost-effective for large designs despite higher assembly complexity.

Chapter 4 — Section 4.1: Open Loop

Practice problems covering open-loop control system analysis and steady-state response.

Problem 4.1.1

Given: An open-loop conveyor belt speed controller has a forward path transfer function G(s) = 120 / (s + 15). A step input of magnitude 2 is applied as R(s) = 2/s.

Find: (a) The output C(s), (b) the time-domain response c(t), (c) the steady-state output, and (d) the time constant.

Solution:

(a) C(s) = G(s) x R(s) = 120 / [s(s + 15)] x 2 = 240 / [s(s + 15)]

Partial fractions: 240/[s(s + 15)] = A/s + B/(s + 15) A = 240/15 = 16, B = 240/(-15) = -16 C(s) = 16/s - 16/(s + 15)

(b) Inverse Laplace: c(t) = 16(1 - e^{-15t}) for t >= 0

(c) Steady-state: c(infinity) = 16 The desired output for a step of magnitude 2 would ideally be 2, but the open-loop gain is G(0) = 120/15 = 8, so the output is 8 x 2 = 16.

(d) Time constant: tau = 1/15 = 0.0667 s = 66.7 ms

Problem 4.1.2

Given: An open-loop heating system has G(s) = 30 / [(s + 0.5)(s + 3)]. A unit step input R(s) = 1/s is applied.

Find: (a) The steady-state output, (b) the steady-state error if the desired output is 1, (c) the time-domain response, and (d) how a 10% increase in the plant gain (from 30 to 33) affects the steady-state output.

Solution:

(a) Steady-state (Final Value Theorem): c_{ss} = lim(s->0) s x G(s) x R(s) = lim(s->0) s x 30/[s(s + 0.5)(s + 3)] = 30/(0.5 x 3) = 20

(b) Steady-state error: $e_{ss} = 1 - 20 = -19$ (massive gain error; the system overshoots by 1,900%)

(c) Partial fractions of C(s) = 30/[s(s + 0.5)(s + 3)]: A/s + B/(s + 0.5) + C/(s + 3) A = 30/(0.5 x 3) = 20 B = 30/((-0.5)(3 - 0.5)) = 30/(-0.5 x 2.5) = -24 C = 30/((-3)(-3 + 0.5)) = 30/((-3)(-2.5)) = 30/7.5 = 4

$c(t) = 20 - 24e^{-0.5t} + 4e^{-3t}$ for t >= 0

(d) With gain increased 10%: $c_{ss} = 33/(0.5 \times 3) = 22$ The 10% gain change produces a 10% output change (from 20 to 22), demonstrating the sensitivity of open-loop systems to parameter variations.

Problem 4.1.3

Given: An open-loop motor positioning system has G(s) = 200 / [s(s + 20)]. The input is a ramp r(t) = 5t, so $R(s) = 5/s^2$.

Find: (a) The output in the Laplace domain, (b) the steady-state output, and (c) whether the system can track the ramp input.

Solution:

(a) $C(s) = G(s) \times R(s) = 200 \times 5 / [s^2(s + 20) \times s] = 1{,}000 / [s^3(s + 20)]$

Wait — $R(s) = 5/s^2$, so: $C(s) = 200 / [s(s + 20)] \times 5/s^2 = 1{,}000 / [s^3(s + 20)]$

(b) Applying the Final Value Theorem: $c_{ss} = \lim(s\text{->}0)\ s \times C(s) = \lim(s\text{->}0)\ 1{,}000 / [s^2(s + 20)]$

As s -> 0, this approaches infinity. The steady-state output is unbounded — it grows without limit.

(c) The system cannot track the ramp in a meaningful open-loop sense. With G(s) containing an integrator (1/s factor), the open-loop output to a ramp input grows as $t^2/2$, diverging from the linear ramp. This illustrates that open-loop systems with integrators are inherently unsuitable for tracking inputs without feedback to regulate the output.

Chapter 4 — Section 4.2: Closed Loop

Practice problems covering closed-loop transfer functions, feedback effects, and steady-state error analysis.

Problem 4.2.1

Given: A unity feedback system has G(s) = 50 / [(s + 1)(s + 10)]. The reference input is a unit step.

Find: (a) The closed-loop transfer function T(s), (b) the closed-loop poles, (c) the steady-state output, and (d) the steady-state error.

Solution:

(a) T(s) = G(s)/(1 + G(s)) = 50/[(s + 1)(s + 10)] / [1 + 50/((s + 1)(s + 10))] = 50 / [(s + 1)(s + 10) + 50] = 50 / [s^2 + 11s + 10 + 50] = 50 / (s^2 + 11s + 60)

(b) Poles: s^2 + 11s + 60 = 0 s = (-11 +/- sqrt(121 - 240))/2 = (-11 +/- sqrt(-119))/2 = (-11 +/- j10.91)/2 Poles: s = -5.5 +/- j5.45 (complex conjugate, stable)

(c) Steady-state: T(0) = 50/60 = 0.833

(d) e_{ss} = 1 - T(0) = 1 - 0.833 = 0.167 (16.7% error)

This is a Type 0 system (no integrator in G(s)), so finite steady-state error to a step is expected.

Problem 4.2.2

Given: A closed-loop system has a forward path G(s) = K / [s(s + 8)] and feedback H(s) = 0.5. The system must have a steady-state error of less than 2% for a unit step input.

Find: (a) The closed-loop transfer function, (b) the minimum value of K to meet the error specification, and (c) the closed-loop poles at that K.

Solution:

(a) T(s) = G(s)/(1 + G(s)H(s)) = [K/(s(s + 8))] / [1 + 0.5K/(s(s + 8))] = K / (s^2 + 8s + 0.5K)

(b) For a unity-feedback-equivalent analysis: e_{ss} = 1/(1 + K_p) where K_p = lim(s->0) G(s)H(s) = lim(s->0) 0.5K/[s(s + 8)]

Since G(s)H(s) has a pole at s = 0, this is Type 1, and K_p = infinity, giving $e_{ss} = 0$ for a step.

Actually, for a non-unity feedback system, we must use: e_{ss} = lim(s->0) s x R(s) / (1 + G(s)H(s))

For unit step R(s) = 1/s: e_{ss} = lim(s->0) 1/(1 + G(s)H(s)) = lim(s->0) 1/(1 + 0.5K/[s(s+8)])

As s -> 0, G(s)H(s) -> infinity, so $e_{ss} = 0$ regardless of K (as long as K > 0).

However, the actual output is Y(s) = T(s) x R(s), and the steady-state output is: y_{ss} = T(0) = K/(0.5K) = 2.0

The reference was 1, so the output is 2.0, meaning the error based on the reference is 1 - 2 = -1 (overshoot in DC). With non-unity feedback, the reference must be scaled. If we define error as r - H x y: e = r - 0.5y -> at steady state: 0 = 1 - 0.5 x y_{ss}, so y_{ss} = 2.0 and e_{ss} = 0.

The system tracks with zero steady-state error for any K > 0 due to the integrator.

(c) For stability, require K > 0. Choose K = 50 for good transient response: s^2 + 8s + 25 = 0 s = (-8 +/- sqrt(64 - 100))/2 = (-8 +/- j6)/2 = -4 +/- j3

Natural frequency omega_n = sqrt(25) = 5 rad/s, damping zeta = 4/5 = 0.8.

Problem 4.2.3

Given: A unity feedback system has G(s) = 200(s + 5) / [s(s + 10)(s + 20)]. A unit step input is applied.

Find: (a) The system type, (b) the position error constant K_p, (c) the steady-state error to a unit step, (d) the velocity error constant K_v, and (e) the steady-state error to a unit ramp.

Solution:

(a) G(s) has one free s in the denominator, so this is a Type 1 system.

(b) K_p = lim(s->0) G(s) = lim(s->0) 200(s + 5)/[s(s + 10)(s + 20)] = infinity (due to the 1/s factor)

(c) e_{ss}(step) = 1/(1 + K_p) = 1/(1 + infinity) = 0 (zero steady-state error to step)

(d) K_v = lim(s->0) s x G(s) = lim(s->0) 200(s + 5)/[(s + 10)(s + 20)] = 200(5)/(10 x 20) = 1,000/200 = 5

(e) e_{ss}(ramp) = 1/K_v = 1/5 = 0.2 (the system lags behind the ramp by 0.2 units)

Chapter 4 — Section 4.3: Control Signals

Practice problems covering step, ramp, impulse, and sinusoidal input responses.

Problem 4.3.1

Given: A first-order system has the transfer function G(s) = 15 / (s + 5). A unit step input is applied.

Find: (a) The time-domain step response, (b) the time constant, (c) the steady-state value, (d) the 2% settling time, and (e) the 10-90% rise time.

Solution: Rewrite in standard form: G(s) = 3 / (0.2s + 1), so K = 3, tau = 0.2 s.

(a) C(s) = 15/[s(s + 5)] = 3/s - 3/(s + 5) c(t) = 3(1 - e^{-5t}) for t >= 0

(b) Time constant: tau = 1/5 = 0.2 s

(c) Steady-state: c(infinity) = 3

(d) Settling time (2%): t_s = 4 x tau = 4 x 0.2 = 0.8 s

(e) Rise time: t_r = 2.2 x tau = 2.2 x 0.2 = 0.44 s

Problem 4.3.2

Given: A unity feedback system has G(s) = 500 / [s(s + 5)(s + 50)]. A ramp input r(t) = 2t is applied.

Find: (a) The system type, (b) the velocity error constant K_v, and (c) the steady-state error.

Solution:

(a) One free integrator in G(s), so this is a Type 1 system.

(b) K_v = lim(s->0) s x G(s) = lim(s->0) 500/[(s + 5)(s + 50)] = 500/(5 x 50) = 2.0

(c) For ramp input r(t) = 2t, R(s) = 2/s^2: e_{ss} = A/K_v where A = 2 (ramp slope) e_{ss} = 2/2.0 = 1.0

The system lags behind the ramp by a constant 1.0 unit.

Problem 4.3.3

Given: A system has the transfer function G(s) = 24 / [(s + 2)(s + 4)(s + 6)]. A unit impulse is applied.

Find: (a) The impulse response g(t) using partial fractions, and (b) the peak value and time of the peak.

Solution:

(a) For an impulse input, C(s) = G(s) x 1 = 24/[(s + 2)(s + 4)(s + 6)].

Partial fractions: A/(s + 2) + B/(s + 4) + C/(s + 6)

A = 24/[(4 - 2)(6 - 2)] = 24/[2 x 4] = 3 B = 24/[(-4 + 2)(6 - 4)] = 24/[(-2)(2)] = -6 C = 24/[(-6 + 2)(-6 + 4)] = 24/[(-4)(-2)] = 3

g(t) = $3e^{-2t}$ - $6e^{-4t}$ + $3e^{-6t}$ for t >= 0

(b) At t = 0: g(0) = 3 - 6 + 3 = 0. Find peak by setting dg/dt = 0: dg/dt = $-6e^{-2t}$ + $24e^{-4t}$ - $18e^{-6t}$ = 0

Dividing by $-6e^{-6t}$: e^{4t} - $4e^{2t}$ + 3 = 0 Let u = e^{2t}: u^2 - 4u + 3 = 0 -> (u - 1)(u - 3) = 0 u = 1 (t = 0) or u = 3 (t = ln(3)/2 = 0.549 s)

At t = 0.549 s: g(0.549) = 3 x $e^{-1.099}$ - 6 x $e^{-2.197}$ + 3 x $e^{-3.296}$ = 3 x 0.333 - 6 x 0.111 + 3 x 0.037 = 1.0 - 0.667 + 0.111 = 0.444

Peak value is 0.444 at t = 0.549 s.

Problem 4.3.4

Given: A stable system has G(s) = 20 / [(s + 1)(s + 5)]. A sinusoidal input r(t) = 4 sin(3t) is applied.

Find: (a) The magnitude |G(j3)| and phase angle, (b) the steady-state output, and (c) the output amplitude and phase at omega = 10 rad/s for the same input amplitude.

Solution:

(a) G(j3) = 20/[(j3 + 1)(j3 + 5)] = 20/[(1 + j3)(5 + j3)] = 20/[(5 - 9) + j(3 + 15)] = 20/[-4 + j18]

|G(j3)| = 20/sqrt(16 + 324) = 20/sqrt(340) = 20/18.44 = 1.085

Phase: angle(-4 + j18) = 180 degrees - arctan(18/4) = 180 - 77.3 = 102.7 degrees Since the denominator has positive imaginary and negative real parts (second quadrant): Phase of G = -102.7 degrees

Alternatively, compute directly: Phase = -arctan(3/1) - arctan(3/5) = -71.57 - 30.96 = -102.5 degrees

(b) Steady-state output: $y_{ss}(t)$ = 4 x 1.085 x sin(3t - 102.5 degrees) = 4.34 sin(3t - 102.5 degrees)

(c) At omega = 10: G(j10) = 20/[(1 + j10)(5 + j10)] = 20/[(5 - 100) + j(10 + 50)] = 20/[-95 + j60] |G(j10)| = 20/sqrt(9,025 + 3,600) = 20/sqrt(12,625) = 20/112.4 = 0.178 Phase = -arctan(10/1) - arctan(10/5) = -84.3 - 63.4 = -147.7 degrees

Output: y_{ss} = 4 x 0.178 x sin(10t - 147.7 degrees) = 0.712 sin(10t - 147.7 degrees)

The amplitude drops from 4.34 at omega = 3 to 0.712 at omega = 10, showing the low-pass filtering effect.

Problem 4.3.5

Given: A second-order system has $G(s) = 100 / (s^2 + 6s + 100)$. A unit step input is applied.

Find: (a) The natural frequency and damping ratio, (b) the damped frequency, (c) the peak time, (d) the percent overshoot, and (e) the 2% settling time.

Solution:

(a) Comparing with $\text{omega}_n^2/(s^2 + 2\textit{zeta}\,\text{omega}_n \textit{s} + \textit{omega}_n^2)$: $\textit{omega}_n^2 = 100$ -> $\textit{omega}_n = 10$ rad/s $2\text{zeta}*\text{omega}_n = 6$ -> zeta = 6/20 = 0.3 (underdamped)

(b) $\text{omega}_d = \text{omega}_n$ x $\text{sqrt}(1 - \text{zeta}^2)$ = 10 x sqrt(1 - 0.09) = 10 x 0.9539 = 9.54 rad/s

(c) Peak time: $t_p = \text{pi}/\text{omega}_d$ = pi/9.54 = 0.329 s

(d) %OS = 100 x $e^{-\text{zeta}\textit{pi/sqrt(1 - zeta}^2\textit{) = 100 x e}^{-0.3}\text{pi}/0.9539}$ = 100 x $e^{-0.988}$ = 100 x 0.3730 = 37.3%

(e) $t_s = 4/(\text{zeta x omega}_n)$ = 4/(0.3 x 10) = 4/3 = 1.33 s

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Problem 4.3.6

Given: A unity feedback system with $G(s) = K/[s(s + 4)]$ requires a 2% settling time of 2 seconds and a percent overshoot no greater than 20%.

Find: (a) The required damping ratio for the overshoot specification, (b) the required sigma = zeta x omega_n for the settling time, (c) the required omega_n, and (d) the value of K.

Solution:

(a) %OS = 20%: 20 = 100 x $e^{-\text{zeta}\textit{pi/sqrt(1 - zeta}^2\textit{) ln(0.20) = -zeta}\text{pi/sqrt(1 - zeta2)}}$ -1.609 = -zeta*pi/sqrt(1 - zeta*^{2}*)* *zeta*pi/sqrt(1 - zeta^2) = 1.609 zeta^2 x pi^2 = 1.609^2 x (1 - zeta^2) = 2.589 - 2.589*zeta^2 zeta^2(9.870 + 2.589) = 2.589 zeta^2 = 2.589/12.459 = 0.2078 zeta = 0.456

(b) sigma = $4/t_s$ = 4/2 = 2 (for 2% criterion: t_s = 4/sigma)

(c) sigma = zeta x omega_n -> omega_n = sigma/zeta = 2/0.456 = 4.386 rad/s

(d) Closed-loop: $T(s) = K/(s^2 + 4s + K)$ $\text{omega}_n^2 = K$ -> K = 4.386^2 = 19.2

Verify: 2*zeta*omega_n = 4 -> zeta = 4/(2 x 4.386) = 0.456. Confirmed.

Chapter 4 — Section 4.4: Transfer Functions and Block Diagrams

Practice problems covering transfer function representation, poles and zeros, DC gain, block diagram algebra, series/parallel/feedback reduction, signal flow graphs, and Mason's gain formula.

Problem 4.4.1

Given: A system has the transfer function $H(s) = 120(s + 4) / (s^2 + 12s + 36)$.

Find: (a) The poles and zeros, (b) the natural frequency and damping ratio, (c) the DC gain, and (d) whether the system is stable.

Solution:

(a) Zero: $s + 4 = 0 \rightarrow s = -4$ (one real zero). Poles: $s^2 + 12s + 36 = (s + 6)^2 = 0 \rightarrow s = -6$ (repeated real pole, multiplicity 2).

(b) Comparing $s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 12s + 36$: $\omega_n = \sqrt{36} = 6$ rad/s $2\zeta\omega_n = 12 \rightarrow \zeta = 12 / (2 \times 6) = 1.0$ (critically damped)

(c) DC gain: $H(0) = 120(4) / 36 = 480 / 36 = 13.33$

(d) Both poles are at $s = -6$ (negative real part). The system is stable.

Problem 4.4.2

Given: A system has the transfer function $G(s) = 200 / (s^3 + 9s^2 + 26s + 24)$.

Find: (a) The poles of the system, (b) the DC gain, and (c) the partial fraction expansion of G(s).

Solution:

(a) Factor the denominator. Test $s = -1$: $(-1)^3 + 9(-1)^2 + 26(-1) + 24 = -1 + 9 - 26 + 24 = 6$ (not a root). Test $s = -2$: $(-2)^3 + 9(-2)^2 + 26(-2) + 24 = -8 + 36 - 52 + 24 = 0$ (root). Factor out $(s + 2)$: $s^3 + 9s^2 + 26s + 24 = (s + 2)(s^2 + 7s + 12) = (s + 2)(s + 3)(s + 4)$. Poles: $s = -2, -3, -4$ (all real, all stable).

(b) DC gain: $G(0) = 200 / (2 \times 3 \times 4) = 200 / 24 = 8.33$

(c) $G(s) = A/(s + 2) + B/(s + 3) + C/(s + 4)$. $A = 200 / [(s + 3)(s + 4)]$ at $s = -2 = 200 / (1 \times 2) = 100$ $B = 200 / [(s + 2)(s + 4)]$ at $s = -3 = 200 / ((-1)(1)) = -200$ $C = 200 / [(s + 2)(s + 3)]$ at $s = -4 = 200 / ((-2)(-1)) = 100$

$G(s) = 100/(s + 2) - 200/(s + 3) + 100/(s + 4)$

Problem 4.4.3

Given: A control system has three blocks in the forward path: $G_1(s) = 5$, $G_2(s) = 1/(s + 3)$, and $G_3(s) = 1/(s + 8)$. The system uses unity feedback $H(s) = 1$.

Find: (a) The open-loop transfer function, (b) the closed-loop transfer function, (c) the closed-loop poles, and (d) the steady-state error to a unit step input.

Solution:

(a) Open-loop: $G(s) = G_1G_2G_3 = 5 / [(s + 3)(s + 8)] = 5 / (s^2 + 11s + 24)$

(b) Closed-loop: $T(s) = G(s) / (1 + G(s)) = 5 / (s^2 + 11s + 24 + 5) = 5 / (s^2 + 11s + 29)$

(c) Poles: $s = (-11 \pm \sqrt{(121 - 116)}) / 2 = (-11 \pm \sqrt{5}) / 2 = (-11 \pm 2.236) / 2$ $s_1 = -4.382$, $s_2 = -6.618$ (both real, stable, overdamped)

(d) This is a Type 0 system. $K_p = G(0) = 5/24 = 0.2083$. $e_{ss} = 1 / (1 + K_p) = 1 / 1.2083 = 0.828$ (82.8% steady-state error)

Problem 4.4.4

Given: A feedback control system has forward path $G(s) = K / [s(s + 6)]$ and feedback path $H(s) = 2$.

Find: (a) The closed-loop transfer function, (b) the value of K that produces a damping ratio $\zeta = 0.5$, (c) the natural frequency at that K, and (d) the steady-state error to a unit step for the non-unity feedback system.

Solution:

(a) Closed-loop: $T(s) = G(s) / (1 + G(s)H(s)) = [K / s(s + 6)] / [1 + 2K / s(s + 6)]$ $T(s) = K / (s^2 + 6s + 2K)$

(b) Comparing with $s^2 + 2\zeta\omega_n s + \omega_n^2$: $\omega_n^2 = 2K$ and $2\zeta\omega_n = 6$. $\omega_n = 6 / (2 \times 0.5) = 6$ rad/s. $2K = \omega_n^2 = 36 \rightarrow K = 18$

(c) $\omega_n = 6$ rad/s

(d) For a non-unity feedback system, $e_{ss} = 1 - T(0)$ where $T(0) = K / (2K) = 1/2$. $e_{ss} = 1 - 0.5 = 0.5$ (50% error due to the feedback gain of 2)

Problem 4.4.5

Given: A control system has two forward paths and two feedback loops. The forward path contains $G_1(s) = 4/(s + 1)$ and $G_2(s) = 3/(s + 5)$. There is a minor loop with feedback $H_1(s) = 0.5$ around G_2, and unity feedback $H_2(s) = 1$ around the entire system.

Find: (a) The inner-loop closed-loop transfer function, (b) the overall open-loop transfer function, (c) the overall closed-loop transfer function, and (d) the DC gain.

Solution:

(a) Inner loop (G_2 with feedback H_1): $G_{2CL}(s) = G_2 / (1 + G_2H_1) = [3/(s + 5)] / [1 + 1.5/(s + 5)] = 3 / (s + 5 + 1.5) = 3 / (s + 6.5)$

(b) Overall open-loop: $G_{OL}(s) = G_1 \times G_{2CL} = [4/(s + 1)] \times [3/(s + 6.5)] = 12 / [(s + 1)(s + 6.5)]$

(c) Overall closed-loop: $T(s) = G_{OL} / (1 + G_{OL}) = 12 / [(s + 1)(s + 6.5) + 12] = 12 / (s^2 + 7.5s + 6.5 + 12) = 12 / (s^2 + 7.5s + 18.5)$

(d) DC gain: $T(0) = 12 / 18.5 = 0.649$

Problem 4.4.6

Given: A second-order system has poles at $s = -3 + j4$ and $s = -3 - j4$, a zero at $s = -10$, and a DC gain of 5.

Find: (a) The transfer function $H(s)$, (b) ω_n and ζ, (c) the damped natural frequency, and (d) the expected percent overshoot (ignoring the zero's effect).

Solution:

(a) Denominator: $(s + 3 - j4)(s + 3 + j4) = (s + 3)^2 + 16 = s^2 + 6s + 25$. Transfer function form: $H(s) = K(s + 10) / (s^2 + 6s + 25)$. DC gain: $H(0) = K(10) / 25 = 5 \rightarrow K = 12.5$. $H(s) = 12.5(s + 10) / (s^2 + 6s + 25)$

(b) From the denominator: $\omega_n^2 = 25 \rightarrow \omega_n = 5$ rad/s. $2\zeta\omega_n = 6 \rightarrow \zeta = 6 / 10 = 0.6$

(c) $\omega_d = \omega_n\sqrt{(1 - \zeta^2)} = 5\sqrt{(1 - 0.36)} = 5 \times 0.8 = 4$ rad/s

(d) $\%OS = 100 \times e^{-\zeta\pi/\sqrt{(1-\zeta^2)}} = 100 \times e^{-0.6\pi/0.8} = 100 \times e^{-2.356} = 100 \times 0.0948 = 9.48\%$

Note: The zero at $s = -10$ will increase the actual overshoot slightly above this value.

Problem 4.4.7

Given: A system has the signal flow graph with nodes R, X_1, X_2, X_3, and Y. The branch gains are: R to $X_1 = 1$, X_1 to $X_2 = G_1 = 4$, X_2 to $X_3 = G_2 = 3$, X_3 to $Y = G_3 = 2$, X_2 to $X_1 = H_1 = -0.25$ (feedback), and X_3 to $X_2 = H_2 = -0.5$ (feedback).

Find: The overall transfer function Y/R using Mason's gain formula.

Solution:

Forward path: $P_1 = 1 \times G_1 \times G_2 \times G_3 = 1 \times 4 \times 3 \times 2 = 24$.

Loop gains: $L_1 = G_1 \times H_1 = 4 \times (-0.25) = -1.0$ (loop: $X_1 \rightarrow X_2 \rightarrow X_1$) $L_2 = G_2 \times H_2 = 3 \times (-0.5) = -1.5$ (loop: $X_2 \rightarrow X_3 \rightarrow X_2$)

Non-touching loops: L_1 and L_2 share node X_2, so they are touching. No non-touching pairs exist.

Graph determinant: $\Delta = 1 - (L_1 + L_2) = 1 - (-1.0 - 1.5) = 1 + 2.5 = 3.5$

Cofactor: $\Delta_1 = 1$ (all loops touch the forward path).

$Y/R = P_1\Delta_1 / \Delta = 24 \times 1 / 3.5 = 6.857$

Problem 4.4.8

Given: A signal flow graph has two forward paths and three loops. Forward paths: $P_1 = 5 \times 3 = 15$ ($R \rightarrow X_1 \rightarrow X_3 \rightarrow Y$), $P_2 = 2 \times 4 = 8$ ($R \rightarrow X_2 \rightarrow X_3 \rightarrow Y$). Loop gains: $L_1 = -2$ ($X_1 \rightarrow X_3 \rightarrow X_1$), $L_2 = -3$ ($X_2 \rightarrow X_3 \rightarrow X_2$), $L_3 = -0.5$ ($X_1 \rightarrow X_2 \rightarrow X_1$). L_1 and L_2 are touching. L_1 and L_3 are touching. L_2 and L_3 are touching.

Find: (a) The graph determinant Δ, (b) the cofactors Δ_1 and Δ_2, and (c) the overall transfer function Y/R.

Solution:

(a) Since all loop pairs are touching (no non-touching pairs): $\Delta = 1 - (L_1 + L_2 + L_3) = 1 - (-2 - 3 - 0.5) = 1 + 5.5 = 6.5$

(b) Cofactor Δ_1: Remove all loops touching P_1. Path P_1 goes through X_1 and X_3, which touches L_1 (X_1, X_3), L_3 (X_1, X_2). L_2 (X_2, X_3) touches P_1 through X_3. All loops touch P_1, so $\Delta_1 = 1$.

Cofactor Δ_2: Remove all loops touching P_2. Path P_2 goes through X_2 and X_3, which touches L_2 (X_2, X_3), L_3 (X_1, X_2). L_1 (X_1, X_3) touches P_2 through X_3. All loops touch P_2, so $\Delta_2 = 1$.

(c) $Y/R = (P_1\Delta_1 + P_2\Delta_2) / \Delta = (15 \times 1 + 8 \times 1) / 6.5 = 23 / 6.5 = 3.538$

Problem 4.4.9

Given: Two systems are connected in parallel. System 1: $G_1(s) = 8/(s + 2)$. System 2: $G_2(s) = -3/(s + 7)$. The combined output feeds into a unity feedback loop.

Find: (a) The combined parallel transfer function, (b) the closed-loop transfer function, (c) the closed-loop poles, and (d) the DC gain.

Solution:

(a) Parallel combination: $G(s) = G_1(s) + G_2(s) = 8/(s + 2) + (-3)/(s + 7) = [8(s + 7) - 3(s + 2)] / [(s + 2)(s + 7)]$ $G(s) = (8s + 56 - 3s - 6) / (s^2 + 9s + 14) = (5s + 50) / (s^2 + 9s + 14)$

(b) Closed-loop: $T(s) = G / (1 + G) = (5s + 50) / (s^2 + 9s + 14 + 5s + 50) = (5s + 50) / (s^2 + 14s + 64)$

(c) Poles: $s = (-14 \pm \sqrt{(196 - 256)}) / 2 = (-14 \pm \sqrt{(-60)}) / 2 = (-14 \pm j7.746) / 2$ $s = -7 \pm j3.873$ (complex conjugate, underdamped, stable)

(d) DC gain: $T(0) = 50 / 64 = 0.781$

Problem 4.4.10

Given: A control system has a disturbance D(s) entering at the plant input. The controller is $G_c(s) = 10(s + 3)/(s + 15)$, the plant is $G_p(s) = 4/[s(s + 2)]$, and the system uses unity feedback.

Find: (a) The closed-loop transfer function Y(s)/R(s), (b) the disturbance transfer function Y(s)/D(s), (c) the steady-state output due to a unit step disturbance (with R = 0), and (d) the steady-state error to a unit step reference (with D = 0).

Solution:

(a) Forward path: $G(s) = G_c(s) \times G_p(s) = [10(s + 3) / (s + 15)] \times [4 / s(s + 2)] = 40(s + 3) / [s(s + 2)(s + 15)]$ Closed-loop: $Y/R = G / (1 + G) = 40(s + 3) / [s(s + 2)(s + 15) + 40(s + 3)]$

Expand denominator: $s^3 + 17s^2 + 30s + 40s + 120 = s^3 + 17s^2 + 70s + 120$. $Y/R = 40(s + 3) / (s^3 + 17s^2 + 70s + 120)$

(b) Disturbance enters at plant input: $Y/D = G_p / (1 + G_cG_p) = [4/s(s + 2)] / [1 + 40(s + 3)/s(s + 2)(s + 15)]$ $Y/D = 4(s + 15) / [s(s + 2)(s + 15) + 40(s + 3)] = 4(s + 15) / (s^3 + 17s^2 + 70s + 120)$

(c) Steady-state output to unit step disturbance: $y_{ss} = \lim(s\to0)\ s \times [4(s + 15) / (s^3 + 17s^2 + 70s + 120)] \times (1/s) = 4(15) / 120 = 60/120 = 0.5$

(d) The open-loop transfer function has one free integrator (Type 1 system). $K_p = \lim(s\to0)\ G(s) = \lim(s\to0)\ 40(s + 3) / [s(s + 2)(s + 15)] = \infty$ (Type 1). $e_{ss} = 1 / (1 + K_p) = 1 / \infty = 0$ (zero steady-state error to a step input)

Chapter 4 — Section 4.5: Time-Domain Performance

Practice problems covering first-order system response, second-order system response, time constants, rise time, settling time, peak time, percent overshoot, damping ratio, natural frequency, and steady-state error analysis.

Problem 4.5.1

Given: A thermal process has a first-order transfer function $G(s) = 150 / (40s + 1)$, where the input is valve position (0-100%) and the output is temperature in degrees C.

Find: (a) The DC gain and time constant, (b) the steady-state temperature for a 30% valve input, (c) the time to reach 63.2% of final temperature, (d) the 2% settling time, and (e) the 10-90% rise time.

Solution:

(a) Comparing $G(s) = K / (\tau s + 1)$: DC gain $K = 150$, time constant $\tau = 40$ seconds

(b) Steady-state temperature: $T_{ss} = K \times \text{input} = 150 \times 0.30 = 45$ degrees C

(c) Time to 63.2%: $t = \tau = 40$ seconds

(d) 2% settling time: $t_s = 4\tau = 4 \times 40 = 160$ seconds

(e) Rise time (10-90%): $t_r = 2.2\tau = 2.2 \times 40 = 88$ seconds

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Problem 4.5.2

Given: A position control servo has a second-order closed-loop transfer function $G(s) = 400 / (s^2 + 16s + 400)$.

Find: (a) ω_n and ζ, (b) the damped natural frequency ω_d, (c) the percent overshoot, (d) the peak time, (e) the 2% settling time, and (f) the pole locations.

Solution:

(a) Comparing with $\omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$: $\omega_n = \sqrt{400} = 20$ rad/s $2\zeta\omega_n = 16 \rightarrow \zeta = 16 / (2 \times 20) = 0.4$ (underdamped)

(b) $\omega_d = \omega_n\sqrt{(1 - \zeta^2)} = 20\sqrt{(1 - 0.16)} = 20 \times 0.9165 = 18.33$ rad/s

(c) $\%OS = 100 \times e^{-\zeta\pi/\sqrt{(1-\zeta^2)}} = 100 \times e^{-0.4\pi/0.9165} = 100 \times e^{-1.371} = 100 \times 0.254 = 25.4\%$

(d) Peak time: $t_p = \pi / \omega_d = \pi / 18.33 = 0.171$ s

(e) 2% settling time: $t_s = 4 / (\zeta\omega_n) = 4 / (0.4 \times 20) = 4 / 8 = 0.5$ s

(f) Poles: $s = -\zeta\omega_n \pm j\omega_d = -8 \pm j18.33$

Problem 4.5.3

Given: A control system must meet the following time-domain specifications: percent overshoot no greater than 10%, 2% settling time no greater than 0.8 s, and peak time no greater than 0.3 s.

Find: (a) The minimum damping ratio, (b) the minimum $\sigma = \zeta\omega_n$ (real part of poles), (c) the minimum ω_d (damped frequency), and (d) the minimum ω_n.

Solution:

(a) From $\%OS = 100 \times e^{-\zeta\pi/\sqrt{(1-\zeta^2)}} \leq 10$: $e^{-\zeta\pi/\sqrt{(1-\zeta^2)}} \leq 0.10 \rightarrow -\zeta\pi/\sqrt{(1-\zeta^2)} \leq \ln(0.10) = -2.303$ $\zeta\pi/\sqrt{(1-\zeta^2)} \geq 2.303 \rightarrow \zeta^2\pi^2 \geq 2.303^2(1-\zeta^2) \rightarrow \zeta^2(\pi^2 + 5.304) = 5.304$ $\zeta^2 = 5.304 / 15.17 = 0.3497 \rightarrow \zeta \geq 0.591$

(b) From $t_s = 4/(\zeta\omega_n) \leq 0.8$: $\sigma = \zeta\omega_n \geq 4 / 0.8 = 5\ s^{-1}$

(c) From $t_p = \pi/\omega_d \leq 0.3$: $\omega_d \geq \pi / 0.3 = 10.47$ rad/s

(d) $\omega_n = \sqrt{(\sigma^2 + \omega_d^2)} = \sqrt{(25 + 109.7)} = \sqrt{134.7} = 11.6$ rad/s

Check: $\zeta = \sigma/\omega_n = 5/11.6 = 0.431 < 0.591$, so the overshoot constraint is more restrictive. Using $\zeta = 0.591$: $\omega_n = \sigma/\zeta = 5/0.591 = 8.46$ rad/s, and $\omega_d = 8.46 \times \sqrt{(1 - 0.349)} = 8.46 \times 0.807 = 6.83$ rad/s. But $t_p = \pi/6.83 = 0.46\ s > 0.3$ s, which violates the peak time spec.

So ω_d must be at least 10.47 rad/s. With $\zeta = 0.591$: $\omega_n = \omega_d/\sqrt{(1 - \zeta^2)} = 10.47/0.807 = 12.98$ rad/s. $\sigma = \zeta\omega_n = 0.591 \times 12.98 = 7.67 > 5$, which satisfies the settling time spec.

Final answer: $\zeta \geq 0.591$, $\omega_n \geq 12.98$ rad/s, with poles at $s = -7.67 \pm j10.47$.

Problem 4.5.4

Given: A unity feedback system has the open-loop transfer function $G(s) = 1000(s + 8) / [s^2(s + 15)(s + 40)]$.

Find: (a) The system type, (b) the position error constant K_p, (c) the velocity error constant K_v, (d) the acceleration error constant K_a, and (e) the steady-state errors for a unit step, unit ramp, and unit parabolic input.

Solution:

(a) G(s) has two free integrators (s^2 in the denominator). This is a Type 2 system.

(b) $K_p = \lim(s\to 0)\ G(s) = \lim(s\to 0)\ 1000(s + 8) / [s^2(s + 15)(s + 40)] = \infty$

(c) $K_v = \lim(s\to 0)\ sG(s) = \lim(s\to 0)\ 1000(s + 8) / [s(s + 15)(s + 40)] = \infty$

(d) $K_a = \lim(s\to 0)\ s^2G(s) = \lim(s\to 0)\ 1000(s + 8) / [(s + 15)(s + 40)] = 1000 \times 8 / (15 \times 40) = 8000 / 600 = 13.33$

(e) Steady-state errors:

- Unit step: $e_{ss} = 1/(1 + K_p) = 1/\infty = 0$
- Unit ramp: $e_{ss} = 1/K_v = 1/\infty = 0$
- Unit parabola $r(t) = t^2/2$: $e_{ss} = 1/K_a = 1/13.33 = 0.075$

Problem 4.5.5

Given: A second-order system has a step response that exhibits the first peak at t = 0.25 s with a value of 1.35 (the final value is 1.0).

Find: (a) The percent overshoot, (b) the damping ratio ζ, (c) the damped natural frequency ω_d, (d) the natural frequency ω_n, and (e) the 2% settling time.

Solution:

(a) %OS = (peak - final) / final × 100 = (1.35 - 1.0) / 1.0 × 100 = 35%

(b) From $\%OS = 100 \times e^{-\zeta\pi/\sqrt{(1-\zeta^2)}}$: $0.35 = e^{-\zeta\pi/\sqrt{(1-\zeta^2)}} \to \ln(0.35) = -1.0498 = -\zeta\pi/\sqrt{(1-\zeta^2)}$ $\zeta\pi/\sqrt{(1-\zeta^2)} = 1.0498 \to \zeta^2\pi^2 = 1.0498^2(1-\zeta^2) = 1.1021(1-\zeta^2)$ $\zeta^2(\pi^2 + 1.1021) = 1.1021 \to \zeta^2 = 1.1021/10.97 = 0.1005$ $\zeta = 0.317$

(c) Peak time $t_p = \pi/\omega_d = 0.25$ s. $\omega_d = \pi / 0.25 = 12.57$ rad/s

(d) $\omega_n = \omega_d / \sqrt{(1 - \zeta^2)} = 12.57 / \sqrt{(1 - 0.1005)} = 12.57 / 0.9487 = 13.25$ rad/s

(e) $t_s = 4 / (\zeta\omega_n) = 4 / (0.317 \times 13.25) = 4 / 4.20 = 0.952$ s

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Problem 4.5.6

Given: An overdamped second-order system has the transfer function $G(s) = 36 / (s^2 + 13s + 36)$.

Find: (a) The poles, (b) ω_n and ζ, (c) the unit step response c(t), and (d) the 2% settling time (using the exact response, not the approximate formula).

Solution:

(a) $s^2 + 13s + 36 = (s + 4)(s + 9) = 0$. Poles: s = -4 and s = -9 (both real, overdamped).

(b) $\omega_n = \sqrt{36} = 6$ rad/s. $2\zeta\omega_n = 13 \to \zeta = 13/12 = 1.083$ (overdamped).

(c) Step response: $C(s) = 36 / [s(s + 4)(s + 9)]$. Partial fractions: $C(s) = A/s + B/(s + 4) + C/(s + 9)$. $A = 36/(4 \times 9) = 1$ $B = 36/[(-4)(-4 + 9)] = 36/(-4 \times 5) = -1.8$ $C = 36/[(-9)(-9 + 4)] = 36/(-9 \times (-5)) = 0.8$

$c(t) = 1 - 1.8e^{-4t} + 0.8e^{-9t}$

(d) The dominant pole at $s = -4$ controls the settling. The 2% criterion requires $|c(t) - 1| \leq 0.02$. At large t, the e^{-9t} term is negligible: $1.8e^{-4t} = 0.02 \rightarrow e^{-4t} = 0.0111 \rightarrow -4t = \ln(0.0111) = -4.50$. $t_s \approx 4.50/4 = 1.125$ s

Note: Using the approximate formula $t_s = 4/(\zeta\omega_n) = 4/6.5 = 0.615$ s would significantly underestimate the settling time for overdamped systems.

Problem 4.5.7

Given: A unity feedback system has open-loop transfer function $G(s) = 50 / [(s + 5)(s + 10)]$. This is a Type 0 system with steady-state error to a unit step.

Find: (a) The steady-state error to a unit step, (b) the value of a proportional gain K placed in the forward path that reduces the steady-state error to 2%, (c) the closed-loop poles with that gain, and (d) whether the system is underdamped or overdamped at that gain.

Solution:

(a) $K_p = G(0) = 50 / (5 \times 10) = 1$. $e_{ss} = 1/(1 + K_p) = 1/(1 + 1) = 0.5$ (50% error)

(b) For 2% error: $e_{ss} = 1/(1 + KK_{p,plant}) = 0.02$, where $K_{p,plant} = 50/50 = 1$. $1/(1 + K) = 0.02 \rightarrow 1 + K = 50 \rightarrow K = 49$ (Verification: open-loop DC gain $= 49 \times 50/50 = 49$, $e_{ss} = 1/50 = 0.02$)

Actually, with gain K in forward path: $G_{OL}(s) = 50K/[(s+5)(s+10)]$. $K_p = 50K/50 = K$. For $e_{ss} = 0.02$: $1/(1+K) = 0.02 \rightarrow K = 49$.

(c) Closed-loop characteristic: $s^2 + 15s + 50 + 50K = s^2 + 15s + 50 + 2450 = s^2 + 15s + 2500 = 0$. $s = (-15 \pm \sqrt{(225 - 10000)})/2 = (-15 \pm \sqrt{(-9775)})/2 = (-15 \pm j98.87)/2$ Poles: $s = -7.5 \pm j49.44$

(d) The poles are complex conjugate → underdamped. $\zeta = 7.5/50 = 0.15$ (very low damping). %OS $= 100 \times e^{-0.15\pi/0.989} = 100 \times e^{-0.476} = 62.1\%$. The gain needed for low steady-state error causes very poor transient response — this illustrates the fundamental trade-off in proportional-only control.

Problem 4.5.8

Given: A unity feedback system has open-loop transfer function $G(s) = 300 / [s(s + 12)(s + 25)]$. Determine the steady-state error performance.

Find: (a) The system type, (b) K_p, K_v, and K_a, (c) the steady-state error for $r(t) = 5u(t)$ (step of magnitude 5), (d) the steady-state error for $r(t) = 2t$ (ramp of slope 2), and (e) the steady-state error for $r(t) = 0.5t^2$ (parabolic input).

Solution:

(a) One free integrator in G(s) → Type 1 system.

(b) K_p = lim(s→0) G(s) = lim(s→0) 300/[s(s+12)(s+25)] = ∞ K_v = lim(s→0) sG(s) = 300/(12 × 25) = 300/300 = 1.0 K_a = lim(s→0) s^2G(s) = lim(s→0) 300s/[(s+12)(s+25)] = 0

(c) Step input of magnitude 5: e_{ss} = 5/(1 + K_p) = 5/∞ = 0

(d) Ramp input r(t) = 2t: e_{ss} = 2/K_v = 2/1.0 = 2.0

(e) Parabolic input r(t) = 0.5t^2 (A = 1): e_{ss} = A/K_a = 1/0 = ∞ (unbounded error; a Type 1 system cannot track a parabolic input)

Problem 4.5.9

Given: A second-order system has the closed-loop transfer function T(s) = ω_n^2 / (s^2 + $2\zeta\omega_n s$ + ω_n^2) with ω_n = 10 rad/s. The damping ratio is varied from ζ = 0.1 to ζ = 2.0.

Find: The percent overshoot, peak time, and 2% settling time for: (a) ζ = 0.1, (b) ζ = 0.3, (c) ζ = 0.707, (d) ζ = 1.0, and (e) ζ = 2.0.

Solution:

(a) ζ = 0.1: ω_d = 10$\sqrt{}$(1-0.01) = 9.95 rad/s. %OS = 100 × $e^{-0.1\pi/0.995}$ = 100 × $e^{-0.316}$ = 72.9% t_p = π/9.95 = 0.316 s t_s = 4/(0.1 × 10) = 4.0 s

(b) ζ = 0.3: ω_d = 10$\sqrt{}$(1-0.09) = 9.54 rad/s. %OS = 100 × $e^{-0.3\pi/0.954}$ = 100 × $e^{-0.987}$ = 37.3% t_p = π/9.54 = 0.329 s t_s = 4/(0.3 × 10) = 1.333 s

(c) ζ = 0.707: ω_d = 10$\sqrt{}$(1-0.5) = 7.07 rad/s. %OS = 100 × $e^{-0.707\pi/0.707}$ = 100 × $e^{-\pi}$ = 100 × 0.0432 = 4.3% t_p = π/7.07 = 0.444 s t_s = 4/(0.707 × 10) = 0.566 s

(d) ζ = 1.0 (critically damped): No overshoot → %OS = 0%. No oscillation, so t_p = ∞ (no peak; monotonic approach to final value). t_s = 4/(1.0 × 10) = 0.4 s

(e) ζ = 2.0 (overdamped): Poles at s = -10(2 ± $\sqrt{}$(4-1)) = -10(2 ± 1.732) → s = -2.68 and -37.32. %OS = 0%, t_p = ∞ (no peak). Settling dominated by slow pole: t_s ≈ 4/2.68 = 1.49 s

Note: ζ = 0.707 provides the best balance — fast settling (0.566 s) with minimal overshoot (4.3%).

Problem 4.5.10

Given: A unity feedback system with open-loop transfer function G(s) = K(s + 6) / [s(s + 3)(s + 10)] is required to have a steady-state error of no more than 0.1 for a ramp input r(t) = t.

Find: (a) The system type, (b) the minimum K to meet the ramp error specification, (c) the closed-loop characteristic equation at that K, and (d) verify stability using the Routh criterion.

Solution:

(a) One free integrator → Type 1 system.

(b) $K_v = \lim(s \to 0)\ sG(s) = \lim(s \to 0)\ K(s+6)/[(s+3)(s+10)] = 6K/(3 \times 10) = K/5$. For $e_{ss} = 1/K_v \leq 0.1$: $K_v \geq 10 \to K/5 \geq 10 \to K \geq 50$

(c) At $K = 50$: closed-loop characteristic equation: $s(s + 3)(s + 10) + 50(s + 6) = 0$ $s^3 + 13s^2 + 30s + 50s + 300 = 0$ $s^3 + 13s^2 + 80s + 300 = 0$

(d) Routh array: | s^3 | 1 | 80 | | s^2 | 13 | 300 | | s^1 | $(13 \times 80 - 300)/13$ | 0 | | s^0 | 300 | |

s^1 row: $(1040 - 300)/13 = 740/13 = 56.9 > 0$. All first-column entries are positive: 1, 13, 56.9, 300 → system is stable at $K = 50$.

Chapter 4 — Section 4.6: PID Control

Practice problems covering proportional control, integral control, derivative control, PID controller design, Ziegler-Nichols tuning, steady-state error elimination, and transient response improvement.

Problem 4.6.1

Given: A proportional controller with gain K_p controls a plant G(s) = 5 / (s + 8) in a unity feedback configuration.

Find: (a) The closed-loop transfer function, (b) the steady-state error to a unit step for K_p = 10, (c) the K_p required to limit the steady-state error to 5%, (d) the closed-loop pole and time constant at that K_p.

Solution:

(a) Closed-loop: $T(s) = K_pG(s) / (1 + K_pG(s)) = 5K_p / (s + 8 + 5K_p) = 5K_p / (s + 8 + 5K_p)$

(b) $K_{p,position} = \lim(s\to 0)\ K_p \times 5/(s + 8) = 5K_p/8 = 5(10)/8 = 6.25$. $e_{ss} = 1/(1 + 6.25) = 1/7.25 = 0.138$ (13.8% error)

(c) For 5% error: $e_{ss} = 1/(1 + 5K_p/8) = 0.05$. $1 + 5K_p/8 = 20 \rightarrow 5K_p/8 = 19 \rightarrow K_p = 30.4$

(d) Closed-loop pole: $s = -(8 + 5 \times 30.4) = -(8 + 152) = -160$. Time constant: $\tau = 1/160 = 6.25$ ms. The high gain makes the system very fast but leaves a 5% residual error.

Problem 4.6.2

Given: A PI controller ($K_p = 3$, $K_i = 6$) controls a plant G(s) = 4/(s + 2) with unity feedback.

Find: (a) The open-loop transfer function, (b) the system type, (c) the steady-state error to a unit step, (d) the steady-state error to a unit ramp, and (e) the closed-loop characteristic equation and its roots.

Solution:

(a) PI controller: $G_c(s) = K_p + K_i/s = (3s + 6)/s$. Open-loop: $G_{OL}(s) = G_c(s)G(s) = 4(3s + 6)/[s(s + 2)] = (12s + 24) / [s(s + 2)]$

(b) One free integrator from the PI controller → Type 1 system.

(c) Type 1 system: e_{ss} for step = 0 (zero steady-state error to a step).

(d) $K_v = \lim(s \to 0)\ s \times (12s + 24)/[s(s + 2)] = 24/2 = 12$. $e_{ss} = 1/K_v = 1/12 = 0.0833$ (8.33% ramp lag)

(e) Characteristic equation: $s(s + 2) + 12s + 24 = s^2 + 14s + 24 = 0$. $s = (-14 \pm \sqrt{(196 - 96)})/2 = (-14 \pm 10)/2$. Roots: $s = -2$ and $s = -12$ (both real, stable, overdamped).

Problem 4.6.3

Given: A PD controller ($K_p = 40$, $K_d = 4$) controls a plant $G(s) = 1/(s^2 + 6s)$ with unity feedback. Compare the performance with proportional-only control ($K_d = 0$).

Find: (a) The closed-loop transfer function with PD control, (b) ω_n and ζ with PD control, (c) the percent overshoot with PD control, (d) ω_n and ζ with P-only control, and (e) the percent overshoot with P-only control.

Solution:

(a) PD controller: $G_c(s) = 40 + 4s$. Closed-loop: $T(s) = (4s + 40)/(s^2 + 6s + 4s + 40) = (4s + 40) / (s^2 + 10s + 40)$

(b) $\omega_n = \sqrt{40} = 6.32$ rad/s, $2\zeta\omega_n = 10 \to \zeta = 10/(2 \times 6.32) = 0.791$

(c) $\%OS_{PD} = 100 \times e^{-0.791\pi/\sqrt{(1-0.626)}} = 100 \times e^{-0.791\pi/0.612} = 100 \times e^{-4.06} = 100 \times 0.0173 = 1.7\%$

(Note: The zero in the numerator at $s = -10$ will increase actual overshoot slightly above this value computed from the denominator alone.)

(d) P-only ($K_d = 0$): $T(s) = 40/(s^2 + 6s + 40)$. $\omega_n = \sqrt{40} = 6.32$ rad/s, $2\zeta\omega_n = 6 \to \zeta = 6/12.65 = 0.474$

(e) $\%OS_P = 100 \times e^{-0.474\pi/\sqrt{(1-0.225)}} = 100 \times e^{-0.474\pi/0.881} = 100 \times e^{-1.690} = 100 \times 0.185 = 18.5\%$

The derivative action reduced overshoot from 18.5% to 1.7% by increasing ζ from 0.474 to 0.791.

Problem 4.6.4

Given: A temperature control loop produces sustained oscillations at $K_u = 12$ with a period $P_u = 8$ s when using proportional-only control. Use the Ziegler-Nichols ultimate gain method.

Find: (a) P-only controller parameters, (b) PI controller parameters, (c) PID controller parameters, and (d) the PID controller transfer function.

Solution:

(a) P-only: $K_p = 0.5 \times K_u = 0.5 \times 12 = 6.0$

(b) PI: $K_p = 0.45 \times K_u = 0.45 \times 12 = 5.4$ $K_i = 0.54 \times K_u/P_u = 0.54 \times 12/8 = 0.81\ s^{-1}$ ($T_i = K_p/K_i = 5.4/0.81 = 6.67$ s)

(c) PID: $K_p = 0.6 \times K_u = 0.6 \times 12 = 7.2$ $K_i = 1.2 \times K_u/P_u = 1.2 \times 12/8 = 1.8\ s^{-1}$ $K_d = 0.075 \times K_u \times P_u = 0.075 \times 12 \times 8 = 7.2$ s

(d) $G_c(s) = K_p + K_i/s + K_d s = 7.2 + 1.8/s + 7.2s = (7.2s^2 + 7.2s + 1.8) / s$

Verification: $T_i = K_p/K_i = 7.2/1.8 = 4.0$ s $= P_u/2 = 4.0$ s. $T_d = K_d/K_p = 7.2/7.2 = 1.0$ s $= P_u/8 = 1.0$ s. Both check out.

Problem 4.6.5

Given: A process plant has the first-order-plus-dead-time (FOPDT) model $G(s) = 2e^{-3s} / (10s + 1)$, identified from an open-loop step response. The reaction curve parameters are: DC gain $K_p = 2$, dead time L = 3 s, and time constant T = 10 s. Use the Ziegler-Nichols step response method.

Find: The PID parameters using the Ziegler-Nichols first method (reaction curve): $K_p = 1.2T/(KL)$, $T_i = 2L$, $T_d = 0.5L$.

Solution:

Proportional gain: $K_p = 1.2T / (K \times L) = 1.2 \times 10 / (2 \times 3) = 12/6 = 2.0$

Integral time: $T_i = 2L = 2 \times 3 = 6.0$ s $K_i = K_p/T_i = 2.0/6.0 = 0.333\ s^{-1}$

Derivative time: $T_d = 0.5L = 0.5 \times 3 = 1.5$ s $K_d = K_p \times T_d = 2.0 \times 1.5 = 3.0$ s

PID transfer function: $G_c(s) = 2.0(1 + 1/(6s) + 1.5s) = 2.0 + 0.333/s + 3.0s$

The ratio $T/L = 10/3 = 3.33$. Ziegler-Nichols works best when $3 < T/L < 10$, so this plant is at the lower end of the useful range. Some detuning may be needed due to the relatively large dead time.

Problem 4.6.6

Given: A PI controller is used to eliminate steady-state error in a Type 0 plant $G(s) = 10 / [(s + 1)(s + 5)]$. The PI gains are $K_p = 4$ and $K_i = 2$.

Find: (a) The open-loop transfer function, (b) the closed-loop transfer function, (c) the closed-loop poles, (d) the percent overshoot (approximate, using dominant poles), and (e) the 2% settling time.

Solution:

(a) $G_c(s) = (4s + 2)/s$. $G_{OL}(s) = 10(4s + 2)/[s(s + 1)(s + 5)] = (40s + 20) / [s(s + 1)(s + 5)]$

(b) Closed-loop characteristic: $s(s + 1)(s + 5) + 40s + 20 = 0$. $s^3 + 6s^2 + 5s + 40s + 20 = s^3 + 6s^2 + 45s + 20 = 0$.

$T(s) = (40s + 20) / (s^3 + 6s^2 + 45s + 20)$

(c) Using numerical methods, the real root is approximately $s \approx -0.472$. Factor: $s^3 + 6s^2 + 45s + 20 \approx (s + 0.472)(s^2 + 5.528s + 42.37)$. Verification: expanding gives $s^3 + 6.0s^2 + (2.609 + 42.37)s + 20.0 = s^3 + 6s^2 + 44.98s + 20 \approx$ original. Complex roots: $s = (-5.528 \pm \sqrt{(30.56 - 169.48)})/2 = (-5.528 \pm j11.79)/2$. $s = -2.764 \pm j5.894$

(d) Dominant complex poles: $\zeta = 2.764/\sqrt{(2.764^2 + 5.894^2)} = 2.764/6.510 = 0.425$. $\%OS \approx 100 \times e^{-0.425\pi/\sqrt{(1-0.180)}} = 100 \times e^{-0.425\pi/0.905} = 100 \times e^{-1.474} = 22.9\%$

(e) $t_s \approx 4/\sigma = 4/2.764 = 1.448$ s

Problem 4.6.7

Given: A PID controller with $K_p = 8$, $K_i = 15$, and $K_d = 1.5$ controls a plant $G(s) = 1/(s + 3)$ with unity feedback.

Find: (a) The open-loop transfer function, (b) the closed-loop characteristic equation, (c) the system type, (d) K_v, and (e) the steady-state error for a ramp input $r(t) = 2t$.

Solution:

(a) $G_c(s) = K_p + K_i/s + K_d s = (1.5s^2 + 8s + 15)/s$. $G_{OL}(s) = (1.5s^2 + 8s + 15)/[s(s + 3)] = (1.5s^2 + 8s + 15) / [s(s + 3)]$

(b) Characteristic equation: $s(s + 3) + 1.5s^2 + 8s + 15 = 0$. $s^2 + 3s + 1.5s^2 + 8s + 15 = 2.5s^2 + 11s + 15 = 0$.

Divide by 2.5: $s^2 + 4.4s + 6 = 0$

(c) One free integrator in the open-loop (from the integral term) → Type 1 system.

(d) $K_v = \lim(s\to0)\ sG_{OL}(s) = \lim(s\to0)\ (1.5s^2 + 8s + 15)/(s + 3) = 15/3 = 5.0$

(e) For ramp $r(t) = 2t$: $e_{ss} = 2/K_v = 2/5.0 = 0.4$

Problem 4.6.8

Given: An industrial pressure loop with plant $G(s) = 3 / [(s + 1)(s + 4)]$ uses a PI controller. The system must have zero steady-state error to a step and a damping ratio of at least $\zeta = 0.5$. The PI controller has $K_i = 2K_p$ (the integral gain is twice the proportional gain).

Find: (a) The open-loop transfer function in terms of K_p, (b) the closed-loop characteristic equation, (c) the value of K_p that yields $\zeta = 0.5$, and (d) the resulting K_i.

Solution:

(a) $G_c(s) = K_p + 2K_p/s = K_p(s + 2)/s$. $G_{OL}(s) = 3K_p(s + 2) / [s(s + 1)(s + 4)] = 3K_p(s + 2) / [s(s + 1)(s + 4)]$

(b) Characteristic equation: $s(s + 1)(s + 4) + 3K_p(s + 2) = 0$. $s^3 + 5s^2 + 4s + 3K_p s + 6K_p = 0$. $s^3 + 5s^2 + (4 + 3K_p)s + 6K_p = 0$

(c) For a third-order system, exact damping ratio assignment requires numerical methods. Using the dominant second-order pole approximation, assume the characteristic equation factors as $(s + a)(s^2 + 2\zeta\omega_n s + \omega_n^2)$ where a is a distant real pole.

Expanding: $s^3 + (a + 2\zeta\omega_n)s^2 + (2a\zeta\omega_n + \omega_n^2)s + a\omega_n^2$. Match: $a + 2\zeta\omega_n = 5$, $2a\zeta\omega_n + \omega_n^2 = 4 + 3K_p$, $a\omega_n^2 = 6K_p$.

With $\zeta = 0.5$, try $K_p = 2$: $s^3 + 5s^2 + 10s + 12 = 0$. Test s = -1: -1 + 5 - 10 + 12 = 6 (no). Test s = -2: -8 + 20 - 20 + 12 = 4 (no). Test s = -3: -27 + 45 - 30 + 12 = 0 (yes!). Factor: $(s + 3)(s^2 + 2s + 4) = 0$. From $s^2 + 2s + 4$: $\omega_n = 2$, $\zeta = 1/2 = 0.5$.

$K_p = 2$

(d) $K_i = 2K_p = 4\ s^{-1}$

Verification: The closed-loop poles are s = -3 and $s = -1 \pm j\sqrt{3} = -1 \pm j1.732$. The real pole at -3 is 3× the real part of the complex poles (-1), so the dominant second-order approximation is reasonable.

Problem 4.6.9

Given: A PID controller must be designed for a plant $G(s) = 1/[s(s + 10)]$ in a unity feedback system. The specifications require: zero steady-state error to a ramp input (guaranteed by Type 2 system), $\zeta \geq 0.6$ for the dominant poles, and $\omega_n \geq 8$ rad/s.

Find: (a) The required system type with PID control, (b) design K_p, K_i, and K_d to place the closed-loop poles at $s = -5 \pm j6.33$ and s = -20 (a fast real pole).

Solution:

(a) The plant has one integrator (1/s). The PID integral term adds another, making the open-loop Type 2. This guarantees zero ramp error.

(b) Desired characteristic polynomial: $(s + 20)(s + 5 - j6.33)(s + 5 + j6.33) = (s + 20)(s^2 + 10s + 65.06)$. Expanding: $s^3 + 30s^2 + 265.06s + 1301.2$.

PID controller: $G_c(s) = (K_ds^2 + K_ps + K_i)/s$. Open-loop: $G_{OL}(s) = (K_ds^2 + K_ps + K_i)/[s^2(s + 10)]$.

Closed-loop characteristic: $s^2(s + 10) + K_ds^2 + K_ps + K_i = 0$. $s^3 + (10 + K_d)s^2 + K_ps + K_i = 0$.

Match with desired: $s^3 + 30s^2 + 265.06s + 1301.2$: $10 + K_d = 30 \rightarrow K_d = 20$ $K_p = 265.06$ $K_i = 1301.2$

Verification: $\omega_n = \sqrt{(25 + 40.07)} = \sqrt{65.06} = 8.07$ rad/s ≥ 8. $\zeta = 5/8.07 = 0.620 \geq 0.6$. Both specifications met.

Problem 4.6.10

Given: A motor speed control system uses a PID controller. An integral windup test is performed: the setpoint is 1000 RPM, the motor saturates at 500 RPM due to a mechanical load, and the error accumulates for 10 seconds. PID parameters are $K_p = 0.5$, $K_i = 2.0$, $K_d = 0.02$, with sample period T = 0.01 s.

Find: (a) The steady-state error during saturation, (b) the accumulated integral term after 10 seconds (assuming constant error), (c) the total control signal at t = 10 s, and (d) the time to unwind the integral if the load is suddenly removed and error reverses at 500 RPM.

Solution:

(a) Error during saturation: e = 1000 - 500 = 500 RPM

(b) Accumulated integral term after 10 s with constant error: Integral sum = $\Sigma e[k] \times T$ for k = 0 to 999 (10 s / 0.01 s = 1000 samples). Each sample contributes $e \times T = 500 \times 0.01 = 5$. Total integral sum = $1000 \times 5 = 5000$. Integral contribution: $K_i \times$ integral sum = $2.0 \times 5000 = 10{,}000$

(c) At t = 10 s (error still 500 RPM, de/dt = 0): $u = K_p \times 500 + 10{,}000 + K_d \times 0 = 250 + 10{,}000 = 10{,}250$

This is far beyond any reasonable actuator range, illustrating the windup problem.

(d) If the error reverses to -500 RPM (load removed, motor overshoots to 1500 RPM): The integral decreases at $K_i \times (-500) \times T = 2.0 \times (-500) \times 0.01 = -10$ per sample. The accumulated integral is 10,000. Time to unwind to zero: 10,000 / 10 = 1000 samples = 10 seconds.

During this 10-second unwind, the controller output remains positive despite a negative error, causing prolonged overshoot. This demonstrates why anti-windup mechanisms (clamping, back-calculation) are essential in practical PID implementations.

Chapter 4 — Section 4.7: Stability Analysis

Practice problems covering Routh-Hurwitz criterion, root locus construction, gain margins, stability boundaries, special Routh cases, and root locus design.

Problem 4.7.1

Given: A unity feedback system has open-loop transfer function $G(s) = K / [s(s + 4)(s + 12)]$.

Find: (a) The closed-loop characteristic equation, (b) the Routh array, (c) the range of K for stability, and (d) the frequency of oscillation at the stability boundary.

Solution:

(a) Characteristic equation: $s(s + 4)(s + 12) + K = 0$. $s^3 + 16s^2 + 48s + K = 0$.

(b) Routh array: | s^3 | 1 | 48 | | s^2 | 16 | K | | s^1 | $(16 \times 48 - K)/16$ | 0 | | s^0 | K | |

s^1 row: $(768 - K)/16$

(c) For stability, all first-column entries must be positive:

- s^3: $1 > 0$
- s^2: $16 > 0$
- s^1: $(768 - K)/16 > 0 \rightarrow K < 768$
- s^0: $K > 0$

Range: $0 < K < 768$

(d) At $K = 768$, the s^1 row is zero. Form the auxiliary equation from the s^2 row: $16s^2 + 768 = 0 \rightarrow s^2 = -48 \rightarrow s = \pm j\sqrt{48} = \pm j6.93$. Frequency of oscillation: $\omega = 6.93$ rad/s

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Problem 4.7.2

Given: A control system has the characteristic equation $s^4 + 3s^3 + 5s^2 + 9s + K = 0$.

Find: (a) The Routh array in terms of K, (b) the range of K for stability, and (c) the value of K at which the system is marginally stable and the corresponding oscillation frequency.

Solution:

(a) Routh array: | s^4 | 1 | 5 | K | | s^3 | 3 | 9 | 0 | | s^2 | (15 - 9)/3 = 2 | K | | | s^1 | $(2 \times 9 - 3K)/2 = (18 - 3K)/2$ | 0 | | | s^0 | K | | |

(b) For stability:

- s^4: $1 > 0$
- s^3: $3 > 0$
- s^2: $2 > 0$
- s^1: $(18 - 3K)/2 > 0 \rightarrow K < 6$
- s^0: $K > 0$

Range: $0 < K < 6$

(c) At K = 6, the s^1 row is zero. Auxiliary equation from s^2 row: $2s^2 + 6 = 0 \rightarrow s^2 = -3 \rightarrow s = \pm j\sqrt{3} = \pm j1.732$. Frequency of oscillation: $\omega = 1.732$ rad/s

The system is marginally stable at K = 6 with sustained oscillations at 1.732 rad/s.

Problem 4.7.3

Given: A characteristic equation has the form $s^3 + 2s^2 + (K + 3)s + 2K = 0$, where K is a positive real parameter.

Find: (a) The Routh array in terms of K, (b) the range of K for stability, (c) whether K = 4 produces a stable system, and (d) the number of RHP poles at K = 10.

Solution:

(a) Routh array: | s^3 | 1 | K + 3 | | s^2 | 2 | 2K | | s^1 | $(2(K + 3) - 2K)/2 = 6/2 = 3$ | 0 | | s^0 | 2K | |

(b) First-column entries: 1, 2, 3, 2K. For stability: $2K > 0 \rightarrow K > 0$. All other entries are always positive. Range: $K > 0$ (stable for all positive K).

(c) At K = 4: all first-column entries are 1, 2, 3, 8 — all positive. Yes, the system is stable.

(d) At K = 10: first-column entries are 1, 2, 3, 20 — all positive. Zero RHP poles (the system is stable). Note that the s^1 row entry is 3 regardless of K, so the system is stable for all $K > 0$.

Problem 4.7.4

Given: The open-loop transfer function is $G(s)H(s) = K / [s(s + 1)(s + 3)(s + 6)]$.

Find: (a) The number of root locus branches, (b) the real-axis segments of the root locus, (c) the asymptote angles and centroid, and (d) the value of K at the imaginary axis crossing.

Solution:

(a) Poles: s = 0, -1, -3, -6 (n = 4 poles, m = 0 zeros). Number of branches: 4

(b) Real-axis segments lie to the left of an odd number of poles/zeros:

- Between 0 and -1: 1 pole to the right (at 0) → odd → on locus

- Between -1 and -3: 2 poles to the right → even → not on locus
- Between -3 and -6: 3 poles to the right → odd → on locus
- Left of -6: 4 poles to the right → even → not on locus

Segments: [0, -1] and [-3, -6]

(c) Asymptote angles: $(2k+1) \times 180°/(n\text{-}m) = (2k+1) \times 45°$. Angles: 45°, 135°, 225°, 315° Centroid: $\sigma_a = (0 - 1 - 3 - 6)/(4 - 0) = -10/4 = -2.5$

(d) Characteristic equation: $s^4 + 10s^3 + 27s^2 + 18s + K = 0$. Routh array: | s^4 | 1 | 27 | K | | s^3 | 10 | 18 | 0 | | s^2 | (270 - 18)/10 = 25.2 | K | | | s^1 | $(25.2 \times 18 - 10K)/25.2 = (453.6 - 10K)/25.2$ | 0 | | | s^0 | K | | |

s^1 row: $(453.6 - 10K)/25.2 = 0 \rightarrow K = 45.36$

Auxiliary equation at K = 45.36: $25.2s^2 + 45.36 = 0 \rightarrow s^2 = -1.8 \rightarrow s = \pm j1.342$. Imaginary axis crossing at $\omega = 1.342$ rad/s with K = 45.36.

Problem 4.7.5

Given: A root locus has open-loop transfer function $G(s)H(s) = K(s + 5) / [s(s + 2)(s + 8)]$.

Find: (a) The number of branches, start/end points, (b) real-axis segments, (c) asymptote angles and centroid, (d) the breakaway point (between 0 and -2), and (e) whether the root locus crosses the imaginary axis.

Solution:

(a) n = 3 poles (0, -2, -8), m = 1 zero (-5). Branches: 3. Start at s = 0, -2, -8 (K = 0). Two branches end at the zero s = -5 and at infinity. Actually: 1 branch ends at the zero s = -5, and 2 branches go to infinity along asymptotes.

(b) Real-axis segments (to the left of an odd total of poles+zeros):

- Between 0 and -2: 1 pole (at 0) to the right → odd → on locus
- Between -2 and -5: 2 poles to the right → even → not on locus
- Between -5 and -8: 2 poles + 1 zero to the right = 3 → odd → on locus
- Left of -8: 3 poles + 1 zero = 4 → even → not on locus

Segments: [0, -2] and [-5, -8]

(c) Asymptotes (n - m = 2 branches go to infinity): Angles: $(2k+1) \times 180°/2 = 90°$ and 270° Centroid: $\sigma_a = (0 - 2 - 8 - (-5))/(3 - 1) = (-10 + 5)/2 = -2.5$

(d) Breakaway point between 0 and -2: $K = -s(s + 2)(s + 8)/(s + 5)$. Set dK/ds = 0. Using the simplified method: $1/s + 1/(s+2) + 1/(s+8) = 1/(s+5)$. Numerically: at s = -0.8: 1/(-0.8) + 1/(1.2) + 1/(7.2) = -1.25 + 0.833 + 0.139 = -0.278. 1/(s+5): 1/4.2 = 0.238. Difference = -0.516. At s = -1.0: 1/(-1) + 1/(1) + 1/(7) = -1 + 1 + 0.143 = 0.143. 1/4 = 0.25. Difference = -0.107. At s = -1.1: 1/(-1.1) + 1/(0.9) + 1/(6.9) = -0.909 + 1.111 + 0.145 = 0.347. 1/3.9 = 0.256. Difference = 0.091. By interpolation, breakaway at $s \approx -1.05$

(e) Characteristic equation: $s(s+2)(s+8) + K(s+5) = s^3 + 10s^2 + 16s + Ks + 5K = s^3 + 10s^2 + (16+K)s + 5K = 0$. Routh s^1: $[10(16+K) - 5K]/10 = (160 + 10K - 5K)/10 = (160 + 5K)/10$. Since

$K > 0$: $160 + 5K > 0$ always. s^0: $5K > 0$ for $K > 0$. The root locus never crosses the imaginary axis — the system is stable for all $K > 0$.

Problem 4.7.6

Given: A characteristic equation with a zero row in the Routh array: $s^4 + 6s^3 + 11s^2 + 6s + K = 0$.

Find: (a) The Routh array, (b) the value of K that produces a zero row, (c) the auxiliary polynomial and its roots at that K, and (d) all four closed-loop poles at that K.

Solution:

(a) Routh array: | s^4 | 1 | 11 | K | | s^3 | 6 | 6 | 0 | | s^2 | (66-6)/6 = 10 | K | | | s^1 | (60 - 6K)/10 | 0 | | | s^0 | K | | |

(b) Zero s^1 row: $60 - 6K = 0 \rightarrow K = 10$

(c) At $K = 10$, the auxiliary polynomial is from the s^2 row: $10s^2 + 10 = 0$. $s^2 = -1 \rightarrow s = \pm j1$ (purely imaginary poles — sustained oscillations at $\omega = 1$ rad/s).

(d) At $K = 10$: $s^4 + 6s^3 + 11s^2 + 6s + 10 = 0$. Factor out $(s^2 + 1)$: $s^4 + 6s^3 + 11s^2 + 6s + 10 = (s^2 + 1)(s^2 + 6s + 10) = 0$. Remaining roots: $s = (-6 \pm \sqrt{(36 - 40)})/2 = (-6 \pm j2)/2 = -3 \pm j1$.

All four poles: $s = \pm j1$ (marginally stable) and $s = -3 \pm j1$ (stable).

Problem 4.7.7

Given: A unity feedback system has $G(s) = K / [(s + 1)(s^2 + 4s + 13)]$. The denominator $s^2 + 4s + 13$ has complex poles at $s = -2 \pm j3$.

Find: (a) All open-loop poles, (b) the root locus real-axis segments, (c) asymptote angles and centroid, (d) the range of K for stability, and (e) the K value at the imaginary axis crossing.

Solution:

(a) Open-loop poles: $s = -1$, $s = -2 + j3$, $s = -2 - j3$ ($n = 3$, $m = 0$).

(b) Complex poles are not on the real axis, so only the real pole at -1 matters for the real-axis test. To the left of -1: 1 real pole to the right → odd → on locus. Segment: $(-\infty, -1]$ — however, this is only valid to the left of -1 with no other real poles/zeros.

Actually: Between 0 and -1: 0 poles/zeros to the right → even → not on locus (the pole at -1 is at the boundary). Left of -1: 1 real pole to the right → odd → on locus. Real-axis segment: $(-\infty, -1]$

(c) Asymptote angles: $(2k+1) \times 180°/3 = 60°, 180°, 300°$ Centroid: $\sigma_a = (-1 - 2 - 2)/3 = -5/3 = -1.667$

(d) Characteristic equation: $(s + 1)(s^2 + 4s + 13) + K = s^3 + 5s^2 + 17s + 13 + K = 0$. Routh array: | s^3 | 1 | 17 | | s^2 | 5 | 13 + K | | s^1 | (85 - 13 - K)/5 = (72 - K)/5 | 0 | | s^0 | 13 + K | |

Stability: $K > -13$ (always true for positive K) and $K < 72$. Range: $0 < K < 72$ (or $-13 < K < 72$ allowing negative gain)

(e) At K = 72: auxiliary equation from s^2 row: $5s^2 + 85 = 0 \rightarrow s^2 = -17 \rightarrow s = \pm j\sqrt{17} = \pm j4.123$. Imaginary axis crossing at $\omega = 4.123$ rad/s with K = 72.

Problem 4.7.8

Given: A root locus with open-loop transfer function $G(s)H(s) = K / [s^2(s + 4)]$ (double pole at origin).

Find: (a) Number of branches and asymptotes, (b) the asymptote angles and centroid, (c) the breakaway angle at the double pole at the origin, (d) whether the system is stable for any K > 0, and (e) the number of RHP poles for K = 10.

Solution:

(a) n = 3 poles (double at 0, one at -4), m = 0 zeros. Branches: 3. All 3 go to infinity along asymptotes.

(b) Asymptote angles: $(2k+1) \times 180°/3 = 60°, 180°, 300°$ Centroid: $\sigma_a = (0 + 0 - 4)/3 = -1.333$

(c) The two branches starting from the double pole at the origin depart at ±90° (perpendicular to the real axis) because they must be symmetric about the real axis and cannot continue along the real axis (the segment to the left of -4 is on the locus, but between 0 and -4 there are 2 poles to the right — even — so it is not on the locus). Departure angle: ±90°

(d) Characteristic equation: $s^3 + 4s^2 + K = 0$. Routh array: | s^3 | 1 | 0 | | s^2 | 4 | K | | s^1 | (0 - K)/4 = -K/4 | 0 | | s^0 | K | |

s^1 row: -K/4 < 0 for all K > 0. The system is unstable for all K > 0.

(e) At K = 10: first-column entries are 1, 4, -2.5, 10. There are two sign changes (4 to -2.5, then -2.5 to 10), so there are 2 RHP poles (with 1 LHP pole).

Problem 4.7.9

Given: A system has the characteristic equation $s^5 + 2s^4 + 4s^3 + 8s^2 + 3s + 6 = 0$.

Find: (a) The Routh array, (b) the number of RHP poles, and (c) whether the system is stable.

Solution:

(a) Routh array: | s^5 | 1 | 4 | 3 | | s^4 | 2 | 8 | 6 | | s^3 | (8 - 8)/2 = 0 | (6 - 6)/2 = 0 | |

The entire s^3 row is zero. This indicates symmetric root patterns. Use the auxiliary polynomial from the s^4 row: $P(s) = 2s^4 + 8s^2 + 6$. Take the derivative: $dP/ds = 8s^3 + 16s$.

Replace the s^3 row with coefficients of $8s^3 + 16s$: | s^3 | 8 | 16 | | | s^2 | (64 - 32)/8 = 4 | 6 | | | s^1 | (64 - 48)/4 = 4 | 0 | | | s^0 | 6 | | |

(b) First column (from s^5 down): 1, 2, 8, 4, 4, 6. No sign changes → 0 RHP poles.

(c) Although no poles are in the RHP, the zero row indicates poles on the imaginary axis. Solve $2s^4 + 8s^2 + 6 = 0$: let $u = s^2$, then $2u^2 + 8u + 6 = 0 \rightarrow u = (-8 \pm \sqrt{(64-48)})/4 = (-8 \pm 4)/4$. $u = -1 \rightarrow s = \pm j1$, and $u = -3 \rightarrow s = \pm j\sqrt{3}$.

The system has poles on the imaginary axis at $s = \pm j1$ and $s = \pm j\sqrt{3}$, making it marginally stable (sustained oscillations, not asymptotically stable).

Problem 4.7.10

Given: A unity feedback system has $G(s) = K(s + 3) / [s(s + 1)(s + 2)(s + 6)]$.

Find: (a) The number of root locus branches and asymptotes, (b) the asymptote angles and centroid, (c) the breakaway point between $s = -1$ and $s = -2$, (d) the break-in point between $s = -3$ and $s = -6$, and (e) the maximum K for stability.

Solution:

(a) n = 4 poles (0, -1, -2, -6), m = 1 zero (-3). Branches: 4. Asymptotic branches: n - m = 3.

(b) Asymptote angles: $(2k+1) \times 180°/3 = 60°, 180°, 300°$ Centroid: $\sigma_a = (0 - 1 - 2 - 6 - (-3))/3 = (-9 + 3)/3 = -2.0$

(c) $1/(s+3) = 1/s + 1/(s+1) + 1/(s+2) + 1/(s+6)$. This is derived from $dK/ds = 0$. Between $s = -1$ and $s = -2$, try $s = -1.5$: LHS: $1/(-1.5+3) = 1/1.5 = 0.667$ RHS: $1/(-1.5) + 1/(-0.5) + 1/(0.5) + 1/(4.5) = -0.667 - 2.0 + 2.0 + 0.222 = -0.445$ Not equal. Try $s = -1.4$: LHS: $1/1.6 = 0.625$. RHS: $1/(-1.4) + 1/(-0.4) + 1/(0.6) + 1/(4.6) = -0.714 - 2.5 + 1.667 + 0.217 = -1.330$ Try $s = -1.2$: LHS: $1/1.8 = 0.556$. RHS: $1/(-1.2) + 1/(-.2) + 1/(0.8) + 1/(4.8) = -0.833 - 5.0 + 1.25 + 0.208 = -4.375$

The breakaway actually occurs very close to the midpoint. Using $K = -s(s+1)(s+2)(s+6)/(s+3)$: $dK/ds = 0$ is solved numerically. At $s = -0.446$: $K = 0.446 \times 0.554 \times 1.554 \times 5.554 / 2.554 = 0.834$. Breakaway point $\approx s = -0.45$

(d) Break-in between -3 and -6 by numerical evaluation: at $s = -4.3$, $K = 4.3 \times 3.3 \times 2.3 \times 1.7 / 1.3 = 42.7$. This is a local minimum of K. Break-in point $\approx s = -4.3$

(e) Characteristic equation: $s(s+1)(s+2)(s+6) + K(s+3) = 0$. Expand: $s^4 + 9s^3 + 20s^2 + 12s + Ks + 3K = s^4 + 9s^3 + 20s^2 + (12+K)s + 3K = 0$.

Routh array: | s^4 | 1 | 20 | 3K | | s^3 | 9 | 12+K | 0 | | s^2 | (180-12-K)/9 = (168-K)/9 | 3K | | | s^1 | [(168-K)(12+K)/9 - 27K] / [(168-K)/9] | 0 | | | s^0 | 3K | | |

s^1 numerator: $(168-K)(12+K) - 243K = 2016 + 168K - 12K - K^2 - 243K = 2016 - 87K - K^2$. Set to zero: $K^2 + 87K - 2016 = 0 \rightarrow K = (-87 + \sqrt{(7569 + 8064)})/2 = (-87 + \sqrt{15633})/2 = (-87 + 125.03)/2 = 19.0$

Maximum K for stability: K = 19.0

Chapter 4 — Section 4.8: Frequency Response Design

Practice problems covering Bode plots, gain and phase margins, Nyquist criterion, lead compensator design, lag compensator design, lead-lag compensation, and bandwidth analysis.

Problem 4.8.1

Given: A system has the open-loop transfer function G(s) = 200 / [s(s + 5)(s + 20)].

Find: (a) The magnitude in dB and phase at ω = 1, 5, 10, and 20 rad/s, (b) the gain crossover frequency (where |G(jω)| = 0 dB), and (c) the low-frequency and high-frequency asymptotic slopes.

Solution:

(a) $|G(j\omega)| = 200 / [\omega \times \sqrt{(\omega^2 + 25)} \times \sqrt{(\omega^2 + 400)}]$

At ω = 1: $|G| = 200/(1 \times \sqrt{26} \times \sqrt{401}) = 200/(1 \times 5.099 \times 20.025) = 200/102.1 = 1.96 \rightarrow 5.84$ dB Phase: $-90° - \arctan(1/5) - \arctan(1/20) = -90° - 11.31° - 2.86° = -104.2°$

At ω = 5: $|G| = 200/(5 \times \sqrt{50} \times \sqrt{425}) = 200/(5 \times 7.071 \times 20.616) = 200/728.9 = 0.274 \rightarrow -11.24$ dB Phase: $-90° - \arctan(5/5) - \arctan(5/20) = -90° - 45° - 14.04° = -149.0°$

At ω = 10: $|G| = 200/(10 \times \sqrt{125} \times \sqrt{500}) = 200/(10 \times 11.18 \times 22.36) = 200/2499 = 0.0800 \rightarrow -21.94$ dB Phase: $-90° - \arctan(10/5) - \arctan(10/20) = -90° - 63.43° - 26.57° = -180.0°$

At ω = 20: $|G| = 200/(20 \times \sqrt{425} \times \sqrt{800}) = 200/(20 \times 20.616 \times 28.284) = 200/11{,}664 = 0.01714 \rightarrow -35.32$ dB Phase: $-90° - \arctan(20/5) - \arctan(20/20) = -90° - 75.96° - 45° = -210.96°$

(b) From the data, gain crosses 0 dB between ω = 1 and ω = 5. Solving numerically: At ω = 3: $|G| = 200/(3 \times \sqrt{34} \times \sqrt{409}) = 200/(3 \times 5.831 \times 20.224) = 200/353.8 = 0.565 \rightarrow -4.96$ dB. At ω = 2: $|G| = 200/(2 \times \sqrt{29} \times \sqrt{404}) = 200/(2 \times 5.385 \times 20.1) = 200/216.5 = 0.924 \rightarrow -0.69$ dB. At ω = 1.9: $|G| = 200/(1.9 \times \sqrt{28.61} \times \sqrt{403.61}) = 200/(1.9 \times 5.349 \times 20.09) = 200/204.1 = 0.980 \rightarrow -0.18$ dB. Gain crossover: $\omega_{gc} \approx 1.95$ rad/s

(c) Low frequency (ω « 5): $|G| \approx 200/(\omega \times 5 \times 20) = 2/\omega \rightarrow$ slope of -20 dB/decade. High frequency (ω » 20): $|G| \approx 200/(\omega^3) \rightarrow$ slope of -60 dB/decade.

Problem 4.8.2

Given: A unity feedback system has G(s) = 40 / [(s + 2)(s + 10)].

Find: (a) The gain crossover frequency, (b) the phase at gain crossover, (c) the phase margin, (d) the phase crossover frequency (where phase = -180°), and (e) the gain margin.

Solution:

(a) $|G(j\omega)| = 40 / [\sqrt{(\omega^2 + 4)} \times \sqrt{(\omega^2 + 100)}] = 1$. $40^2 = (\omega^2 + 4)(\omega^2 + 100)$. Let $u = \omega^2$: $1600 = u^2 + 104u + 400 \rightarrow u^2 + 104u - 1200 = 0$. $u = (-104 + \sqrt{(10816 + 4800)})/2 = (-104 + \sqrt{15616})/2 = (-104 + 124.96)/2 = 10.48$. $\omega_{gc} = \sqrt{10.48} = 3.24$ rad/s

(b) Phase at ω_{gc}: $\angle G$ = -arctan(3.24/2) - arctan(3.24/10) = -58.3° - 17.95° = -76.3°

(c) Phase margin: PM = 180° - 76.3° = 103.7° (very large — this is a Type 0 system with no integrators, hence very stable)

(d) Phase = -180° requires -arctan(ω/2) - arctan(ω/10) = -180°. Since the maximum phase lag from two real poles is -180° only as $\omega \rightarrow \infty$, there is no finite phase crossover frequency.

(e) Since the phase never reaches -180° at any finite frequency, the gain margin is infinite (∞ dB). The system is stable for all finite gains (though with increasing steady-state error as gain decreases).

Problem 4.8.3

Given: An open-loop transfer function G(s) = 100 / [s(s + 5)(s + 20)] with unity feedback.

Find: (a) The phase crossover frequency, (b) the gain at the phase crossover frequency, (c) the gain margin in dB, (d) the gain crossover frequency, and (e) the phase margin.

Solution:

(a) Phase = -180°: -90° - arctan(ω/5) - arctan(ω/20) = -180°. arctan(ω/5) + arctan(ω/20) = 90°. Using the identity: $\tan^{-1}(a) + \tan^{-1}(b) = 90°$ when ab = 1. $(\omega/5)(\omega/20) = 1 \rightarrow \omega^2/100 = 1 \rightarrow \omega = 10$ rad/s

(b) $|G(j10)| = 100/(10 \times \sqrt{(100+25)} \times \sqrt{(100+400)}) = 100/(10 \times 11.18 \times 22.36) = 100/2499 = 0.04002$

(c) $GM = 20 \log_{10}(1/0.04002) = 20 \log_{10}(24.99) = 20 \times 1.398 = 27.96$ dB

(d) $|G(j\omega)| = 1$: $100/(\omega\sqrt{(\omega^2+25)}\sqrt{(\omega^2+400)}) = 1$. $10{,}000 = \omega^2(\omega^2+25)(\omega^2+400)$. Try ω = 1.5: 2.25 × 27.25 × 402.25 = 24,659 (too high). Try ω = 2: 4 × 29 × 404 = 46,864 (too high). Try ω = 1: 1 × 26 × 401 = 10,426 (close). Try ω = 0.99: 0.98 × 25.98 × 400.98 = 10,205 (close). $\omega_{gc} \approx 1.0$ rad/s

(e) Phase at $\omega_{gc} = 1$: -90° - arctan(1/5) - arctan(1/20) = -90° - 11.31° - 2.86° = -104.2°. PM = 180° - 104.2° = 75.8° (very comfortable margin)

Problem 4.8.4

Given: A system has $G(j\omega)H(j\omega)$ that, when plotted as a Nyquist diagram, crosses the negative real axis at $-0.6 + j0$ at $\omega = 8$ rad/s. The system has no open-loop RHP poles ($P = 0$) and makes zero clockwise encirclements of the (-1, 0) point.

Find: (a) Whether the closed-loop system is stable, (b) the gain margin, (c) the factor by which the gain could be increased before instability, and (d) the gain margin in dB.

Solution:

(a) $Z = P + N = 0 + 0 = 0$. No closed-loop RHP poles → stable.

(b) The Nyquist plot crosses the real axis at -0.6. Gain margin = 1/|real axis crossing| = 1/0.6 = 1.667

(c) The gain can be multiplied by 1.667 before the Nyquist plot passes through (-1, 0).

(d) $GM = 20 \log_{10}(1.667) = 20 \times 0.2218 = 4.44$ dB

This is below the typical recommended minimum of 6 dB, indicating the system has marginal stability.

Problem 4.8.5

Given: A unity feedback system has $G(s) = 15 / [s(s + 3)]$. The current phase margin is PM = 40°. A lead compensator is needed to increase the phase margin to 55°.

Find: (a) The current gain crossover frequency, (b) the additional phase needed from the lead compensator, (c) the lead compensator parameters (zero z, pole p, gain K_c), and (d) the new gain crossover frequency.

Solution:

(a) $|G(j\omega)| = 15/(\omega\sqrt{(\omega^2+9)}) = 1 \rightarrow 225 = \omega^2(\omega^2+9) \rightarrow \omega^4 + 9\omega^2 - 225 = 0$. $\omega^2 = (-9 + \sqrt{(81+900)})/2 = (-9 + 31.32)/2 = 11.16$. $\omega_{gc} = \sqrt{11.16} = 3.34$ rad/s

Phase: -90° - arctan(3.34/3) = -90° - 48.1° = -138.1°. PM = 180° - 138.1° = 41.9° (confirms ≈ 40°).

(b) Additional phase: 55° - 41.9° + 8° (safety margin for crossover shift) = 21.1°

(c) $\sin(\varphi_{max}) = \sin(21.1°) = 0.360$. $\alpha = (1 - 0.360)/(1 + 0.360) = 0.640/1.360 = 0.471$

New gain crossover frequency (slightly shifted; place the lead compensator peak here): $\omega_{gc,new} \approx \omega_{gc} / \sqrt{(\sqrt{\alpha})}$ — actually, we target the lead peak at the new crossover. Use $\omega_m = \omega_{gc}$ as a first approximation. The lead compensator attenuation at ω_m is $\sqrt{\alpha} = 0.686$, and $|G(j\omega_{gc})| = 1$, so $K_c = 1/\sqrt{\alpha} = 1/0.686 = 1.458$

Compensator zero and pole: $z = \omega_{gc} \times \sqrt{\alpha} = 3.34 \times 0.686 = 2.29$ rad/s $p = \omega_{gc} / \sqrt{\alpha} = 3.34 / 0.686 = 4.87$ rad/s

Lead compensator: $G_c(s) = 1.458(s + 2.29) / (s + 4.87)$

(d) The new gain crossover frequency is approximately $\omega_{gc,new} \approx 3.34$ rad/s (the lead compensator is centered at this frequency to provide maximum phase boost there).

Problem 4.8.6

Given: A unity feedback system has $G(s) = 10 / [s(s + 1)(s + 5)]$. The steady-state error to a ramp input must be reduced from the current value to $e_{ss} = 0.1$, while maintaining the current phase margin (approximately 50°).

Find: (a) The current K_v and ramp error, (b) the required gain increase, and (c) the lag compensator parameters to achieve the gain increase without significantly affecting the phase margin.

Solution:

(a) $K_v = \lim(s\to 0)\ sG(s) = 10/(1 \times 5) = 2$. $e_{ss} = 1/K_v = 1/2 = 0.5$

(b) Required $K_v = 1/0.1 = 10$. Gain increase needed: $10/2 = 5$ (or 13.98 dB).

(c) A lag compensator $G_c(s) = (s + z_{lag})/(s + p_{lag})$ where $z_{lag}/p_{lag} = 5$ (the required gain increase).

First, find the current gain crossover: $|G(j\omega)| = 10/(\omega\sqrt{(\omega^2+1)}\sqrt{(\omega^2+25)}) = 1$. Try $\omega = 1.3$: $10/(1.3 \times \sqrt{2.69} \times \sqrt{26.69}) = 10/(1.3 \times 1.640 \times 5.166) = 10/11.01 = 0.908$. Close to 1. $\omega_{gc} \approx 1.25$ rad/s

Place the lag compensator corner frequencies well below ω_{gc} (at least one decade below): $z_{lag} = \omega_{gc}/10 = 0.125$ rad/s $p_{lag} = z_{lag}/5 = 0.025$ rad/s

Lag compensator: $G_c(s) = (s + 0.125) / (s + 0.025)$

At DC: $G_c(0) = 0.125/0.025 = 5$. This provides the required 5× gain increase.

At $\omega = \omega_{gc} = 1.25$: the lag compensator magnitude ≈ 1 (since both corners are far below crossover) and phase ≈ -arctan(1.25/0.125) + arctan(1.25/0.025) ≈ -84.3° + 88.9° = +4.6°. The net phase contribution is small, preserving the phase margin.

Problem 4.8.7

Given: A unity feedback system has $G(s) = K / [s(s + 4)]$. The specifications are: steady-state ramp error ≤ 0.05 and phase margin $\geq 50°$.

Find: (a) The minimum K from the ramp error requirement, (b) the phase margin at that K, (c) whether lead compensation is needed, and (d) if so, design a lead compensator.

Solution:

(a) $K_v = \lim(s\to 0)\ sG(s) = K/4$. For $e_{ss} \leq 0.05$: $K_v \geq 20 \rightarrow K/4 \geq 20 \rightarrow K \geq 80$

(b) At K = 80: $|G(j\omega)| = 80/(\omega\sqrt{(\omega^2+16)}) = 1 \rightarrow 6400 = \omega^2(\omega^2+16) \rightarrow \omega^4+16\omega^2-6400 = 0$. $\omega^2 = (-16+\sqrt{(256+25600)})/2 = (-16+160.8)/2 = 72.4$. $\omega_{gc} = 8.51$ rad/s. Phase: -90° - arctan(8.51/4) = -90° - 64.8° = -154.8°. PM = 180° - 154.8° = 25.2° (below the 50° requirement).

(c) PM = 25.2° < 50° → yes, lead compensation is needed.

(d) Additional phase: 50° - 25.2° + 7° (safety) = 31.8°. $\alpha = (1 - \sin 31.8°)/(1 + \sin 31.8°) = (1 - 0.527)/(1 + 0.527) = 0.473/1.527 = 0.310$

The lead compensator attenuates by $\sqrt{\alpha} = 0.557$ at ω_m, so the new crossover shifts. Find ω where $|KG| = 1/\sqrt{\alpha}$: $80/(\omega\sqrt{(\omega^2+16)}) = 1/0.557 = 1.796 \rightarrow \omega\sqrt{(\omega^2+16)} = 44.54$. Try $\omega = 6$: $6\sqrt{52} = 43.27$ (close). $\omega_m \approx 6.1$ rad/s.

$z = \omega_m\sqrt{\alpha} = 6.1 \times 0.557 = 3.40$ rad/s $p = \omega_m/\sqrt{\alpha} = 6.1/0.557 = 10.95$ rad/s

Lead compensator: $G_c(s) = (s + 3.40) / (s + 10.95)$

Problem 4.8.8

Given: A system has transfer function $G(s) = 50(s + 2) / [s(s + 10)(s + 50)]$.

Find: (a) The Bode magnitude plot corner frequencies and slopes, (b) the low-frequency asymptote magnitude at $\omega = 1$, (c) the high-frequency asymptotic slope, and (d) the approximate gain crossover frequency from the asymptotic plot.

Solution:

(a) Rewrite in standard form: $G(s) = 50 \times 2 \times (s/2 + 1) / [s \times 10 \times 50 \times (s/10 + 1)(s/50 + 1)]$ $G(s) = 0.2(s/2 + 1) / [s(s/10 + 1)(s/50 + 1)]$

Corner frequencies: $\omega = 2$ rad/s (zero), $\omega = 10$ rad/s (pole), $\omega = 50$ rad/s (pole).

Slopes: - $\omega < 2$: -20 dB/dec (from 1/s) - $2 < \omega < 10$: -20 + 20 = 0 dB/dec (zero cancels the 1/s slope) - $10 < \omega < 50$: 0 - 20 = -20 dB/dec - $\omega > 50$: -20 - 20 = -40 dB/dec

(b) At $\omega = 1$ (low frequency): $|G| \approx 0.2/\omega = 0.2/1 = 0.2 \rightarrow 20\log_{10}(0.2) = -14$ dB

(c) High-frequency slope: -40 dB/decade

(d) From the asymptotic plot, the magnitude is -14 dB at $\omega = 1$ and rises at -20 dB/dec, so it reaches 0 dB: $-14 + 20\log_{10}(\omega) = 0$ is wrong since slope is -20 dB/dec. At $\omega = 1$: -14 dB. The magnitude decreases at -20 dB/dec until $\omega = 2$ where the zero flattens it. At $\omega = 2$: $-14 + 20\log_{10}(1) = -14$ dB… Actually, magnitude at $\omega = 2$ on the -20 dB/dec line: $|G|$ at $\omega = 2 = 0.2/2 = 0.1 \rightarrow -20$ dB. But the zero starts at $\omega = 2$, flattening to 0 dB/dec. So the magnitude stays at -20 dB until $\omega = 10$. Then pole at $\omega = 10$ resumes -20 dB/dec. The magnitude never reaches 0 dB.

Gain crossover: does not cross 0 dB (the system gain is too low). The maximum asymptotic magnitude is about -14 dB at $\omega = 1$, decreasing from there. With additional gain of at least 14 dB (factor of 5), gain crossover would occur.

Problem 4.8.9

Given: The open-loop Nyquist plot of a system with two open-loop RHP poles ($P = 2$) makes two counterclockwise (CCW) encirclements of the (-1, 0) point. CCW encirclements are counted as $N = -2$ (negative).

Find: (a) The number of closed-loop RHP poles, (b) whether the closed-loop system is stable, and (c) what would happen if the gain were increased so that the plot made only one CCW encirclement.

Solution:

(a) $Z = P + N = 2 + (-2) = 0$ closed-loop RHP poles.

(b) $Z = 0 \rightarrow$ the closed-loop system is stable. The feedback has stabilized an open-loop unstable plant.

(c) With $N = -1$: $Z = P + N = 2 + (-1) = 1$ closed-loop RHP pole. The closed-loop system would be unstable with one RHP pole. This shows that the system is conditionally stable — it is stable only within a specific range of gain.

Problem 4.8.10

Given: A lead-lag compensator is needed for $G(s) = 5 / [s(s + 1)]$ with unity feedback. Requirements: $K_v \geq 50$ and $PM \geq 45°$.

Find: (a) The gain K needed for the K_v requirement, (b) the uncompensated phase margin at that gain, (c) the lag compensator ratio to provide the necessary gain, and (d) the lead compensator to restore the phase margin.

Solution:

(a) $K_v = \lim(s \rightarrow 0)\ s \times K \times 5/[s(s+1)] = 5K$. For $K_v \geq 50$: $5K \geq 50 \rightarrow K \geq 10$.

(b) At $K = 10$: $G(s) = 50/[s(s+1)]$. $|G(j\omega)| = 50/(\omega\sqrt{(\omega^2+1)}) = 1$. $2500 = \omega^2(\omega^2+1) \rightarrow \omega^4 + \omega^2 - 2500 = 0$. $\omega^2 = (-1+\sqrt{(1+10000)})/2 = (-1+100.005)/2 = 49.5$. $\omega_{gc} = 7.04$ rad/s. Phase: $-90° - \arctan(7.04) = -90° - 81.9° = -171.9°$. $PM = 180° - 171.9° = 8.1°$ (very poor).

(c) Choose a new crossover frequency where the uncompensated plant has sufficient phase. For $PM = 45°$ + lead contribution, target the compensated crossover at a lower frequency. After lag compensation reduces the crossover to $\omega_{gc,new} \approx 3$ rad/s:

Uncompensated gain at $\omega = 3$: $|G(j3)| = 50/(3\sqrt{10}) = 50/9.487 = 5.27 \rightarrow 14.44$ dB. The lag compensator must attenuate by 14.44 dB (factor of 5.27) at $\omega = 3$ but maintain DC gain.

Lag: $z_{lag}/p_{lag} = 1$ (no DC gain change needed — the $K = 10$ already provides it). Actually, we need the lag to reduce gain at $\omega = 3$. Use a lag with high-frequency attenuation: $z_{lag} = 0.3$ rad/s, $p_{lag} = 0.3/5.27 = 0.057$ rad/s

(d) Phase at $\omega = 3$ without lead: $-90° - \arctan(3) = -90° - 71.6° = -161.6°$. Phase from lag at $\omega = 3$: $\approx -5°$ (small lag penalty). Total: $-166.6°$. PM would be $13.4°$ — still needs lead.

Additional phase needed: $45° - 13.4° + 5° = 36.6°$. $\alpha = (1 - \sin 36.6°)/(1 + \sin 36.6°) = (1 - 0.596)/(1 + 0.596) = 0.404/1.596 = 0.253$. $z_{lead} = 3\sqrt{0.253} = 1.51$ rad/s, $p_{lead} = 3/\sqrt{0.253} = 5.97$ rad/s.

Lead compensator: $G_{lead}(s) = (s + 1.51) / (s + 5.97)$ Lag compensator: $G_{lag}(s) = (s + 0.3) / (s + 0.057)$

Chapter 4 — Section 4.9: State-Space Representation

Practice problems covering state-space models, matrix formulation, eigenvalue analysis, controllability, observability, state feedback pole placement, and Luenberger observer design.

Problem 4.9.1

Given: A second-order mechanical system (mass-spring-damper) has mass m = 2 kg, spring constant k = 18 N/m, and damping coefficient b = 8 N·s/m. The state variables are x_1 = position (m) and x_2 = velocity (m/s), the input u is an applied force (N), and the output y is the position.

Find: (a) The state-space matrices A, B, C, and D, (b) the eigenvalues of A, and (c) whether the system is stable.

Solution:

From Newton's second law: $m\ddot{x} + b\dot{x} + kx = u$, which gives $\ddot{x} = -(b/m)\dot{x} - (k/m)x + (1/m)u$.

Defining $\dot{x}_1 = x_2$ and $\dot{x}_2 = -(k/m)x_1 - (b/m)x_2 + (1/m)u$:

(a) A = [0, 1; -9, -4], B = [0; 0.5], C = [1, 0], D = [0]

where k/m = 18/2 = 9, b/m = 8/2 = 4, and 1/m = 1/2 = 0.5.

(b) Eigenvalues from det(sI - A) = 0:

$s^2 + 4s + 9 = 0$

$s = (-4 \pm \sqrt{(16 - 36)}) / 2 = (-4 \pm \sqrt{(-20)}) / 2 = (-4 \pm j4.472) / 2 = -2 \pm j2.236$

(c) Both eigenvalues have negative real parts (Re = -2), so the system is stable. The natural frequency is $\omega_n = \sqrt{9} = 3$ rad/s and the damping ratio is $\zeta = 4/(2 \times 3) = 0.667$.

Problem 4.9.2

Given: A DC motor is modeled with state variables x_1 = armature current (A) and x_2 = angular velocity (rad/s). The parameters are armature resistance $R_a = 2\ \Omega$, armature inductance $L_a = 0.5$ H, back-EMF constant $K_b = 0.1$ V·s/rad, torque constant $K_t = 0.1$ N·m/A, moment of inertia $J = 0.01$ kg·m^2, and friction coefficient $B_f = 0.05$ N·m·s/rad. The input u is the applied armature voltage, and the output y is the angular velocity.

Find: (a) The state-space matrices A, B, C, D, (b) the eigenvalues of A, and (c) the transfer function from voltage to angular velocity.

Solution:

From circuit and mechanical equations: $L_a(dx_1/dt) = u - R_a x_1 - K_b x_2$ $J(dx_2/dt) = K_t x_1 - B_f x_2$

(a) $\dot{x}_1 = -(R_a/L_a)x_1 - (K_b/L_a)x_2 + (1/L_a)u = -4x_1 - 0.2x_2 + 2u$ $\dot{x}_2 = (K_t/J)x_1 - (B_f/J)x_2 = 10x_1 - 5x_2$

A = [-4, -0.2; 10, -5], B = [2; 0], C = [0, 1], D = [0]

(b) $\det(sI - A) = (s + 4)(s + 5) + 2 = s^2 + 9s + 22 = 0$

$s = (-9 \pm \sqrt{(81 - 88)}) / 2 = (-9 \pm \sqrt{(-7)}) / 2 = -4.5 \pm j1.323$

Both poles have negative real parts — the system is stable.

(c) $H(s) = C(sI - A)^{-1}B + D$

$(sI - A)^{-1} = (1/(s^2 + 9s + 22)) \times [s + 5, -0.2; 10, s + 4]$

$H(s) = [0, 1] \times (1/(s^2 + 9s + 22)) \times [s + 5, -0.2; 10, s + 4] \times [2; 0]$

$= (1/(s^2 + 9s + 22)) \times [0, 1] \times [2(s + 5); 20] = 20 / (s^2 + 9s + 22)$

Problem 4.9.3

Given: A system has state-space matrices: A = [0, 1; -6, -5], B = [0; 1], C = [1, 0], D = [0]

Find: (a) Whether the system is controllable, (b) whether the system is observable, and (c) the transfer function.

Solution:

(a) Controllability matrix: $C_M = [B, AB]$

AB = [0, 1; -6, -5] × [0; 1] = [1; -5]

C_M = [[0, 1]; [1, -5]]

$\det(C_M) = 0 \times (-5) - 1 \times 1 = -1 \neq 0$

Rank = 2 = n → fully controllable

(b) Observability matrix: $O_M = [C; CA]$

CA = [1, 0] × [0, 1; -6, -5] = [0, 1]

O_M = [[1, 0]; [0, 1]]

$\det(O_M) = 1 \times 1 - 0 \times 0 = 1 \neq 0$

Rank = 2 = n → fully observable

(c) $H(s) = C(sI - A)^{-1}B + D$

Characteristic polynomial: $s^2 + 5s + 6 = (s + 2)(s + 3)$

$H(s) = [1, 0] \times (1/(s^2 + 5s + 6)) \times [s + 5, 1; -6, s] \times [0; 1]$

$= (1/(s^2 + 5s + 6)) \times [1, 0] \times [1; s] = 1 / (s^2 + 5s + 6) = 1 / ((s + 2)(s + 3))$

Problem 4.9.4

Given: A system has A = [0, 1; -2, -3] and B = [1; 0].

Find: Whether the system is controllable, and explain the physical implication.

Solution:

Controllability matrix: $C_M = [B, AB]$

$AB = [0, 1; -2, -3] \times [1; 0] = [0; -2]$

$C_M = [[1, 0]; [0, -2]]$

$\det(C_M) = 1 \times (-2) - 0 \times 0 = -2 \neq 0$

Rank = 2 = n → fully controllable

Since $\det(C_M) \neq 0$, arbitrary pole placement via state feedback is possible. The eigenvalues of A are found from $s^2 + 3s + 2 = (s + 1)(s + 2) = 0$, giving open-loop poles at s = -1 and s = -2. A state feedback gain K can move these poles to any desired closed-loop locations.

Now consider the alternative B' = [1; 1]. Then AB' = [1; -5], and C_M' = [[1, 1]; [1, -5]], det = -5 - 1 = -6 ≠ 0 — still controllable.

But if B'' = [1; -2], then AB'' = [-2; 4] and C_M'' = [[1, -2]; [-2, 4]], det = 4 - 4 = 0 — not controllable. The input aligns with the eigenvector of the s = -2 mode, so only one mode can be influenced.

Problem 4.9.5

Given: A third-order system has: A = [-1, 0, 0; 0, -3, 0; 0, 0, -5], B = [1; 1; 0], C = [1, 1, 1]

Find: (a) The system eigenvalues, (b) whether the system is controllable, and (c) whether the system is observable.

Solution:

(a) Since A is diagonal, the eigenvalues are the diagonal entries: $s_1 = -1$, $s_2 = -3$, $s_3 = -5$. All are negative, so the system is stable.

(b) Controllability matrix (n = 3): $C_M = [B, AB, A^2B]$

$AB = [-1, 0, 0; 0, -3, 0; 0, 0, -5] \times [1; 1; 0] = [-1; -3; 0]$

$A^2B = A \times AB = [-1, 0, 0; 0, -3, 0; 0, 0, -5] \times [-1; -3; 0] = [1; 9; 0]$

$C_M = [[1, -1, 1]; [1, -3, 9]; [0, 0, 0]]$

The third row is all zeros, so $\text{rank}(C_M) \leq 2 < 3$. The system is not controllable.

The $s = -5$ mode (x_3) has $B_3 = 0$, meaning the input cannot excite this state — it is an uncontrollable mode.

(c) Observability matrix: $O_M = [C; CA; CA^2]$

$CA = [1, 1, 1] \times A = [-1, -3, -5]$ $CA^2 = [-1, -3, -5] \times A = [1, 9, 25]$

$O_M = [[1, 1, 1]; [-1, -3, -5]; [1, 9, 25]]$

$\det(O_M) = 1(-75 + 45) - 1(-25 + 5) + 1(-9 + 3) = 1(-30) - 1(-20) + 1(-6) = -30 + 20 - 6 = -16 \neq 0$

Rank = 3 = n → fully observable

All three modes are visible in the output, even though x_3 cannot be controlled by the input.

Problem 4.9.6

Given: A system with $A = [0, 1; -4, -5]$ and $B = [0; 1]$. Design a state feedback controller $u = -Kx + r$ to place the closed-loop poles at $s = -10$ and $s = -10$ (repeated real poles for critically damped response).

Find: The feedback gain vector $K = [k_1, k_2]$ and the closed-loop characteristic polynomial.

Solution:

Desired characteristic polynomial: $(s + 10)(s + 10) = s^2 + 20s + 100$

Open-loop $A - BK = [0, 1; -4 - k_1, -5 - k_2]$

Characteristic polynomial of A_{CL}:

$\det(sI - A_{CL}) = s(s + 5 + k_2) + (4 + k_1) = s^2 + (5 + k_2)s + (4 + k_1)$

Matching coefficients with $s^2 + 20s + 100$:

$5 + k_2 = 20 \rightarrow k_2 = 15$

$4 + k_1 = 100 \rightarrow k_1 = 96$

$K = [96, 15]$. The control law is $u = -96x_1 - 15x_2 + r$.

Verification: $A_{CL} = [0, 1; -100, -20]$. Eigenvalues from $s^2 + 20s + 100 = (s + 10)^2 = 0 \rightarrow s = -10, -10$. The critically damped response has a time constant of $\tau = 1/10 = 0.1$ s with no overshoot.

Problem 4.9.7

Given: A system with $A = [-2, 1; 0, -1]$ and $B = [0; 1]$. A state feedback controller places the poles at $s = -5 \pm j5$.

Find: (a) The required gain $K = [k_1, k_2]$, (b) the resulting damping ratio and natural frequency, and (c) the expected percent overshoot for a step input.

Solution:

(a) Desired characteristic polynomial: $(s + 5 - j5)(s + 5 + j5) = s^2 + 10s + 50$

$A_{CL} = A - BK = [-2, 1; -k_1, -1 - k_2]$

$\det(sI - A_{CL}) = (s + 2)(s + 1 + k_2) + k_1 = s^2 + (3 + k_2)s + (2 + 2k_2 + k_1)$

Matching coefficients: $3 + k_2 = 10 \rightarrow k_2 = 7$ $2 + 2(7) + k_1 = 50 \rightarrow 16 + k_1 = 50 \rightarrow k_1 = 34$

$K = [34, 7]$

(b) From $s^2 + 10s + 50$: $\omega_n = \sqrt{50} = 7.07$ rad/s, $\zeta = 10/(2 \times 7.07) = 0.707$

(c) Percent overshoot $= 100 \times e^{-\pi\zeta/\sqrt{(1-\zeta^2)}} = 100 \times e^{-\pi(0.707)/\sqrt{(1-0.5)}} = 100 \times e^{-2.221/0.707} = 100 \times e^{-3.142}$

Correcting: $-\pi\zeta/\sqrt{(1 - \zeta^2)} = -\pi(0.707)/\sqrt{(0.5)} = -2.221/0.707 = -3.142$

$\%OS = 100 \times e^{-3.142} = 100 \times 0.0432 = 4.3\%$

Problem 4.9.8

Given: A system with $A = [0, 1; -3, -4]$, $B = [0; 1]$, $C = [1, 0]$ (only position x_1 is measured). The controller poles have been placed at $s = -6 \pm j6$. Design a Luenberger observer with poles 4× faster than the controller poles.

Find: The observer gain vector $L = [l_1; l_2]$ and the observer pole locations.

Solution:

Controller poles at $s = -6 \pm j6$ have $\omega_n = \sqrt{(36 + 36)} = 8.49$ rad/s.

Observer poles placed 4× faster: $s = -24 \pm j24$.

Desired observer characteristic polynomial: $(s + 24 - j24)(s + 24 + j24) = s^2 + 48s + 1152$

Observer error dynamics: $A - LC$ where $L = [l_1; l_2]$:

$A - LC = [0 - l_1, 1; -3 - l_2, -4]$

$\det(sI - (A - LC)) = (s + l_1)(s + 4) + (3 + l_2) = s^2 + (4 + l_1)s + (4l_1 + 3 + l_2)$

Matching coefficients: $4 + l_1 = 48 \rightarrow l_1 = 44$ $4(44) + 3 + l_2 = 1152 \rightarrow 179 + l_2 = 1152 \rightarrow l_2 = 973$

$L = [44; 973]$

The observer estimation error decays with a time constant of approximately $1/24 = 0.042$ s, compared to the controller time constant of $1/6 = 0.167$ s. This ensures the state estimates converge well before the controller response settles.

Problem 4.9.9

Given: A thermal system has two states: x_1 = room temperature (°C above ambient) and x_2 = wall temperature (°C above ambient). The system matrices are: A = [-0.1, 0.05; 0.02, -0.04], B = [0.08; 0], C = [1, 0], D = [0] The input u is the heater power (kW), and only the room temperature is measured.

Find: (a) The eigenvalues and their physical meaning, (b) the controllability and observability of the system, and (c) an observer with poles at s = -0.5 and s = -0.6.

Solution:

(a) $\det(sI - A) = (s + 0.1)(s + 0.04) - (0.05)(0.02) = s^2 + 0.14s + 0.004 - 0.001 = s^2 + 0.14s + 0.003 = 0$

$s = (-0.14 \pm \sqrt{(0.0196 - 0.012)}) / 2 = (-0.14 \pm \sqrt{0.0076}) / 2 = (-0.14 \pm 0.0872) / 2$

$s_1 = (-0.14 + 0.0872) / 2 = -0.0264$ (slow thermal mode, $\tau = 37.9$ s) $s_2 = (-0.14 - 0.0872) / 2 = -0.1136$ (fast thermal mode, $\tau = 8.8$ s)

The slow mode represents the overall building thermal mass, while the fast mode represents the temperature difference between room and wall equalizing.

(b) C_M = [B, AB] = [[0.08, -0.008 + 0]; [0, 0.0016 + 0]]

AB = [-0.1(0.08) + 0.05(0), 0.02(0.08) + (-0.04)(0)] = [-0.008; 0.0016]

C_M = [[0.08, -0.008]; [0, 0.0016]]

$\det(C_M) = 0.08 \times 0.0016 - (-0.008) \times 0 = 0.000128 \neq 0 \rightarrow$ controllable

O_M = [C; CA] = [[1, 0]; [-0.1, 0.05]]

$\det(O_M) = 1 \times 0.05 - 0 \times (-0.1) = 0.05 \neq 0 \rightarrow$ observable

(c) Desired observer polynomial: $(s + 0.5)(s + 0.6) = s^2 + 1.1s + 0.3$

A - LC = [$-0.1 - l_1$, 0.05; $0.02 - l_2$, -0.04]

$\det(sI - (A - LC)) = (s + 0.1 + l_1)(s + 0.04) - 0.05(0.02 - l_2)$

$= s^2 + (0.14 + l_1)s + (0.1 + l_1)(0.04) - 0.001 + 0.05l_2$

$= s^2 + (0.14 + l_1)s + 0.004 + 0.04l_1 - 0.001 + 0.05l_2$

$= s^2 + (0.14 + l_1)s + 0.003 + 0.04l_1 + 0.05l_2$

Matching: $0.14 + l_1 = 1.1 \rightarrow l_1 = 0.96$ $0.003 + 0.04(0.96) + 0.05l_2 = 0.3 \rightarrow 0.003 + 0.0384 + 0.05l_2 = 0.3 \rightarrow 0.05l_2 = 0.2586 \rightarrow l_2 = 5.172$

L = [0.96; 5.172]

Problem 4.9.10

Given: A combined controller-observer system for the RLC circuit from §4.9.1 has: A = [0, 10; -2, -6], B = [0; 2], C = [1, 0] Controller gain K = [1.5, 2] (poles at $s = -5 \pm j5$) and observer gain L = [34; 57.6] (poles at $s = -20 \pm j20$).

Find: (a) The four closed-loop poles of the combined controller-observer system, (b) the system matrix of the combined (4th-order) system, and (c) whether the separation principle holds.

Solution:

(a) By the separation principle, the combined system poles are the union of:

- Controller poles: eigenvalues of (A - BK) = s = -5 ± j5
- Observer poles: eigenvalues of (A - LC) = s = -20 ± j20

The four poles are: s = -5 + j5, s = -5 - j5, s = -20 + j20, s = -20 - j20

(b) The combined system with state $[x; e]^T$ (where $e = x - \hat{x}$) has the block triangular form:

$A_{combined}$ = [A - BK, BK; 0, A - LC]

A - BK = [0, 10; -2 - 2(1.5), -6 - 2(2)] = [0, 10; -5, -10]

A - LC = [0 - 34, 10; -2 - 57.6, -6] = [-34, 10; -59.6, -6]

$A_{combined}$ = [[0, 10, 0, -20]; [-5, -10, 3, 4]; [0, 0, -34, 10]; [0, 0, -59.6, -6]]

Note: BK = [0; 2] × [1.5, 2] = [0, 0; 3, 4]

(c) The separation principle holds: the characteristic polynomial of $A_{combined}$ is:

det(sI - $A_{combined}$) = det(sI - (A - BK)) × det(sI - (A - LC))

= $(s^2 + 10s + 50)(s^2 + 40s + 800)$

The controller and observer can be designed independently — the controller poles and observer poles are the eigenvalues of their respective subsystems, and neither design affects the other's pole locations.

Chapter 4 — Section 4.10: Digital Control Systems

Practice problems covering discrete-time system analysis, zero-order hold discretization, Tustin (bilinear) transformation, z-plane stability, digital PID implementation, sample period selection, and anti-windup strategies.

Problem 4.10.1

Given: A continuous-time first-order plant $G(s) = 20/(s + 10)$ is preceded by a zero-order hold (ZOH) with sample period $T = 0.02$ s.

Find: (a) The discrete-time transfer function G(z) using the ZOH method, (b) the location of the discrete pole and whether it is stable, and (c) the DC gain of G(z) and verify it matches G(0).

Solution:

(a) $G(s)/s = 20/[s(s + 10)]$. Partial fractions: $20/[s(s + 10)] = 2/s - 2/(s + 10)$

Z-transform: $Z\{2/s - 2/(s + 10)\} = 2z/(z - 1) - 2z/(z - e^{-10T})$

$e^{-10(0.02)} = e^{-0.2} = 0.8187$

$G(z) = (1 - z^{-1})[2z/(z - 1) - 2z/(z - 0.8187)]$

$= 2 - 2(z - 1)/(z - 0.8187)$

$= [2(z - 0.8187) - 2(z - 1)] / (z - 0.8187)$

$= 2(1 - 0.8187) / (z - 0.8187) = 0.3625 / (z - 0.8187)$

(b) The discrete pole is at $z = 0.8187$. Since $|0.8187| < 1$, the pole is inside the unit circle and the system is stable.

The continuous pole at $s = -10$ maps to $z = e^{sT} = e^{-0.2} = 0.8187$, confirming the relationship.

(c) DC gain: $G(1) = 0.3625 / (1 - 0.8187) = 0.3625 / 0.1813 = 2.0$

Continuous DC gain: $G(0) = 20/10 = 2.0$ — the DC gains match.

Problem 4.10.2

Given: A continuous-time integrator $G(s) = 1/s$ is discretized using the Tustin (bilinear) approximation with sample period $T = 0.1$ s.

Find: (a) The discrete transfer function $G(z)$, (b) the step response for the first 4 samples, and (c) comparison with the exact discrete integrator (ZOH method).

Solution:

(a) Tustin approximation: $s = (2/T)(z - 1)/(z + 1) = 20(z - 1)/(z + 1)$

$G(z) = 1/s = (z + 1) / [20(z - 1)] = (T/2)(z + 1)/(z - 1) = 0.05(z + 1)/(z - 1)$

(b) For a unit step input $U(z) = z/(z - 1)$:

$Y(z) = G(z) \times U(z) = 0.05z(z + 1) / (z - 1)^2$

Expanding by long division or partial fractions: $y[0] = 0.05$, $y[1] = 0.05 + 0.1 = 0.15$, $y[2] = 0.15 + 0.1 = 0.25$, $y[3] = 0.25 + 0.1 = 0.35$

The output increments by $T = 0.1$ per step (after the first half-step), as expected for an integrator with unit input.

(c) The ZOH discrete integrator is $G_{ZOH}(z) = Tz^{-1}/(1 - z^{-1}) = T/(z - 1) = 0.1/(z - 1)$.

ZOH step response: $y[0] = 0$, $y[1] = 0.1$, $y[2] = 0.2$, $y[3] = 0.3$.

The Tustin method introduces a half-sample advance (trapezoidal integration averages current and previous inputs), giving a more accurate approximation of the continuous integral at each sample instant. At $k = 3$ the exact continuous value is 0.3, Tustin gives 0.35 (slight lead), and ZOH gives 0.3 (exact for step input). The Tustin method better preserves frequency response while ZOH better preserves step response.

Problem 4.10.3

Given: A second-order continuous plant $G(s) = 100/(s^2 + 6s + 100)$ has a closed-loop bandwidth of $f_{BW} = 1.6$ Hz ($\omega_{BW} = 10$ rad/s).

Find: (a) The maximum sample period T using the rule $T \leq 1/(10 \times f_{BW})$, (b) the location of the continuous poles, (c) the corresponding z-plane pole locations for the chosen T, and (d) whether a sample period of $T = 0.2$ s would be adequate.

Solution:

(a) Maximum sample period: $T_{max} = 1/(10 \times f_{BW}) = 1/(10 \times 1.6) = 0.0625$ s (minimum sample rate = 16 Hz)

(b) Continuous poles from $s^2 + 6s + 100 = 0$: $s = (-6 \pm \sqrt{(36 - 400)}) / 2 = (-6 \pm j19.08) / 2 = -3 \pm j9.54$

$\omega_n = \sqrt{100} = 10$ rad/s, $\zeta = 6/(2 \times 10) = 0.3$

(c) For $T = 0.0625$ s, the z-plane poles are: $z = e^{sT} = e^{(-3 \pm j9.54)(0.0625)} = e^{-0.1875} \times e^{\pm j0.5963}$

$|z| = e^{-0.1875} = 0.829$

$\angle z = \pm 0.5963$ rad $= \pm 34.2°$

$z = 0.829\angle\pm 34.2° = 0.685 \pm j0.466$

Both poles are inside the unit circle ($|z| = 0.829 < 1$) → stable.

(d) For T = 0.2 s: $f_s = 1/0.2 = 5$ Hz. The rule requires $f_s \geq 16$ Hz. Since 5 Hz < 16 Hz, T = 0.2 s is not adequate — significant phase lag would be introduced, and the natural frequency of 10 rad/s (1.59 Hz) would be above the Nyquist frequency of 2.5 Hz, potentially causing aliasing and instability.

Problem 4.10.4

Given: A digital PID controller has gains $K_p = 3.0$, $K_i = 8.0$, $K_d = 0.2$, and sample period T = 0.02 s. The error sequence is e[0] = 5.0, e[1] = 3.5, e[2] = 2.0, e[3] = 0.8. Assume e[-1] = 0 and prior integral sum = 0.

Find: The control output u[k] for k = 0 through 3 using the position-form digital PID.

Solution:

Position form: $u[k] = K_p e[k] + K_i T \times \Sigma e[j] + K_d(e[k] - e[k-1])/T$

k = 0: P = 3.0 × 5.0 = 15.0 I = 8.0 × 0.02 × 5.0 = 0.8 D = 0.2 × (5.0 - 0)/0.02 = 50.0 u[0] = 15.0 + 0.8 + 50.0 = 65.8

k = 1: P = 3.0 × 3.5 = 10.5 Integral sum = 5.0 + 3.5 = 8.5; I = 8.0 × 0.02 × 8.5 = 1.36 D = 0.2 × (3.5 - 5.0)/0.02 = -15.0 u[1] = 10.5 + 1.36 + (-15.0) = -3.14

k = 2: P = 3.0 × 2.0 = 6.0 Integral sum = 8.5 + 2.0 = 10.5; I = 8.0 × 0.02 × 10.5 = 1.68 D = 0.2 × (2.0 - 3.5)/0.02 = -15.0 u[2] = 6.0 + 1.68 + (-15.0) = -7.32

k = 3: P = 3.0 × 0.8 = 2.4 Integral sum = 10.5 + 0.8 = 11.3; I = 8.0 × 0.02 × 11.3 = 1.808 D = 0.2 × (0.8 - 2.0)/0.02 = -12.0 u[3] = 2.4 + 1.808 + (-12.0) = -7.792

The large derivative term at k = 0 (50.0) is characteristic of the initial step — in practice, derivative-on-measurement (rather than derivative-on-error) is used to avoid this "derivative kick."

Problem 4.10.5

Given: A motor speed controller uses a velocity-form digital PID with $K_p = 1.5$, $K_i = 4.0$, $K_d = 0.08$, T = 0.01 s. The error sequence is e[0] = 100, e[1] = 80, e[2] = 55, e[3] = 30. Assume e[-1] = e[-2] = 0 and u[-1] = 0.

Find: The incremental control output Δu[k] and the total control output u[k] for k = 0 through 3.

Solution:

Velocity form: $\Delta u[k] = K_p(e[k] - e[k-1]) + K_i T e[k] + K_d(e[k] - 2e[k-1] + e[k-2])/T$

k = 0: $\Delta u[0] = 1.5(100 - 0) + 4.0(0.01)(100) + 0.08(100 - 0 + 0)/0.01 = 150 + 4.0 + 800 = 954.0$ $u[0] = 0 + 954.0 = 954.0$

k = 1: $\Delta u[1] = 1.5(80 - 100) + 4.0(0.01)(80) + 0.08(80 - 200 + 0)/0.01 = -30 + 3.2 + (-960) = -986.8$ $u[1] = 954.0 + (-986.8) = -32.8$

k = 2: $\Delta u[2] = 1.5(55 - 80) + 4.0(0.01)(55) + 0.08(55 - 160 + 100)/0.01 = -37.5 + 2.2 + (-40) = -75.3$ $u[2] = -32.8 + (-75.3) = -108.1$

k = 3: $\Delta u[3] = 1.5(30 - 55) + 4.0(0.01)(30) + 0.08(30 - 110 + 55)/0.01 = -37.5 + 1.2 + (-200) = -236.3$ $u[3] = -108.1 + (-236.3) = -344.4$

The negative control outputs indicate the controller is reducing the drive signal as the motor approaches the setpoint. The velocity form naturally prevents integral windup because it computes incremental changes rather than accumulating the integral sum.

Problem 4.10.6

Given: A temperature controller uses a digital PID with $K_p = 5.0$, $K_i = 0.5$, $K_d = 1.0$, and $T = 1.0$ s. The actuator (heater) saturates at $u_{max} = 100\%$ and $u_{min} = 0\%$. The error sequence is e[0] = 50, e[1] = 48, e[2] = 45, e[3] = 40. Assume e[-1] = 0 and initial integral sum = 0. Anti-windup uses the clamping method (freeze integrator when output saturates).

Find: The unclamped and clamped control outputs for k = 0 through 3.

Solution:

Position form: $u[k] = K_p e[k] + K_i T \times \Sigma e + K_d(e[k] - e[k-1])/T$

k = 0: $P = 5.0 \times 50 = 250$; $I = 0.5 \times 1.0 \times 50 = 25$; $D = 1.0 \times (50 - 0)/1.0 = 50$ $u_{unclamped} = 250 + 25 + 50 = 325$ $u_{clamped} = \min(325, 100) = 100$ (saturated — freeze integrator at $\Sigma e = 50$)

k = 1: Integrator frozen, Σe remains at 50 (clamping active since output saturated at k = 0) $P = 5.0 \times 48 = 240$; $I = 0.5 \times 1.0 \times 50 = 25$; $D = 1.0 \times (48 - 50)/1.0 = -2$ $u_{unclamped} = 240 + 25 + (-2) = 263$ $u_{clamped} = 100$ (still saturated — integrator remains frozen at 50)

k = 2: Σe still frozen at 50 $P = 5.0 \times 45 = 225$; $I = 25$; $D = 1.0 \times (45 - 48)/1.0 = -3$ $u_{unclamped} = 225 + 25 + (-3) = 247$ $u_{clamped} = 100$ (still saturated)

k = 3: Σe still frozen at 50 $P = 5.0 \times 40 = 200$; $I = 25$; $D = 1.0 \times (40 - 45)/1.0 = -5$ $u_{unclamped} = 200 + 25 + (-5) = 220$ $u_{clamped} = 100$ (still saturated)

Without clamping, the integral sum would grow to 50 + 48 + 45 + 40 = 183, and the integral term alone would be 91.5 — causing severe overshoot when the error finally reverses sign. The clamping method prevents this accumulation.

Problem 4.10.7

Given: A continuous-time PD controller $C(s) = 2.0 + 0.15s$ is to be discretized using the Tustin (bilinear) method with $T = 0.005$ s.

Find: (a) The discrete transfer function C(z), (b) the difference equation relating u[k] to e[k], and (c) the controller gain at 0 Hz and at the Nyquist frequency.

Solution:

(a) Substitute $s = (2/T)(z - 1)/(z + 1) = 400(z - 1)/(z + 1)$:

$C(z) = 2.0 + 0.15 \times 400(z - 1)/(z + 1) = 2.0 + 60(z - 1)/(z + 1)$

$= [2.0(z + 1) + 60(z - 1)] / (z + 1) = (2z + 2 + 60z - 60) / (z + 1)$

$= (62z - 58) / (z + 1)$

(b) Cross-multiplying: $uk = (62z - 58)e[k]$

$u[k] + u[k-1] = 62e[k] - 58e[k-1]$

$u[k] = -u[k-1] + 62e[k] - 58e[k-1]$

(c) DC gain (z = 1): $C(1) = (62 - 58)/(1 + 1) = 4/2 = 2.0$

This matches $C(0) = 2.0 + 0.15(0) = 2.0$.

Nyquist frequency gain (z = -1): $C(-1) = (-62 - 58)/(-1 + 1) \rightarrow$ undefined (pole at z = -1).

In practice, $|C(z)|$ at $z = e^{j\pi} = -1$ approaches infinity, meaning the derivative term has very high gain at the Nyquist frequency. This is why practical digital PD controllers include a derivative filter: $C(s) = K_p + K_d s/(1 + \tau_f s)$ with $\tau_f = K_d/(N \times K_p)$ to limit high-frequency gain.

Problem 4.10.8

Given: A continuous second-order system $G(s) = 50/(s^2 + 4s + 50)$ is discretized using the ZOH method with T = 0.05 s. The continuous poles are at $s = -2 \pm j6.782$.

Find: (a) The z-plane pole locations, (b) the magnitude of the z-plane poles and verify stability, and (c) the angle of the z-plane poles in degrees.

Solution:

(a) $z = e^{sT} = e^{(-2 \pm j6.782)(0.05)} = e^{-0.1} \times e^{\pm j0.3391}$

$e^{-0.1} = 0.9048$

$z = 0.9048 \times (\cos(0.3391) \pm j \sin(0.3391))$

$\cos(0.3391) = 0.9430$, $\sin(0.3391) = 0.3327$

$z = 0.9048 \times (0.9430 \pm j0.3327)$

$z = 0.8532 \pm j0.3011$

(b) $|z| = 0.9048$, which equals $e^{-0.1} = e^{-\sigma T}$ where $\sigma = 2$ is the real part of the continuous pole.

Since $|z| = 0.9048 < 1$, both poles are inside the unit circle $\rightarrow$ stable.

Verification: $|z|^2 = 0.8532^2 + 0.3011^2 = 0.7280 + 0.0907 = 0.8187$. $|z| = \sqrt{0.8187} = 0.9048$.

(c) Angle: $\theta = \arctan(0.3011/0.8532) = \arctan(0.3529) = 19.44°$

This equals $\omega_d T$ in degrees: $6.782 \times 0.05 \times (180/\pi) = 0.3391 \times 57.30 = 19.43°$ — confirmed.

The z-plane poles preserve the damping information in the magnitude and the oscillation frequency in the angle.

Problem 4.10.9

Given: A discrete-time system has the transfer function $G(z) = 0.2z / (z^2 - 1.2z + 0.52)$. A unit step input is applied.

Find: (a) The z-plane pole locations and stability, (b) the equivalent continuous-time poles (assuming T = 0.1 s), (c) the steady-state output, and (d) the first 3 output samples y[0], y[1], y[2] (assume zero initial conditions).

Solution:

(a) Poles from $z^2 - 1.2z + 0.52 = 0$: $z = (1.2 \pm \sqrt{(1.44 - 2.08)}) / 2 = (1.2 \pm \sqrt{(-0.64)}) / 2 = (1.2 \pm j0.8) / 2$

$z = 0.6 \pm j0.4$

$|z| = \sqrt{(0.36 + 0.16)} = \sqrt{0.52} = 0.7211$

Since $|z| = 0.7211 < 1 \rightarrow$ stable

(b) $z = e^{sT}$, so $s = \ln(z)/T$:

$|z| = e^{\sigma T} \rightarrow \sigma = \ln(0.7211)/0.1 = -0.3268/0.1 = -3.27$ rad/s (real part)

$\angle z = \omega_d T \rightarrow \omega_d = \arctan(0.4/0.6)/0.1 = 0.5880/0.1 = 5.88$ rad/s (imaginary part)

Continuous poles: $s = -3.27 \pm j5.88$

(c) Steady-state (Final Value Theorem): $y_{ss} = \lim_{z\rightarrow 1} (z - 1) \times G(z) \times z/(z - 1) = G(1) = 0.2(1)/(1 - 1.2 + 0.52) = 0.2/0.32 = 0.625$

(d) $G(z) = Y(z)/U(z) \rightarrow Y(z)(z^2 - 1.2z + 0.52) = 0.2z \times U(z)$

Difference equation: $y[k] = 1.2y[k-1] - 0.52y[k-2] + 0.2u[k-1]$

For step input $u[k] = 1$: $y[0] = 1.2(0) - 0.52(0) + 0.2(0) = 0$ $y[1] = 1.2(0) - 0.52(0) + 0.2(1) = 0.2$ $y[2] = 1.2(0.2) - 0.52(0) + 0.2(1) = 0.24 + 0.2 = 0.44$

Problem 4.10.10

Given: A continuous PID controller $C(s) = 4.0(1 + 1/(2s) + 0.3s)$ is to be implemented digitally. The plant has a closed-loop bandwidth of $f_{BW} = 25$ Hz.

Find: (a) The K_p, K_i, and K_d gains in standard form, (b) the maximum sample period T, (c) the discrete PID in position form for the selected T, and (d) the first two control outputs for a unit step error (e[k] = 1 for k ≥ 0, e[-1] = 0).

Solution:

(a) Expanding: $C(s) = 4.0 + 4.0/(2s) + 4.0(0.3)s = 4.0 + 2.0/s + 1.2s$

$K_p = 4.0$, $K_i = 2.0$, $K_d = 1.2$

(b) $T_{max} = 1/(10 \times f_{BW}) = 1/(10 \times 25) = 0.004$ s (4 ms, sample rate = 250 Hz)

Select T = 0.004 s.

(c) Position-form digital PID: $u[k] = K_p e[k] + K_i T \, \Sigma e[j] + K_d(e[k] - e[k-1])/T$

$u[k] = 4.0 \times e[k] + 2.0 \times 0.004 \times \Sigma e[j] + 1.2 \times (e[k] - e[k-1])/0.004$

$u[k] = 4.0 \times e[k] + 0.008 \times \Sigma e[j] + 300 \times (e[k] - e[k-1])$

(d) k = 0: e[0] = 1, Σe = 1 P = 4.0(1) = 4.0; I = 0.008(1) = 0.008; D = 300(1 - 0) = 300 u[0] = 4.0 + 0.008 + 300 = 304.008

k = 1: e[1] = 1, Σe = 2 P = 4.0(1) = 4.0; I = 0.008(2) = 0.016; D = 300(1 - 1) = 0 u[1] = 4.0 + 0.016 + 0 = 4.016

The enormous derivative kick at k = 0 (300) is a practical concern. In implementation, derivative-on-measurement should be used: replace e[k] - e[k-1] with -(y[k] - y[k-1]) in the derivative term, eliminating the spike caused by a step change in the reference signal.

Chapter 5 — Section 5.1: Microcontrollers

Practice problems covering microcontroller architecture, common MCU families, and clock/PLL configuration.

Problem 5.1.1

Given: A Von Neumann architecture MCU runs at 64 MHz. Due to bus contention between instruction fetches and data accesses, 40% of instructions require an additional bus cycle. A subroutine consists of 200 instructions, of which 80 are load/store instructions that trigger the extra cycle and the remaining 120 are single-cycle instructions.

Find: (a) The total execution time for the subroutine. (b) How much faster would the same subroutine run on a Harvard architecture MCU at the same clock frequency, assuming all instructions execute in one cycle?

Solution:

(a) Clock period = 1 / 64 MHz = 15.625 ns

Single-cycle instructions: 120 instructions x 1 cycle = 120 cycles Two-cycle instructions (bus contention): 80 instructions x 2 cycles = 160 cycles Total cycles = 120 + 160 = 280 cycles

Execution time = 280 x 15.625 ns = 4.375 us

(b) On Harvard architecture, all 200 instructions execute in 1 cycle each: Total cycles = 200 Execution time = 200 x 15.625 ns = 3.125 us

Speedup = 4.375 / 3.125 = 1.40x (40% faster)

The Harvard architecture eliminates the bus contention penalty by providing separate instruction and data buses.

Problem 5.1.2

Given: An embedded system must read 4 analog channels at 10 kHz each, run a PID control loop every 1 ms (requiring approximately 500 multiply-accumulate operations per iteration), communicate over UART at 115200 baud, and operate from a 3.3 V coin cell with a target battery life of 2 years. The system must cost under $2 per unit in volume.

Find: Select the most appropriate MCU family (8-bit AVR, ARM Cortex-M0+, or ARM Cortex-M4) and justify the choice.

Solution: ADC requirement: 4 channels x 10,000 samples/s = 40,000 samples/s total — achievable by all three families.

PID computation: 500 MAC operations per ms = 500,000 MAC/s. - 8-bit AVR (16 MHz): software multiply takes ~2 cycles, total = 500 x 2 = 1000 cycles/ms. At 16 MHz, that is 1000/16,000 = 6.25% CPU — feasible but leaves little headroom. - Cortex-M0+ (48 MHz): single-cycle 32-bit multiply, 500 cycles/ms = 500/48,000 = 1.04% CPU — comfortable. - Cortex-M4 (up to 168 MHz): single-cycle MAC with FPU — vastly overqualified.

Power: Cortex-M0+ devices (e.g., STM32L0 series) are specifically designed for ultra-low-power applications with sleep currents as low as 0.3 uA and active current of ~100 uA/MHz.

Cost: 8-bit AVR and Cortex-M0+ both fall well under $2; Cortex-M4 devices typically cost $3-5.

The ARM Cortex-M0+ is the best choice. It provides sufficient computation for the PID loop, meets the ADC and UART requirements, offers best-in-class low-power modes for battery life, and stays within the cost target.

Problem 5.1.3

Given: A microcontroller uses a 16 MHz external crystal (HSE) with a PLL configured as follows: input divider M = 4, multiplication factor N = 192, output divider P = 4. The AHB prescaler is 1, the APB1 prescaler is 2, and the APB2 prescaler is 1.

Find: (a) The system clock (PLL output), (b) the AHB bus frequency, (c) the APB1 peripheral clock, (d) the APB1 timer clock, and (e) the APB2 peripheral clock.

Solution:

(a) f_{PLL} = (HSE / M) x N / P = (16 MHz / 4) x 192 / 4 = 4 MHz x 48 = 192 MHz

Wait — let's recalculate: (16 / 4) = 4 MHz VCO input; 4 x 192 = 768 MHz VCO output; 768 / 4 = 192 MHz system clock

(b) $f_{AHB} = f_{PLL}$ / AHB prescaler = 192 / 1 = 192 MHz

(c) $f_{APB1} = f_{AHB}$ / APB1 prescaler = 192 / 2 = 96 MHz

(d) When the APB prescaler is not 1, the timer clock is doubled: $f_{APB1\_timer} = 2 \times f_{APB1}$ = 2 x 96 = 192 MHz

(e) $f_{APB2} = f_{AHB}$ / APB2 prescaler = 192 / 1 = 192 MHz

Since APB2 prescaler is 1, APB2 timers also run at 192 MHz (no doubling needed when prescaler = 1).

Problem 5.1.4

Given: An engineer needs a timer to generate precise 1-second interrupts for a real-time clock function. The available clock sources are: HSI at 16 MHz (+/- 1% accuracy) and LSE at 32.768 kHz (+/- 20 ppm accuracy).

Find: (a) The maximum timing error per day using each clock source. (b) Which source is appropriate for the RTC function?

Solution:

(a) HSI at +/- 1%: Error per second = 1% of 1 s = 10 ms Error per day = 0.01 x 86,400 s = 864 seconds/day = 14.4 minutes/day

LSE at +/- 20 ppm: Error per second = 20 x 10^{-6} x 1 s = 20 us Error per day = 20 x 10^{-6} x 86,400 = 1.728 seconds/day

(b) The LSE (32.768 kHz crystal) is the correct choice for RTC. Its 20 ppm accuracy yields less than 2 seconds of drift per day, while the HSI would drift nearly 15 minutes per day. The 32.768 kHz frequency also divides evenly by powers of 2 (32,768 = 2^{15}), producing an exact 1 Hz output with a 15-bit prescaler.

Problem 5.1.5

Given: A Cortex-M7 MCU at 216 MHz has a 5-stage pipeline. A branch misprediction flushes the pipeline. In a particular function, 15% of instructions are branches, and the branch predictor has a 90% accuracy rate.

Find: (a) The number of wasted cycles per 1000 instructions due to mispredictions. (b) The effective CPI (cycles per instruction) if the ideal CPI is 1.0.

Solution:

(a) Branches per 1000 instructions = 0.15 x 1000 = 150 branches Mispredictions = 150 x (1 - 0.90) = 150 x 0.10 = 15 mispredictions Each misprediction wastes the pipeline depth - 1 = 5 - 1 = 4 cycles (flushing 4 in-flight instructions) Wasted cycles = 15 x 4 = 60 cycles per 1000 instructions

(b) Total cycles for 1000 instructions = 1000 (ideal) + 60 (penalty) = 1060 cycles Effective CPI = 1060 / 1000 = 1.06 cycles per instruction

This represents a 6% performance degradation from the ideal case. Improving branch prediction accuracy to 95% would reduce wasted cycles to 30, giving CPI = 1.03.

Problem 5.1.6

Given: An MCU system clock is configured at 100 MHz. A UART peripheral requires a 48 MHz clock derived from the PLL. The available PLL output dividers for the UART clock are integers from 1 to 16.

Find: Can the PLL provide exactly 48 MHz for the UART? If not, determine the closest achievable frequency and the resulting baud rate error for a target of 115200 baud.

Solution: The PLL output is 100 MHz. Available UART clocks = 100 / N for N = 1 to 16: 100, 50, 33.33, 25, 20, 16.67, 14.29, 12.5, 11.11, 10, 9.09, 8.33, 7.69, 7.14, 6.67, 6.25 MHz

The closest to 48 MHz is 50 MHz (N = 2). Exact 48 MHz is not achievable with integer division from 100 MHz.

Baud rate generation at 115200 from 50 MHz: Divider = 50,000,000 / 115,200 = 434.028 Using integer divider = 434: actual baud = 50,000,000 / 434 = 115,207.4 baud Error = (115,207.4 - 115,200) / 115,200 x 100 = 0.0064%

This error is well within the +/- 3% tolerance for UART communication, so 50 MHz is acceptable as the UART peripheral clock.

Chapter 5 — Section 5.2: Memory

Practice problems covering flash memory, SRAM allocation, and EEPROM endurance calculations.

Problem 5.2.1

Given: A firmware image is 192 KB. The MCU flash is organized as 8 sectors: sector 0 is 16 KB, sector 1 is 16 KB, sector 2 is 16 KB, sector 3 is 16 KB, sector 4 is 64 KB, and sectors 5-7 are each 128 KB. The flash endurance is 10,000 write-erase cycles. A bootloader occupies sector 0. Firmware updates are performed via OTA 3 times per day.

Find: (a) Which sectors are needed for the application image? (b) Time to reach the flash endurance limit. (c) If a dual-bank A/B update scheme is used, how does this change the answer?

Solution:

(a) Available sectors (excluding sector 0 for bootloader): Sector 1: 16 KB, Sector 2: 16 KB, Sector 3: 16 KB, Sector 4: 64 KB, Sector 5: 128 KB, Sector 6: 128 KB, Sector 7: 128 KB

192 KB firmware requires: Sector 1 (16) + Sector 2 (16) + Sector 3 (16) + Sector 4 (64) + Sector 5 (128) = 240 KB total capacity, using sectors 1 through 5 (5 sectors).

Alternatively, Sector 5 (128 KB) + Sector 4 (64 KB) = 192 KB exactly, using sectors 4 and 5 for a more efficient layout.

(b) At 3 updates/day: 3 x 365 = 1,095 erase cycles per year Time to endurance limit = 10,000 / 1,095 = 9.13 years

(c) With dual-bank (A/B): each slot is updated on alternating cycles, so each bank sees half the updates: Effective cycles per bank per year = 1,095 / 2 = 547.5 Time to endurance limit = 10,000 / 547.5 = 18.26 years

The dual-bank scheme doubles the effective flash lifetime and also provides rollback protection.

Problem 5.2.2

Given: An RTOS-based embedded application on an MCU with 64 KB SRAM has the following memory requirements: - RTOS kernel: 2 KB - Task 1 stack: 512 bytes - Task 2 stack: 1024 bytes - Task 3 stack: 2048 bytes - Idle task stack: 256 bytes - Global variables: 4.5 KB - DMA receive buffer (UART): 256

bytes (double-buffered) - ADC sample buffer: 2048 samples at 16 bits each (circular, double-buffered) - Display frame buffer: 128 x 64 pixels, 1 bit per pixel

Find: (a) Total SRAM usage. (b) Remaining SRAM. (c) Maximum additional task stack size that could be allocated while keeping 25% SRAM free as safety margin.

Solution:

(a) Itemized memory usage:

- RTOS kernel: 2,048 bytes
- Task 1 stack: 512 bytes
- Task 2 stack: 1,024 bytes
- Task 3 stack: 2,048 bytes
- Idle task stack: 256 bytes
- Global variables: 4,608 bytes (4.5 x 1024)
- UART DMA buffer: 2 x 256 = 512 bytes (double-buffered)
- ADC buffer: 2 x 2048 x 2 = 8,192 bytes (double-buffered, 16-bit samples)
- Display frame buffer: (128 x 64) / 8 = 1,024 bytes

Total = 2,048 + 512 + 1,024 + 2,048 + 256 + 4,608 + 512 + 8,192 + 1,024 = 20,224 bytes = 19.75 KB

(b) Remaining = 64 KB - 19.75 KB = 44.25 KB

(c) 25% safety margin = 0.25 x 64 = 16 KB reserved Available for tasks = 64 - 19.75 - 16 = 28.25 KB maximum additional stack

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Problem 5.2.3

Given: A data logger stores sensor readings in EEPROM. Each reading is a 6-byte record (4-byte timestamp + 2-byte sensor value). The EEPROM is 4 KB (4096 bytes) with 1,000,000 write cycles per byte endurance. Readings are taken every 30 seconds. The logger uses sequential writes with wrap-around (circular buffer).

Find: (a) Maximum number of records that fit in the EEPROM. (b) Time before the buffer wraps around. (c) Time to reach the endurance limit on the first byte.

Solution:

(a) Records = 4096 / 6 = 682 complete records (682 x 6 = 4092 bytes used, 4 bytes unused)

Maximum records = 682 records

(b) Readings per hour = 3600 / 30 = 120 readings/hour Time to fill = 682 / 120 = 5.683 hours = 5 hours 41 minutes

After this, the buffer wraps and begins overwriting the oldest records.

(c) Each wrap-around writes 1 new record (6 bytes) to the first location. Wraps occur every 5.683 hours. Writes per year to byte 0 = (365 x 24) / 5.683 = 8,760 / 5.683 = 1,541 writes/year Time to endurance limit = 1,000,000 / 1,541 = 649 years

The EEPROM endurance is not a concern for this application.

Problem 5.2.4

Given: An MCU has 256 KB of flash. The linker output reports the following section sizes: - .text (code): 87,432 bytes - .rodata (constants): 12,288 bytes - .data (initialized variables, stored in flash): 3,072 bytes - .bss (zero-initialized variables): 8,192 bytes (not stored in flash)

Find: (a) Total flash usage. (b) Flash utilization percentage. (c) If a 16 KB flash sector is reserved for non-volatile parameter storage and 32 KB for a bootloader, what is the remaining space for the application?

Solution:

(a) Flash-resident sections: .text + .rodata + .data (initial values) Flash usage = 87,432 + 12,288 + 3,072 = 102,792 bytes = 100.38 KB

Note: .bss is not stored in flash; it is zeroed in SRAM at startup.

(b) Utilization = 102,792 / (256 x 1024) x 100 = 102,792 / 262,144 x 100 = 39.2%

(c) Reserved space = 32 KB (bootloader) + 16 KB (parameters) = 48 KB Available for application = 256 - 48 = 208 KB Remaining after current application = 208 - 100.38 = 107.62 KB free

The application uses 100.38 / 208 = 48.3% of its allocated space.

Problem 5.2.5

Given: A wear-leveling algorithm distributes writes evenly across 64 EEPROM pages of 32 bytes each (2 KB total). The application writes a 4-byte counter value once per minute. Without wear leveling, only one page is written. With wear leveling, writes rotate across all 64 pages. The EEPROM endurance is 100,000 cycles per page.

Find: (a) Time to endurance limit without wear leveling. (b) Time to endurance limit with wear leveling. (c) The improvement factor.

Solution:

(a) Without wear leveling: Writes per year = 60 x 24 x 365 = 525,600 writes/year Time = 100,000 / 525,600 = 0.190 years = 69.4 days

(b) With wear leveling across 64 pages: Each page sees 525,600 / 64 = 8,212.5 writes/year Time = 100,000 / 8,212.5 = 12.18 years

(c) Improvement factor = 12.18 / 0.190 = 64x improvement

The wear-leveling factor equals the number of pages, as expected. This demonstrates why wear leveling is critical for high-frequency write applications.

Chapter 5 — Section 5.3: Peripherals

Practice problems covering GPIO, timers/PWM, ADC, DAC, DMA, and watchdog timers.

Problem 5.3.1

Given: A 3.3 V MCU GPIO pin (maximum sink current 12 mA) is used to drive a common-anode RGB LED. Each LED color element has a forward voltage of 2.1 V (red), 3.0 V (green), and 3.0 V (blue), and the desired forward current for each is 8 mA. The GPIO pins drive low (sink current) to turn on each LED element, with the common anode connected to 3.3 V through individual current-limiting resistors.

Find: (a) The resistor value for each color. (b) Whether the GPIO pins can directly drive the LEDs. (c) The total power dissipated in the resistors when all three colors are on (white).

Solution:

(a) R = (V_{CC} - V_f - V_{OL}) / I_f, where V_{OL} ~ 0.1 V for CMOS output sinking current.

Red: R = (3.3 - 2.1 - 0.1) / 0.008 = 1.1 / 0.008 = 137.5 ohm -> use 150 ohm (standard value) Actual I_{red} = 1.1 / 150 = 7.33 mA

Green: R = (3.3 - 3.0 - 0.1) / 0.008 = 0.2 / 0.008 = 25 ohm -> use 27 ohm (standard value) Actual I_{green} = 0.2 / 27 = 7.41 mA

Blue: R = (3.3 - 3.0 - 0.1) / 0.008 = 25 ohm -> use 27 ohm (standard value) Actual I_{blue} = 7.41 mA

(b) Each GPIO pin sinks at most 7.41 mA, well below the 12 mA maximum. Yes, direct drive is feasible.

(c) Total resistor power: P_{red} = (7.33 x 10^{-3})^2 x 150 = 8.06 mW P_{green} = (7.41 x 10^{-3})^2 x 27 = 1.48 mW P_{blue} = (7.41 x 10^{-3})^2 x 27 = 1.48 mW Total = 11.02 mW

Problem 5.3.2

Given: An MCU running at 84 MHz uses a 16-bit timer to generate a 50 Hz PWM signal for a hobby servo motor. Servos require a 20 ms period with a pulse width ranging from 1.0 ms (0 degrees) to 2.0 ms (180 degrees).

Find: (a) The minimum prescaler value to fit the 20 ms period within the 16-bit counter (max 65535). (b) The auto-reload register (ARR) value with this prescaler. (c) The compare register (CCR) values for 0, 90, and 180 degrees. (d) The angular resolution in degrees.

Solution:

(a) Timer ticks for 20 ms at 84 MHz (no prescaler) = 84,000,000 x 0.020 = 1,680,000 ticks. This exceeds 65,535, so a prescaler is needed. Minimum prescaler = ceil(1,680,000 / 65,536) = ceil(25.63) = 26.

But prescaler values are typically N-1, so prescaler register = 26 - 1 = 25 (divides by 26). Timer frequency = 84 MHz / 26 = 3,230,769.2 Hz. Not a clean number.

Better: use prescaler = 84. Timer freq = 84 MHz / 84 = 1 MHz (1 us per tick). Ticks for 20 ms = 20,000. This fits in 16 bits. Prescaler = 84 (register value = 83)

(b) ARR = 20,000 - 1 = 19,999

(c) Pulse widths at 1 us per tick: 0 degrees: 1.0 ms = 1000 ticks -> CCR = 1000 90 degrees: 1.5 ms = 1500 ticks -> CCR = 1500 180 degrees: 2.0 ms = 2000 ticks -> CCR = 2000

(d) Servo range = 2000 - 1000 = 1000 ticks over 180 degrees Angular resolution = 180 / 1000 = 0.18 degrees per tick

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Problem 5.3.3

Given: A 10-bit ADC with V_{ref} = 5.0 V is used to measure a 4-20 mA current loop signal through a 250 ohm precision sense resistor. The current range of 4-20 mA corresponds to a process variable range of 0-100 PSI.

Find: (a) The voltage range at the ADC input. (b) The ADC digital output range. (c) The pressure resolution in PSI per ADC count. (d) The ADC reading for a pressure of 62.5 PSI.

Solution:

(a) Voltage range: V_{min} = 4 mA x 250 ohm = 1.0 V V_{max} = 20 mA x 250 ohm = 5.0 V

(b) ADC LSB = 5.0 / 1024 = 4.883 mV ADC_{min} = 1.0 / 0.004883 = 204.8 -> 205 counts ADC_{max} = 5.0 / 0.004883 = 1023.9 -> 1023 counts

Digital output range = 205 to 1023 (819 usable counts)

(c) Pressure resolution = 100 PSI / 819 counts = 0.122 PSI per count

(d) 62.5 PSI corresponds to a current of: I = 4 + (62.5 / 100) x (20 - 4) = 4 + 10 = 14 mA V = 14 mA x 250 = 3.5 V ADC = 3.5 / 0.004883 = 716.8 -> 717 counts

Verification: (717 - 205) / 819 x 100 = 512 / 819 x 100 = 62.5 PSI.

Problem 5.3.4

Given: A 12-bit DAC with V_{ref} = 3.3 V generates a sawtooth waveform at 500 Hz. The waveform ramps linearly from 0 V to 3.3 V over one period, then resets to 0 V. A DMA channel transfers values from a lookup table in SRAM to the DAC data register.

Find: (a) The number of lookup table entries for a voltage step size equal to the DAC's LSB. (b) The required DMA transfer rate. (c) The SRAM consumed by the lookup table (16-bit entries). (d) If only 256 entries are used instead, what is the voltage step size?

Solution:

(a) The DAC has 2^{12} = 4096 levels. For one full ramp from 0 to maximum: Lookup table entries = 4096 entries (codes 0 through 4095)

(b) DMA transfer rate = 4096 entries x 500 Hz = 2,048,000 transfers/s = 2.048 MSPS Transfer interval = 1 / 2,048,000 = 488 ns

(c) SRAM = 4096 x 2 bytes = 8,192 bytes = 8 KB

(d) With 256 entries, the step size between consecutive DAC codes: Code step = 4095 / 255 = 16.06, round to 16 codes per step Voltage step = 16 x (3.3 / 4096) = 16 x 0.8057 mV = 12.89 mV

DMA rate = 256 x 500 = 128,000 transfers/s = 128 kSPS (much more manageable) SRAM = 256 x 2 = 512 bytes

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Problem 5.3.5

Given: An audio system uses DMA to stream 16-bit PCM audio samples from SRAM to a DAC at 44.1 kHz (CD quality). The DMA uses a ping-pong (double) buffer scheme: while the DAC reads from buffer A, the CPU fills buffer B, and vice versa. Each buffer holds 512 samples. The MCU runs at 120 MHz.

Find: (a) The DMA transfer rate in bytes per second. (b) The time available for the CPU to fill each buffer. (c) The minimum CPU time to fill one buffer if a sample generation algorithm takes 15 cycles per sample. (d) The CPU utilization percentage for audio generation.

Solution:

(a) DMA rate = 44,100 samples/s x 2 bytes/sample = 88,200 bytes/s = 88.2 KB/s

(b) Buffer duration = 512 samples / 44,100 samples/s = 11.61 ms The CPU has 11.61 ms to fill the next 512-sample buffer before the DMA needs it.

(c) CPU cycles per buffer = 512 x 15 = 7,680 cycles Time = 7,680 / 120,000,000 = 64.0 us

(d) CPU utilization = 64.0 us / 11,610 us x 100 = 0.55%

The DMA-based audio streaming uses negligible CPU time, leaving over 99% of the processor available for other tasks such as UI, communication, or audio effects processing.

Problem 5.3.6

Given: A safety-critical industrial controller uses a window watchdog (WWDG) clocked at 36 MHz with a prescaler of 8. The WWDG 7-bit downcounter is loaded with a value of 127 (0x7F) and the window register is set to 100. The WWDG generates a reset if the counter reaches 63 (0x3F) or if the counter is refreshed while the counter value is greater than the window value.

Find: (a) The watchdog tick period. (b) The timeout period (time from reload to counter reaching 0x3F). (c) The earliest allowed refresh time after a reload. (d) The valid refresh window in milliseconds.

Solution:

(a) WWDG uses a fixed additional divide-by-4096 internally: f_{WWDG} = 36 MHz / (8 x 4096) = 36,000,000 / 32,768 = 1,098.63 Hz Tick period = 1 / 1,098.63 = 910.2 us per tick

(b) The counter counts down from 127 to 63 (reset threshold): Ticks to timeout = 127 - 63 = 64 ticks Timeout period = 64 x 910.2 us = 58.25 ms

(c) Refresh is forbidden while counter > window value (100). Ticks before window opens = 127 - 100 = 27 ticks Earliest refresh time = 27 x 910.2 us = 24.58 ms after reload

(d) Valid refresh window = timeout - earliest refresh = 58.25 - 24.58 = 33.67 ms The firmware must refresh the WWDG between 24.58 ms and 58.25 ms after each reload.

This window prevents both stuck firmware (timeout) and runaway firmware (refreshing too early).

Problem 5.3.7

Given: A 12-bit ADC samples at 1 MSPS with V_{ref} = 2.5 V. The signal of interest is a 0-1 V sensor output. An op-amp gain stage amplifies the sensor output to span the full ADC range before digitization.

Find: (a) The required amplifier gain. (b) The effective number of bits (ENOB) for the original 0-1 V range without amplification. (c) The ENOB with the gain stage (assuming the amplifier adds no noise). (d) The improvement in voltage resolution.

Solution:

(a) Gain = V_{ref} / V_{max_signal} = 2.5 / 1.0 = 2.5 V/V The amplifier maps the 0-1 V sensor range to 0-2.5 V, spanning the full ADC range.

(b) Without amplification: Usable codes = 1.0 V / (2.5 V / 4096) = 1.0 / 0.000610 = 1638 codes ENOB = $\log_2(1638)$ = 10.68 bits

(c) With amplification, the signal spans all 4096 codes: ENOB = $\log_2(4096)$ = 12.0 bits (full ADC resolution utilized)

(d) Resolution without gain: 2.5 / 4096 = 610.4 uV Resolution with gain: (1.0 / 4096) = 244.1 uV (referred to sensor input: 2.5 V / 4096 / 2.5 = 244.1 uV)

Improvement = 610.4 / 244.1 = 2.5x finer resolution, matching the gain factor.

Chapter 5 — Section 5.4: Communication Interfaces

Practice problems covering UART, SPI, I2C, CAN bus, and USB communication protocols.

Problem 5.4.1

Given: An MCU UART is configured at 9600 baud with 8E1 framing (8 data bits, even parity, 1 stop bit). The MCU must transmit a 128-byte data packet with a 2-byte CRC appended.

Find: (a) The total bits per byte including overhead. (b) The total transmission time. (c) The effective data throughput (data bytes only, excluding CRC). (d) The efficiency compared to a raw 8N1 configuration.

Solution:

(a) Each byte frame: 1 start + 8 data + 1 parity + 1 stop = 11 bits per byte

(b) Total bytes = 128 (data) + 2 (CRC) = 130 bytes Total bits = 130 x 11 = 1,430 bits Transmission time = 1,430 / 9,600 = 148.96 ms

(c) Effective data throughput = 128 x 8 / 0.14896 = 1,024 / 0.14896 = 6,874 bps

(d) At 8N1 (10 bits/byte), the same 130 bytes would take: 130 x 10 / 9,600 = 135.42 ms Efficiency of 8E1 vs 8N1: 135.42 / 148.96 = 90.9%, meaning the parity bit adds 9.1% overhead

Effective throughput as percentage of baud rate: 6,874 / 9,600 = 71.6%

Problem 5.4.2

Given: An MCU communicates with 3 SPI slave devices: a flash memory (max 20 MHz), an accelerometer (max 8 MHz), and a display controller (max 4 MHz). The MCU's SPI peripheral is clocked from a 48 MHz APB bus. The SPI clock can only be divided by powers of 2 (2, 4, 8, 16, 32, 64, 128, 256).

Find: (a) The optimal SPI prescaler and actual clock frequency for each device. (b) The time to read 4096 bytes from the flash at its maximum speed. (c) The time to write a 128 x 160 pixel, 16-bit color frame to the display.

Solution:

(a) Flash (max 20 MHz): 48 / 2 = 24 MHz (too fast), 48 / 4 = 12 MHz (safe). Prescaler = 4, f_{SPI} = 12 MHz

Accelerometer (max 8 MHz): 48 / 8 = 6 MHz (safe). Prescaler = 8, f_{SPI} = 6 MHz

Display (max 4 MHz): 48 / 16 = 3 MHz (safe). Prescaler = 16, f_{SPI} = 3 MHz

(b) Flash read: 1 byte opcode + 3 byte address + 4096 data bytes = 4100 bytes Time = (4100 x 8) / 12,000,000 = 32,800 / 12,000,000 = 2.733 ms

(c) Display frame: 128 x 160 x 2 = 40,960 bytes Time = (40,960 x 8) / 3,000,000 = 327,680 / 3,000,000 = 109.2 ms

At 109.2 ms per frame, the maximum display refresh rate is 1000 / 109.2 = 9.2 FPS. For higher frame rates, DMA would be used to free the CPU during the transfer.

Problem 5.4.3

Given: An I2C bus running at 400 kHz (Fast mode) has 5 slave devices connected. The master polls each device by writing a 1-byte register address and then reading 4 bytes of data. Each complete read transaction consists of: START + address_W(1 byte) + ACK + register(1 byte) + ACK + RESTART + address_R(1 byte) + ACK + data(4 bytes with ACK/NACK) + STOP.

Find: (a) The total bit count for one device read transaction. (b) The time for one device read. (c) The total polling time for all 5 devices. (d) The maximum polling rate if I2C bus utilization must stay below 50%.

Solution:

(a) Transaction breakdown: Write phase: START(1) + address_W(8) + ACK(1) + register(8) + ACK(1) = 19 bits Read phase: RESTART(1) + address_R(8) + ACK(1) + data1(8) + ACK(1) + data2(8) + ACK(1) + data3(8) + ACK(1) + data4(8) + NACK(1) + STOP(1) = 47 bits Total = 19 + 47 = 66 bits per transaction

(b) Time per device = 66 / 400,000 = 165 us

(c) Total for 5 devices = 5 x 165 = 825 us

(d) At 50% bus utilization, 825 us of every polling period must leave 825 us idle: Minimum period = 825 / 0.50 = 1,650 us = 1.65 ms Maximum polling rate = 1 / 0.00165 = 606 Hz

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Problem 5.4.4

Given: A CAN FD bus operates with a nominal bit rate of 500 kbps for the arbitration phase and 2 Mbps for the data phase. A CAN FD frame carries 32 bytes of payload. The arbitration overhead (SOF + ID + control bits) is 29 bits at the nominal rate, and the data + CRC + ACK + EOF is 280 bits at the data rate, plus 3 bits for IFS at the nominal rate.

Find: (a) The time for the arbitration phase. (b) The time for the data phase. (c) The total frame time. (d) The maximum message throughput and effective data rate.

Solution:

(a) Arbitration phase: 29 bits at 500 kbps t_{arb} = 29 / 500,000 = 58.0 us

(b) Data phase: 280 bits at 2 Mbps t_{data} = 280 / 2,000,000 = 140.0 us

(c) IFS: 3 bits at 500 kbps = 6.0 us Total frame time = 58.0 + 140.0 + 6.0 = 204.0 us

(d) Maximum throughput = 1 / 204.0 us = 4,902 frames/s Effective data rate = 4,902 x 32 bytes = 156,863 bytes/s = 1.255 Mbps

Compared to classical CAN (244 kbps effective from Example 5.4.4), CAN FD provides a 5.14x improvement in data throughput by using a higher bit rate during the data phase and a larger payload.

Problem 5.4.5

Given: An embedded device uses USB 2.0 High Speed (480 Mbps) with a bulk transfer endpoint. The maximum bulk packet size for High Speed is 512 bytes. Protocol overhead (token packets, handshakes, inter-packet gaps, and microframe scheduling) reduces the usable bandwidth to approximately 53% for bulk transfers. The device must stream 24-bit audio at 96 kHz in stereo.

Find: (a) The required data rate for the audio stream. (b) The maximum achievable bulk transfer throughput. (c) The percentage of USB bandwidth used by the audio stream. (d) How many additional audio channels could be supported simultaneously?

Solution:

(a) Audio data rate = 96,000 samples/s x 2 channels x 3 bytes/sample = 576,000 bytes/s = 4.608 Mbps

(b) Usable bandwidth = 480 Mbps x 0.53 = 254.4 Mbps Maximum bulk throughput = 254.4 / 8 = 31.8 MB/s

(c) USB bandwidth used = 0.576 / 31.8 x 100 = 1.81%

(d) Available bandwidth for additional audio = 31.8 - 0.576 = 31.224 MB/s Additional stereo channels = floor(31.224 / 0.576) = 54 additional stereo pairs

Or equivalently, 108 additional mono channels at 24-bit/96 kHz. USB High Speed has vastly more bandwidth than needed for audio, which is why professional multi-channel audio interfaces commonly use USB 2.0 High Speed.

Problem 5.4.6

Given: An RS-485 multi-drop network connects one master and 8 slave devices on a shared bus at 19200 baud (8N1). The master polls each slave in round-robin fashion. Each poll consists of a 5-byte command from the master and a 10-byte response from the slave. There is a 1 ms turnaround delay after each direction change (master-to-slave and slave-to-master) for bus driver enable/disable.

Find: (a) The time for one poll-response cycle to a single slave. (b) The total time to poll all 8 slaves. (c) The maximum polling rate. (d) The bus efficiency (data bytes / total time).

Solution:

(a) At 19200 baud, 8N1: 10 bits/byte, so 1920 bytes/s.

Master command: 5 bytes x (10/19200) = 2.604 ms Turnaround delay: 1.0 ms Slave response: 10 bytes x (10/19200) = 5.208 ms Turnaround delay: 1.0 ms Total per slave = 2.604 + 1.0 + 5.208 + 1.0 = 9.812 ms

(b) Total for 8 slaves = 8 x 9.812 = 78.5 ms

(c) Maximum polling rate = 1 / 0.0785 = 12.74 polls/s per slave

(d) Data bytes per cycle = 8 x (5 + 10) = 120 bytes in 78.5 ms Throughput = 120 / 0.0785 = 1,529 bytes/s Bus efficiency = 1,529 / 1,920 = 79.6%

The turnaround delays consume 2.0 ms out of 9.812 ms per slave (20.4%), which is the dominant source of inefficiency.

Chapter 5 — Section 5.5: Interrupts

Practice problems covering interrupt priority, latency, NVIC behavior, and ISR design.

Problem 5.5.1

Given: A Cortex-M3 MCU at 72 MHz has the following interrupt configuration: - Timer interrupt: priority 0 (highest), ISR takes 50 cycles - UART RX interrupt: priority 1, ISR takes 80 cycles - ADC complete interrupt: priority 2, ISR takes 120 cycles - SPI transfer complete: priority 3 (lowest), ISR takes 40 cycles

The NVIC requires 12 cycles for context save (entry) and 12 cycles for context restore (exit). The SPI ISR is currently executing when the Timer, UART, and ADC interrupts all trigger simultaneously.

Find: (a) The sequence of ISR execution. (b) The total time from the simultaneous trigger until all four ISRs have completed and the main program resumes.

Solution:

(a) The SPI ISR (priority 3) is preempted by the Timer interrupt (priority 0, highest of the three new interrupts). After Timer completes, UART (priority 1) runs next, then ADC (priority 2), then the remainder of SPI (already executing, 0 cycles remaining since it was preempted at an arbitrary point — assume worst case: full SPI ISR re-runs after restoration).

Execution sequence: 1. SPI ISR is interrupted (context save for preemption) 2. Timer ISR (priority 0) runs 3. UART ISR (priority 1) runs 4. ADC ISR (priority 2) runs 5. SPI ISR resumes (context restore to SPI) 6. Return to main program

(b) Using ARM tail-chaining (no full context save/restore between consecutive ISRs of different priority):

- Initial preemption of SPI: 12 cycles (save SPI context)
- Timer ISR: 50 cycles
- Tail-chain to UART (6 cycles, reduced overhead): 6 cycles
- UART ISR: 80 cycles
- Tail-chain to ADC: 6 cycles
- ADC ISR: 120 cycles
- Restore SPI context: 12 cycles (late arrival optimization)
- SPI ISR completes remaining work: 40 cycles (worst case)
- Restore main context: 12 cycles

Total cycles = 12 + 50 + 6 + 80 + 6 + 120 + 12 + 40 + 12 = 338 cycles Total time = 338 / 72,000,000 = 4.69 us

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Problem 5.5.2

Given: An MCU handles a high-speed ADC sampling at 500 kHz using an interrupt-driven approach. Each ADC ISR reads the 16-bit result and stores it in a circular buffer (30 CPU cycles per ISR, including context save/restore). The MCU runs at 144 MHz.

Find: (a) The CPU utilization consumed by the ADC interrupt. (b) The maximum number of simultaneous 500 kHz ADC channels the CPU can support before exceeding 50% utilization. (c) The advantage of switching to DMA (assume DMA interrupt fires once per 256-sample buffer).

Solution:

(a) ISR frequency = 500,000 interrupts/s Cycles per second = 500,000 x 30 = 15,000,000 cycles/s CPU utilization = 15,000,000 / 144,000,000 = 10.42%

(b) At 50% utilization: max cycles = 0.50 x 144,000,000 = 72,000,000 cycles/s Max channels = 72,000,000 / 15,000,000 = 4.8, so 4 channels

(c) With DMA, the interrupt fires every 256 samples: DMA ISR frequency = 500,000 / 256 = 1,953 interrupts/s Assuming 50 cycles per DMA ISR (slightly longer for buffer management): Cycles per second = 1,953 x 50 = 97,656 cycles/s CPU utilization = 97,656 / 144,000,000 = 0.068%

Reduction = 10.42 / 0.068 = 153x less CPU overhead with DMA.

Problem 5.5.3

Given: A real-time system requires that a motor control ISR always completes within 10 us of its trigger. The MCU runs at 200 MHz. The ISR itself takes 150 cycles. There are two higher-priority interrupts: a safety shutdown ISR (priority 0, 80 cycles) and a communication ISR (priority 1, 200 cycles). The motor ISR is priority 2. Context switch overhead is 12 cycles.

Find: (a) The worst-case latency for the motor ISR (from trigger to completion). (b) Does the system meet the 10 us deadline? (c) What changes could bring it within deadline if not?

Solution:

(a) Worst case: motor ISR triggers just as the safety ISR begins, and the communication ISR also triggers.

Worst-case timeline: - Wait for safety ISR to complete: 12 (entry) + 80 (execution) = 92 cycles - Communication ISR runs (higher priority than motor): 6 (tail-chain) + 200 = 206 cycles - Motor ISR entry and execution: 6 (tail-chain) + 150 = 156 cycles

Worst-case total from motor trigger to motor completion = 92 + 206 + 156 = 454 cycles Time = 454 / 200,000,000 = 2.27 us

(b) 2.27 us < 10 us. Yes, the system meets the deadline with 7.73 us of margin.

(c) Not needed in this case. However, if the deadline were tighter (e.g., 2 us), options would include:

- Elevating motor ISR to priority 0 (above communication)
- Reducing communication ISR execution time
- Splitting the communication ISR into a short critical section and deferred processing

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Problem 5.5.4

Given: An MCU application uses a bare-metal super-loop architecture. The main loop takes 800 us per iteration. A GPIO interrupt detects a button press and sets a flag (ISR takes 0.5 us). The main loop checks the flag and performs a 5 ms debounce verification. The button press must be acknowledged within 50 ms.

Find: (a) The worst-case response time from button press to acknowledgment. (b) Whether the 50 ms deadline is met. (c) The response time if the main loop period were increased to 20 ms (e.g., by adding a complex computation).

Solution:

(a) Worst-case response:

- ISR latency: negligible (< 1 us for flag set)
- Worst-case time until main loop checks flag: the press occurs just after the flag check point, so the flag isn't seen until the next iteration = 800 us
- Debounce wait: 5 ms = 5,000 us
- Processing after debounce: negligible

Total worst-case = 0.5 + 800 + 5,000 = 5,800.5 us = 5.8 ms

(b) 5.8 ms « 50 ms. Yes, the deadline is easily met.

(c) With 20 ms main loop: Worst-case = 0.5 + 20,000 + 5,000 = 25,000.5 us = 25.0 ms

Still within the 50 ms deadline, with 25 ms of margin. If the main loop grew to 45 ms, the worst case would be 50 ms, reaching the limit and suggesting the need for an RTOS or performing the debounce in a timer ISR instead.

Chapter 5 — Section 5.6: Real-Time Operating Systems (RTOS)

Practice problems covering RTOS task scheduling, CPU utilization, and Rate Monotonic Analysis.

Problem 5.6.1

Given: A FreeRTOS application on a Cortex-M4 at 120 MHz has five tasks:

Task	Priority	Period	Execution Time
Motor control	5 (highest)	500 us	80 us
Sensor sampling	4	2 ms	300 us
PID computation	3	5 ms	1.2 ms
Communication	2	20 ms	3 ms
Display update	1 (lowest)	100 ms	15 ms

Find: (a) The CPU utilization for each task and the total. (b) Whether the task set is schedulable using Rate Monotonic Analysis (RMA). (c) The idle time percentage.

Solution:

(a) Utilization per task $U_i = C_i / T_i$:

Motor: 80 / 500 = 0.160 = 16.0% Sensor: 300 / 2,000 = 0.150 = 15.0% PID: 1,200 / 5,000 = 0.240 = 24.0% Communication: 3,000 / 20,000 = 0.150 = 15.0% Display: 15,000 / 100,000 = 0.150 = 15.0%

Total utilization = 0.160 + 0.150 + 0.240 + 0.150 + 0.150 = 0.850 = 85.0%

(b) RMA schedulability bound for n = 5 tasks: $U_{bound} = n \text{ x } (2^{1/n} - 1) = 5 \text{ x } (2^{0.2} - 1) = 5 \text{ x } (1.1487 - 1) = 5 \text{ x } 0.1487 = 0.7435$ = 74.35%

Since 85.0% > 74.35%, the RMA sufficient condition is not satisfied.

However, the RMA bound is a sufficient condition, not a necessary one. The task set may still be schedulable. A detailed response-time analysis would be needed to confirm. For guaranteed schedulability, the total utilization should be reduced below 74.35%.

(c) Idle time = 100% - 85.0% = 15.0%

Problem 5.6.2

Given: Two RTOS tasks share a resource protected by a mutex. Task A (priority 3, high) and Task C (priority 1, low) both need the mutex. Task B (priority 2, medium) does not use the mutex. Task C acquires the mutex and begins its critical section (2 ms long). Task A becomes ready and needs the mutex.

Find: (a) Describe the priority inversion problem that occurs without mitigation. (b) Calculate the worst-case delay for Task A if Task B runs for 10 ms. (c) How does priority inheritance solve this problem?

Solution:

(a) Priority inversion scenario:

1. Task C (priority 1) acquires the mutex and enters its critical section.
2. Task A (priority 3) preempts C and tries to acquire the mutex — it blocks because C holds it.
3. Task B (priority 2) becomes ready. Since A is blocked and B has higher priority than C, B preempts C.
4. Task C cannot complete its critical section to release the mutex until B finishes.
5. Task A (highest priority) is effectively blocked by Task B (medium priority), which doesn't even use the shared resource. This is priority inversion.

(b) Worst-case delay for Task A:

- B runs for 10 ms (preempting C)
- C finishes critical section: 2 ms
- Total delay for A = 10 + 2 = 12 ms

Task A, the highest-priority task, waits 12 ms for a 2 ms critical section.

(c) Priority inheritance protocol: When Task A blocks on the mutex held by Task C, the RTOS temporarily raises C's priority to match A's priority (3). Now C cannot be preempted by B (priority 2), so C completes its 2 ms critical section immediately, releases the mutex, and its priority reverts to 1. Task A acquires the mutex with only 2 ms of delay instead of 12 ms.

Problem 5.6.3

Given: An RTOS tick timer runs at 1 kHz (1 ms tick). A task uses vTaskDelay(10) to sleep for 10 ticks. The task's processing before the delay takes 0.3 ms.

Find: (a) The actual delay range (minimum and maximum). (b) The total period of the task (execution + delay). (c) How vTaskDelayUntil() improves periodicity, and what period it would produce.

Solution:

(a) vTaskDelay(10) delays for 10 complete tick periods from the time the call is made. However, the first tick may occur anywhere from 0 to 1 ms after the call (depending on where in the current tick period the call lands).

Minimum delay: 10 ticks x 1 ms - 1 ms (nearly aligned with tick) = 9 ms (just barely misses a tick boundary and counts 10 from the next tick minus almost-full first tick)

More precisely: minimum delay = 9.0+ ms (10 ticks, first is partial), maximum delay = 10.0 ms (call coincides with tick edge).

Delay range: 9 to 10 ms

(b) Total period = processing + delay = 0.3 ms + (9 to 10 ms) = 9.3 to 10.3 ms The period jitters by up to 1 ms (one tick period).

(c) vTaskDelayUntil() delays until an absolute tick count, removing the jitter. If the wake time is set to a period of 10 ticks: Period = exactly 10 ms (10 ticks), regardless of when processing finishes. The task always wakes at the same phase relative to the tick timer, providing deterministic periodicity with zero jitter (assuming processing completes within the period).

Problem 5.6.4

Given: An RTOS message queue has a depth of 16 entries, each holding a 32-byte message. A producer task generates messages at a rate of 200 messages/second. A consumer task processes each message in 3 ms.

Find: (a) The consumer's processing rate. (b) Whether the queue is stable (consumer keeps up with producer). (c) The maximum burst of messages the queue can absorb if the consumer is temporarily blocked for 50 ms. (d) Whether the queue overflows during the burst.

Solution:

(a) Consumer rate = 1 / 3 ms = 333.3 messages/s

(b) Producer rate = 200 msg/s, Consumer rate = 333.3 msg/s Since 333.3 > 200, the queue is stable — the consumer keeps up with the producer.

(c) During a 50 ms blockage: Messages arriving = 200 x 0.050 = 10 messages

(d) Queue depth = 16 entries. During the blockage, 10 messages accumulate. 10 < 16, so the queue does not overflow.

After the blockage, the consumer processes the backlog at its net drain rate: Net rate = 333.3 - 200 = 133.3 msg/s Time to drain backlog = 10 / 133.3 = 75 ms

If the consumer were blocked for 80 ms, the burst would be 200 x 0.080 = 16 messages, exactly filling the queue.

Problem 5.6.5

Given: An RTOS application must allocate memory for tasks. Each task requires a stack plus a Task Control Block (TCB) of 88 bytes. The system has 32 KB of RTOS heap. Tasks and their minimum stack requirements:

Task	Stack Size
Main	2048 bytes
Sensor	512 bytes
Comm	1024 bytes
Logger	4096 bytes
Idle (automatic)	128 bytes
Timer daemon (automatic)	256 bytes

Find: (a) Total memory for all tasks. (b) RTOS heap utilization. (c) Maximum number of additional 512-byte-stack tasks.

Solution:

(a) Memory per task = stack + TCB (88 bytes)

Main: 2048 + 88 = 2,136 bytes Sensor: 512 + 88 = 600 bytes Comm: 1024 + 88 = 1,112 bytes Logger: 4096 + 88 = 4,184 bytes Idle: 128 + 88 = 216 bytes Timer: 256 + 88 = 344 bytes

Total = 2,136 + 600 + 1,112 + 4,184 + 216 + 344 = 8,592 bytes

(b) Heap utilization = 8,592 / 32,768 = 26.2%

(c) Remaining heap = 32,768 - 8,592 = 24,176 bytes Memory per additional task = 512 + 88 = 600 bytes Max additional tasks = floor(24,176 / 600) = 40 additional tasks

Chapter 5 — Section 5.7: Power Management

Practice problems covering sleep modes, battery life calculations, and low-power design techniques.

Problem 5.7.1

Given: A wildlife tracking collar runs on a 3.7 V, 2500 mAh lithium-polymer battery. The MCU has three power modes: - Active (GPS acquisition): 45 mA for 30 seconds - Active (data processing + satellite uplink): 120 mA for 5 seconds - Stop mode: 8 uA

The collar wakes every 4 hours to acquire a GPS fix and transmit location data.

Find: (a) The average current consumption. (b) The expected battery life. (c) The battery life if the wake interval is reduced to 1 hour for tracking an endangered animal.

Solution:

(a) Period T = 4 hours = 14,400 seconds.

Charge per cycle: GPS: 45 mA x 30 s = 1,350 mA-s = 0.375 mAh Uplink: 120 mA x 5 s = 600 mA-s = 0.1667 mAh Sleep: 0.008 mA x (14,400 - 35) s = 0.008 x 14,365 = 114.92 mA-s = 0.03192 mAh

Total per cycle = 0.375 + 0.1667 + 0.03192 = 0.5736 mAh Average current = 0.5736 mAh / 4 h = 0.1434 mA = 143.4 uA

(b) Battery life = 2500 / 0.1434 = 17,434 hours = 726 days = 1.99 years

(c) At 1-hour intervals (T = 3600 s): Sleep charge: 0.008 x (3600 - 35) = 28.52 mA-s = 0.007922 mAh Total per cycle = 0.375 + 0.1667 + 0.007922 = 0.5496 mAh Average current = 0.5496 / 1 = 0.5496 mA = 549.6 uA

Battery life = 2500 / 0.5496 = 4,548 hours = 189 days = 6.2 months

The 4x increase in wake frequency increases average current by 3.83x because the active-mode energy dominates.

Problem 5.7.2

Given: An MCU at V_{DD} = 1.8 V runs at 80 MHz, drawing 12 mA in active mode. The dynamic power follows $P = C_L \times V_{DD}^2 \times f$, and the voltage regulator supports 1.2 V operation at frequencies up to 48 MHz.

Find: (a) The estimated load capacitance C_L. (b) The active current at 1.2 V / 48 MHz. (c) The power savings as a percentage. (d) The energy to execute a fixed task of 10 million cycles at each operating point.

Solution:

(a) $P_{active} = V_{DD} \times I = 1.8 \times 0.012 = 0.0216$ W = 21.6 mW $C_L = P / (V_{DD}^2 \times f) = 0.0216 / (1.8^2 \times 80 \times 10^6) = 0.0216 / (3.24 \times 8 \times 10^7)$ $C_L = 0.0216 / 2.592 \times 10^8 = 83.3$ pF

(b) P at 1.2 V / 48 MHz = $83.3 \times 10^{-12} \times 1.2^2 \times 48 \times 10^6 = 83.3 \times 10^{-12} \times 1.44 \times 4.8 \times 10^7 = 83.3 \times 10^{-12} \times 6.912 \times 10^7 = 5.76 \times 10^{-3}$ W = 5.76 mW $I = P / V_{DD} = 5.76 / 1.2 = 4.8$ mA

(c) Power savings = (21.6 - 5.76) / 21.6 x 100 = 73.3%

(d) At 80 MHz: time = $10^7 / 80 \times 10^6 = 0.125$ s Energy = 21.6 mW x 0.125 s = 2.70 mJ

At 48 MHz: time = $10^7 / 48 \times 10^6 = 0.2083$ s Energy = 5.76 mW x 0.2083 s = 1.20 mJ

Energy savings = (2.70 - 1.20) / 2.70 x 100 = 55.6%

Running slower at lower voltage saves significant energy per task because energy scales as V^2, even though the task takes longer.

Problem 5.7.3

Given: A battery-powered sensor hub polls 3 external I2C sensors. Each sensor has a standby current of 50 uA and an active current of 2 mA during measurement. Measurements take 10 ms each. The hub polls every 60 seconds. An external load switch (I_Q = 0.5 uA) can cut power to all three sensors between measurements.

Find: (a) Average sensor current without the load switch (sensors in standby between measurements). (b) Average sensor current with the load switch. (c) Battery life improvement (battery = 3.3 V, 500 mAh, MCU sleep current = 5 uA).

Solution:

(a) Without load switch (3 sensors always powered): Active time per poll = 3 x 10 ms = 30 ms at 3 x 2 mA = 6 mA Standby time = 60,000 - 30 = 59,970 ms at 3 x 50 uA = 150 uA

$I_{avg\_sensors}$ = (6 x 30 + 0.150 x 59,970) / 60,000 = (180 + 8,995.5) / 60,000 = 9,175.5 / 60,000 = 0.1529 mA = 152.9 uA

(b) With load switch: Active time = 30 ms at 6 mA (sensors powered) Off time = 59,970 ms at 0.5 uA (load switch quiescent only)

$I_{avg\_sensors}$ = (6 x 30 + 0.0005 x 59,970) / 60,000 = (180 + 29.985) / 60,000 = 209.985 / 60,000 = 0.003500 mA = 3.50 uA

(c) Total system current: Without switch: 5 (MCU) + 152.9 (sensors) = 157.9 uA With switch: 5 (MCU) + 3.5 (sensors) = 8.5 uA

Battery life without switch = 500 / 0.1579 = 3,167 hours = 132 days Battery life with switch = 500 / 0.0085 = 58,824 hours = 6.7 years

Improvement factor = 58,824 / 3,167 = 18.6x

The load switch nearly eliminates the sensor standby current, which dominated the power budget.

Problem 5.7.4

Given: An IoT gateway MCU processes incoming packets. The packet arrival rate follows a Poisson distribution with an average of 5 packets per second. Each packet requires 2 ms of active processing at 25 mA. Between packets, the MCU enters Sleep mode at 1 mA.

Find: (a) The average active duty cycle. (b) The average current. (c) If Stop mode (10 uA) is used instead of Sleep mode and waking takes an additional 0.5 ms at 25 mA, what is the new average current?

Solution:

(a) Active time per second = 5 packets x 2 ms = 10 ms Duty cycle = 10 / 1000 = 1.0%

(b) I_{avg} = 25 x 0.01 + 1.0 x 0.99 = 0.25 + 0.99 = 1.24 mA

(c) With Stop mode and 0.5 ms wake penalty: Active per packet = 2.0 + 0.5 = 2.5 ms at 25 mA Active time per second = 5 x 2.5 = 12.5 ms Sleep time = 987.5 ms at 0.010 mA

I_{avg} = 25 x 0.0125 + 0.010 x 0.9875 = 0.3125 + 0.009875 = 0.322 mA

Improvement = 1.24 / 0.322 = 3.85x reduction in average current

Despite the wake-up penalty, Stop mode saves significant power because the sleep current drops from 1 mA to 10 uA, and the MCU spends 99% of its time sleeping.

Chapter 5 — Section 5.8: Development and Debugging

Practice problems covering debug interfaces, firmware download, and bootloader design.

Problem 5.8.1

Given: A development team uses a CMSIS-DAP debug probe with SWD at 10 MHz to program a Cortex-M7 target. The firmware image is 512 KB. The MCU flash has 8 sectors of 64 KB each. Each sector erase takes 250 ms. Flash programming speed (after erase) is 80 KB/s. After programming, a full read-back verification is performed at the SWD link speed.

Find: (a) The total sector erase time. (b) The flash programming time. (c) The SWD raw throughput (assuming 50% protocol efficiency). (d) The verification read-back time. (e) The total firmware download-and-verify time.

Solution:

(a) Sectors to erase = 512 / 64 = 8 sectors Total erase time = 8 x 250 ms = 2.0 seconds

(b) Programming time = 512 KB / 80 KB/s = 6.4 seconds

(c) SWD at 10 MHz, 1-bit data line, 50% efficiency: Raw throughput = 10 MHz x 0.50 / 8 bits = 625 KB/s

(d) Verification reads back the entire 512 KB at SWD speed: Read-back time = 512 / 625 = 0.819 seconds

(e) Total = erase + program + verify = 2.0 + 6.4 + 0.819 = 9.22 seconds

The programming step is the bottleneck (69% of total time), limited by the flash write speed rather than the SWD link bandwidth.

Problem 5.8.2

Given: A custom bootloader for a medical device uses dual-bank (A/B) firmware slots. The MCU has 1 MB of flash organized as: - Bootloader: 64 KB (sectors 0-3, 16 KB each) - Configuration/keys: 16 KB (sector 4) - Slot A: 464 KB (sectors 5-11) - Slot B: 464 KB (sectors 12-18)

The active firmware is 320 KB. Updates arrive over BLE (Bluetooth Low Energy) at an effective throughput of 20 KB/s. Flash erase for Slot B takes 3.5 seconds. Flash programming runs at 40 KB/s. A SHA-256 hash verification of the programmed image takes 800 ms.

Find: (a) The BLE transfer time for the firmware image. (b) The total update time (receive + erase + program + verify). (c) If the device must remain operational during the update, explain how the A/B scheme enables this.

Solution:

(a) BLE transfer time = 320 KB / 20 KB/s = 16.0 seconds

(b) Total update time:

- BLE receive: 16.0 s
- Flash erase (Slot B): 3.5 s
- Flash program: 320 / 40 = 8.0 s
- SHA-256 verify: 0.8 s

Total = 16.0 + 3.5 + 8.0 + 0.8 = 28.3 seconds

Note: BLE receive and flash erase can overlap if the bootloader erases sectors progressively while receiving data. In that case: Overlapped total = max(16.0, 3.5) + 8.0 + 0.8 = 16.0 + 8.0 + 0.8 = 24.8 seconds

(c) A/B update scheme operation: The device runs its current firmware from Slot A throughout the entire update process. The new firmware is written to Slot B. If the download or verification fails, Slot A remains untouched and the device continues operating normally. After successful verification, the bootloader updates a flag indicating Slot B is the new active image. On the next reboot, the bootloader loads from Slot B. If Slot B fails to boot (detected by a watchdog or boot counter), the bootloader automatically reverts to Slot A.

This ensures the device is never bricked by a failed update — critical for medical devices under IEC 62304.

Problem 5.8.3

Given: A Cortex-M4 MCU supports 6 hardware breakpoints and 4 data watchpoints through the CoreSight debug architecture. A developer is debugging a firmware with 150 functions. The bug manifests as a corrupted 4-byte variable at address 0x2000_1A00 that changes unexpectedly during normal execution.

Find: (a) How many hardware breakpoints can be simultaneously active? (b) How a data watchpoint can identify the source of the corruption. (c) If the developer needs to monitor 5 different variables simultaneously, what limitation is encountered? (d) An alternative approach using ITM trace.

Solution:

(a) The MCU supports 6 simultaneous hardware breakpoints. Each breakpoint halts the processor when execution reaches a specific address. These are a limited hardware resource — unlike software breakpoints (which replace an instruction with a breakpoint opcode), hardware breakpoints don't modify the code and work on read-only flash memory.

(b) Configure a data watchpoint at address 0x2000_1A00 with a 4-byte size, triggering on write access. When any instruction writes to this address, the processor halts immediately after the write. The debugger then shows:

- The program counter (PC) of the instruction that performed the write
- The call stack showing which function path led to the write
- Register values at the point of the write

This directly identifies the offending code without needing to know which of the 150 functions is responsible.

(c) With only 4 data watchpoints available, the developer cannot monitor all 5 variables simultaneously. Options:

- Monitor the 4 most likely candidates first, then swap in the 5th
- Use conditional watchpoints to filter specific write patterns
- Combine adjacent variables into a single watchpoint if addresses are within a watchpoint's address range

(d) ITM trace approach: Instrument the code to output a trace message (via ITM stimulus port) whenever the variable is written. The ITM provides printf-style debug output at minimal CPU overhead (writing a single word to an ITM stimulus register takes 1 cycle). The SWO (Serial Wire Output) pin streams the trace data to the debug probe at up to 2 Mbit/s, allowing real-time monitoring without halting the processor.

Problem 5.8.4

Given: A bootloader receives firmware over UART at 921600 baud (8N1). The incoming data is framed in packets of 128 bytes payload + 2 bytes CRC-16 + 1 byte sequence number = 131 bytes per packet. The MCU acknowledges each packet with a 1-byte ACK/NACK. The round-trip latency (packet send + ACK return + host processing) adds 5 ms per packet.

Find: (a) The raw UART throughput. (b) The effective firmware transfer rate including protocol overhead and latency. (c) The time to transfer a 384 KB firmware image. (d) How a sliding window protocol with window size 4 would improve throughput.

Solution:

(a) Raw UART throughput at 921600 baud, 8N1 (10 bits/byte): Throughput = 921,600 / 10 = 92,160 bytes/s = 90.0 KB/s

(b) Per packet: Packet size = 131 bytes at 92,160 B/s -> transmit time = 131 / 92,160 = 1.422 ms ACK = 1 byte at 92,160 B/s -> 0.011 ms Round-trip overhead = 5 ms Total per packet = 1.422 + 0.011 + 5.0 = 6.433 ms Payload per packet = 128 bytes

Effective rate = 128 / 0.006433 = 19,898 bytes/s = 19.4 KB/s

(c) Packets needed = ceil(384 x 1024 / 128) = ceil(3072) = 3072 packets Transfer time = 3072 x 6.433 ms = 19.76 seconds

(d) With sliding window of 4: the host sends 4 packets before waiting for the first ACK. Transmit 4 packets = 4 x 1.422 = 5.688 ms Wait for ACK of first packet (includes 5 ms round-trip) = 5.0 ms But by the time the ACK arrives, 4 packets have been sent, so the pipeline stays full.

Effective time per packet (once pipeline is full) = max(1.422 ms, 5.0/4) = max(1.422, 1.25) = 1.422 ms Effective rate = 128 / 0.001422 = 90,014 bytes/s = 87.9 KB/s

Transfer time = 3072 x 1.422 ms = 4.37 seconds

The sliding window provides a 4.52x speedup by hiding the round-trip latency.

Chapter 6 — Section 6.1: Number Systems

Practice problems covering binary, hexadecimal, and octal number conversions.

Problem 6.1.1

Given: The decimal number 4013.

Find: (a) The binary representation. (b) The hexadecimal representation. (c) Verify by converting the hex result back to decimal.

Solution:

(a) Successive division by 2: 4013 / 2 = 2006 R 1 2006 / 2 = 1003 R 0 1003 / 2 = 501 R 1 501 / 2 = 250 R 1 250 / 2 = 125 R 0 125 / 2 = 62 R 1 62 / 2 = 31 R 0 31 / 2 = 15 R 1 15 / 2 = 7 R 1 7 / 2 = 3 R 1 3 / 2 = 1 R 1 1 / 2 = 0 R 1

Reading MSB to LSB: 1111 1010 1101 binary

(b) Group into 4-bit nibbles from LSB: 1111 1010 1101 F = 1111, A = 1010, D = 1101 Hexadecimal: 0xFAD

(c) Verification: F x 16^2 + A x 16^1 + D x 16^0 = 15 x 256 + 10 x 16 + 13 = 3840 + 160 + 13 = 4013. Confirmed.

Problem 6.1.2

Given: A 16-bit microcontroller register contains the hexadecimal value 0xB7E3.

Find: (a) The full binary representation. (b) The decimal value. (c) The value of bits [11:8] (upper nibble of the lower byte). (d) The octal representation.

Solution:

(a) Convert each hex digit to 4 bits: B = 1011, 7 = 0111, E = 1110, 3 = 0011 Binary: 1011 0111 1110 0011

(b) Decimal: B x 16^3 + 7 x 16^2 + E x 16^1 + 3 x 16^0 = 11 x 4096 + 7 x 256 + 14 x 16 + 3 = 45,056 + 1,792 + 224 + 3 = 47,075

(c) Bits [11:8] correspond to the second nibble from the left: 0111 = 7 (decimal)

(d) Group binary into 3-bit groups from LSB: 1 011 011 111 100 011 = 1 3 3 7 4 3 Octal: 133743

Verification: $1 \times 8^5 + 3 \times 8^4 + 3 \times 8^3 + 7 \times 8^2 + 4 \times 8 + 3 = 32{,}768 + 12{,}288 + 1{,}536 + 448 + 32 + 3 = 47{,}075$. Confirmed.

Problem 6.1.3

Given: Two 8-bit unsigned binary numbers: A = 1100 1010 and B = 0011 1001.

Find: (a) A + B in binary (show carry chain). (b) A AND B. (c) A OR B. (d) A XOR B. (e) The decimal values of A, B, and A + B.

Solution:

(a) Binary addition with carry:

```
  1100 1010
+ 0011 1001
-----------
```

Bit 0: 0 + 1 = 1, C = 0 Bit 1: 1 + 0 = 1, C = 0 Bit 2: 0 + 0 = 0, C = 0 Bit 3: 1 + 1 = 0, C = 1 Bit 4: 0 + 1 + 1 = 0, C = 1 Bit 5: 0 + 1 + 1 = 0, C = 1 Bit 6: 1 + 0 + 1 = 0, C = 1 Bit 7: 1 + 0 + 1 = 0, C = 1 (carry out)

A + B = (1) 0000 0011 = 9-bit result 1 0000 0011

(b) A AND B = 1100 1010 AND 0011 1001 = 0000 1000

(c) A OR B = 1100 1010 OR 0011 1001 = 1111 1011

(d) A XOR B = 1100 1010 XOR 0011 1001 = 1111 0011

(e) A = 128 + 64 + 8 + 2 = 202 B = 32 + 16 + 8 + 1 = 57 A + B = 202 + 57 = 259 = 259 (requires 9 bits; carry out indicates overflow for 8-bit unsigned)

Problem 6.1.4

Given: A signed 8-bit number uses two's complement representation. The bit pattern is 1101 0110.

Find: (a) The decimal value. (b) The two's complement of this number (i.e., the negation). (c) The range of representable values in 8-bit two's complement.

Solution:

(a) The MSB is 1, so the number is negative. Two's complement to magnitude: invert all bits and add 1. Invert: 0010 1001 Add 1: 0010 1010 = 32 + 8 + 2 = 42

The decimal value is -42.

(b) The two's complement (negation) of -42 is +42. +42 in binary = 0010 1010 Verification: 0010 1010 = +42

(c) 8-bit two's complement range: Minimum = -2^7 = -128 (bit pattern 1000 0000) Maximum = 2^7 - 1 = +127 (bit pattern 0111 1111)

Range: -128 to +127 (256 total values)

Problem 6.1.5

Given: A 32-bit IEEE 754 single-precision floating-point number has the hex representation 0x41C80000.

Find: (a) The sign, exponent, and mantissa fields. (b) The decimal value.

Solution:

(a) Convert to binary: 0x41C80000 = 0100 0001 1100 1000 0000 0000 0000 0000

Sign bit (bit 31): 0 (positive) Exponent (bits [30:23]): 1000 0011 = 131 Mantissa (bits [22:0]): 100 1000 0000 0000 0000 0000

(b) Biased exponent = 131, bias = 127 Actual exponent = 131 - 127 = 4

Mantissa with implicit leading 1: 1.1001000... = 1 + 0.5 + 0.0625 = 1.5625

Value = $(-1)^0$ x 1.5625 x 2^4 = 1 x 1.5625 x 16 = 25.0

Verification: 25.0 = 11001.0 binary = 1.1001 x 2^4. Confirmed.

Problem 6.1.6

Given: A BCD (Binary-Coded Decimal) counter displays the value 947 on three 7-segment displays. Each digit is encoded as a 4-bit BCD nibble.

Find: (a) The 12-bit BCD representation. (b) The equivalent binary value. (c) The number of bits saved by using binary instead of BCD.

Solution:

(a) Each decimal digit is encoded as 4 bits: 9 = 1001, 4 = 0100, 7 = 0111 BCD: 1001 0100 0111 (12 bits)

(b) 947 in binary: successive division by 2: 947 = 512 + 256 + 128 + 32 + 16 + 2 + 1 = 1110110011 Verification: 512 + 256 + 128 + 32 + 16 + 2 + 1 = 947

947 in binary: 11 1011 0011 (10 bits)

(c) BCD requires 12 bits; binary requires 10 bits. Savings = 12 - 10 = 2 bits (16.7% more efficient)

For larger numbers the savings grow: a 6-digit BCD number needs 24 bits, while binary can represent up to 999,999 in $\text{ceil}(\log_2(1{,}000{,}000))$ = 20 bits, saving 4 bits.

Chapter 6 — Section 6.2: Boolean Algebra

Practice problems covering truth tables and Karnaugh map simplification.

Problem 6.2.1

Given: The Boolean function F(A, B, C) = AB + A'C + BC.

Find: (a) Construct the complete truth table. (b) Express F as a sum of minterms. (c) Simplify using Boolean algebra theorems.

Solution:

(a) Truth table (evaluate each term):

A	B	C	AB	A'C	BC	F = AB + A'C + BC
0	0	0	0	0	0	0
0	0	1	0	1	0	1
0	1	0	0	0	0	0
0	1	1	0	1	1	1
1	0	0	0	0	0	0
1	0	1	0	0	0	0
1	1	0	1	0	0	1
1	1	1	1	0	1	1

(b) F = 1 for rows 1, 3, 6, 7 (zero-indexed), which are minterms: F = sum(1, 3, 6, 7)

(c) Simplification: F = AB + A'C + BC Apply the consensus theorem: XY + X'Z + YZ = XY + X'Z (since YZ is redundant). Here X = A, Y = B, Z = C: F = AB + A'C

The term BC is the consensus term and can be eliminated. Verification: AB + A'C covers all four minterms (AB covers m6, m7; A'C covers m1, m3).

Problem 6.2.2

Given: The Boolean function F(A, B, C, D) = sum of minterms (0, 2, 5, 7, 8, 10, 13, 15).

Find: Simplify using a 4-variable Karnaugh map.

Solution: K-map layout (AB on rows, CD on columns, Gray code order):

```
        CD=00  CD=01  CD=11  CD=10
AB=00 |  1    |  0    |  0    |  1    |
AB=01 |  0    |  1    |  1    |  0    |
AB=11 |  0    |  1    |  1    |  0    |
AB=10 |  1    |  0    |  0    |  1    |
```

Identify groups: Group 1 (quad): minterms 0, 2, 8, 10 (corners of the map). These are at CD=00 and CD=10 for AB=00 and AB=10. Common variables: B' and D'. Term: B'D'

Group 2 (quad): minterms 5, 7, 13, 15. These are at CD=01 and CD=11 for AB=01 and AB=11. Common variables: B and D. Term: BD

All 8 minterms are covered.

F = B'D' + BD

This can also be written as F = (B XNOR D), meaning the function is true whenever B and D have the same value.

Problem 6.2.3

Given: F(A, B, C, D) = sum of minterms (1, 3, 4, 5, 9, 11, 12, 13) with don't-care conditions d(6, 14).

Find: Simplify using a K-map, taking advantage of don't-care terms.

Solution: K-map with 1s for minterms and Xs for don't-cares:

```
        CD=00  CD=01  CD=11  CD=10
AB=00 |  0    |  1    |  1    |  0    |
AB=01 |  1    |  1    |  0    |  X    |
AB=11 |  1    |  1    |  0    |  X    |
AB=10 |  0    |  1    |  1    |  0    |
```

Groups: Group 1 (octet): minterms 1, 3, 5, 9, 11, 13 plus don't-cares can we find a larger group?

Let's identify systematically: Group 1: columns CD=01 (all rows): cells 1, 5, 13, 9 -> all are 1. Quad with common variable: D and C'. Term: C'D

Group 2: cells 1, 3, 9, 11 (AB=00 row CD=01,11 and AB=10 row CD=01,11): common variables B' and D. Term: B'D

Group 3: cells 4, 5, 12, 13 (AB=01 row CD=00,01 and AB=11 row CD=00,01): common variables C' and B. Wait — checking: 4=AB01,CD00; 5=AB01,CD01; 12=AB11,CD00; 13=AB11,CD01. Common: B=1, C=0. Term: BC'

Include don't-cares 6 and 14 to enlarge groups: Cell 6 (AB=01, CD=10) and cell 14 (AB=11, CD=10): with cells 4 and 12 forms a quad: 4, 6, 12, 14 -> common: B and D'. Term: BD'

Now check coverage: C'D covers {1,5,9,13}. B'D covers {1,3,9,11}. BD' covers {4,6,12,14}. Missing: minterm 3 is covered by B'D. Minterm 5 is covered by C'D. All covered.

But BD' uses both don't-cares. Can we do better?

Simpler solution: C'D covers {1,5,9,13}. B'D covers {1,3,9,11}. BC' + don't-cares: cells 4,5,12,13,6,14 -> BC' covers {4,5,12,13} and with X at 6,14, we can use BD' for {4,6,12,14}.

Essential terms: C'D and B'D cover all D=1 minterms. BC' covers {4,5,12,13} (5 and 13 already covered). Actually we just need to cover 4 and 12: these share B=1, C=0, D=0 -> BCD' doesn't simplify unless we use don't-cares.

Using don't-care 6: group {4, 6, 12, 14} = BD'. This covers 4 and 12 with a simpler term.

F = B'D + C'D + BD'

Simplify further: B'D + C'D = D(B' + C') = D(BC)' by De Morgan's. And BD' is separate.

F = B'D + C'D + BD'

Problem 6.2.4

Given: Prove De Morgan's theorem: (A + B)' = A' * B' using a truth table.

Find: Construct truth tables for both sides and verify they are identical.

Solution:

A	B	A + B	(A + B)'	A'	B'	A' * B'
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

The columns (A + B)' and A' * B' are identical for all input combinations: Row 0: 1 = 1 Row 1: 0 = 0 Row 2: 0 = 0 Row 3: 0 = 0

De Morgan's theorem (A + B)' = A' * B' is proven.

This theorem is fundamental to digital design because it allows converting between OR/AND forms, which is essential for implementing circuits using only NAND or only NOR gates.

Problem 6.2.5

Given: A Boolean function is specified by the product-of-sums (POS) expression: F(A, B, C) = (A + B + C) * (A + B' + C) * (A' + B + C) * (A' + B + C')

Find: (a) The equivalent sum-of-minterms form. (b) Simplify using a K-map.

Solution:

(a) A maxterm (sum term) equals 0 for one specific input combination. Each maxterm corresponds to the complement of a minterm: (A + B + C) = 0 when A=0, B=0, C=0 -> maxterm M_0 -> minterm m_0 is absent (A + B' + C) = 0 when A=0, B=1, C=0 -> maxterm M_2 -> minterm m_2 is absent (A' + B + C) = 0 when A=1, B=0, C=0 -> maxterm M_4 -> minterm m_4 is absent (A' + B + C') = 0 when A=1, B=0, C=1 -> maxterm M_5 -> minterm m_5 is absent

The function is 0 for minterms 0, 2, 4, 5; therefore it is 1 for minterms 1, 3, 6, 7: F = sum(1, 3, 6, 7)

(b) K-map for 3 variables:

```
       C=0  C=1
AB=00 | 0 | 1 |
AB=01 | 0 | 1 |
AB=11 | 1 | 1 |
AB=10 | 0 | 0 |
```

Groups: Group 1: minterms 1, 3 (AB=00,01; C=1) -> A' and C, but B varies. Common: A'C. Group 2: minterms 6, 7 (AB=11; C=0,1) -> AB. Group 3: minterms 3, 7 (AB=01,11; C=1) -> BC.

Essential: A'C covers {1,3}. AB covers {6,7}. BC is redundant (3 covered by A'C, 7 by AB).

F = A'C + AB

Problem 6.2.6

Given: A 5-variable Boolean function F(A, B, C, D, E) = sum of minterms (0, 2, 4, 6, 16, 18, 20, 22).

Find: Identify the pattern and write the simplified expression without a K-map.

Solution: List the minterms in binary (ABCDE): m0 = 00000 m2 = 00010 m4 = 00100 m6 = 00110 m16 = 10000 m18 = 10010 m20 = 10100 m22 = 10110

Pattern analysis: - Bit E (LSB) is always 0 in all minterms - Bit B is always 0 in all minterms - Bits A, C, D vary freely

E = 0 -> E' is present in all terms B = 0 -> B' is present in all terms

F = B'E'

Verification: B'E' is true when B = 0 and E = 0. There are $2^3 = 8$ combinations of A, C, D, giving minterms where ABCDE has B=0, E=0: {0, 2, 4, 6, 16, 18, 20, 22}. All 8 minterms match. Confirmed.

Chapter 6 — Section 6.3: Logic Gates

Practice problems covering AND, OR, NOT, NAND, NOR, XOR gates, logic families, voltage levels, noise margins, and universal gate implementations.

Problem 6.3.1

Given: A digital system requires the function Y = ABCD (4-input AND). The only available components are 2-input AND gates with a propagation delay of t_{pd} = 4 ns each.

Find: (a) The minimum number of 2-input AND gates required. (b) Two different gate arrangements (cascaded chain vs. balanced tree). (c) The propagation delay of each arrangement.

Solution:

(a) A 4-input AND requires three 2-input AND gates (each gate reduces two inputs to one; 4 inputs need 3 reductions).

Minimum gates: 3

(b) Arrangement 1 — cascaded chain: Gate 1: G_1 = A * B Gate 2: G_2 = G_1 * C = ABC Gate 3: Y = G_2 * D = ABCD Delay = 3 stages x 4 ns = 12 ns

Arrangement 2 — balanced tree: Gate 1: G_1 = A * B (level 1) Gate 2: G_2 = C * D (level 1, in parallel) Gate 3: Y = G_1 * G_2 = ABCD (level 2) Delay = 2 levels x 4 ns = 8 ns

(c) The balanced tree is faster by 12 - 8 = 4 ns (33% reduction) because gates at the same level operate in parallel. The tree structure is preferred in high-speed designs to minimize critical path depth.

Problem 6.3.2

Given: A 3-input OR gate has inputs A, B, C. The gate is built from CMOS transistors operating at V_{DD} = 3.3 V. The CMOS implementation uses a NOR gate (3 series PMOS + 3 parallel NMOS) followed by an inverter. Each NMOS transistor has on-resistance R_n = 500 Ω and each PMOS transistor has on-resistance R_p = 1000 Ω.

Find: (a) The truth table output for all 8 input combinations. (b) The number of input combinations that produce a high output. (c) The worst-case pull-down resistance of the NOR stage. (d) The worst-case pull-up resistance of the NOR stage.

Solution:

(a) Truth table for Y = A + B + C:

A	B	C	Y
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

(b) Number of high outputs = 2^3 - 1 = 7 out of 8 combinations

(c) NOR stage pull-down: the NMOS transistors are in parallel. Worst case for pull-down is when only one NMOS is on (one input high): $R_{pd,worst} = R_n = 500\ \Omega$

Best case (all three on): $R_{pd,best} = R_n / 3 = 500 / 3 = 167\ \Omega$

(d) NOR stage pull-up: the PMOS transistors are in series (all three must be on, meaning all inputs low): $R_{pu} = 3 \text{ x } R_p = 3 \text{ x } 1000 = 3000\ \Omega$

The high pull-up resistance makes the low-to-high transition slower than the high-to-low transition, a characteristic of multi-input CMOS NOR gates.

Problem 6.3.3

Given: A CMOS inverter has $V_{DD} = 5.0$ V. The switching threshold is at $V_{DD}/2 = 2.5$ V. The inverter drives a load capacitance $C_L = 15$ pF. The output drive current during switching is $I_{avg} = 2$ mA.

Find: (a) The approximate output transition time (0 to V_{DD}). (b) The dynamic power dissipation at a switching frequency of 50 MHz. (c) The propagation delay if t_{pd} is approximated as half the transition time.

Solution:

(a) Transition time using I = C dV/dt, so dt = C dV / I: $t_{transition} = C_L \text{ x } V_{DD} / I_{avg} = 15 \text{ x } 10^{-12} \text{ x } 5.0 / (2 \text{ x } 10^{-3})$ $t_{transition} = 75 \text{ x } 10^{-12} / 2 \text{ x } 10^{-3} = 37.5 \text{ x } 10^{-9}$ s = 37.5 ns

(b) Dynamic power: $P_{dyn} = C_L \text{ x } V_{DD}^2 \text{ x } f = 15 \text{ x } 10^{-12} \text{ x } (5.0)^2 \text{ x } 50 \text{ x } 10^6$ $P_{dyn} = 15 \text{ x } 10^{-12} \text{ x } 25 \text{ x } 50 \text{ x } 10^6 = 18{,}750 \text{ x } 10^{-6} = 18.75$ mW

(c) Propagation delay: $t_{pd} \approx t_{transition} / 2 = 37.5 / 2 = 18.75$ ns

This is the delay from input crossing the threshold to the output crossing the threshold.

Problem 6.3.4

Given: An engineer needs to implement the function Y = A * B using only 2-input NAND gates. The NAND gate propagation delay is t_{pd} = 3.5 ns.

Find: (a) The circuit implementation showing all gates. (b) The total number of NAND gates required. (c) The total propagation delay from input to output.

Solution:

(a) Since Y = A * B = ((A * B)')' = NAND(NAND(A,B), NAND(A,B)): Gate 1: N_1 = (A * B)' [NAND of A and B] Gate 2: Y = (N_1 * N_1)' = (N_1)' = A * B [NAND used as inverter with both inputs tied to N_1]

Alternatively, using the identity: Gate 1: N_1 = NAND(A, B) = (AB)' Gate 2: Y = NAND(N_1, N_1) = (N_1)' = ((AB)')' = AB

(b) Total NAND gates: 2

(c) Total propagation delay = 2 x t_{pd} = 2 x 3.5 = 7.0 ns

Note: A single NAND gate produces (AB)', so one additional NAND-as-inverter stage is needed to restore the true AND function. This is why NAND gates are preferred as the basic building block in CMOS — the AND function always costs one extra gate delay.

Problem 6.3.5

Given: A system must implement the XOR function Y = A XOR B using only 2-input NOR gates. Each NOR gate has a propagation delay of 4 ns.

Find: (a) The Boolean derivation showing how to express XOR using NOR operations. (b) The circuit implementation with gate count. (c) The total propagation delay.

Solution:

(a) Derivation: Y = A XOR B = A'B + AB'

Using NOR gates, first generate complements: A' = NOR(A, A) B' = NOR(B, B)

Now express A'B + AB': Note that NOR(A', B') = (A' + B')' = AB (by De Morgan's). That gives AND, not what we need.

Instead, use the identity: A'B + AB' = ((A'B + AB')')' The inner expression: (A'B + AB')' = NOR(A'B, AB')

We need to form A'B and AB'. Using NOR: A'B: NOR(A, B') gives (A + B')'. That is not A'B.

A better approach: Let P = NOR(A, B) = (A + B)' = A'B' Let Q = NOR(A', B') = (A' + B')' = AB Then Y = A'B + AB' and note: NOR(P, Q) = (P + Q)' = (A'B' + AB)' = ((A XNOR B))' = A XOR B

So: Gate 1: A' = NOR(A, A) Gate 2: B' = NOR(B, B) Gate 3: P = NOR(A, B) = A'B' Gate 4: Q = NOR(A', B') = AB Gate 5: Y = NOR(P, Q) = (A'B' + AB)' = A'B + AB' = A XOR B

(b) Total NOR gates: 5

(c) Critical path: A -> Gate 1 (A') -> Gate 4 (Q) -> Gate 5 (Y) = 3 gate levels. Also: A -> Gate 3 (P) -> Gate 5 (Y) = 2 gate levels. The longest path is 3 levels (through the inverter).

Total propagation delay = 3 x 4 = 12 ns

Problem 6.3.6

Given: A 4-bit even parity generator uses XOR gates to compute parity bit P from data bits $D_3D_2D_1D_0$. Each 2-input XOR gate has a propagation delay of 6 ns. The data word is $D_3D_2D_1D_0$ = 1001.

Find: (a) The parity bit P. (b) The complete 5-bit transmitted word. (c) The propagation delay using a cascaded (chain) structure. (d) The propagation delay using a balanced tree structure. (e) Verify that the received word has even parity.

Solution:

(a) P = D_3 XOR D_2 XOR D_1 XOR D_0

Using a tree structure: Level 1: X_1 = D_3 XOR D_2 = 1 XOR 0 = 1 Level 1: X_2 = D_1 XOR D_0 = 0 XOR 1 = 1 Level 2: P = X_1 XOR X_2 = 1 XOR 1 = 0

Parity bit: P = 0

(b) Transmitted word = $D_3D_2D_1D_0P$ = 10010

(c) Cascaded chain: 3 XOR gates in series (D_3 XOR D_2, then XOR D_1, then XOR D_0). Delay = 3 x 6 = 18 ns

(d) Balanced tree: 2 levels (two XOR gates in parallel at level 1, one at level 2). Delay = 2 x 6 = 12 ns

(e) Verification: count the 1s in 10010 -> positions 4 and 1 have 1s -> two 1s total. Two is even, so even parity is confirmed.

Problem 6.3.7

Given: A 74HC04 CMOS hex inverter operates at V_{DD} = 5 V with the following specifications: V_{OH} = 4.9 V, V_{OL} = 0.1 V, V_{IH} = 3.5 V, V_{IL} = 1.5 V. It drives a 74LVC04 CMOS inverter operating at V_{DD} = 3.3 V with specifications: V_{IH} = 2.0 V, V_{IL} = 0.8 V, absolute maximum input voltage = 4.3 V.

Find: (a) The noise margins of the 74HC04. (b) Whether the 74HC04 can directly drive the 74LVC04 for logic-low outputs. (c) Whether the 74HC04 can directly drive the 74LVC04 for logic-high outputs. (d) A solution if direct driving is not safe.

Solution:

(a) 74HC04 noise margins: NM_H = V_{OH} - V_{IH} = 4.9 - 3.5 = 1.4 V NM_L = V_{IL} - V_{OL} = 1.5 - 0.1 = 1.4 V

(b) Logic-low: V_{OL}(74HC04) = 0.1 V < V_{IL}(74LVC04) = 0.8 V. The low output is well within the acceptable range. Compatible for logic low.

(c) Logic-high: V_{OH}(74HC04) = 4.9 V > V_{max}(74LVC04) = 4.3 V. The 5 V output exceeds the absolute maximum rating of the 3.3 V device by 0.6 V. This will damage the 74LVC04. Not safe for direct connection.

(d) Solutions:

1. Use a resistive voltage divider to reduce the high output to below 3.3 V.
2. Insert a bidirectional level shifter (e.g., TXS0108E) between the two devices.
3. Use a series resistor (e.g., 1 kΩ) to limit current through the 74LVC04 input clamping diodes, provided the data sheet allows this approach.

A level-shifting buffer is the recommended solution.

Problem 6.3.8

Given: A 74HC00 CMOS NAND gate operates at V_{DD} = 3.3 V with C_{pd} = 24 pF (internal power dissipation capacitance) and drives an external load of C_L = 30 pF. The quiescent supply current is I_{CC} = 4 μA. The gate switches at f = 10 MHz.

Find: (a) The static power dissipation. (b) The dynamic power dissipation due to output load switching. (c) The dynamic power dissipation due to internal switching. (d) The total power dissipation.

Solution:

(a) Static power: $P_{static} = V_{DD} \text{ x } I_{CC} = 3.3 \text{ x } 4 \text{ x } 10^{-6} = 13.2\ \mu W$

(b) Dynamic power from external load: $P_{load} = C_L \text{ x } V_{DD}^2 \text{ x } f = 30 \text{ x } 10^{-12} \text{ x } (3.3)^2 \text{ x } 10 \text{ x } 10^6$ $P_{load} = 30 \text{ x } 10^{-12} \text{ x } 10.89 \text{ x } 10 \text{ x } 10^6 = 3{,}267 \text{ x } 10^{-6} = 3.267$ mW

(c) Dynamic power from internal capacitance: $P_{internal} = C_{pd} \text{ x } V_{DD}^2 \text{ x } f = 24 \text{ x } 10^{-12} \text{ x } 10.89 \text{ x } 10 \text{ x } 10^6$ $P_{internal} = 2{,}613.6 \text{ x } 10^{-6} = 2.614$ mW

(d) Total power: $P_{total} = P_{static} + P_{load} + P_{internal} = 0.013 + 3.267 + 2.614 = 5.894$ mW

The static power is negligible compared to dynamic power at 10 MHz. At higher frequencies dynamic power dominates even more, which is why reducing V_{DD} is so effective (power scales as V_{DD}^2).

Problem 6.3.9

Given: A CMOS inverter at V_{DD} = 1.8 V drives a fan-out of 8 identical gate inputs. Each gate input has an input capacitance of C_{in} = 5 pF. The driving inverter has a maximum output current of I_{OH} = 4 mA (sourcing) and I_{OL} = 8 mA (sinking). The system operates at f = 100 MHz.

Find: (a) The total load capacitance. (b) The rise time of the output (10% to 90% of V_{DD}) assuming constant current charging. (c) The fall time of the output. (d) The dynamic power dissipated driving the fan-out load. (e) Whether the fan-out is acceptable if the maximum allowable rise/fall time is 5 ns.

Solution:

(a) Total load capacitance: C_{total} = 8 x C_{in} = 8 x 5 = 40 pF

(b) Rise time (charging from 0.1 V_{DD} to 0.9 V_{DD}): ΔV = 0.8 x V_{DD} = 0.8 x 1.8 = 1.44 V t_r = C_{total} x ΔV / I_{OH} = 40 x 10^{-12} x 1.44 / (4 x 10^{-3}) t_r = 57.6 x 10^{-12} / 4 x 10^{-3} = 14.4 x 10^{-9} = 14.4 ns

(c) Fall time (discharging from 0.9 V_{DD} to 0.1 V_{DD}): t_f = C_{total} x ΔV / I_{OL} = 40 x 10^{-12} x 1.44 / (8 x 10^{-3}) t_f = 57.6 x 10^{-12} / 8 x 10^{-3} = 7.2 x 10^{-9} = 7.2 ns

(d) Dynamic power: P_{dyn} = C_{total} x V_{DD}^2 x f = 40 x 10^{-12} x $(1.8)^2$ x 100 x 10^6 P_{dyn} = 40 x 10^{-12} x 3.24 x 10^8 = 12.96 mW

(e) The rise time of 14.4 ns exceeds the 5 ns limit. The fall time of 7.2 ns also exceeds the limit. The fan-out of 8 is not acceptable at this speed. The maximum fan-out for t_r <= 5 ns: N_{max} = I_{OH} x $t_{r,max}$ / (ΔV x C_{in}) = 4 x 10^{-3} x 5 x 10^{-9} / (1.44 x 5 x 10^{-12}) = 20 x 10^{-12} / 7.2 x 10^{-12} = 2.78

Maximum fan-out for the rise time constraint is 2 gates. A buffer or larger driver is needed for a fan-out of 8.

Problem 6.3.10

Given: A 5 V TTL system (74LS series) must interface with a 3.3 V LVCMOS device. The TTL gate has V_{OH} = 2.7 V, V_{OL} = 0.5 V, I_{OH} = -0.4 mA, I_{OL} = 8 mA. The LVCMOS device requires V_{IH} = 2.0 V and V_{IL} = 0.8 V. Each LVCMOS input draws I_{IH} = 1 μA and I_{IL} = -1 μA.

Find: (a) The noise margins for the TTL-to-LVCMOS interface. (b) The DC fan-out of the TTL gate driving LVCMOS inputs. (c) Whether a pull-up resistor to 3.3 V is needed, and if so, the appropriate value. (d) The noise margins with the pull-up in place.

Solution:

(a) Without pull-up: NM_H = V_{OH}(TTL) - V_{IH}(LVCMOS) = 2.7 - 2.0 = 0.7 V NM_L = V_{IL}(LVCMOS) - V_{OL}(TTL) = 0.8 - 0.5 = 0.3 V

Both margins are positive, so the interface technically works, but the margins are thin.

(b) DC fan-out: For high state: N = $|I_{OH}|$ / I_{IH} = 0.4 x 10^{-3} / 1 x 10^{-6} = 400 For low state: N = I_{OL} / $|I_{IL}|$ = 8 x 10^{-3} / 1 x 10^{-6} = 8000

DC fan-out is limited by the high state: 400 gates (effectively unlimited for practical circuits).

(c) A pull-up resistor to 3.3 V improves the high-state noise margin. When the TTL output is high, the pull-up helps raise V_{OH} closer to 3.3 V. When the TTL output is low, the pull-up resistor must not exceed the sink current capability: R_{min} = V_{pullup} / I_{OL} = 3.3 / 8 x 10^{-3} = 412.5 Ω

Use a standard value of R_{pullup} = 4.7 kΩ (draws only 3.3/4700 = 0.70 mA in low state, well within the 8 mA sink capability).

Yes, a pull-up resistor is recommended to improve the high-level noise margin.

(d) With the 4.7 kΩ pull-up to 3.3 V, the high output approaches 3.3 V: NM_H = 3.3 - 2.0 = 1.3 V NM_L = 0.8 - 0.5 = 0.3 V (unchanged)

The pull-up nearly doubles the high noise margin from 0.7 V to 1.3 V.

Chapter 6 — Section 6.4: Combinational Circuits

Practice problems covering adders, multiplexers, decoders, encoders, carry-lookahead logic, and combinational circuit design.

Problem 6.4.1

Given: A full adder has inputs A = 1, B = 1, and C_{in} = 1.

Find: (a) The sum bit S and carry-out C_{out}. (b) The Boolean expressions for S and C_{out} in terms of A, B, C_{in}. (c) The gate count required to implement one full adder using AND, OR, and XOR gates.

Solution:

(a) S = A XOR B XOR C_{in} = 1 XOR 1 XOR 1 = 0 XOR 1 = 1 C_{out} = AB + AC_{in} + BC_{in} = (1)(1) + (1)(1) + (1)(1) = 1 + 1 + 1 = 1

The result represents 1 + 1 + 1 = 3 in decimal = 11 in binary (S = 1, C_{out} = 1).

(b) Boolean expressions: S = A XOR B XOR C_{in} C_{out} = AB + C_{in}(A XOR B)

The second form of C_{out} uses the generate (G = AB) and propagate (P = A XOR B) signals: C_{out} = G + P * C_{in}.

(c) Gate count:

- 2 XOR gates (for S: A XOR B, then result XOR C_{in})
- 2 AND gates (for AB and (A XOR B) * C_{in})
- 1 OR gate (for G + P * C_{in})

Total: 2 XOR + 2 AND + 1 OR = 5 gates

Problem 6.4.2

Given: A 16-bit ripple-carry adder is built from full adders. Each full adder has a carry propagation delay of t_{carry} = 5 ns and a sum generation delay of t_{sum} = 7 ns (measured from C_{in} to S).

Find: (a) The worst-case delay for the carry to propagate from C_{in} of bit 0 to C_{out} of bit 15. (b) The total delay for the MSB sum bit S_{15}. (c) The maximum operating frequency. (d) How much faster a 4-bit carry-lookahead adder (CLA) block with t_{CLA} = 8 ns would be if four CLA blocks are cascaded for 16 bits.

Solution:

(a) Carry ripple delay through 16 stages: $t_{carry,total}$ = 16 x t_{carry} = 16 x 5 = 80 ns

(b) MSB sum delay = carry propagation through 15 stages + sum generation at stage 15: t_{S15} = 15 x t_{carry} + t_{sum} = 15 x 5 + 7 = 75 + 7 = 82 ns

(c) Maximum frequency (limited by worst-case carry-out delay): f_{max} = 1 / $t_{carry,total}$ = 1 / 80 ns = 12.5 MHz

(d) With 4-bit CLA blocks, the carry output of each block is available after one CLA delay. Four cascaded CLA blocks: $t_{CLA,total}$ = 4 x t_{CLA} = 4 x 8 = 32 ns Add t_{sum} for the final bit: 32 + 7 = 39 ns

Speed improvement: 82 / 39 = 2.10x Time saved: 82 - 39 = 43 ns (2.1x faster)

Problem 6.4.3

Given: A 4-bit carry-lookahead adder adds A = 1011 and B = 0111. The generate and propagate signals for each bit position are $G_i = A_i * B_i$ and $P_i = A_i$ XOR B_i. The input carry is $C_0 = 0$.

Find: (a) The generate and propagate signals for each bit. (b) The carry signals C_1 through C_4 using the CLA equations. (c) The sum bits $S_3S_2S_1S_0$. (d) Verify the result in decimal.

Solution:

(a) Generate and propagate for each bit: Bit 0: $G_0 = A_0 * B_0 = 1 * 1 = 1$, $P_0 = A_0$ XOR B_0 = 1 XOR 1 = 0 Bit 1: $G_1 = A_1 * B_1 = 1 * 1 = 1$, $P_1 = A_1$ XOR B_1 = 1 XOR 1 = 0 Bit 2: $G_2 = A_2 * B_2 = 0 * 1 = 0$, $P_2 = A_2$ XOR B_2 = 0 XOR 1 = 1 Bit 3: $G_3 = A_3 * B_3 = 1 * 0 = 0$, $P_3 = A_3$ XOR B_3 = 1 XOR 0 = 1

(b) CLA carry equations: $C_1 = G_0 + P_0C_0 = 1 + 0 \text{ x } 0 = 1$ $C_2 = G_1 + P_1G_0 + P_1P_0C_0 = 1 + 0 + 0 = 1$ $C_3 = G_2 + P_2G_1 + P_2P_1G_0 + P_2P_1P_0C_0 = 0 + 1 \text{ x } 1 + 0 + 0 = 1$ $C_4 = G_3 + P_3G_2 + P_3P_2G_1 + P_3P_2P_1G_0 + P_3P_2P_1P_0C_0$ $C_4 = 0 + 0 + 1 \text{ x } 1 \text{ x } 1 + 0 + 0 = 1$

(c) Sum bits: $S_i = P_i$ XOR C_i $S_0 = P_0$ XOR C_0 = 0 XOR 0 = 0 $S_1 = P_1$ XOR C_1 = 0 XOR 1 = 1 $S_2 = P_2$ XOR C_2 = 1 XOR 1 = 0 $S_3 = P_3$ XOR C_3 = 1 XOR 1 = 0

Result: $C_4S_3S_2S_1S_0$ = 1 0010 = 10010 binary

(d) Decimal verification: A = 1011 = 11, B = 0111 = 7 11 + 7 = 18 = 10010 binary. Confirmed.

Problem 6.4.4

Given: A 4:1 multiplexer with select lines S_1S_0 is used to implement the Boolean function F(A, B, C) = sum of minterms (0, 2, 5, 7). Variables A and B are connected to S_1 and S_0 respectively. Variable C is available as a data input.

Find: (a) The data input connections D_0 through D_3 in terms of C, C', 0, or 1. (b) Verify the implementation produces the correct output for all 8 input combinations.

Solution:

(a) Group minterms by the select line values (A, B):

S_1S_0 = AB = 00: minterms 0 (C=0) and 1 (C=1). F is 1 for m_0 only -> D_0 = C' S_1S_0 = AB = 01: minterms 2 (C=0) and 3 (C=1). F is 1 for m_2 only -> D_1 = C' S_1S_0 = AB = 10: minterms 4 (C=0) and 5 (C=1). F is 1 for m_5 only -> D_2 = C S_1S_0 = AB = 11: minterms 6 (C=0) and 7 (C=1). F is 1 for m_7 only -> D_3 = C

(b) Verification:

A	B	C	S_1S_0	D selected	F
0	0	0	00	D_0 = C' = 1	1 (m_0)
0	0	1	00	D_0 = C' = 0	0 (m_1)
0	1	0	01	D_1 = C' = 1	1 (m_2)
0	1	1	01	D_1 = C' = 0	0 (m_3)
1	0	0	10	D_2 = C = 0	0 (m_4)
1	0	1	10	D_2 = C = 1	1 (m_5)
1	1	0	11	D_3 = C = 0	0 (m_6)
1	1	1	11	D_3 = C = 1	1 (m_7)

F = 1 for minterms {0, 2, 5, 7}. Confirmed — matches the specification.

Problem 6.4.5

Given: A 3-to-8 decoder with active-low outputs and an active-low enable (EN') is used to implement two Boolean functions: F_1(A, B, C) = sum of minterms (1, 3, 5) F_2(A, B, C) = sum of minterms (0, 2, 6) Inputs A, B, C are connected to the decoder inputs I_2, I_1, I_0 respectively. EN' is tied to ground (always enabled).

Find: (a) Which decoder outputs are needed for F_1. (b) Which decoder outputs are needed for F_2. (c) The external gate type and connections for each function. (d) The total gate count (decoder + external gates).

Solution:

(a) F_1 uses minterms 1, 3, 5. With active-low outputs, output Y_i' goes low when input equals i. The needed outputs are: Y_1', Y_3', Y_5' (active low for minterms 1, 3, 5)

(b) F_2 uses minterms 0, 2, 6: Y_0', Y_2', Y_6' (active low for minterms 0, 2, 6)

(c) Since the decoder outputs are active-low, each output $Y_i' = 0$ when minterm i is selected. To OR the minterms, use NAND gates on the active-low outputs (because NAND of active-low signals implements OR of the active-high equivalents):

$F_1 = \text{NAND}(Y_1', Y_3', Y_5') = (Y_1' * Y_3' * Y_5')'$

When any $Y_i' = 0$, the NAND output goes high, which is the desired OR behavior.

F_1: 3-input NAND gate with inputs Y_1', Y_3', Y_5' F_2: 3-input NAND gate with inputs Y_0', Y_2', Y_6'

(d) Gate count:

- 1 decoder (3-to-8): typically contains 8 AND gates + 3 inverters internally
- 2 external NAND gates (one 3-input for F_1, one 3-input for F_2)

Total external gates: 2 NAND gates plus the decoder

Problem 6.4.6

Given: An 8-to-3 priority encoder has 8 inputs I_7 (highest priority) through I_0 (lowest priority), a 3-bit output $Y_2Y_1Y_0$, and a valid output V. At a given instant, inputs I_6, I_4, I_2, and I_0 are all asserted (logic high) while I_7, I_5, I_3, and I_1 are deasserted.

Find: (a) The output code $Y_2Y_1Y_0$ and the valid bit V. (b) The Boolean expression for the MSB output Y_2 as a function of I_7 through I_4. (c) If the priority encoder is used as an interrupt controller handling 8 interrupt sources, how many bits are needed to mask (enable/disable) individual interrupts, and sketch the logic for one masked input.

Solution:

(a) Active inputs: I_6, I_4, I_2, I_0. The highest priority among these is I_6. Output code encodes 6: $Y_2Y_1Y_0 = 110$ Valid bit: $V = 1$ (at least one input is active)

(b) $Y_2 = 1$ when the highest active input is I_4, I_5, I_6, or I_7: $Y_2 = I_7 + I_7'I_6 + I_7'I_6'I_5 + I_7'I_6'I_5'I_4$ Simplified: $Y_2 = I_7 + I_6 + I_5 + I_4$ (Y_2 is high if any of the upper 4 inputs is active, since their codes all have bit 2 = 1)

Wait — this overcounts. Y_2 should reflect the highest-priority active input's code bit 2. For a standard priority encoder: $Y_2 = I_7 + I_7'I_6 + I_7'I_6'I_5 + I_7'I_6'I_5'I_4$

This simplifies to: $Y_2 = I_4 + I_5 + I_6 + I_7$ because if any of I_4-I_7 is active, the encoded output is 4-7 (all have $Y_2 = 1$), and if the highest active input is I_3 or below, $Y_2 = 0$.

(c) An 8-bit mask register is needed, one bit per interrupt source: 8 bits. For masked input i: $I_{i,\text{masked}} = I_i$ AND M_i, where M_i is the mask bit (1 = enabled, 0 = disabled). This requires one 2-input AND gate per interrupt input, placed between the interrupt source and the priority encoder input.

Problem 6.4.7

Given: A BCD-to-7-segment decoder drives a common-cathode display. The BCD input is 0111 (decimal 7). The seven segments are labeled a through g (top, upper-right, lower-right, bottom, lower-left, upper-left, middle).

Find: (a) Which segments should be lit for decimal 7. (b) The truth table entries for segments a, b, and c for BCD inputs 0 through 9. (c) The Boolean expression for segment a using a K-map (treating BCD values 10-15 as don't-cares).

Solution:

(a) For decimal 7, the active segments are: Segment a (top): ON Segment b (upper-right): ON Segment c (lower-right): ON Segments d, e, f, g: OFF

Display shows: three segments lit (a, b, c) forming the digit 7.

(b) Truth table for segments a, b, c (1 = lit):

BCD ($D_3D_2D_1D_0$)	Decimal	a	b	c
0000	0	1	1	1
0001	1	0	1	1
0010	2	1	1	0
0011	3	1	1	1
0100	4	0	1	1
0101	5	1	0	1
0110	6	1	0	1
0111	7	1	1	1
1000	8	1	1	1
1001	9	1	1	1

(c) K-map for segment a (D_3D_2 on rows, D_1D_0 on columns), with don't-cares for 10-15:

```
          D₁D₀=00  D₁D₀=01  D₁D₀=11  D₁D₀=10
D₃D₂=00 |   1    |   0    |   1    |   1    |
D₃D₂=01 |   0    |   1    |   1    |   1    |
D₃D₂=11 |   X    |   X    |   X    |   X    |
D₃D₂=10 |   1    |   1    |   X    |   X    |
```

Groups: - Quad: cells (0,2,8,10) using don't-cares -> $D_1' * D_0'$ ... checking: these are at D_1D_0=00 and D_1D_0=10 for D_3D_2=00 and D_3D_2=10. Common: D_2' and D_0'. Term: $D_2'D_0'$. Wait, cell 4 (D_3D_2=01, D_1D_0=00) = 0, so we can't group 0 and 4.

Let me redo: minterms where a=1: {0, 2, 3, 5, 6, 7, 8, 9}, don't-cares: {10,11,12,13,14,15}.

Group 1: {0, 2, 8, 10} (D_3D_2=00/10, D_1D_0=00/10) -> common: $D_2'D_0'$. Includes don't-care 10. $D_2'D_0'$ Group 2: {2, 3, 6, 7} (D_3D_2=00/01, D_1D_0=11/10) -> common: D_1. D_1 Group 3: {5, 7, 13, 15} -> common: D_2D_0. Using don't-cares. D_2D_0 Group 4: {8, 9, 10, 11} -> D_3 covers all. D_3

Check coverage: $D_2'D_0'$ covers {0, 8, 2, *10*} — wait, let me be more careful.

Minterm 0 = 0000: $D_2'D_0'$ covers it (D_2=0, D_0=0). Yes. Minterm 5 = 0101: D_2D_0 covers it (D_2=1, D_0=1). Yes. Minterm 3 = 0011: D_1 covers it (D_1=1). Yes. All minterms covered.

$a = D_3 + D_1 + D_2'D_0' + D_2D_0$

This can also be written as: $a = D_3 + D_1 + (D_2 \text{ XNOR } D_0)$

Problem 6.4.8

Given: A 4-bit binary subtractor is built using full adders and the two's complement method. The minuend is A = 1100 (12 decimal) and the subtrahend is B = 0101 (5 decimal). The subtractor computes A - B by adding A to the two's complement of B.

Find: (a) The one's complement of B. (b) The two's complement of B using the adder (invert B and set C_{in} = 1). (c) The result of A + B' + 1 showing the carry chain. (d) The decimal result and the overflow status.

Solution:

(a) One's complement of B: invert each bit. B = 0101, so B' = 1010

(b) Two's complement is formed by the adder: A + B' with C_{in} = 1. This computes A + B' + 1 = A + (-B) = A - B.

(c) Addition with carry chain:

```
  1100   (A = 12)
+ 1010   (B' = one's complement of 5)
+    1   (C_in = 1)
------
```

Bit 0: 0 + 0 + 1 = 1, C_1 = 0 Bit 1: 0 + 1 + 0 = 1, C_2 = 0 Bit 2: 1 + 0 + 0 = 1, C_3 = 0 Bit 3: 1 + 1 + 0 = 0, C_4 = 1

Result: C_4 = 1, Sum = 0111

(d) In two's complement subtraction, C_{out} = 1 indicates no borrow (result is positive). The 4-bit result is 0111 = 7 decimal. A - B = 12 - 5 = 7. Confirmed.

Overflow check: C_3 XOR C_4 = 0 XOR 1 = 1? No — C_3 = 0 (carry into bit 3) and C_4 = 1 (carry out of bit 3). Overflow = C_3 XOR C_4 = 0 XOR 1 = 1. But this indicates signed overflow.

For unsigned subtraction: C_{out} = 1 means no borrow. For signed: both operands are positive and the result (7) is positive and fits in 4 bits (range -8 to +7). Since the MSB of the result is 0, there is no signed overflow. The XOR check applies to the carries into and out of the MSB: C_3 (carry into bit 3) = 0, C_4 (carry out of bit 3) = 1, so C_3 XOR C_4 = 1 would suggest overflow, but we must recalculate.

Rechecking carries: Bit 2 gives C_3 = 0 (correct). Bit 3: 1 + 1 + 0 = 10, C_4 = 1. Overflow = C_3 XOR C_4 = 0 XOR 1 = 1. However, the result 0111 = +7 is the correct answer for 12 - 5 in unsigned. The overflow flag is meaningful only for signed interpretation. In 4-bit signed, A = 1100 = -4 (not +12), so signed A - B = -4 - 5 = -9, which overflows the range [-8, +7]. Unsigned result is correct (7); signed interpretation would overflow.

Problem 6.4.9

Given: A combinational circuit must compare two 4-bit unsigned numbers $A = A_3A_2A_1A_0$ and $B = B_3B_2B_1B_0$ and produce three outputs: G (A > B), E (A = B), and L (A < B). The comparison is done starting from the MSB.

Find: (a) The Boolean expression for the equality output E. (b) The expression for G (A greater than B) for a 1-bit comparator. (c) How 1-bit comparator stages are cascaded for the full 4-bit comparison. (d) Compute G, E, L for A = 1010 and B = 1001.

Solution:

(a) Equality requires all bit pairs to match: $E_i = A_i$ XNOR $B_i = (A_i$ XOR $B_i)'$ for each bit position.

$E = E_3 * E_2 * E_1 * E_0 = (A_3$ XNOR $B_3)(A_2$ XNOR $B_2)(A_1$ XNOR $B_1)(A_0$ XNOR $B_0)$

(b) For a single bit position i, $A_i > B_i$ when $A_i = 1$ and $B_i = 0$: $G_i = A_i * B_i'$

Similarly, $L_i = A_i' * B_i$.

(c) Cascading from MSB to LSB: $G = G_3 + E_3G_2 + E_3E_2G_1 + E_3E_2E_1G_0$

In words: A > B if $A_3 > B_3$, or if $A_3 = B_3$ and $A_2 > B_2$, etc. $L = L_3 + E_3L_2 + E_3E_2L_1 + E_3E_2E_1L_0$

(d) For A = 1010 and B = 1001: Bit 3: $A_3 = 1, B_3 = 1$ -> $G_3 = 0, E_3 = 1, L_3 = 0$ Bit 2: $A_2 = 0, B_2 = 0$ -> $G_2 = 0, E_2 = 1, L_2 = 0$ Bit 1: $A_1 = 1, B_1 = 0$ -> $G_1 = 1, E_1 = 0, L_1 = 0$ Bit 0: $A_0 = 0, B_0 = 1$ -> $G_0 = 0, E_0 = 0, L_0 = 1$

G = 0 + 1 x 0 + 1 x 1 x 1 + 1 x 1 x 0 x 0 = 0 + 0 + 1 + 0 = 1 E = 1 x 1 x 0 x 0 = 0 L = 0 + 1 x 0 + 1 x 1 x 0 + 1 x 1 x 0 x 1 = 0

Result: G = 1, E = 0, L = 0 (A > B). Verification: A = 1010 = 10, B = 1001 = 9, and 10 > 9. Confirmed.

Problem 6.4.10

Given: A combinational lock circuit accepts a 4-bit BCD input representing digits 0-9. The lock opens (output UNLOCK = 1) when the correct 3-digit code 4-7-2 is entered sequentially. However, the sequencing is handled externally — this problem concerns only the single-digit combinational comparator. The circuit compares the 4-bit BCD input $D_3D_2D_1D_0$ against a fixed 4-bit key value $K_3K_2K_1K_0$.

Find: (a) The Boolean expression for the MATCH output. (b) Implement the MATCH function for key digit 7 (K = 0111) using the minimum number of gates. (c) The gate count and gate types. (d) How to extend this to a 3-digit comparator with separate key registers.

Solution:

(a) MATCH is true when all 4 bits of D equal the corresponding bits of K: MATCH = $(D_3$ XNOR $K_3)(D_2$ XNOR $K_2)(D_1$ XNOR $K_1)(D_0$ XNOR $K_0)$

(b) For K = 0111 (key digit 7): $K_3 = 0$: D_3 XNOR 0 = D_3' (match when $D_3 = 0$) $K_2 = 1$: D_2 XNOR 1 = D_2 (match when $D_2 = 1$) $K_1 = 1$: D_1 XNOR 1 = D_1 (match when $D_1 = 1$) $K_0 = 1$: D_0 XNOR 1 = D_0 (match when $D_0 = 1$)

MATCH = D_3' * D_2 * D_1 * D_0

Implementation: Gate 1: Inverter on D_3 -> D_3' Gate 2: 4-input AND gate on D_3', D_2, D_1, D_0 Or with 2-input gates: Gate 1: NOT D_3 -> D_3' Gate 2: AND(D_3', D_2) -> W_1 Gate 3: AND(D_1, D_0) -> W_2 Gate 4: AND(W_1, W_2) -> MATCH

(c) Using 2-input gates: 1 NOT gate + 3 AND gates = 4 gates total Using a 4-input AND gate with one inverter: 1 NOT + 1 AND = 2 gates total

(d) For a 3-digit comparator:

- Three separate 4-bit comparators, one per digit position
- Each comparator has its own 4-bit key register (programmable or hardwired)
- $MATCH_1$ compares digit 1 input against $K^{(1)} = 0100$ (digit 4)
- $MATCH_2$ compares digit 2 input against $K^{(2)} = 0111$ (digit 7)
- $MATCH_3$ compares digit 3 input against $K^{(3)} = 0010$ (digit 2)
- UNLOCK = $MATCH_1$ AND $MATCH_2$ AND $MATCH_3$ (all three digits must match)

Total: 3 comparators + 1 AND gate for the final UNLOCK output.

Chapter 6 — Section 6.5: Sequential Circuits

Practice problems covering latches, flip-flops, counters, shift registers, and state machines.

Problem 6.5.1

Given: An SR latch is built from two cross-coupled NOR gates. The current state is Q = 0, Q' = 1. The following input sequence is applied:

Time	S	R
t_1	1	0
t_2	0	0
t_3	0	1
t_4	0	0
t_5	1	1

Find: (a) The output Q after each time step. (b) Why the S = R = 1 condition is forbidden. (c) What happens to Q and Q' during the forbidden state if both S and R return to 0 simultaneously.

Solution:

(a) NOR-based SR latch: S = 1 sets Q to 1; R = 1 resets Q to 0; S = R = 0 holds the previous state.

t_1: S = 1, R = 0 -> SET. Q = 1, Q' = 0. t_2: S = 0, R = 0 -> HOLD. Q = 1 (retains set value). t_3: S = 0, R = 1 -> RESET. Q = 0, Q' = 1. t_4: S = 0, R = 0 -> HOLD. Q = 0 (retains reset value). t_5: S = 1, R = 1 -> FORBIDDEN. Both NOR gates output 0: Q = 0, Q' = 0.

(b) The S = R = 1 condition is forbidden because it forces both Q = 0 and Q' = 0, violating the fundamental requirement that Q and Q' be complementary. The latch is no longer in a valid bistable state.

(c) When both S and R return to 0 simultaneously, both NOR gates see inputs (0, 0). The output depends on which gate's propagation delay is slightly shorter — a race condition. The latch will settle into either Q = 1 or Q = 0, but the outcome is unpredictable (metastable). This is why the forbidden state must be avoided in reliable designs.

Problem 6.5.2

Given: A gated D latch (D latch with enable) is used in an 8-bit register. The enable signal goes high for 12 ns. During this window, the data input transitions from 0 to 1 at 4 ns after enable goes high, and back to 0 at 9 ns after enable goes high. The latch propagation delay is t_{pd} = 2 ns.

Find: (a) The output Q waveform with timestamps (assume Q = 0 initially). (b) The value stored when enable returns low at t = 12 ns. (c) Why this transparency behavior is problematic in multi-stage pipeline designs.

Solution:

(a) Timeline (t = 0 is when enable goes high):

t = 0 ns: Enable goes high. D = 0. Q remains 0 (transparent, follows D). t = 4 ns: D transitions to 1. After t_{pd} = 2 ns: t = 6 ns: Q transitions to 1. t = 9 ns: D transitions to 0. After t_{pd} = 2 ns: t = 11 ns: Q transitions to 0. t = 12 ns: Enable goes low. D = 0 at this instant. Q latches at 0.

(b) The stored value when enable goes low: Q = 0 (the latch captured D = 0 at the falling edge of enable).

(c) Transparency is problematic in pipelines because while the enable is high, any input change propagates through the latch to the output. In a multi-stage pipeline, data from stage N could ripple through to stage N+1 and even N+2 during the same clock phase, corrupting pipeline timing. This is why edge-triggered flip-flops are preferred for synchronous designs — they sample the input only at the clock edge, preventing transparency-related timing violations.

Problem 6.5.3

Given: A positive-edge-triggered D flip-flop has: - Setup time: t_{su} = 3 ns - Hold time: t_h = 1.5 ns - Clock-to-Q delay: t_{cq} = 2.5 ns

The clock period is T_{clk} = 8 ns and the clock frequency is 125 MHz. The flip-flop drives combinational logic that feeds another identical flip-flop.

Find: (a) The maximum combinational logic delay between the two flip-flops. (b) The minimum combinational logic delay to satisfy the hold time constraint. (c) The slack (timing margin) if the actual logic delay is 3.8 ns. (d) Whether the design meets timing at 150 MHz.

Solution:

(a) Setup time constraint: $t_{cq} + t_{logic,max} + t_{su} <= T_{clk}$ $t_{logic,max} = T_{clk} - t_{cq} - t_{su}$ = 8 - 2.5 - 3 = 2.5 ns

(b) Hold time constraint: $t_{cq} + t_{logic,min} >= t_h$ $t_{logic,min} = t_h - t_{cq}$ = 1.5 - 2.5 = -1.0 ns

Since this is negative, the constraint is automatically satisfied even with zero logic delay: $t_{logic,min}$ = 0 ns (hold time is met by t_{cq} alone).

(c) With t_{logic} = 3.8 ns: Total path delay = $t_{cq} + t_{logic} + t_{su}$ = 2.5 + 3.8 + 3.0 = 9.3 ns Slack = T_{clk} - total = 8.0 - 9.3 = -1.3 ns (negative slack)

The design fails timing with 3.8 ns of logic delay at 125 MHz. The logic must be reduced to 2.5 ns or less, or the clock frequency must be lowered.

(d) At 150 MHz: T_{clk} = 1 / 150 x 10^6 = 6.667 ns $t_{logic,max}$ = 6.667 - 2.5 - 3.0 = 1.167 ns

At 150 MHz, only 1.167 ns of combinational logic is allowed, which is extremely tight. The design does not meet timing at 150 MHz with the actual logic delay of 3.8 ns (slack = 6.667 - 9.3 = -2.633 ns).

Problem 6.5.4

Given: A JK flip-flop has inputs J and K and a clock input. The characteristic equation is $Q^+ = JQ' + K'Q$. The initial state is Q = 0. The following input sequence is applied over 6 clock cycles:

Clock	J	K
1	1	0
2	1	1
3	0	0
4	0	1
5	1	1
6	1	0

Find: (a) The output Q after each clock edge. (b) Which clock cycles correspond to Set, Reset, Toggle, and Hold operations. (c) The advantages of JK flip-flops over D flip-flops.

Solution:

(a) Apply $Q^+ = JQ' + K'Q$ at each rising edge:

Clock 1: J=1, K=0. Q^+ = 1 x 1 + 1 x 0 = 1. Q = 1 (SET) Clock 2: J=1, K=1. Q^+ = 1 x 0 + 0 x 1 = 0. Q = 0 (TOGGLE) Clock 3: J=0, K=0. Q^+ = 0 x 1 + 1 x 0 = 0. Q = 0 (HOLD) Clock 4: J=0, K=1. Q^+ = 0 x 1 + 0 x 0 = 0. Q = 0 (RESET, already 0) Clock 5: J=1, K=1. Q^+ = 1 x 1 + 0 x 0 = 1. Q = 1 (TOGGLE) Clock 6: J=1, K=0. Q^+ = 1 x 0 + 1 x 1 = 1. Q = 1 (SET, already 1)

(b) Operation summary:

- Set (J=1, K=0): clocks 1 and 6
- Toggle (J=1, K=1): clocks 2 and 5
- Hold (J=0, K=0): clock 3
- Reset (J=0, K=1): clock 4

(c) Advantages of JK over D: The JK flip-flop can toggle without external feedback (D flip-flops require connecting Q' back to D). The JK eliminates the forbidden state of the SR latch. However, D flip-flops are simpler (one input vs. two) and are more common in modern synchronous designs because synthesis tools optimize for D flip-flops.

Problem 6.5.5

Given: A synchronous 4-bit binary up-counter is clocked at f_{clk} = 20 MHz. It counts from 0000 to 1111 and then wraps around to 0000.

Find: (a) The total number of states. (b) The output frequency at each bit (Q_0, Q_1, Q_2, Q_3). (c) The time for one complete count cycle. (d) The counter state after 25 clock pulses starting from 0000. (e) The excitation equations for T flip-flop implementation.

Solution:

(a) A 4-bit counter has $2^4 = 16$ states (0 through 15).

(b) Each bit toggles at half the frequency of the previous bit: Q_0 (LSB): $f_0 = f_{clk} / 2 = 20 / 2 = 10$ MHz Q_1: $f_1 = f_{clk} / 4 = 20 / 4 = 5$ MHz Q_2: $f_2 = f_{clk} / 8 = 20 / 8 = 2.5$ MHz Q_3 (MSB): $f_3 = f_{clk} / 16 = 20 / 16 = 1.25$ MHz

(c) One complete cycle = 16 clock pulses: $T_{cycle} = 16 / f_{clk} = 16 / (20 \times 10^6) = 800 \times 10^{-9} = 800$ ns = 0.8 μs

(d) After 25 clock pulses: 25 mod 16 = 9. State = 9 = 1001 binary

(e) T flip-flop excitation ($T_i = 1$ means flip-flop i toggles): $T_0 = 1$ (Q_0 toggles every clock) $T_1 = Q_0$ (Q_1 toggles when $Q_0 = 1$) $T_2 = Q_0 * Q_1$ (Q_2 toggles when $Q_0 = Q_1 = 1$) $T_3 = Q_0 * Q_1 * Q_2$ (Q_3 toggles when $Q_0 = Q_1 = Q_2 = 1$)

General: $T_i = Q_0 * Q_1 * \ldots * Q_{i-1}$ for $i > 0$, and $T_0 = 1$.

Problem 6.5.6

Given: A modulo-10 (decade) counter is needed for a digital clock's seconds-ones digit. The counter uses 4 D flip-flops ($Q_3Q_2Q_1Q_0$) and must count from 0000 to 1001, then reset to 0000. The clock frequency is 1 Hz (one pulse per second).

Find: (a) The count sequence. (b) The reset detection logic. (c) The output frequency of Q_3 (MSB). (d) The output frequency that would drive the seconds-tens counter. (e) The total number of decade counters needed to display hours, minutes, and seconds on a 24-hour clock.

Solution:

(a) Count sequence: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, 1, ... In binary: 0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, (reset) 0000.

(b) Detect count 10 = 1010 and reset. However, in a well-designed synchronous counter, we detect when the next state would be 10 and instead load 0.

For an asynchronous-reset approach: detect the state after 9, which would momentarily be 1010. Reset = Q_3 AND Q_1 (since 1010 has $Q_3 = 1$ and $Q_1 = 1$, and this combination does not occur in any valid state 0-9).

Reset = $Q_3 * Q_1$

(c) Q_3 goes high at count 8 and returns low at count 0 (after 9). It is high for 2 clock periods out of 10. f_{Q3} is not a clean square wave, but the frequency of its rising edge is: $f = f_{clk} / 10 = 1 / 10 = 0.1$ Hz

(d) The carry/ripple output to the tens counter triggers once every 10 counts: $f_{carry} = 1$ Hz / 10 = 0.1 Hz (one pulse every 10 seconds)

(e) A 24-hour clock displays HH:MM:SS:

- Seconds ones: 1 decade counter (mod-10, 0-9)
- Seconds tens: 1 mod-6 counter (0-5)
- Minutes ones: 1 decade counter (mod-10, 0-9)
- Minutes tens: 1 mod-6 counter (0-5)
- Hours ones: 1 decade counter (mod-10, 0-9, reset at 4 when tens = 2)
- Hours tens: 1 mod-3 counter (0-2)

Total: 3 decade counters + 2 mod-6 counters + 1 mod-3 counter = 6 counters total (If counting only decade counters: 3 decade counters)

Problem 6.5.7

Given: An 8-bit Linear Feedback Shift Register (LFSR) uses the polynomial $x^8 + x^6 + x^5 + x^4 + 1$, meaning feedback taps are at bit positions 8, 6, 5, and 4 (XOR of these positions feeds back to the input). The initial seed value is 0000 0001.

Find: (a) The maximum sequence length. (b) The first 5 states of the LFSR (show the XOR feedback computation). (c) One application of this LFSR. (d) Why the all-zeros state must be avoided.

Solution:

(a) An n-bit LFSR with a maximal-length (primitive) polynomial produces a sequence of length $2^n - 1$ before repeating. Maximum sequence length = $2^8 - 1 = 255$ states

(b) The polynomial $x^8 + x^6 + x^5 + x^4 + 1$ means the feedback is: new bit = Q_8 XOR Q_6 XOR Q_5 XOR Q_4 (using 1-indexed positions where Q_8 is the MSB being shifted out and Q_1 is the LSB).

Using notation [Q_8 Q_7 Q_6 Q_5 Q_4 Q_3 Q_2 Q_1], shift right, feedback enters Q_8:

State 0: [0 0 0 0 0 0 0 1] Feedback = Q_8 XOR Q_6 XOR Q_5 XOR Q_4 = 0 XOR 0 XOR 0 XOR 0 = 0 State 1: [0 0 0 0 0 0 1 0] (shifted right, 0 enters MSB)

Wait — let me clarify the shift direction. In a standard LFSR, bits shift from MSB toward LSB (right), and feedback enters the MSB.

State 0: 0000 0001. $Q_8Q_7Q_6Q_5Q_4Q_3Q_2Q_1$ = 00000001. Feedback = Q_8 XOR Q_6 XOR Q_5 XOR Q_4 = 0 XOR 0 XOR 0 XOR 0 = 0. Shift right, feedback enters Q_8: State 1: 0000 0000 — but this is the all-zeros state, which is a lockup state.

The issue is that seed 00000001 with the MSB-out shifting convention produces zero feedback. Let me use the Fibonacci LFSR convention where Q_1 is the output bit:

Feedback = Q_8 XOR Q_6 XOR Q_5 XOR Q_4. Shift left, output exits Q_1, feedback enters Q_8... Actually, let's use a cleaner convention:

Register [b_1 b_2 b_3 b_4 b_5 b_6 b_7 b_8], shift right (b_1 is MSB input side): Feedback to b_1 = b_8 XOR b_6 XOR b_5 XOR b_4.

State 0: [0 0 0 0 0 0 0 1] (b_8 = 1) Feedback = 1 XOR 0 XOR 0 XOR 0 = 1 State 1: [1 0 0 0 0 0 0 0] Feedback = 0 XOR 0 XOR 0 XOR 0 = 0 State 2: [0 1 0 0 0 0 0 0] Feedback = 0 XOR 0 XOR 0 XOR 0 = 0 State 3: [0 0 1 0 0 0 0 0] Feedback = 0 XOR 0 XOR 0 XOR 0 = 0 State 4: [0 0 0 1 0 0 0 0] Feedback = 0 XOR 0 XOR 1 XOR 0 = 1

First 5 states: 00000001, 10000000, 01000000, 00100000, 00010000

(c) Applications: CRC (Cyclic Redundancy Check) calculation for error detection in communication protocols, pseudo-random number generation for built-in self-test (BIST), and data scrambling in serial communication links.

(d) The all-zeros state must be avoided because XOR of all zeros produces zero feedback, so the LFSR would remain stuck at 00000000 forever. This is why LFSRs are initialized with a nonzero seed value. The all-zeros state is the only excluded state in a maximal-length LFSR sequence.

Problem 6.5.8

Given: A 4-bit serial-in/parallel-out (SIPO) shift register receives data at 1 Mbps (1 bit per microsecond). The serial input sequence is: 1, 0, 1, 1, 0, 0, 1, 0 (transmitted LSB first). The register is initially cleared to 0000.

Find: (a) The register contents after each of the first 4 clock pulses. (b) The parallel output after 4 clocks (first nibble). (c) The parallel output after 8 clocks (second nibble). (d) The total time to receive one byte. (e) The equivalent parallel data rate.

Solution:

(a) Data shifts in from the left (MSB position), LSB first:

Clock 1: shift in 1 -> register = 1000 Clock 2: shift in 0 -> register = 0100

Wait — for SIPO with LSB first, the first bit received occupies the LSB position after the full word is loaded. Let me reconsider: as bits shift in, the first bit ends up in the rightmost position.

Shift right convention (new data enters from the left): Clock 1: shift in 1 -> [1 x x x] = 1000 Clock 2: shift in 0 -> [0 1 x x] = 0100 Clock 3: shift in 1 -> [1 0 1 x] = 1010 Clock 4: shift in 0 -> wait, the sequence is 1, 0, 1, 1... Clock 3: shift in 1 -> [1 0 1 x] = 1010 Clock 4: shift in 1 -> [1 1 0 1] = 1101

(b) After 4 clocks, parallel output = 1101 The first bit (1) is now in the LSB, and the fourth bit (1) is in the MSB. In decimal: 1101 = 13 (or interpreting as received LSB-first: the nibble is 1011 = 0xB = 11).

(c) Continuing with bits 5-8 (0, 0, 1, 0): Clock 5: shift in 0 -> [0 1 1 0] = 0110 Clock 6: shift in 0 -> [0 0 1 1] = 0011 Clock 7: shift in 1 -> [1 0 0 1] = 1001 Clock 8: shift in 0 -> [0 1 0 0] = 0100

Second nibble parallel output: 0100

(d) Total time for 8 bits at 1 Mbps: t = 8 x 1 μs = 8 μs

(e) Parallel data rate: one byte every 8 μs: Data rate = 8 bits / 8 μs = 1 Mbps serial = 125 kBytes/s The parallel output is available every 4 μs (per nibble) or 8 μs (per byte), with the same aggregate bit rate.

Problem 6.5.9

Given: A Moore state machine controls a traffic light at an intersection. The light cycles through Green (30 s), Yellow (5 s), and Red (35 s) for the main road. The cross street has the complementary pattern (Red when main is Green or Yellow, Green when main is Red, etc.). A 1 Hz clock drives a counter that generates timing signals.

Find: (a) The state diagram with states, outputs, and transitions. (b) The number of states and flip-flops required. (c) The state encoding (binary). (d) The counter terminal count values for each state. (e) The total cycle time.

Solution:

(a) State diagram for the main road:

State S0 (Main Green, Cross Red): Output = Main Green / Cross Red. After 30 s -> transition to S1.
State S1 (Main Yellow, Cross Red): Output = Main Yellow / Cross Red. After 5 s -> transition to S2.
State S2 (Main Red, Cross Green): Output = Main Red / Cross Green. After 30 s -> transition to S3.
State S3 (Main Red, Cross Yellow): Output = Main Red / Cross Yellow. After 5 s -> transition to S0.

(b) Number of states: 4 Flip-flops required: $\text{ceil}(\log_2(4)) = 2$ flip-flops (Q_1Q_0)

(c) State encoding: S0 = 00 (Main Green / Cross Red) S1 = 01 (Main Yellow / Cross Red) S2 = 10 (Main Red / Cross Green) S3 = 11 (Main Red / Cross Yellow)

(d) Using a counter clocked at 1 Hz: S0: count to 30, terminal count = 29 (counts 0 to 29) S1: count to 5, terminal count = 4 (counts 0 to 4) S2: count to 30, terminal count = 29 S3: count to 5, terminal count = 4

A 6-bit counter (max count 63) is sufficient for the longest interval.

(e) Total cycle time: $T_{cycle} = 30 + 5 + 30 + 5 = 70$ seconds

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Problem 6.5.10

Given: A Mealy state machine detects the bit sequence “1010” on a serial input line X. The output Z = 1 during the clock cycle in which the final bit of the sequence is received. The detector is overlapping (the last bits of one detection can be part of the next). The clock rate is 10 MHz.

Find: (a) The state diagram with 4 states. (b) The state transition table. (c) The next-state equations using D flip-flops with binary state encoding. (d) The output equation. (e) The maximum detection rate.

Solution:

(a) States based on progress toward "1010":

S0 = no progress (idle). S1 = received "1". S2 = received "10". S3 = received "101".

When in S3 and X = 0, the full sequence "1010" is detected (Z = 1), and the machine transitions to S2 (overlapping: "10" is a prefix of the next potential "1010").

(b) State transition table (Mealy: output depends on state and input):

Current State	X = 0	Z (X=0)	X = 1	Z (X=1)
S0 (00)	S0	0	S1	0
S1 (01)	S2	0	S1	0
S2 (10)	S0	0	S3	0
S3 (11)	S2	1	S1	0

(c) State encoding: S0=00, S1=01, S2=10, S3=11 (Q_1Q_0).

Next-state equations from the table: Q_1^+: is 1 in transitions to S2 (10) and S3 (11). - S1,X=0 -> S2: $Q_1'Q_0X'$ -> $Q_1^+ = 1$ - S2,X=1 -> S3: $Q_1Q_0'X$ -> $Q_1^+ = 1$ - S3,X=0 -> S2: Q_1Q_0X' -> $Q_1^+ = 1$

$Q_1^+ = Q_1'Q_0X' + Q_1Q_0'X + Q_1Q_0X'$ Simplify: $Q_1^+ = Q_0X' + Q_1Q_0'X = Q_0X' + Q_1X(Q_0')$ Further: $= X'Q_0 + XQ_1Q_0'$

Q_0^+: is 1 in transitions to S1 (01) and S3 (11). - S0,X=1 -> S1: $Q_1'Q_0'X$ - S1,X=1 -> S1: $Q_1'Q_0X$ - S2,X=1 -> S3: $Q_1Q_0'X$ - S3,X=1 -> S1: Q_1Q_0X

All transitions to states with Q_0=1 have X=1: $Q_0^+ = X$

(d) Output equation (Mealy — depends on state and input): Z = 1 only when in S3 ($Q_1Q_0 = 11$) and X = 0: $Z = Q_1 * Q_0 * X'$

(e) The minimum time between detections in an overlapping detector occurs with the sequence "…10101010…": The pattern "1010" is detected every 2 clock cycles (the next "1010" starts 2 bits after the previous one since they overlap by 2 bits). Maximum detection rate = f_{clk} / 2 = 10 MHz / 2 = 5 million detections per second

Chapter 6 — Section 6.6: Programmable Logic

Practice problems covering FPGAs, CPLDs, timing analysis, LUT utilization, clock domain crossing, and design trade-offs.

Problem 6.6.1

Given: An FPGA contains 10,000 6-input lookup tables (LUTs) and 10,000 flip-flops. A digital design requires the following resources: - 32-bit ALU: 180 LUTs, 32 flip-flops - 16 KB block RAM: 0 LUTs, 0 flip-flops (uses dedicated BRAM) - UART controller: 85 LUTs, 48 flip-flops - SPI master: 60 LUTs, 35 flip-flops - I2C controller: 70 LUTs, 40 flip-flops - PWM generator (8 channels): 120 LUTs, 64 flip-flops - Interrupt controller (16 sources): 95 LUTs, 32 flip-flops - GPIO block (32 pins): 96 LUTs, 64 flip-flops

Find: (a) The total LUT and flip-flop usage. (b) The utilization percentages. (c) Whether the design fits. (d) The remaining resources available for additional logic. (e) The recommended maximum utilization for routability.

Solution:

(a) Total LUT usage: 180 + 85 + 60 + 70 + 120 + 95 + 96 = 706 LUTs

Total flip-flop usage: 32 + 48 + 35 + 40 + 64 + 32 + 64 = 315 flip-flops

(b) Utilization: LUT utilization = 706 / 10,000 = 7.06% -> 7.1% Flip-flop utilization = 315 / 10,000 = 3.15% -> 3.2%

(c) Both resources are well below 100%. The design fits comfortably.

(d) Remaining resources: LUTs available = 10,000 - 706 = 9,294 LUTs Flip-flops available = 10,000 - 315 = 9,685 flip-flops

(e) FPGA designs should generally stay below 70-80% utilization to ensure the place-and-route tool can find valid routing paths and meet timing. Beyond 80%, routing congestion increases dramatically, timing closure becomes difficult, and compilation times grow significantly. For this design at 7.1% LUT usage, there is ample margin.

Problem 6.6.2

Given: A CPLD has 256 macrocells with a guaranteed maximum pin-to-pin delay of 5.0 ns. It is used to implement address decoding for a microprocessor system with the following requirements: - 4 chip select outputs for SRAM (each 64 KB, starting at 0x00000) - 2 chip select outputs for flash memory (each 256 KB, starting at 0x40000) - 1 chip select for an I/O peripheral block (4 KB at 0xC0000) - Wait-state generation logic (8 macrocells) - Bus arbitration logic for 2 bus masters (12 macrocells)

The address bus is 20 bits wide (A_{19} through A_0).

Find: (a) The address bits used to decode each chip select. (b) The macrocell count for address decoding. (c) The total macrocell usage. (d) The utilization percentage. (e) The maximum bus speed supported.

Solution:

(a) Address decoding:

SRAM (4 x 64 KB = 256 KB total, 0x00000 to 0x3FFFF): - Each 64 KB block uses 16 address bits (A_{15} to A_0) internally. - Decode using $A_{19}A_{18} = 00$, then $A_{17}A_{16}$ selects the chip: - CS_0: $A_{19}A_{18} = 00$, $A_{17}A_{16} = 00$ -> 0x00000 to 0x0FFFF - CS_1: $A_{19}A_{18} = 00$, $A_{17}A_{16} = 01$ -> 0x10000 to 0x1FFFF - CS_2: $A_{19}A_{18} = 00$, $A_{17}A_{16} = 10$ -> 0x20000 to 0x2FFFF - CS_3: $A_{19}A_{18} = 00$, $A_{17}A_{16} = 11$ -> 0x30000 to 0x3FFFF - Decode bits: A_{19}, A_{18}, A_{17}, A_{16}

Flash (2 x 256 KB = 512 KB, 0x40000 to 0xBFFFF): - Each 256 KB block uses 18 address bits (A_{17} to A_0) internally. - Decode using $A_{19}A_{18}$: flash region starts at 0x40000 ($A_{19}A_{18} = 01$) through 0xBFFFF ($A_{19}A_{18} = 10$). - CS_4: $A_{19} = 0$, $A_{18} = 1$ -> 0x40000 to 0x7FFFF - CS_5: $A_{19} = 1$, $A_{18} = 0$ -> 0x80000 to 0xBFFFF - Decode bits: A_{19}, A_{18}

I/O peripheral (4 KB at 0xC0000 to 0xC0FFF): - 4 KB = 12 address bits (A_{11} to A_0) for internal decoding. - Decode bits: A_{19} through A_{12} (8 bits must match 0xC0 = 1100 0000). - CS_6: $A_{19}A_{18}A_{17}A_{16} = 1100$, $A_{15}A_{14}A_{13}A_{12} = 0000$

(b) Macrocells for address decoding:

- 4 SRAM chip selects: 4 macrocells (each is a simple AND of 2-4 address bits)
- 2 flash chip selects: 2 macrocells
- 1 I/O chip select: 1 macrocell (AND of 8 address bits, fits in one macrocell with product term sharing)

Total decoding macrocells: 7 macrocells

(c) Total macrocell usage: Decoding: 7 Wait-state generation: 8 Bus arbitration: 12 Total = 7 + 8 + 12 = 27 macrocells

(d) Utilization = 27 / 256 = 10.5%

(e) The pin-to-pin delay of 5.0 ns allows address decode to complete within one bus cycle. For a synchronous bus, the maximum speed: $f_{max} = 1 / (2 \times t_{pd}) = 1 / (2 \times 5.0 \text{ ns}) = 1 / 10 \text{ ns} = 100$ MHz

For an asynchronous bus where decode must complete within the address setup time (typically one-half cycle): $f_{bus,max} = 100$ MHz

Problem 6.6.3

Given: A synchronous digital circuit in an FPGA has the following timing parameters: - Clock period: T_{clk} = 6.25 ns (160 MHz) - Source flip-flop t_{cq} = 0.5 ns - Destination flip-flop t_{su} = 0.3 ns, t_h = 0.15 ns - Routing delay (source FF to logic): t_{route1} = 0.8 ns - Combinational logic: 3 LUTs in series, each with t_{LUT} = 0.4 ns - Routing delay (logic to destination FF): t_{route2} = 0.6 ns - Clock skew: t_{skew} = 0.2 ns (destination clock arrives late)

Find: (a) The total data path delay. (b) The setup slack. (c) The hold slack (minimum path delay through logic is 0.3 ns with 0.2 ns routing). (d) Whether the design meets timing. (e) The maximum achievable frequency.

Solution:

(a) Total data path delay: $t_{data} = t_{cq} + t_{route1} + 3 \text{ x } t_{LUT} + t_{route2}$ t_{data} = 0.5 + 0.8 + 3 x 0.4 + 0.6 = 0.5 + 0.8 + 1.2 + 0.6 = 3.1 ns

(b) Setup time analysis (with clock skew — late arrival at destination means less time for data): Setup slack = T_{clk} - t_{data} - t_{su} - t_{skew} Setup slack = 6.25 - 3.1 - 0.3 - 0.2 = 2.65 ns (positive — met)

(c) Hold time analysis (minimum path delay): $t_{data,min} = t_{cq} + t_{route,min} + t_{logic,min}$ = 0.5 + 0.2 + 0.3 = 1.0 ns Hold slack = $t_{data,min}$ - t_h - t_{skew}

With late-arriving destination clock, hold is easier to meet (the data has more time): Hold slack = $t_{data,min}$ + t_{skew} - t_h = 1.0 + 0.2 - 0.15 = 1.05 ns (positive — met)

(d) Both setup slack (2.65 ns) and hold slack (1.05 ns) are positive. The design meets timing at 160 MHz.

(e) Maximum frequency (setup-limited): $T_{clk,min} = t_{data} + t_{su} + t_{skew}$ = 3.1 + 0.3 + 0.2 = 3.6 ns f_{max} = 1 / $T_{clk,min}$ = 1 / 3.6 ns = 277.8 MHz

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Problem 6.6.4

Given: An FPGA design must cross between two asynchronous clock domains: a 100 MHz domain (source) and a 75 MHz domain (destination). A two-flip-flop synchronizer is used to safely transfer a single-bit control signal from the source to the destination domain. The destination flip-flops have t_{su} = 0.3 ns and a metastability resolution time constant τ = 0.2 ns.

Find: (a) The destination clock period. (b) The time available for metastability resolution in the synchronizer (settling window). (c) The mean time between failures (MTBF) if the metastability window t_w = 0.04 ns and the data change rate is 100 million transitions/second. Use MTBF = $e^{tr/\tau}$ / (t_w x f_{clk} x f_{data}). (d) Whether this MTBF is acceptable for a system requiring 10 years of reliable operation.

Solution:

(a) Destination clock period: T_{dst} = 1 / 75 x 10^6 = 13.33 ns

(b) The settling window is the time between when the first synchronizer flip-flop captures data and when the second flip-flop samples it (one full destination clock period, minus setup time): t_r = T_{dst} - t_{su} = 13.33 - 0.3 = 13.03 ns

(c) MTBF calculation: MTBF = $e^{tr/\tau}$ / (t_w x f_{clk} x f_{data})

Numerator: $e^{13.03/0.2} = e^{65.15}$

$e^{65.15}$ is extremely large. Using ln(10) = 2.3026: 65.15 / 2.3026 = 28.3, so $e^{65.15} \approx 10^{28.3} \approx 2.0 \times 10^{28}$

Denominator: t_w x f_{clk} x f_{data} = $0.04 \times 10^{-9} \times 75 \times 10^6 \times 100 \times 10^6 = 0.04 \times 10^{-9} \times 7.5 \times 10^{15} = 0.3 \times 10^6 = 3.0 \times 10^5$

MTBF = $2.0 \times 10^{28} / 3.0 \times 10^5 = 6.67 \times 10^{22}$ seconds

Converting to years: $6.67 \times 10^{22} / (3.156 \times 10^7) = 2.11 \times 10^{15}$ years

(d) The required lifetime is 10 years. The MTBF of 2.11×10^{15} years exceeds this by a factor of 2.11×10^{14}. The two-flip-flop synchronizer is more than adequate. This enormous margin is typical for two-stage synchronizers at moderate clock frequencies.

Problem 6.6.5

Given: An FPGA has 6-input LUTs. A designer needs to implement several combinational functions: - Function A: 4-input Boolean function (1 minterm table) - Function B: 6-input Boolean function - Function C: 8-input Boolean function - Function D: 3-input Boolean function combined with a 2-input function (sharing inputs)

Find: (a) The number of LUTs required for each function. (b) The total LUT count. (c) How a 6-input LUT implements a function (truth table size). (d) The technique used to implement Function C across multiple LUTs.

Solution:

(a) LUT requirements:

Function A (4 inputs): A 6-input LUT can implement any function of up to 6 variables. A 4-input function fits in one LUT with 2 inputs unused (tied to constant values). LUTs for A: 1 LUT

Function B (6 inputs): exactly matches the LUT size. LUTs for B: 1 LUT

Function C (8 inputs): exceeds the 6-input LUT capacity. The function must be decomposed. The synthesis tool uses Shannon decomposition: $F(x_1,\ldots,x_8) = x_7' * F(x_1,\ldots,x_6,0,x_8) + x_7 * F(x_1,\ldots,x_6,1,x_8)$ Wait — this still has 7 variables per sub-function. For 8 inputs decomposed across 6-input LUTs:

Method: Fix 2 variables (e.g., x_7, x_8) and implement 4 sub-functions of 6 variables, then use a 4:1 MUX (another LUT) to select. - 4 LUTs for the sub-functions of x_1 through x_6 (one for each combination of x_7, x_8) - 1 LUT for the 4:1 MUX (inputs: 4 sub-function outputs + x_7 + x_8 = 6 inputs)

LUTs for C: 5 LUTs

Function D (3+2 inputs, 5 total): fits in one 6-input LUT since 5 < 6. LUTs for D: 1 LUT

(b) Total LUT count = 1 + 1 + 5 + 1 = 8 LUTs

(c) A 6-input LUT stores a truth table with $2^6 = 64$ entries (one bit per entry). The 6 inputs serve as address lines to this 64-bit memory, and the stored bit at the addressed location becomes the output. This is equivalent to a 64x1 ROM.

(d) Function C uses Shannon decomposition (also called function decomposition or cofactor-based decomposition). The function is split into sub-functions of fewer variables, each fitting in a single LUT, with a multiplexer LUT combining the results.

Problem 6.6.6

Given: A combinational logic path in an FPGA consists of: - Source register: t_{cq} = 0.4 ns - 4 levels of LUT logic: t_{LUT} = 0.3 ns each - 5 routing segments between elements: t_{route} = 0.5 ns, 0.3 ns, 0.7 ns, 0.4 ns, 0.6 ns - Destination register: t_{su} = 0.2 ns

The designer considers pipelining by inserting a register after the second LUT to split the path into two stages.

Find: (a) The critical path delay without pipelining. (b) The maximum frequency without pipelining. (c) The critical path delay of each stage with pipelining. (d) The maximum frequency with pipelining. (e) The throughput improvement. (f) The latency increase.

Solution:

(a) Without pipelining: $t_{path} = t_{cq} + t_{route1} + t_{LUT1} + t_{route2} + t_{LUT2} + t_{route3} + t_{LUT3} + t_{route4} + t_{LUT4} + t_{route5} + t_{su}$ $t_{path} = 0.4 + 0.5 + 0.3 + 0.3 + 0.3 + 0.7 + 0.3 + 0.4 + 0.3 + 0.6 + 0.2$ $t_{path} = 4.3$ ns

(b) f_{max} = 1 / 4.3 ns = 232.6 MHz

(c) With pipeline register after LUT2: Stage 1: $t_{cq} + t_{route1} + t_{LUT1} + t_{route2} + t_{LUT2} + t_{route3}$(to pipeline reg) + t_{su} Assume the pipeline register is placed at the route3 boundary. The routing splits approximately: Stage 1 = 0.4 + 0.5 + 0.3 + 0.3 + 0.3 + 0.35 + 0.2 = 2.35 ns

Stage 2: t_{cq}(pipe reg) + $t_{route3b} + t_{LUT3} + t_{route4} + t_{LUT4} + t_{route5} + t_{su}$ Stage 2 = 0.4 + 0.35 + 0.3 + 0.4 + 0.3 + 0.6 + 0.2 = 2.55 ns

Critical path = max(Stage 1, Stage 2) = 2.55 ns

(d) $f_{max,pipelined}$ = 1 / 2.55 ns = 392.2 MHz

(e) Throughput improvement = 392.2 / 232.6 = 1.69x (69% improvement)

(f) Latency without pipelining: 1 clock cycle = 4.3 ns Latency with pipelining: 2 clock cycles x 2.55 ns = 5.1 ns Latency increase: 5.1 - 4.3 = 0.8 ns increase (or 1 additional clock cycle of latency)

The trade-off: pipelining increases throughput at the cost of slightly higher latency and one additional flip-flop per data bit.

Problem 6.6.7

Given: A static timing analysis report for an FPGA design operating at 200 MHz shows the following 5 critical paths:

Path	Source	Destination	Data Delay (ns)	Required (ns)	Slack (ns)
1	REG_A	REG_B	4.8	4.7	-0.1
2	REG_C	REG_D	4.5	4.7	+0.2
3	REG_E	REG_F	4.6	4.7	+0.1
4	REG_A	REG_G	4.9	4.7	-0.2
5	REG_B	REG_H	4.3	4.7	+0.4

The required time is T_{clk} - t_{su} = 5.0 - 0.3 = 4.7 ns.

Find: (a) Which paths fail timing. (b) The worst negative slack (WNS). (c) The total negative slack (TNS). (d) The maximum safe operating frequency. (e) Three strategies to fix the timing violations.

Solution:

(a) Paths with negative slack fail timing: Path 1: slack = -0.1 ns (FAIL) Path 4: slack = -0.2 ns (FAIL)

(b) Worst negative slack (WNS): the most negative slack value. WNS = -0.2 ns (Path 4, REG_A to REG_G)

(c) Total negative slack (TNS): sum of all negative slacks. TNS = (-0.1) + (-0.2) = -0.3 ns

(d) Maximum safe frequency: determined by the worst path. Worst path delay = 4.9 ns (Path 4). $T_{clk,min} = 4.9 + t_{su} = 4.9 + 0.3 = 5.2$ ns $f_{max} = 1 / 5.2$ ns = 192.3 MHz

(e) Three strategies to fix timing violations:

1. Retiming/pipelining: Insert a pipeline register in the combinational logic between REG_A and REG_B/REG_G. Since REG_A is the common source of both failing paths, pipelining its output path will fix both violations.
2. Logic optimization: Restructure the combinational logic to reduce the number of logic levels. Replace cascaded logic with parallel structures (tree reduction), or use FPGA-specific primitives (carry chains, DSP blocks) that have faster propagation.
3. Physical constraints: Apply placement constraints (floorplanning) to place REG_A closer to REG_B and REG_G, reducing routing delay. Use the FPGA tool's physical optimization features such as post-route optimization or register duplication to reduce fanout on critical nets.

Problem 6.6.8

Given: A designer must choose between an FPGA and a CPLD for a glue logic application. The requirements are: - 45 I/O pins - 180 macrocells equivalent of logic - Must be operational within 10 ms of power-up - Maximum pin-to-pin delay: 8 ns - Production volume: 500 units/year - Power budget: 200 mW maximum

Available options: - CPLD: 256 macrocells, 64 I/O pins, t_{pd} = 5 ns, non-volatile, P_{static} = 50 mW, unit cost $3.50 - FPGA: 1,500 LUTs, 100 I/O pins, t_{pd} = 3 ns (after configuration), requires external flash, configuration time = 50 ms, P_{static} = 150 mW, unit cost $8.00 (+ $1.50 for config flash)

Find: (a) Whether each device meets all requirements. (b) A comparison table. (c) The recommended choice with justification. (d) The annual cost difference.

Solution:

(a) Requirements check:

CPLD: - I/O pins: 64 >= 45. PASS - Logic: 256 macrocells >= 180. PASS - Power-up time: instant (non-volatile). PASS (< 10 ms) - Pin-to-pin delay: 5 ns <= 8 ns. PASS - Power: 50 mW <= 200 mW. PASS

FPGA: - I/O pins: 100 >= 45. PASS - Logic: 1,500 LUTs » 180. PASS (more than enough) - Power-up time: 50 ms > 10 ms. FAIL - Pin-to-pin delay: 3 ns <= 8 ns. PASS - Power: 150 mW <= 200 mW. PASS

(b) Comparison table:

Parameter	CPLD	FPGA	Requirement
I/O pins	64	100	45
Logic capacity	256 MC	1,500 LUTs	180 MC
Power-up	Instant	50 ms	< 10 ms
Delay	5 ns	3 ns	< 8 ns
Power	50 mW	150 mW	< 200 mW
Unit cost	$3.50	$9.50	—
Config memory	Built-in	External	—

(c) Recommended choice: CPLD

Justification: The FPGA fails the 10 ms power-up requirement since it needs 50 ms to load its configuration from external flash. The CPLD meets all requirements and has lower power (50 mW vs. 150 mW), lower cost ($3.50 vs. $9.50), and simpler BOM (no external configuration flash). The CPLD's deterministic timing also simplifies design validation.

(d) Annual cost difference: CPLD: 500 x $3.50 = $1,750/year FPGA: 500 x ($8.00 + $1.50) = 500 x $9.50 = $4,750/year Difference = $4,750 - $1,750 = $3,000/year savings with CPLD

Problem 6.6.9

Given: An FPGA-based design has a static hazard in a combinational output. The Boolean function is F = AC + A'B, and a transition occurs where A changes from 1 to 0 while B = 1 and C = 1 (input changes from ABC = 111 to ABC = 011). The gate delays are: AND gate = 3 ns, OR gate = 2 ns, inverter = 2 ns.

Find: (a) The expected output value before and after the transition. (b) The timing of each signal during the transition (show the glitch). (c) The type of hazard. (d) The redundant term needed to eliminate the hazard. (e) The corrected Boolean expression.

Solution:

(a) Before: ABC = 111. F = (1)(1) + (0)(1) = 1 + 0 = 1 After: ABC = 011. F = (0)(1) + (1)(1) = 0 + 1 = 1 The output should remain at 1 throughout the transition.

(b) Signal timing (A changes from 1 to 0 at t = 0):

t = 0 ns: A transitions from 1 to 0.

Path through AC (direct): - AC: A goes to 0, C = 1. AC drops from 1 to 0 at t = 3 ns (AND gate delay).

Path through A'B (through inverter): - A' rises from 0 to 1 at t = 2 ns (inverter delay). - A'B: rises from 0 to 1 at t = 2 + 3 = 5 ns (inverter + AND gate delay).

OR gate output F: - At t = 3 ns: AC = 0, A'B still = 0 (hasn't risen yet). F = 0 + 0 = 0. - At t = 5 ns: AC = 0, A'B = 1 at AND output. F updates at t = 5 + 2 = 7 ns: F = 0 + 1 = 1.

Glitch: F drops from 1 to 0 during t = 3+2 = 5 ns to t = 7 ns. Actually, let's be more precise:

F (OR gate output) responds to its inputs: - F was 1 (from AC=1). At t = 3 ns, AC input to OR drops. OR delay = 2 ns, so F drops at t = 3 + 2 = 5 ns. - A'B rises at t = 5 ns. OR responds at t = 5 + 2 = 7 ns, F returns to 1.

The glitch lasts from t = 5 ns to t = 7 ns (2 ns wide).

(c) This is a static-1 hazard (output should remain at 1 but momentarily dips to 0).

(d) The hazard occurs because BC = 1 during the transition, but there is no term in F covering BC. On the K-map, the hazard exists between the two prime implicants AC and A'B when B = C = 1.

The redundant consensus term is: BC

(e) Corrected expression: F = AC + A'B + BC

The added BC term holds the output at 1 during the transition, since B = C = 1 throughout. This eliminates the static-1 hazard at the cost of one additional AND gate.

Problem 6.6.10

Given: An FPGA implements a digital FIR filter that processes 8-bit audio samples at a sample rate of 48 kHz. The filter has 64 taps with 12-bit coefficients. Each tap requires one multiply-accumulate (MAC) operation. The FPGA has dedicated DSP blocks that can perform one 18x18 bit multiply + 48-bit accumulate per clock cycle with a DSP block delay of 4 ns. The FPGA fabric clock is 100 MHz.

Find: (a) The number of clock cycles available per sample. (b) The minimum number of DSP blocks needed to process all 64 taps within one sample period (serial approach). (c) The number of DSP blocks needed for a fully parallel implementation. (d) The minimum DSP clock frequency for a single-DSP serial approach. (e) The trade-off between serial and parallel implementations.

Solution:

(a) Clock cycles per sample: Cycles = f_{clk} / f_{sample} = 100 x 10^6 / 48 x 10^3 = 2,083 clock cycles per sample

(b) Serial approach with minimum DSP blocks: Each DSP block can perform 2,083 MAC operations per sample period. The filter needs 64 MACs per sample. Since 64 < 2,083, a single DSP block can easily handle all 64 taps.

Minimum DSP blocks: 1 DSP block (performing 64 MACs sequentially out of 2,083 available cycles)

DSP utilization: 64 / 2,083 = 3.1%

(c) Fully parallel: each tap gets its own DSP block, all operating simultaneously. DSP blocks needed: 64 DSP blocks All MACs complete in 1 clock cycle (10 ns). The remaining 2,082 cycles are idle.

(d) For a single DSP, minimum clock frequency to complete 64 MACs in one sample period: $T_{sample} = 1 / 48{,}000 = 20.833\ \mu s$ $f_{min} = 64 / T_{sample} = 64 / 20.833 \times 10^{-6} = 3.072$ MHz

Since the DSP block delay is 4 ns, $f_{max} = 1/4$ ns = 250 MHz, so 3.072 MHz is easily achievable.

(e) Trade-off summary:

Approach	DSP Blocks	Latency (clocks)	Utilization
Serial (1 DSP)	1	64 cycles = 640 ns	3.1%
Parallel (64 DSPs)	64	1 cycle = 10 ns	0.048%
Semi-parallel (4 DSPs)	4	16 cycles = 160 ns	0.77%

Serial uses minimal resources but has higher latency (640 ns, still far below the 20.8 μs sample period). Parallel has the lowest latency but uses 64x more DSP resources. A semi-parallel approach (e.g., 4 DSPs each processing 16 taps) balances resource usage and latency. For audio at 48 kHz, the serial approach is optimal because the sample period is so long relative to the processing time that resource efficiency far outweighs the negligible latency difference.

Chapter 7 — Section 7.1: Electric Charge and Current

Practice problems covering electric charge, current, voltage, energy, drift velocity, and power relationships.

Problem 7.1.1

Given: A lightning bolt transfers 5 C of charge from a cloud to the ground in 0.001 seconds.

Find: (a) The average current during the lightning strike. (b) The number of electrons transferred.

Solution:

(a) Current is the rate of charge flow: I = Q / t

I = 5 C / 0.001 s = 5,000 A (5 kA)

(b) Number of electrons: N = Q / e

N = 5 / (1.602 x 10^{-19}) = 3.12 x 10^{19} electrons

Problem 7.1.2

Given: An aluminum conductor has a cross-sectional area of 53.5 mm^2 (1/0 AWG). Aluminum has a free electron density of n = 18.1 x 10^{28} electrons/m^3. The conductor carries a current of 100 A.

Find: (a) The drift velocity of the electrons. (b) The time it takes an electron to travel 1 meter along the conductor.

Solution:

(a) Drift velocity: v_d = I / (n x A x e)

v_d = 100 / (18.1 x 10^{28} x 53.5 x 10^{-6} x 1.602 x 10^{-19})

v_d = 100 / (18.1 x 10^{28} x 53.5 x 10^{-6} x 1.602 x 10^{-19})

Denominator = 18.1 x 53.5 x 1.602 x $10^{28-6-19}$ = 1550.5 x 10^3 = 1.551 x 10^6

$v_d = 100 / 1.551 \times 10^6 = 6.45 \times 10^{-5}$ m/s = 0.0645 mm/s

(b) Time to travel 1 meter: $t = d / v_d$

$t = 1 / 6.45 \times 10^{-5} = 15{,}504$ s = 4.31 hours

Problem 7.1.3

Given: A smartphone battery is rated at 3,000 mAh at a nominal voltage of 3.7 V. The phone draws an average current of 250 mA during normal use.

Find: (a) The total charge stored in the battery in coulombs. (b) The total energy stored in watt-hours and joules. (c) The expected battery life. (d) The average power consumption.

Solution:

(a) Total charge: Q = I x t = capacity in Ah x 3600 s/h

Q = 3.0 Ah x 3600 = 10,800 C

(b) Energy: W = capacity x voltage = 3.0 Ah x 3.7 V = 11.1 Wh

In joules: W = 11.1 x 3600 = 39,960 J = 40.0 kJ

(c) Battery life: t = capacity / I = 3,000 mAh / 250 mA = 12 hours

(d) Average power: P = V x I = 3.7 x 0.250 = 0.925 W

Problem 7.1.4

Given: A residential electric water heater has two 4,500 W heating elements operating at 240 V. The utility charges $0.12 per kWh. The heater runs an average of 3 hours per day.

Find: (a) The current drawn by one heating element. (b) The resistance of one heating element. (c) The daily energy consumption in kWh. (d) The monthly energy cost (30 days).

Solution:

(a) Current per element: I = P / V = 4,500 / 240 = 18.75 A

(b) Resistance: $R = V^2 / P = 240^2 / 4{,}500 = 57{,}600 / 4{,}500 = 12.8$ ohm

(c) Daily energy (assuming one element operates at a time, total on-time 3 hours):

W = P x t = 4.5 kW x 3 h = 13.5 kWh per day

(d) Monthly cost: Cost = 13.5 x 30 x $0.12 = $48.60 per month

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Problem 7.1.5

Given: A solar panel produces 8.5 A at a terminal voltage of 32 V under full sun conditions. The panel is connected to a charge controller and battery system.

Find: (a) The power output of the panel. (b) The total energy produced during 5 peak sun hours. (c) The total charge delivered to the battery in one day of 5 peak sun hours.

Solution:

(a) Power: P = V x I = 32 x 8.5 = 272 W

(b) Energy: W = P x t = 272 W x 5 h = 1,360 Wh = 1.36 kWh

(c) Charge: Q = I x t = 8.5 A x 5 x 3600 s = 153,000 C = 153 kC

Equivalently, Q = 8.5 A x 5 h = 42.5 Ah

Chapter 7 — Section 7.2: Passive Components

Practice problems covering resistors, capacitors, inductors, and wire/cable characteristics.

Problem 7.2.1

Given: A precision voltage divider uses a 10 kohm metal film resistor (tolerance +/-0.1%, temperature coefficient +/-50 ppm/degC). The resistor operates at 25 degC in a lab environment, but the circuit must also function at 75 degC.

Find: (a) The resistance range at 25 degC due to tolerance. (b) The resistance change due to temperature at 75 degC. (c) The total worst-case resistance at 75 degC. (d) The power dissipated if 15 V is applied across the resistor.

Solution:

(a) Tolerance range at 25 degC:

R_{min} = 10,000 x (1 - 0.001) = 9,990 ohm

R_{max} = 10,000 x (1 + 0.001) = 10,010 ohm

(b) Temperature change: deltaT = 75 - 25 = 50 degC

deltaR = R x TC x deltaT = 10,000 x 50 x 10^{-6} x 50 = 25 ohm

(c) Worst-case resistance at 75 degC:

$R_{max,total}$ = 10,010 + 25 = 10,035 ohm

$R_{min,total}$ = 9,990 - 25 = 9,965 ohm

Total uncertainty: +/-35 ohm or +/-0.35%

(d) Power: P = V^2 / R = 15^2 / 10,000 = 225 / 10,000 = 22.5 mW

Problem 7.2.2

Given: A timing circuit uses a 47 uF aluminum electrolytic capacitor (ESR = 0.8 ohm, voltage rating 25 V) charged to 12 V.

Find: (a) The energy stored in the capacitor. (b) If the capacitor is discharged through a 100 ohm load, the peak discharge current. (c) The peak power dissipated in the ESR during discharge. (d) If the capacitor must hold its charge for 10 minutes with less than 1 V droop, the minimum allowable leakage resistance.

Solution:

(a) Energy: $W = 1/2 \text{ x } C \text{ x } V^2 = 0.5 \text{ x } 47 \text{ x } 10^{-6} \text{ x } 12^2 = 0.5 \text{ x } 47 \text{ x } 10^{-6} \text{ x } 144 = 3.38$ mJ

(b) Peak discharge current (total resistance = R_{load} + ESR):

$I_{peak} = V / (R_{load} + ESR) = 12 / (100 + 0.8) = 12 / 100.8 = 119$ mA

(c) Peak power in ESR: $P_{ESR} = I_{peak}{}^2 \text{ x } ESR = 0.119^2 \text{ x } 0.8 = 0.01416 \text{ x } 0.8 = 11.3$ mW

(d) For voltage droop < 1 V in 10 minutes, using $V(t) = V_0 \text{ x } e^{-t/(Rleak \text{ x } C)}$:

$11 = 12 \text{ x } e^{-600/(Rleak \text{ x } 47 \text{ x } 10\text{-}6})$

$e^{-600/(Rleak \text{ x } 47 \text{ x } 10\text{-}6}) = 11/12 = 0.9167$

$-600 / (R_{leak} \text{ x } 47 \text{ x } 10^{-6}) = \ln(0.9167) = -0.0870$

$R_{leak} = 600 / (0.0870 \text{ x } 47 \text{ x } 10^{-6}) = 600 / (4.089 \text{ x } 10^{-6}) = 146.7$ Mohm

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Problem 7.2.3

Given: A switching power supply uses a 22 uH toroidal inductor with a DC resistance of 0.015 ohm. The inductor operates at 250 kHz with a DC current of 5 A and a ripple current of 1.5 A peak-to-peak.

Find: (a) The peak inductor current. (b) The energy stored at peak current. (c) The quality factor Q at the switching frequency. (d) The minimum saturation current rating needed with a 20% safety margin. (e) The minimum self-resonant frequency (SRF) for reliable operation.

Solution:

(a) Peak current: $I_{peak} = I_{DC} + \text{deltaI}/2 = 5 + 1.5/2 = 5 + 0.75 = 5.75$ A

(b) Energy at peak: $W = 1/2 \text{ x } L \text{ x } I_{peak}{}^2 = 0.5 \text{ x } 22 \text{ x } 10^{-6} \text{ x } 5.75^2 = 0.5 \text{ x } 22 \text{ x } 10^{-6} \text{ x } 33.06 = 363.7$ uJ

(c) Quality factor: $Q = \text{omega x } L / R_{DC}$

omega = 2 x pi x 250,000 = 1,570,796 rad/s

$Q = 1{,}570{,}796 \text{ x } 22 \text{ x } 10^{-6} / 0.015 = 34.56 / 0.015 = 2{,}304$

(d) Minimum saturation rating: $I_{sat} >= 1.2 \text{ x } I_{peak} = 1.2 \text{ x } 5.75 = 6.9$ A

(e) Minimum SRF $>= 3 \text{ x } f_{sw} = 3 \text{ x } 250$ kHz = 750 kHz

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Problem 7.2.4

Given: A 240 V, 30 A branch circuit uses 10 AWG copper conductors (resistance = 3.277 ohm/km) in a 200-foot (61 m) run from the panel to a workshop subpanel.

Find: (a) The one-way conductor resistance. (b) The total voltage drop at full load (single-phase, accounting for both conductors). (c) The voltage drop as a percentage. (d) Whether it meets the NEC 3% recommendation for feeders. (e) What conductor size would meet the 3% recommendation.

Solution:

(a) One-way resistance: R = 3.277 ohm/km x 0.061 km = 0.200 ohm

(b) Voltage drop (both conductors): V_{drop} = 2 x I x R = 2 x 30 x 0.200 = 12.0 V

(c) Percentage: V_{drop}% = 12.0 / 240 x 100 = 5.0%

(d) The NEC recommends <= 3% for feeders. At 5.0%, this circuit does not meet the recommendation. The maximum allowable drop is 240 x 0.03 = 7.2 V.

(e) Required maximum resistance per conductor: R_{max} = 7.2 / (2 x 30) = 0.12 ohm

Required resistance per km: 0.12 / 0.061 = 1.967 ohm/km

From the AWG table: 6 AWG has 1.296 ohm/km. Check: V_{drop} = 2 x 30 x 1.296 x 0.061 = 4.74 V = 1.97%

6 AWG copper meets the 3% recommendation.

Problem 7.2.5

Given: A DC motor circuit requires a 0.1 ohm, 50 W wirewound power resistor for current sensing. The resistor has a temperature coefficient of +30 ppm/degC and operates in an ambient of 40 degC. At full power, the resistor surface temperature rises by 80 degC above ambient.

Find: (a) The current at which the resistor dissipates its full rated power. (b) The voltage across the resistor at full power. (c) The resistance shift at full operating temperature. (d) The actual current-sense voltage error percentage due to resistance change.

Solution:

(a) Current: P = I^2 x R, so I = sqrt(P/R) = sqrt(50/0.1) = sqrt(500) = 22.36 A

(b) Voltage: V = I x R = 22.36 x 0.1 = 2.236 V

(c) Operating temperature: T = 40 + 80 = 120 degC. Temperature rise above reference (25 degC): deltaT = 120 - 25 = 95 degC.

deltaR = R x TC x deltaT = 0.1 x 30 x 10^{-6} x 95 = 0.000285 ohm

R_{hot} = 0.1 + 0.000285 = 0.100285 ohm

(d) Voltage error: The sense voltage at the same current would be V_{hot} = 22.36 x 0.100285 = 2.2424 V.

Error = (0.100285 - 0.1) / 0.1 x 100 = 0.285%

This error may be acceptable for motor current sensing but would need compensation for precision measurements.

Problem 7.2.6

Given: A decoupling network uses a 100 nF ceramic capacitor (X7R, rated 50 V) in parallel with a 10 uF tantalum capacitor (ESR = 1.5 ohm) for a 3.3 V digital supply. The X7R capacitor loses 40% of its capacitance at its rated voltage. A load transient of 500 mA occurs.

Find: (a) The effective capacitance of the ceramic capacitor at 3.3 V and at 50 V. (b) The ESR voltage spike from the tantalum alone during the load transient. (c) The total capacitance available for energy storage at 3.3 V. (d) The voltage droop if the regulator takes 20 us to respond.

Solution:

(a) At 3.3 V (well below the 50 V rating), derating is minimal: C_{eff} = 100 nF (essentially full capacitance).

At 50 V (rated voltage), C_{eff} = 100 x (1 - 0.40) = 60 nF

(b) ESR spike from tantalum: deltaV = I x ESR = 0.5 x 1.5 = 0.75 V (22.7% of 3.3 V – unacceptable alone)

(c) Total capacitance at 3.3 V: C_{total} = 100 nF + 10 uF = 10 uF + 0.1 uF = 10.1 uF

(d) Voltage droop: deltaV = I x dt / C = 0.5 x 20 x 10^{-6} / (10.1 x 10^{-6}) = 0.99 V

This 30% droop on a 3.3 V rail is excessive. Additional bulk capacitance or a faster regulator is needed.

Chapter 7 — Section 7.3: Fundamental Laws

Practice problems covering Ohm's Law, Kirchhoff's Voltage Law (KVL), and Kirchhoff's Current Law (KCL).

Problem 7.3.1

Given: A heating element in a toaster oven draws 10 A from a 120 V supply.

Find: (a) The resistance of the heating element. (b) The power dissipated. (c) If the supply voltage drops to 108 V (a 10% sag), the new current and power. (d) The percentage reduction in heating power during the voltage sag.

Solution:

(a) Resistance: R = V / I = 120 / 10 = 12 ohm

(b) Power: P = V x I = 120 x 10 = 1,200 W

(c) At 108 V (resistance assumed constant):

I_{new} = 108 / 12 = 9 A

$P_{new} = V^2 / R = 108^2 / 12$ = 11,664 / 12 = 972 W

(d) Percentage reduction: (1200 - 972) / 1200 x 100 = 19%

A 10% voltage reduction causes a 19% power reduction because power is proportional to V^2.

Problem 7.3.2

Given: A series circuit contains a 36 V battery, a 1.5 kohm resistor R_1, a 2.2 kohm resistor R_2, and a 3.3 kohm resistor R_3.

Find: (a) The total series resistance. (b) The loop current. (c) The voltage drop across each resistor. (d) Verify KVL around the loop.

Solution:

(a) $R_{total} = R_1 + R_2 + R_3 = 1{,}500 + 2{,}200 + 3{,}300 = 7{,}000$ ohm = 7 kohm

(b) Loop current: $I = V / R_{total} = 36 / 7{,}000 = 5.143$ mA

(c) Voltage drops:

$V_1 = I \times R_1 = 5.143 \times 10^{-3} \times 1{,}500 = 7.714$ V

$V_2 = I \times R_2 = 5.143 \times 10^{-3} \times 2{,}200 = 11.314$ V

$V_3 = I \times R_3 = 5.143 \times 10^{-3} \times 3{,}300 = 16.971$ V

(d) KVL check: $V_1 + V_2 + V_3 = 7.714 + 11.314 + 16.971 = 36.0 \text{ V} = V_{source}$ (verified)

Problem 7.3.3

Given: At a node in a circuit, five branches meet. Branch 1 carries 2.5 A into the node. Branch 2 carries 1.8 A out of the node. Branch 3 carries 3.2 A into the node. Branch 4 carries 0.7 A out of the node.

Find: (a) The current in branch 5 (magnitude and direction). (b) Verify KCL at the node.

Solution:

(a) By KCL, the sum of currents entering a node equals the sum of currents leaving:

$I_{in} = I_1 + I_3 = 2.5 + 3.2 = 5.7$ A

$I_{out} = I_2 + I_4 = 1.8 + 0.7 = 2.5$ A

$I_5 = I_{in} - I_{out} = 5.7 - 2.5 = 3.2$ A leaving the node

(b) KCL check: Total in $= I_1 + I_3 = 2.5 + 3.2 = 5.7$ A. Total out $= I_2 + I_4 + I_5 = 1.8 + 0.7 + 3.2 = 5.7$ A. Verified.

Problem 7.3.4

Given: A series-parallel circuit is powered by a 48 V source. A 200 ohm resistor R_1 is in series with the source. Two parallel branches connect from the junction of R_1 to ground: Branch A has a 600 ohm resistor R_2, and Branch B has a 300 ohm resistor R_3 in series with a 100 ohm resistor R_4.

Find: (a) The equivalent resistance of the parallel combination. (b) The total circuit resistance. (c) The source current. (d) The voltage at the junction node (after R_1). (e) The current through each parallel branch.

Solution:

(a) Branch B resistance: $R_B = R_3 + R_4 = 300 + 100 = 400$ ohm

Parallel combination: $R_{parallel} = (R_2 \times R_B) / (R_2 + R_B) = (600 \times 400) / (600 + 400) = 240{,}000 / 1{,}000 = 240$ ohm

(b) Total resistance: $R_{total} = R_1 + R_{parallel} = 200 + 240 = 440$ ohm

(c) Source current: I_{source} = 48 / 440 = 109.1 mA

(d) Voltage at junction: $V_{junction} = I_{source} \times R_{parallel}$ = 0.1091 x 240 = 26.18 V

Or: $V_{junction} = 48 - I_{source} \times R_1$ = 48 - 0.1091 x 200 = 48 - 21.82 = 26.18 V

(e) Branch currents:

$I_A = V_{junction} / R_2$ = 26.18 / 600 = 43.6 mA

$I_B = V_{junction} / R_B$ = 26.18 / 400 = 65.5 mA

Check: $I_A + I_B$ = 43.6 + 65.5 = 109.1 mA = I_{source} (verified)

Problem 7.3.5

Given: An LED circuit uses a red LED (forward voltage V_{LED} = 2.0 V, desired current I_{LED} = 20 mA) powered by a 5 V supply.

Find: (a) The required current-limiting resistor value. (b) The power dissipated by the resistor. (c) The power dissipated by the LED. (d) The minimum resistor power rating.

Solution:

(a) By KVL: $V_{supply} = V_R + V_{LED}$

V_R = 5 - 2.0 = 3.0 V

$R = V_R / I$ = 3.0 / 0.020 = 150 ohm

(b) Power in resistor: $P_R = V_R \times I$ = 3.0 x 0.020 = 60 mW

(c) Power in LED: $P_{LED} = V_{LED} \times I$ = 2.0 x 0.020 = 40 mW

(d) A standard 1/8 W (125 mW) resistor provides a 2:1 derating margin and is the minimum standard rating above 60 mW.

Check total power: $P_{total} = P_R + P_{LED}$ = 60 + 40 = 100 mW = $V_{supply} \times I$ = 5 x 0.020 = 100 mW (verified).

Problem 7.3.6

Given: Three 120 V branch circuits share a common neutral in a residential panel (single-phase, 120/240 V system). Circuit A draws 12 A, Circuit B draws 8 A, and Circuit C draws 15 A. Circuits A and C are on phase A (same hot leg), and Circuit B is on phase B (opposite hot leg).

Find: (a) The neutral current if all three circuits are active simultaneously. (b) Verify using KCL at the neutral bus.

Solution:

(a) In a 120/240 V single-phase system, currents on opposite phases partially cancel on the neutral.

Phase A contribution to neutral: $I_{A\text{-phase}} = I_A + I_C = 12 + 15 = 27$ A

Phase B contribution to neutral: $I_{B\text{-phase}} = 8$ A

Neutral current: $I_N = |I_{A\text{-phase}} - I_{B\text{-phase}}| = |27 - 8| = 19$ A

(b) KCL verification at the neutral bus: The neutral carries the unbalanced current between the two phases. Current flowing from phase A loads into the neutral = 27 A. Current flowing from neutral to phase B loads = 8 A. Net neutral current returning to the transformer = 27 - 8 = 19 A (verified).

Chapter 7 — Section 7.4: DC Circuit Analysis

Practice problems covering series circuits, parallel circuits, and series-parallel combinations.

Problem 7.4.1

Given: A series string of five identical 1 kohm resistors is connected across a 25 V supply. A voltmeter is connected across the third resistor in the string.

Find: (a) The total resistance. (b) The series current. (c) The voltmeter reading using the voltage divider rule. (d) The power dissipated by each resistor and the total power.

Solution:

(a) R_{total} = 5 x 1,000 = 5,000 ohm = 5 kohm

(b) Series current: $I = V / R_{total}$ = 25 / 5,000 = 5 mA

(c) Voltage across one resistor: $V_3 = V_{source} \times (R_3 / R_{total})$ = 25 x (1,000 / 5,000) = 25 x 0.2 = 5 V

(d) Power per resistor: $P = I^2 \times R = (0.005)^2 \times 1{,}000 = 25 \times 10^{-6} \times 1{,}000$ = 25 mW

Total power: P_{total} = 5 x 25 mW = 125 mW

Check: $P_{total} = V^2 / R_{total}$ = 625 / 5,000 = 125 mW (verified).

Problem 7.4.2

Given: Four resistors are connected in parallel across a 24 V source: R_1 = 100 ohm, R_2 = 220 ohm, R_3 = 470 ohm, R_4 = 1 kohm.

Find: (a) The total parallel resistance. (b) The total current from the source. (c) The current through each branch. (d) The total power consumed.

Solution:

(a) $1/R_{total} = 1/100 + 1/220 + 1/470 + 1/1000$

$1/R_{total} = 0.01000 + 0.004545 + 0.002128 + 0.001000 = 0.017673$ S

$R_{total} = 1 / 0.017673 = 56.58$ ohm

(b) Total current: $I_{total} = 24 / 56.58 = 424.2$ mA

(c) Branch currents:

$I_1 = 24 / 100 = 240$ mA

$I_2 = 24 / 220 = 109.1$ mA

$I_3 = 24 / 470 = 51.1$ mA

$I_4 = 24 / 1000 = 24.0$ mA

Check: 240 + 109.1 + 51.1 + 24.0 = 424.2 mA (verified).

(d) Total power: $P = V \times I_{total} = 24 \times 0.4242 = 10.18$ W

Problem 7.4.3

Given: A circuit consists of a 9 V battery with an internal resistance of 0.5 ohm, connected to an external network. The external network has $R_1 = 10$ ohm in series with a parallel combination of $R_2 = 30$ ohm and $R_3 = 60$ ohm.

Find: (a) The equivalent external resistance. (b) The total circuit resistance. (c) The current drawn from the battery. (d) The terminal voltage of the battery (voltage across the external network). (e) The current through R_2 and R_3.

Solution:

(a) Parallel combination: $R_{23} = (30 \times 60) / (30 + 60) = 1{,}800 / 90 = 20$ ohm

External resistance: $R_{ext} = R_1 + R_{23} = 10 + 20 = 30$ ohm

(b) Total resistance: $R_{total} = R_{int} + R_{ext} = 0.5 + 30 = 30.5$ ohm

(c) Current: $I = 9 / 30.5 = 295.1$ mA

(d) Terminal voltage: $V_{terminal} = V_{battery} - I \times R_{int} = 9 - 0.2951 \times 0.5 = 9 - 0.148 = 8.852$ V

Or: $V_{terminal} = I \times R_{ext} = 0.2951 \times 30 = 8.852$ V (verified).

(e) Voltage across parallel combination: $V_{23} = I \times R_{23} = 0.2951 \times 20 = 5.902$ V

$I_2 = V_{23} / R_2 = 5.902 / 30 = 196.7$ mA

$I_3 = V_{23} / R_3 = 5.902 / 60 = 98.4$ mA

Check: $I_2 + I_3 = 196.7 + 98.4 = 295.1$ mA $= I$ (verified).

Problem 7.4.4

Given: A Wheatstone bridge has R_1 = 1 kohm, R_2 = 2 kohm, R_3 = 1.5 kohm, and R_4 is unknown. The bridge is powered by a 10 V source. When balanced, the galvanometer reads zero current.

Find: (a) The value of R_4 for a balanced bridge. (b) Verify the balance condition. (c) If R_4 is actually 3.1 kohm (slightly unbalanced), find the open-circuit voltage across the galvanometer terminals.

Solution:

(a) Balance condition: $R_1/R_2 = R_3/R_4$

$R_4 = R_2$ x R_3 / R_1 = 2,000 x 1,500 / 1,000 = 3,000 ohm = 3 kohm

(b) Check: R_1/R_2 = 1/2 = 0.5. R_3/R_4 = 1.5/3 = 0.5. Balanced.

(c) With R_4 = 3.1 kohm, the voltage at each midpoint:

$V_A = V_{source}$ x R_2 / ($R_1 + R_2$) = 10 x 2,000 / (1,000 + 2,000) = 10 x 2/3 = 6.667 V

$V_B = V_{source}$ x R_4 / ($R_3 + R_4$) = 10 x 3,100 / (1,500 + 3,100) = 10 x 3,100/4,600 = 6.739 V

$V_{galvanometer} = V_B - V_A$ = 6.739 - 6.667 = 72.5 mV

Problem 7.4.5

Given: A voltage divider supplies 3.3 V to a microcontroller from a 12 V source. The microcontroller input has an input impedance of 10 Mohm and draws negligible current. The divider uses R_1 (upper, connected to 12 V) and R_2 (lower, connected to ground) with R_2 = 10 kohm.

Find: (a) The required value of R_1. (b) The current through the divider (quiescent current). (c) The power dissipated by the divider. (d) If the microcontroller input impedance drops to 100 kohm (a different MCU), the new output voltage.

Solution:

(a) Voltage divider: $V_{out} = V_{in}$ x R_2 / ($R_1 + R_2$)

3.3 = 12 x 10,000 / (R_1 + 10,000)

R_1 + 10,000 = 12 x 10,000 / 3.3 = 120,000 / 3.3 = 36,364

R_1 = 36,364 - 10,000 = 26,364 ohm (use standard value 27 kohm)

With R_1 = 27 kohm: V_{out} = 12 x 10,000 / (27,000 + 10,000) = 120,000 / 37,000 = 3.24 V

(b) Quiescent current: I_q = 12 / (27,000 + 10,000) = 12 / 37,000 = 324 uA

(c) Power: $P = V_{in}$ x I_q = 12 x 0.000324 = 3.89 mW

(d) With 100 kohm load in parallel with R_2:

R_{2eff} = (10,000 x 100,000) / (10,000 + 100,000) = 1,000,000,000 / 110,000 = 9,091 ohm

V_{out} = 12 x 9,091 / (27,000 + 9,091) = 109,091 / 36,091 = 3.02 V

The output drops from 3.24 V to 3.02 V due to loading, which may be below the MCU minimum voltage.

Chapter 7 — Section 7.5: Analysis Methods

Practice problems covering nodal analysis, mesh analysis, superposition, and dependent sources.

Problem 7.5.1

Given: A circuit has three non-reference nodes (V_1, V_2, V_3). A 5 mA current source feeds into node V_1. A 1 kohm resistor connects V_1 to ground, a 2 kohm resistor connects V_1 to V_2, a 2 kohm resistor connects V_2 to ground, a 3 kohm resistor connects V_2 to V_3, and a 1 kohm resistor connects V_3 to ground.

Find: V_1, V_2, and V_3 using nodal analysis.

Solution: KCL at node V_1: $0.005 = V_1/1000 + (V_1 - V_2)/2000$

Multiply by 2000: $10 = 2V_1 + V_1 - V_2 = 3V_1 - V_2$ ... (1)

KCL at node V_2: $(V_1 - V_2)/2000 = V_2/2000 + (V_2 - V_3)/3000$

Multiply by 6000: $3(V_1 - V_2) = 3V_2 + 2(V_2 - V_3)$

$3V_1 - 3V_2 = 3V_2 + 2V_2 - 2V_3$

$3V_1 - 8V_2 + 2V_3 = 0$ ... (2)

KCL at node V_3: $(V_2 - V_3)/3000 = V_3/1000$

Multiply by 3000: $V_2 - V_3 = 3V_3$, so $V_2 = 4V_3$ ... (3)

From (3): $V_2 = 4V_3$. Substitute into (2): $3V_1 - 32V_3 + 2V_3 = 0$, so $3V_1 = 30V_3$, $V_1 = 10V_3$ ... (4)

Substitute (3) and (4) into (1): $10 = 3(10V_3) - 4V_3 = 30V_3 - 4V_3 = 26V_3$

$V_3 = 10/26 = 0.385$ V

$V_2 = 4 \times 0.385 = 1.538$ V

$V_1 = 10 \times 0.385 = 3.846$ V

Problem 7.5.2

Given: A circuit has two meshes. The left mesh contains a 20 V source and resistors R_1 = 4 kohm and R_2 = 8 kohm (shared). The right mesh contains R_2 (shared), R_3 = 6 kohm, and a 10 V source. Both mesh currents are defined clockwise.

Find: The mesh currents I_1 and I_2 using mesh analysis.

Solution: KVL for mesh 1 (clockwise): $20 = 4000I_1 + 8000(I_1 - I_2)$

$20 = 12000I_1 - 8000I_2$ ... (1)

KVL for mesh 2 (clockwise): $-10 = 6000I_2 + 8000(I_2 - I_1)$

$-10 = -8000I_1 + 14000I_2$ ... (2)

From (2): $8000I_1 = 14000I_2 + 10$, so $I_1 = 1.75I_2 + 0.00125$

Substitute into (1): $20 = 12000(1.75I_2 + 0.00125) - 8000I_2$

$20 = 21000I_2 + 15 - 8000I_2$

$5 = 13000I_2$

$I_2 = 5/13000 = 0.385$ mA

$I_1 = 1.75 \text{ x } 0.000385 + 0.00125 = 0.000673 + 0.00125 = 1.923$ mA

Current through R_2: $I_{R_2} = I_1 - I_2 = 1.923 - 0.385 = 1.538$ mA (flowing downward)

Problem 7.5.3

Given: A circuit has a 12 V voltage source on the left, a 4 mA current source on the right, and three resistors: R_1 = 3 kohm (connected from the voltage source to a central node), R_2 = 6 kohm (from the central node to ground), and R_3 = 2 kohm (from the central node to the current source, which connects to ground).

Find: The voltage at the central node using superposition.

Solution: Source 1 alone (12 V active, current source open):

R_2 and R_3 are in series (since the current source is open): R_{23} = 6,000 + 2,000 = 8,000 ohm.

Wait – if the current source is opened, R_3 is disconnected. R_2 is from the node to ground.

$V_{node,1} = 12 \text{ x } R_2 / (R_1 + R_2) = 12 \text{ x } 6{,}000 / (3{,}000 + 6{,}000) = 12 \text{ x } 2/3 = 8.0$ V

Source 2 alone (4 mA active, voltage source shorted):

With the 12 V source shorted, R_1 connects the node to ground. R_1 and R_2 are in parallel:

$R_{12} = (3{,}000 \text{ x } 6{,}000) / (3{,}000 + 6{,}000) = 18{,}000{,}000 / 9{,}000 = 2{,}000$ ohm

The 4 mA current source drives current through R_3 into the node. The current splits between R_{12} to ground.

$V_{node,2} = I_{source} \text{ x } R_{12} =$ ... Actually, the voltage at the node with the current source is:

KCL: $4 \times 10^{-3} = V_{node}/3000 + V_{node}/6000 +$ (V_{node} is connected through R_3)

Let me reconsider. The current source pushes 4 mA into the central node through R_3. The node connects to ground through R_1 (3 kohm) and R_2 (6 kohm) in parallel.

Actually, R_3 is between the current source and the node. The voltage at the node:

$V_{node,2} = I_{source} \times (R_1 \,||\, R_2) = 0.004 \times 2{,}000 = 8.0$ V

But the current also flows through R_3, so the voltage at the node depends on R_3 placement. If R_3 is between the node and the current source:

The current source forces 4 mA. The node voltage is determined by KCL at the node:

$(V_{node})/3000 + (V_{node})/6000 = 4 \times 10^{-3}$ (current entering from current source through R_3)

Wait – current source in series with R_3 means the branch current is fixed at 4 mA. So 4 mA enters the node, and exits through R_1 and R_2:

$V_{node,2} = 4 \times 10^{-3} \times (R_1 \,||\, R_2) = 0.004 \times 2{,}000 = 8.0$ V

Superposition result:

$V_{node} = V_{node,1} + V_{node,2} = 8.0 + 8.0 = 16.0$ V

Problem 7.5.4

Given: A circuit contains a VCVS (voltage-controlled voltage source). A 10 V independent source is in series with a 5 kohm resistor R_1 connected to node A. A dependent voltage source $V_x = 3V_A$ is in series with a 2 kohm resistor R_2, and this branch connects node A to ground.

Find: The voltage V_A at node A.

Solution: The branch from node A through the dependent source and R_2 to ground has a KVL equation:

$V_A = V_x + I_2 \times R_2 = 3V_A + I_2 \times 2000$

Also, $I_2 = (V_A - 3V_A) / 2000 = -2V_A / 2000 = -V_A/1000$

KCL at node A: current from 10 V source = current through R_2 branch

$(10 - V_A) / 5000 = -(-V_A/1000)$

Wait – let me set up KCL properly. Current entering node A from the 10 V source through R_1:

$I_1 = (10 - V_A) / 5000$

Current leaving node A through the dependent source branch. The voltage across R_2 is $V_A - 3V_A = -2V_A$ (from node A, through the VCVS with + at the node side, then R_2 to ground):

$I_2 = (V_A - 3V_A) / 2000 = -2V_A / 2000 = -V_A/1000$

KCL at node A: $I_1 = I_2$ (current entering = current leaving)

$(10 - V_A) / 5000 = -V_A / 1000$

Multiply by 5000: $10 - V_A = -5V_A$

$10 = -5V_A + V_A = -4V_A$

$V_A = 10 / (-4) = -2.5$ V

The negative voltage indicates the dependent source forces a reversal of the expected polarity. With $V_A = -2.5$ V, the dependent source produces $V_x = 3(-2.5) = -7.5$ V.

Problem 7.5.5

Given: A circuit has three voltage sources (6 V, 12 V, and 18 V) and four resistors forming a two-mesh network. Mesh 1 (left): 6 V source, $R_1 = 2$ kohm, $R_2 = 4$ kohm (shared). Mesh 2 (right): $R_2 = 4$ kohm (shared), $R_3 = 3$ kohm, 12 V source. Additionally, an 18 V source is in series with $R_4 = 6$ kohm between the top node (junction of R_1 and R_2) and ground, forming a third branch.

Find: The voltage at the top node using superposition (considering each source independently).

Solution: Let V_N be the voltage at the top node. Using nodal analysis with superposition:

18 V source alone (6 V and 12 V replaced by short circuits):

The 18 V source is in series with $R_4 = 6$ kohm. The other branches present $R_1 = 2$ kohm and $R_3 = 3$ kohm (since the voltage sources are shorted, R_2 connects directly through the shorted sources).

Wait – R_2 is shared between the two meshes. With both 6 V and 12 V sources shorted:

From the top node: R_1 to ground (through shorted 6 V), R_3 to ground (through shorted 12 V), and the 18 V source with R_4.

$R_{parallel} = R_1 || R_3 = (2000 \times 3000) / (2000 + 3000) = 6{,}000{,}000 / 5{,}000 = 1{,}200$ ohm

$V_{N,18} = 18 \times R_{parallel} / (R_4 + R_{parallel}) = 18 \times 1200 / (6000 + 1200) = 21{,}600 / 7{,}200 = 3.0$ V

6 V source alone (12 V and 18 V shorted):

From the top node: R_3 to ground (through shorted 12 V), R_4 to ground (through shorted 18 V).

$R_{parallel} = R_3 || R_4 = (3000 \times 6000) / (3000 + 6000) = 18{,}000{,}000 / 9{,}000 = 2{,}000$ ohm

$V_{N,6} = 6 \times R_{parallel} / (R_1 + R_{parallel}) = 6 \times 2000 / (2000 + 2000) = 12{,}000 / 4{,}000 = 3.0$ V

12 V source alone (6 V and 18 V shorted):

From the top node: R_1 to ground (through shorted 6 V), R_4 to ground (through shorted 18 V).

$R_{parallel} = R_1 || R_4 = (2000 \times 6000) / (2000 + 6000) = 12{,}000{,}000 / 8{,}000 = 1{,}500$ ohm

$V_{N,12} = 12 \times R_{parallel} / (R_3 + R_{parallel}) = 12 \times 1500 / (3000 + 1500) = 18{,}000 / 4{,}500 = 4.0$ V

Total by superposition:

$V_N = V_{N,18} + V_{N,6} + V_{N,12} = 3.0 + 3.0 + 4.0 = 10.0$ V

Problem 7.5.6

Given: A MOSFET amplifier small-signal model contains a VCCS (voltage-controlled current source) with g_m = 10 mA/V. The gate-to-source voltage V_{gs} appears across a 50 kohm input resistance. The drain current source $g_m V_{gs}$ drives into a 5 kohm drain resistor R_D in parallel with a 20 kohm load R_L.

Find: (a) The effective load resistance seen by the current source. (b) The voltage gain $A_v = V_{out}/V_{gs}$. (c) If V_{gs} = 0.1 V, the output voltage.

Solution:

(a) Effective load: $R_{eff} = R_D \,||\, R_L$ = (5,000 x 20,000) / (5,000 + 20,000) = 100,000,000 / 25,000 = 4,000 ohm = 4 kohm

(b) Output voltage: $V_{out} = -g_m \text{ x } V_{gs} \text{ x } R_{eff}$ (negative because drain current flows into the resistor, creating an inverting gain)

$A_v = V_{out} / V_{gs} = -g_m \text{ x } R_{eff}$ = -0.010 x 4,000 = -40 V/V

(c) $V_{out} = A_v \text{ x } V_{gs}$ = -40 x 0.1 = -4.0 V

The minus sign indicates the amplifier is inverting – a positive input produces a negative output.

Chapter 7 — Section 7.6: Circuit Theorems

Practice problems covering Thevenin's theorem, Norton's theorem, and maximum power transfer.

Problem 7.6.1

Given: A linear network consists of a 50 V source in series with $R_1 = 10$ ohm, connected to a node. From that node, $R_2 = 15$ ohm connects to ground, and two output terminals are taken across R_2.

Find: (a) The Thevenin equivalent voltage V_{Th}. (b) The Thevenin equivalent resistance R_{Th}. (c) The current delivered to a 5 ohm load connected across the output terminals.

Solution:

(a) V_{Th} (open-circuit voltage across R_2): No load current flows through any additional path.

$V_{Th} = V_{source} \times R_2 / (R_1 + R_2) = 50 \times 15 / (10 + 15) = 750 / 25 = 30$ V

(b) R_{Th} (deactivate 50 V source by shorting): R_1 and R_2 are in parallel.

$R_{Th} = (R_1 \times R_2) / (R_1 + R_2) = (10 \times 15) / (10 + 15) = 150 / 25 = 6$ ohm

(c) With a 5 ohm load: $I_{load} = V_{Th} / (R_{Th} + R_{load}) = 30 / (6 + 5) = 30 / 11 = 2.727$ A

Problem 7.6.2

Given: A circuit has a 2 A current source in parallel with a 12 ohm resistor, which is in series with an 8 ohm resistor. The output terminals are across the 8 ohm resistor.

Find: (a) The Norton equivalent current I_N. (b) The Norton equivalent resistance R_N. (c) The Thevenin equivalent circuit. (d) The load current for a 24 ohm load.

Solution: Topology: Node A at top. Current source (2 A upward) from ground to node A. $R_1 = 12$ ohm from node A to ground (parallel with the current source). $R_2 = 8$ ohm from node A to node B. Output terminals: node B to ground.

(a) Finding I_N via the Thevenin approach.

Open-circuit voltage V_{Th}: With the output open, no current flows through R_2. KCL at node A: $2 = V_A/12$, so $V_A = 24$ V. Since no current flows through R_2, $V_B = V_A = 24$ V.

$V_{Th} = 24$ V

R_{Th}: Open the current source. Looking from B to ground: R_2 is in series with R_1 (from A to ground).

$R_{Th} = R_2 + R_1 = 8 + 12 = 20$ ohm

$I_N = V_{Th} / R_{Th} = 24 / 20 = 1.2$ A

(b) $R_N = R_{Th} = 20$ ohm

(c) Thevenin equivalent: 24 V source in series with 20 ohm.

(d) Load current with $R_L = 24$ ohm:

$I_{load} = V_{Th} / (R_{Th} + R_L) = 24 / (20 + 24) = 24 / 44 = 0.545$ A

Problem 7.6.3

Given: A signal generator has a Thevenin equivalent of $V_{Th} = 2$ V and $R_{Th} = 50$ ohm. It drives a transmission line terminated in a load.

Find: (a) The load resistance for maximum power transfer. (b) The maximum power delivered to the load. (c) The power delivered if the load is 75 ohm (a common mismatch). (d) The percentage of maximum power lost due to the mismatch.

Solution:

(a) For maximum power transfer: $R_{load} = R_{Th} = 50$ ohm

(b) Maximum power: $P_{max} = V_{Th}^2 / (4 \text{ x } R_{Th}) = 2^2 / (4 \text{ x } 50) = 4 / 200 = 20$ mW

(c) With $R_{load} = 75$ ohm:

$I = V_{Th} / (R_{Th} + R_{load}) = 2 / (50 + 75) = 2 / 125 = 16$ mA

$P_{load} = I^2 \text{ x } R_{load} = (0.016)^2 \text{ x } 75 = 2.56 \text{ x } 10^{-4} \text{ x } 75 = 19.2$ mW

(d) Power loss relative to maximum: $(20 - 19.2) / 20 \text{ x } 100 = 4\%$

A 50-to-75 ohm mismatch only loses 4% of the maximum deliverable power, which explains why moderate impedance mismatches are often tolerable.

Problem 7.6.4

Given: A bridge circuit is powered by a 100 V source. The four bridge arms are: $R_1 = 100$ ohm (top-left), $R_2 = 200$ ohm (top-right), $R_3 = 150$ ohm (bottom-left), $R_4 = 300$ ohm (bottom-right). A load R_L is connected across the bridge diagonal (between the left midpoint and right midpoint).

Find: (a) The Thevenin equivalent seen by R_L. (b) The current through R_L if $R_L = 50$ ohm. (c) Whether the bridge is balanced.

Solution:

(a) V_{Th}: Open-circuit voltage across the diagonal.

Left midpoint voltage: $V_L = 100 \times R_3 / (R_1 + R_3) = 100 \times 150 / (100 + 150) = 100 \times 150/250 = 60$ V

Right midpoint voltage: $V_R = 100 \times R_4 / (R_2 + R_4) = 100 \times 300 / (200 + 300) = 100 \times 300/500 = 60$ V

$V_{Th} = V_L - V_R = 60 - 60 = 0$ V

R_{Th}: Short the voltage source. Looking into the diagonal:

$R_{Th} = (R_1 || R_3) + (R_2 || R_4) = (100 \times 150)/(100 + 150) + (200 \times 300)/(200 + 300) = 60 + 120 = 180$ ohm

(b) Since $V_{Th} = 0$ V, the current through any load R_L is: $I_L = 0 / (180 + 50) = 0$ A

(c) The bridge is balanced because $R_1/R_3 = 100/150 = 2/3$ and $R_2/R_4 = 200/300 = 2/3$. When $R_1/R_3 = R_2/R_4$, no current flows through the bridge diagonal regardless of load resistance.

Problem 7.6.5

Given: A battery-powered sensor system has a battery with $V_{oc} = 3.0$ V and internal resistance $R_{int} =$ 2 ohm. The sensor draws a constant 50 mA.

Find: (a) The terminal voltage under load. (b) The power delivered to the sensor. (c) The power lost in the internal resistance. (d) The efficiency (ratio of load power to total power). (e) Compare to the maximum power transfer case.

Solution:

(a) Terminal voltage: $V_t = V_{oc} - I \times R_{int} = 3.0 - 0.050 \times 2 = 3.0 - 0.1 = 2.9$ V

(b) Power to sensor: $P_{load} = V_t \times I = 2.9 \times 0.050 = 145$ mW

(c) Power lost internally: $P_{int} = I^2 \times R_{int} = (0.050)^2 \times 2 = 0.0025 \times 2 = 5$ mW

(d) Efficiency: eta = $P_{load} / (P_{load} + P_{int}) = 145 / (145 + 5) = 145 / 150 = 96.7\%$

(e) At maximum power transfer, $R_{load} = R_{int} = 2$ ohm:

$I_{max} = 3.0 / (2 + 2) = 0.75$ A

$P_{max} = I_{max}^2 \times R_{load} = 0.5625 \times 2 = 1.125$ W

But efficiency would be only 50%, and the battery would drain 30x faster. The sensor's operating point ($R_{load} = 2.9/0.050 = 58$ ohm » R_{int}) achieves much higher efficiency at the cost of lower power extraction – appropriate for a battery-powered device.

Chapter 7 — Section 7.7: AC Circuit Analysis

Practice problems covering impedance, resonance, AC power, and mutual inductance/coupled circuits.

Problem 7.7.1

Given: A series RLC circuit has R = 47 ohm, L = 100 mH, and C = 10 uF, driven by a 120 V_{rms}, 50 Hz source.

Find: (a) The inductive reactance. (b) The capacitive reactance. (c) The total impedance (magnitude and angle). (d) The current magnitude. (e) The phase angle between voltage and current.

Solution:

(a) omega = 2 x pi x 50 = 314.16 rad/s

X_L = omega x L = 314.16 x 0.100 = 31.42 ohm

(b) X_C = 1 / (omega x C) = 1 / (314.16 x 10 x 10^{-6}) = 1 / 0.003142 = 318.3 ohm

(c) Net reactance: X = X_L - X_C = 31.42 - 318.3 = -286.9 ohm (capacitive)

|Z| = sqrt(R^2 + X^2) = sqrt(47^2 + 286.9^2) = sqrt(2,209 + 82,312) = sqrt(84,521) = 290.7 ohm

theta = arctan(X/R) = arctan(-286.9/47) = arctan(-6.105) = -80.7 deg

Z = 47 - j286.9 ohm

(d) Current: I = V / |Z| = 120 / 290.7 = 0.413 A_{rms}

(e) The current leads the voltage by 80.7 deg (capacitive circuit, since X_C > X_L).

Problem 7.7.2

Given: A parallel RLC circuit has R = 1 kohm, L = 5 mH, and C = 20 nF. It is driven by a current source.

Find: (a) The resonant frequency. (b) The quality factor. (c) The bandwidth. (d) The impedance at resonance.

Solution:

(a) Resonant frequency: $f_0 = 1 / (2 \times pi \times sqrt(LC))$

$f_0 = 1 / (2 \times pi \times sqrt(5 \times 10^{-3} \times 20 \times 10^{-9}))$

$f_0 = 1 / (2 \times pi \times sqrt(10^{-10}))$

$f_0 = 1 / (2 \times pi \times 10^{-5}) = 1 / (6.283 \times 10^{-5}) = 15{,}915$ Hz = 15.92 kHz

(b) For a parallel RLC circuit: Q = R x sqrt(C/L)

$Q = 1000 \times sqrt(20 \times 10^{-9} / 5 \times 10^{-3}) = 1000 \times sqrt(4 \times 10^{-6}) = 1000 \times 2 \times 10^{-3} = 2.0$

Or equivalently: $Q = R / (2 \times pi \times f_0 \times L)$ ... Let's verify:

$Q = R / (omega_0 \times L) = 1000 / (2 \times pi \times 15915 \times 0.005) = 1000 / 500 = 2.0$ (verified)

(c) Bandwidth: $BW = f_0 / Q = 15{,}915 / 2.0 = 7{,}958$ Hz = 7.96 kHz

(d) At resonance, the impedance of a parallel RLC circuit equals R (the reactive branches cancel):

$Z_{resonance} = 1{,}000$ ohm = 1 kohm

Problem 7.7.3

Given: A single-phase motor draws 8 A_{rms} from a 240 V_{rms}, 60 Hz supply with a power factor of 0.65 lagging. A capacitor bank is to be added in parallel to correct the power factor to 0.95 lagging.

Find: (a) The real power P, reactive power Q, and apparent power S before correction. (b) The reactive power after correction. (c) The required capacitor reactive power. (d) The capacitor value needed. (e) The new supply current after correction.

Solution:

(a) Before correction:

S = V x I = 240 x 8 = 1,920 VA

P = S x cos(phi) = 1,920 x 0.65 = 1,248 W

$phi_1 = arccos(0.65) = 49.46$ deg

$Q_1 = S \times sin(phi_1) = 1{,}920 \times sin(49.46) = 1{,}920 \times 0.7599 = 1{,}459$ VAR (inductive)

(b) After correction (pf = 0.95), the real power remains the same:

$phi_2 = arccos(0.95) = 18.19$ deg

$Q_2 = P \times tan(phi_2) = 1{,}248 \times tan(18.19) = 1{,}248 \times 0.3287 = 410.3$ VAR

(c) Capacitor reactive power: $Q_C = Q_1 - Q_2 = 1{,}459 - 410.3 = 1{,}048.7$ VAR

(d) Capacitor value: $Q_C = V^2 \times omega \times C$

$C = Q_C$ / (V^2 x omega) = 1,048.7 / (240^2 x 2 x pi x 60)

C = 1,048.7 / (57,600 x 376.99) = 1,048.7 / 21,714,624 = 48.3 uF

(e) New apparent power: S_2 = P / cos(phi_2) = 1,248 / 0.95 = 1,313.7 VA

New current: $I_2 = S_2$ / V = 1,313.7 / 240 = 5.47 A_{rms}

The supply current drops from 8 A to 5.47 A – a 31.6% reduction – reducing I^2R losses in the supply wiring.

Problem 7.7.4

Given: An audio transformer has a turns ratio of N_1/N_2 = 10:1, primary inductance L_1 = 500 mH, and coupling coefficient k = 0.97. The transformer connects a 600 ohm microphone line to an 8 ohm preamplifier input.

Find: (a) The secondary inductance L_2. (b) The mutual inductance M. (c) The reflected impedance seen by the microphone line. (d) Whether this transformer provides a good impedance match.

Solution:

(a) $L_2 = L_1$ / $(N_1/N_2)^2$ = 500 mH / 10^2 = 500 / 100 = 5 mH

(b) M = k x sqrt(L_1 x L_2) = 0.97 x sqrt(0.500 x 0.005) = 0.97 x sqrt(0.0025) = 0.97 x 0.05 = 48.5 mH

(c) Reflected impedance: $Z_{reflected}$ = $(N_1/N_2)^2$ x Z_{load} = 10^2 x 8 = 800 ohm

(d) The microphone line sees 800 ohm instead of the desired 600 ohm – a mismatch ratio of 800/600 = 1.33:1.

For a perfect match: N_1/N_2 = sqrt(Z_{source}/Z_{load}) = sqrt(600/8) = sqrt(75) = 8.66:1

The 10:1 transformer is reasonably close but not optimal. A custom 8.66:1 ratio would provide exact impedance matching.

Problem 7.7.5

Given: A series RL circuit with R = 200 ohm and L = 0.5 H carries a current of 3 A_{rms} at 60 Hz.

Find: (a) The impedance. (b) The voltage across the resistor. (c) The voltage across the inductor. (d) The total source voltage. (e) The real, reactive, and apparent power.

Solution:

(a) omega = 2 x pi x 60 = 376.99 rad/s

X_L = omega x L = 376.99 x 0.5 = 188.5 ohm

Z = R + jX_L = 200 + j188.5 ohm

|Z| = sqrt(200^2 + 188.5^2) = sqrt(40,000 + 35,532) = sqrt(75,532) = 274.8 ohm

(b) $V_R = I \times R = 3 \times 200 = 600\ V_{rms}$

(c) $V_L = I \times X_L = 3 \times 188.5 = 565.5\ V_{rms}$

(d) $V_{source} = I \times |Z| = 3 \times 274.8 = 824.4\ V_{rms}$

Note: V_{source} is not equal to $V_R + V_L$ arithmetically (600 + 565.5 = 1165.5) because the voltages are 90 deg out of phase. The phasor sum gives sqrt($600^2 + 565.5^2$) = sqrt(360,000 + 319,790) = sqrt(679,790) = 824.5 V (verified).

(e) Phase angle: phi = arctan(188.5/200) = arctan(0.9425) = 43.3 deg

P = V x I x cos(phi) = 824.4 x 3 x cos(43.3) = 2473.2 x 0.7275 = 1,800 W

Or: P = I^2 x R = 9 x 200 = 1,800 W (verified)

Q = I^2 x X_L = 9 x 188.5 = 1,696.5 VAR

S = V x I = 824.4 x 3 = 2,473.2 VA

Check: S = sqrt($P^2 + Q^2$) = sqrt(3,240,000 + 2,878,140) = sqrt(6,118,140) = 2,473.5 VA (verified).

Problem 7.7.6

Given: A series RLC tuning circuit in an AM radio receiver must select a station at 1,000 kHz. The inductor is L = 250 uH with a quality factor Q = 50 at the resonant frequency.

Find: (a) The required capacitance. (b) The bandwidth of the tuned circuit. (c) The resistance of the inductor coil. (d) Whether the circuit can reject an adjacent channel at 1,010 kHz (10 kHz spacing).

Solution:

(a) f_0 = 1 / (2 x pi x sqrt(LC)), solving for C:

C = 1 / (4 x pi^2 x f_0^2 x L) = 1 / (4 x pi^2 x $(10^6)^2$ x 250 x 10^{-6})

C = 1 / (4 x 9.8696 x 10^{12} x 2.5 x 10^{-4}) = 1 / (9.8696 x 10^9)

C = 101.3 pF

(b) Bandwidth: BW = f_0 / Q = 1,000,000 / 50 = 20,000 Hz = 20 kHz

(c) At resonance: Q = $omega_0$ x L / R

R = $omega_0$ x L / Q = 2 x pi x 10^6 x 250 x 10^{-6} / 50 = 6,283,185 x 2.5 x 10^{-4} / 50 = 1,570.8 / 50 = 31.4 ohm

(d) The adjacent channel at 1,010 kHz is 10 kHz away from the center frequency. The 3 dB bandwidth is 20 kHz, so the 10 kHz offset is at the half-bandwidth point.

At f = 1,010 kHz, the attenuation relative to the peak is approximately:

Attenuation = 1 / sqrt(1 + $(2 \times delta\_f / BW)^2$) where delta_f = 10 kHz

= 1 / sqrt(1 + $(20/20)^2$) = 1 / sqrt(2) = 0.707 = -3 dB

Only 3 dB of adjacent channel rejection is insufficient for good selectivity. A higher-Q circuit or a multi-stage filter would be needed. Doubling Q to 100 would narrow the bandwidth to 10 kHz, providing better selectivity.

Chapter 7 — Section 7.8: Transient Analysis

Practice problems covering RC circuits, RL circuits, and RLC circuits.

Problem 7.8.1

Given: A 220 uF capacitor charged to 50 V is discharged through a 10 kohm resistor at t = 0.

Find: (a) The time constant. (b) The capacitor voltage at t = 1 s, 2 s, and 5 s. (c) The initial discharge current. (d) The time at which the voltage drops to 10 V.

Solution:

(a) tau = R x C = 10,000 x 220 x 10^{-6} = 2.2 s

(b) Discharge: $V_c(t) = V_0 \times e^{-t/tau}$

At t = 1 s: $V_c = 50 \times e^{-1/2.2} = 50 \times e^{-0.4545} = 50 \times 0.6346 = 31.73$ V

At t = 2 s: $V_c = 50 \times e^{-2/2.2} = 50 \times e^{-0.9091} = 50 \times 0.4029 = 20.14$ V

At t = 5 s: $V_c = 50 \times e^{-5/2.2} = 50 \times e^{-2.2727} = 50 \times 0.1030 = 5.15$ V

(c) Initial current: $I_0 = V_0 / R = 50 / 10{,}000 = 5$ mA

(d) Time to reach 10 V: $10 = 50 \times e^{-t/2.2}$

$e^{-t/2.2} = 0.2$

$-t/2.2 = \ln(0.2) = -1.6094$

$t = 2.2 \times 1.6094 = 3.54$ s

Problem 7.8.2

Given: A relay coil has an inductance of 50 mH and a resistance of 25 ohm. It is suddenly connected to a 12 V DC supply.

Find: (a) The time constant. (b) The steady-state current. (c) The current at t = 1 ms. (d) The time to reach 90% of the steady-state current. (e) The initial rate of change of current (di/dt at t = 0).

Solution:

(a) tau = L / R = 0.050 / 25 = 2 ms

(b) Steady-state current: I_{ss} = V / R = 12 / 25 = 480 mA

(c) At t = 1 ms: I(t) = I_{ss} x (1 - $e^{-t/tau}$)

I(0.001) = 0.480 x (1 - $e^{-1/2}$) = 0.480 x (1 - 0.6065) = 0.480 x 0.3935 = 188.9 mA

(d) For 90% of I_{ss}: 0.9 = 1 - $e^{-t/tau}$

$e^{-t/tau}$ = 0.1

-t/tau = ln(0.1) = -2.3026

t = 2 x 10^{-3} x 2.3026 = 4.61 ms

(e) Initial di/dt: di/dt$|_{t=0}$ = V / L = 12 / 0.050 = 240 A/s

This is the maximum rate of current rise. It decreases exponentially as the back-EMF decreases.

Problem 7.8.3

Given: A series RLC circuit has R = 200 ohm, L = 50 mH, and C = 0.5 uF. A 20 V DC step is applied at t = 0 with zero initial conditions.

Find: (a) The natural frequency $omega_0$. (b) The damping ratio zeta. (c) The type of response (underdamped, critically damped, or overdamped). (d) The damped natural frequency $omega_d$ (if applicable). (e) The final (steady-state) capacitor voltage.

Solution:

(a) $omega_0$ = 1 / sqrt(LC) = 1 / sqrt(0.050 x 0.5 x 10^{-6})

= 1 / sqrt(25 x 10^{-9}) = 1 / (5 x $10^{-4.5}$)

Wait: sqrt(25 x 10^{-9}) = 5 x $10^{-4.5}$ = 5 x 3.162 x 10^{-5} = 1.581 x 10^{-4}

$omega_0$ = 1 / (1.581 x 10^{-4}) = 6,325 rad/s

(b) zeta = R / (2 x sqrt(L/C)) = 200 / (2 x sqrt(0.050 / 0.5 x 10^{-6}))

= 200 / (2 x sqrt(100,000)) = 200 / (2 x 316.2) = 200 / 632.5 = 0.316

(c) Since zeta = 0.316 < 1, the response is underdamped (oscillatory with exponentially decaying envelope).

(d) $omega_d$ = $omega_0$ x sqrt(1 - $zeta^2$) = 6,325 x sqrt(1 - 0.1) = 6,325 x sqrt(0.9) = 6,325 x 0.9487 = 5,999 rad/s

f_d = $omega_d$ / (2 x pi) = 5,999 / 6.283 = 955 Hz

(e) At DC steady state, the capacitor is fully charged and no current flows:

$V_{C,ss}$ = V_{source} = 20 V (all voltage appears across the capacitor since the inductor acts as a short and the resistor has no current through it)

Problem 7.8.4

Given: An RC timing circuit in a 555 timer uses R = 100 kohm and C = 4.7 uF. The capacitor charges from 1/3 V_{CC} to 2/3 V_{CC} (where V_{CC} = 12 V) and then discharges back to 1/3 V_{CC} to set the oscillation frequency.

Find: (a) The time constant. (b) The charge time from 4 V to 8 V (1/3 to 2/3 of 12 V). (c) The approximate oscillation frequency (using the 555 formula f = 1.44 / (R x C) for a basic astable with $R_A = R_B$ = R).

Solution:

(a) tau = R x C = 100,000 x 4.7 x 10^{-6} = 0.47 s

(b) The capacitor charges from V_1 = 4 V toward V_{CC} = 12 V. Using the charging equation:

$V(t) = V_{CC} - (V_{CC} - V_1) \times e^{-t/tau} = 12 - 8 \times e^{-t/0.47}$

At V(t) = 8 V: $8 = 12 - 8 \times e^{-t/0.47}$

$8 \times e^{-t/0.47} = 4$

$e^{-t/0.47} = 0.5$

-t/0.47 = ln(0.5) = -0.6931

t_{charge} = 0.47 x 0.6931 = 0.326 s

(c) For a basic 555 astable with $R_A = R_B$ = R:

f = 1.44 / ((R_A + 2R_B) x C) = 1.44 / ((100,000 + 200,000) x 4.7 x 10^{-6})

= 1.44 / (300,000 x 4.7 x 10^{-6}) = 1.44 / 1.41 = 1.02 Hz

Period: T = 1/f = 0.98 s

Problem 7.8.5

Given: A series RLC circuit has R = 400 ohm, L = 25 mH, and C = 1 uF. The circuit is initially at rest and a 10 V step is applied.

Find: (a) $omega_0$ and zeta. (b) Classify the response. (c) If overdamped, find the two natural frequencies s_1 and s_2.

Solution:

(a) $omega_0$ = 1 / sqrt(LC) = 1 / sqrt(25 x 10^{-3} x 10^{-6}) = 1 / sqrt(25 x 10^{-9}) = 1 / (5 x $10^{-4.5}$)

= 1 / (1.581 x 10^{-4}) = 6,325 rad/s

zeta = R / (2 x sqrt(L/C)) = 400 / (2 x sqrt(0.025 / 10^{-6})) = 400 / (2 x sqrt(25,000))

= 400 / (2 x 158.1) = 400 / 316.2 = 1.265

(b) Since zeta = 1.265 > 1, the response is overdamped (no oscillation, exponential decay).

(c) The characteristic equation roots:

$s_{1,2}$ = -zeta x $omega_0$ +/- $omega_0$ x sqrt($zeta^2$ - 1)

= -1.265 x 6,325 +/- 6,325 x sqrt(1.6002 - 1)

= -8,001 +/- 6,325 x sqrt(0.6002)

= -8,001 +/- 6,325 x 0.7748

= -8,001 +/- 4,901

s_1 = -8,001 + 4,901 = -3,100 rad/s (slower mode, tau_1 = 1/3100 = 0.323 ms)

s_2 = -8,001 - 4,901 = -12,902 rad/s (faster mode, tau_2 = 1/12902 = 0.0775 ms)

The response is a sum of two decaying exponentials: $V_C(t) = 10 + A_1e^{s_1t} + A_2e^{s_2t}$, where A_1 and A_2 are determined by initial conditions.

Problem 7.8.6

Given: An RL circuit consists of a 500 mH inductor in series with a 100 ohm resistor, carrying a steady-state current of 2 A from a DC source. At t = 0, the source is suddenly disconnected and the inductor discharges through the resistor.

Find: (a) The time constant. (b) The initial energy stored in the inductor. (c) The voltage across the resistor at $t = 0^+$. (d) The time at which 95% of the stored energy has been dissipated.

Solution:

(a) tau = L / R = 0.500 / 100 = 5 ms

(b) Initial energy: W = 1/2 x L x I^2 = 0.5 x 0.500 x 2^2 = 0.5 x 0.500 x 4 = 1.0 J

(c) At $t = 0^+$, the inductor current cannot change instantaneously and equals 2 A. The voltage across the resistor:

$V_R(0^+) = I_0$ x R = 2 x 100 = 200 V

This demonstrates why disconnecting an inductive circuit can create dangerous voltage spikes.

(d) The energy remaining at time t: W(t) = 1/2 x L x $I(t)^2$ = 1/2 x L x $(I_0e^{-t/tau})^2$ = W_0 x $e^{-2t/tau}$

For 95% dissipated (5% remaining): 0.05 = $e^{-2t/tau}$

-2t/tau = ln(0.05) = -2.996

t = tau x 2.996 / 2 = 0.005 x 1.498 = 7.49 ms

This is approximately 1.5 time constants, at which point 95% of the magnetic energy has been converted to heat in the resistor.

Chapter 7 — Section 7.9: Two-Port Networks

Practice problems covering two-port parameters (Z, Y, h, ABCD) and two-port interconnections.

Problem 7.9.1

Given: A pi-network has $Z_a = 100$ ohm as a shunt element from port 1 to ground, $Z_b = 50$ ohm as the series element between port 1 and port 2, and $Z_c = 200$ ohm as a shunt element from port 2 to ground.

Find: The Y-parameters of the pi-network.

Solution: For a pi-network, the Y-parameters are:

$Y_{11} = 1/Z_a + 1/Z_b$ (short-circuit input admittance with $V_2 = 0$)

$Y_{11} = 1/100 + 1/50 = 0.01 + 0.02 = 0.03$ S (30 mS)

$Y_{12} = -1/Z_b$ (mutual admittance)

$Y_{12} = -1/50 = -0.02$ S (-20 mS)

$Y_{21} = -1/Z_b = -0.02$ S (-20 mS)

$Y_{22} = 1/Z_b + 1/Z_c = 1/50 + 1/200 = 0.02 + 0.005 = 0.025$ S (25 mS)

Since $Y_{12} = Y_{21}$, the network is reciprocal, as expected for a passive network.

Problem 7.9.2

Given: A transistor amplifier is modeled as a two-port with h-parameters: $h_{11} = 2.5$ kohm (input impedance), $h_{12} = 2 \times 10^{-4}$ (reverse voltage ratio), $h_{21} = 150$ (current gain), $h_{22} = 25$ uS (output admittance). The amplifier is driven by a 1 kohm source and loaded with a 10 kohm collector resistor.

Find: (a) The voltage gain $A_v = V_2/V_1$. (b) The current gain $A_i = I_2/I_1$. (c) The input impedance Z_{in}. (d) The output impedance Z_{out}.

Solution:

(a) Voltage gain: $A_v = -h_{21} / (h_{22} \times R_L + 1)$

Wait – more precisely, for the h-parameter model with a load R_L:

$A_v = -h_{21} \times R_L / (h_{11} \times (1 + h_{22} \times R_L) - h_{12} \times h_{21} \times R_L)$

Denominator = $h_{11} + (h_{11}h_{22} - h_{12}h_{21}) \times R_L$

Delta_h = $h_{11} \times h_{22} - h_{12} \times h_{21} = 2{,}500 \times 25 \times 10^{-6} - 2 \times 10^{-4} \times 150 = 0.0625 - 0.03 = 0.0325$

$A_v = -h_{21} \times R_L / (h_{11} + \text{Delta\_h} \times R_L) = -150 \times 10{,}000 / (2{,}500 + 0.0325 \times 10{,}000)$

$= -1{,}500{,}000 / (2{,}500 + 325) = -1{,}500{,}000 / 2{,}825 = -531$ V/V

(b) Current gain: $A_i = h_{21} / (1 + h_{22} \times R_L) = 150 / (1 + 25 \times 10^{-6} \times 10{,}000)$

$= 150 / (1 + 0.25) = 150 / 1.25 = 120$ A/A

(c) Input impedance: $Z_{in} = h_{11} - h_{12} \times h_{21} \times R_L / (1 + h_{22} \times R_L)$

$= 2{,}500 - (2 \times 10^{-4} \times 150 \times 10{,}000) / (1.25) = 2{,}500 - 300/1.25 = 2{,}500 - 240 = 2{,}260$ ohm

(d) Output impedance: $Z_{out} = 1 / (h_{22} - h_{12} \times h_{21} / (h_{11} + R_S))$

$= 1 / (25 \times 10^{-6} - 2 \times 10^{-4} \times 150 / (2{,}500 + 1{,}000))$

$= 1 / (25 \times 10^{-6} - 0.03/3{,}500) = 1 / (25 \times 10^{-6} - 8.571 \times 10^{-6})$

$= 1 / (16.43 \times 10^{-6}) = 60.9$ kohm

Problem 7.9.3

Given: A transmission line segment is modeled as an ABCD network with parameters: A = cosh(gamma x l), B = Z_0 x sinh(gamma x l), C = sinh(gamma x l)/Z_0, D = cosh(gamma x l). For a lossless 50 ohm line that is lambda/4 long at the operating frequency, gamma x l = j x pi/2.

Find: (a) The ABCD parameters for the quarter-wave line. (b) The input impedance when terminated in a 200 ohm load. (c) The voltage gain V_2/V_1.

Solution:

(a) cosh(j x pi/2) = cos(pi/2) = 0

sinh(j x pi/2) = j x sin(pi/2) = j

A = 0, B = Z_0 x j = j50 ohm, C = j/Z_0 = j0.02 S, D = 0

Check reciprocity: AD - BC = 0 - (j50)(j0.02) = $-j^2$ x 1 = 1. Verified.

(b) Input impedance: $Z_{in} = (AZ_L + B) / (CZ_L + D)$

= (0 x 200 + j50) / (j0.02 x 200 + 0) = j50 / j4 = 12.5 ohm

This demonstrates the quarter-wave transformer property: $Z_{in} = Z_0^2 / Z_L = 50^2/200 = 2{,}500/200 =$ 12.5 ohm.

(c) Voltage gain: $V_2/V_1 = Z_L / (AZ_L + B)$ = 200 / (0 + j50) = 200 / (j50)

$|V_2/V_1| = 200/50 = 4.0$ (with a phase shift of -90 deg)

The quarter-wave transformer produces a 4:1 voltage step-up from 12.5 ohm to 200 ohm.

Problem 7.9.4

Given: Three identical filter sections are cascaded. Each section has ABCD parameters A = 1, B = 100 ohm, C = 0.01 S, D = 1.

Find: (a) The overall ABCD parameters for the cascade of three sections. (b) The insertion loss (in dB) when the cascade is terminated in Z_L = 600 ohm and driven by a Z_S = 600 ohm source.

Solution:

(a) For a cascade, $[ABCD]_{total} = [ABCD]_1 \times [ABCD]_2 \times [ABCD]_3$.

First, cascade two sections:

$A_{12} = 1 \times 1 + 100 \times 0.01 = 1 + 1 = 2$

$B_{12} = 1 \times 100 + 100 \times 1 = 200$ ohm

$C_{12} = 0.01 \times 1 + 1 \times 0.01 = 0.02$ S

$D_{12} = 0.01 \times 100 + 1 \times 1 = 2$

Now cascade the two-section result with the third section:

$A_T = 2 \times 1 + 200 \times 0.01 = 2 + 2 = 4$

$B_T = 2 \times 100 + 200 \times 1 = 200 + 200 = 400$ ohm

$C_T = 0.02 \times 1 + 2 \times 0.01 = 0.02 + 0.02 = 0.04$ S

$D_T = 0.02 \times 100 + 2 \times 1 = 2 + 2 = 4$

Check: AD - BC = 4 x 4 - 400 x 0.04 = 16 - 16 = 0.

Per-section: AD - BC = 1 - 1 = 0. Since $(0)^3 = 0$, but this should equal $(AD - BC)_{single}{}^3$...

Actually, for non-reciprocal check: each section has AD - BC = 1 x 1 - 100 x 0.01 = 1 - 1 = 0. This means the network is not reciprocal (or it's a degenerate case). The cascade determinant should be $0^3 = 0$. And 4 x 4 - 400 x 0.04 = 16 - 16 = 0. Consistent.

(b) Insertion loss: The voltage across the load with the cascade inserted:

$V_2/V_{source} = Z_L / ((A_T + B_T/Z_L) \times Z_L + \ldots)$

Using the full formula: $V_2 = V_S \times Z_L / (A_T Z_L + B_T + C_T Z_S Z_L + D_T Z_S)$

$= V_S \times 600 / (4 \times 600 + 400 + 0.04 \times 600 \times 600 + 4 \times 600)$

$= V_S \times 600 / (2{,}400 + 400 + 14{,}400 + 2{,}400)$

$= V_S \times 600 / 19{,}600 = V_S \times 0.03061$

Without the cascade (direct connection): $V_2 = V_S \times Z_L / (Z_S + Z_L) = V_S \times 600/1200 = 0.5 \times V_S$

Insertion loss = 20 x $\log_{10}$(0.5 / 0.03061) = 20 x $\log_{10}$(16.33) = 20 x 1.213 = 24.3 dB

Problem 7.9.5

Given: Two two-port networks are connected in parallel (port voltages shared). Network A has Y-parameters: Y_{11A} = 0.05 S, Y_{12A} = -0.02 S, Y_{21A} = -0.02 S, Y_{22A} = 0.04 S. Network B has Y-parameters: Y_{11B} = 0.03 S, Y_{12B} = -0.01 S, Y_{21B} = -0.01 S, Y_{22B} = 0.02 S.

Find: (a) The overall Y-parameters. (b) The voltage gain V_2/V_1 with port 2 terminated in a 20 ohm load.

Solution:

(a) For parallel-connected two-ports, Y-parameters add:

$Y_{11} = Y_{11A} + Y_{11B} = 0.05 + 0.03 = 0.08$ S

$Y_{12} = Y_{12A} + Y_{12B} = -0.02 + (-0.01) = -0.03$ S

$Y_{21} = Y_{21A} + Y_{21B} = -0.02 + (-0.01) = -0.03$ S

$Y_{22} = Y_{22A} + Y_{22B} = 0.04 + 0.02 = 0.06$ S

(b) With a 20 ohm load at port 2: $Y_L = 1/20 = 0.05$ S

KCL at port 2 with the load: $I_2 = Y_{21}V_1 + Y_{22}V_2$, and $I_2 = -Y_LV_2$ (current into the load)

$-0.05\ V_2 = -0.03\ V_1 + 0.06\ V_2$

$-0.03\ V_1 = -0.05\ V_2 - 0.06\ V_2 = -0.11\ V_2$

$V_2/V_1 = 0.03/0.11 = 0.273$ V/V (-11.3 dB)

Chapter 8 — Section 8.1: Signals and Systems

Practice problems covering signal classification, LTI systems, convolution, correlation, the sampling theorem, and aliasing.

Problem 8.1.1

Given: A signal $x(t) = 3\sin(2\pi \times 500t) + 7\cos(2\pi \times 1500t)$ is measured across a 1 Ω resistor.

Find: (a) Whether the signal is periodic, and if so its fundamental period, (b) the average power of each component, and (c) the total average power.

Solution:

(a) The first component has frequency $f_1 = 500$ Hz (period $T_1 = 2$ ms). The second has $f_2 = 1500$ Hz (period $T_2 = 0.667$ ms). Since $f_2/f_1 = 1500/500 = 3$, a rational ratio, the composite signal is periodic with fundamental period $T = T_1 = 2$ ms and fundamental frequency $f_0 = 500$ Hz.

(b) For sinusoidal signals into 1 Ω, $P = A^2/2$: $P_1 = 3^2/2 = 4.5$ W $P_2 = 7^2/2 = 24.5$ W

(c) Since the two components are at different frequencies (orthogonal), total power is the sum: $P_{total} = P_1 + P_2 = 4.5 + 24.5 = 29.0$ W

Problem 8.1.2

Given: An LTI system has impulse response $h(t) = 10e^{-500t}u(t)$, where $u(t)$ is the unit step function.

Find: (a) The frequency response $H(\omega)$, (b) the magnitude $|H(\omega)|$ and phase $\angle H(\omega)$ at $\omega = 500$ rad/s, and (c) the 3 dB bandwidth of the system.

Solution:

(a) The Fourier transform of $h(t) = ae^{-at}u(t)$ is $H(\omega) = a/(a + j\omega)$. With $a = 500$ and the leading coefficient of 10: $H(\omega) = 10 \times 500 / (500 + j\omega) = 5000 / (500 + j\omega)$

(b) At $\omega = 500$ rad/s: $|H(500)| = 5000 / \sqrt{(500^2 + 500^2)} = 5000 / \sqrt{(500{,}000)} = 5000 / 707.1 = 7.07$ $\angle H(500) = -\arctan(500/500) = -\arctan(1) = -45°$

(c) The DC gain is $|H(0)| = 5000/500 = 10$. The 3 dB point occurs where $|H(\omega)| = 10/\sqrt{2} = 7.07$: $5000 / \sqrt{(500^2 + \omega^2)} = 7.07$ $\sqrt{(250{,}000 + \omega^2)} = 707.1$ $250{,}000 + \omega^2 = 500{,}000$ $\omega_{3dB} = \sqrt{250{,}000} = 500$ rad/s $\rightarrow f_{3dB} = 500/(2\pi) = 79.6$ Hz

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Problem 8.1.3

Given: Compute the continuous-time convolution $y(t) = x(t) * h(t)$ where $x(t) = u(t) - u(t - 3)$ (a rectangular pulse of width 3 s and height 1) and $h(t) = 2e^{-t}u(t)$.

Find: The output $y(t)$ for all values of t.

Solution:

$y(t) = \int_0^\infty x(\tau) \times h(t - \tau)\, d\tau$

For $t < 0$: $x(\tau)$ and $h(t - \tau)$ do not overlap, so $y(t) = 0$.

For $0 \leq t < 3$: $x(\tau) = 1$ for $0 \leq \tau \leq t$, and $h(t - \tau) = 2e^{-(t-\tau)}$: $y(t) = \int_0^t 2e^{-(t-\tau)}\, d\tau = 2e^{-t} \int_0^t e^\tau\, d\tau = 2e^{-t}(e^t - 1) = 2(1 - e^{-t})$

For $t \geq 3$: $x(\tau) = 1$ for $0 \leq \tau \leq 3$: $y(t) = \int_0^3 2e^{-(t-\tau)}\, d\tau = 2e^{-t} \int_0^3 e^\tau\, d\tau = 2e^{-t}(e^3 - 1) = 2(e^3 - 1)e^{-t} \approx 38.17e^{-t}$

The output rises exponentially toward a steady value of 2 during the pulse, then decays exponentially after the pulse ends.

Problem 8.1.4

Given: Two discrete-time sequences are $x[n] = \{2, -1, 3, 1\}$ for $n = 0, 1, 2, 3$ and $h[n] = \{1, 2, 1\}$ for $n = 0, 1, 2$.

Find: (a) The convolution $y[n] = x[n] * h[n]$, and (b) the total energy of the output signal.

Solution:

(a) The output length is $4 + 3 - 1 = 6$ samples. Apply $y[n] = \Sigma\, x[k]h[n - k]$:

$y[0] = x[0]h[0] = 2(1) = 2$ $y[1] = x[0]h[1] + x[1]h[0] = 2(2) + (-1)(1) = 4 - 1 = 3$ $y[2] = x[0]h[2] + x[1]h[1] + x[2]h[0] = 2(1) + (-1)(2) + 3(1) = 2 - 2 + 3 = 3$ $y[3] = x[1]h[2] + x[2]h[1] + x[3]h[0] = (-1)(1) + 3(2) + 1(1) = -1 + 6 + 1 = 6$ $y[4] = x[2]h[2] + x[3]h[1] = 3(1) + 1(2) = 3 + 2 = 5$ $y[5] = x[3]h[2] = 1(1) = 1$

Therefore $y[n] = \{2, 3, 3, 6, 5, 1\}$ for $n = 0, 1, 2, 3, 4, 5$.

(b) Total energy: $E = \Sigma\, |y[n]|^2 = 4 + 9 + 9 + 36 + 25 + 1 = 84$

Problem 8.1.5

Given: A sonar system transmits a pulse and receives an echo. The cross-correlation $R_{xy}(\tau)$ between the transmitted and received signals peaks at a lag of $\tau = 12$ ms. The speed of sound in water is $c = 1500$ m/s.

Find: (a) The round-trip distance, (b) the one-way range to the target, and (c) the minimum pulse duration required to distinguish two targets separated by 5 m.

Solution:

(a) Round-trip distance: $d_{\text{round-trip}} = c \times \tau = 1500 \times 0.012 = 18$ m

(b) One-way range: $R = d_{\text{round-trip}} / 2 = 18 / 2 = 9$ m

(c) Two targets separated by $\Delta R = 5$ m have a round-trip time difference of: $\Delta\tau = 2\Delta R / c = 2 \times 5 / 1500 = 6.667$ ms

To resolve them, the pulse duration T_p must satisfy $T_p \leq \Delta\tau$: $T_p \leq 6.67$ ms

A shorter pulse provides finer range resolution. The range resolution is $\delta R = cT_p/2$.

Problem 8.1.6

Given: A biomedical signal has a maximum frequency content of 150 Hz. The signal is sampled at $f_s = 500$ Hz. Due to electromagnetic interference, a 260 Hz component is also present in the signal path.

Find: (a) The Nyquist rate for the 150 Hz signal, (b) the Nyquist frequency at the chosen sampling rate, (c) whether the 260 Hz interference will alias, and (d) the aliased frequency if it does.

Solution:

(a) Nyquist rate: $f_N = 2 \times f_{max} = 2 \times 150 = 300$ Hz

(b) Nyquist frequency at $f_s = 500$ Hz: $f_{\text{Nyquist}} = f_s / 2 = 500 / 2 = 250$ Hz

(c) The 260 Hz interference exceeds the Nyquist frequency of 250 Hz, so aliasing will occur if not filtered before sampling.

(d) Aliased frequency: $f_{\text{alias}} = f_s - f_{\text{interference}} = 500 - 260 = 240$ Hz

This alias falls within the 0-250 Hz passband and will corrupt the biomedical signal. An anti-aliasing lowpass filter with cutoff at 250 Hz (or preferably 150 Hz to reject all content above the signal bandwidth) must be placed before the ADC.

Problem 8.1.7

Given: A signal $x(t) = 4\cos(2\pi \times 200t) + \cos(2\pi \times 800t)$ is sampled at $f_s = 1000$ Hz without an anti-aliasing filter.

Find: (a) The Nyquist frequency, (b) whether each component aliases, (c) the frequencies that appear in the sampled signal, and (d) a suitable sampling rate to avoid aliasing with a practical guard band of 25%.

Solution:

(a) Nyquist frequency: $f_{Nyquist} = f_s / 2 = 1000 / 2 = 500$ Hz

(b) The 200 Hz component is below 500 Hz, so it does not alias. The 800 Hz component exceeds 500 Hz, so it aliases.

(c) The 200 Hz component appears at 200 Hz in the sampled signal. The 800 Hz component aliases to: $f_{alias} = f_s - 800 = 1000 - 800 = 200$ Hz. Both components appear at 200 Hz and cannot be separated. The sampled signal appears as a single 200 Hz tone with amplitude between 3 and 5 (depending on phase alignment).

(d) To sample both components without aliasing: $f_s \geq 2 \times f_{max} \times 1.25 = 2 \times 800 \times 1.25 = 2000$ Hz

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Problem 8.1.8

Given: An LTI system has impulse response $h[n] = \{1, -0.5\}$ for $n = 0, 1$. The input is $x[n] = \cos(\pi n/4)$ for $n = 0, 1, 2, \ldots$, which is a discrete-time sinusoid at normalized frequency $\omega_0 = \pi/4$ rad/sample.

Find: (a) The frequency response $H(e^{j\omega})$ of the system, (b) the magnitude and phase at $\omega = \pi/4$, and (c) the steady-state output.

Solution:

(a) The frequency response of an FIR filter with $h[n] = \{h_0, h_1\}$ is: $H(e^{j\omega}) = h_0 + h_1 e^{-j\omega} = 1 - 0.5e^{-j\omega}$

(b) At $\omega = \pi/4$: $H(e^{j\pi/4}) = 1 - 0.5e^{-j\pi/4} = 1 - 0.5(\cos(\pi/4) - j\sin(\pi/4)) = 1 - 0.5(0.7071 - j0.7071) = 1 - 0.3536 + j0.3536 = 0.6464 + j0.3536$

$|H(e^{j\pi/4})| = \sqrt{(0.6464^2 + 0.3536^2)} = \sqrt{(0.4178 + 0.1250)} = \sqrt{0.5428} = 0.737$ $\angle H(e^{j\pi/4}) = \arctan(0.3536 / 0.6464) = \arctan(0.5472) = 28.7°$

(c) For an LTI system with sinusoidal input, the steady-state output is: $y[n] = |H| \times \cos(\omega_0 n + \angle H) = 0.737 \cos(\pi n/4 + 28.7°)$

Problem 8.1.9

Given: A discrete-time signal $x[n]$ has total energy $E_x = 50$. Its autocorrelation $R_{xx}[0]$ is measured, and the autocorrelation at lag 1 is $R_{xx}[1] = 30$.

Find: (a) The value of $R_{xx}[0]$, (b) the normalized autocorrelation coefficient at lag 1, and (c) whether the signal has significant correlation between adjacent samples.

Solution:

(a) By definition, the autocorrelation at zero lag equals the signal energy: $R_{xx}[0] = E_x = 50$

(b) The normalized autocorrelation coefficient: $\rho[1] = R_{xx}[1] / R_{xx}[0] = 30 / 50 = 0.6$

(c) The normalized autocorrelation coefficient of 0.6 indicates significant positive correlation between adjacent samples. This means the signal changes slowly relative to the sampling rate, with each sample being moderately predictable from its neighbor. A value of 0.6 is typical of oversampled signals or signals with most energy at low frequencies. This correlation can be exploited for data compression (predictive coding) or noise reduction (averaging).

Problem 8.1.10

Given: A bandpass signal is centered at a carrier frequency $f_c = 70$ MHz with a bandwidth of $B = 5$ MHz (occupying 67.5 to 72.5 MHz). Instead of sampling at the Nyquist rate of $2 \times 72.5 = 145$ MHz, bandpass sampling is used.

Find: (a) The minimum bandpass sampling rate, (b) a valid sampling rate that avoids spectral overlap, and (c) the frequency shift of the signal in the sampled spectrum.

Solution:

(a) The minimum bandpass sampling rate is: $f_{s,min} = 2B = 2 \times 5 = 10$ MHz

(b) For bandpass sampling, the sampling rate must satisfy $f_s = 2f_{upper}/m$ for integer m, where the spectral replicas do not overlap. With $f_{upper} = 72.5$ MHz:

Check m = 5: $f_s = 2 \times 72.5 / 5 = 29.0$ MHz Verify: the signal at 67.5-72.5 MHz replicates. The baseband image is at $f_s - f_{upper} = 29.0 - 72.5 = \ldots$ Using the formula, the folded band: 72.5 mod 29.0 = 72.5 - 2(29.0) = 14.5 MHz (upper edge), and 67.5 mod 29.0 = 67.5 - 2(29.0) = 9.5 MHz (lower edge). The band 9.5-14.5 MHz fits within 0 to $f_s/2 = 14.5$ MHz with no overlap. A valid sampling rate is $f_s = 29.0$ MHz.

(c) In the sampled spectrum, the signal occupies 9.5 to 14.5 MHz (bandwidth still 5 MHz). The center frequency shifts from 70 MHz to $(9.5 + 14.5)/2 = 12.0$ MHz. This is a reduction from 145 MHz Nyquist rate to 29 MHz, a factor of 5× reduction in sampling rate.

Chapter 8 — Section 8.2: Fourier Analysis

Practice problems covering Fourier series, Fourier transform, DFT, FFT, DCT, Hilbert transform, and the Goertzel algorithm.

Problem 8.2.1

Given: A periodic sawtooth wave has a peak-to-peak amplitude of 10 V (ranging from -5 V to +5 V) and a fundamental frequency of 200 Hz.

Find: (a) The Fourier series coefficients for the first four harmonics, (b) the DC component, and (c) the fraction of total power contained in the fundamental and first three harmonics.

Solution:

(a) A sawtooth wave with amplitude A = 5 V has the Fourier series: $x(t) = \Sigma\ (2A/(n\pi)) \times (-1)^{n+1} \times \sin(2\pi n f_0 t)$. The coefficients $b_n = 2A/(n\pi) \times (-1)^{n+1}$:

$b_1 = 2(5)/(1\pi) \times 1 = 10/\pi = 3.183$ V (at 200 Hz) $b_2 = 2(5)/(2\pi) \times (-1) = -10/(2\pi) = -1.592$ V (at 400 Hz) $b_3 = 2(5)/(3\pi) \times 1 = 10/(3\pi) = 1.061$ V (at 600 Hz) $b_4 = 2(5)/(4\pi) \times (-1) = -10/(4\pi) = -0.796$ V (at 800 Hz)

(b) The sawtooth wave is an odd function symmetric about zero, so the DC component is: $a_0 = 0$ V

(c) Power of each harmonic (into 1 Ω): $P_n = b_n^2/2$. $P_1 = 3.183^2/2 = 5.066$ W $P_2 = 1.592^2/2 = 1.267$ W $P_3 = 1.061^2/2 = 0.563$ W $P_4 = 0.796^2/2 = 0.317$ W Sum = 7.213 W

Total power: $P_{total} = A^2/3 = 25/3 = 8.333$ W (for a sawtooth). Fraction: 7.213 / 8.333 = 86.6%

Problem 8.2.2

Given: A Gaussian pulse is defined as $x(t) = e^{-\pi t^2/\tau^2}$ with $\tau = 2$ ms.

Find: (a) The Fourier transform X(f), (b) the 3 dB bandwidth (half-power bandwidth), and (c) the time-bandwidth product.

Solution:

(a) The Fourier transform of a Gaussian is also a Gaussian: $X(f) = \tau \times e^{-\pi\tau^2 f^2} = 0.002 \times e^{-\pi(0.002)^2 f^2} = 0.002 \times e^{-\pi \times 4 \times 10^{-6} \times f^2}$

(b) The 3 dB bandwidth is where $|X(f)|^2 = |X(0)|^2/2$: $e^{-2\pi\tau^2 f^2} = 0.5$ $-2\pi\tau^2 f^2 = \ln(0.5) = -0.6931$ $f_{3dB} = \sqrt{(0.6931 / (2\pi \times (0.002)^2))} = \sqrt{(0.6931 / (2.513 \times 10^{-5}))}$ $f_{3dB} = \sqrt{(27{,}573)} = 166.1$ Hz

The two-sided 3 dB bandwidth is $2 \times 166.1 = 332.2$ Hz.

(c) Time-bandwidth product: The pulse duration at the 3 dB level: setting $e^{-\pi t^2/\tau^2} = 1/\sqrt{2}$ gives $t_{3dB} = \tau\sqrt{(\ln 2/\pi)} = 0.002 \times \sqrt{(0.2206)} = 0.002 \times 0.4697 = 0.939$ ms. Two-sided pulse width $= 2 \times 0.939 = 1.879$ ms. TBP $= 1.879 \times 10^{-3} \times 332.2 = 0.624$

This is close to the theoretical minimum of 0.44 (for a Gaussian), confirming that Gaussian pulses have near-optimal time-bandwidth product.

Problem 8.2.3

Given: A signal consists of two tones at 1000 Hz and 1100 Hz, sampled at $f_s = 10{,}000$ Hz. A 200-point DFT is computed.

Find: (a) The frequency resolution, (b) the DFT bin indices for each tone, (c) whether the two tones can be resolved, and (d) the minimum DFT length to resolve the two tones with a Hanning window.

Solution:

(a) Frequency resolution: $\Delta f = f_s / N = 10{,}000 / 200 = 50$ Hz

(b) Bin indices: $k_1 = f_1 / \Delta f = 1000 / 50 = 20$ $k_2 = f_2 / \Delta f = 1100 / 50 = 22$

(c) The two tones are separated by 100 Hz = 2 bins. With a rectangular window (main lobe width = 2 bins), the tones are barely resolvable — they fall in adjacent main lobes with a visible dip between them.

(d) A Hanning window has a main lobe width of approximately 4 bins. To resolve two tones separated by $\Delta f_{sep} = 100$ Hz, we need $4 \times (f_s/N) \leq 100$: $N \geq 4 \times f_s / 100 = 4 \times 10{,}000 / 100 = 400$ points

With N = 400: $\Delta f = 25$ Hz, and the Hanning main lobe width $= 4 \times 25 = 100$ Hz, just sufficient to resolve the two tones.

Problem 8.2.4

Given: A real-time spectrum analyzer processes audio data at $f_s = 48$ kHz using overlapping FFT frames. Each frame is 2048 samples with a Hanning window and 75% overlap.

Find: (a) The number of complex multiplications per FFT frame, (b) the hop size in samples and milliseconds, (c) the number of FFT frames per second, and (d) the total computational rate in millions of multiplications per second (MMPS).

Solution:

(a) Radix-2 FFT multiplications: Mults $= (N/2) \times \log_2(N) = (2048/2) \times \log_2(2048) = 1024 \times 11 =$ 11,264 complex multiplications per frame

(b) Hop size with 75% overlap: Hop = N × (1 - 0.75) = 2048 × 0.25 = 512 samples = 512 / 48,000 = 10.67 ms

(c) Frames per second: Frames/s = f_s / hop = 48,000 / 512 = 93.75 frames/s

(d) Total computational rate: Rate = 11,264 × 93.75 = 1,056,000 complex multiplications/s Each complex multiplication requires 4 real multiplications and 2 real additions, so: Real multiplications = 1,056,000 × 4 = 4,224,000 = 4.22 MMPS

This is well within the capability of even low-cost embedded processors.

Problem 8.2.5

Given: An 8×8 block of pixel values from a grayscale image is transformed using the 2-D DCT. The resulting DCT coefficient matrix has these magnitudes: DC coefficient = 1200, the next 5 largest coefficients have magnitudes of 85, 62, 41, 28, and 15, and the remaining 58 coefficients all have magnitudes below 10.

Find: (a) The total number of DCT coefficients, (b) the compression ratio if coefficients below magnitude 10 are quantized to zero, (c) the percentage of energy retained, and (d) the JPEG quality implication.

Solution:

(a) Total DCT coefficients in an 8×8 block: N = 8 × 8 = 64 coefficients

(b) Coefficients retained: DC (1) + 5 significant AC + 0 below threshold = 6 nonzero coefficients. Compression ratio = 64 / 6 = 10.7:1

(c) Energy retained: $E_{retained} = 1200^2 + 85^2 + 62^2 + 41^2 + 28^2 + 15^2$ = 1,440,000 + 7,225 + 3,844 + 1,681 + 784 + 225 = 1,453,759 $E_{discarded} \leq 58 \times 10^2$ = 5,800 (upper bound, assuming all are at magnitude 10) $E_{total} \approx$ 1,453,759 + 5,800 = 1,459,559 Fraction retained = 1,453,759 / 1,459,559 = 99.6%

(d) Retaining 99.6% of energy with 10.7:1 compression corresponds to a high-quality JPEG setting (approximately quality 85-90). The dominant DC coefficient carries 98.7% of the energy alone, which is typical for smooth image regions. Textured regions would have more energy spread across AC coefficients.

Problem 8.2.6

Given: A signal $x(t) = \cos(2\pi \times 100t) \times \cos(2\pi \times 5000t)$ represents a double-sideband suppressed carrier (DSB-SC) AM signal with carrier f_c = 5000 Hz and modulating frequency f_m = 100 Hz.

Find: (a) The frequency components present in x(t), (b) the Hilbert transform $\hat{x}(t)$, (c) the analytic signal z(t), and (d) the instantaneous envelope A(t).

Solution:

(a) Using the product-to-sum identity: $\cos(A)\cos(B) = 0.5[\cos(A-B) + \cos(A+B)]$: $x(t) = 0.5\cos(2\pi \times 4900t) + 0.5\cos(2\pi \times 5100t)$ The signal contains frequencies at 4900 Hz and 5100 Hz.

(b) The Hilbert transform shifts each frequency component by -90°: $\hat{x}(t) = 0.5\sin(2\pi \times 4900t) + 0.5\sin(2\pi \times 5100t)$ Using the sum-to-product identity: $\hat{x}(t) = \cos(2\pi \times 100t) \times \sin(2\pi \times 5000t) = \sin(2\pi \times 5000t) \times \cos(2\pi \times 100t)$

(c) The analytic signal: $z(t) = x(t) + j\hat{x}(t) = \cos(2\pi \times 100t)[\cos(2\pi \times 5000t) + j\sin(2\pi \times 5000t)]$ $z(t) = \cos(2\pi \times 100t) \times e^{j2\pi\times5000t}$

(d) The instantaneous envelope: $A(t) = |z(t)| = |\cos(2\pi \times 100t)| \times |e^{j2\pi\times5000t}| = |\cos(2\pi \times 100t)|$

The envelope varies between 0 and 1 at twice the modulation frequency (200 Hz) due to the absolute value, which is characteristic of DSB-SC (the envelope goes through zero, unlike standard AM).

Problem 8.2.7

Given: A DTMF decoder must detect the digit "9," which produces tones at 852 Hz and 1477 Hz. The system samples at $f_s = 8000$ Hz using an $N = 205$ sample block.

Find: (a) The Goertzel bin indices for 852 Hz and 1477 Hz, (b) the actual frequencies corresponding to those bins, (c) the frequency error for each tone, and (d) the number of real multiplications for the Goertzel algorithm versus a 256-point FFT.

Solution:

(a) Bin indices: $k_1 = \text{round}(852 \times 205 / 8000) = \text{round}(21.83) = 22$ $k_2 = \text{round}(1477 \times 205 / 8000) = \text{round}(37.85) = 38$

(b) Actual frequencies at those bins: $f_1 = 22 \times 8000 / 205 = 858.5$ Hz $f_2 = 38 \times 8000 / 205 = 1482.9$ Hz

(c) Frequency errors: $\text{Error}_1 = 858.5 - 852 = 6.5$ Hz $\rightarrow 6.5/852 \times 100 = 0.76\%$ $\text{Error}_2 = 1482.9 - 1477 = 5.9$ Hz $\rightarrow 5.9/1477 \times 100 = 0.40\%$

Both errors are well within the DTMF detection tolerance of ±1.5%.

(d) Computational cost: Goertzel for 2 bins: $2 \times N = 2 \times 205 = 410$ real multiplications (plus 2 complex multiplications for the final output = 8 more → 418 total) In the full DTMF detector (8 bins): $8 \times 205 + 32 = 1{,}672$ real multiplications 256-point FFT: $(N/2) \times \log_2(N) = 128 \times 8 = 1{,}024$ complex multiplications = 4,096 real multiplications Goertzel is 2.45× more efficient for detecting 8 specific frequencies.

Problem 8.2.8

Given: A periodic triangular wave has amplitude A = 3 V and period T = 5 ms. The Fourier series contains only odd harmonics with coefficients $a_n = 8A/(n^2\pi^2)$ for odd n.

Find: (a) The fundamental frequency, (b) the amplitudes of the first three nonzero harmonics, (c) the total harmonic distortion (THD) considering only these three harmonics, and (d) the Fourier series approximation at t = T/4.

Solution:

(a) Fundamental frequency: $f_0 = 1/T = 1/0.005 = 200$ Hz

(b) Harmonic amplitudes (odd n only): $a_1 = 8(3)/(1^2 \times \pi^2) = 24/\pi^2 = 24/9.8696 = 2.432$ V (at 200 Hz) $a_3 = 8(3)/(9 \times \pi^2) = 24/(9 \times 9.8696) = 0.270$ V (at 600 Hz) $a_5 = 8(3)/(25 \times \pi^2) = 24/(25 \times 9.8696) = 0.097$ V (at 1000 Hz)

(c) THD considering these harmonics: THD = $\sqrt{(a_3^2 + a_5^2)} / a_1 = \sqrt{(0.0730 + 0.00944)} / 2.432 = \sqrt{0.0824} / 2.432$ THD $= 0.287 / 2.432 = 11.8\%$

(d) At t = T/4 = 1.25 ms, the triangular wave reaches its peak of A = 3 V. Fourier approximation: $x(T/4) = a_1\sin(\pi/2) + a_3\sin(3\pi/2) + a_5\sin(5\pi/2) = 2.432(1) + 0.270(-1) + 0.097(1) = 2.432 - 0.270 + 0.097 = 2.259$ V

The 3-term approximation gives 2.259 V versus the exact value of 3.0 V, showing 24.7% error at the peak — more terms are needed for accurate peak representation.

Problem 8.2.9

Given: A rectangular pulse x(t) of width $\tau = 0.5$ ms and amplitude A = 2 V is centered at t = 0.

Find: (a) The Fourier transform X(f), (b) the spectral null frequencies, (c) the 3 dB bandwidth of the main lobe, and (d) the fraction of total energy in the main lobe.

Solution:

(a) The Fourier transform of a rectangular pulse of width τ and amplitude A: $X(f) = A\tau \times \mathrm{sinc}(f\tau) = 2 \times 0.0005 \times \mathrm{sinc}(0.0005f) = 0.001 \times \mathrm{sinc}(0.0005f)$ V·s

(b) Spectral nulls occur where $\mathrm{sinc}(f\tau) = 0$, i.e., $f\tau = n$ for integer $n \neq 0$: $f_{null,n} = n/\tau = n/0.0005 = 2000n$ Hz First null: 2000 Hz, second null: 4000 Hz, third null: 6000 Hz

(c) The 3 dB bandwidth of the main lobe: $\mathrm{sinc}(f\tau) = 1/\sqrt{2}$ occurs at approximately $f\tau = 0.4429$: $f_{3dB} = 0.4429/\tau = 0.4429/0.0005 = 885.8$ Hz Two-sided 3 dB bandwidth = 2 × 885.8 = 1,771.6 Hz

(d) The fraction of total energy in the main lobe (between $\pm f_{null,1} = \pm 2000$ Hz): For a sinc spectrum, the main lobe contains approximately 90.3% of the total energy (this is a well-known property of the sinc function — the integral of $\mathrm{sinc}^2(x)$ from -1 to 1 divided by the integral from $-\infty$ to ∞ equals 0.903).

Problem 8.2.10

Given: A 512-point FFT is used to analyze a signal sampled at $f_s = 20$ kHz. The signal contains a 3 kHz tone at 0 dBFS and a 3.5 kHz tone at -40 dBFS. A rectangular window is applied.

Find: (a) The frequency resolution, (b) the bin indices for both tones, (c) whether the weaker tone is detectable given the rectangular window's sidelobe level of -13 dB, and (d) a window choice that would allow detection.

Solution:

(a) Frequency resolution: $\Delta f = f_s$ / N = 20,000 / 512 = 39.06 Hz

(b) Bin indices: k_1 = 3000 / 39.06 = 76.8 → nearest bin 77 (actual f = 3007.8 Hz) k_2 = 3500 / 39.06 = 89.6 → nearest bin 90 (actual f = 3515.6 Hz)

The two tones are separated by 90 - 77 = 13 bins.

(c) The rectangular window has its highest sidelobe at -13 dB. The sidelobes decay at approximately 6 dB/octave. At 13 bins from the strong tone's peak: Sidelobe level ≈ -13 - $20\log_{10}(13)$ ≈ -13 - 22.3 = -35.3 dB Since the weak tone is at -40 dB relative to the strong tone, and the sidelobe leakage at that bin is only -35.3 dB, the weak tone is masked by the spectral leakage from the strong tone (it falls within the sidelobe floor).

(d) A Blackman window has sidelobes at -58 dB, which is well below the -40 dB weak tone level, allowing detection. A Hamming window (-43 dB sidelobes) would marginally work. A Blackman-Harris window (-92 dB sidelobes) provides the most margin.

The trade-off: the Blackman window widens the main lobe to approximately 6 bins (234 Hz), but since the tones are 13 bins apart, resolution is not an issue.

Chapter 8 — Section 8.3: Laplace Transform

Practice problems covering the Laplace transform definition, properties, transfer functions, inverse Laplace transform, and s-domain circuit analysis.

Problem 8.3.1

Given: A signal $x(t) = 5te^{-4t}u(t)$, where $u(t)$ is the unit step function.

Find: (a) The Laplace transform $X(s)$, (b) the region of convergence, and (c) the initial value $x(0^+)$ and the final value $x(\infty)$ using the value theorems.

Solution:

(a) Using the transform pair $L\{te^{-at}u(t)\} = 1/(s + a)^2$: $X(s) = 5 \times 1/(s + 4)^2 = 5 / (s + 4)^2$

(b) The region of convergence is: $Re\{s\} > -4$

(c) Initial value theorem: $x(0^+) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} 5s/(s + 4)^2 = \lim_{s\to\infty} 5s/(s^2 + 8s + 16) = 0$ (Confirmed: $x(0) = 5 \times 0 \times e^0 = 0$.)

Final value theorem: $x(\infty) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} 5s/(s + 4)^2 = 0/16 = 0$ (Confirmed: as $t \to \infty$, $te^{-4t} \to 0$ since the exponential decay dominates.)

Problem 8.3.2

Given: A system has the transfer function $H(s) = 50(s + 2) / (s^2 + 6s + 25)$.

Find: (a) The poles and zeros, (b) whether the system is stable, (c) the natural frequency ω_n and damping ratio ζ, (d) the DC gain, and (e) the peak resonant frequency.

Solution:

(a) Zeros: $s + 2 = 0 \to z_1 = -2$ Poles: $s^2 + 6s + 25 = 0 \to s = (-6 \pm \sqrt{(36 - 100)})/2 = (-6 \pm \sqrt{(-64)})/2 = (-6 \pm j8)/2$ $p_1 = -3 + j4$, $p_2 = -3 - j4$

(b) Both poles have negative real parts ($Re = -3 < 0$), so the system is stable.

(c) From the standard form $s^2 + 2\zeta\omega_n s + \omega_n^2$: $\omega_n^2 = 25 \to \omega_n = 5$ rad/s $2\zeta\omega_n = 6 \to \zeta = 6/(2 \times 5) = 0.6$

(d) DC gain: $H(0) = 50(2)/25 = 100/25 = 4.0$

(e) The peak resonant frequency for an underdamped second-order system: $\omega_r = \omega_n\sqrt{(1 - 2\zeta^2)} = 5\sqrt{(1 - 2(0.36))} = 5\sqrt{(0.28)} = 5 \times 0.5292 = 2.65$ rad/s ($f_r = 0.421$ Hz)

Since $1 - 2\zeta^2 = 0.28 > 0$, a resonant peak exists.

Problem 8.3.3

Given: Find the inverse Laplace transform of $X(s) = (7s + 11) / (s^2 + 5s + 6)$.

Find: The time-domain signal x(t) for $t \geq 0$.

Solution:

Factor the denominator: $s^2 + 5s + 6 = (s + 2)(s + 3)$.

Partial fraction expansion: $X(s) = A/(s + 2) + B/(s + 3)$. $7s + 11 = A(s + 3) + B(s + 2)$

Setting $s = -2$: $7(-2) + 11 = A(1) \rightarrow A = -14 + 11 = -3$ Setting $s = -3$: $7(-3) + 11 = B(-1) \rightarrow -21 + 11 = -B \rightarrow B = 10$

$X(s) = -3/(s + 2) + 10/(s + 3)$

Inverse transform: $x(t) = -3e^{-2t} + 10e^{-3t}$ for $t \geq 0$

Verification: $x(0) = -3 + 10 = 7$. From the initial value theorem: $sX(s)$ as $s \rightarrow \infty$: $s(7s + 11)/(s^2 + 5s + 6) \rightarrow 7$. Confirmed.

Problem 8.3.4

Given: A series RL circuit has $R = 200\ \Omega$ and $L = 50$ mH. A step voltage $V_{in}(t) = 24u(t)$ V is applied with zero initial current.

Find: (a) The transfer function $H(s) = I(s)/V_{in}(s)$, (b) the current I(s) in the s-domain, (c) the time-domain current i(t), and (d) the time constant and the time to reach 95% of steady-state current.

Solution:

(a) The s-domain impedance is $Z(s) = R + sL = 200 + 0.05s$. $H(s) = I(s)/V_{in}(s) = 1/Z(s) = 1/(0.05s + 200) = 20/(s + 4000)$

(b) With $V_{in}(s) = 24/s$: $I(s) = H(s) \times V_{in}(s) = 20 \times 24 / [s(s + 4000)] = 480 / [s(s + 4000)]$

Partial fractions: $480/[s(s + 4000)] = A/s + B/(s + 4000)$ $A = 480/4000 = 0.12$, $B = 480/(-4000) = -0.12$ $I(s) = 0.12/s - 0.12/(s + 4000)$

(c) Inverse transform: $i(t) = 0.12(1 - e^{-4000t})$ A $= 120(1 - e^{-4000t})$ mA for $t \geq 0$

(d) Time constant: $\tau = L/R = 0.05/200 = 250\ \mu s$ Time to 95% of steady state: $t = 3\tau = 3 \times 250 = 750\ \mu s$ Steady-state current: $I_{ss} = V/R = 24/200 = 120$ mA

Problem 8.3.5

Given: A second-order system has the transfer function $H(s) = 900 / (s^2 + 12s + 900)$. A unit step input is applied.

Find: (a) The natural frequency and damping ratio, (b) the damped natural frequency, (c) the percent overshoot of the step response, and (d) the settling time (2% criterion).

Solution:

(a) Comparing with the standard form $\omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$: $\omega_n^2 = 900 \rightarrow \omega_n = 30$ rad/s ($f_n = 4.77$ Hz) $2\zeta\omega_n = 12 \rightarrow \zeta = 12/60 = 0.2$

(b) Damped natural frequency: $\omega_d = \omega_n\sqrt{(1 - \zeta^2)} = 30\sqrt{(1 - 0.04)} = 30\sqrt{0.96} = 30 \times 0.9798 = 29.4$ rad/s

(c) Percent overshoot: $\%OS = 100 \times e^{-\pi\zeta/\sqrt{(1-\zeta^2)}} = 100 \times e^{-\pi(0.2)/\sqrt{(0.96)}} = 100 \times e^{-0.6283/0.9798} = 100 \times e^{-0.6413} = 100 \times 0.5267 = 52.7\%$

(d) Settling time (2% criterion): $t_s = 4/(\zeta\omega_n) = 4/(0.2 \times 30) = 4/6 = 0.667$ s

The low damping ratio ($\zeta = 0.2$) results in significant overshoot and oscillation before settling.

Problem 8.3.6

Given: A parallel RLC circuit has R = 10 kΩ, L = 100 mH, and C = 100 nF. The input is a current step $I_{in}(t) = 1$ mA $\times$ u(t) with zero initial conditions.

Find: (a) The transfer function $H(s) = V(s)/I_{in}(s)$, (b) the resonant frequency, (c) the quality factor Q, and (d) the bandwidth.

Solution:

(a) The parallel impedance: $Z(s) = 1/(1/R + 1/sL + sC) = 1/(sC + 1/R + 1/(sL)) = s/(s^2C + s/R + 1/L) = (s/C)/(s^2 + s/(RC) + 1/(LC))$

$H(s) = V(s)/I_{in}(s) = Z(s) = (s/C) / (s^2 + s/(RC) + 1/(LC))$

Substituting values: $1/(RC) = 1/(10^4 \times 10^{-7}) = 1000$, $1/(LC) = 1/(0.1 \times 10^{-7}) = 10^8$ $H(s) = (10^7 s) / (s^2 + 1000s + 10^8)$

(b) Resonant frequency: $\omega_0 = 1/\sqrt{(LC)} = 1/\sqrt{(0.1 \times 10^{-7})} = 1/\sqrt{(10^{-8})} = 10^4$ rad/s ($f_0 = 1{,}592$ Hz)

(c) Quality factor: $Q = R\sqrt{(C/L)} = 10{,}000 \times \sqrt{(10^{-7}/0.1)} = 10{,}000 \times \sqrt{(10^{-6})} = 10{,}000 \times 10^{-3} = 10$

(d) Bandwidth: $BW = \omega_0/Q = 10^4/10 = 1{,}000$ rad/s ($f_{BW} = 159.2$ Hz)

Alternatively: $BW = 1/(RC) = 1/(10^4 \times 10^{-7}) = 1000$ rad/s. Confirmed.

Problem 8.3.7

Given: A system has transfer function $H(s) = (s + 10) / (s^3 + 6s^2 + 11s + 6)$.

Find: (a) The poles and their locations, (b) whether the system is stable, (c) the partial fraction expansion, and (d) the impulse response h(t).

Solution:

(a) Factor the denominator. Testing $s = -1$: $(-1)^3 + 6(1) - 11 + 6 = -1 + 6 - 11 + 6 = 0$. So $(s + 1)$ is a factor. Dividing: $s^3 + 6s^2 + 11s + 6 = (s + 1)(s^2 + 5s + 6) = (s + 1)(s + 2)(s + 3)$ Poles: $p_1 = -1$, $p_2 = -2$, $p_3 = -3$

(b) All poles are in the left half-plane (negative real parts), so the system is stable.

(c) Partial fraction expansion: $H(s) = A/(s + 1) + B/(s + 2) + C/(s + 3)$ $s + 10 = A(s + 2)(s + 3) + B(s + 1)(s + 3) + C(s + 1)(s + 2)$

$s = -1$: $-1 + 10 = A(1)(2) \rightarrow 9 = 2A \rightarrow A = 4.5$ $s = -2$: $-2 + 10 = B(-1)(1) \rightarrow 8 = -B \rightarrow B = -8$ $s = -3$: $-3 + 10 = C(-2)(-1) \rightarrow 7 = 2C \rightarrow C = 3.5$

$H(s) = 4.5/(s + 1) - 8/(s + 2) + 3.5/(s + 3)$

(d) Impulse response: $h(t) = 4.5e^{-t} - 8e^{-2t} + 3.5e^{-3t}$ for $t \geq 0$

Problem 8.3.8

Given: A series RLC circuit has $R = 50\ \Omega$, $L = 20$ mH, and $C = 2\ \mu$F. A 10 V step voltage is applied. The output is taken across the capacitor.

Find: (a) The transfer function $H(s) = V_C(s)/V_{in}(s)$, (b) whether the response is underdamped, critically damped, or overdamped, (c) the complete step response $v_C(t)$, and (d) the peak capacitor voltage.

Solution:

(a) $H(s) = (1/sC)/(R + sL + 1/sC) = 1/(s^2LC + sRC + 1) = (1/LC)/(s^2 + (R/L)s + 1/(LC))$

$R/L = 50/0.02 = 2500$, $1/(LC) = 1/(0.02 \times 2 \times 10^{-6}) = 1/(4 \times 10^{-8}) = 2.5 \times 10^7$ $H(s) = 2.5 \times 10^7 / (s^2 + 2500s + 2.5 \times 10^7)$

(b) $\omega_n = \sqrt{(2.5 \times 10^7)} = 5000$ rad/s $2\zeta\omega_n = 2500 \rightarrow \zeta = 2500/10{,}000 = 0.25$ (underdamped, since $\zeta < 1$)

(c) $\omega_d = \omega_n\sqrt{(1 - \zeta^2)} = 5000\sqrt{(1 - 0.0625)} = 5000 \times 0.9682 = 4841$ rad/s $\sigma = \zeta\omega_n = 0.25 \times 5000 = 1250$

The step response is: $v_C(t) = 10[1 - (e^{-1250t}/\sqrt{(1 - \zeta^2)}) \times \sin(\omega_d t + \varphi)]$ where $\varphi = \arccos(\zeta) = \arccos(0.25) = 75.5°$ $v_C(t) = 10[1 - 1.033e^{-1250t}\sin(4841t + 75.5°)]$ V for $t \geq 0$

(d) The percent overshoot is: $\%OS = e^{-\pi\zeta/\sqrt{(1-\zeta^2)}} = e^{-\pi(0.25)/0.9682} = e^{-0.8109} = 0.4443 \rightarrow 44.4\%$

Peak voltage: $V_{peak} = 10 \times (1 + 0.4443) = 14.44$ V This occurs at $t_{peak} = \pi/\omega_d = \pi/4841 = 0.649$ ms

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Problem 8.3.9

Given: Use the final value theorem to find the steady-state output of a system with transfer function $H(s) = 20/(s + 5)$ when the input is (a) a unit step, (b) a unit ramp $r(t) = tu(t)$, and (c) a sinusoidal input $x(t) = \sin(3t)u(t)$.

Find: The steady-state value for each input.

Solution:

(a) Unit step input: $X(s) = 1/s$, $Y(s) = 20/[s(s + 5)]$ $y(\infty) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} 20/(s + 5) = 20/5 = 4.0$

(b) Unit ramp input: $X(s) = 1/s^2$, $Y(s) = 20/[s^2(s + 5)]$ $y(\infty) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} 20/[s(s + 5)] = \infty$ The final value theorem indicates the output grows without bound (the system has zero steady-state error for a step but infinite error for a ramp when there is no integrator in the loop).

(c) Sinusoidal input: $X(s) = 3/(s^2 + 9)$ The final value theorem does not apply because the output $Y(s)$ has poles on the $j\omega$-axis (at $s = \pm j3$), violating the requirement that $sY(s)$ have all poles in the left half-plane. The steady-state output is found from the frequency response: $y_{ss}(t) = |H(j3)| \times \sin(3t + \angle H(j3))$ $|H(j3)| = 20/|j3 + 5| = 20/\sqrt{(25 + 9)} = 20/\sqrt{34} = 3.43$ $\angle H(j3) = -\arctan(3/5) = -30.96°$ $y_{ss}(t) = 3.43 \sin(3t - 30.96°)$

Problem 8.3.10

Given: An active lowpass filter has transfer function $H(s) = K / [(s + 100)(s + 5000)]$, where K is an adjustable gain constant. The filter must have a DC gain of exactly 20 dB.

Find: (a) The value of K, (b) the magnitude response at f = 500 Hz, (c) the phase at f = 500 Hz, and (d) the approximate high-frequency roll-off rate.

Solution:

(a) DC gain = 20 dB = 10 (linear). $H(0) = K/(100 \times 5000) = K/500{,}000 = 10$ $K = 5{,}000{,}000$ (or 5×10^6)

(b) At f = 500 Hz, $\omega = 2\pi \times 500 = 3141.6$ rad/s: $H(j\omega) = 5 \times 10^6 / [(j\omega + 100)(j\omega + 5000)]$ $|j\omega + 100| = \sqrt{(3141.6^2 + 100^2)} = \sqrt{(9{,}869{,}645 + 10{,}000)} = \sqrt{9{,}879{,}645} = 3143.2$ $|j\omega + 5000| = \sqrt{(3141.6^2 + 5000^2)} = \sqrt{(9{,}869{,}645 + 25{,}000{,}000)} = \sqrt{34{,}869{,}645} = 5905.1$ $|H| = 5 \times 10^6 / (3143.2 \times 5905.1) = 5 \times 10^6 / 18{,}560{,}586 = 0.2694$ In dB: $20 \log_{10}(0.2694) = -11.39$ dB

(c) Phase: $\angle H = -\arctan(3141.6/100) - \arctan(3141.6/5000) = -\arctan(31.42) - \arctan(0.6283) = -88.18° - 32.14° = -120.3°$

(d) The system has two real poles, so the high-frequency roll-off is: -40 dB/decade (second-order system, -20 dB/decade per pole)

The two corner frequencies are at $f_1 = 100/(2\pi) = 15.9$ Hz and $f_2 = 5000/(2\pi) = 795.8$ Hz. Between these frequencies, the roll-off is -20 dB/decade; above 795.8 Hz, it steepens to -40 dB/decade.

Chapter 8 — Section 8.4: Z-Transform

Practice problems covering Z-transform definition and properties, discrete-time transfer functions, inverse Z-transform, bilinear transform, and stability analysis in the z-plane.

Problem 8.4.1

Given: A discrete-time sequence $x[n] = 3(0.6)^n u[n] + 2(-0.4)^n u[n]$, where $u[n]$ is the unit step.

Find: (a) The Z-transform $X(z)$, (b) the region of convergence, and (c) the value of $x[0]$ and $x[1]$ (verify using the initial value theorem).

Solution:

(a) Using the Z-transform pair $a^n u[n] \leftrightarrow z/(z - a)$: $X(z) = 3z/(z - 0.6) + 2z/(z - (-0.4)) = 3z/(z - 0.6) + 2z/(z + 0.4)$

Combining over a common denominator: $X(z) = [3z(z + 0.4) + 2z(z - 0.6)] / [(z - 0.6)(z + 0.4)] = [3z^2 + 1.2z + 2z^2 - 1.2z] / [(z - 0.6)(z + 0.4)] = 5z^2 / [(z - 0.6)(z + 0.4)] = 5z^2 / (z^2 - 0.2z - 0.24)$

(b) The ROC is the intersection of $|z| > 0.6$ and $|z| > 0.4$: ROC: $|z| > 0.6$ Since the ROC includes the unit circle, the signal has a valid DTFT.

(c) $x[0] = 3(0.6)^0 + 2(-0.4)^0 = 3 + 2 = 5$ $x[1] = 3(0.6)^1 + 2(-0.4)^1 = 1.8 - 0.8 = 1.0$

Initial value theorem: $x[0] = \lim_{z\to\infty} X(z) = \lim_{z\to\infty} 5z^2/(z^2 - 0.2z - 0.24) = 5$. Confirmed.

Problem 8.4.2

Given: A digital filter is described by the difference equation $y[n] = x[n] - 0.5x[n-1] + 0.8y[n-1] - 0.25y[n-2]$.

Find: (a) The transfer function $H(z)$, (b) the poles and zeros, (c) whether the system is stable, and (d) the magnitude response at DC and at the Nyquist frequency.

Solution:

(a) Taking the Z-transform: $Y(z) = X(z) - 0.5z^{-1}X(z) + 0.8z^{-1}Y(z) - 0.25z^{-2}Y(z)$ $Y(z)[1 - 0.8z^{-1} + 0.25z^{-2}] = X(z)[1 - 0.5z^{-1}]$ $H(z) = (1 - 0.5z^{-1})/(1 - 0.8z^{-1} + 0.25z^{-2}) = z(z - 0.5) / (z^2 - 0.8z + 0.25)$

(b) Zeros: $z = 0$ and $z = 0.5$ Poles: $z = (0.8 \pm \sqrt{(0.64 - 1.0)})/2 = (0.8 \pm \sqrt{(-0.36)})/2 = (0.8 \pm j0.6)/2$ $p_1 = 0.4 + j0.3$, $p_2 = 0.4 - j0.3$

(c) Pole magnitude: $|p| = \sqrt{(0.4^2 + 0.3^2)} = \sqrt{(0.16 + 0.09)} = \sqrt{0.25} = 0.5$ Since $|p| = 0.5 < 1$, the system is stable.

(d) At DC ($z = 1$): $|H(1)| = |1 - 0.5|/|1 - 0.8 + 0.25| = 0.5/0.45 = 1.111$ (0.92 dB) At Nyquist ($z = -1$): $|H(-1)| = |-1 - 0.5|/|1 + 0.8 + 0.25| = 1.5/2.05 = 0.732$ (-2.71 dB)

Problem 8.4.3

Given: Find the inverse Z-transform of $X(z) = (4z^2 + 2z) / (z^2 - 0.5z - 0.5)$.

Find: (a) The partial fraction expansion, (b) the time-domain sequence x[n], and (c) the first four samples x[0] through x[3].

Solution:

(a) Factor the denominator: $z^2 - 0.5z - 0.5 = 0$ $z = (0.5 \pm \sqrt{(0.25 + 2)})/2 = (0.5 \pm \sqrt{2.25})/2 = (0.5 \pm 1.5)/2$ $z_1 = 1.0$, $z_2 = -0.5$

So $z^2 - 0.5z - 0.5 = (z - 1)(z + 0.5)$.

First, perform polynomial long division since the degree of the numerator equals the denominator: $(4z^2 + 2z) / (z^2 - 0.5z - 0.5) = 4 + (4z + 2)/(z^2 - 0.5z - 0.5)$

Partial fraction expansion of $(4z + 2)/[(z - 1)(z + 0.5)]$: $X(z)/z$ for the remainder: $(4z + 2)/[z(z - 1)(z + 0.5)] = A/z + B/(z - 1) + C/(z + 0.5)$

Actually, let's expand $(4z + 2)/[(z - 1)(z + 0.5)] = B/(z - 1) + C/(z + 0.5)$: $4z + 2 = B(z + 0.5) + C(z - 1)$ $z = 1$: $6 = 1.5B \rightarrow B = 4$ $z = -0.5$: $0 = -1.5C \rightarrow C = 0$

So $X(z) = 4 + 4z/(z - 1) + 0/(z + 0.5) = 4\delta[n] + 4(1)^n u[n]$

Wait, let me redo this more carefully. We need $X(z)/z$: $X(z)/z = (4z + 2)/[(z - 1)(z + 0.5)] = A/(z - 1) + B/(z + 0.5)$ $4z + 2 = A(z + 0.5) + B(z - 1)$ $z = 1$: $6 = 1.5A \rightarrow A = 4$ $z = -0.5$: $0 = -1.5B \rightarrow B = 0$

$X(z) = 4z/(z - 1) = 4 \times z/(z - 1)$

(b) $x[n] = 4u[n]$ (a constant sequence of value 4 for $n \geq 0$)

(c) Verification by long division of $X(z) = (4z^2 + 2z)/(z^2 - 0.5z - 0.5)$ in powers of z^{-1}: Rewrite as $(4 + 2z^{-1})/(1 - 0.5z^{-1} - 0.5z^{-2})$

$x[0] = 4$, $x[1] = 2 + 0.5(4) = 2 + 2 = 4$, $x[2] = 0.5(4) + 0.5(4) = 2 + 2 = 4$, $x[3] = 0.5(4) + 0.5(4) = 4$

The sequence is indeed a constant value of 4 for all $n \geq 0$.

Problem 8.4.4

Given: Design a first-order digital highpass filter using the bilinear transform applied to the analog prototype $H_a(s) = s/(s + \Omega_c)$. The desired -3 dB cutoff is 1 kHz and the sampling rate is $f_s = 8$ kHz.

Find: (a) The pre-warped analog cutoff frequency, (b) the digital transfer function H(z), (c) the difference equation, and (d) the magnitude response at DC and at 4 kHz (Nyquist).

Solution:

(a) Digital cutoff frequency: $\omega_c = 2\pi \times 1000/8000 = 0.25\pi$ rad/sample Pre-warped analog frequency: $\Omega_c = (2/T)\tan(\omega_c T/2) = 2 \times 8000 \times \tan(0.25\pi/2) = 16{,}000 \times \tan(0.3927) = 16{,}000 \times 0.4142 =$ 6,627 rad/s

(b) Analog prototype: $H_a(s) = s/(s + 6627)$ Bilinear substitution $s = (2/T)(z - 1)/(z + 1) = 16{,}000(z - 1)/(z + 1)$:

$H(z) = [16{,}000(z - 1)/(z + 1)] / [16{,}000(z - 1)/(z + 1) + 6627] = 16{,}000(z - 1) / [16{,}000(z - 1) + 6627(z + 1)] = 16{,}000(z - 1) / [16{,}000z - 16{,}000 + 6627z + 6627] = 16{,}000(z - 1) / [22{,}627z - 9{,}373]$

Dividing by 22,627: $H(z) = 0.7071(z - 1)/(z - 0.4142) = 0.7071(1 - z^{-1})/(1 - 0.4142z^{-1})$

(c) Difference equation: $y[n] = 0.4142y[n-1] + 0.7071x[n] - 0.7071x[n-1]$

(d) At DC ($z = 1$): $H(1) = 0.7071(1 - 1)/(1 - 0.4142) = 0$ (blocks DC, as expected for a highpass filter)

At Nyquist ($z = -1$): $|H(-1)| = 0.7071|-1 - 1|/|-1 - 0.4142| = 0.7071 \times 2/1.4142 = 1.0$ (unity gain at Nyquist)

Problem 8.4.5

Given: A second-order IIR filter has transfer function $H(z) = 1/(1 - 1.2z^{-1} + 0.8z^{-2})$. The coefficients are $a_1 = -1.2$ and $a_2 = 0.8$.

Find: (a) Whether the filter is stable using the stability triangle conditions, (b) the pole locations and their magnitudes, (c) the resonant frequency if the sampling rate is $f_s = 44.1$ kHz, and (d) the maximum value of a_2 that would maintain stability.

Solution:

(a) Stability triangle check ($a_1 = -1.2$, $a_2 = 0.8$): Condition 1: $1 + a_1 + a_2 = 1 - 1.2 + 0.8 = 0.6 > 0$ ✓ Condition 2: $1 - a_1 + a_2 = 1 + 1.2 + 0.8 = 3.0 > 0$ ✓ Condition 3: $|a_2| = 0.8 < 1$ ✓ All conditions met → Stable

(b) Poles from $z^2 - 1.2z + 0.8 = 0$: $z = (1.2 \pm \sqrt{(1.44 - 3.2)})/2 = (1.2 \pm \sqrt{(-1.76)})/2 = (1.2 \pm j1.3266)/2$ $z = 0.6 \pm j0.6633$

$|z| = \sqrt{(0.36 + 0.44)} = \sqrt{0.80} = 0.8944$

Both poles have magnitude 0.8944, confirming they are inside the unit circle.

(c) Pole angle: $\theta = \arctan(0.6633/0.6) = \arctan(1.1055) = 47.87°$ Resonant frequency: $f_r = \theta/(360°) \times f_s = (47.87/360) \times 44{,}100 = 5{,}865$ Hz

(d) From the stability triangle: Condition 1: $a_2 > -1 - a_1 = -1 + 1.2 = 0.2$ Condition 3: $a_2 < 1$ Therefore the maximum a_2 for stability (with $a_1 = -1.2$) is $a_2 < 1.0$. At $a_2 = 1.0$, the poles would be exactly on the unit circle ($|z| = \sqrt{a_2} = 1$), making the system marginally stable.

Problem 8.4.6

Given: A causal system has the Z-transform $H(z) = (z^2 - 1) / (z^2 - 0.5z + 0.06)$.

Find: (a) The poles and zeros, (b) the partial fraction expansion of H(z), (c) the impulse response h[n], and (d) the steady-state response to a unit step.

Solution:

(a) Zeros: $z^2 - 1 = 0 \rightarrow z = +1$ and $z = -1$ Poles: $z^2 - 0.5z + 0.06 = 0 \rightarrow z = (0.5 \pm \sqrt{(0.25 - 0.24)})/2 = (0.5 \pm 0.1)/2$ $p_1 = 0.3$, $p_2 = 0.2$

(b) Since numerator and denominator have the same degree, perform long division first: $(z^2 - 1)/(z^2 - 0.5z + 0.06) = 1 + (0.5z - 1.06)/(z^2 - 0.5z + 0.06)$

Partial fraction of remainder $R(z)/z = (0.5z - 1.06)/[z(z - 0.3)(z - 0.2)]$: Wait, let's use $H(z)/z = (z^2 - 1)/[z(z - 0.3)(z - 0.2)]$ for the full expression.

Better approach — expand the remainder term: $(0.5z - 1.06)/[(z - 0.3)(z - 0.2)] = A/(z - 0.3) + B/(z - 0.2)$ $0.5z - 1.06 = A(z - 0.2) + B(z - 0.3)$ $z = 0.3$: $0.15 - 1.06 = 0.1A \rightarrow A = -9.1$ $z = 0.2$: $0.10 - 1.06 = -0.1B \rightarrow B = 9.6$

$H(z) = 1 + (-9.1z)/(z - 0.3) + (9.6z)/(z - 0.2)$ … actually this doesn't look right. Let me redo:

$(0.5z - 1.06)/[(z - 0.3)(z - 0.2)]$: multiply by z/z to get partial fractions involving $z/(z - a)$: $= [-9.1(z - 0.3) + \dots]$ — let's just do it directly for the time-domain sequence.

$H(z)/z = (z^2 - 1)/[z(z - 0.3)(z - 0.2)] = A/z + B/(z - 0.3) + C/(z - 0.2)$ $z^2 - 1 = A(z - 0.3)(z - 0.2) + Bz(z - 0.2) + Cz(z - 0.3)$ $z = 0$: $-1 = A(0.06) \rightarrow A = -16.667$ $z = 0.3$: $0.09 - 1 = B(0.3)(0.1) \rightarrow -0.91 = 0.03B \rightarrow B = -30.333$ $z = 0.2$: $0.04 - 1 = C(0.2)(-0.1) \rightarrow -0.96 = -0.02C \rightarrow C = 48.0$

$H(z) = -16.667 + (-30.333)z/(z - 0.3) + 48.0z/(z - 0.2)$

(c) $h[n] = -16.667\delta[n] - 30.333(0.3)^n u[n] + 48.0(0.2)^n u[n]$ $h[0] = -16.667 - 30.333 + 48.0 = 1.0$ $h[1] = -30.333(0.3) + 48.0(0.2) = -9.1 + 9.6 = 0.5$

Verify: h[0] should equal the leading coefficient, which from long division is 1. Confirmed.

(d) For a unit step $(z/(z-1))$, using the final value theorem: $y(\infty) = \lim_{z \to 1} (z-1) \times H(z) \times z/(z-1) = H(1) = (1 - 1)/(1 - 0.5 + 0.06) = 0/0.56 = 0$

The zero at $z = 1$ blocks the DC component of the step, so the steady-state output is zero.

Problem 8.4.7

Given: Design a second-order digital bandpass filter using the bilinear transform. The center frequency is 2 kHz, the 3 dB bandwidth is 400 Hz, and the sampling rate is $f_s = 16$ kHz. The analog prototype is $H_a(s) = (BW \times s)/(s^2 + BW \times s + \omega_0^2)$.

Find: (a) The pre-warped center frequency and bandwidth, (b) the analog prototype transfer function, and (c) the digital transfer function coefficients.

Solution:

(a) Digital frequencies: $\omega_0 = 2\pi \times 2000/16{,}000 = 0.25\pi$ rad/sample $\omega_L = 2\pi \times 1800/16{,}000 = 0.225\pi$ (lower edge) $\omega_H = 2\pi \times 2200/16{,}000 = 0.275\pi$ (upper edge)

Pre-warped frequencies: $\Omega_0 = (2/T)\tan(\omega_0/2) = 2 \times 16{,}000 \times \tan(0.125\pi) = 32{,}000 \times 0.4142 = 13{,}255$ rad/s $\Omega_L = 32{,}000 \times \tan(0.1125\pi) = 32{,}000 \times 0.3640 = 11{,}647$ rad/s $\Omega_H = 32{,}000 \times \tan(0.1375\pi) = 32{,}000 \times 0.4680 = 14{,}977$ rad/s Pre-warped BW $= \Omega_H - \Omega_L = 14{,}977 - 11{,}647 = 3{,}330$ rad/s

(b) Analog prototype: $H_a(s) = 3{,}330s / (s^2 + 3{,}330s + 13{,}255^2) = 3{,}330s / (s^2 + 3{,}330s + 1.757 \times 10^8)$

(c) Applying the bilinear transform $s = 32{,}000(z - 1)/(z + 1)$ and simplifying (normalizing by the coefficient of z^2 in the denominator):

The denominator becomes: $[32{,}000(z-1)/(z+1)]^2 + 3{,}330[32{,}000(z-1)/(z+1)] + 1.757 \times 10^8$

After algebraic simplification (expanding and collecting terms in z^2, z, and constant): Numerator $\propto$ $(z^2 - 1)$ (bandpass: zero at DC and Nyquist) Denominator coefficients yield:

$H(z) \approx 0.0940(1 - z^{-2}) / (1 - 1.6629z^{-1} + 0.8120z^{-2})$

The pole magnitude is $\sqrt{0.8120} = 0.9011$ (stable, inside unit circle). The Q factor is $f_0/BW = 2000/400 = 5$, producing sharp resonance at 2 kHz.

Problem 8.4.8

Given: A third-order IIR filter has denominator polynomial $D(z) = z^3 - 1.8z^2 + 1.2z - 0.3$.

Find: (a) Apply the Jury stability test necessary conditions, (b) determine if the filter is stable, and (c) find the approximate pole locations.

Solution:

(a) The polynomial is $D(z) = z^3 + a_1z^2 + a_2z + a_3$ where $a_1 = -1.8$, $a_2 = 1.2$, $a_3 = -0.3$, and the leading coefficient $a_0 = 1$.

Necessary conditions for the Jury test ($n = 3$): Condition 1: $D(1) > 0 \rightarrow 1 - 1.8 + 1.2 - 0.3 = 0.1 > 0$ ✓ Condition 2: $(-1)^nD(-1) > 0 \rightarrow (-1)^3(-1 - 1.8 - 1.2 + 0.3) = (-1)(-2.7) = 2.7 > 0$ ✓ Condition 3: $|a_3| < |a_0| \rightarrow |-0.3| < |1| \rightarrow 0.3 < 1$ ✓

(b) Construct the Jury array: Row 1: 1, -1.8, 1.2, -0.3 Row 2 (reversed): -0.3, 1.2, -1.8, 1

Compute $b_k = |a_0, a_{3-k}; a_3, a_k|$ (determinant): $b_0 = 1(1) - (-0.3)(-0.3) = 1 - 0.09 = 0.91$ $b_1 = 1(-1.8) - (-0.3)(1.2) = -1.8 + 0.36 = -1.44$ $b_2 = 1(1.2) - (-0.3)(-1.8) = 1.2 - 0.54 = 0.66$

Check $|b_0| > |b_2|$: $|0.91| > |0.66|$ ✓

All conditions satisfied → Stable

(c) Trying $z = 0.5$: $D(0.5) = 0.125 - 0.45 + 0.6 - 0.3 = -0.025 \approx 0$ Refining, one real pole near $z \approx 0.48$. The remaining quadratic: $z^2 - 1.32z + 0.625 = 0 \rightarrow z = (1.32 \pm \sqrt{(1.7424 - 2.5)})/2 = (1.32 \pm j0.870)/2$ $z = 0.66 \pm j0.435$, $|z| = \sqrt{(0.4356 + 0.1892)} = \sqrt{0.6248} = 0.791$ All poles inside the unit circle — confirmed stable.

Problem 8.4.9

Given: A digital system has transfer function $H(z) = (1 + 2z^{-1} + z^{-2}) / (1 - 0.5z^{-1})$. The sampling rate is $f_s = 10$ kHz.

Find: (a) The poles and zeros, (b) whether this is FIR or IIR, (c) the impulse response h[n] for n = 0 to 4, and (d) the frequency response magnitude at 1 kHz and 5 kHz.

Solution:

(a) Numerator: $1 + 2z^{-1} + z^{-2} = (1 + z^{-1})^2 \rightarrow$ zeros at z = -1 (double zero) Denominator: $1 - 0.5z^{-1} \rightarrow$ pole at z = 0.5 This is an IIR filter (has a non-trivial pole at z = 0.5).

(b) This is IIR — the feedback pole at z = 0.5 creates an infinite impulse response.

(c) Difference equation: y[n] = 0.5y[n-1] + x[n] + 2x[n-1] + x[n-2] With impulse input (x[0] = 1, x[n] = 0 for $n \neq 0$): h[0] = 0 + 1 + 0 + 0 = 1 h[1] = 0.5(1) + 0 + 2(1) + 0 = 2.5 h[2] = 0.5(2.5) + 0 + 0 + 1 = 2.25 h[3] = 0.5(2.25) + 0 + 0 + 0 = 1.125 h[4] = 0.5(1.125) = 0.5625

(d) At f = 1 kHz: $\omega = 2\pi \times 1000/10{,}000 = 0.2\pi$ rad/sample Numerator: $|1 + 2e^{-j0.2\pi} + e^{-j0.4\pi}| = |(1 + e^{-j0.2\pi})^2| = |1 + \cos(0.2\pi) - j\sin(0.2\pi)|^2$ … let me compute directly: $|1 + e^{-j\omega}|^2 = (1 + \cos\omega)^2 + \sin^2\omega = 2 + 2\cos\omega = 2(1 + \cos(0.2\pi)) = 2(1 + 0.809) = 3.618$ So |numerator| = 3.618

Denominator: $|1 - 0.5e^{-j0.2\pi}| = \sqrt{((1 - 0.5\cos(0.2\pi))^2 + (0.5\sin(0.2\pi))^2)} = \sqrt{((1 - 0.4045)^2 + (0.2939)^2)} = \sqrt{(0.3544 + 0.0864)} = \sqrt{0.4408} = 0.6639$

|H| = 3.618/0.6639 = 5.45

At f = 5 kHz (Nyquist): $\omega = \pi$, z = -1: |Numerator| = |1 - 2 + 1| = 0 (double zero at z = -1) |H| = 0 (complete null at Nyquist, as expected from the double zero at z = -1)

Problem 8.4.10

Given: A digital control system has open-loop transfer function G(z) = K/(z - 0.7) in a unity feedback configuration. The closed-loop transfer function is T(z) = G(z)/(1 + G(z)).

Find: (a) The closed-loop transfer function in terms of K, (b) the closed-loop pole location, (c) the maximum value of K for stability, and (d) the value of K that places the closed-loop pole at z = 0 (deadbeat response).

Solution:

(a) T(z) = [K/(z - 0.7)] / [1 + K/(z - 0.7)] = K / (z - 0.7 + K) = K / (z - 0.7 + K)

(b) The closed-loop pole is at: $z = 0.7 - K \rightarrow z_{pole} = 0.7 - K$

(c) For stability, the pole must be inside the unit circle: $|z_{pole}| < 1$ $-1 < 0.7 - K < 1$ From the left inequality: K < 1.7 From the right inequality: K > -0.3 (always true for positive K) Maximum K for stability: K < 1.7

(d) For deadbeat response (pole at z = 0): $0.7 - K = 0 \rightarrow K = 0.7$

With K = 0.7, $T(z) = 0.7/z = 0.7z^{-1}$. The impulse response is h[0] = 0, h[1] = 0.7, h[n] = 0 for $n \geq 2$ — the system settles in exactly one sample period (deadbeat response).

Chapter 8 — Section 8.5: Digital Filters

Practice problems covering FIR filters, IIR filters, filter design specifications, multirate signal processing, fixed-point implementation, polyphase filters, and allpass group delay equalization.

Problem 8.5.1

Given: A 5-tap FIR filter has symmetric coefficients b = {0.1, 0.25, 0.3, 0.25, 0.1}. The filter is sampled at $f_s = 20$ kHz.

Find: (a) Whether the filter has linear phase, (b) the group delay in samples and milliseconds, (c) the DC gain, (d) the gain at the Nyquist frequency, and (e) the type of frequency response (lowpass, highpass, bandpass).

Solution:

(a) The coefficients are symmetric: b[0] = b[4] = 0.1 and b[1] = b[3] = 0.25. A symmetric FIR filter with odd length has Type I linear phase — yes, the filter has exactly linear phase.

(b) For a Type I FIR filter with N = 5 taps: Group delay = (N - 1)/2 = (5 - 1)/2 = 2 samples = 2/20,000 = 0.1 ms

(c) DC gain (z = 1): H(1) = 0.1 + 0.25 + 0.3 + 0.25 + 0.1 = 1.0 (0 dB)

(d) Gain at Nyquist (z = -1): H(-1) = 0.1 - 0.25 + 0.3 - 0.25 + 0.1 = 0.0 ($-\infty$ dB)

(e) The filter passes DC with unity gain and completely rejects the Nyquist frequency. This is a lowpass filter. The null at Nyquist confirms the lowpass characteristic.

Problem 8.5.2

Given: A second-order IIR bandpass filter is defined by: y[n] = 1.2728y[n-1] - 0.81y[n-2] + 0.095x[n] - 0.095x[n-2]

The sampling rate is $f_s = 8000$ Hz.

Find: (a) The transfer function H(z), (b) the pole and zero locations, (c) the center frequency, and (d) the 3 dB bandwidth.

Solution:

(a) Taking the Z-transform: $H(z) = (0.095 - 0.095z^{-2})/(1 - 1.2728z^{-1} + 0.81z^{-2}) = 0.095(z^2 - 1)/(z^2 - 1.2728z + 0.81)$

(b) Zeros: $z^2 - 1 = 0 \rightarrow z = +1$ and $z = -1$ (zeros at DC and Nyquist — bandpass characteristic) Poles: $z^2 - 1.2728z + 0.81 = 0$ $z = (1.2728 \pm \sqrt{(1.6200 - 3.24)})/2 = (1.2728 \pm \sqrt{(-1.62)})/2$ Wait: $1.2728^2 = 1.6200$, $4 \times 0.81 = 3.24$ $z = (1.2728 \pm j\sqrt{(3.24 - 1.62)})/2 = (1.2728 \pm j\sqrt{1.62})/2 = (1.2728 \pm j1.2728)/2$

Hmm, let me recalculate: discriminant $= 1.2728^2 - 4(0.81) = 1.6200 - 3.24 = -1.62$ $z = (1.2728 \pm j1.2728)/2 = 0.6364 \pm j0.6364$

Pole magnitude: $|z| = \sqrt{(0.6364^2 + 0.6364^2)} = 0.6364\sqrt{2} = 0.9$ $(= \sqrt{0.81})$

(c) Pole angle: $\theta = \arctan(0.6364/0.6364) = \arctan(1) = 45° = \pi/4$ rad Center frequency: $f_0 = (\theta/2\pi) \times f_s = (45/360) \times 8000 = 1000$ Hz

(d) The 3 dB bandwidth for a second-order bandpass filter: $BW \approx (1 - r^2) \times f_s / (2\pi)$ where r = pole radius = 0.9 $BW \approx (1 - 0.81) \times 8000 / (2\pi) = 0.19 \times 8000 / 6.283 = 241.9$ Hz

The Q factor is $Q = f_0/BW = 1000/241.9 \approx 4.1$.

Problem 8.5.3

Given: A digital lowpass filter is required with passband edge $f_p = 3$ kHz (1 dB ripple), stopband edge $f_s = 4.5$ kHz (50 dB attenuation), and sampling rate $f_s = 20$ kHz. Compare the required orders for Butterworth and Chebyshev Type I IIR designs.

Find: (a) The normalized frequencies, (b) the pre-warped analog frequencies, (c) the Butterworth order, and (d) the Chebyshev Type I order.

Solution:

(a) Normalized digital frequencies: $\Omega_p = 2\pi \times 3000/20{,}000 = 0.3\pi$ rad/sample $\Omega_s = 2\pi \times 4500/20{,}000 = 0.45\pi$ rad/sample

(b) Pre-warped analog frequencies: $\omega_p = \tan(\Omega_p/2) = \tan(0.15\pi) = \tan(27°) = 0.5095$ $\omega_s = \tan(\Omega_s/2) = \tan(0.225\pi) = \tan(40.5°) = 0.8541$ Selectivity ratio: $k = \omega_p/\omega_s = 0.5095/0.8541 = 0.5965$

(c) Butterworth order: $n \geq \log[(10^{As/10} - 1)/(10^{Ap/10} - 1)] / (2 \times \log(1/k)) = \log[(10^5 - 1)/(10^{0.1} - 1)] / (2 \times \log(1.676)) = \log[99{,}999/0.2589] / (2 \times 0.2245) = \log(386{,}247) / 0.4490 = 5.587 / 0.4490 = 12.4 \rightarrow n = 13$

(d) Chebyshev Type I order: $n \geq \cosh^{-1}(\sqrt{[(10^{As/10} - 1)/(10^{Ap/10} - 1)]}) / \cosh^{-1}(1/k) = \cosh^{-1}(\sqrt{(386{,}247)}) / \cosh^{-1}(1.676) = \cosh^{-1}(621.5) / \cosh^{-1}(1.676) = 7.125 / 1.114 = 6.4 \rightarrow n = 7$

The Chebyshev filter requires roughly half the order of the Butterworth (7 vs 13) at the cost of passband ripple. This translates to approximately half the computational cost for the IIR implementation.

Problem 8.5.4

Given: A digital audio signal at f_s = 44.1 kHz must be converted to 48 kHz for a professional audio interface.

Find: (a) The rational sample rate conversion factor L/M, (b) the required anti-aliasing/anti-imaging filter cutoff, (c) the number of output samples for 1 second of input, and (d) the computational implications.

Solution:

(a) Sample rate ratio: 48,000/44,100 = 480/441 = 160/147 (after reducing by GCD = 3). L = 160 (interpolation factor), M = 147 (decimation factor).

(b) The anti-aliasing filter must prevent aliasing from the lower sampling rate: $f_{cutoff} = \min(f_{s,in}, f_{s,out})/2 = \min(44{,}100, 48{,}000)/2 = 44{,}100/2 = 22{,}050$ Hz In the intermediate (upsampled) rate of $44{,}100 \times 160 = 7{,}056{,}000$ Hz, the cutoff is 22,050 Hz.

(c) Input samples for 1 second: 44,100 Output samples: $44{,}100 \times 160/147 = 7{,}056{,}000/147 = 48{,}000$ samples ✓

(d) Direct implementation at the intermediate rate of 7.056 MHz is impractical. A polyphase implementation operates the filter at the output rate of 48 kHz. If the anti-aliasing filter has N = 1440 taps, the polyphase structure uses 160 phases of 9 taps each, requiring only 9 multiplications per output sample — a dramatic reduction from 1440 at the intermediate rate.

Problem 8.5.5

Given: A second-order IIR notch filter centered at 60 Hz (for power line rejection) is designed for f_s = 1000 Hz. The ideal coefficients are a_1 = -1.61803 and a_2 = 0.95. The filter is implemented in Q15 fixed-point (16-bit signed).

Find: (a) The Q15 representations of a_1 and a_2, (b) the quantized coefficient values, (c) the quantization error for each coefficient, and (d) the shift in notch frequency due to quantization.

Solution:

(a) Q15 represents values from -1.0 to +0.99997 with resolution $2^{-15} = 3.052 \times 10^{-5}$. However, $|a_1|$ = 1.61803 > 1.0, which exceeds Q15 range.

Use Q14 scaling (divide by 2): $a_{1Q14} = \text{round}(-1.61803 \times 2^{14}) = \text{round}(-26{,}510.4) = -26{,}510$ $a_{2Q15} = \text{round}(0.95 \times 2^{15}) = \text{round}(31{,}129.6) = 31{,}130$

(b) Quantized values: $a_{1quant} = -26{,}510 / 2^{14} = -26{,}510 / 16{,}384 = -1.61798$ $a_{2quant} = 31{,}130 / 2^{15} = 31{,}130 / 32{,}768 = 0.94998$

(c) Quantization errors: $\Delta a_1 = -1.61798 - (-1.61803) = +5 \times 10^{-5}$ (0.003%) $\Delta a_2 = 0.94998 - 0.95 = -2 \times 10^{-5}$ (0.002%)

(d) The notch frequency depends on a_1: for a second-order notch, $\omega_0 = \arccos(-a_1/2)$. Ideal: $\omega_0 = \arccos(1.61803/2) = \arccos(0.80902) = 0.37699$ rad/sample → f_0 = 60.000 Hz Quantized: ω_0 =

arccos(1.61798/2) = arccos(0.80899) = 0.37703 rad/sample → f_0 = 60.005 Hz Shift = 0.005 Hz, which is negligible for a 60 Hz notch application.

Problem 8.5.6

Given: A 120-tap FIR lowpass filter is used for decimation by M = 4 from 40 kS/s to 10 kS/s. Compare direct and polyphase implementations.

Find: (a) The polyphase subfilter length, (b) multiplications per output sample for each approach, (c) total multiplications per second, and (d) the memory requirement for the polyphase structure.

Solution:

(a) Polyphase subfilter length: Each phase has N/M = 120/4 = 30 taps

(b) Multiplications per output sample: Direct: the filter runs at the input rate. For each output sample, M = 4 input samples are processed, each requiring 120 multiplications. But only 1 output per 4 inputs is kept: Direct = 120 × 4 = 480 multiplications per output sample

Polyphase: M subfilters of 30 taps each run at the output rate: Polyphase = 4 × 30 = 120 multiplications per output sample

(c) Multiplications per second: Direct: 120 × 40,000 = 4,800,000/s (filter at input rate) Polyphase: 120 × 10,000 = 1,200,000/s (filter at output rate) Savings: 4× reduction = factor of M, as expected.

(d) Memory for polyphase: Coefficients: 120 values (same as direct — just rearranged into 4 phases) Delay line: 4 delay lines of 30 samples each = 120 samples State variable for commutator: 1 variable Total: 120 coefficient + 120 delay line = 240 values

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Problem 8.5.7

Given: A 6th-order Elliptic lowpass IIR filter (3 biquad sections) has group delay that varies from 5.8 samples at DC to 31.2 samples at the passband edge. A cascade of two second-order allpass sections is designed to equalize the group delay to approximately 32 samples.

Find: (a) The group delay variation before equalization, (b) the total system order after adding the equalizer, (c) the added latency in milliseconds at f_s = 48 kHz, and (d) the total number of multiply-accumulate operations per sample.

Solution:

(a) Group delay variation: $\Delta\tau = \tau_{max} - \tau_{min}$ = 31.2 - 5.8 = 25.4 samples At f_s = 48 kHz: $\Delta\tau$ = 25.4/48,000 = 0.529 ms

(b) Total system order: Original Elliptic filter: 6th order (3 biquad sections) Allpass equalizer: 2 × 2nd order = 4th order (2 biquad sections) Total: 10th order (5 biquad sections)

(c) Added latency: Before: minimum group delay = 5.8 samples After: approximately flat at 32 samples Added latency = 32 - 5.8 = 26.2 samples = 26.2/48,000 = 0.546 ms

(d) Each biquad section requires 5 multiply-accumulate operations (2 feedforward + 2 feedback + 1 input scaling, using Direct Form II Transposed): Total MACs = 5 × 5 sections = 25 multiply-accumulate operations per sample

At f_s = 48 kHz: 25 × 48,000 = 1,200,000 MACs/s = 1.2 MMAC/s — easily handled by any modern DSP.

Problem 8.5.8

Given: A Parks-McClellan (equiripple) FIR lowpass filter is designed with passband edge 4 kHz, stopband edge 5 kHz, passband ripple $\delta_p = 0.01$ (0.087 dB), and stopband attenuation $\delta_s = 0.001$ (-60 dB). The sampling rate is f_s = 44.1 kHz.

Find: (a) The transition bandwidth, (b) an estimate of the required filter order using the Bellanger formula, (c) the group delay, and (d) the computational cost in MMAC/s.

Solution:

(a) Transition bandwidth: $\Delta f = f_{stop} - f_{pass}$ = 5000 - 4000 = 1000 Hz

(b) Bellanger formula: $N \approx (-2/3)\log_{10}(10\delta_p\delta_s) / (\Delta f/f_s) = (-2/3)\log_{10}(10 \times 0.01 \times 0.001) / (1000/44{,}100) = (-2/3)\log_{10}(10^{-4}) / 0.02268 = (-2/3)(-4) / 0.02268 = 2.667 / 0.02268 = 117.6 \rightarrow N \approx 118$ taps

(c) Group delay for linear-phase FIR: $\tau = (N - 1)/2 = 117/2 = 58.5$ samples = 58.5/44,100 = 1.33 ms

(d) Computational cost: MACs per sample = N = 118 (or 59 using symmetry) At f_s = 44.1 kHz: 59 × 44,100 = 2,601,900 = 2.60 MMAC/s

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Problem 8.5.9

Given: A half-band FIR filter has the property that every other coefficient (except the center tap) is exactly zero. A 15-tap half-band filter has nonzero coefficients at positions 0, 2, 4, 6, 7, 8, 10, 12, 14 with values b = {-0.025, 0, 0.075, 0, -0.30, 0.5, -0.30, 0, 0.075, 0, -0.025, 0, 0, 0, 0}.

Wait — let me reconsider. A proper 15-tap half-band has nonzero taps at indices 0, 2, 4, 6, 7, 8, 10, 12, 14. Actually, a half-band filter with N = 15 has (N+1)/2 = 8 nonzero coefficients.

Find: (a) How many multiplications per output sample are needed (exploiting zero coefficients and symmetry), (b) the cutoff frequency relative to f_s, (c) why half-band filters are ideal for decimation by 2, and (d) the magnitude response at $f_s/4$.

Solution:

Let the actual half-band coefficients be: h = {-0.0215, 0, 0.0718, 0, -0.3066, 0.5, -0.3066, 0, 0.0718, 0, -0.0215, 0, 0, 0, 0} — but this has 15 entries. A standard 11-tap half-band: h = {c_0, 0, c_2, 0, c_4, 0.5, c_4, 0, c_2, 0, c_0}.

Using an 11-tap half-band with coefficients h = {-0.025, 0, 0.15, 0, -0.375, 0.5, -0.375, 0, 0.15, 0, -0.025}:

(a) Total taps = 11. Zero coefficients: 4 (at odd positions except center). Nonzero: 7. Using symmetry (h[0]=h[10], h[2]=h[8], h[4]=h[6]): 3 symmetric pairs + 1 center tap = 4 multiplications per output sample (instead of 11).

(b) A half-band filter has its -6 dB point at exactly: $f_{-6dB} = f_s/4$

This is the defining property — the transition band is centered at $f_s/4$, and the passband and stopband ripples are equal ($\delta_p = \delta_s$).

(c) Half-band filters are ideal for decimation/interpolation by 2 because:

- The cutoff is naturally at $f_s/4 = f_{new}/2$ (the new Nyquist frequency after decimation)
- The zero coefficients reduce computation by nearly 50%
- Combined with polyphase decomposition, each output sample requires only (N+1)/4 multiplications

(d) At $f = f_s/4$, by the half-band property: $|H(e^{j\pi/2})| = 0.5$ (-6.02 dB)

This is exact by construction, not an approximation.

Problem 8.5.10

Given: A 3-stage cascaded integrator-comb (CIC) decimation filter reduces the sampling rate from 10 MHz to 100 kHz (decimation factor M = 100). The CIC filter has N = 3 stages and differential delay D = 1.

Find: (a) The transfer function of the CIC filter, (b) the first null frequency, (c) the passband droop at 40 kHz, and (d) whether a compensation filter is needed.

Solution:

(a) The CIC decimation filter transfer function (before downsampling): $H(z) = [(1 - z^{-M})/(1 - z^{-1})]^N = [(1 - z^{-100})/(1 - z^{-1})]^3$

The magnitude response: $|H(f)| = |\sin(\pi Mf/f_s)/\sin(\pi f/f_s)|^N$

(b) First null frequency: The first null occurs when $\sin(\pi Mf/f_s) = 0$ and $\sin(\pi f/f_s) \neq 0$: $\pi Mf/f_s = \pi \rightarrow f = f_s/M = 10{,}000{,}000/100 = 100$ kHz

This conveniently falls at the decimated sampling rate, so the first null acts as the anti-aliasing boundary.

(c) Passband droop at 40 kHz (the edge of the desired passband, which is $f_{out}/2 = 50$ kHz, so 40 kHz is within the passband): $|H(40\text{ kHz})| = |\sin(\pi \times 100 \times 40{,}000/10^7)/\sin(\pi \times 40{,}000/10^7)|^3 = |\sin(0.4\pi)/\sin(0.004\pi)|^3 = |\sin(72°)/\sin(0.72°)|^3 = |0.9511/0.01257|^3 = |75.66|^3 \ldots$

This is the unnormalized gain. The DC gain is M^N = $100^3 = 10^6$. Normalized: $|H(40\text{ kHz})|_{norm} = |\sin(0.4\pi)/(100 \times \sin(0.004\pi))|^3 = (0.9511/(100 \times 0.01257))^3 = (0.9511/1.257)^3 = (0.7565)^3 = 0.4328$

Droop in dB: $20\log_{10}(0.4328) = -7.28$ dB

(d) A passband droop of 7.28 dB at 40 kHz is severe — a compensation (droop correction) FIR filter operating at the decimated rate of 100 kHz is required. The compensator has an inverse-sinc response that boosts frequencies near the passband edge to flatten the overall response. Typically a short (5-15 tap) FIR filter with a mild highpass tilt suffices, adding negligible computational cost at the low output rate.

Chapter 8 — Section 8.6: Spectral Analysis

Practice problems covering power spectral density, Welch's method, windowing trade-offs, time-frequency analysis (STFT), parametric spectral estimation (AR models, MUSIC), cepstral analysis, MFCCs, and wavelet transforms with denoising.

Problem 8.6.1

Given: A thermal noise source has a flat (white) power spectral density of $S(f) = 4.0 \times 10^{-9}\ V^2/Hz$. The signal is measured through a system with a noise bandwidth of 200 kHz.

Find: (a) The total noise power, (b) the RMS noise voltage, and (c) the RMS noise voltage if a bandpass filter restricts the measurement to 50 kHz bandwidth centered at 1 MHz.

Solution:

(a) Total noise power: $P = S(f) \times BW = 4.0 \times 10^{-9} \times 200{,}000 = 8.0 \times 10^{-4}\ V^2 = 0.8\ mV^2$

(b) RMS noise voltage: $V_{rms} = \sqrt{P} = \sqrt{(8.0 \times 10^{-4})} = 0.02828\ V = 28.28\ mV$

(c) With the 50 kHz bandpass filter: $P_{BP} = S(f) \times 50{,}000 = 4.0 \times 10^{-9} \times 50{,}000 = 2.0 \times 10^{-4}\ V^2$ $V_{rms,BP} = \sqrt{(2.0 \times 10^{-4})} = 0.01414\ V = 14.14\ mV$

The bandpass filter reduces the noise power by a factor of 200,000/50,000 = 4, and the RMS voltage by $\sqrt{4} = 2$. This demonstrates that narrowing the measurement bandwidth is the primary technique for reducing noise in precision measurements.

Problem 8.6.2

Given: A signal is sampled at $f_s = 48$ kHz, and a 2048-point FFT is used for spectral analysis. The signal contains two tones at 5,000 Hz and 5,080 Hz with equal amplitudes.

Find: (a) The frequency resolution with a rectangular window, (b) whether the two tones can be resolved with a rectangular window, (c) the frequency resolution with a Hanning window and whether the tones can be resolved, and (d) the minimum FFT length required to resolve the tones using a Hanning window.

Solution:

(a) Rectangular window frequency resolution: $\Delta f = f_s$ / N = 48,000 / 2,048 = 23.44 Hz

Main lobe width (rectangular) = $2 \times \Delta f$ = 46.88 Hz.

(b) The tone separation is 5,080 - 5,000 = 80 Hz. Since 80 Hz > 46.88 Hz (main lobe width), the two tones can be resolved with the rectangular window. However, sidelobes at -13 dB may cause artifacts.

(c) Hanning window main lobe width = $4 \times \Delta f = 4 \times 23.44$ = 93.75 Hz.

Since 80 Hz < 93.75 Hz, the two tones cannot be resolved with a Hanning window at N = 2048.

(d) For the Hanning window to resolve the tones, the main lobe width must be less than the separation: $4 \times (f_s/N) < 80 \rightarrow N > 4 \times 48{,}000/80 = 2{,}400$

The minimum FFT length is N = 2,400 (next power of 2 would be N = 4,096 for computational efficiency).

With N = 4,096: Δf = 48,000/4,096 = 11.72 Hz; Hanning main lobe = 46.88 Hz < 80 Hz → resolved.

Problem 8.6.3

Given: Welch's method is applied to estimate the PSD of a 10-second recording sampled at f_s = 8 kHz. The data is divided into segments of L = 1024 samples with 50% overlap, and each segment is windowed with a Hamming window.

Find: (a) The total number of samples, (b) the number of overlapping segments, (c) the frequency resolution, (d) the variance reduction factor compared to a single periodogram, and (e) the number of frequency bins in the output PSD.

Solution:

(a) Total samples: N = $10 \times 8{,}000$ = 80,000 samples

(b) With 50% overlap, the hop size is L/2 = 512 samples. Number of segments: K = (N - L)/(L/2) + 1 = (80,000 - 1,024)/512 + 1 = 78,976/512 + 1 = 154.25 + 1 $\approx$ 155 segments

(c) Frequency resolution: $\Delta f = f_s$ / L = 8,000 / 1,024 = 7.8125 Hz

(d) For Welch's method with 50% overlap and Hamming window, the effective number of independent segments is approximately K $\times$ (1 - overlap_correlation). For Hamming with 50% overlap, the correlation between adjacent segments is about 0.5, giving an effective independent segment count of approximately $K \times 0.667 \approx 155 \times 0.667 \approx 103$.

Variance reduction factor $\approx$ 103$\times$ compared to a single periodogram (the PSD estimate variance is reduced by a factor of ~103).

In the simplified case (assuming independent segments): variance reduction $\approx$ K = 155$\times$.

(e) Number of frequency bins: N_{bins} = L/2 + 1 = 1,024/2 + 1 = 513 bins (covering 0 to 4,000 Hz)

Problem 8.6.4

Given: An STFT is performed on a 5-second speech signal sampled at $f_s = 22.05$ kHz. A 1024-sample Hamming window with 75% overlap is used.

Find: (a) The window duration in milliseconds, (b) the frequency resolution, (c) the time resolution (hop size), (d) the total number of STFT frames, and (e) the spectrogram dimensions (time frames × frequency bins).

Solution:

(a) Window duration: $T_w = 1024 / 22{,}050 = 0.04644$ s $= 46.44$ ms

(b) Frequency resolution: $\Delta f = f_s / N = 22{,}050 / 1{,}024 = 21.53$ Hz

(c) Hop size with 75% overlap: Hop $= 1024 \times (1 - 0.75) = 256$ samples $\Delta t = 256 / 22{,}050 = 0.01161$ s $= 11.61$ ms

(d) Total samples: $5 \times 22{,}050 = 110{,}250$ Number of frames: $(110{,}250 - 1{,}024) / 256 + 1 = 109{,}226 / 256 + 1 = 426.7 + 1 \approx 427$ frames

(e) Frequency bins: $N/2 + 1 = 513$ Spectrogram dimensions: 427×513 (427 time frames by 513 frequency bins, covering 0 to 11.025 kHz)

The spectrogram contains $427 \times 513 = 219{,}051$ complex-valued entries. At 75% overlap, the time resolution is approximately 4× finer than with no overlap, at the cost of 4× more computation.

Problem 8.6.5

Given: Two sinusoids at $f_1 = 2{,}000$ Hz and $f_2 = 2{,}030$ Hz (30 Hz apart) are present in a signal sampled at $f_s = 8$ kHz. Only $N = 128$ samples are available. Both sinusoids have equal amplitude ($A = 1.0$) with additive white Gaussian noise at SNR = 20 dB.

Find: (a) The FFT frequency resolution and whether classical FFT can resolve the tones, (b) the required AR model order for parametric estimation, and (c) the required data length for FFT-based resolution of the two tones.

Solution:

(a) FFT frequency resolution: $\Delta f = f_s / N = 8{,}000 / 128 = 62.5$ Hz

Since the tone separation is 30 Hz < 62.5 Hz, the FFT cannot resolve the two sinusoids. They will appear as a single peak at approximately 2,015 Hz.

Even with zero-padding to 1,024 points (interpolated $\Delta f = 7.8$ Hz), the fundamental resolution remains 62.5 Hz — zero-padding smooths the spectral shape but does not improve resolving power.

(b) For two sinusoids, the signal can be modeled as an AR process with 2 poles per sinusoid (representing the complex conjugate pair). The minimum AR model order is:

$p = 2 \times (\text{number of sinusoids}) = 2 \times 2 = 4$

In practice, $p = 6$ to 8 is chosen to account for noise. The Burg method with $p = 6$ applied to the 128 samples will place spectral peaks near 2,000 Hz and 2,030 Hz.

Alternatively, the MUSIC algorithm with a 32×32 correlation matrix and 2 assumed signal components can achieve super-resolution, resolving the 30 Hz separation from only 128 samples at 20 dB SNR.

(c) For FFT-based resolution with a rectangular window: $N \geq f_s / \Delta f_{required} = 8{,}000 / 30 = 267$ samples (minimum)

With a Hanning window (main lobe $= 4 \times \Delta f$): $N \geq 4 \times 8{,}000 / 30 = 1{,}067$ samples

At $f_s = 8$ kHz, this corresponds to $267/8{,}000 = 33$ ms (rectangular) or $1{,}067/8{,}000 = 133$ ms (Hanning) of data.

Problem 8.6.6

Given: A speech signal is sampled at $f_s = 16$ kHz. A 20 ms Hamming window is used for frame analysis ($N = 320$ samples, zero-padded to 512). The speaker's fundamental frequency is $f_0 = 200$ Hz (male speaker). A 24-channel mel filter bank is applied from 0 to 8 kHz.

Find: (a) The mel frequency range, (b) the pitch period in samples, (c) the quefrency index of the pitch peak in the cepstrum, and (d) the number of MFCC coefficients typically retained and their significance.

Solution:

(a) Mel frequency at 8 kHz: $f_{mel} = 2595 \times \log_{10}(1 + 8000/700) = 2595 \times \log_{10}(12.43) = 2595 \times 1.0943 = 2840$ mel

The mel range spans 0 to 2840 mel, with the 24 triangular filters uniformly spaced across this range.

(b) Pitch period: $T_0 = 1/f_0 = 1/200 = 5$ ms

In samples: $T_0 = 5 \times 10^{-3} \times 16{,}000 = 80$ samples

(c) In the 512-point cepstrum, the pitch appears as a peak at quefrency index: $n = T_0 \times f_s / 1 = 80$ (corresponding to $80/16{,}000 = 5$ ms)

This peak is clearly separated from the vocal tract information, which is concentrated in cepstral indices 0 to about 50 (0 to 3.125 ms).

(d) Typically 13 MFCCs (c_0 through c_{12}) are retained. These coefficients capture the smooth spectral envelope that encodes phoneme identity:

- c_0 represents the overall log energy of the frame
- c_1 through c_{12} capture the spectral shape (formant locations and bandwidths)
- Higher-order coefficients (c_{13} and above) represent fine spectral detail (pitch harmonics) and are discarded

In practice, delta (velocity) and delta-delta (acceleration) coefficients are appended, giving a total of 39 features per frame (13 static + 13 delta + 13 delta-delta) for speech recognition systems.

Problem 8.6.7

Given: A vibration signal from a rotating machine is sampled at $f_s = 4$ kHz for 2 seconds. The signal contains a known shaft frequency at $f_{shaft} = 25$ Hz and harmonics, plus broadband bearing noise. The PSD is estimated using Welch's method with L = 512 segments and 50% overlap.

Find: (a) The frequency resolution, (b) the number of segments, (c) the total noise power if the broadband PSD level is $S_n = 5.0 \times 10^{-6}$ V²/Hz from 0 to 2 kHz, (d) the signal power in a 10 Hz band centered on the shaft frequency if the peak PSD is $S_{peak} = 2.0 \times 10^{-3}$ V²/Hz, and (e) the SNR in that band.

Solution:

(a) Frequency resolution: $\Delta f = f_s / L = 4{,}000 / 512 = 7.8125$ Hz

(b) Total samples: $N = 2 \times 4{,}000 = 8{,}000$ Hop size = L/2 = 256 Number of segments: K = (8,000 - 512)/256 + 1 = 7,488/256 + 1 = 29.25 + 1 ≈ 30 segments

(c) Total broadband noise power: $P_n = S_n \times BW = 5.0 \times 10^{-6} \times 2{,}000 = 0.01\ V^2 = 10\ mV^2$

$V_{rms,noise} = \sqrt{0.01} = 0.1\ V = 100\ mV$

(d) Signal power in 10 Hz band around shaft frequency: $P_{signal} = S_{peak} \times 10 = 2.0 \times 10^{-3} \times 10 = 0.02\ V^2 = 20\ mV^2$

(e) Noise power in the same 10 Hz band: $P_{n,band} = S_n \times 10 = 5.0 \times 10^{-6} \times 10 = 5.0 \times 10^{-5}\ V^2$

$SNR = P_{signal} / P_{n,band} = 0.02 / 5.0 \times 10^{-5} = 400$

$SNR_{dB} = 10 \times \log_{10}(400) = 26.0$ dB

The shaft frequency is clearly detectable above the broadband noise floor.

Problem 8.6.8

Given: A noisy ECG signal (1024 samples, $f_s = 500$ Hz) is decomposed using a 4-level DWT with Daubechies db6 wavelets. The noise standard deviation is estimated at $\sigma = 0.15$ mV.

Find: (a) The frequency bands at each decomposition level, (b) the number of coefficients at each level, (c) the universal threshold λ, and (d) the expected effect of soft thresholding on the ECG signal.

Solution:

(a) At each DWT level, the bandwidth is halved: Level 1 detail (d_1): 125–250 Hz (highest frequency noise, muscle artifacts) Level 2 detail (d_2): 62.5–125 Hz (high-frequency noise) Level 3 detail (d_3): 31.25–62.5 Hz (includes some high-frequency ECG components) Level 4 detail (d_4): 15.625–31.25 Hz (QRS complex energy band) Level 4 approximation (a_4): 0–15.625 Hz (baseline, P and T waves)

(b) Coefficients at each level (downsampled by 2 at each level): d_1: 1024/2 = 512 coefficients d_2: 512/2 = 256 coefficients d_3: 256/2 = 128 coefficients d_4: 128/2 = 64 coefficients a_4: 64 coefficients

Total wavelet coefficients = 512 + 256 + 128 + 64 + 64 = 1024 (same as original signal length — DWT preserves information).

(c) Universal threshold: $\lambda = \sigma\sqrt{(2 \ln N)} = 0.15 \times \sqrt{(2 \times \ln(1024))} = 0.15 \times \sqrt{(2 \times 6.931)} = 0.15 \times \sqrt{13.863} = 0.15 \times 3.724 = 0.559$ mV

(d) Soft thresholding is applied to the detail coefficients d_1 through d_4. The approximation coefficients a_4 are left unchanged (they contain the baseline ECG waveform). Detail coefficients with magnitudes below 0.559 mV are set to zero (removing noise), while larger coefficients are shrunk by λ toward zero. The QRS complex, which produces large coefficients in d_3 and d_4, is preserved because its amplitude (typically 1–3 mV) exceeds the threshold. High-frequency muscle noise in d_1 and d_2 is largely removed. This method preserves sharp QRS edges better than a conventional lowpass filter.

Problem 8.6.9

Given: A signal of length N = 512 samples at $f_s = 10$ kHz contains 3 sinusoidal components in white Gaussian noise. The MUSIC algorithm is applied using a correlation matrix of size M × M, where M = 64.

Find: (a) The number of signal and noise subspace eigenvectors, (b) the minimum SNR typically required for MUSIC to resolve two closely spaced sinusoids, (c) the theoretical frequency resolution advantage over FFT, and (d) the frequency resolution of a standard 512-point FFT for comparison.

Solution:

(a) With 3 sinusoidal components (each producing a complex conjugate pair in the correlation matrix but counted as one signal dimension for real sinusoids observed via real-valued data — for the general complex case, each sinusoid uses one eigenvector):

Signal subspace dimension: 3 eigenvectors (one per sinusoidal component, for the complex data model) Noise subspace dimension: M - 3 = 64 - 3 = 61 eigenvectors

MUSIC forms the pseudospectrum by projecting the frequency steering vector onto the noise subspace. The pseudospectrum peaks sharply where the steering vector is orthogonal to the noise subspace, indicating the signal frequencies.

(b) MUSIC typically requires SNR > 0 to 10 dB for reliable super-resolution, with performance degrading rapidly below 0 dB. For closely spaced sinusoids (separation $< \Delta f_{FFT}$), SNR > 10 dB is generally needed.

(c) MUSIC's resolution is not limited by the data record length N or the FFT bin width. It is determined by the correlation matrix size M and the SNR. For M = 64 and adequate SNR, MUSIC can resolve sinusoids separated by as little as:

$\Delta f_{MUSIC} \approx f_s / (M \times \sqrt{SNR_{linear}})$

At SNR = 20 dB (linear = 100): $\Delta f \approx 10{,}000 / (64 \times 10) = 15.6$ Hz

(d) Standard FFT resolution: $\Delta f_{FFT} = f_s / N = 10{,}000 / 512 = 19.53$ Hz

MUSIC achieves approximately 15.6 Hz resolution from only 64-sample windows, slightly better than the 19.53 Hz FFT resolution from the full 512 samples. The real advantage of MUSIC is when N is small — for N = 64 samples, FFT resolution would be 156.25 Hz while MUSIC still achieves ~15.6 Hz.

Problem 8.6.10

Given: A 3-level DWT is performed on a 2048-sample signal at $f_s = 16$ kHz using Haar wavelets. The signal contains a transient event (a 1 ms pulse) occurring at sample 1000 and a 200 Hz sinusoidal component.

Find: (a) The frequency bands at each DWT level, (b) which decomposition level best captures the transient event and why, (c) the time resolution at each level, and (d) which level captures the 200 Hz sinusoid.

Solution:

(a) Frequency bands: Level 1 detail (d_1): 4–8 kHz (1024 coefficients) Level 2 detail (d_2): 2–4 kHz (512 coefficients) Level 3 detail (d_3): 1–2 kHz (256 coefficients) Level 3 approximation (a_3): 0–1 kHz (256 coefficients)

(b) The 1 ms transient pulse has a bandwidth of approximately 1/0.001 = 1 kHz, meaning its spectral content extends from 0 to several kHz.

The transient will appear in all detail levels (d_1, d_2, d_3) because its broadband energy spans all frequency bands. However, the transient is best localized in level 1 (d_1) because: - d_1 has the finest time resolution (1 sample = 0.0625 ms per Haar wavelet coefficient at this level) - The transient's energy in the 4–8 kHz band produces a sharp spike in d_1 near coefficient index 500 (corresponding to sample 1000)

(c) Time resolution at each level (Haar wavelet width): Level 1: 2 samples = 2/16,000 = 0.125 ms Level 2: 4 samples = 4/16,000 = 0.25 ms Level 3: 8 samples = 8/16,000 = 0.5 ms

(d) The 200 Hz sinusoid falls in the 0–1 kHz band, which is captured by the level 3 approximation coefficients (a_3). The a_3 coefficients contain all energy from 0 to 1 kHz, including the 200 Hz component. With 256 coefficients representing this band, the effective frequency resolution is 1,000/256 = 3.91 Hz — sufficient to represent the 200 Hz sinusoid.

This illustrates the DWT's multi-resolution property: the transient is well-localized in time at level 1, while the sinusoid is well-localized in frequency at level 3.

Chapter 8 — Section 8.7: Adaptive Filtering

Practice problems covering the LMS algorithm, Normalized LMS, adaptive noise cancellation, echo cancellation, channel equalization, Recursive Least Squares (RLS), Kalman filtering, Wiener filter design, and compressive sensing.

Problem 8.7.1

Given: An LMS adaptive filter of order $M = 8$ is used for system identification. The input signal power is $P_x = 2.0$ W.

Find: (a) The maximum step size μ_{max} for convergence, (b) a practical step size at 5% of the maximum, (c) the approximate number of iterations to converge within 10% of the optimal solution using the time constant formula $\tau = 1/(4\mu MP_x)$, and (d) the steady-state excess mean squared error (MSE) relative to the minimum MSE, given by the misadjustment factor $M_{adj} = \mu MP_x$.

Solution:

(a) Maximum step size: $\mu_{max} = 1/(M \times P_x) = 1/(8 \times 2.0) = 0.0625$

(b) Practical step size: $\mu = 0.05 \times \mu_{max} = 0.05 \times 0.0625 = 0.003125$

(c) Convergence time constant: $\tau = 1/(4\mu MP_x) = 1/(4 \times 0.003125 \times 8 \times 2.0) = 1/0.2 = 5$ iterations (time constant)

To converge within 10% of optimal (2.3 time constants): $2.3 \times 5 = \sim 12$ iterations

To converge within 1% of optimal (4.6 time constants): $4.6 \times 5 = \sim 23$ iterations

(d) Misadjustment factor: $M_{adj} = \mu MP_x = 0.003125 \times 8 \times 2.0 = 0.05 = 5\%$

This means the steady-state MSE is 5% above the minimum achievable MSE (Wiener solution). The excess MSE is the penalty for using a stochastic gradient instead of the true gradient.

Problem 8.7.2

Given: An LMS adaptive filter has $M = 6$ taps, step size $\mu = 0.01$, current weight vector w = [0.5, -0.3, 0.2, 0.1, -0.1, 0.4], input vector x = [0.8, -0.6, 0.4, 1.0, -0.2, 0.5], and desired signal $d = 1.2$.

Find: (a) The filter output y, (b) the error signal e, (c) the updated weight vector after one LMS iteration, and (d) the squared error before and expected after the update.

Solution:

(a) Filter output: $y = w^T x = 0.5(0.8) + (-0.3)(-0.6) + 0.2(0.4) + 0.1(1.0) + (-0.1)(-0.2) + 0.4(0.5) = 0.40 + 0.18 + 0.08 + 0.10 + 0.02 + 0.20 = 0.98$

(b) Error: $e = d - y = 1.2 - 0.98 = 0.22$

(c) Weight update: $w[n+1] = w[n] + 2\mu e \times x$

$2\mu e = 2 \times 0.01 \times 0.22 = 0.0044$

Update vector $= 0.0044 \times [0.8, -0.6, 0.4, 1.0, -0.2, 0.5] = [0.00352, -0.00264, 0.00176, 0.00440, -0.00088, 0.00220]$

$w[n+1] = [0.50352, -0.30264, 0.20176, 0.10440, -0.10088, 0.40220]$

Rounded: $w = [0.5035, -0.3026, 0.2018, 0.1044, -0.1009, 0.4022]$

(d) Squared error before update: $e^2 = 0.22^2 = 0.0484$

After the update, the new output would be approximately: $y' = w[n+1]^T x \approx 0.98 + 2\mu e \times x^T x = 0.98 + 0.0044 \times (0.64 + 0.36 + 0.16 + 1.00 + 0.04 + 0.25) = 0.98 + 0.0044 \times 2.45 = 0.98 + 0.01078 = 0.9908$

Expected new error: $1.2 - 0.9908 = 0.2092$, $e^2 \approx 0.0438$ (a 9.5% reduction in squared error)

Problem 8.7.3

Given: An NLMS (Normalized LMS) adaptive filter has $M = 10$ taps, normalized step size $\tilde{\mu} = 0.5$, and regularization parameter $\delta = 0.001$. The input signal has instantaneous power $||x[n]||^2 = 3.2$.

Find: (a) The effective step size for this sample, (b) comparison with a standard LMS step size of $\mu = 0.02$, and (c) the effective step size when $||x[n]||^2$ drops to 0.1 (low-power segment).

Solution:

(a) NLMS update: $w[n+1] = w[n] + (\tilde{\mu} / (||x[n]||^2 + \delta)) \times e[n] \times x[n]$

Effective step size: $\mu_{eff} = \tilde{\mu} / (||x[n]||^2 + \delta) = 0.5 / (3.2 + 0.001) = 0.5 / 3.201 = 0.1562$

(b) The standard LMS with $\mu = 0.02$ applies the same step size regardless of input power. The NLMS effective step size of 0.1562 is much larger, meaning NLMS converges faster when the input power is high.

Maximum stable LMS step size: $\mu_{max} = 1/(MP_x) = 1/(10 \times 0.32) = 0.3125$ (using average power $= ||x||^2/M = 3.2/10 = 0.32$).

The NLMS effective step size of 0.1562 is 50% of μ_{max}, which is within the stable range.

(c) When $||x[n]||^2 = 0.1$ (quiet segment): $\mu_{eff} = 0.5 / (0.1 + 0.001) = 0.5 / 0.101 = 4.95$

Without the regularization parameter, $\mu_{eff} = 0.5/0.1 = 5.0$, which would cause instability. The regularization $\delta = 0.001$ limits the effective step size during very low-power segments but does not prevent the large value here. A more robust choice would be $\delta = 0.01$ or using a power-estimated NLMS variant.

In practice, the NLMS step size $\tilde{\mu}$ should satisfy $0 < \tilde{\mu} < 2$ for stability, and the normalization automatically prevents the update from becoming too large relative to the input energy — the weight correction magnitude is bounded by $\tilde{\mu} \times |e[n]| / ||x[n]||$ regardless of the input power level.

Problem 8.7.4

Given: A telephone echo canceller uses a 512-tap LMS adaptive filter at $f_s = 8$ kHz. The echo path has a flat-delay component of 30 ms followed by an exponentially decaying room impulse response with a time constant of 20 ms. The near-end speech power is $P_s = 0.01$ W and the far-end speech power is $P_x = 0.05$ W.

Find: (a) The duration of the echo path that can be modeled, (b) whether the filter length is sufficient, (c) the maximum LMS step size for convergence, (d) the computational cost per sample in multiply-accumulate (MAC) operations, and (e) the echo return loss enhancement (ERLE) target and its relation to the misadjustment.

Solution:

(a) Echo path duration covered by the filter: $T_{filter} = M / f_s = 512 / 8{,}000 = 64$ ms

(b) The echo path requires: flat delay (30 ms) + decay time (at least $3\tau = 60$ ms for 95% decay) = 90 ms total.

Since 64 ms < 90 ms, the 512-tap filter is not sufficient. Minimum required taps: $90 \times 10^{-3} \times 8{,}000 = 720$ taps. A practical choice would be 1024 taps (128 ms), providing margin for the echo tail.

(c) Maximum step size: $\mu_{max} = 1/(M \times P_x) = 1/(512 \times 0.05) = 1/25.6 = 0.0391$

A practical step size: $\mu = 0.1 \times \mu_{max} = 0.00391$

(d) Computational cost per sample: LMS requires 2M + 1 MAC operations per sample = 2 × 512 + 1 = 1,025 MACs (512 for filter output, 512 for weight update, 1 for error computation)

At $f_s = 8$ kHz: 1,025 × 8,000 = 8.2 MMAC/s (easily handled by any modern DSP)

(e) The ERLE measures how well the adaptive filter cancels the echo: $ERLE = 10 \times \log_{10}(P_{echo} / P_{residual})$

For LMS with misadjustment $M_{adj} = \mu M P_x = 0.00391 \times 512 \times 0.05 = 0.1$: $ERLE \approx -10 \times \log_{10}(M_{adj}) = -10 \times \log_{10}(0.1) = 10$ dB

Typical targets are 20–30 dB ERLE, requiring a smaller step size ($\mu \approx 0.0004$) or a normalized algorithm.

Problem 8.7.5

Given: An RLS adaptive equalizer has M = 4 taps and forgetting factor $\lambda = 0.98$. The initial inverse correlation matrix is $P[0] = 100 \times I$ (4 × 4 identity scaled by 100). At time n = 1, the input vector is $x = [1.0, 0.5, -0.3, 0.2]^T$, the desired signal is d = 0.8, and the current weight vector is $w = [0, 0, 0, 0]^T$.

Find: (a) The a priori error, (b) the gain vector k, (c) the updated weight vector, and (d) the updated P matrix.

Solution:

(a) A priori error: $y = w^T x = [0,0,0,0] \times [1.0, 0.5, -0.3, 0.2]^T = 0$ $e = d - y = 0.8 - 0 = 0.8$

(b) Gain vector: $k = P[0]x / (\lambda + x^T P[0]x)$

$P[0]x = 100I \times x = 100 \times [1.0, 0.5, -0.3, 0.2]^T = [100, 50, -30, 20]^T$

$x^T P[0]x = x^T \times 100x = 100 \times (1.0^2 + 0.5^2 + 0.3^2 + 0.2^2) = 100 \times (1.0 + 0.25 + 0.09 + 0.04) = 100 \times 1.38 = 138$

$k = [100, 50, -30, 20]^T / (0.98 + 138) = [100, 50, -30, 20]^T / 138.98$

$k = [0.7195, 0.3598, -0.2159, 0.1439]^T$

(c) Updated weights: $w[1] = w[0] + k \times e = [0,0,0,0] + 0.8 \times [0.7195, 0.3598, -0.2159, 0.1439]$

$w[1] = [0.5756, 0.2878, -0.1727, 0.1152]^T$

(d) Updated P matrix: $P[1] = (P[0] - k \times x^T \times P[0]) / \lambda$

$k \times x^T \times P[0] = k \times [100, 50, -30, 20]^T$

This is the outer product of k (4×1) with $P[0]x^T$ (1×4):

$k \times (P[0]x)^T = [0.7195; 0.3598; -0.2159; 0.1439] \times [100, 50, -30, 20]$

The diagonal entries of $P[0] - k(P[0]x)^T$: $P_{11} = 100 - 0.7195 \times 100 = 100 - 71.95 = 28.05$ $P_{22} = 100 - 0.3598 \times 50 = 100 - 17.99 = 82.01$ $P_{33} = 100 - (-0.2159)(-30) = 100 - 6.477 = 93.52$ $P_{44} = 100 - 0.1439 \times 20 = 100 - 2.878 = 97.12$

Dividing by $\lambda = 0.98$: $P_{11} = 28.62$, $P_{22} = 83.68$, $P_{33} = 95.43$, $P_{44} = 99.10$

The diagonal entries show that uncertainty has decreased most for the first tap (which had the largest input component), demonstrating how RLS allocates learning proportional to the input signal strength.

Problem 8.7.6

Given: A 1-D Kalman filter tracks the voltage of a slowly drifting battery. The state is x = battery voltage (V). The state transition model is $x[n] = x[n-1] + w[n]$ (random walk), the measurement model is $z[n] = x[n] + v[n]$, the process noise variance is $Q = 0.001\ V^2$, and the measurement noise variance is $R = 0.25\ V^2$. The initial estimate is $\hat{x}[0] = 12.0$ V with error variance $P[0] = 1.0\ V^2$. Three measurements arrive: $z[1] = 11.8$ V, $z[2] = 12.1$ V, $z[3] = 11.9$ V.

Find: The Kalman gain, updated state estimate, and error variance after each measurement.

Solution:

Measurement 1 ($z[1] = 11.8$ V): Prediction: $\hat{x}[1|0] = \hat{x}[0] = 12.0$ V, $P[1|0] = P[0] + Q = 1.0 + 0.001 = 1.001$

Kalman gain: $K[1] = P[1|0]/(P[1|0] + R) = 1.001/(1.001 + 0.25) = 1.001/1.251 = 0.800$

Update: $\hat{x}[1] = 12.0 + 0.800 \times (11.8 - 12.0) = 12.0 + 0.800 \times (-0.2) = 12.0 - 0.16 = 11.84$ V $P[1] = (1 - 0.800) \times 1.001 = 0.200 \times 1.001 = 0.2002$ V^2

Measurement 2 ($z[2] = 12.1$ V): Prediction: $\hat{x}[2|1] = 11.84$ V, $P[2|1] = 0.2002 + 0.001 = 0.2012$

$K[2] = 0.2012/(0.2012 + 0.25) = 0.2012/0.4512 = 0.446$

Update: $\hat{x}[2] = 11.84 + 0.446 \times (12.1 - 11.84) = 11.84 + 0.446 \times 0.26 = 11.84 + 0.116 = 11.956$ V $P[2] = (1 - 0.446) \times 0.2012 = 0.554 \times 0.2012 = 0.1115$ V^2

Measurement 3 ($z[3] = 11.9$ V): Prediction: $\hat{x}[3|2] = 11.956$ V, $P[3|2] = 0.1115 + 0.001 = 0.1125$

$K[3] = 0.1125/(0.1125 + 0.25) = 0.1125/0.3625 = 0.310$

Update: $\hat{x}[3] = 11.956 + 0.310 \times (11.9 - 11.956) = 11.956 + 0.310 \times (-0.056) = 11.956 - 0.017 = 11.939$ V $P[3] = (1 - 0.310) \times 0.1125 = 0.690 \times 0.1125 = 0.0776$ V^2

The Kalman gain decreases ($0.800 \to 0.446 \to 0.310$) as the estimate becomes more confident. The error variance drops from 1.0 to 0.0776 V^2 (standard deviation from 1.0 V to 0.279 V). The filter converges toward a steady-state gain of $K_{ss} \approx Q/(Q + Q) \times$ (something), which can be found by solving $P_{ss} = (P_{ss} + Q)R / (P_{ss} + Q + R)$. Eventually, the Kalman gain will settle at approximately $K_{ss} \approx 0.06$, balancing the slow process drift against measurement noise.

Problem 8.7.7

Given: A Kalman filter tracks a falling object under gravity. The state vector is x = [height (m), velocity (m/s)]^T. The discrete state transition ($\Delta t = 0.1$ s) is A = [[1, 0.1], [0, 1]], the control input is B = [-0.5(0.1)^2, -0.1]^T $\times$ g = [-0.049, -0.981]^T (free fall with g = 9.81 m/s^2), H = [1, 0], Q = [[0.01, 0], [0, 0.1]], and R = 4 m^2. At n = 0: $\hat{x}$ = [1000, 0]^T, P = [[10, 0], [0, 1]]. A radar measurement z = 994 m arrives at n = 1.

Find: (a) The predicted state, (b) the predicted covariance, (c) the Kalman gain, and (d) the updated state estimate.

Solution:

(a) Predicted state (free fall, no control input u — gravity is incorporated in B): $\hat{x}[1|0] = A\hat{x}[0] + B$

$\hat{x}[1|0]$ = [[1, 0.1], [0, 1]] $\times$ [1000, 0]^T + [-0.049, -0.981]^T

= [1000 + 0, 0]^T + [-0.049, -0.981]^T = [999.951, -0.981]^T

The object has fallen 0.049 m and has velocity -0.981 m/s (downward) after 0.1 s.

(b) Predicted covariance: $P[1|0] = APA^T + Q$

APA^T = [[1, 0.1], [0, 1]] $\times$ [[10, 0], [0, 1]] $\times$ [[1, 0], [0.1, 1]]

First: AP = [[10, 0.1], [0, 1]]

APA^T = [[10, 0.1], [0, 1]] $\times$ [[1, 0], [0.1, 1]] = [[10 + 0.01, 0 + 0.1], [0 + 0.1, 0 + 1]] = [[10.01, 0.1], [0.1, 1]]

P[1|0] = [[10.01, 0.1], [0.1, 1]] + [[0.01, 0], [0, 0.1]] = [[10.02, 0.1], [0.1, 1.1]]

(c) Kalman gain: $S = HP[1|0]H^T + R = [1, 0] \times [[10.02, 0.1], [0.1, 1.1]] \times [1, 0]^T + 4 = 10.02 + 4 = 14.02$

$K = P[1|0]H^T / S = [10.02, 0.1]^T / 14.02 = [0.715, 0.00713]^T$

(d) Innovation: $z - H\hat{x}[1|0] = 994 - 999.951 = -5.951$ m

Updated state: $\hat{x}[1] = [999.951, -0.981]^T + [0.715, 0.00713]^T \times (-5.951)$

$= [999.951 - 4.255, -0.981 - 0.0424]^T = [995.696, -1.023]^T$

The Kalman filter corrects the predicted height (999.95 m) significantly toward the measurement (994 m), pulling it to 995.7 m. The velocity estimate also increases slightly (from -0.981 to -1.023 m/s) because the lower-than-expected altitude implies the object is falling faster.

Problem 8.7.8

Given: A signal d(t) with PSD $S_{dd}(f) = 10^{-3} / (1 + (f/500)^2)$ V^2/Hz (Lorentzian spectrum centered at DC with 3 dB bandwidth of 500 Hz) is observed in additive white noise with $S_{nn}(f) = 5 \times 10^{-6}$ V^2/Hz. The measurement bandwidth extends from 0 to 5 kHz.

Find: (a) The Wiener filter frequency response H(f) at f = 0, 500, 1000, and 5000 Hz, (b) the input SNR at each of these frequencies, and (c) the overall noise reduction in dB.

Solution:

(a) Wiener filter: $H(f) = S_{dd}(f) / (S_{dd}(f) + S_{nn}(f))$

At f = 0: $S_{dd}(0) = 10^{-3}$ V^2/Hz $H(0) = 10^{-3} / (10^{-3} + 5 \times 10^{-6}) = 10^{-3} / 1.005 \times 10^{-3} = 0.995$

At f = 500 Hz: $S_{dd}(500) = 10^{-3} / (1 + 1) = 5 \times 10^{-4}$ V^2/Hz $H(500) = 5 \times 10^{-4} / (5 \times 10^{-4} + 5 \times 10^{-6}) = 5 \times 10^{-4} / 5.05 \times 10^{-4} = 0.990$

At f = 1000 Hz: $S_{dd}(1000) = 10^{-3} / (1 + 4) = 2 \times 10^{-4}$ V^2/Hz $H(1000) = 2 \times 10^{-4} / (2 \times 10^{-4} + 5 \times 10^{-6}) = 2 \times 10^{-4} / 2.05 \times 10^{-4} = 0.976$

At f = 5000 Hz: $S_{dd}(5000) = 10^{-3} / (1 + 100) = 9.9 \times 10^{-6}$ V^2/Hz $H(5000) = 9.9 \times 10^{-6} / (9.9 \times 10^{-6} + 5 \times 10^{-6}) = 9.9 \times 10^{-6} / 14.9 \times 10^{-6} = 0.664$

(b) Input SNR at each frequency: $SNR(0) = 10^{-3} / 5 \times 10^{-6} = 200 = 23.0$ dB $SNR(500) = 5 \times 10^{-4} / 5 \times 10^{-6} = 100 = 20.0$ dB $SNR(1000) = 2 \times 10^{-4} / 5 \times 10^{-6} = 40 = 16.0$ dB $SNR(5000) = 9.9 \times 10^{-6} / 5 \times 10^{-6} = 1.98 = 3.0$ dB

The Wiener filter passes frequencies with high SNR nearly unchanged and attenuates frequencies with low SNR.

(c) Total input noise power: $P_{n,in} = S_{nn} \times BW = 5 \times 10^{-6} \times 5000 = 0.025$ V^2

Total output noise power: $P_{n,out} = \int_0^{5000} H^2(f) \times S_{nn}\, df$

Since H(f) varies, we approximate by integrating numerically. For the Lorentzian signal spectrum, $H(f) \approx 1$ for f « 5000 Hz and drops toward 0.664 near 5 kHz. The signal energy is concentrated below 1 kHz, so $H^2(f) \times S_{nn} \approx S_{nn}$ for f < 1 kHz and progressively attenuated beyond.

Approximate: $P_{n,out} \approx S_{nn} \times [1000 \times 0.99^2 + 2000 \times 0.95^2 + 2000 \times 0.75^2] = 5 \times 10^{-6} \times [980 + 1805 + 1125] = 5 \times 10^{-6} \times 3910 = 0.01955\ V^2$

Noise reduction = $10 \times \log_{10}(0.025 / 0.01955) = 10 \times \log_{10}(1.279) = 1.1$ dB

The modest noise reduction reflects that the signal spectrum is broadband (500 Hz bandwidth) relative to the measurement bandwidth (5 kHz), leaving the Wiener filter unable to fully separate signal from noise at most frequencies.

Problem 8.7.9

Given: A signal of length N = 512 is K = 10 sparse in the wavelet domain. A random Gaussian measurement matrix of size $M \times 512$ is used for compressive sensing. The practical constant C = 3 is used for the measurement bound.

Find: (a) The minimum number of measurements M, (b) the compression ratio, (c) the percentage of measurements saved compared to Nyquist, (d) the measurement matrix dimensions, and (e) the expected number of OMP iterations for recovery.

Solution:

(a) Minimum measurements: $M \geq C \times K \times \log(N/K) = 3 \times 10 \times \log(512/10) = 30 \times \log(51.2) = 30 \times 3.934$

Using natural logarithm: $M \geq 30 \times 3.934 = 118.0$

M = 118 measurements (minimum)

(b) Compression ratio: CR = N/M = 512/118 = 4.34:1

(c) Percentage saved: Savings = $(1 - M/N) \times 100 = (1 - 118/512) \times 100 = (1 - 0.2305) \times 100 = 76.9\%$

(d) Measurement matrix Φ is 118×512. Each of the 118 measurements is a random linear combination of all 512 signal values, with entries drawn from a Gaussian distribution N(0, 1/M).

(e) OMP (Orthogonal Matching Pursuit) iteratively identifies one support element per iteration. Since the signal has K = 10 nonzero coefficients, OMP requires exactly K = 10 iterations to identify the full support. At each iteration, OMP:

1. Correlates the residual with all 512 columns of Φ (512 inner products)
2. Selects the column with maximum correlation
3. Solves a least-squares problem over the selected columns (growing from 1 to 10 columns)

Total computation: approximately $10 \times 512 \times 118 \approx 604{,}160$ multiply-accumulate operations, plus the growing least-squares solutions.

Problem 8.7.10

Given: An MRI scan acquires a 256×256 image with sparsity K = 3,000 in the wavelet domain. The current full-scan acquisition takes 8 minutes. Compressive sensing is applied with a practical constant

$C = 4$.

Find: (a) The total number of pixels (Nyquist samples), (b) the minimum CS measurements required, (c) the scan time reduction factor, (d) the new scan time, and (e) the trade-off considerations for choosing the actual number of measurements.

Solution:

(a) Total pixels (Nyquist): $N = 256 \times 256 = 65{,}536$ samples

(b) Minimum CS measurements: $M \geq C \times K \times \log(N/K) = 4 \times 3{,}000 \times \log(65{,}536/3{,}000) = 12{,}000 \times \log(21.85)$

$\log(21.85) = 3.085$ (natural logarithm)

$M \geq 12{,}000 \times 3.085 = 37{,}016$ measurements

(c) Scan time reduction: Reduction factor $= N/M = 65{,}536/37{,}016 = 1.77\times$

(d) New scan time: $t_{CS} = 8 / 1.77 = 4.52$ minutes

This is a modest reduction because the sparsity ratio $K/N = 3{,}000/65{,}536 = 4.6\%$ is not extremely sparse.

(e) Trade-off considerations:

- Minimum $M = 37{,}016$: fastest scan but highest reconstruction error, potential artifacts near edges and fine structures
- Practical $M = 45{,}000$–$50{,}000$: 20–35% safety margin, scan time ≈ 5.5–6.1 minutes, reliable reconstruction at moderate SNR
- $M = 55{,}000$: only 16% scan time reduction but near-Nyquist quality — diminishing returns

For clinical MRI, the typical choice is $M \approx 1.5 \times M_{min} = 55{,}500$ measurements (scan time ≈ 6.8 minutes, 1.2 minutes saved), providing robust image quality. Research applications may accept M closer to the theoretical minimum for faster scanning of dynamic processes (cardiac MRI, functional MRI).

The actual k-space sampling pattern also matters: random undersampling of k-space phase-encode lines (maintaining full readout) produces incoherent aliasing artifacts that CS algorithms can remove, whereas structured undersampling produces coherent artifacts that violate the RIP and degrade reconstruction.

Chapter 9 — Section 9.1: Electrostatics

Practice problems covering electric charge, Coulomb's law, electric fields, potential, capacitance, dielectric materials, boundary conditions, and Laplace's/Poisson's equations.

Problem 9.1.1

Given: Three point charges are arranged along the x-axis: $q_1 = +6\ \mu C$ at $x = 0$, $q_2 = -3\ \mu C$ at $x = 20$ cm, and $q_3 = +2\ \mu C$ at $x = 50$ cm. The Coulomb constant $k = 8.99 \times 10^9\ N{\cdot}m^2/C^2$.

Find: The net electrostatic force on q_2 (magnitude and direction).

Solution:

Force on q_2 due to q_1: $F_{21} = k|q_1||q_2| / r_{12}^2 = (8.99 \times 10^9)(6 \times 10^{-6})(3 \times 10^{-6}) / (0.20)^2 = (8.99 \times 10^9)(18 \times 10^{-12}) / 0.04 = 0.16182 / 0.04 = 4.046$ N

Since q_1 is positive and q_2 is negative, the force is attractive, pulling q_2 toward q_1 (in the -x direction).

Force on q_2 due to q_3: $r_{23} = 0.50 - 0.20 = 0.30$ m $F_{23} = k|q_2||q_3| / r_{23}^2 = (8.99 \times 10^9)(3 \times 10^{-6})(2 \times 10^{-6}) / (0.30)^2 = (8.99 \times 10^9)(6 \times 10^{-12}) / 0.09 = 0.05394 / 0.09 = 0.5993$ N

Since q_2 is negative and q_3 is positive, the force is attractive, pulling q_2 toward q_3 (in the +x direction).

Net force on q_2: $F_{net} = -4.046 + 0.5993 = -3.447$ N (in the -x direction).

The net force on q_2 is 3.45 N directed toward q_1 (in the -x direction).

Problem 9.1.2

Given: A uniformly charged thin spherical shell of radius $R = 15$ cm carries a total charge $Q = +10$ nC. The permittivity of free space is $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m.

Find: (a) The electric field at $r = 10$ cm (inside the shell). (b) The electric field at $r = 25$ cm (outside the shell). (c) The electric potential at $r = 25$ cm and at $r = 10$ cm.

Solution:

(a) By Gauss's Law, a spherical Gaussian surface at $r = 10\text{ cm} < R$ encloses no charge. $E(r = 10\text{ cm}) = 0$ V/m.

(b) At $r = 25$ cm > R, the shell appears as a point charge: $E = kQ / r^2 = (8.99 \times 10^9)(10 \times 10^{-9}) / (0.25)^2 = 89.9 / 0.0625 = 1{,}438$ V/m directed radially outward.

(c) Potential at $r = 25$ cm: $V = kQ / r = (8.99 \times 10^9)(10 \times 10^{-9}) / 0.25 = 89.9 / 0.25 = 359.6$ V

Potential at $r = 10$ cm: Inside a conducting shell, the potential is constant and equal to the potential at the surface. $V = kQ / R = (8.99 \times 10^9)(10 \times 10^{-9}) / 0.15 = 89.9 / 0.15 = 599.3$ V

Problem 9.1.3

Given: Two charges $q_1 = +8\ \mu C$ and $q_2 = -5\ \mu C$ are located at positions (0, 0) and (0.6 m, 0), respectively. The Coulomb constant $k = 8.99 \times 10^9\ N{\cdot}m^2/C^2$.

Find: The electric potential at point P located at (0.3 m, 0.4 m).

Solution:

Distance from q_1 to P: $r_1 = \sqrt{(0.3^2 + 0.4^2)} = \sqrt{(0.09 + 0.16)} = \sqrt{0.25} = 0.50$ m

Distance from q_2 to P: $r_2 = \sqrt{((0.3 - 0.6)^2 + 0.4^2)} = \sqrt{(0.09 + 0.16)} = \sqrt{0.25} = 0.50$ m

$V_P = kq_1/r_1 + kq_2/r_2 = (8.99 \times 10^9)(8 \times 10^{-6})/0.50 + (8.99 \times 10^9)(-5 \times 10^{-6})/0.50$ $V_P = 143{,}840 + (-89{,}900) = 53{,}940\ V \approx 53.9$ kV

Problem 9.1.4

Given: A parallel-plate capacitor with plate area $A = 50\ cm^2$ and plate separation $d = 1$ mm uses a ceramic dielectric with $\varepsilon_r = 200$. The capacitor is charged to 50 V and then disconnected from the source.

Find: (a) The capacitance. (b) The stored energy. (c) The charge on the plates. (d) If the dielectric is removed while the capacitor remains disconnected, find the new voltage and stored energy.

Solution:

(a) $C = \varepsilon_0\varepsilon_r A / d = (8.854 \times 10^{-12} \times 200 \times 50 \times 10^{-4}) / (1 \times 10^{-3}) = (8.854 \times 10^{-9}) / 10^{-3} = 8.854$ nF

(b) $W = ½CV^2 = 0.5 \times 8.854 \times 10^{-9} \times 50^2 = 0.5 \times 8.854 \times 10^{-9} \times 2500 = 11.07\ \mu J$

(c) $Q = CV = 8.854 \times 10^{-9} \times 50 = 442.7$ nC

(d) With the dielectric removed, Q remains constant (disconnected). New capacitance: $C_{new} = \varepsilon_0 A / d = 8.854 \times 10^{-12} \times 50 \times 10^{-4} / 10^{-3} = 44.27$ pF

New voltage: $V_{new} = Q / C_{new} = 442.7 \times 10^{-9} / 44.27 \times 10^{-12} = 10{,}000\ V = 10$ kV

New energy: $W_{new} = ½C_{new}V_{new}^2 = 0.5 \times 44.27 \times 10^{-12} \times (10{,}000)^2 = 2.214$ mJ

The energy increased by a factor of 200 ($= \varepsilon_r$). The work to remove the dielectric against the attractive electric force was converted to additional stored energy.

Problem 9.1.5

Given: A coaxial capacitor has inner radius a = 2 mm, outer radius b = 6 mm, and length l = 10 cm. The space between the conductors is filled with two concentric dielectric layers: the inner half (2 mm to 4 mm) has $\varepsilon_{r1} = 3.0$, and the outer half (4 mm to 6 mm) has $\varepsilon_{r2} = 6.0$.

Find: The total capacitance of the structure.

Solution:

For a coaxial geometry, the two concentric dielectric layers act as two capacitors in series (since the voltage adds radially).

$C_1 = 2\pi\varepsilon_0\varepsilon_{r1}l / \ln(r_{mid}/a) = 2\pi(8.854 \times 10^{-12})(3.0)(0.10) / \ln(4/2) = 2\pi \times 2.656 \times 10^{-12} / 0.6931 = 16.69 \times 10^{-12} / 0.6931 = 24.08$ pF

$C_2 = 2\pi\varepsilon_0\varepsilon_{r2}l / \ln(b/r_{mid}) = 2\pi(8.854 \times 10^{-12})(6.0)(0.10) / \ln(6/4) = 2\pi \times 5.313 \times 10^{-12} / 0.4055 = 33.38 \times 10^{-12} / 0.4055 = 82.31$ pF

Series combination: $1/C = 1/C_1 + 1/C_2 = 1/24.08 + 1/82.31 = 0.04153 + 0.01215 = 0.05368$ pF^{-1}

C = 18.63 pF

Problem 9.1.6

Given: A parallel-plate capacitor has plate separation d = 4 mm. The bottom plate is grounded (V = 0) and the top plate is at V = 200 V. A 1 mm thick conductor (floating, unconnected) is inserted parallel to the plates with its bottom surface 1.5 mm above the grounded plate. The medium is air ($\varepsilon_r = 1$) everywhere except within the conductor.

Find: Using Laplace's equation, find the electric field in each air gap and the surface charge densities on the conductor surfaces.

Solution:

The conductor is an equipotential body, so V is constant throughout it. The air gaps are: lower gap d_1 = 1.5 mm, upper gap d_2 = 4 - 1.5 - 1 = 1.5 mm.

In each air region, $\nabla^2 V = 0$ reduces to $d^2V/dz^2 = 0$ (1D problem), so the potential varies linearly.

Since both air gaps have the same thickness (1.5 mm each) and the same dielectric (air), the electric field is the same in both gaps: $E = V_{total} / (d_1 + d_2) = 200 / (1.5 \times 10^{-3} + 1.5 \times 10^{-3}) = 200 / 3.0 \times 10^{-3}$ = 66,667 V/m = 66.7 kV/m in both air gaps

The conductor floats at $V_{cond} = E \times d_1 = 66{,}667 \times 1.5 \times 10^{-3} = 100$ V (exactly midway, as expected from symmetry).

Surface charge density on the bottom surface of the conductor: $\sigma_{bottom} = \varepsilon_0 E = 8.854 \times 10^{-12} \times 66{,}667$ = 590.3 nC/m^2 (positive, facing the grounded plate)

Surface charge density on the top surface: $\sigma_{top} = -\varepsilon_0 E$ = -590.3 nC/m^2 (negative, facing the +200 V plate)

The conductor redistributes the field but does not change its magnitude in this symmetric geometry. The conducting slab reduces the effective capacitor gap from 4 mm to 3 mm, increasing capacitance by a factor of 4/3.

Problem 9.1.7

Given: A point charge Q = +20 nC is located at the center of a dielectric sphere of radius R = 5 cm with relative permittivity $\varepsilon_r = 4$.

Find: (a) The electric displacement D, electric field E, and polarization P at r = 3 cm (inside the dielectric). (b) The electric field just outside the sphere at r = 5.01 cm (in free space).

Solution:

(a) By Gauss's Law for D (unaffected by dielectrics): $D = Q / (4\pi r^2)$ At r = 3 cm: $D = 20 \times 10^{-9} / (4\pi \times (0.03)^2) = 20 \times 10^{-9} / (4\pi \times 9 \times 10^{-4}) = 20 \times 10^{-9} / 1.131 \times 10^{-2} = 1.769\ \mu C/m^2$

$E = D / (\varepsilon_0\varepsilon_r) = 1.769 \times 10^{-6} / (8.854 \times 10^{-12} \times 4) = 1.769 \times 10^{-6} / 3.542 \times 10^{-11} = 49{,}950$ V/m ≈ 50.0 kV/m

$P = D - \varepsilon_0 E = D(1 - 1/\varepsilon_r) = 1.769 \times 10^{-6} \times (1 - 0.25) = 1.327\ \mu C/m^2$

(b) Just outside the sphere (r = 5.01 cm, free space): $D = Q / (4\pi r^2) = 20 \times 10^{-9} / (4\pi \times (0.0501)^2) = 20 \times 10^{-9} / 3.155 \times 10^{-2} = 0.6339\ \mu C/m^2$ $E = D / \varepsilon_0 = 0.6339 \times 10^{-6} / 8.854 \times 10^{-12} = 71{,}590$ V/m ≈ 71.6 kV/m

The field is 1.43× stronger just outside the sphere than at the same radius inside, because the dielectric reduces the internal field by a factor of ε_r. At the boundary, the normal D is continuous (no free surface charge), but E jumps by a factor of ε_r.

Problem 9.1.8

Given: Two capacitors, $C_1 = 10\ \mu F$ charged to 100 V and $C_2 = 22\ \mu F$ initially uncharged, are connected in parallel (positive to positive).

Find: (a) The final voltage across the combination. (b) The energy stored before and after connection. (c) The energy lost and explain where it went.

Solution:

(a) Total charge: $Q = C_1V_1 + C_2V_2 = 10 \times 10^{-6} \times 100 + 0 = 1.0$ mC Final voltage: $V_f = Q / (C_1 + C_2) = 1.0 \times 10^{-3} / (32 \times 10^{-6}) = 31.25$ V

(b) Energy before: $W_{before} = ½C_1V_1^2 = 0.5 \times 10 \times 10^{-6} \times 100^2 = 50.0$ mJ Energy after: $W_{after} = ½(C_1 + C_2)V_f^2 = 0.5 \times 32 \times 10^{-6} \times 31.25^2 = 0.5 \times 32 \times 10^{-6} \times 976.6 = 15.63$ mJ

(c) Energy lost: $\Delta W = 50.0 - 15.63 = 34.37$ mJ

This energy was dissipated as heat in the resistance of the connecting wires and contact resistance during the transient equalization current. Even with ideal (zero resistance) wires, the energy is radiated as an electromagnetic pulse. Energy is always lost when capacitors at different voltages are connected in parallel; the fraction lost is $C_1C_2(V_1 - V_2)^2 / [2(C_1 + C_2)]$.

Problem 9.1.9

Given: A parallel-plate capacitor has plates of area A = 200 cm² separated by d = 3 mm of air. A voltage of V = 1,000 V is applied. The dielectric strength of air is 3 MV/m.

Find: (a) The capacitance. (b) The electric field between the plates. (c) The surface charge density on the plates. (d) Whether the air will break down. (e) The maximum voltage that can be applied without breakdown.

Solution:

(a) $C = \varepsilon_0 A / d = 8.854 \times 10^{-12} \times 200 \times 10^{-4} / 3 \times 10^{-3} = 8.854 \times 10^{-12} \times 6.667 = 59.0$ pF

(b) $E = V / d = 1{,}000 / 3 \times 10^{-3} = 333{,}333$ V/m = 333 kV/m

(c) $\sigma = \varepsilon_0 E = 8.854 \times 10^{-12} \times 333{,}333 = 2.951\ \mu C/m^2$

(d) The dielectric strength of air is 3 MV/m = 3,000 kV/m. Since E = 333 kV/m < 3,000 kV/m, the air will not break down. The field is only 11% of the breakdown value.

(e) $V_{max} = E_{breakdown} \times d = 3 \times 10^6 \times 3 \times 10^{-3} = 9{,}000$ V = 9 kV

Problem 9.1.10

Given: A spherical capacitor has inner radius a = 10 cm and outer radius b = 15 cm. The space between the shells is filled with oil having $\varepsilon_r = 2.5$.

Find: (a) The capacitance. (b) If the capacitor stores 5 μJ, find the voltage between the shells and the maximum electric field (which occurs at r = a).

Solution:

(a) $C = 4\pi\varepsilon_0\varepsilon_r ab / (b - a) = 4\pi(8.854 \times 10^{-12})(2.5)(0.10)(0.15) / (0.05) = 4\pi \times 8.854 \times 10^{-12} \times 2.5 \times 0.015 / 0.05 = 4\pi \times 3.320 \times 10^{-13} / 0.05 = 4.175 \times 10^{-12} / 0.05 = 83.5$ pF

(b) From $W = ½CV^2$: $V = \sqrt{(2W/C)} = \sqrt{(2 \times 5 \times 10^{-6} / 83.5 \times 10^{-12})} = \sqrt{(1.198 \times 10^5)} = 346.1$ V

Maximum electric field at r = a: $E_{max} = V \times b / [a(b - a)] = 346.1 \times 0.15 / [0.10 \times 0.05] = 51.92 / 0.005 = 10{,}383$ V/m ≈ 10.4 kV/m

Alternatively: $E_{max} = Q / (4\pi\varepsilon_0\varepsilon_r a^2)$ where $Q = CV = 83.5 \times 10^{-12} \times 346.1 = 28.9$ nC. $E_{max} = 28.9 \times 10^{-9} / (4\pi \times 8.854 \times 10^{-12} \times 2.5 \times 0.01) = 28.9 \times 10^{-9} / 2.78 \times 10^{-12} = 10{,}396$ V/m, confirming the result.

Chapter 9 — Section 9.2: Magnetostatics

Practice problems covering magnetic fields, Ampere's law, magnetic force, inductance, magnetic materials, hysteresis, magnetic circuits, and eddy currents.

Problem 9.2.1

Given: Two long, parallel conductors are separated by 8 cm. Conductor A carries 20 A and conductor B carries 30 A, both in the same direction. The permeability of free space is $\mu_0 = 4\pi \times 10^{-7}$ H/m.

Find: (a) The magnetic field at the midpoint between the conductors. (b) The force per unit length between the conductors and whether it is attractive or repulsive.

Solution:

(a) At the midpoint (r = 0.04 m from each conductor): $B_A = \mu_0 I_A / (2\pi r) = (4\pi \times 10^{-7} \times 20) / (2\pi \times 0.04) = (8\pi \times 10^{-6}) / (0.08\pi) = 100\ \mu T$ $B_B = \mu_0 I_B / (2\pi r) = (4\pi \times 10^{-7} \times 30) / (2\pi \times 0.04) = (12\pi \times 10^{-6}) / (0.08\pi) = 150\ \mu T$

By the right-hand rule, at the midpoint, B_A and B_B point in opposite directions (each field circles its conductor, and at the midpoint between same-direction currents, the fields oppose). $B_{net} = |B_B - B_A| = 150 - 100 = 50\ \mu T$

(b) Force per unit length: $F/l = \mu_0 I_A I_B / (2\pi d) = (4\pi \times 10^{-7} \times 20 \times 30) / (2\pi \times 0.08) = (4\pi \times 10^{-7} \times 600) / (0.16\pi) = (2400\pi \times 10^{-7}) / (0.16\pi) = 2400 \times 10^{-7} / 0.16 = 1.5 \times 10^{-3}$ N/m = 1.5 mN/m

Since the currents flow in the same direction, the force is attractive.

Problem 9.2.2

Given: A toroidal coil has N = 400 turns, a mean radius of 12 cm (mean path length $l = 2\pi \times 0.12$ m), and a cross-sectional area A = 6 cm^2. The core material has relative permeability $\mu_r = 500$. The current is I = 1.5 A.

Find: (a) The magnetic field intensity H. (b) The magnetic flux density B. (c) The total magnetic flux. (d) The inductance of the toroid.

Solution:

Mean path length: $l = 2\pi \times 0.12 = 0.7540$ m

(a) $H = NI / l = 400 \times 1.5 / 0.7540 = 795.8$ A/m

(b) $B = \mu_0\mu_r H = 4\pi \times 10^{-7} \times 500 \times 795.8 = 6.283 \times 10^{-4} \times 795.8 = 0.500$ T

(c) $\Phi = BA = 0.500 \times 6 \times 10^{-4} = 3.0 \times 10^{-4}$ Wb $= 0.30$ mWb

(d) $L = N\Phi / I = 400 \times 3.0 \times 10^{-4} / 1.5 = 0.12 / 1.5 = 80.0$ mH

Alternatively: $L = \mu_0\mu_r N^2 A / l = 4\pi \times 10^{-7} \times 500 \times 400^2 \times 6 \times 10^{-4} / 0.7540 = 6.283 \times 10^{-4} \times 160{,}000 \times 6 \times 10^{-4} / 0.7540 = 0.06032 / 0.7540 = 80.0$ mH, confirming the result.

Problem 9.2.3

Given: A rectangular current loop (dimensions 15 cm × 10 cm) carries a current of 5 A and is placed in a uniform magnetic field $B = 0.4$ T. The plane of the loop makes an angle of 60° with the magnetic field direction.

Find: (a) The magnetic dipole moment. (b) The torque on the loop.

Solution:

(a) $m = NIA = 1 \times 5 \times (0.15 \times 0.10) = 5 \times 0.015 = 0.075$ A·m²

(b) The torque is $\tau = mB \sin\theta$, where θ is the angle between the magnetic moment (normal to the loop plane) and the field. If the loop plane makes 60° with B, then the normal to the plane makes 90° - 60° = 30° with B.

$\tau = mB \sin(30°) = 0.075 \times 0.4 \times 0.5 = 0.015$ N·m $= 15$ mN·m

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Problem 9.2.4

Given: Two coils wound on a common ferrite core have $N_1 = 200$ turns and $N_2 = 50$ turns. The self-inductances are $L_1 = 40$ mH and $L_2 = 2.5$ mH. The coupling coefficient $k = 0.95$. The current in coil 1 changes at a rate of $dI_1/dt = 500$ A/s.

Find: (a) The mutual inductance M. (b) The voltage induced in coil 2. (c) The energy stored in the system when $I_1 = 3$ A and $I_2 = 0$ A.

Solution:

(a) $M = k\sqrt{(L_1L_2)} = 0.95 \times \sqrt{(40 \times 10^{-3} \times 2.5 \times 10^{-3})} = 0.95 \times \sqrt{(100 \times 10^{-6})} = 0.95 \times 10 \times 10^{-3} = 9.5$ mH

(b) $V_2 = M \times dI_1/dt = 9.5 \times 10^{-3} \times 500 = 4.75$ V

(c) $W = ½L_1I_1^2 + ½L_2I_2^2 + MI_1I_2 = ½ \times 40 \times 10^{-3} \times 9 + 0 + 0 = 180$ mJ

Problem 9.2.5

Given: A ferrite E-core inductor for a flyback converter has a mean magnetic path length $l_e = 10$ cm, effective cross-sectional area $A_e = 2$ cm^2, and relative permeability $\mu_r = 2{,}000$. The core saturates at $B_{sat} = 0.38$ T. The winding has N = 30 turns.

Find: (a) The core reluctance. (b) The inductance without an air gap. (c) The maximum current before saturation. (d) If a 0.5 mm air gap is introduced, find the new inductance and the new saturation current.

Solution:

(a) $\mathcal{R}_{core} = l_e / (\mu_0\mu_r A_e) = 0.10 / (4\pi \times 10^{-7} \times 2000 \times 2 \times 10^{-4}) = 0.10 / (5.027 \times 10^{-7}) = 1.989 \times 10^5$ A-turns/Wb

(b) $L = N^2 / \mathcal{R} = 30^2 / 1.989 \times 10^5 = 900 / 1.989 \times 10^5 = 4.525$ mH

(c) $B_{max} = \mu_0\mu_r NI / l_e$, so $I_{max} = B_{sat} l_e / (\mu_0\mu_r N) = 0.38 \times 0.10 / (4\pi \times 10^{-7} \times 2000 \times 30) = 0.038 / 0.07540 = 0.504$ A

(d) Air gap reluctance: $\mathcal{R}_{gap} = l_g / (\mu_0 A_e) = 0.5 \times 10^{-3} / (4\pi \times 10^{-7} \times 2 \times 10^{-4}) = 5 \times 10^{-4} / 2.513 \times 10^{-10} = 1.989 \times 10^6$ A-turns/Wb

Total reluctance: $\mathcal{R}_{total} = \mathcal{R}_{core} + \mathcal{R}_{gap} = 1.989 \times 10^5 + 1.989 \times 10^6 = 2.188 \times 10^6$ A-turns/Wb

New inductance: $L = N^2 / \mathcal{R}_{total} = 900 / 2.188 \times 10^6 = 411$ μH

New saturation current: The gap dominates. $B_{max} = LI / (NA_e)$, so $I_{sat} = B_{sat} NA_e / L = 0.38 \times 30 \times 2 \times 10^{-4} / 411 \times 10^{-6} = 2.28 \times 10^{-3} / 4.11 \times 10^{-4} = 5.55$ A

The air gap reduced the inductance by a factor of 11 but increased the saturation current by a factor of 11, which is exactly the purpose of gapping inductor cores.

Problem 9.2.6

Given: A transformer core has two paths: a center leg (length 10 cm, area 4 cm^2, $\mu_r = 3{,}000$) and two outer legs (each 20 cm, area 4 cm^2, $\mu_r = 3{,}000$). The center leg has an N = 100-turn coil carrying 2 A. The two outer legs are in parallel.

Find: (a) The total reluctance of the magnetic circuit. (b) The flux in the center leg and in each outer leg.

Solution:

(a) Center leg reluctance: $\mathcal{R}_{center} = 0.10 / (4\pi \times 10^{-7} \times 3000 \times 4 \times 10^{-4}) = 0.10 / (1.508 \times 10^{-6}) = 6.631 \times 10^4$ A-turns/Wb

Each outer leg reluctance: $\mathcal{R}_{outer} = 0.20 / (4\pi \times 10^{-7} \times 3000 \times 4 \times 10^{-4}) = 0.20 / 1.508 \times 10^{-6} = 1.326 \times 10^5$ A-turns/Wb

The two outer legs are in parallel: $\mathcal{R}_{parallel} = \mathcal{R}_{outer} / 2 = 1.326 \times 10^5 / 2 = 6.631 \times 10^4$ A-turns/Wb

Total reluctance: $\mathcal{R}_{total} = \mathcal{R}_{center} + \mathcal{R}_{parallel} = 6.631 \times 10^4 + 6.631 \times 10^4 = 1.326 \times 10^5$ A-turns/Wb

(b) MMF = NI = $100 \times 2 = 200$ A-turns. $\Phi_{center} = \mathcal{F} / \mathcal{R}_{total} = 200 / 1.326 \times 10^5 = 1.508$ mWb

By symmetry, the flux splits equally between the two outer legs: $\Phi_{each\ outer} = \Phi_{center} / 2 = 0.754$ mWb

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Problem 9.2.7

Given: An induction heating system operates at f = 50 kHz to heat a steel cylindrical workpiece ($\rho = 1.2 \times 10^{-6}\ \Omega\cdot$m, $\mu_r = 200$, density = 7,800 kg/m³, specific heat = 460 J/(kg·°C)). The workpiece has diameter 25 mm and length 100 mm. The system delivers 5 kW to the workpiece.

Find: (a) The skin depth. (b) The time to raise the surface temperature by 500°C (assuming uniform heating for a simplified estimate). (c) Whether through-heating or surface-only heating occurs.

Solution:

(a) $\delta = \sqrt{(2\rho / (\omega\mu))} = \sqrt{(2 \times 1.2 \times 10^{-6} / (2\pi \times 50{,}000 \times 200 \times 4\pi \times 10^{-7}))} = \sqrt{(2.4 \times 10^{-6} / (2\pi \times 5 \times 10^4 \times 2.513 \times 10^{-4}))} = \sqrt{(2.4 \times 10^{-6} / 78.96)} = \sqrt{(3.039 \times 10^{-8})} = 0.174$ mm

(b) Workpiece volume: $V = \pi(0.0125)^2 \times 0.10 = \pi \times 1.5625 \times 10^{-4} \times 0.10 = 4.909 \times 10^{-5}$ m³ Mass: $m = 7800 \times 4.909 \times 10^{-5} = 0.3829$ kg Energy for $\Delta T = 500$°C: $Q = mc\Delta T = 0.3829 \times 460 \times 500 = 88{,}067$ J Time: $t = Q / P = 88{,}067 / 5{,}000 = 17.6$ seconds

(c) The skin depth is 0.174 mm while the workpiece radius is 12.5 mm. The ratio radius/δ = 12.5/0.174 = 71.8. Since the skin depth is only 1.4% of the radius, this is surface-only heating. The surface heats first and heat conducts inward. At 50 kHz, this setup is suitable for surface hardening (case hardening) of steel, not through-heating. For through-heating, a much lower frequency (1-3 kHz, giving δ of several mm) would be needed.

Problem 9.2.8

Given: A Helmholtz coil pair consists of two identical circular coils of radius R = 20 cm, each with N = 50 turns, separated by a distance equal to R (20 cm). The current in each coil is I = 4 A, flowing in the same direction.

Find: The magnetic field at the midpoint between the coils.

Solution:

The field at the center of a single coil on its axis at distance x is: $B = \mu_0 NIR^2 / [2(R^2 + x^2)^{3/2}]$

At the midpoint, each coil is at x = R/2 from the center point. $B_{each} = \mu_0 NIR^2 / [2(R^2 + R^2/4)^{3/2}] = \mu_0 NIR^2 / [2(5R^2/4)^{3/2}]$

$(5R^2/4)^{3/2} = (5/4)^{3/2} \times R^3 = (1.25)^{1.5} \times R^3 = 1.3975 \times R^3$

$B_{each} = \mu_0 NI / (2 \times 1.3975 \times R) = \mu_0 NI / (2.795R)$

Total field (both coils contribute equally): $B = 2 \times B_{each} = \mu_0 NI / (1.3975R)$

$B = (4\pi \times 10^{-7} \times 50 \times 4) / (1.3975 \times 0.20) = (2.513 \times 10^{-4}) / 0.2795 = 8.99 \times 10^{-4}\ T \approx 0.899$ mT

The Helmholtz configuration produces a highly uniform field in the region between the coils, with the first and second spatial derivatives of B vanishing at the midpoint.

Chapter 9 — Section 9.3: Maxwell's Equations

Practice problems covering Gauss's law for electricity, Gauss's law for magnetism, Faraday's law, the Ampere-Maxwell law, electromagnetic boundary conditions, and electromagnetic potentials.

Problem 9.3.1

Given: A solid insulating sphere of radius R = 10 cm carries a uniformly distributed total charge of Q = +50 nC. The permittivity of free space is $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m.

Find: Using Gauss's Law, find the electric field at (a) r = 5 cm (inside) and (b) r = 30 cm (outside). (c) Find the radius at which E is maximum.

Solution:

(a) Inside (r = 5 cm): The enclosed charge is $Q_{enc} = Q(r/R)^3 = 50 \times 10^{-9} \times (0.05/0.10)^3 = 50 \times 10^{-9} \times 0.125 = 6.25$ nC. By Gauss's Law: $E \times 4\pi r^2 = Q_{enc} / \varepsilon_0$ $E = Q_{enc} / (4\pi\varepsilon_0 r^2) = (8.99 \times 10^9)(6.25 \times 10^{-9}) / (0.05)^2 = 56.19 / 0.0025 = 22{,}475$ V/m ≈ 22.5 kV/m

Alternatively: $E = Qr / (4\pi\varepsilon_0 R^3) = (8.99 \times 10^9)(50 \times 10^{-9})(0.05) / (0.10)^3 = 22.475 / 0.001 = 22{,}475$ V/m, confirming the result.

(b) Outside (r = 30 cm): $E = kQ / r^2 = (8.99 \times 10^9)(50 \times 10^{-9}) / (0.30)^2 = 449.5 / 0.09 = 4{,}994$ V/m ≈ 5.0 kV/m

(c) Inside the sphere, E increases linearly with r: $E = Qr/(4\pi\varepsilon_0 R^3)$. Outside, E decreases as $1/r^2$. E is maximum at the surface (r = R): $E_{max} = kQ / R^2 = (8.99 \times 10^9)(50 \times 10^{-9}) / 0.01 = 44{,}950$ V/m ≈ 45.0 kV/m

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Problem 9.3.2

Given: A closed cylindrical surface (radius 5 cm, height 10 cm) is placed in a non-uniform magnetic field. The flux entering through the top circular face is $\Phi_{top} = 2.5$ mWb (inward). The flux through the curved side is $\Phi_{side} = -1.8$ mWb (outward).

Find: The magnetic flux through the bottom circular face (magnitude and direction).

Solution:

By Gauss's Law for magnetism, the total flux through any closed surface is zero: $\Phi_{top} + \Phi_{side} + \Phi_{bottom} = 0$

Taking inward as positive: $2.5 + (-1.8) + \Phi_{bottom} = 0$ $\Phi_{bottom} = -0.7$ mWb

The flux through the bottom face is 0.7 mWb directed outward (leaving the surface).

The total outward flux (1.8 + 0.7 = 2.5 mWb) equals the inward flux (2.5 mWb), as required by the absence of magnetic monopoles.

Problem 9.3.3

Given: A square loop of wire (side length 25 cm, resistance $R = 5\ \Omega$) lies in a uniform magnetic field. The field is perpendicular to the loop and varies as $B(t) = 0.8 - 0.3t$ (in tesla), where t is in seconds.

Find: (a) The induced EMF. (b) The induced current. (c) The power dissipated in the loop at t = 1 s.

Solution:

(a) Area: $A = (0.25)^2 = 0.0625\ m^2$ $\Phi = BA = (0.8 - 0.3t) \times 0.0625$

EMF $= -d\Phi/dt = -A \times dB/dt = -0.0625 \times (-0.3) = 0.01875$ V = 18.75 mV

The EMF is constant because $dB/dt = -0.3$ T/s is constant.

(b) I = EMF / R = 0.01875 / 5 = 3.75 mA

By Lenz's law, the current opposes the decrease in flux, so it flows in a direction that creates a field in the same direction as the original field.

(c) $P = I^2R = (3.75 \times 10^{-3})^2 \times 5 = 1.406 \times 10^{-5} \times 5 = 70.3\ \mu W$

This is constant and independent of time since the rate of change of B is constant.

Problem 9.3.4

Given: A parallel-plate capacitor with circular plates of radius 10 cm and gap d = 2 mm is being charged. The electric field between the plates is increasing at a rate of $dE/dt = 2 \times 10^{13}$ V/(m·s).

Find: (a) The displacement current between the plates. (b) The magnetic field at the edge of the plates (r = 10 cm) due to the displacement current. (c) The conduction current flowing in the external circuit.

Solution:

(a) Plate area: $A = \pi(0.10)^2 = 0.03142\ m^2$ $I_d = \varepsilon_0 A(dE/dt) = 8.854 \times 10^{-12} \times 0.03142 \times 2 \times 10^{13} = 8.854 \times 10^{-12} \times 6.283 \times 10^{11} = 5.564$ A

(b) Using Ampere-Maxwell law with a circular Amperian loop of radius r = 10 cm around the axis: $\oint B \cdot dl = \mu_0 I_{d(enclosed)}$ $B \times 2\pi r = \mu_0 I_d$ (the full displacement current is enclosed at r = plate radius) $B = \mu_0 I_d / (2\pi r) = (4\pi \times 10^{-7} \times 5.564) / (2\pi \times 0.10) = 6.991 \times 10^{-6} / 0.6283 = 11.13\ \mu T$

(c) By Kirchhoff's current law (continuity of current), the conduction current in the external circuit equals the displacement current between the plates: $I_c = 5.564$ A

Problem 9.3.5

Given: An electromagnetic wave in free space ($\mu_1 = \mu_0$, $\varepsilon_1 = \varepsilon_0$) strikes a flat glass surface ($\mu_2 = \mu_0$, $\varepsilon_2 = 4\varepsilon_0$, non-magnetic, lossless) at normal incidence. The incident electric field amplitude is $E_i = 50$ V/m.

Find: (a) The intrinsic impedances of both media. (b) The reflection and transmission coefficients. (c) The reflected and transmitted electric field amplitudes. (d) The fraction of incident power transmitted.

Solution:

(a) $\eta_1 = \sqrt{(\mu_0/\varepsilon_0)} = 377\ \Omega$ $\eta_2 = \sqrt{(\mu_0/4\varepsilon_0)} = 377/\sqrt{4} = 377/2 = 188.5\ \Omega$

(b) $\Gamma = (\eta_2 - \eta_1)/(\eta_2 + \eta_1) = (188.5 - 377)/(188.5 + 377) = -188.5/565.5 = -0.3333$ $\tau = 2\eta_2/(\eta_1 + \eta_2) = 2 \times 188.5/565.5 = 377/565.5 = 0.6667$

(c) $E_r = \Gamma E_i = -0.3333 \times 50 = -16.67$ V/m (180° phase reversal) $E_t = \tau E_i = 0.6667 \times 50 = 33.33$ V/m

(d) Power transmitted: $1 - |\Gamma|^2 = 1 - (1/3)^2 = 1 - 1/9 = 8/9 \approx 88.9\%$

Verification: $(\eta_1/\eta_2)|\tau|^2 = (377/188.5)(0.6667)^2 = 2 \times 0.4444 = 0.889 = 88.9\%$.

Problem 9.3.6

Given: A generator coil has 200 turns and area $A = 0.05\ m^2$. It rotates at 3,600 RPM in a uniform magnetic field $B = 1.2$ T. The axis of rotation is perpendicular to the field.

Find: (a) The angular frequency of rotation. (b) The peak EMF. (c) The RMS EMF. (d) The frequency of the generated voltage.

Solution:

(a) $\omega = 2\pi \times (RPM/60) = 2\pi \times (3600/60) = 2\pi \times 60 = 376.99$ rad/s

(b) The flux is $\Phi = NBA\cos(\omega t)$. The EMF $= -d\Phi/dt = NBA\omega \sin(\omega t)$. $EMF_{peak} = NBA\omega = 200 \times 1.2 \times 0.05 \times 377.0 = 4{,}523$ V

(c) $EMF_{rms} = EMF_{peak} / \sqrt{2} = 4{,}523 / 1.414 = 3{,}199\ V \approx 3.2$ kV

(d) $f = RPM / 60 = 3600 / 60 = 60$ Hz

Problem 9.3.7

Given: A Hertzian dipole (short current element) of length $dl = 0.02\lambda$ carries a peak current $I_0 = 5$ A at frequency $f = 1$ GHz.

Find: (a) The radiation resistance. (b) The total radiated power. (c) The directivity in dBi.

Solution:

(a) $R_{rad} = 80\pi^2(dl/\lambda)^2 = 80 \times 9.8696 \times (0.02)^2 = 80 \times 9.8696 \times 4 \times 10^{-4} = 0.3158\ \Omega$

(b) $P_{rad} = ½I_0^2R_{rad} = 0.5 \times 25 \times 0.3158 = 3.95$ W

(c) The directivity of a Hertzian dipole is always D = 1.5 (3/2), regardless of length. $D = 10 \log_{10}(1.5) = 1.76$ dBi

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Problem 9.3.8

Given: A transformer has a primary coil ($N_1 = 500$ turns) and a secondary coil ($N_2 = 100$ turns) wound on a common iron core. The primary is connected to a 120 V_{rms}, 60 Hz source. The peak magnetic flux in the core is 0.5 mWb.

Find: (a) The secondary voltage. (b) Whether the core flux is consistent with the applied primary voltage (using Faraday's law). (c) The magnetizing current if the core reluctance is $\mathcal{R} = 5 \times 10^5$ A-turns/Wb.

Solution:

(a) $V_2 = V_1 \times N_2/N_1 = 120 \times 100/500 = 24\ V_{rms}$

(b) By Faraday's law: $V_1(\text{rms}) = 4.44 \times f \times N_1 \times \Phi_{max} = 4.44 \times 60 \times 500 \times 0.5 \times 10^{-3} = 4.44 \times 60 \times 0.25 = 66.6$ V

This is less than 120 V, which means the actual peak flux must be higher: $\Phi_{max} = V_1 / (4.44 \times f \times N_1) = 120 / (4.44 \times 60 \times 500) = 120 / 133{,}200 = 0.9009$ mWb

The stated 0.5 mWb is inconsistent; the actual peak flux for 120 V at 60 Hz with 500 turns is 0.90 mWb.

(c) Peak MMF = $\mathcal{R} \times \Phi_{max} = 5 \times 10^5 \times 0.9009 \times 10^{-3} = 450.5$ A-turns Peak magnetizing current: $I_{m(peak)} = \text{MMF} / N_1 = 450.5 / 500 = 0.901$ A RMS magnetizing current: $I_{m(rms)} = 0.901 / \sqrt{2} = 0.637$ A

Chapter 9 — Section 9.4: Electromagnetic Waves

Practice problems covering the wave equation, the electromagnetic spectrum, polarization, skin effect, wave reflection at interfaces, the Poynting vector, and wave propagation in lossy media.

Problem 9.4.1

Given: An electromagnetic wave propagates through a lossless, non-magnetic dielectric medium with relative permittivity $\varepsilon_r = 9.0$ and $\mu_r = 1$.

Find: (a) The wave velocity. (b) The intrinsic impedance. (c) The wavelength and period at 2.4 GHz. (d) If the electric field amplitude is $E_0 = 20$ V/m, find the magnetic field amplitude.

Solution:

(a) $v = c / \sqrt{(\varepsilon_r)} = 3 \times 10^8 / \sqrt{9} = 3 \times 10^8 / 3 = 1.0 \times 10^8$ m/s

(b) $\eta = \eta_0 / \sqrt{(\varepsilon_r)} = 377 / 3 = 125.7\ \Omega$

(c) $\lambda = v / f = 1.0 \times 10^8 / 2.4 \times 10^9 = 0.04167$ m $= 41.7$ mm $T = 1 / f = 1 / 2.4 \times 10^9 = 4.167 \times 10^{-10}$ s $= 0.417$ ns

(d) $H_0 = E_0 / \eta = 20 / 125.7 = 0.159$ A/m $= 159$ mA/m

Problem 9.4.2

Given: A 77 GHz automotive radar system (FMCW radar for adaptive cruise control) operates in the millimeter-wave band.

Find: (a) The free-space wavelength. (b) The region of the electromagnetic spectrum. (c) If the radar antenna is a circular aperture of diameter D = 25 mm, estimate the far-field boundary distance.

Solution:

(a) $\lambda = c / f = 3 \times 10^8 / 77 \times 10^9 = 3.896 \times 10^{-3}$ m $= 3.9$ mm

(b) This falls in the millimeter-wave band (30-300 GHz, wavelengths 1-10 mm), at the boundary of EHF (Extremely High Frequency) in the radio spectrum.

(c) Far-field distance: $r_{ff} = 2D^2 / \lambda = 2 \times (0.025)^2 / 3.896 \times 10^{-3} = 2 \times 6.25 \times 10^{-4} / 3.896 \times 10^{-3} = 1.25 \times 10^{-3} / 3.896 \times 10^{-3} = 0.321 \text{ m} \approx 32 \text{ cm}$

Since automotive targets are at distances » 32 cm, the far-field assumption is always valid for this radar.

Problem 9.4.3

Given: A satellite downlink transmitter sends a right-hand circularly polarized (RHCP) signal. The receiving ground station antenna is left-hand circularly polarized (LHCP).

Find: (a) The polarization loss factor. (b) The received power if the incident power density is 10 pW/m² and the receive antenna has an effective aperture of 5 m². (c) If the ground antenna is switched to RHCP, find the new received power.

Solution:

(a) RHCP and LHCP are orthogonal polarizations. The polarization loss factor (PLF) = 0. In dB: PLF = -∞ dB (complete polarization mismatch; no power is received).

(b) $P_r = S \times A_e \times PLF = 10 \times 10^{-12} \times 5 \times 0 = 0$ W (no signal received)

(c) With matched RHCP polarization: PLF = 1 (0 dB). $P_r = S \times A_e \times 1 = 10 \times 10^{-12} \times 5 = 50$ pW = 50×10^{-12} W In dBm: $P_r = 10 \log_{10}(50 \times 10^{-12}/10^{-3}) = 10 \log_{10}(5 \times 10^{-8}) = -73.0$ dBm

This illustrates why polarization matching is critical in satellite communications.

Problem 9.4.4

Given: Calculate the skin depth and AC resistance increase for the following metals at 1 GHz. All are non-magnetic ($\mu_r = 1$). - Copper: $\sigma = 5.8 \times 10^7$ S/m - Aluminum: $\sigma = 3.5 \times 10^7$ S/m - Gold: $\sigma = 4.1 \times 10^7$ S/m

A copper PCB trace is 35 μm thick (1 oz copper) and 0.5 mm wide.

Find: (a) The skin depth for each metal at 1 GHz. (b) Whether the 35 μm copper trace is thick enough for effective RF conduction at 1 GHz. (c) The sheet resistance of the copper trace at 1 GHz.

Solution:

(a) $\delta = 1 / \sqrt{(\pi f \mu_0 \sigma)}$

Copper: $\delta = 1 / \sqrt{(\pi \times 10^9 \times 4\pi \times 10^{-7} \times 5.8 \times 10^7)} = 1 / \sqrt{(\pi \times 10^9 \times 72.88)} = 1 / \sqrt{(2.289 \times 10^{11})} = 2.09\ \mu\text{m}$

Aluminum: $\delta = 1 / \sqrt{(\pi \times 10^9 \times 4\pi \times 10^{-7} \times 3.5 \times 10^7)} = 1 / \sqrt{(1.382 \times 10^{11})} = 2.69\ \mu\text{m}$

Gold: $\delta = 1 / \sqrt{(\pi \times 10^9 \times 4\pi \times 10^{-7} \times 4.1 \times 10^7)} = 1 / \sqrt{(1.619 \times 10^{11})} = 2.49\ \mu\text{m}$

(b) The copper trace is 35 μm thick = 35/2.09 = 16.7 skin depths thick. Since current penetrates approximately 3δ = 6.3 μm from each side, the trace is far thicker than needed. The trace is effectively infinitely thick for RF at 1 GHz.

(c) Sheet resistance: $R_s = 1/(\sigma\delta) = 1/(5.8 \times 10^7 \times 2.09 \times 10^{-6}) = 1/121.2 = 8.25$ mΩ/square

For the 0.5 mm wide trace, current flows on both top and bottom surfaces, so the effective resistance per unit length is approximately $R_s/(2w) = 8.25 \times 10^{-3} / (2 \times 0.5 \times 10^{-3}) = 8.25$ Ω/m.

Problem 9.4.5

Given: A 900 MHz cellular signal in free space ($\eta_1 = 377$ Ω) strikes a brick wall ($\varepsilon_r = 4.0$, $\mu_r = 1$, lossless approximation) at normal incidence.

Find: (a) The intrinsic impedance of the brick. (b) The reflection coefficient. (c) The percentage of power reflected and transmitted. (d) If 1 W/m² is incident, find the power density transmitted into the brick.

Solution:

(a) $\eta_2 = \eta_0 / \sqrt{(\varepsilon_r)} = 377 / \sqrt{4} = 377 / 2 = 188.5$ Ω

(b) $\Gamma = (\eta_2 - \eta_1) / (\eta_2 + \eta_1) = (188.5 - 377) / (188.5 + 377) = -188.5 / 565.5 = -0.3333$

(c) Power reflected: $|\Gamma|^2 = (1/3)^2 = 11.1\%$ Power transmitted: $1 - |\Gamma|^2 = 88.9\%$

(d) Transmitted power density: $S_t = 0.889 \times 1 = 0.889$ W/m²

Note: This calculation neglects the absorption within the brick and the second reflection at the exit surface. A real brick wall (with loss tangent and finite thickness) would have additional absorption loss, making the total wall attenuation significantly higher.

Problem 9.4.6

Given: A 2.4 GHz Wi-Fi access point transmits 200 mW (23 dBm) with an antenna gain of 6 dBi. An isotropic receiver is located 50 m away.

Find: (a) The EIRP (Effective Isotropic Radiated Power). (b) The power density at the receiver using the Poynting vector. (c) The electric field strength at the receiver. (d) Compare the power density to the ICNIRP general public exposure limit of 10 W/m² at 2.4 GHz.

Solution:

(a) $\text{EIRP} = P_t \times G_t(\text{linear}) = 200 \times 10^{-3} \times 10^{6/10} = 0.2 \times 3.981 = 796$ mW In dBm: EIRP = 23 + 6 = 29 dBm

(b) $S = \text{EIRP} / (4\pi r^2) = 0.796 / (4\pi \times 50^2) = 0.796 / 31{,}416 = 2.533 \times 10^{-5}$ W/m² = 25.3 μW/m²

(c) $S = E^2 / (2\eta_0)$, so $E = \sqrt{(2\eta_0 S)} = \sqrt{(2 \times 377 \times 2.533 \times 10^{-5})} = \sqrt{(0.01910)} = 0.138$ V/m = 138 mV/m

(d) The power density of 25.3 μW/m² is far below the 10 W/m² limit — by a factor of 395,000 (56 dB). The exposure is completely safe, even accounting for multiple access points in the environment.

Problem 9.4.7

Given: A 5 GHz Wi-Fi signal must pass through a concrete floor slab. The concrete has $\varepsilon_r = 5.5$, $\tan\delta = 0.03$, $\mu_r = 1$, and the slab is 150 mm thick.

Find: (a) The attenuation constant α. (b) The total absorption loss through the slab. (c) The reflection loss at each air-concrete interface. (d) The total signal attenuation through the slab.

Solution:

(a) For low-loss dielectric ($\tan\delta \ll 1$): $\alpha = (\pi f \tan\delta \sqrt{\varepsilon_r}) / c = (\pi \times 5 \times 10^9 \times 0.03 \times \sqrt{5.5}) / (3 \times 10^8) = (\pi \times 5 \times 10^9 \times 0.03 \times 2.345) / (3 \times 10^8) = (1.104 \times 10^9) / (3 \times 10^8) = 3.68$ Np/m

(b) Absorption loss: $A = \alpha \times d = 3.68 \times 0.15 = 0.552$ Np $= 0.552 \times 8.686 = 4.79$ dB

(c) Reflection coefficient at each surface: $\eta_2 = 377 / \sqrt{5.5} = 377 / 2.345 = 160.8\ \Omega$ $\Gamma = (160.8 - 377) / (160.8 + 377) = -216.2 / 537.8 = -0.402$

Reflection loss per surface: $-10 \log_{10}(1 - |\Gamma|^2) = -10 \log_{10}(1 - 0.1616) = -10 \log_{10}(0.8384) = 0.77$ dB Total reflection loss (two surfaces): $2 \times 0.77 = 1.53$ dB

(d) Total attenuation: A_{total} = absorption + reflection = 4.79 + 1.53 = 6.32 dB

A single concrete floor attenuates the 5 GHz signal by approximately 6.3 dB (factor of 4.3× in power). Multiple floors would multiply this loss, which is why multi-story buildings often need access points on every floor.

Problem 9.4.8

Given: An electromagnetic shield made of 0.1 mm thick copper foil ($\sigma = 5.8 \times 10^7$ S/m, $\mu_r = 1$) is used to shield a sensitive receiver from interference at 100 kHz.

Find: (a) The skin depth at 100 kHz. (b) The absorption loss. (c) Whether the shield provides adequate attenuation (target: >40 dB).

Solution:

(a) $\delta = 1 / \sqrt{(\pi f \mu_0 \sigma)} = 1 / \sqrt{(\pi \times 10^5 \times 4\pi \times 10^{-7} \times 5.8 \times 10^7)} = 1 / \sqrt{(\pi \times 10^5 \times 72.88)} = 1 / \sqrt{(2.289 \times 10^7)} = 0.209$ mm = 209 μm

(b) $A = 8.686 \times (t/\delta) = 8.686 \times (0.1/0.209) = 8.686 \times 0.4785 = 4.16$ dB

The absorption loss alone (4.16 dB) is far below the 40 dB target. At this low frequency, the copper foil is less than one skin depth thick, so absorption is minimal.

However, the reflection loss for a good conductor at low frequency is very high: $R \approx 168 - 10 \log_{10}(f \times \mu_r/\sigma_r)$ where $\sigma_r = \sigma/\sigma_{Cu} = 1.0$ $R = 168 - 10 \log_{10}(10^5 \times 1/1) = 168 - 50 = 118$ dB

Total SE = A + R = 4.16 + 118 = 122 dB » 40 dB

(c) Yes, the shield is more than adequate. At low frequencies, the shielding is dominated by reflection (impedance mismatch between the low-impedance conductor and the high-impedance wave in free space), not absorption. Even at 100 kHz where the foil is thin relative to the skin depth, the copper reflects virtually all incident energy.

Chapter 9 — Section 9.5: Transmission Lines

Practice problems covering characteristic impedance, reflection and standing waves, the Smith Chart, waveguides, microstrip and stripline, transmission line transients and TDR, impedance matching, and coupled lines/crosstalk.

Problem 9.5.1

Given: A coaxial cable has inner conductor diameter 2a = 1.02 mm and outer conductor inner diameter 2b = 3.66 mm. The dielectric is solid polyethylene with $\varepsilon_r = 2.25$ and $\mu_r = 1$.

Find: (a) The characteristic impedance. (b) The propagation velocity. (c) The propagation delay per meter. (d) The electrical length in wavelengths for a 10 m cable at 100 MHz.

Solution:

(a) $Z_0 = (1/2\pi) \times (\eta_0/\sqrt{\varepsilon_r}) \times \ln(b/a) = (1/2\pi) \times (377/1.5) \times \ln(1.83/0.51) = (1/6.283) \times 251.3 \times \ln(3.588) = (1/6.283) \times 251.3 \times 1.278 = (1/6.283) \times 321.2 = 51.1\ \Omega$

This is a standard 50 Ω coaxial cable (RG-58 type); the slight deviation from 50 Ω is within manufacturing tolerance.

(b) $v = c / \sqrt{\varepsilon_r} = 3 \times 10^8 / 1.5 = 2.0 \times 10^8$ m/s (velocity factor = 0.667 = 66.7%)

(c) Delay $= 1/v = 1/(2.0 \times 10^8) = 5.0$ ns/m

(d) $\lambda = v/f = 2.0 \times 10^8 / 10^8 = 2.0$ m. Electrical length = 10 / 2.0 = 5.0 wavelengths

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Problem 9.5.2

Given: A 75 Ω coaxial cable is terminated with an antenna having impedance $Z_L = 150 + j100\ \Omega$.

Find: (a) The complex reflection coefficient (magnitude and phase). (b) The VSWR. (c) The return loss. (d) The fraction of incident power delivered to the load.

Solution:

(a) $\Gamma = (Z_L - Z_0) / (Z_L + Z_0) = (150 + j100 - 75) / (150 + j100 + 75) = (75 + j100) / (225 + j100)$

$|75 + j100| = \sqrt{(75^2 + 100^2)} = \sqrt{(5625 + 10000)} = \sqrt{15625} = 125$ $|225 + j100| = \sqrt{(225^2 + 100^2)} = \sqrt{(50625 + 10000)} = \sqrt{60625} = 246.2$

$|\Gamma| = 125 / 246.2 = 0.5077$

Phase: $\angle\Gamma = \arctan(100/75) - \arctan(100/225) = 53.13° - 23.96° = 29.17°$

So $\Gamma = 0.508 \angle 29.2°$

(b) $VSWR = (1 + |\Gamma|) / (1 - |\Gamma|) = 1.508 / 0.492 = 3.06:1$

(c) Return loss $= -20 \log_{10}(|\Gamma|) = -20 \log_{10}(0.508) = -20 \times (-0.294) = 5.88$ dB

(d) Power delivered: $1 - |\Gamma|^2 = 1 - 0.258 = 0.742 = 74.2\%$

Problem 9.5.3

Given: An antenna with impedance $Z_L = 36 + j20\ \Omega$ must be matched to a 50 Ω system at 2.4 GHz.

Find: (a) The normalized impedance. (b) The reflection coefficient magnitude. (c) The VSWR. (d) Design a quarter-wave transformer match (assuming the reactive part is first tuned out with a series capacitor).

Solution:

(a) $z = Z_L / Z_0 = (36 + j20) / 50 = 0.72 + j0.40$

(b) $|\Gamma| = |z - 1| / |z + 1| = |(-0.28 + j0.40)| / |(1.72 + j0.40)| = \sqrt{(0.0784 + 0.16)} / \sqrt{(2.9584 + 0.16)} = \sqrt{0.2384} / \sqrt{3.1184} = 0.4883 / 1.7659 = 0.2766$

(c) $VSWR = (1 + 0.277) / (1 - 0.277) = 1.277 / 0.723 = 1.77:1$

(d) First, tune out the $+j20\ \Omega$ reactance with a series capacitor: $C = 1 / (2\pi fX) = 1 / (2\pi \times 2.4 \times 10^9 \times 20) = 1 / (3.016 \times 10^{11}) = 3.32$ pF

Now the load is purely resistive at 36 Ω. Quarter-wave transformer: $Z_T = \sqrt{(Z_0 \times R_L)} = \sqrt{(50 \times 36)} = \sqrt{1800} = 42.43\ \Omega$

Transformer length: $\lambda/4$ at 2.4 GHz. $\lambda = c/f = 3 \times 10^8 / 2.4 \times 10^9 = 0.125$ m in free space. On a PCB ($\varepsilon_{eff} \approx 3.0$): $\lambda_{eff} = 0.125 / \sqrt{3} = 0.0722$ m. Length $= \lambda_{eff}/4 = 18.0$ mm

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Problem 9.5.4

Given: A WR-62 rectangular waveguide has interior dimensions a = 15.80 mm and b = 7.90 mm.

Find: (a) The cutoff frequency of the dominant TE_{10} mode. (b) The cutoff frequency of the next mode (TE_{20}). (c) The single-mode operating range. (d) The guide wavelength at 15 GHz.

Solution:

(a) $f_{c(TE_{10})} = c / (2a) = 3 \times 10^8 / (2 \times 0.01580) = 3 \times 10^8 / 0.03160 = 9.494$ GHz

(b) $f_{c(TE_{20})} = c / a = 3 \times 10^8 / 0.01580 = 18.99$ GHz

Also check TE_{01}: $f_{c(TE_{01})} = c / (2b) = 3 \times 10^8 / (2 \times 0.00790) = 18.99$ GHz — same as TE_{20}.

(c) Single-mode range: from ~$1.25 \times f_{c(TE_{10})}$ to $f_{c(TE_{20})}$: 11.9 GHz to 19.0 GHz (the Ku-band). Practical usable range is 12.4-18.0 GHz.

(d) At 15 GHz: $\lambda_0 = c/f = 3 \times 10^8 / 15 \times 10^9 = 20.0$ mm $\lambda_g = \lambda_0 / \sqrt{(1 - (f_c/f)^2)} = 20.0 / \sqrt{(1 - (9.494/15)^2)} = 20.0 / \sqrt{(1 - 0.4007)} = 20.0 / \sqrt{0.5993} = 20.0 / 0.7742 = 25.83$ mm

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Problem 9.5.5

Given: A 50 Ω microstrip line is designed on Rogers RO4003C substrate ($\varepsilon_r = 3.55$, h = 0.508 mm, t = 0.035 mm copper).

Find: (a) The approximate trace width for 50 Ω impedance. (b) The effective dielectric constant. (c) The propagation delay per unit length. (d) The maximum frequency for which a 100 mm trace length is less than λ/10 (electrically short).

Solution:

(a) Using the microstrip impedance approximation $Z_0 \approx (87/\sqrt{(\varepsilon_r + 1.41)}) \times \ln(5.98h/(0.8w + t))$: Solving iteratively for $Z_0 = 50\ \Omega$: $w \approx 1.12$ mm ($w/h \approx 2.2$)

(b) $\varepsilon_{eff} \approx (\varepsilon_r + 1)/2 + (\varepsilon_r - 1)/(2\sqrt{(1 + 12h/w)}) = (3.55 + 1)/2 + (3.55 - 1)/(2\sqrt{(1 + 12 \times 0.508/1.12)}) = 2.275 + 2.55/(2\sqrt{(1 + 5.44)}) = 2.275 + 2.55/(2 \times 2.539) = 2.275 + 0.502 = 2.777$

(c) $v = c/\sqrt{\varepsilon_{eff}} = 3 \times 10^8 / \sqrt{2.777} = 3 \times 10^8 / 1.666 = 1.801 \times 10^8$ m/s Delay = 1/v = 5.55 ns/m = 5.55 ps/mm

(d) For a 100 mm trace to be < λ/10: $\lambda > 10 \times 0.1 = 1.0$ m. $f_{max} = v / \lambda = 1.801 \times 10^8 / 1.0 = 180$ MHz

Above 180 MHz, the trace must be treated as a transmission line and impedance-controlled routing is required.

Problem 9.5.6

Given: A 50 Ω transmission line (velocity factor 0.66, length 2 m) connects a 50 Ω source ($V_S = 2$ V step) to a load $Z_L = 200\ \Omega$.

Find: Using a lattice diagram, find (a) the initial voltage wave, (b) the voltage at the load after the first transit, (c) whether any re-reflection occurs, and (d) the final steady-state voltage at the load.

Solution:

(a) Propagation velocity: $v = 0.66 \times 3 \times 10^8 = 1.98 \times 10^8$ m/s One-way delay: $t_d = l/v = 2 / 1.98 \times 10^8 = 10.1$ ns

Initial voltage wave: $V^+ = V_S \times Z_0 / (Z_S + Z_0) = 2 \times 50 / (50 + 50) = 1.0$ V

(b) Load reflection coefficient: $\Gamma_L = (200 - 50)/(200 + 50) = 150/250 = 0.6$ At t = 10.1 ns: $V_L = V^+(1 + \Gamma_L) = 1.0 \times (1 + 0.6) = 1.6$ V

Source reflection coefficient: $\Gamma_S = (50 - 50)/(50 + 50) = 0$ (matched source)

(c) The reflected wave $V^- = \Gamma_L V^+ = 0.6 \times 1.0 = 0.6$ V travels back. Since $\Gamma_S = 0$, the source absorbs this wave entirely. No re-reflection occurs.

(d) Steady-state: $V_L = V_S \times Z_L/(Z_S + Z_L) = 2 \times 200/(50 + 200) = 400/250 = 1.6$ V

The line reaches steady state after just one round trip (20.2 ns) because the source is matched to Z_0.

Problem 9.5.7

Given: A 433 MHz ISM-band transmitter has a 50 Ω output and must drive a loop antenna with impedance $Z_L = 10\ \Omega$ (purely resistive at the design frequency).

Find: (a) Design an L-network match. (b) Calculate component values. (c) Estimate the matching bandwidth.

Solution:

(a) Since $R_L = 10\ \Omega < R_S = 50\ \Omega$, the shunt element goes across the high-impedance (source) side. $Q = \sqrt{(R_{high}/R_{low} - 1)} = \sqrt{(50/10 - 1)} = \sqrt{4} = 2.0$

Shunt reactance: $X_{shunt} = R_{high}/Q = 50/2 = 25\ \Omega$ Series reactance: $X_{series} = Q \times R_{low} = 2 \times 10 = 20\ \Omega$

(b) Low-pass solution (preferred for harmonic rejection): shunt inductor + series capacitor. $L_{shunt} = X_{shunt}/(2\pi f) = 25/(2\pi \times 433 \times 10^6) = 9.19$ nH $C_{series} = 1/(2\pi f X_{series}) = 1/(2\pi \times 433 \times 10^6 \times 20) = 18.4$ pF

High-pass solution: shunt capacitor + series inductor. $C_{shunt} = 1/(2\pi f \times 25) = 14.7$ pF $L_{series} = 20/(2\pi f) = 7.35$ nH

(c) Bandwidth: $BW \approx f_0/Q = 433/2.0 = 216.5$ MHz

The VSWR < 2:1 bandwidth spans approximately ±108 MHz around 433 MHz, which is far wider than the ISM band (433.05-434.79 MHz). The match easily covers the full band.

Problem 9.5.8

Given: Two 50 Ω microstrip traces on a 4-layer PCB (h = 0.15 mm, $\varepsilon_{eff} = 3.0$) run parallel with edge-to-edge spacing s = 0.15 mm for a coupled length of 50 mm. The aggressor carries a 1.8 V LVCMOS signal with 0.3 ns rise time. The coupling parameters are: $C_m = 35$ pF/m, $L_m = 80$ nH/m, $C_0 = 110$ pF/m, $L_0 = 275$ nH/m.

Find: (a) The NEXT coefficient K_b. (b) The critical length. (c) The NEXT voltage. (d) The FEXT coefficient and voltage.

Solution:

(a) $K_b = (C_m/C_0 + L_m/L_0) / 4 = (35/110 + 80/275) / 4 = (0.3182 + 0.2909) / 4 = 0.6091 / 4 = 0.1523$

(b) $v = 1/\sqrt{(L_0C_0)} = 1/\sqrt{(275 \times 10^{-9} \times 110 \times 10^{-12})} = 1/\sqrt{(3.025 \times 10^{-17})} = 1.818 \times 10^8$ m/s Critical length: $l_{crit} = t_r \times v / 2 = 0.3 \times 10^{-9} \times 1.818 \times 10^8 / 2 = 27.3$ mm

(c) Coupled length (50 mm) > l_{crit} (27.3 mm), so NEXT saturates: $V_{NEXT} = K_b \times V_{step} = 0.1523 \times 1.8 = 274$ mV (15.2% of signal amplitude)

This is dangerously high for LVCMOS (noise margin ~0.4 V), and could cause false switching.

(d) $K_f = (C_m/C_0 - L_m/L_0) / 2 = (0.3182 - 0.2909) / 2 = 0.02727 / 2 = 0.01364$

Propagation delay for 50 mm: $t_d = 0.05 / 1.818 \times 10^8 = 0.275$ ns $V_{FEXT} = K_f \times V_{step} \times (t_d/t_r) = 0.01364 \times 1.8 \times (0.275/0.3) = 0.01364 \times 1.8 \times 0.917 = 22.5$ mV

FEXT is much smaller than NEXT because C_m/C_0 and L_m/L_0 are nearly equal. The design should increase spacing to at least 3h = 0.45 mm (the "3W" rule) to reduce NEXT to acceptable levels.

Problem 9.5.9

Given: A quarter-wave transformer must match a 100 Ω load to a 50 Ω line at 5.8 GHz. The transformer is fabricated as a microstrip on FR-4 ($\varepsilon_{eff} = 3.1$).

Find: (a) The required transformer impedance. (b) The physical length of the transformer. (c) The bandwidth over which VSWR remains below 1.5:1.

Solution:

(a) $Z_T = \sqrt{(Z_0 \times Z_L)} = \sqrt{(50 \times 100)} = \sqrt{5000} = 70.71\ \Omega$

(b) $\lambda_{eff} = c / (f \times \sqrt{\varepsilon_{eff}}) = 3 \times 10^8 / (5.8 \times 10^9 \times \sqrt{3.1}) = 3 \times 10^8 / (5.8 \times 10^9 \times 1.761) = 3 \times 10^8 / 1.021 \times 10^{10} = 29.38$ mm Length $= \lambda_{eff} / 4 = 7.35$ mm

(c) For a single-section quarter-wave transformer matching $Z_L/Z_0 = 2:1$, the fractional bandwidth for VSWR < 1.5:1 is approximately:

For VSWR = 1.5, $|\Gamma_{max}| = (1.5 - 1)/(1.5 + 1) = 0.2$ At center frequency: $|\Gamma_0| = (Z_L - Z_0)/(Z_L + Z_0) = (100 - 50)/(100 + 50) = 1/3$

The bandwidth where $|\Gamma| < 0.2$: $BW/f_0 \approx (2/\pi) \times \arccos(|\Gamma_{max}|/|\Gamma_0| \times 1/\sqrt{(1 - |\Gamma_{max}|^2)})$

A simplified estimate: fractional BW $\approx 2 - 4f_c/f_0$, giving approximately 40-50% bandwidth, or roughly 2.3-3.5 GHz around 5.8 GHz. The quarter-wave transformer provides reasonably broadband matching for a 2:1 impedance ratio.

Chapter 9 — Section 9.6: Antennas

Practice problems covering antenna fundamentals, dipole antennas, antenna arrays, and near-field/far-field regions.

Problem 9.6.1

Given: A satellite transmitter radiates 20 W at 12 GHz (Ku-band) through a parabolic dish antenna with a gain of 35 dBi. A ground station 36,000 km away has a receive antenna with gain 42 dBi.

Find: (a) The free-space path loss. (b) The received power using the Friis equation. (c) The effective aperture of the receive antenna.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 12 \times 10^9 = 0.025$ m FSPL = $(4\pi d/\lambda)^2 = (4\pi \times 3.6 \times 10^7 / 0.025)^2 = (1.8096 \times 10^{10})^2$

In dB: FSPL = $20 \log_{10}(4\pi d/\lambda) = 20 \log_{10}(1.8096 \times 10^{10}) = 20 \times 10.258 = 205.2$ dB

(b) $P_r(\text{dBW}) = P_t(\text{dBW}) + G_t + G_r - \text{FSPL} = 10 \log_{10}(20) + 35 + 42 - 205.2 = 13.0 + 35 + 42 - 205.2 = -115.2$ dBW

$P_r = 10^{-115.2/10} = 3.02 \times 10^{-12}$ W = 3.02 pW

(c) $A_e = G_r\lambda^2 / (4\pi)$ where $G_r = 10^{42/10} = 15{,}849$ $A_e = 15{,}849 \times (0.025)^2 / (4\pi) = 15{,}849 \times 6.25 \times 10^{-4} / 12.566 = 9.906 / 12.566 = 0.788$ m^2

Problem 9.6.2

Given: A half-wave dipole antenna is designed for the 2-meter amateur radio band at 146 MHz.

Find: (a) The theoretical full length. (b) The practical length with 5% shortening. (c) The radiation resistance. (d) The gain in dBi and dBd. (e) The beamwidth.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 146 \times 10^6 = 2.055$ m $L = \lambda/2 = 1.027$ m (each arm = 0.514 m)

(b) $L_{practical} = 0.95 \times 1.027 = 0.976$ m (each arm = 0.488 m)

(c) The radiation resistance of a half-wave dipole is $R_{rad} = 73.1\ \Omega$

(d) Gain: In dBi: G = 2.15 dBi (linear gain = 1.64) In dBd: G = 0 dBd (by definition, since dBd uses the half-wave dipole as reference)

(e) The E-plane (containing the dipole axis) half-power beamwidth of a half-wave dipole is 78°. The H-plane pattern is omnidirectional (360°).

Problem 9.6.3

Given: A broadside uniform linear array of N = 16 isotropic elements is spaced at $d = \lambda/2$ and operates at 10 GHz.

Find: (a) The wavelength and element spacing. (b) The half-power beamwidth. (c) The array directivity. (d) The total array length.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 10 \times 10^9 = 0.03$ m = 30 mm $d = \lambda/2 = 15$ mm

(b) $\theta_{3dB} \approx 0.886\lambda / (Nd) = 0.886 \times 0.03 / (16 \times 0.015) = 0.02658 / 0.24 = 0.1108$ rad = 6.35°

(c) For a uniform linear array of isotropic elements with $\lambda/2$ spacing: $D \approx N = 16 = 12.04$ dBi

(d) $L = (N - 1) \times d = 15 \times 0.015 = 0.225$ m = 225 mm

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Problem 9.6.4

Given: A parabolic reflector antenna has a diameter D = 1.2 m and operates at 6 GHz (C-band satellite uplink). The aperture efficiency is $\eta_{ap} = 0.55$.

Find: (a) The wavelength. (b) The physical aperture area. (c) The effective aperture. (d) The antenna gain. (e) The far-field boundary distance. (f) The half-power beamwidth.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 6 \times 10^9 = 0.05$ m = 50 mm

(b) $A_{physical} = \pi(D/2)^2 = \pi(0.6)^2 = 1.131\ m^2$

(c) $A_e = \eta_{ap} \times A_{physical} = 0.55 \times 1.131 = 0.622\ m^2$

(d) $G = 4\pi A_e/\lambda^2 = 4\pi \times 0.622 / (0.05)^2 = 7.817 / 0.0025 = 3127 = 34.95$ dBi

Alternatively: $G = \eta_{ap}(\pi D/\lambda)^2 = 0.55 \times (\pi \times 1.2/0.05)^2 = 0.55 \times (75.4)^2 = 0.55 \times 5685 = 3127$, confirming.

(e) Far-field boundary: $r_{ff} = 2D^2/\lambda = 2 \times (1.2)^2 / 0.05 = 2.88 / 0.05 = 57.6$ m

(f) Half-power beamwidth: $\theta_{3dB} \approx 70\lambda/D = 70 \times 0.05 / 1.2 = 3.5 / 1.2 = 2.92°$

Problem 9.6.5

Given: A 5G millimeter-wave base station uses a planar phased array antenna panel of dimensions 0.3 m × 0.3 m operating at 39 GHz. An NFC reader at 13.56 MHz uses a 40 mm × 40 mm loop antenna.

Find: For each system: (a) The far-field boundary. (b) The reactive near-field extent. (c) Classify the typical operating range as near-field or far-field.

Solution:

5G mmWave (39 GHz): $\lambda = 3 \times 10^8 / 39 \times 10^9 = 7.69$ mm, D = 0.3 m

(a) $r_{ff} = 2D^2/\lambda = 2 \times 0.09 / 0.00769 = 23.4$ m

(b) $r_{reactive} = 0.62\sqrt{(D^3/\lambda)} = 0.62\sqrt{(0.027/0.00769)} = 0.62\sqrt{3.512} = 0.62 \times 1.874 = 1.16$ m

(c) A 5G base station communicates at typical distances of 50-300 m, which is well into the far-field region. Antenna measurements require at least 23.4 m of clear range.

NFC (13.56 MHz): $\lambda = 3 \times 10^8 / 13.56 \times 10^6 = 22.12$ m, D = 0.04 m

(a) $r_{ff} = 2D^2/\lambda = 2 \times 0.0016 / 22.12 = 0.145$ mm — essentially zero.

(b) $r_{reactive} = 0.62\sqrt{(D^3/\lambda)} = 0.62\sqrt{(6.4 \times 10^{-5} / 22.12)} = 0.62\sqrt{(2.89 \times 10^{-6})} = 0.62 \times 1.70 \times 10^{-3} = 1.05$ mm

(c) NFC operates at ranges up to ~100 mm, which is deep in the reactive near-field. At 100 mm, the field decays as approximately $(1.05/100)^3 \approx 10^{-6}$ relative to the reactive boundary, providing the security advantage of NFC's extremely limited range.

Problem 9.6.6

Given: A Yagi-Uda antenna has 6 elements (1 driven dipole, 1 reflector, 4 directors) and operates at 432 MHz (70 cm amateur band). The measured gain is 11.5 dBi and the front-to-back ratio is 20 dB. The antenna input impedance is 28 Ω.

Find: (a) The wavelength. (b) The VSWR when fed with 50 Ω coax. (c) The effective aperture. (d) If the transmitter power is 50 W, find the EIRP and the power density at 10 km.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 432 \times 10^6 = 0.694$ m = 69.4 cm

(b) $\Gamma = (28 - 50)/(28 + 50) = -22/78 = -0.282$ VSWR = (1 + 0.282)/(1 - 0.282) = 1.282/0.718 = 1.79:1

(c) $G = 10^{11.5/10} = 14.13$ $A_e = G\lambda^2/(4\pi) = 14.13 \times (0.694)^2 / (4\pi) = 14.13 \times 0.4816 / 12.566 = 6.805 / 12.566 = 0.541\ m^2$

(d) EIRP = $P_t \times G = 50 \times 14.13 = 706.5$ W = 28.49 dBW = 58.49 dBm

Power density at 10 km: $S = EIRP/(4\pi r^2) = 706.5/(4\pi \times (10^4)^2) = 706.5 / (1.257 \times 10^9) = 5.62 \times 10^{-7}$ W/m² = 0.562 μW/m²

Chapter 9 — Section 9.7: Electromagnetic Compatibility

Practice problems covering shielding effectiveness, EMI filtering, conducted emissions, PCB layout for EMC, ESD protection, and grounding/bonding.

Problem 9.7.1

Given: A steel enclosure ($\sigma = 6.0 \times 10^6$ S/m, $\mu_r = 200$) with wall thickness t = 2 mm must shield against a 1 MHz interference source. The target shielding effectiveness is 80 dB.

Find: (a) The skin depth at 1 MHz. (b) The absorption loss. (c) The approximate reflection loss. (d) Whether the target SE is met. (e) The maximum permissible ventilation slot length.

Solution:

(a) $\delta = 1/\sqrt{(\pi f \mu_0 \mu_r \sigma)} = 1/\sqrt{(\pi \times 10^6 \times 4\pi \times 10^{-7} \times 200 \times 6.0 \times 10^6)} = 1/\sqrt{(\pi \times 10^6 \times 1.508 \times 10^{-3})}$ $= 1/\sqrt{(4{,}736)} = 14.5\ \mu$m

(b) $A = 8.686 \times (t/\delta) = 8.686 \times (2 \times 10^{-3} / 14.5 \times 10^{-6}) = 8.686 \times 137.9 = 1{,}198$ dB

(c) $\sigma_r = \sigma/\sigma_{Cu} = 6.0 \times 10^6 / 5.8 \times 10^7 = 0.1034$ $R \approx 168 - 10 \log_{10}(f \times \mu_r/\sigma_r) = 168 - 10 \log_{10}(10^6 \times 200/0.1034) = 168 - 10 \log_{10}(1.934 \times 10^9) = 168 - 92.9 = 75.1$ dB

(d) Total SE = A + R = 1,198 + 75.1 = 1,273 dB » 80 dB. The target is easily met by the solid walls.

However, any aperture in the enclosure is the limiting factor.

(e) At 1 MHz: $\lambda = c/f = 3 \times 10^8 / 10^6 = 300$ m. For SE > 80 dB, the maximum slot length should be less than $\lambda/20$ for good practice, but the actual limit depends on the required attenuation. A slot of length l resonates at $\lambda = 2l$ and degrades SE dramatically.

Maximum slot: $l < \lambda/20 = 300/20 = 15$ m — at 1 MHz, even large ventilation openings are not a concern.

At higher frequencies, slots become the bottleneck. At 300 MHz ($\lambda = 1$ m), $l_{max} = 50$ mm; at 1 GHz ($\lambda = 0.3$ m), $l_{max} = 15$ mm.

Problem 9.7.2

Given: A switching power supply produces the following conducted emissions measured with a LISN: - At 150 kHz: 82 dBμV (quasi-peak) - At 500 kHz: 70 dBμV - At 5 MHz: 55 dBμV

The CISPR 32 Class B quasi-peak limits are: - 150 kHz – 500 kHz: 66-56 dBμV (linearly decreasing in log scale) - 500 kHz – 5 MHz: 56 dBμV - 5 MHz – 30 MHz: 60 dBμV

A 6 dB design margin is required.

Find: (a) The amount by which each frequency exceeds the limit. (b) The required filter attenuation at each frequency. (c) Design an EMI filter (choose L and C values) to achieve compliance.

Solution:

(a) At 150 kHz: limit = 66 dBμV, emission = 82 dBμV → exceeds by 16 dB At 500 kHz: limit = 56 dBμV, emission = 70 dBμV → exceeds by 14 dB At 5 MHz: limit = 60 dBμV, emission = 55 dBμV → within limit by 5 dB (but need 6 dB margin, so need 1 dB reduction)

(b) Required attenuation (including 6 dB margin): At 150 kHz: 16 + 6 = 22 dB At 500 kHz: 14 + 6 = 20 dB At 5 MHz: 1 dB (already close to margin)

(c) The critical requirement is 22 dB at 150 kHz. A single-stage LC filter with -40 dB/decade roll-off: Attenuation at f: $A(f) = (f/f_0)^2$ above corner frequency. For 22 dB at 150 kHz: $10^{22/20} = 12.59 = (150{,}000/f_0)^2$ $f_0 = 150{,}000/\sqrt{12.59} = 150{,}000/3.548 = 42.3$ kHz

Choose $L_{DM} = 470$ μH. Then: $C = 1/(4\pi^2 f_0^2 L) = 1/(4\pi^2 \times (42{,}300)^2 \times 470 \times 10^{-6}) = 1/(4 \times 9.87 \times 1.789 \times 10^9 \times 4.7 \times 10^{-4}) = 1/(3.333 \times 10^7) = 30.0$ nF

Use a 33 nF X2 capacitor (standard value).

Verify at 500 kHz: $A = (500/42.3)^2 = (11.82)^2 = 139.7 = 21.45$ dB (needs 20 dB — passes). At 5 MHz: $A = (5000/42.3)^2 = (118.2)^2 = 13{,}971 = 41.5$ dB (far exceeds the 1 dB needed).

Add a 10 mH common-mode choke and 2.2 nF Y-capacitors for common-mode filtering.

Final filter: 10 mH CM choke + 33 nF X2 cap + 470 μH DM inductor + 33 nF X2 cap.

Problem 9.7.3

Given: A 6-layer PCB stackup has the following layer assignments: Sig1 – GND – Sig2 – PWR – GND – Sig3. The substrate height between Sig1 and GND is $h_1 = 0.1$ mm. A 3.3 V, 100 MHz clock trace runs 80 mm on Sig1. The clock signal has a 50% duty cycle and the current is approximately 15 mA for the fundamental.

Find: (a) The loop area formed by the clock trace and its return current. (b) Compare to the same trace on Sig2 ($h_2 = 0.2$ mm to GND). (c) Estimate whether the radiated emission from the Sig1 trace is likely to pass FCC Class B at 3 m (limit ~100 μV/m at 100 MHz).

Solution:

(a) On Sig1 (0.1 mm above GND): Loop area: $A_1 = l \times h_1 = 80 \times 0.1 = 8.0\ \text{mm}^2 = 8.0 \times 10^{-6}\ \text{m}^2$

(b) On Sig2 (0.2 mm above GND): $A_2 = 80 \times 0.2 = 16.0$ mm^2 $= 16.0 \times 10^{-6}$ m^2 Ratio: $A_2/A_1 = 2.0$ — the Sig2 trace radiates 6 dB more than the Sig1 trace.

(c) For an electrically small loop antenna, the radiated E-field at distance r: $E \approx 1.316 \times 10^{-14} \times f^2 \times I \times A / r$

At 100 MHz from Sig1: $E \approx 1.316 \times 10^{-14} \times (10^8)^2 \times 0.015 \times 8 \times 10^{-6} / 3 = 1.316 \times 10^{-14} \times 10^{16} \times 1.2 \times 10^{-7} / 3 = 131.6 \times 1.2 \times 10^{-7} / 3 = 5.26 \times 10^{-6}$ V/m $= 5.26$ μV/m

This is well below the FCC Class B limit of ~100 μV/m at 3 m. The design is likely to pass.

However, harmonics (300 MHz, 500 MHz) will have higher radiation due to the f^2 factor, and actual PCB emissions include contributions from all signal traces, vias, and decoupling loops. A margin of 10-20 dB above the single-trace estimate is prudent.

Problem 9.7.4

Given: A USB-C port on a consumer device must survive ±8 kV contact discharge and ±15 kV air discharge per IEC 61000-4-2. The USB 3.2 data lines (10 Gbps) must be protected. A TVS diode array is selected with: $V_{BR} = 5.5$ V, $V_C = 9$ V at 8 A, $C_J = 0.25$ pF per line.

Find: (a) The peak current during an 8 kV contact discharge. (b) The signal integrity impact on USB 3.2 (5 GHz fundamental frequency). (c) Whether a series resistor can be used and what value is acceptable.

Solution:

(a) Per IEC 61000-4-2, an 8 kV contact discharge produces a peak current of approximately 30 A (with 0.7-1 ns rise time and characteristic double-exponential waveform).

The 8 kV voltage is applied through a 330 Ω + 150 pF network defined by the standard. The initial peak is $I_{peak} = V / Z_{contact} \approx 8000/330 \approx 24$ A for the first pulse, followed by a 30 A peak from the discharge network resonance.

(b) The TVS capacitance of 0.25 pF per line is critical for USB 3.2. With a 45 Ω differential line impedance: the 3 dB bandwidth due to the TVS capacitance: $f_{3dB} = 1/(\pi \times Z_0 \times C_J) = 1/(\pi \times 45 \times 0.25 \times 10^{-12}) = 28.3$ GHz

Since USB 3.2 Gen 2 operates at 10 Gbps (5 GHz fundamental), the TVS capacitance causes: Insertion loss at 5 GHz $\approx (f/f_{3dB})^2 = (5/28.3)^2 = 0.031 = 0.14$ dB — negligible impact.

The S11 (return loss) is more important: $|S11| \approx 2\pi f C_J Z_0 = 2\pi \times 5 \times 10^9 \times 0.25 \times 10^{-12} \times 45 = 0.354$ → return loss = 9.0 dB. This is marginal but acceptable for ESD protection.

(c) A series resistor increases insertion loss: at 10 Gbps, even 5 Ω of series resistance degrades the eye diagram significantly due to the high data rate and the equalizer margin required.

For USB 3.2: no series resistor is recommended. The TVS alone must handle the ESD event. The low 0.25 pF capacitance TVS was specifically chosen to avoid needing a series resistor.

For lower-speed USB 2.0 (480 Mbps): a 10-22 Ω series resistor is acceptable and improves clamping voltage seen by the IC.

Problem 9.7.5

Given: An audio measurement system has a preamplifier (sensitivity 50 μV full scale, bandwidth 20 Hz - 20 kHz) connected to a microphone 30 m away via a shielded cable. The building has a 3 V, 60 Hz ground potential difference between the microphone location and the preamplifier location. The cable shield has 0.2 Ω resistance.

Find: (a) The ground loop current if the shield is grounded at both ends. (b) The noise voltage induced in the signal conductor (assume transfer impedance $Z_T = 15$ mΩ/m). (c) The SNR. (d) Propose a practical solution.

Solution:

(a) $I_{loop} = V_{ground} / R_{shield} = 3 / 0.2 = 15$ A

(b) $V_{noise} = I_{loop} \times Z_T \times \text{length} = 15 \times 0.015 \times 30 = 6.75$ V

(c) SNR = 20 $\log_{10}(V_{signal}/V_{noise})$ = 20 $\log_{10}(50 \times 10^{-6} / 6.75)$ = 20 $\log_{10}(7.41 \times 10^{-6})$ = 20 × (-5.13) = -102.6 dB

The noise completely buries the signal. The measurement is impossible in this configuration.

(d) Solution: Use a combination of techniques:

1. Ground the shield at one end only (receiver end): This breaks the ground loop. The 3 V appears as common-mode voltage on the signal pair.
2. Use a balanced (differential) input with an instrumentation amplifier having CMRR $\geq$ 100 dB at 60 Hz: $V_{noise} = 3\text{ V} / 10^{100/20} = 3 / 100{,}000 = 30\ \mu$V
3. New SNR = 20 $\log_{10}(50 \times 10^{-6} / 30 \times 10^{-6})$ = 20 $\log_{10}(1.67)$ = 4.4 dB — barely usable.
4. Add galvanic isolation (audio isolation transformer or isolated preamp at the microphone): This completely eliminates the ground loop, reducing noise to the preamp's intrinsic noise (typically 1-5 μV_{rms} in a 20 kHz bandwidth).

With isolation: SNR = 20 $\log_{10}(50 \times 10^{-6} / 3 \times 10^{-6})$ = 20 $\log_{10}(16.7)$ = 24.4 dB — acceptable for the measurement.

Problem 9.7.6

Given: A 4-layer PCB design routes a 25 MHz clock signal (10 mA, 1 ns rise time) across a 15 mm gap (split) in the ground plane on layer 2. The trace is 40 mm long on the top signal layer. The substrate height is 0.15 mm.

Find: (a) The loop area with an intact ground plane. (b) The effective loop area when the return current detours around the 15 mm split. (c) The increase in radiated emissions (in dB). (d) A design remedy.

Solution:

(a) Intact ground plane: A_1 = trace length $\times$ substrate height = $40 \times 0.15 = 6.0$ mm^2

(b) With the 15 mm split, the return current must detour around the gap. The detour path adds roughly the split width to the effective loop height. Effective loop area: $A_2 \approx 40 \times 15 = 600$ mm^2 (conservative estimate — the actual loop is the area enclosed by the signal trace going forward and the return current going around the split)

More precisely, the detour creates a slot antenna of dimensions approximately 40 mm $\times$ 15 mm. $A_2 =$ 600 mm^2

(c) Ratio: $A_2/A_1 = 600/6 = 100$. Since radiated emissions scale with loop area: $\Delta E = 20 \log_{10}(100)$ $= 40$ dB increase in radiated emissions

A 40 dB increase can easily cause an EMC failure. If the intact design had 10 dB margin, the split ground plane would exceed the limit by 30 dB.

(d) Design remedies:

- Never route high-speed signals across ground plane splits. Route around the split or use a different signal layer.
- If a split is unavoidable, place stitching capacitors (100 nF) across the split every 5-10 mm to provide a low-impedance AC return path.
- Redesign the stackup so that the ground plane is continuous under all high-speed signals, or move the split to a power plane layer with decoupling capacitors bridging it.

Chapter 10 — Section 10.1: Switching Devices

Practice problems covering power diodes, MOSFETs, IGBTs, thyristors, wide-bandgap semiconductors (SiC and GaN), and gate driver design for power electronic converters.

Problem 10.1.1

Given: A fast-recovery silicon power diode has a forward voltage drop $V_F = 1.1$ V and carries a pulsed forward current with an RMS value of 18 A and an average value of 12 A. The reverse recovery charge is $Q_{rr} = 800$ nC, the DC bus voltage is $V_R = 600$ V, and the switching frequency is $f_{sw} = 50$ kHz.

Find: (a) The conduction power loss, (b) the reverse recovery energy per switching event, (c) the reverse recovery power loss, and (d) the total diode power loss.

Solution:

(a) Conduction power loss (using average current for the V_F drop model): $P_{cond} = V_F \times I_{avg} = 1.1 \times 12 = 13.2$ W

(b) Reverse recovery energy per event: $E_{rr} = ½ \times Q_{rr} \times V_R = 0.5 \times 800 \times 10^{-9} \times 600 = 240$ μJ

(c) Reverse recovery power loss: $P_{rr} = E_{rr} \times f_{sw} = 240 \times 10^{-6} \times 50 \times 10^3 = 12.0$ W

(d) Total diode power loss: $P_{total} = P_{cond} + P_{rr} = 13.2 + 12.0 = 25.2$ W

Problem 10.1.2

Given: A power MOSFET with $R_{DS(on)} = 80$ mΩ at 25°C has a temperature coefficient of +0.5%/°C. It operates in a synchronous buck converter at $f_{sw} = 400$ kHz with a drain current $I_D = 15$ A, bus voltage $V_{DS} = 48$ V, turn-on time $t_{on} = 18$ ns, and turn-off time $t_{off} = 25$ ns. The junction temperature is 110°C and the duty cycle is $D = 0.35$.

Find: (a) $R_{DS(on)}$ at 110°C, (b) the RMS current through the MOSFET, (c) the conduction loss, (d) the switching loss, and (e) the total MOSFET power loss.

Solution:

(a) $R_{DS(on)}$ at 110°C: $R_{DS(on)} = 80 \times [1 + 0.005 \times (110 - 25)] = 80 \times [1 + 0.425] = 80 \times 1.425 = 114$ mΩ

(b) RMS current through the high-side MOSFET: $I_{rms} = I_D \times \sqrt{D} = 15 \times \sqrt{0.35} = 15 \times 0.5916 =$ 8.87 A

(c) Conduction loss: $P_{cond} = I_{rms}{}^2 \times R_{DS(on)} = 8.87^2 \times 0.114 = 78.7 \times 0.114 = 8.97$ W

(d) Switching loss: $P_{sw} = ½ \times V_{DS} \times I_D \times (t_{on} + t_{off}) \times f_{sw}$ $P_{sw} = 0.5 \times 48 \times 15 \times (18 + 25) \times 10^{-9} \times 400 \times 10^3$ $P_{sw} = 0.5 \times 48 \times 15 \times 43 \times 10^{-9} \times 4 \times 10^5 = 6.19$ W

(e) Total MOSFET loss: $P_{total} = 8.97 + 6.19 = 15.16$ W

Problem 10.1.3

Given: An IGBT module rated for 1,200 V / 300 A has $V_{CE(sat)} = 2.4$ V at rated current, turn-on energy $E_{on} = 45$ mJ, and turn-off energy $E_{off} = 35$ mJ (at $V_{DC} = 600$ V, $I_C = 300$ A). The module drives a three-phase motor at $f_{sw} = 8$ kHz. The modulation index is $m_a = 0.9$ and the load power factor angle is $\varphi = 30°$.

Find: (a) The average conduction loss per IGBT (using the approximation for sinusoidal PWM), (b) the switching loss per IGBT, (c) the total loss per IGBT, and (d) the total loss for all six IGBTs in the three-phase bridge.

Solution:

(a) Average conduction loss per IGBT (sinusoidal PWM approximation): $P_{cond} = V_{CE(sat)} \times I_C \times (1/(2\pi)) \times (1 + m_a \times \cos\varphi \times \pi/4)$

More precisely, for sinusoidal modulation: $P_{cond} = V_{CE(sat)} \times I_{peak}/(2\pi) \times (\pi/4 + m_a \times \cos\varphi\ /\ 3)$

Using $I_{peak} = 300$ A: $P_{cond} = 2.4 \times 300\ /\ (2\pi) \times (\pi/4 + 0.9 \times \cos 30°\ /\ 3)$ $P_{cond} = 2.4 \times 300\ /\ 6.283 \times (0.7854 + 0.9 \times 0.866\ /\ 3)$ $P_{cond} = 114.6 \times (0.7854 + 0.2598)$ $P_{cond} = 114.6 \times 1.0452 = 119.8$ W

(b) Switching loss per IGBT: $P_{sw} = (E_{on} + E_{off}) \times f_{sw}\ /\ \pi$ $P_{sw} = (45 + 35) \times 10^{-3} \times 8{,}000\ /\ \pi = 80 \times 10^{-3} \times 8{,}000\ /\ 3.1416$ $P_{sw} = 640\ /\ 3.1416 = 203.7$ W

(c) Total loss per IGBT: $P_{total} = 119.8 + 203.7 = 323.5$ W

(d) Total loss for six IGBTs: $P_{6\text{-}IGBT} = 6 \times 323.5 = 1{,}941$ W

Note: This does not include the antiparallel diode losses, which typically add 20–40% to the total module loss.

Problem 10.1.4

Given: A thyristor-controlled single-phase full-wave bridge rectifier is fed from a 240 V_{rms}, 50 Hz source. The load is a DC motor with back-EMF E = 180 V and armature resistance $R_a = 0.5\ \Omega$. The thyristor forward voltage drop is $V_T = 1.5$ V (two thyristors conduct at any time).

Find: (a) The firing angle α required to produce an average motor current of 50 A, (b) the average output voltage, (c) the power delivered to the motor, and (d) the total thyristor conduction losses.

Solution:

(a) The average output voltage of a fully controlled single-phase bridge: $V_{dc} = (2V_{peak}/\pi) \times \cos\alpha$

The required V_{dc} must overcome the back-EMF, armature resistance drop, and thyristor drops: $V_{dc} = E + I_a \times R_a + 2V_T = 180 + 50 \times 0.5 + 2 \times 1.5 = 180 + 25 + 3 = 208$ V

$V_{peak} = 240 \times \sqrt{2} = 339.4$ V $2V_{peak}/\pi = 2 \times 339.4 / \pi = 216.1$ V

$\cos\alpha = V_{dc} / (2V_{peak}/\pi) = 208 / 216.1 = 0.9625$ $\alpha = \cos^{-1}(0.9625) = 15.7°$

(b) Average output voltage: $V_{dc} = 208$ V

(c) Power delivered to the motor (mechanical plus armature heating): $P_{motor} = E \times I_a + I_a{}^2 \times R_a = 180 \times 50 + 50^2 \times 0.5 = 9{,}000 + 1{,}250 = 10{,}250$ W

(d) Total thyristor conduction losses (two thyristors in series): $P_{thyristor} = 2 \times V_T \times I_{avg} = 2 \times 1.5 \times 50 = 150$ W

Problem 10.1.5

Given: A 1,200 V silicon IGBT module has $V_{CE(sat)} = 2.1$ V and switching energies $E_{on} = 12$ mJ, $E_{off} = 8$ mJ at 100 A. A 1,200 V SiC MOSFET has $R_{DS(on)} = 13$ mΩ and switching energies $E_{on} = 1.5$ mJ, $E_{off} = 0.8$ mJ at 100 A. Both devices operate at $I_D = 100$ A with a duty cycle of 0.5 in a DC-DC converter with $V_{bus} = 800$ V.

Find: (a) The conduction loss for each device, (b) the switching loss for each at $f_{sw} = 20$ kHz, (c) the total losses and efficiency improvement of SiC, and (d) the maximum switching frequency at which the SiC device dissipates the same total power as the Si IGBT at 20 kHz.

Solution:

(a) Conduction losses (D = 0.5): Si IGBT: $P_{cond} = V_{CE(sat)} \times I_D \times D = 2.1 \times 100 \times 0.5 = 105$ W SiC MOSFET: $I_{rms} = I_D \times \sqrt{D} = 100 \times \sqrt{0.5} = 70.7$ A $P_{cond} = I_{rms}{}^2 \times R_{DS(on)} = 70.7^2 \times 0.013 = 5{,}000 \times 0.013 = 65.0$ W

(b) Switching losses at 20 kHz: Si IGBT: $P_{sw} = (12 + 8) \times 10^{-3} \times 20{,}000 = 400$ W SiC MOSFET: $P_{sw} = (1.5 + 0.8) \times 10^{-3} \times 20{,}000 = 46.0$ W

(c) Total losses: Si IGBT: $P_{total} = 105 + 400 = 505$ W SiC MOSFET: $P_{total} = 65.0 + 46.0 = 111.0$ W

At $P_{out} = V_{bus} \times I_D \times D = 800 \times 100 \times 0.5 = 40{,}000$ W: Si IGBT efficiency: $\eta = 40{,}000 / (40{,}000 + 505) = 98.75\%$ SiC efficiency: $\eta = 40{,}000 / (40{,}000 + 111) = 99.72\%$ Efficiency improvement: 0.97 percentage points

(d) For SiC total loss = 505 W (same as Si IGBT at 20 kHz): $505 = 65.0 + (1.5 + 0.8) \times 10^{-3} \times f_{sw}$ $440 = 2.3 \times 10^{-3} \times f_{sw}$ $f_{sw} = 440 / 0.0023 = 191.3$ kHz

The SiC device can switch at nearly 10× the frequency of the Si IGBT at equal total loss.

Problem 10.1.6

Given: A bootstrap gate driver circuit drives the high-side MOSFET in a half-bridge. The MOSFET has total gate charge $Q_g = 85$ nC and requires $V_{GS} \geq 10$ V. The driver IC quiescent current is $I_Q = 3$ mA, and the level shifter leakage is $I_{leak} = 100$ μA. The switching frequency is $f_{sw} = 200$ kHz with a maximum duty cycle of 95%. The supply voltage is $V_{CC} = 12$ V and the bootstrap diode has $V_F = 0.7$ V. The allowable bootstrap voltage droop is $\Delta V = 0.8$ V.

Find: (a) The maximum high-side on-time, (b) the total charge drawn from the bootstrap capacitor per cycle, (c) the minimum bootstrap capacitance, (d) the effective gate drive voltage at end of the high-side on-time, and (e) whether the design provides adequate gate drive voltage.

Solution:

(a) Maximum high-side on-time: $T_{sw} = 1/f_{sw} = 1/200{,}000 = 5$ μs $t_{on(max)} = D_{max} \times T_{sw} = 0.95 \times 5 = 4.75$ μs

(b) Total charge per cycle: $Q_{total} = Q_g + (I_Q + I_{leak}) \times t_{on(max)}$ $Q_{total} = 85 \times 10^{-9} + (3 \times 10^{-3} + 100 \times 10^{-6}) \times 4.75 \times 10^{-6}$ $Q_{total} = 85$ nC $+ 3.1 \times 10^{-3} \times 4.75 \times 10^{-6} = 85$ nC $+ 14.7$ nC $= 99.7$ nC

(c) Minimum bootstrap capacitance: $C_{boot} = Q_{total} / \Delta V = 99.7 \times 10^{-9} / 0.8 = 124.6$ nF

Use a standard 220 nF ceramic capacitor for margin.

(d) Bootstrap voltage when fully charged: $V_{boot} = V_{CC} - V_F = 12 - 0.7 = 11.3$ V

Voltage at end of on-time (using 220 nF): $V_{boot,end} = 11.3 - Q_{total}/C = 11.3 - 99.7 \times 10^{-9} / 220 \times 10^{-9} = 11.3 - 0.45 = 10.85$ V

(e) Since $V_{boot,end} = 10.85$ V $> V_{GS(min)} = 10$ V, the design provides adequate gate drive voltage with a margin of 0.85 V. This is a slim margin; using a 470 nF capacitor would give $V_{boot,end} = 11.3 - 0.21 = 11.09$ V with a more comfortable 1.09 V margin.

Problem 10.1.7

Given: A SiC Schottky diode ($V_F = 1.35$ V, $Q_{rr} \approx 0$) and a silicon fast-recovery diode ($V_F = 0.95$ V, $t_{rr} = 35$ ns) both operate as the freewheeling diode in a boost converter with $V_{bus} = 400$ V, $I_{avg} = 8$ A, $I_{peak} = 10$ A, and $f_{sw} = 200$ kHz. The silicon diode's reverse recovery current peak is estimated as $I_{rr} \approx 0.5 \times I_{peak} = 5$ A.

Find: (a) The conduction loss for each diode, (b) the reverse recovery loss for the silicon diode, (c) the total loss for each diode, and (d) the percentage reduction in total diode loss using SiC.

Solution:

(a) Conduction losses: SiC: $P_{cond} = V_F \times I_{avg} = 1.35 \times 8 = 10.8$ W Si: $P_{cond} = V_F \times I_{avg} = 0.95 \times 8 = 7.6$ W

(b) Silicon diode reverse recovery loss: $Q_{rr} = ½ \times I_{rr} \times t_{rr} = 0.5 \times 5 \times 35 \times 10^{-9} = 87.5$ nC $E_{rr} = ½ \times Q_{rr} \times V_R = 0.5 \times 87.5 \times 10^{-9} \times 400 = 17.5$ μJ $P_{rr} = E_{rr} \times f_{sw} = 17.5 \times 10^{-6} \times 200 \times 10^3 = 3.5$ W

(c) Total losses: SiC: $P_{total} = 10.8 + 0 = 10.8$ W Si: $P_{total} = 7.6 + 3.5 = 11.1$ W

(d) Reduction: (11.1 − 10.8) / 11.1 × 100 = 2.7%

At this frequency, the SiC advantage is modest. At 500 kHz, the Si recovery loss would be 8.75 W, making total Si loss 16.35 W versus 10.8 W for SiC — a 34% reduction, illustrating the growing SiC advantage at higher frequencies.

Problem 10.1.8

Given: A half-bridge gate driver must provide dead time to prevent shoot-through. The high-side MOSFET has turn-off delay $t_{d(off)} = 40$ ns and fall time $t_f = 18$ ns. The low-side MOSFET has turn-on delay $t_{d(on)} = 25$ ns and rise time $t_r = 12$ ns. The gate driver propagation delay mismatch between channels is ±8 ns. The switching frequency is 500 kHz and $V_{bus} = 48$ V.

Find: (a) The minimum dead time required to prevent shoot-through, (b) the recommended dead time with a 20% safety margin, (c) the body diode conduction time during dead time, (d) the body diode conduction loss if $I_{load} = 20$ A and $V_{SD} = 0.8$ V, and (e) the dead time loss as a percentage of output power at 24 V output.

Solution:

(a) Minimum dead time must ensure the turning-off device is fully off before the turning-on device starts conducting. Worst case: the turn-off command arrives late (−8 ns mismatch) and the turn-on command arrives early (+8 ns mismatch).

$t_{dead(min)} = t_{d(off)} + t_f + 2 \times \Delta t_{prop} - t_{d(on)}$ $t_{dead(min)} = 40 + 18 + 16 - 25 = 49$ ns

(b) Recommended dead time with 20% margin: $t_{dead} = 49 \times 1.20 = 58.8$ ns → use 60 ns

(c) Body diode conduction time ≈ dead time: $t_{BD} = 60$ ns (per transition, two transitions per switching cycle)

(d) Body diode conduction loss: $P_{BD} = V_{SD} \times I_{load} \times 2 \times t_{BD} \times f_{sw}$ $P_{BD} = 0.8 \times 20 \times 2 \times 60 \times 10^{-9} \times 500 \times 10^3 = 0.96$ W

(e) Output power: $P_{out} = V_{out} \times I_{load} = 24 \times 20 = 480$ W Dead time loss as percentage: 0.96 / 480 × 100 = 0.20%

Problem 10.1.9

Given: A GaN HEMT (650 V, $R_{DS(on)} = 55$ mΩ, $C_{oss} = 45$ pF, $Q_g = 6.2$ nC) and a silicon superjunction MOSFET (650 V, $R_{DS(on)} = 120$ mΩ, $C_{oss} = 80$ pF, $Q_g = 52$ nC) are compared for a 400 V, 600 W totem-pole PFC application at $f_{sw} = 500$ kHz. The RMS switch current is 2.5 A. Gate drive voltage is 6 V for GaN and 12 V for Si.

Find: (a) The conduction loss for each device, (b) the output capacitance (C_{oss}) switching loss for each, (c) the gate drive loss for each, (d) the total device loss for each, and (e) the efficiency impact at 600 W output.

Solution:

(a) Conduction losses: GaN: $P_{cond} = I_{rms}{}^2 \times R_{DS(on)} = 2.5^2 \times 0.055 = 6.25 \times 0.055 = 0.344$ W Si: $P_{cond} = 2.5^2 \times 0.120 = 6.25 \times 0.120 = 0.750$ W

(b) C_{oss} switching loss: GaN: $P_{oss} = ½ \times C_{oss} \times V^2 \times f_{sw} = 0.5 \times 45 \times 10^{-12} \times 400^2 \times 500 \times 10^3$ $P_{oss} = 0.5 \times 45 \times 10^{-12} \times 160{,}000 \times 500{,}000 = 1.80$ W Si: $P_{oss} = 0.5 \times 80 \times 10^{-12} \times 160{,}000 \times 500{,}000 = 3.20$ W

(c) Gate drive loss: GaN: $P_{gate} = Q_g \times V_{GS} \times f_{sw} = 6.2 \times 10^{-9} \times 6 \times 500 \times 10^3 = 0.019$ W Si: $P_{gate} = 52 \times 10^{-9} \times 12 \times 500 \times 10^3 = 0.312$ W

(d) Total device losses: GaN: $P_{total} = 0.344 + 1.80 + 0.019 = 2.16$ W Si: $P_{total} = 0.750 + 3.20 + 0.312 = 4.26$ W

(e) Efficiency impact: GaN: $\eta = 600 / (600 + 2.16) = 99.64\%$ Si: $\eta = 600 / (600 + 4.26) = 99.29\%$ Efficiency improvement with GaN: 0.35 percentage points

The GaN device reduces total losses by 49%, enabling higher power density or elimination of the heat sink in a compact charger design.

Problem 10.1.10

Given: An IGBT-based three-phase inverter drives a 150 kW induction motor. The DC bus is 650 V. Each IGBT has $V_{CE(sat)} = 1.8$ V at 200 A, $E_{on} = 18$ mJ, and $E_{off} = 14$ mJ. The antiparallel diode has $V_F = 1.2$ V and $E_{rr} = 8$ mJ. The motor power factor is $\cos \varphi = 0.87$ (lagging) and the modulation index is $m_a = 0.92$. The switching frequency is $f_{sw} = 10$ kHz and the peak phase current is $I_{peak} = 200$ A.

Find: (a) The conduction loss per IGBT, (b) the switching loss per IGBT, (c) the diode conduction loss per antiparallel diode, (d) the diode reverse recovery loss, and (e) the total inverter loss for all six IGBT/diode pairs.

Solution:

(a) IGBT conduction loss (sinusoidal PWM approximation per IGBT): $P_{cond,IGBT} = V_{CE(sat)} \times I_{peak} / (2\pi) \times (\pi/4 + m_a \times \cos \varphi / 3)$ $P_{cond,IGBT} = 1.8 \times 200 / (2\pi) \times (0.7854 + 0.92 \times 0.87 / 3)$ $P_{cond,IGBT} = 57.30 \times (0.7854 + 0.2668)$ $P_{cond,IGBT} = 57.30 \times 1.0522 = 60.3$ W

(b) IGBT switching loss: $P_{sw,IGBT} = (E_{on} + E_{off}) \times f_{sw} / \pi$ $P_{sw,IGBT} = (18 + 14) \times 10^{-3} \times 10{,}000 / \pi = 320 / \pi = 101.9$ W

(c) Diode conduction loss (sinusoidal PWM, antiparallel diode conducts during the complementary interval): $P_{cond,D} = V_F \times I_{peak} / (2\pi) \times (\pi/4 - m_a \times \cos \varphi / 3)$ $P_{cond,D} = 1.2 \times 200 / (2\pi) \times (0.7854 - 0.2668)$ $P_{cond,D} = 38.20 \times 0.5186 = 19.8$ W

(d) Diode reverse recovery loss: $P_{rr} = E_{rr} \times f_{sw} / \pi = 8 \times 10^{-3} \times 10{,}000 / \pi = 80 / \pi = 25.5$ W

(e) Total inverter loss (6 IGBTs + 6 diodes): $P_{total} = 6 \times (P_{cond,IGBT} + P_{sw,IGBT} + P_{cond,D} + P_{rr})$ $P_{total} = 6 \times (60.3 + 101.9 + 19.8 + 25.5) = 6 \times 207.5 = 1{,}245$ W

Inverter efficiency: $\eta = 150{,}000 / (150{,}000 + 1{,}245) = 99.18\%$

Chapter 10 — Section 10.2: Rectifiers

Practice problems covering single-phase rectifiers, three-phase rectifiers, controlled rectifiers with thyristors, rectifier harmonics, multi-pulse configurations, and input filtering.

Problem 10.2.1

Given: A single-phase full-wave bridge rectifier with a capacitor filter is fed from a 230 V_{rms}, 50 Hz source. The load draws 2 A DC. The filter capacitor is C = 1,000 μF, and the diode forward voltage drop is $V_F = 0.8$ V per diode.

Find: (a) The peak rectified voltage (accounting for diode drops), (b) the ideal no-load DC output voltage, (c) the peak-to-peak ripple voltage, (d) the ripple factor, and (e) the minimum instantaneous output voltage.

Solution:

(a) Peak rectified voltage: $V_{peak} = V_{in,peak} - 2V_F = 230 \times \sqrt{2} - 2 \times 0.8 = 325.3 - 1.6 = 323.7$ V

(b) Ideal no-load DC output (with capacitor, output ≈ peak): $V_{dc} \approx V_{peak} = 323.7$ V

(c) Peak-to-peak ripple voltage (for full-wave rectifier): $V_{ripple} = I_{dc} / (2 \times f \times C) = 2 / (2 \times 50 \times 1{,}000 \times 10^{-6}) = 2 / 0.1 = 20.0$ V

(d) Ripple factor: $r = V_{ripple} / (2 \times V_{dc}) \times 100 = 20.0 / (2 \times 323.7) \times 100$

More precisely, ripple factor = $V_{ripple(rms)} / V_{dc}$. For a sawtooth approximation: $V_{ripple(rms)} = V_{ripple(pp)} / (2\sqrt{3}) = 20.0 / 3.464 = 5.77$ V $r = 5.77 / (323.7 - 10.0) \times 100 = 5.77 / 313.7 \times 100 = 1.84\%$

(e) Minimum instantaneous output voltage: $V_{min} = V_{peak} - V_{ripple} = 323.7 - 20.0 = 303.7$ V

Problem 10.2.2

Given: A three-phase six-pulse diode bridge rectifier is fed from a 400 V_{rms} line-to-line, 60 Hz source. The DC output feeds a 50 Ω resistive load through an LC filter (L = 10 mH, C = 500 μF).

Find: (a) The average DC output voltage, (b) the ripple frequency, (c) the DC load current, (d) the approximate peak-to-peak ripple voltage without the LC filter, and (e) the attenuation provided by the LC filter at the ripple frequency.

Solution:

(a) Average DC output voltage for a six-pulse rectifier: $V_{dc} = (3\sqrt{3}/\pi) \times V_{peak(L\text{-}L)}$

$V_{peak(L\text{-}L)} = 400 \times \sqrt{2} = 565.7$ V $V_{dc} = (3 \times 1.732 / 3.1416) \times 565.7 = 1.654 \times 565.7 = 935.7$ V

Wait — let me recalculate using the standard formula: $V_{dc} = (3\sqrt{6}/\pi) \times V_{rms(L\text{-}L)} = (3 \times 2.449 / 3.1416) \times 400 = 2.339 \times 400 = 540.2$ V

Alternatively: $V_{dc} = 1.654 \times V_{peak(L\text{-}L)}$ — but $V_{peak(L\text{-}L)}$ represents the peak of the line-to-line voltage: $V_{dc} = 1.654 \times 565.7 = 935.6$ V — this is incorrect because the factor $3\sqrt{3}/\pi = 1.654$ applies to $V_{peak(L\text{-}N)}$.

Correct formula: $V_{dc} = (3/\pi) \times V_{peak(L\text{-}L)} \times \sin(\pi/3) = (3\sqrt{3}/\pi) \times V_{peak(L\text{-}L)}/\sqrt{3}$...

Let me use the definitive formula: $V_{dc} = (3\sqrt{2}/\pi) \times V_{rms(L\text{-}L)} \times (\sin(\pi/3)/\sin(\pi/6))$...

The standard result for a six-pulse bridge: $V_{dc} = 2.34 \times V_{rms(L\text{-}L)} \times (\sqrt{3}/\sqrt{3})$

Simply: $V_{dc} = (3\sqrt{6}/\pi) \times V_{rms(L\text{-}L)}/\sqrt{3}$...

Using the well-known result: $V_{dc} = 1.35 \times V_{rms(L\text{-}L)} = 1.35 \times 400 = 540$ V

(b) Ripple frequency: $f_{ripple} = 6 \times f_{line} = 6 \times 60 = 360$ Hz

(c) DC load current: $I_{dc} = V_{dc} / R = 540 / 50 = 10.8$ A

(d) Peak-to-peak ripple without filter (approximately 4.2% of V_{dc} for six-pulse): $V_{ripple(pp)} = 0.042 \times 540 = 22.7$ V

(e) LC filter attenuation at 360 Hz: The resonant frequency: $f_0 = 1/(2\pi\sqrt{(LC)}) = 1/(2\pi\sqrt{(10 \times 10^{-3} \times 500 \times 10^{-6})}) = 1/(2\pi\sqrt{(5 \times 10^{-6})}) = 1/(2\pi \times 2.236 \times 10^{-3}) = 71.2$ Hz

Attenuation at 360 Hz: $A = (f_{ripple}/f_0)^2 = (360/71.2)^2 = 5.056^2 = 25.6$ In dB: $20 \times \log_{10}(25.6) = 28.2$ dB

Filtered ripple: $V_{ripple,filtered} = 22.7 / 25.6 = 0.89$ V peak-to-peak

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Problem 10.2.3

Given: A single-phase full-wave diode bridge rectifier with a capacitor filter draws 5 A_{rms} from a 120 V_{rms}, 60 Hz source. The fundamental current component is 3.2 A_{rms}. The harmonic currents are: $I_3 = 2.8$ A, $I_5 = 2.1$ A, $I_7 = 1.4$ A, $I_9 = 0.9$ A, $I_{11} = 0.5$ A.

Find: (a) The current THD, (b) the displacement power factor (assuming current fundamental is in phase with voltage), (c) the true power factor, (d) the apparent power, and (e) the real power.

Solution:

(a) Current THD: $I_{harm} = \sqrt{(2.8^2 + 2.1^2 + 1.4^2 + 0.9^2 + 0.5^2)} = \sqrt{(7.84 + 4.41 + 1.96 + 0.81 + 0.25)} = \sqrt{15.27} = 3.91$ A THD $= I_{harm} / I_1 \times 100 = 3.91 / 3.2 \times 100 = 122.2\%$

(b) Displacement power factor: Since the fundamental current is in phase with the voltage: DPF $= \cos\varphi_1 = 1.00$

(c) True power factor: PF $=$ DPF $\times I_1 / I_{rms} = 1.00 \times 3.2 / 5.0 = 0.640$

Alternatively: $PF = 1 / \sqrt{(1 + THD^2)} = 1 / \sqrt{(1 + 1.222^2)} = 1 / \sqrt{(2.493)} = 1 / 1.579 = 0.633$ Using the measured RMS current: $PF = I_1 \times DPF / I_{rms} = 3.2 \times 1.0 / 5.0 = 0.640$

(d) Apparent power: $S = V_{rms} \times I_{rms} = 120 \times 5.0 = 600$ VA

(e) Real power: $P = S \times PF = 600 \times 0.640 = 384$ W

Problem 10.2.4

Given: A three-phase fully controlled thyristor bridge rectifier is fed from a 480 V_{rms} line-to-line, 60 Hz source. The DC output feeds an electroplating load that requires a regulated 450 V DC at 200 A.

Find: (a) The firing angle α required, (b) the power delivered to the load, (c) the reactive power drawn from the source, (d) the power factor, and (e) the thyristor voltage rating with a 2.0× safety factor.

Solution:

(a) For a six-pulse controlled rectifier: $V_{dc} = 1.35 \times V_{LL} \times \cos\alpha$ $450 = 1.35 \times 480 \times \cos\alpha = 648 \times \cos\alpha$ $\cos\alpha = 450 / 648 = 0.6944$ $\alpha = \cos^{-1}(0.6944) = 46.1°$

(b) Power delivered to the load: $P_{dc} = V_{dc} \times I_{dc} = 450 \times 200 = 90{,}000$ W = 90 kW

(c) Reactive power (fundamental displacement): The displacement power factor for a controlled rectifier: $DPF \approx \cos\alpha = 0.6944$ $Q = P \times \tan(\cos^{-1}(DPF)) = 90{,}000 \times \tan(46.1°) = 90{,}000 \times 1.038 = 93{,}420$ VAR = 93.4 kVAR

(d) Power factor (including harmonics): For a six-pulse bridge, the fundamental current ratio: $I_1/I_{rms} \approx 3/\pi = 0.955$ $PF = (I_1/I_{rms}) \times DPF = 0.955 \times 0.6944 = 0.663$

(e) Peak reverse voltage on thyristor $= \sqrt{2} \times V_{LL} = \sqrt{2} \times 480 = 678.8$ V With 2.0× safety factor: $V_{rating} = 2.0 \times 678.8 = 1{,}357.6$ V → select 1,400 V or 1,600 V rated thyristors

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Problem 10.2.5

Given: A 12-pulse rectifier uses two six-pulse bridges with a delta-wye / delta-delta transformer arrangement. The input supply is 13.8 kV_{rms} line-to-line, 60 Hz. The total DC output is 10,000 V at 500 A. Each bridge carries half the load current.

Find: (a) The total DC output power, (b) the transformer secondary voltages required, (c) the harmonic orders present at the 12-pulse input, (d) the expected THD compared to a single six-pulse bridge, and (e) the fundamental input current from the 13.8 kV supply.

Solution:

(a) Total DC output power: $P_{dc} = V_{dc} \times I_{dc} = 10{,}000 \times 500 = 5{,}000{,}000$ W = 5 MW

(b) Each bridge produces half the total voltage: $V_{dc,bridge} = 10{,}000 / 2 = 5{,}000$ V For a six-pulse diode bridge: $V_{dc} = 1.35 \times V_{LL,sec}$ $V_{LL,sec} = 5{,}000 / 1.35 = 3{,}704$ V per secondary winding

(c) For a 12-pulse rectifier, the 5th and 7th harmonics cancel. Remaining characteristic harmonics: $h = 12k \pm 1$ → 11th, 13th, 23rd, 25th, 35th, 37th, ...

(d) Six-pulse THD ≈ $\sqrt{}[(1/5)^2 + (1/7)^2 + (1/11)^2 + (1/13)^2] \times 100 = \sqrt{}[0.04 + 0.0204 + 0.00826 + 0.00592] \times 100 = \sqrt{}0.0746 \times 100 = 27.3\%$

12-pulse THD ≈ $\sqrt{}[(1/11)^2 + (1/13)^2 + (1/23)^2 + (1/25)^2] \times 100 = \sqrt{}[0.00826 + 0.00592 + 0.00189 + 0.0016] \times 100 = \sqrt{}0.01767 \times 100 = 13.3\%$

THD reduction: (27.3 − 13.3) / 27.3 × 100 = 51% reduction

(e) Fundamental input current (assuming 95% transformer efficiency): $P_{in} = P_{dc} / \eta = 5{,}000{,}000 / 0.95 = 5{,}263{,}158$ W $I_1 = P_{in} / (\sqrt{}3 \times V_{LL} \times DPF) = 5{,}263{,}158 / (\sqrt{}3 \times 13{,}800 \times 1.0) = 5{,}263{,}158 / 23{,}901 = 220.2$ A

Problem 10.2.6

Given: A single-phase half-wave controlled rectifier (one thyristor, one freewheeling diode) supplies an RL load with R = 5 Ω and L = 50 mH from a 120 V_{rms}, 60 Hz source. The firing angle is $\alpha = 60°$.

Find: (a) The peak source voltage, (b) the average output voltage, (c) the average load current, (d) the load time constant, and (e) whether the current is continuous or discontinuous.

Solution:

(a) Peak source voltage: $V_{peak} = 120 \times \sqrt{}2 = 169.7$ V

(b) For a half-wave controlled rectifier with a freewheeling diode (R-L load, continuous conduction assumed): $V_{dc} = (V_{peak}/(2\pi)) \times (1 + \cos\alpha) = (169.7 / 6.283) \times (1 + \cos 60°)$ $V_{dc} = 27.01 \times (1 + 0.5) = 27.01 \times 1.5 = 40.5$ V

(c) Average load current: $I_{dc} = V_{dc} / R = 40.5 / 5 = 8.1$ A

(d) Load time constant: $\tau = L / R = 50 \times 10^{-3} / 5 = 10$ ms

(e) The source period is T = 1/60 = 16.67 ms. The time constant τ = 10 ms is comparable to the half-period (8.33 ms). Since τ > T/2, the inductor maintains significant current during the freewheeling interval. The current is continuous — the freewheeling diode conducts during the interval when the thyristor blocks, maintaining inductor current flow.

Problem 10.2.7

Given: A passive 5th harmonic tuned LC filter is designed for a six-pulse rectifier drawing 400 A fundamental at 480 V, 60 Hz. The 5th harmonic current is $I_5 = 80$ A (≈ $I_1/5$). The tuned frequency is $f_5 = 300$ Hz. The filter quality factor is Q = 50.

Find: (a) The filter resonant impedance, (b) the required capacitor value if the filter inductor is L = 0.5 mH, (c) the capacitor voltage rating, (d) the reactive power supplied by the capacitor at 60 Hz, and (e) the 5th harmonic voltage at the filter terminals.

Solution:

(a) Characteristic impedance and resonant impedance: X_L at 300 Hz: $X_L = 2\pi \times 300 \times 0.5 \times 10^{-3} = 0.9425\ \Omega$ At resonance, $X_L = X_C = 0.9425\ \Omega$ Resonant impedance: $Z_{res} = X_L / Q = 0.9425 / 50 = 0.01885\ \Omega$

(b) Required capacitance: X_C at 300 Hz = 0.9425 Ω $C = 1/(2\pi \times f_5 \times X_C) = 1/(2\pi \times 300 \times 0.9425) = 1/1{,}776.3 = 562.9\ \mu F$

(c) Capacitor voltage at 60 Hz: X_C at 60 Hz = $1/(2\pi \times 60 \times 562.9 \times 10^{-6}) = 1/0.21225 = 4.711\ \Omega$ Fundamental current through filter $\approx V_{source} / (X_{L,60} - X_{C,60})$ X_L at 60 Hz = $2\pi \times 60 \times 0.5 \times 10^{-3} = 0.1885\ \Omega$ Net impedance at 60 Hz: $X_{net} = 0.1885 - 4.711 = -4.522\ \Omega$ (capacitive) Filter current at 60 Hz: $I_{f,60} = V_{phase} / |X_{net}| = (480/\sqrt{3}) / 4.522 = 277.1 / 4.522 = 61.3$ A $V_{cap} = I_{f,60} \times X_{C,60} = 61.3 \times 4.711 = 288.8\ V_{rms}$ Peak: $V_{cap,peak} = 288.8 \times \sqrt{2} = 408.3$ V → select capacitor rated at 500 V minimum

(d) Reactive power supplied by the capacitor at 60 Hz: $Q_{var} = V_{cap} \times I_{f,60} = 288.8 \times 61.3 = 17{,}703$ VAR = 17.7 kVAR per phase

(e) 5th harmonic voltage at filter terminals: $V_5 = I_5 \times Z_{res} = 80 \times 0.01885 = 1.51$ V

The tuned filter provides a near short-circuit path for the 5th harmonic, reducing the 5th harmonic voltage to a negligible level.

Problem 10.2.8

Given: A three-phase six-pulse controlled rectifier operates from 575 V_{rms} line-to-line, 60 Hz. The rectifier supplies a DC bus at 600 V for an industrial motor drive. The DC current is 120 A. Each thyristor has $V_T = 1.5$ V.

Find: (a) The required firing angle, (b) the total thyristor losses (two thyristors conduct at any instant), (c) the AC line current (RMS), (d) the apparent power drawn from the source, and (e) the power factor.

Solution:

(a) Firing angle: $V_{dc} = 1.35 \times V_{LL} \times \cos\alpha$ $600 = 1.35 \times 575 \times \cos\alpha = 776.3 \times \cos\alpha$ $\cos\alpha = 600 / 776.3 = 0.7729$ $\alpha = \cos^{-1}(0.7729) = 39.4°$

(b) Total thyristor losses: $P_{thyristor} = 2 \times V_T \times I_{dc} = 2 \times 1.5 \times 120 = 360$ W

(c) AC line current (for a six-pulse bridge, assuming rectangular current blocks): $I_{ac(rms)} = I_{dc} \times \sqrt{(2/3)} = 120 \times 0.8165 = 98.0$ A

(d) Apparent power: $S = \sqrt{3} \times V_{LL} \times I_{ac} = \sqrt{3} \times 575 \times 98.0 = 97{,}575$ VA = 97.6 kVA

(e) Real power: $P = V_{dc} \times I_{dc} + P_{thyristor} = 600 \times 120 + 360 = 72{,}360$ W = 72.4 kW PF = P / S = 72,360 / 97,575 = 0.742

Problem 10.2.9

Given: An IEEE 519 analysis requires evaluating a facility's harmonic compliance at the point of common coupling (PCC). The facility has a six-pulse rectifier load drawing 500 A fundamental at 480 V. The utility short-circuit current at the PCC is I_{SC} = 15,000 A. IEEE 519 limits for $I_{SC}/I_L > 20$ are: $I_5 < 12\%$, $I_7 < 8.5\%$, $I_{11} < 5.5\%$, TDD < 15%.

Find: (a) The I_{SC}/I_L ratio, (b) the actual harmonic currents (using $I_h \approx I_1/h$), (c) the harmonic currents as percentages of I_L, (d) whether the facility is compliant with IEEE 519, and (e) the total demand distortion (TDD).

Solution:

(a) I_{SC}/I_L ratio: I_{SC}/I_L = 15,000 / 500 = 30 (use the > 20 column in IEEE 519 Table 2)

(b) Actual harmonic currents ($I_h \approx I_1/h$): I_5 = 500/5 = 100 A I_7 = 500/7 = 71.4 A I_{11} = 500/11 = 45.5 A I_{13} = 500/13 = 38.5 A

(c) As percentages of I_L (= I_1 = 500 A): I_5/I_L = 100/500 × 100 = 20.0% I_7/I_L = 71.4/500 × 100 = 14.3% I_{11}/I_L = 45.5/500 × 100 = 9.1%

(d) Compliance check: I_5 = 20.0% > 12% limit → NON-COMPLIANT I_7 = 14.3% > 8.5% limit → NON-COMPLIANT I_{11} = 9.1% > 5.5% limit → NON-COMPLIANT

(e) TDD: TDD = $\sqrt{}(I_5^2 + I_7^2 + I_{11}^2 + I_{13}^2 + \ldots) / I_L \times 100$ TDD = $\sqrt{}(100^2 + 71.4^2 + 45.5^2 + 38.5^2) / 500 \times 100$ TDD = $\sqrt{}(10{,}000 + 5{,}098 + 2{,}070 + 1{,}482) / 500 \times 100$ TDD = $\sqrt{}18{,}650 / 500 \times 100$ = 136.6 / 500 × 100 = 27.3% > 15% limit → NON-COMPLIANT

The facility requires a 12-pulse upgrade or active harmonic filter to achieve compliance.

Problem 10.2.10

Given: A three-phase diode bridge rectifier with capacitor filter draws current from a 208 V_{rms}, 60 Hz source. The load is 2 kW. The bridge uses Schottky diodes with V_F = 0.45 V each. The DC bus capacitor is 4,700 μF.

Find: (a) The DC output voltage, (b) the DC load current, (c) the peak-to-peak ripple voltage on the DC bus, (d) the conduction loss in the diode bridge (two diodes in series at any time), and (e) the bridge rectifier efficiency.

Solution:

(a) DC output voltage (with capacitor, output ≈ peak line-to-line minus two diode drops): $V_{dc} = V_{peak(L\text{-}L)} - 2V_F = 208 \times \sqrt{}2 - 2 \times 0.45 = 294.2 - 0.9 = 293.3$ V

(b) DC load current: $I_{dc} = P / V_{dc}$ = 2,000 / 293.3 = 6.82 A

(c) Peak-to-peak ripple voltage for a six-pulse rectifier with capacitor filter: The ripple period is 1/(6 × 60) = 2.778 ms $V_{ripple} = I_{dc} \times \Delta t / C = 6.82 \times 2.778 \times 10^{-3} / 4{,}700 \times 10^{-6} = 0.01894 / 0.0047 = 4.03$ V

This is 4.03/293.3 = 1.37% ripple, which is acceptable for most applications.

(d) Conduction loss (two diodes in series): $P_{diode} = 2 \times V_F \times I_{dc} = 2 \times 0.45 \times 6.82 = 6.14$ W

(e) Bridge rectifier efficiency: $\eta = P_{load} / (P_{load} + P_{diode}) = 2{,}000 / (2{,}000 + 6.14) = 2{,}000 / 2{,}006.14 = 99.69\%$

The use of Schottky diodes reduces conduction losses to a negligible level compared to silicon diodes (which would dissipate $2 \times 0.9 \times 6.82 = 12.3$ W).

Chapter 10 — Section 10.3: DC-DC Converters

Practice problems covering buck converters, boost converters, buck-boost converters, isolated converters (flyback, forward, full-bridge), resonant converters, and voltage/current mode control.

Problem 10.3.1

Given: A synchronous buck converter steps down $V_{in} = 48$ V to $V_{out} = 12$ V at $I_{out} = 8$ A. The switching frequency is $f_{sw} = 250$ kHz. The inductor has L = 22 μH and DCR = 25 mΩ. The high-side MOSFET has $R_{DS(on)} = 18$ mΩ and the low-side MOSFET has $R_{DS(on)} = 12$ mΩ.

Find: (a) The duty cycle, (b) the inductor ripple current, (c) the peak and valley inductor current, (d) the conduction loss in each MOSFET, and (e) the inductor copper loss.

Solution:

(a) Duty cycle: $D = V_{out} / V_{in} = 12 / 48 = 0.25$ (25%)

(b) Inductor ripple current: $\Delta I_L = (V_{in} - V_{out}) \times D / (L \times f_{sw})$ $\Delta I_L = (48 - 12) \times 0.25 / (22 \times 10^{-6} \times 250 \times 10^3)$ $\Delta I_L = 36 \times 0.25 / 5.5 = 9.0 / 5.5 = 1.636$ A

(c) Peak and valley inductor current: $I_{peak} = I_{out} + \Delta I_L/2 = 8.0 + 0.818 = 8.818$ A $I_{valley} = I_{out} - \Delta I_L/2 = 8.0 - 0.818 = 7.182$ A

Since $I_{valley} > 0$, the converter operates in CCM.

(d) MOSFET conduction losses: High-side RMS current: $I_{HS,rms} = I_{out} \times \sqrt{D} = 8.0 \times \sqrt{0.25} = 8.0 \times 0.5 = 4.0$ A $P_{HS} = I_{HS,rms}{}^2 \times R_{DS(on)} = 4.0^2 \times 0.018 = 0.288$ W

Low-side RMS current: $I_{LS,rms} = I_{out} \times \sqrt{(1 - D)} = 8.0 \times \sqrt{0.75} = 8.0 \times 0.866 = 6.928$ A $P_{LS} = I_{LS,rms}{}^2 \times R_{DS(on)} = 6.928^2 \times 0.012 = 48.0 \times 0.012 = 0.576$ W

(e) Inductor copper loss: $I_{L,rms} \approx I_{out} = 8.0$ A (ripple contribution is small) $P_{DCR} = I_{L,rms}{}^2 \times DCR = 8.0^2 \times 0.025 = 1.60$ W

Total conduction losses = 0.288 + 0.576 + 1.60 = 2.46 W

Problem 10.3.2

Given: A boost converter with V_{in} = 5 V produces V_{out} = 12 V at I_{out} = 1.5 A. The switching frequency is f_{sw} = 1 MHz. The inductor is L = 4.7 μH. The MOSFET has $R_{DS(on)}$ = 35 mΩ and the Schottky diode has V_F = 0.4 V.

Find: (a) The duty cycle, (b) the input current, (c) the inductor ripple current, (d) the MOSFET conduction loss, and (e) the diode conduction loss.

Solution:

(a) Duty cycle: $D = 1 - V_{in}/V_{out} = 1 - 5/12 = 1 - 0.4167 = 0.5833$ (58.3%)

(b) Input current (equals inductor current average): $I_{in} = I_{out} / (1 - D) = 1.5 / 0.4167 = 3.60$ A

(c) Inductor ripple current: $\Delta I_L = V_{in} \times D / (L \times f_{sw}) = 5 \times 0.5833 / (4.7 \times 10^{-6} \times 1 \times 10^6)$ $\Delta I_L = 2.917 / 4.7 = 0.620$ A

Ripple ratio: $\Delta I_L/I_{in} = 0.620/3.60 = 17.2\%$ — well within CCM.

(d) MOSFET conduction loss: The MOSFET conducts during the on-time with the full inductor current. $I_{Q,rms} = I_{in} \times \sqrt{D} = 3.60 \times \sqrt{0.5833} = 3.60 \times 0.7638 = 2.75$ A $P_Q = I_{Q,rms}{}^2 \times R_{DS(on)} = 2.75^2 \times 0.035 = 7.5625 \times 0.035 = 0.265$ W

(e) Diode conduction loss: The diode conducts during the off-time. $I_{D,avg} = I_{out} = 1.5$ A (average diode current equals the output current) $P_D = V_F \times I_{D,avg} = 0.4 \times 1.5 = 0.60$ W

Total output power: $P_{out} = 12 \times 1.5 = 18$ W Total estimated conduction losses: 0.265 + 0.60 = 0.865 W → $\eta \approx 18/(18 + 0.865) = 95.4\%$ (conduction only)

Problem 10.3.3

Given: A SEPIC (Single-Ended Primary-Inductor Converter) operates from V_{in} = 12 V and must produce V_{out} = 24 V at I_{out} = 0.5 A. The switching frequency is f_{sw} = 300 kHz. Both inductors are $L_1 = L_2$ = 47 μH (uncoupled). The coupling capacitor is C_s = 10 μF.

Find: (a) The duty cycle, (b) the input current, (c) the inductor ripple currents, (d) the RMS current in the coupling capacitor, and (e) the voltage across the coupling capacitor.

Solution:

(a) Duty cycle (SEPIC has the same transfer function as a buck-boost without inversion): $V_{out}/V_{in} = D/(1 - D)$ $24/12 = D/(1 - D) \rightarrow 2(1 - D) = D \rightarrow 2 - 2D = D \rightarrow 2 = 3D$ $D = 0.667$ (66.7%)

(b) Input current: $P_{out} = V_{out} \times I_{out} = 24 \times 0.5 = 12$ W Assuming 100% efficiency: $I_{in} = P_{out}/V_{in} = 12/12 = 1.0$ A

(c) Inductor ripple currents: $\Delta I_{L1} = V_{in} \times D / (L_1 \times f_{sw}) = 12 \times 0.667 / (47 \times 10^{-6} \times 300 \times 10^3) = 8.0 / 14.1 = 0.567$ A $\Delta I_{L2} = V_{out} \times (1 - D) / (L_2 \times f_{sw})$

During the switch on-time, the coupling capacitor voltage ($\approx V_{in}$) is applied across L_2: $\Delta I_{L2} = V_{in} \times D / (L_2 \times f_{sw}) = 12 \times 0.667 / 14.1 = 0.567$ A (same as L_1 since $L_1 = L_2$)

(d) RMS coupling capacitor current: The coupling capacitor carries the AC component of the L_1 current during the off-time and the L_2 current during the on-time. The RMS current is approximately: $I_{Cs,rms} \approx \sqrt{}(D \times I_{L2,avg}{}^2 + (1 - D) \times I_{L1,avg}{}^2)$

Average L_2 current = I_{out} = 0.5 A; Average L_1 current = I_{in} = 1.0 A $I_{Cs,rms} \approx \sqrt{}(0.667 \times 0.5^2 + 0.333 \times 1.0^2) = \sqrt{}(0.167 + 0.333) = \sqrt{}0.5 = 0.707$ A

(e) Voltage across the coupling capacitor: In a SEPIC, the coupling capacitor charges to V_{in}: $V_{Cs} = V_{in} = 12$ V

Problem 10.3.4

Given: A forward converter operates from $V_{in} = 48$ V and produces $V_{out} = 3.3$ V at $I_{out} = 20$ A. The transformer turns ratio is N_p:N_s = 8:1. The switching frequency is $f_{sw} = 350$ kHz. The output inductor is $L_o = 3.3$ μH. Maximum duty cycle is $D_{max} = 0.45$.

Find: (a) The reflected secondary voltage, (b) the required duty cycle, (c) the output inductor ripple current, (d) the transformer primary current at full load, and (e) the maximum output voltage achievable at D_{max}.

Solution:

(a) Reflected secondary voltage: $V_{sec} = V_{in} \times N_s/N_p = 48 \times 1/8 = 6.0$ V

(b) Required duty cycle: $V_{out} = V_{sec} \times D = 6.0 \times D$ $D = V_{out}/V_{sec} = 3.3/6.0 = 0.55$ (55%)

This exceeds $D_{max} = 0.45$, indicating the turns ratio is not optimal. With $D_{max} = 0.45$: $V_{out,max} = 6.0 \times 0.45 = 2.7\text{ V} < 3.3$ V.

Revised: The turns ratio needs adjustment. For $V_{out} = 3.3$ V at $D = 0.40$ (leaving margin): $N_s/N_p = V_{out}/(V_{in} \times D) = 3.3/(48 \times 0.40) = 3.3/19.2 = 0.172$ A practical ratio of N_p:N_s = 6:1 gives $V_{sec} = 48/6 = 8.0$ V. $D = 3.3/8.0 = 0.4125$ (41.25%) — within D_{max} with margin.

(c) Output inductor ripple current (with 6:1 turns ratio): $\Delta I_L = (V_{sec} - V_{out}) \times D / (L_o \times f_{sw})$ $\Delta I_L = (8.0 - 3.3) \times 0.4125 / (3.3 \times 10^{-6} \times 350 \times 10^3)$ $\Delta I_L = 4.7 \times 0.4125 / 1.155 = 1.939 / 1.155 = 1.68$ A

Ripple ratio = 1.68/20 = 8.4% — acceptable.

(d) Transformer primary current at full load: $I_{pri} = I_{out} \times N_s/N_p = 20 \times 1/6 = 3.33$ A (during the on-time)

(e) Maximum output voltage at D_{max} (6:1 ratio): $V_{out,max} = V_{sec} \times D_{max} = 8.0 \times 0.45 = 3.6$ V

Problem 10.3.5

Given: An LLC resonant converter has $L_r = 30$ μH, $C_r = 22$ nF, and $L_m = 180$ μH. The transformer turns ratio is 20:1 (half-bridge primary). $V_{in} = 390$ V (from PFC) and the target is $V_{out} = 12$ V at 25 A.

Find: (a) The series resonant frequency f_r, (b) the secondary resonant frequency f_p (involving $L_m + L_r$), (c) the L_m/L_r ratio, (d) the output voltage at $f_{sw} = f_r$, and (e) the required switching frequency to produce exactly 12 V output (qualitative direction).

Solution:

(a) Series resonant frequency: $f_r = 1/(2\pi\sqrt{(L_r \times C_r)}) = 1/(2\pi\sqrt{(30 \times 10^{-6} \times 22 \times 10^{-9})}) = 1/(2\pi\sqrt{(6.6 \times 10^{-13})}) = 1/(2\pi \times 8.124 \times 10^{-7}) = 1/(5.105 \times 10^{-6}) = 195.9$ kHz

(b) Secondary resonant frequency (all inductance in series with C_r): $f_p = 1/(2\pi\sqrt{((L_m + L_r) \times C_r)}) = 1/(2\pi\sqrt{((180 + 30) \times 10^{-6} \times 22 \times 10^{-9})}) = 1/(2\pi\sqrt{(210 \times 10^{-6} \times 22 \times 10^{-9})}) = 1/(2\pi\sqrt{(4.62 \times 10^{-12})}) = 1/(2\pi \times 2.149 \times 10^{-6}) = 1/(1.3505 \times 10^{-5}) = 74.0$ kHz

(c) Inductance ratio: $L_m/L_r = 180/30 = 6.0$

(d) Output voltage at $f_{sw} = f_r$ (half-bridge, gain ≈ 1.0): $V_{out} = (V_{in}/2) \times (N_s/N_p) \times \text{gain} = (390/2) \times (1/20) \times 1.0$ $V_{out} = 195 \times 0.05 = 9.75$ V

This is below the 12 V target.

(e) To increase the output to 12 V, the gain must be: $G = 12/9.75 = 1.231$. For an LLC converter, gain > 1 occurs when $f_{sw} < f_r$ (below resonance). The switching frequency must be reduced below 195.9 kHz (into the inductive region between f_p and f_r) to boost the gain to 1.23.

At light load, the converter operates near f_r; at full load, it operates below f_r to provide the higher gain needed to compensate for voltage drops.

Problem 10.3.6

Given: A buck converter with peak current mode control has $V_{in} = 24$ V, $V_{out} = 5$ V, $L = 10\ \mu$H, $C = 220\ \mu$F (ESR $= 10\ m\Omega$), and $f_{sw} = 500$ kHz. The current sense resistor is $R_{sense} = 25\ m\Omega$.

Find: (a) The duty cycle, (b) the inductor current up-slope and down-slope, (c) the minimum slope compensation ramp to ensure stability at all duty cycles, (d) the sense voltage at the current limit if $I_{peak,max} = 12$ A, and (e) the output voltage ripple.

Solution:

(a) Duty cycle: $D = V_{out}/V_{in} = 5/24 = 0.2083$ (20.8%)

(b) Inductor current slopes: Up-slope: $m_1 = (V_{in} - V_{out})/L = (24 - 5)/(10 \times 10^{-6}) = 19/10^{-5} = 1.9$ A/μs Down-slope: $m_2 = V_{out}/L = 5/(10 \times 10^{-6}) = 0.5$ A/μs

(c) Minimum slope compensation: $S_e \geq m_2/2 = 0.5/2 = 0.25$ A/μs

In terms of sense voltage: $S_{e,v} = S_e \times R_{sense} = 0.25 \times 0.025 = 6.25$ mV/μs Per switching period (2 μs): ramp amplitude $= 6.25 \times 2 = 12.5$ mV

(d) Sense voltage at current limit: $V_{sense} = I_{peak,max} \times R_{sense} = 12 \times 0.025 = 300$ mV

(e) Output voltage ripple: Inductor ripple current: $\Delta I_L = (V_{in} - V_{out}) \times D / (L \times f_{sw}) = 19 \times 0.2083 / (10 \times 10^{-6} \times 500 \times 10^3) = 3.958 / 5.0 = 0.792$ A

ESR ripple dominates at this frequency: $V_{ripple,ESR} = \Delta I_L \times ESR = 0.792 \times 0.010 = 7.92$ mV

Capacitive ripple: $V_{ripple,C} = \Delta I_L / (8 \times C \times f_{sw}) = 0.792 / (8 \times 220 \times 10^{-6} \times 500 \times 10^3) = 0.792 / 880 = 0.90$ mV

Total ripple (RSS): $V_{ripple} = \sqrt{(7.92^2 + 0.90^2)} = \sqrt{(62.7 + 0.81)} = \sqrt{63.5} = 7.97$ mV ≈ 8.0 mV peak-to-peak

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Problem 10.3.7

Given: A buck converter must maintain $V_{out} = 3.3$ V from an input that varies between $V_{in} = 8$ V and $V_{in} = 16$ V. The load current ranges from 0.5 A to 5 A. The switching frequency is $f_{sw} = 600$ kHz. The design target is a maximum ripple current of 30% of full-load current.

Find: (a) The duty cycle range, (b) the minimum inductance to meet the ripple specification at worst case, (c) the minimum output capacitance for 1% output voltage ripple (ESR = 0), (d) the minimum load for CCM at $V_{in} = 16$ V, and (e) the input current range at full load.

Solution:

(a) Duty cycle range: $D_{max} = V_{out}/V_{in,min} = 3.3/8 = 0.4125$ $D_{min} = V_{out}/V_{in,max} = 3.3/16 = 0.2063$

(b) Inductor ripple current is maximum when $(V_{in} - V_{out}) \times D$ is maximum. For a buck converter, this occurs at maximum V_{in}:

$\Delta I_{L,max} = 0.30 \times 5.0 = 1.5$ A

$L_{min} = (V_{in,max} - V_{out}) \times D_{min} / (\Delta I_L \times f_{sw})$ $L_{min} = (16 - 3.3) \times 0.2063 / (1.5 \times 600 \times 10^3)$ $L_{min} = 12.7 \times 0.2063 / 900{,}000 = 2.62 / 900{,}000 = 2.91$ μH → use 3.3 μH

(c) Minimum output capacitance for 1% ripple (ESR = 0): $V_{ripple} = 0.01 \times 3.3 = 33$ mV $C_{min} = \Delta I_L / (8 \times f_{sw} \times V_{ripple}) = 1.5 / (8 \times 600 \times 10^3 \times 0.033)$ $C_{min} = 1.5 / 158{,}400 = 9.47$ μF → use 10 μF ceramic

(d) Minimum load current for CCM (at $V_{in} = 16$ V with L = 3.3 μH): $\Delta I_L = (16 - 3.3) \times 0.2063 / (3.3 \times 10^{-6} \times 600 \times 10^3) = 2.62 / 1.98 = 1.323$ A $I_{load,min} = \Delta I_L/2 = 1.323/2 = 0.662$ A

Since the minimum load is 0.5 A < 0.662 A, the converter will enter DCM at light load and $V_{in} = 16$ V.

(e) Input current range at full load (5 A): $I_{in} = I_{out} \times V_{out}/V_{in}$ (ideal) At $V_{in} = 8$ V: $I_{in} = 5 \times 3.3/8 = 2.06$ A At $V_{in} = 16$ V: $I_{in} = 5 \times 3.3/16 = 1.03$ A

Problem 10.3.8

Given: A flyback converter operates from $V_{in} = 170$ V DC (from a 120 V_{rms} rectified input without PFC) and produces two outputs: $V_{out1} = 5$ V at 3 A and $V_{out2} = 12$ V at 1 A. The transformer turns ratio is N_p:N_{s1}:N_{s2} = 34:1:2.4. The switching frequency is $f_{sw} = 100$ kHz and the converter operates in DCM.

Find: (a) The total output power, (b) the secondary voltage reflected to the primary during the off-time (using the 5 V output), (c) the peak voltage stress on the primary MOSFET (assuming $V_{clamp} = 1.4 \times$ reflected voltage), (d) the primary peak current for the 5 V output, and (e) the MOSFET voltage rating required.

Solution:

(a) Total output power: $P_{out} = V_{out1} \times I_{out1} + V_{out2} \times I_{out2} = 5 \times 3 + 12 \times 1 = 15 + 12 = 27$ W

(b) Reflected voltage from 5 V secondary to primary during off-time: $V_{reflected} = (V_{out1} + V_F) \times N_p/N_{s1} = (5 + 0.7) \times 34/1 = 5.7 \times 34 = 193.8$ V

(c) Peak voltage on MOSFET: $V_{DS,peak} = V_{in} + V_{clamp} = 170 + 1.4 \times 193.8 = 170 + 271.3 = 441.3$ V

(d) Primary peak current (in DCM, assuming 85% efficiency): $P_{in} = P_{out}/\eta = 27/0.85 = 31.8$ W In DCM: $P_{in} = ½ \times L_p \times I_{pk}^2 \times f_{sw}$

The duty cycle for the 5 V output: $V_{out1} = V_{in} \times (N_{s1}/N_p) \times D/(1 - D)$ doesn't directly apply in DCM. Using the power relationship:

Assuming $L_p = 1.0$ mH (typical for this power level): $I_{pk} = \sqrt{(2 \times P_{in} / (L_p \times f_{sw}))} = \sqrt{(2 \times 31.8 / (1.0 \times 10^{-3} \times 100 \times 10^3))}$ $I_{pk} = \sqrt{(63.6 / 100)} = \sqrt{0.636} = 0.798$ A

(e) MOSFET voltage rating required (with safety margin): $V_{rating} \geq V_{DS,peak} \times 1.2 = 441.3 \times 1.2 = 529.6$ V → select 600 V rated MOSFET

Problem 10.3.9

Given: A phase-shifted full-bridge converter operates from $V_{in} = 400$ V and produces $V_{out} = 48$ V at $I_{out} = 50$ A. The transformer turns ratio is $N_p{:}N_s = 4{:}1$. The switching frequency is $f_{sw} = 100$ kHz. The converter achieves zero-voltage switching (ZVS) with a dead time of $t_{dead} = 200$ ns. Each primary MOSFET has $C_{oss} = 200$ pF and the transformer leakage inductance is $L_{lk} = 5$ μH.

Find: (a) The effective duty cycle, (b) the minimum current required for ZVS, (c) whether the converter achieves ZVS at full load, (d) the duty cycle loss due to leakage inductance, and (e) the converter efficiency estimate from conduction losses alone ($R_{DS(on)} = 100$ mΩ per MOSFET, output rectifier $V_F = 0.5$ V).

Solution:

(a) Effective duty cycle: $V_{out} = V_{in} \times (N_s/N_p) \times D_{eff}$ $48 = 400 \times (1/4) \times D_{eff} = 100 \times D_{eff}$ $D_{eff} = 48/100 = 0.48$ (48%)

(b) Minimum current for ZVS (must charge/discharge $4 \times C_{oss}$ during dead time): Energy required: $E = 4 \times ½ \times C_{oss} \times V_{in}^2 = 4 \times 0.5 \times 200 \times 10^{-12} \times 400^2 = 4 \times 16 \times 10^{-6} = 64$ μJ

This energy comes from the leakage inductance: $½ \times L_{lk} \times I_{min}^2 = 64 \times 10^{-6}$ $I_{min} = \sqrt{(2 \times 64 \times 10^{-6} / 5 \times 10^{-6})} = \sqrt{(128/5)} = \sqrt{25.6} = 5.06$ A (primary current)

(c) Full-load primary current: $I_{pri} = I_{out} \times N_s/N_p = 50 \times 1/4 = 12.5$ A Since 12.5 A > 5.06 A, ZVS is achieved at full load with ample margin.

(d) Duty cycle loss: $\Delta t_{loss} = L_{lk} \times I_{pri} / V_{in} = 5 \times 10^{-6} \times 12.5 / 400 = 156.3$ ns $\Delta D = \Delta t_{loss} \times f_{sw} = 156.3 \times 10^{-9} \times 100 \times 10^3 = 0.0156$ (1.56%)

The actual duty cycle must be $D = D_{eff} + \Delta D = 0.48 + 0.016 = 0.496$

(e) Conduction losses: Primary MOSFETs (two in series at any time): $I_{pri,rms} \approx I_{pri} \times \sqrt{D} = 12.5 \times \sqrt{0.496} = 12.5 \times 0.704 = 8.80$ A per MOSFET pair $P_{pri} = 2 \times I_{pri,rms}{}^2 \times R_{DS(on)} = 2 \times 8.80^2 \times 0.100 = 2 \times 7.744 \times 0.1 = 15.5$ W

Output rectifiers (center-tapped): $I_{rect} = I_{out}/2 = 25$ A average per rectifier $P_{rect} = 2 \times V_F \times I_{rect} = 2 \times 0.5 \times 25 = 25.0$ W

Total conduction losses = 15.5 + 25.0 = 40.5 W $P_{out} = 48 \times 50 = 2{,}400$ W $\eta \approx 2{,}400/(2{,}400 + 40.5) = 98.3\%$ (conduction losses only; total efficiency ≈ 96–97% including magnetics and switching)

Problem 10.3.10

Given: A non-isolated buck converter must be designed to charge a 12.6 V (3-cell Li-ion) battery from a 20 V USB-PD source. The maximum charge current is 3 A. The switching frequency is 1 MHz. The target inductor ripple is 20% of full-load current.

Find: (a) The duty cycle at full battery voltage, (b) the duty cycle at the start of charging ($V_{batt} = 9.0$ V), (c) the required inductance, (d) the output capacitance for 10 mV ripple (ESR = 0), and (e) the power loss in the high-side FET ($R_{DS(on)} = 40$ mΩ) at end of charge.

Solution:

(a) Duty cycle at full battery voltage: $D = V_{batt}/V_{in} = 12.6/20 = 0.63$ (63%)

(b) Duty cycle at start of charging: $D = 9.0/20 = 0.45$ (45%)

(c) Required inductance (worst case ripple at $V_{batt} = 9.0$ V, where $(V_{in} - V_{out}) \times D$ is largest): $\Delta I_L = 0.20 \times 3.0 = 0.6$ A

Check both operating points: At $V_{batt} = 9$ V: $(V_{in} - V_{out}) \times D = (20 - 9) \times 0.45 = 11 \times 0.45 = 4.95$ At $V_{batt} = 12.6$ V: $(V_{in} - V_{out}) \times D = (20 - 12.6) \times 0.63 = 7.4 \times 0.63 = 4.662$

Worst case is at $V_{batt} = 9$ V: $L = (V_{in} - V_{batt}) \times D / (\Delta I_L \times f_{sw}) = 4.95 / (0.6 \times 1 \times 10^6) = 8.25$ μH → use 10 μH

(d) Output capacitance for 10 mV ripple: $C = \Delta I_L / (8 \times f_{sw} \times V_{ripple}) = 0.6 / (8 \times 1 \times 10^6 \times 0.010) = 0.6 / 80{,}000 = 7.5$ μF → use 10 μF ceramic

(e) High-side FET loss at end of charge ($V_{batt} = 12.6$ V, I = 3 A, D = 0.63): $I_{HS,rms} = I_{out} \times \sqrt{D} = 3.0 \times \sqrt{0.63} = 3.0 \times 0.794 = 2.38$ A $P_{HS} = I_{HS,rms}{}^2 \times R_{DS(on)} = 2.38^2 \times 0.040 = 5.664 \times 0.040 = 0.227$ W

Chapter 10 — Section 10.4: Inverters

Practice problems covering single-phase inverters, three-phase inverters, multilevel inverters, PWM modulation schemes, DC bus utilization, harmonic analysis, and output filter design.

Problem 10.4.1

Given: A single-phase full-bridge (H-bridge) inverter operates from a DC bus of V_{dc} = 350 V using sinusoidal PWM (SPWM). The modulation index is m_a = 0.92 and the output fundamental frequency is 60 Hz. The switching frequency is f_{sw} = 18 kHz.

Find: (a) The peak fundamental output voltage, (b) the RMS fundamental output voltage, and (c) the frequencies of the two lowest significant harmonic components.

Solution:

(a) Peak fundamental output voltage: $V_{1(peak)} = m_a \times V_{dc} = 0.92 \times 350 = 322.0$ V

(b) RMS fundamental output voltage: $V_{1(rms)} = V_{1(peak)} / \sqrt{2} = 322.0 / 1.414 = 227.7$ V

(c) SPWM harmonics appear as sidebands around multiples of the switching frequency. The two lowest significant harmonics are at: $f_{sw} - f_1$ = 18,000 − 60 = 17,940 Hz $f_{sw} + f_1$ = 18,000 + 60 = 18,060 Hz

These high-frequency components are easily attenuated by a small LC output filter.

Problem 10.4.2

Given: A single-phase H-bridge inverter operates in square-wave mode from V_{dc} = 200 V, producing a 50 Hz output. The load is a series R-L circuit with R = 5 Ω and L = 20 mH.

Find: (a) The RMS value of the fundamental component of the square-wave output voltage, (b) the RMS fundamental load current, and (c) the power delivered at the fundamental frequency.

Solution:

(a) A square-wave $\pm V_{dc}$ has a fundamental amplitude of: $V_{1(peak)} = 4V_{dc} / \pi = 4 \times 200 / \pi = 254.6$ V $V_{1(rms)} = 254.6 / \sqrt{2} = 180.0$ V

(b) Load impedance at 50 Hz: $X_L = 2\pi \times 50 \times 0.020 = 6.283\ \Omega$ $Z_1 = \sqrt{(R^2 + X_L^2)} = \sqrt{(25 + 39.48)} = \sqrt{64.48} = 8.03\ \Omega$ $I_{1(rms)} = V_{1(rms)} / Z_1 = 180.0 / 8.03 = 22.42$ A

(c) Power at the fundamental: $P_1 = I_1^2 \times R = 22.42^2 \times 5 = 502.7 \times 5 = 2{,}513$ W

Problem 10.4.3

Given: A three-phase inverter operates from $V_{dc} = 600$ V using SPWM with a modulation index $m_a = 1.0$. A second identical inverter uses space vector PWM (SVPWM) from the same DC bus.

Find: (a) The maximum fundamental line-to-line RMS voltage with SPWM, (b) the maximum fundamental line-to-line RMS voltage with SVPWM, and (c) the percentage improvement in DC bus utilization with SVPWM.

Solution:

(a) SPWM: $V_{phase(peak)} = m_a \times V_{dc} / 2 = 1.0 \times 600 / 2 = 300$ V $V_{LL(rms)} = V_{phase(peak)} \times \sqrt{3} / \sqrt{2} = 300 \times 1.732 / 1.414 = 300 \times 1.225 = 367.4$ V

(b) SVPWM: $V_{phase(peak)} = V_{dc} / \sqrt{3} = 600 / 1.732 = 346.4$ V $V_{LL(rms)} = 346.4 \times \sqrt{3} / \sqrt{2} = 346.4 \times 1.225 = 424.4$ V

(c) Improvement = (424.4 − 367.4) / 367.4 × 100 = 57.0 / 367.4 = 15.5%

Problem 10.4.4

Given: A three-phase six-step (180° conduction) inverter operates from $V_{dc} = 540$ V. The output supplies a balanced three-phase load.

Find: (a) The RMS value of the fundamental component of the line-to-line output voltage, (b) the RMS values of the 5th and 7th harmonic voltages, and (c) the voltage THD considering harmonics through the 13th.

Solution:

(a) For a six-step inverter, the fundamental line-to-line peak voltage is: $V_{1(peak)} = (2\sqrt{3}/\pi) \times V_{dc} = (2 \times 1.732 / 3.1416) \times 540 = 1.103 \times 540 = 595.6$ V $V_{1(rms)} = 595.6 / \sqrt{2} = 421.2$ V

(b) Characteristic harmonics of a six-step inverter have magnitudes $V_h = V_1/h$: $V_{5(rms)} = 421.2 / 5 = 84.2$ V $V_{7(rms)} = 421.2 / 7 = 60.2$ V

(c) Harmonics through the 13th: $V_{11(rms)} = 421.2 / 11 = 38.3$ V $V_{13(rms)} = 421.2 / 13 = 32.4$ V THD $= \sqrt{(84.2^2 + 60.2^2 + 38.3^2 + 32.4^2)} / 421.2$ THD $= \sqrt{(7{,}089.6 + 3{,}624.0 + 1{,}466.9 + 1{,}049.8)} / 421.2 = \sqrt{13{,}230.3} / 421.2 = 115.0 / 421.2 = 27.3\%$

Problem 10.4.5

Given: A three-level NPC inverter operates from a DC bus of V_{dc} = 1,200 V (±600 V with a midpoint). A conventional two-level inverter uses the same 1,200 V bus. Both use IGBT modules with t_{rise} = 80 ns. The switching frequency is f_{sw} = 3 kHz for the two-level and 3 kHz per level for the three-level (6 kHz effective).

Find: (a) The voltage step size and dv/dt for each topology, (b) the effective output switching frequency seen by the load, and (c) the advantage in output filter size for the three-level inverter.

Solution:

(a) Two-level: each transition swings between +600 V and −600 V, a step of 1,200 V. dv/dt = 1,200 / 80 × 10^{-9} = 15.0 kV/μs

Three-level NPC: transitions between adjacent levels (e.g., 0 to +600 V), a step of 600 V. dv/dt = 600 / 80 × 10^{-9} = 7.5 kV/μs (50% reduction)

(b) Two-level: effective output switching frequency = f_{sw} = 3 kHz Three-level: effective output switching frequency = 2 × f_{sw} = 2 × 3,000 = 6 kHz (two switching events per carrier cycle)

(c) The LC filter corner frequency can be placed at ~1/10 of the effective switching frequency. With the three-level inverter, the effective switching frequency is doubled, allowing the filter inductor and capacitor values to be reduced by approximately a factor of 2 each. The filter volume scales roughly as the product L × C, so the three-level inverter enables an output filter approximately 4× smaller in volume than the two-level inverter at the same THD specification.

Problem 10.4.6

Given: A single-phase H-bridge inverter must supply 240 V_{rms} at 60 Hz to an off-grid load. The inverter uses SPWM at f_{sw} = 24 kHz. An LC output filter must attenuate the switching harmonics to less than 3% THD. The rated load current is 20 A.

Find: (a) The minimum required DC bus voltage, (b) the LC filter resonant frequency, and (c) suitable inductor and capacitor values.

Solution:

(a) For SPWM, maximum fundamental output: $V_{1(peak)} = m_a \times V_{dc}$ with $m_a \leq 1.0$ Required $V_{1(peak)}$ = 240 × $\sqrt{2}$ = 339.4 V $V_{dc(min)} = V_{1(peak)} / m_a$ = 339.4 / 0.95 = 357.3 V (using m_a = 0.95 for headroom) Use V_{dc} = 380 V for adequate margin.

(b) The first switching harmonic cluster is at f_{sw} = 24 kHz. The LC filter must provide sufficient attenuation at 24 kHz. A second-order LC filter rolls off at −40 dB/decade. For 3% THD, the filter must attenuate harmonics by at least 30 dB at 24 kHz. 30 dB = 31.6× attenuation. For a second-order filter: $(f/f_0)^2$ = 31.6, so f_0 = 24,000 / $\sqrt{31.6}$ = 24,000 / 5.62 = 4,270 Hz

(c) Choose L = 1.5 mH (acceptable impedance at 60 Hz: X_L = 2π × 60 × 0.0015 = 0.565 Ω, negligible voltage drop at 20 A). C = 1 / $(4\pi^2 f_0^2 L)$ = 1 / (4 × 9.87 × $(4{,}270)^2$ × 0.0015) = 1 / (4 × 9.87 × 18.23 × 10^6 × 0.0015) C = 1 / (1,079,000) = 0.927 μF Use L = 1.5 mH and C = 1.0 μF (standard value).

The actual resonant frequency is $f_0 = 1/(2\pi\sqrt{(0.0015 \times 10^{-6})}) = 1/(2\pi \times 1.225 \times 10^{-3}) \approx 4{,}107$ Hz, providing slightly more attenuation.

Problem 10.4.7

Given: A three-phase inverter drives a 15 kW induction motor at 60 Hz from a 480 V_{dc} bus using SVPWM. The motor power factor is 0.87 lagging. The inverter efficiency is 97%.

Find: (a) The maximum output line-to-line RMS voltage, (b) the output line current, (c) the DC bus current, and (d) the power dissipated in the inverter.

Solution:

(a) SVPWM maximum output: $V_{LL(rms)} = V_{dc} / \sqrt{2} \times \sqrt{3} / \sqrt{3}$ ... use standard formula: $V_{phase(peak)} = V_{dc} / \sqrt{3} = 480 / 1.732 = 277.1$ V $V_{LL(rms)} = 277.1 \times \sqrt{3} / \sqrt{2} = 277.1 \times 1.225 = 339.5$ V

(b) Output line current: $I_{line} = P / (\sqrt{3} \times V_{LL} \times PF) = 15{,}000 / (1.732 \times 339.5 \times 0.87) = 15{,}000 / 511.7 = 29.3$ A

(c) DC bus current (input power = output power / η): $P_{in} = 15{,}000 / 0.97 = 15{,}464$ W $I_{dc} = P_{in} / V_{dc} = 15{,}464 / 480 = 32.2$ A

(d) Power dissipated in the inverter: $P_{loss} = P_{in} - P_{out} = 15{,}464 - 15{,}000 = 464$ W

Problem 10.4.8

Given: A five-level cascaded H-bridge (CHB) inverter uses two H-bridge modules per phase, each powered from an isolated 200 V DC source. The inverter operates at 60 Hz with phase-shifted PWM at $f_{sw} = 5$ kHz per module.

Find: (a) The output voltage levels per phase, (b) the peak fundamental phase voltage at maximum modulation, (c) the effective output switching frequency, and (d) the voltage step size between levels.

Solution:

(a) Two H-bridge modules per phase, each capable of $+V_{dc}$, 0, $-V_{dc}$. The combined output can produce: +400 V, +200 V, 0 V, −200 V, −400 V → 5 levels

(b) Peak fundamental phase voltage at maximum modulation ($m_a = 1.0$): $V_{phase(peak)} = 2 \times V_{dc} = 2 \times 200 = 400$ V

(c) With phase-shifted PWM, each module switches at 5 kHz with 90° carrier phase shift between the two modules. The effective output switching frequency: $f_{eff} = N \times f_{sw} = 2 \times 5{,}000 = 10$ kHz

(d) Voltage step between adjacent levels: $\Delta V = V_{dc} = 200$ V (compared to 800 V for a two-level inverter with $V_{dc} = 400$ V, a 75% reduction)

Problem 10.4.9

Given: A grid-tied three-phase inverter rated at 250 kW connects to a 480 V, 60 Hz utility through an LCL filter. The grid code requires THD < 5% at the point of common coupling. The inverter switching frequency is $f_{sw} = 10$ kHz. The LCL filter has $L_1 = 0.5$ mH (inverter-side), $L_2 = 0.2$ mH (grid-side), and $C_f = 15$ μF (with a damping resistor R_d).

Find: (a) The LCL filter resonant frequency, (b) an appropriate damping resistor value, and (c) the attenuation at the switching frequency.

Solution:

(a) LCL resonant frequency: $f_{res} = (1/2\pi) \times \sqrt{}((L_1 + L_2) / (L_1 \times L_2 \times C_f))$ $f_{res} = (1/2\pi) \times \sqrt{}((0.0005 + 0.0002) / (0.0005 \times 0.0002 \times 15 \times 10^{-6}))$ $f_{res} = (1/2\pi) \times \sqrt{}(0.0007 / (1.5 \times 10^{-12}))$ $f_{res} = (1/2\pi) \times \sqrt{}(4.667 \times 10^8) = (1/6.283) \times 21{,}602 = 3{,}439$ Hz

This is between 10% and 50% of f_{sw}, a good design range for effective attenuation without stability issues.

(b) Damping resistor (typically set to 1/3 of the capacitor impedance at resonance): $X_{C(res)} = 1 / (2\pi \times 3{,}439 \times 15 \times 10^{-6}) = 1 / 0.324 = 3.09\ \Omega$ $R_d = X_{C(res)} / 3 = 3.09 / 3 = 1.0\ \Omega$

(c) At $f_{sw} = 10$ kHz, well above resonance, the LCL filter provides approximately −60 dB/decade roll-off. The frequency ratio is 10,000 / 3,439 = 2.91. Attenuation $\approx (f/f_{res})^3 = 2.91^3 = 24.6 \rightarrow 20 \times \log_{10}(24.6) = 27.8$ dB

The 28 dB attenuation at 10 kHz, combined with the SVPWM spectrum characteristics, is typically sufficient to meet the 5% THD requirement.

Problem 10.4.10

Given: A modular multilevel converter (MMC) for an HVDC application has 200 submodules per arm, each with a submodule capacitor voltage of 2.0 kV. The converter has six arms (three phase legs, upper and lower arms). The DC link voltage is V_{dc}.

Find: (a) The DC link voltage, (b) the maximum AC phase voltage (peak, line-to-neutral), (c) the maximum AC line-to-line RMS voltage, and (d) the number of output voltage levels per phase.

Solution:

(a) In an MMC, the DC link voltage equals the sum of submodule voltages in one arm (upper or lower): $V_{dc} = N \times V_{SM} = 200 \times 2.0 = 400$ kV

(b) The AC phase voltage peak (line-to-neutral) for an MMC: $V_{phase(peak)} = V_{dc} / 2 = 400 / 2 = 200$ kV

(c) Maximum line-to-line RMS: $V_{LL(rms)} = V_{phase(peak)} \times \sqrt{3} / \sqrt{2} = 200 \times 1.732 / 1.414 = 200 \times 1.225 = 245.0$ kV

(d) Each arm has 200 submodules, each either inserted (contributes V_{SM}) or bypassed (contributes 0). The number of distinct voltage levels in the phase output: $N_{levels} = N + 1 = 200 + 1 = 201$ levels

With 201 voltage levels, the output waveform is an extremely close approximation of a sine wave, producing negligible harmonic distortion without output filters and voltage steps of only 2.0 kV out of 400 kV total (0.5% per step).

Chapter 10 — Section 10.5: AC-AC Converters

Practice problems covering AC voltage controllers, cycloconverters, matrix converters, phase-angle control, integral cycle control, voltage transfer ratios, and power factor effects.

Problem 10.5.1

Given: A single-phase AC voltage controller uses back-to-back thyristors to supply a 15 Ω resistive heating element from a 208 V_{rms}, 60 Hz source. The firing angle is $\alpha = 60°$.

Find: (a) The RMS output voltage, (b) the power delivered to the load, and (c) the power as a percentage of full (uncontrolled) power.

Solution:

(a) For a resistive load with phase-angle control: $V_{out(rms)} = V_{in} \times \sqrt{}[(\pi - \alpha + \sin(2\alpha)/2) / \pi]$ With $\alpha = 60° = \pi/3$ radians: $\sin(2 \times 60°) = \sin(120°) = 0.8660$ $V_{out(rms)} = 208 \times \sqrt{}[(\pi - \pi/3 + 0.8660/2) / \pi]$ $V_{out(rms)} = 208 \times \sqrt{}[(3.1416 - 1.0472 + 0.4330) / 3.1416]$ $V_{out(rms)} = 208 \times \sqrt{}[2.5274 / 3.1416] = 208 \times \sqrt{}0.8046 = 208 \times 0.8970 = 186.6$ V

(b) Power delivered: $P = V_{out(rms)}^2 / R = 186.6^2 / 15 = 34{,}820 / 15 = 2{,}321$ W

(c) Full power: $P_{full} = V_{in}^2 / R = 208^2 / 15 = 43{,}264 / 15 = 2{,}884$ W Percentage: $2{,}321 / 2{,}884 \times 100 = 80.5\%$

Problem 10.5.2

Given: A single-phase AC voltage controller with phase-angle control supplies a 20 Ω resistive load from a 240 V_{rms}, 50 Hz source. The desired output power is 1,200 W.

Find: (a) The required RMS output voltage, (b) the required firing angle α, and (c) the input power factor.

Solution:

(a) Required RMS output voltage: $V_{out} = \sqrt{}(P \times R) = \sqrt{}(1{,}200 \times 20) = \sqrt{}24{,}000 = 154.9$ V

(b) The output-to-input voltage ratio: $V_{out}/V_{in} = 154.9 / 240 = 0.6454$ $(V_{out}/V_{in})^2 = 0.4165 = (\pi - \alpha + \sin(2\alpha)/2) / \pi$

Solving: $\pi - \alpha + \sin(2\alpha)/2 = 0.4165 \times \pi = 1.3084$ rad

Testing $\alpha = 105° = 1.8326$ rad: $\pi - 1.8326 + \sin(210°)/2 = 1.3090 + (-0.5)/2 = 1.3090 - 0.25 = 1.0590$ (too low)

Testing $\alpha = 90° = \pi/2$: $\pi - \pi/2 + \sin(180°)/2 = 1.5708 + 0 = 1.5708$ (too high)

Testing $\alpha = 100° = 1.7453$ rad: $\pi - 1.7453 + \sin(200°)/2 = 1.3963 + (-0.3420)/2 = 1.3963 - 0.1710 = 1.2253$ (too low)

Testing $\alpha = 95° = 1.6581$ rad: $\pi - 1.6581 + \sin(190°)/2 = 1.4835 + (-0.1736)/2 = 1.4835 - 0.0868 = 1.3967$ (close)

Testing $\alpha = 97° = 1.6930$ rad: $\pi - 1.6930 + \sin(194°)/2 = 1.4486 + (-0.2419)/2 = 1.4486 - 0.1210 = 1.3276$ (close)

Testing $\alpha = 98° = 1.7104$ rad: $\pi - 1.7104 + \sin(196°)/2 = 1.4312 + (-0.2756)/2 = 1.4312 - 0.1378 = 1.2934$

Interpolating between 97° and 95°: $\alpha \approx 96°$

(c) Input power factor = P_{out} / S_{in}. The input apparent power: $S_{in} = V_{in} \times I_{in(rms)} = 240 \times (154.9/20) = 240 \times 7.745 = 1{,}858.8$ VA (Note: $I_{in(rms)} = I_{out(rms)} = V_{out(rms)}/R$ for a resistive load with thyristor control) $PF = P / S_{in} = 1{,}200 / 1{,}858.8 = 0.645$

The low power factor is a characteristic disadvantage of phase-angle control at reduced output.

Problem 10.5.3

Given: An integral cycle (burst-firing) controller operates on a 240 V_{rms}, 50 Hz supply. It uses a duty pattern of 3 cycles ON, 5 cycles OFF to control power to a 30 Ω resistive furnace heating element.

Find: (a) The fraction of power delivered, (b) the average power to the load, (c) the RMS output voltage, and (d) the input power factor.

Solution:

(a) Fraction of power: Duty ratio = cycles ON / total cycles = 3 / (3 + 5) = 3/8 = 0.375 (37.5%)

(b) Full power: $P_{full} = V_{in}^2 / R = 240^2 / 30 = 57{,}600 / 30 = 1{,}920$ W Average power: $P_{avg} = 0.375 \times 1{,}920 = 720$ W

(c) RMS output voltage: $V_{out(rms)} = V_{in} \times \sqrt{\text{(duty ratio)}} = 240 \times \sqrt{0.375} = 240 \times 0.6124 = 146.97$ V

(d) Input power factor: $PF = \sqrt{\text{(duty ratio)}} = \sqrt{0.375} = 0.612$

Integral cycle control produces no sub-line-frequency harmonics at the switching frequency, but introduces a modulation component at the burst rate of 50/(3+5) = 6.25 Hz, which can cause visible flicker in lighting loads.

Problem 10.5.4

Given: A three-phase to single-phase cycloconverter is fed from a 60 Hz, 460 V_{rms} (line-to-line) three-phase supply. The cycloconverter must produce a 12 Hz output for a low-speed grinding mill drive rated at 200 kW.

Find: (a) Whether the 12 Hz output frequency is within the cycloconverter's capability, (b) the maximum output RMS voltage, (c) the output current at rated power, and (d) the input apparent power assuming a displacement power factor of 0.85.

Solution:

(a) Maximum recommended output frequency: $f_{out(max)} = f_{in} / 3 = 60 / 3 = 20$ Hz Since 12 Hz < 20 Hz, the output frequency is achievable with acceptable waveform quality.

(b) Practical maximum output voltage for a cycloconverter (accounting for commutation overlap): $V_{out(rms)} \approx 0.75 \times V_{in(LL)} = 0.75 \times 460 = 345\ V_{rms}$

(c) Output current at rated power: $I_{out} = P / V_{out} = 200{,}000 / 345 = 579.7$ A

(d) Input apparent power: $S_{in} = P / DPF = 200{,}000 / 0.85 = 235.3$ kVA Input line current: $I_{in} = S_{in} / (\sqrt{3} \times V_{in}) = 235{,}300 / (1.732 \times 460) = 235{,}300 / 796.7 = 295.4$ A

Problem 10.5.5

Given: A three-phase matrix converter is supplied from a 400 V_{rms} line-to-line, 50 Hz source and must drive a three-phase induction motor at 35 Hz. The motor is rated at 22 kW with a power factor of 0.85. The converter efficiency is 96%.

Find: (a) The maximum output line-to-line voltage, (b) the output line current, (c) the input line current at unity input displacement power factor, and (d) the input power factor.

Solution:

(a) Maximum voltage transfer ratio for a matrix converter: $q_{max} = \sqrt{3}/2 = 0.866$ $V_{out(LL)} = q_{max} \times V_{in(LL)} = 0.866 \times 400 = 346.4$ V

(b) Output line current: $I_{out} = P / (\sqrt{3} \times V_{out} \times PF) = 22{,}000 / (1.732 \times 346.4 \times 0.85) = 22{,}000 / 509.8 = 43.2$ A

(c) Input power (accounting for efficiency): $P_{in} = P_{out} / \eta = 22{,}000 / 0.96 = 22{,}917$ W $I_{in} = P_{in} / (\sqrt{3} \times V_{in} \times \cos\varphi) = 22{,}917 / (1.732 \times 400 \times 1.0) = 22{,}917 / 692.8 = 33.1$ A

(d) The input displacement power factor is controllable to unity. The true input power factor accounts for harmonic currents: $PF_{in} \approx DPF \times (I_1/I_{rms}) \approx 1.0 \times 0.98 = 0.98$ (matrix converters produce very low input current THD, approximately 3-5%)

Problem 10.5.6

Given: A single-phase AC voltage controller with back-to-back thyristors drives a 5 Ω pure resistive load from a 120 V_{rms}, 60 Hz source. The firing angle is $\alpha = 120°$.

Find: (a) The RMS output voltage, (b) the output power, (c) the RMS of the fundamental component of the input current, and (d) the total harmonic distortion of the input current.

Solution:

(a) $V_{out(rms)} = V_{in} \times \sqrt{}[(\pi - \alpha + \sin(2\alpha)/2) / \pi]$ $\alpha = 120° = 2\pi/3$ rad; $\sin(240°) = -0.866$ $V_{out(rms)} = 120 \times \sqrt{}[(\pi - 2\pi/3 + (-0.866)/2) / \pi]$ $V_{out(rms)} = 120 \times \sqrt{}[(3.1416 - 2.0944 - 0.4330) / 3.1416]$ $V_{out(rms)} = 120 \times \sqrt{}[0.6142 / 3.1416] = 120 \times \sqrt{}0.1955 = 120 \times 0.4421 = 53.1$ V

(b) Output power: $P = V_{out(rms)}{}^2 / R = 53.1^2 / 5 = 2{,}819.6 / 5 = 564$ W

(c) The RMS fundamental input current: $I_{rms(total)} = V_{out(rms)} / R = 53.1 / 5 = 10.62$ A The fundamental Fourier coefficient for a phase-controlled resistive load: $I_{1(rms)}$ can be found from the displacement power factor relationship. For $\alpha = 120°$, the fundamental component amplitude: $I_{1(peak)} = (V_{peak}/(\pi R)) \times \sqrt{}[(\pi - \alpha)^2 + \sin^2(\alpha) + 2(\pi - \alpha)\sin(\alpha)\cos(\alpha)]$

Simplified approach using known power factor relations: The total power = $V_{in} \times I_{1(rms)} \times \cos(\varphi_1)$, where φ_1 is the displacement angle $\approx \alpha/2$ for resistive loads. P = 564 W, $V_{in} = 120$ V $I_{1(rms)} \times \cos(60°) = 564/120 = 4.70$ $I_{1(rms)} = 4.70/0.5 = 9.40$ A

(d) THD of input current: $THD = \sqrt{}(I_{rms}{}^2 - I_1{}^2) / I_1 = \sqrt{}(10.62^2 - 9.40^2) / 9.40 = \sqrt{}(112.8 - 88.4) / 9.40 = \sqrt{}24.4 / 9.40 = 4.94 / 9.40 = 52.5\%$

Problem 10.5.7

Given: A three-phase matrix converter replaces a back-to-back voltage-source converter (VSC) in a wind turbine application. The matrix converter eliminates the DC link capacitor. The back-to-back VSC uses 4,000 μF of DC link capacitance at 700 V.

Find: (a) The stored energy in the DC link capacitors of the back-to-back VSC, (b) the weight savings if electrolytic capacitors have an energy density of 0.08 J/g, (c) the voltage transfer ratio limitation of the matrix converter, and (d) the maximum output voltage if the grid is 690 V_{rms} line-to-line.

Solution:

(a) Stored energy in DC link: $E = ½CV^2 = 0.5 \times 4{,}000 \times 10^{-6} \times 700^2 = 0.5 \times 0.004 \times 490{,}000 = 980$ J

(b) Weight of capacitors: m = E / (energy density) = 980 / 0.08 = 12,250 g = 12.25 kg Eliminating the DC link saves approximately 12.25 kg of capacitor weight (additional savings from capacitor mounting hardware and bus bars).

(c) Matrix converter maximum voltage transfer ratio: $q_{max} = \sqrt{}3/2 = 0.866$ (86.6% of the input voltage)

This is a limitation compared to the back-to-back VSC which can achieve unity or higher voltage ratio through the DC link.

(d) Maximum output line-to-line voltage: $V_{out(LL)} = 0.866 \times 690 = 597.5$ V

Problem 10.5.8

Given: An AC voltage controller uses integral cycle control to regulate the temperature of a 5 kW, 240 V_{rms} industrial oven (pure resistive load, R = 11.52 Ω). The control period is 10 cycles (200 ms at 50 Hz). The oven requires 3.2 kW to maintain the setpoint temperature.

Find: (a) The number of ON cycles required per control period, (b) the RMS output voltage, (c) the modulation frequency, and (d) the sub-harmonic frequency components introduced.

Solution:

(a) Power fraction required: $P_{required} / P_{full} = 3{,}200 / 5{,}000 = 0.64$ Number of ON cycles $= 0.64 \times 10 = 6.4 \rightarrow$ round to 6 cycles ON, 4 cycles OFF (delivering 60% power) or 7 cycles ON, 3 cycles OFF (delivering 70% power)

For a setpoint of 3.2 kW, use 6 ON / 4 OFF (3,000 W) or 7 ON / 3 OFF (3,500 W), alternating between the two patterns. A common approach: use 6 ON for most periods and 7 ON periodically to average 3,200 W.

With 6 ON: $P = (6/10) \times 5{,}000 = 3{,}000$ W

(b) RMS output voltage: $V_{out(rms)} = 240 \times \sqrt{(6/10)} = 240 \times 0.7746 = 185.9$ V

(c) Modulation frequency: $f_{mod} = f_{line}$ / total cycles = 50 / 10 = 5 Hz (the burst repetition rate)

(d) Sub-harmonic components are introduced at multiples of the modulation frequency: Primary sub-harmonic at 5 Hz, with sidebands at $50 \pm 5 = 45$ Hz and 55 Hz, and higher-order sidebands at 100 ± 5 Hz, etc. The 5 Hz modulation is well below the flicker sensitivity peak of the human eye (~8 Hz), making this acceptable for heating loads but unsuitable for lighting.

Problem 10.5.9

Given: A three-phase to three-phase cycloconverter uses six three-phase thyristor bridges to produce a three-phase, 10 Hz output from a 50 Hz, 415 V_{rms} (line-to-line) supply. The load is a 500 kW synchronous motor at 0.90 power factor leading.

Find: (a) The output frequency ratio, (b) the theoretical maximum output voltage, (c) the practical output voltage at 75% voltage ratio, (d) the output current, and (e) the input kVA.

Solution:

(a) Output frequency ratio: $f_{out} / f_{in} = 10 / 50 = 0.20$ (well within the 1/3 limit of 0.333)

(b) Theoretical maximum output voltage: $V_{dc(max)}$ of a three-phase bridge $= (3\sqrt{3}/\pi) \times V_{peak(LL)} = 1.654 \times 415\sqrt{2} = 1.654 \times 586.9 = 970.7$ V $V_{out(rms,max)} = V_{dc(max)} / \sqrt{2} = 970.7 / 1.414 = 686.5$ V

(c) Practical output voltage: $V_{out(rms)} = 0.75 \times 415 = 311.3$ V

(d) Output current: $I_{out} = P / (\sqrt{3} \times V_{out} \times PF) = 500{,}000 / (1.732 \times 311.3 \times 0.90) = 500{,}000 / 484.8 = 1{,}031$ A

(e) Input kVA (assuming cycloconverter input power factor ≈ 0.70 due to thyristor phase control): $P_{in} \approx P_{out} / 0.98 = 500{,}000 / 0.98 = 510{,}204$ W (assuming 98% converter efficiency) $S_{in} = P_{in} / PF_{in} = 510{,}204 / 0.70 = 728.9$ kVA $I_{in} = S_{in} / (\sqrt{3} \times 415) = 728{,}900 / 718.8 = 1{,}014$ A

Problem 10.5.10

Given: A matrix converter feeding a three-phase motor must perform a four-step commutation sequence when switching an output phase from input phase A to input phase B. The output current is $I_{out} = 30$ A, $V_A = 310$ V, $V_B = 185$ V at the switching instant, and the IGBT turn-off time is $t_{off} = 200$ ns. A clamp circuit with a 20 μF capacitor protects against overvoltage.

Find: (a) The voltage across the outgoing switch at turn-off, (b) the time available for each commutation step if the total commutation must complete within 2 μs, (c) the energy stored in the clamp capacitor if it absorbs one commutation event, and (d) the clamp capacitor voltage rise from one event.

Solution:

(a) The voltage across the outgoing switch when it turns off is the difference between the two input phase voltages: $V_{switch} = V_A - V_B = 310 - 185 = 125$ V

(b) With a four-step commutation and total time budget of 2 μs: Time per step = 2,000 ns / 4 = 500 ns per step This exceeds the 200 ns IGBT turn-off time, providing adequate margin.

(c) Energy stored if the clamp absorbs the commutation event: The commutation occurs over approximately $t_{off} = 200$ ns. Energy = ½ × $L_{stray} \times I^2$ (from stray inductance), but for a voltage-driven clamp: $E_{clamp} \approx V_{switch} \times I_{out} \times t_{off} = 125 \times 30 \times 200 \times 10^{-9} = 0.75$ mJ

(d) Voltage rise on the clamp capacitor: $\Delta V = \sqrt{(2 \times E / C + V_{initial}^2)} - V_{initial}$

For a pre-charged clamp at $V_{clamp} = V_{peak(LL)} = 400\sqrt{2} = 565.7$ V: $\Delta E = 0.75 \times 10^{-3}$ J. $C = 20 \times 10^{-6}$ F. $\Delta V \approx E / (C \times V_{clamp}) = 0.75 \times 10^{-3} / (20 \times 10^{-6} \times 565.7) = 0.75 \times 10^{-3} / 0.01131 = 0.066$ V

The voltage rise is negligible, confirming the 20 μF clamp capacitor is adequately sized for commutation protection.

Chapter 10 — Section 10.6: Thermal Management

Practice problems covering power loss calculations, thermal resistance networks, heat sink sizing, junction temperature analysis, thermal interface materials, liquid cooling design, and EMI filter design for power converters.

Problem 10.6.1

Given: A synchronous buck converter uses a high-side MOSFET with $R_{DS(on)} = 22$ mΩ at 25°C (temperature coefficient +0.4%/°C), switching at $f_{sw} = 400$ kHz. Operating conditions: $V_{in} = 48$ V, $V_{out} = 12$ V, $I_{out} = 15$ A. The switching transitions have $t_{rise} = 12$ ns and $t_{fall} = 18$ ns. The junction temperature is 105°C. The gate charge is $Q_g = 45$ nC at $V_{gs} = 10$ V.

Find: (a) The duty cycle, (b) the RMS current through the high-side MOSFET, (c) the conduction loss at operating temperature, (d) the switching loss, (e) the gate drive loss, and (f) the total high-side MOSFET loss.

Solution:

(a) Duty cycle: $D = V_{out} / V_{in} = 12 / 48 = 0.25$ (25%)

(b) RMS current through the high-side MOSFET: $I_{Q1(rms)} = I_{out} \times \sqrt{D} = 15 \times \sqrt{0.25} = 15 \times 0.5 = 7.5$ A

(c) $R_{DS(on)}$ at 105°C: $R_{DS(on)} = 22 \times [1 + 0.004 \times (105 - 25)] = 22 \times [1 + 0.32] = 22 \times 1.32 = 29.04$ mΩ $P_{cond} = I_{Q1(rms)}^2 \times R_{DS(on)} = 7.5^2 \times 0.02904 = 56.25 \times 0.02904 = 1.63$ W

(d) Switching loss (high-side MOSFET sees full V_{in} and I_{out} during transitions): $P_{sw} = ½ \times V_{in} \times I_{out} \times (t_{rise} + t_{fall}) \times f_{sw}$ $P_{sw} = 0.5 \times 48 \times 15 \times (12 + 18) \times 10^{-9} \times 400 \times 10^3 = 0.5 \times 48 \times 15 \times 30 \times 10^{-9} \times 4 \times 10^5$ $P_{sw} = 0.5 \times 48 \times 15 \times 0.012 = 4.32$ W

(e) Gate drive loss: $P_{gate} = Q_g \times V_{gs} \times f_{sw} = 45 \times 10^{-9} \times 10 \times 400 \times 10^3 = 0.18$ W

(f) Total high-side MOSFET loss: $P_{total} = 1.63 + 4.32 + 0.18 = 6.13$ W

Problem 10.6.2

Given: An IGBT module dissipates 350 W total (IGBT: 220 W, freewheeling diode: 130 W). The thermal path has: $R_{\theta JC(IGBT)}$ = 0.12 °C/W, $R_{\theta JC(diode)}$ = 0.20 °C/W, $R_{\theta CS}$ = 0.03 °C/W (shared case-to-sink), and $R_{\theta SA}$ = 0.10 °C/W. Ambient temperature is T_A = 45°C. Maximum T_J = 150°C for both devices.

Find: (a) The heat sink temperature, (b) the case temperature, (c) the IGBT junction temperature, (d) the diode junction temperature, and (e) whether the design meets thermal limits.

Solution:

(a) Heat sink temperature: $T_{HS} = T_A + P_{total} \times R_{\theta SA} = 45 + 350 \times 0.10 = 45 + 35 = 80°C$

(b) Case temperature (total power flows through shared $R_{\theta CS}$): $T_C = T_{HS} + P_{total} \times R_{\theta CS} = 80 + 350 \times 0.03 = 80 + 10.5 = 90.5°C$

(c) IGBT junction temperature: $T_{J(IGBT)} = T_C + P_{IGBT} \times R_{\theta JC(IGBT)} = 90.5 + 220 \times 0.12 = 90.5 + 26.4 = 116.9°C$

(d) Diode junction temperature: $T_{J(diode)} = T_C + P_{diode} \times R_{\theta JC(diode)} = 90.5 + 130 \times 0.20 = 90.5 + 26.0 = 116.5°C$

(e) Both $T_{J(IGBT)}$ = 116.9°C and $T_{J(diode)}$ = 116.5°C are below $T_{J(max)}$ = 150°C. IGBT margin: 150 − 116.9 = 33.1°C margin (adequate) Diode margin: 150 − 116.5 = 33.5°C margin (adequate) The design meets thermal limits.

Problem 10.6.3

Given: A power supply designer must select a heat sink for a MOSFET dissipating 8.5 W. The MOSFET has $R_{\theta JC}$ = 1.2 °C/W and $R_{\theta CS}$ = 0.5 °C/W (with thermal pad). The maximum junction temperature is $T_{J(max)}$ = 150°C, the ambient is 50°C, and the designer wants a 25°C safety margin.

Find: (a) The maximum allowable $R_{\theta SA}$, (b) whether a natural convection heat sink with $R_{\theta SA}$ = 6.5 °C/W is adequate, and (c) the required airflow velocity if a forced-air heat sink has $R_{\theta SA}$ = 12 / (1 + 0.8v) °C/W where v is air velocity in m/s.

Solution:

(a) Target junction temperature: $T_{J(target)}$ = 150 − 25 = 125°C Allowable total temperature rise: ΔT = 125 − 50 = 75°C $R_{\theta JA(max)} = \Delta T / P = 75 / 8.5 = 8.82$ °C/W $R_{\theta SA(max)} = R_{\theta JA(max)} - R_{\theta JC} - R_{\theta CS}$ = 8.82 − 1.2 − 0.5 = 7.12 °C/W

(b) Natural convection heat sink: $R_{\theta SA}$ = 6.5 °C/W < 7.12 °C/W. $T_J = 50 + 8.5 \times (1.2 + 0.5 + 6.5) = 50 + 8.5 \times 8.2 = 50 + 69.7 = 119.7°C$ This is below 125°C with 5.3°C margin. The heat sink is adequate.

(c) Required airflow if forced-air is needed (e.g., in a more constrained design): $R_{\theta SA} \leq 7.12$ °C/W: $12 / (1 + 0.8v) \leq 7.12$ $1 + 0.8v \geq 12 / 7.12 = 1.685$ $0.8v \geq 0.685$ $v \geq 0.856$ m/s ≈ 0.9 m/s (approximately 170 ft/min, achievable with a small fan)

Problem 10.6.4

Given: A SiC MOSFET module dissipates 600 W and is mounted on a liquid-cooled cold plate. The thermal interface uses sintered silver with thermal conductivity k_{TIM} = 200 W/m·K and bond-line thickness t = 30 μm over a 50 mm × 60 mm contact area. The cold plate $R_{\theta(\text{plate-to-coolant})}$ = 0.015 °C/W, the module $R_{\theta JC}$ = 0.06 °C/W, and coolant inlet temperature is 30°C.

Find: (a) The thermal resistance of the sintered silver interface, (b) the total junction-to-coolant thermal resistance, (c) the junction temperature, and (d) the comparison to thermal grease (k = 3.5 W/m·K, t = 50 μm).

Solution:

(a) TIM contact area: $A = 0.050 \times 0.060 = 3.0 \times 10^{-3}$ m² $R_{\theta CS} = t / (k \times A) = 30 \times 10^{-6} / (200 \times 3.0 \times 10^{-3}) = 30 \times 10^{-6} / 0.60 = 0.00005$ °C/W (negligible)

(b) Total thermal resistance: $R_{\theta(\text{total})} = R_{\theta JC} + R_{\theta CS} + R_{\theta(\text{plate})} = 0.06 + 0.00005 + 0.015 = 0.0751$ °C/W

(c) Junction temperature: $T_J = T_{coolant} + P \times R_{\theta(\text{total})} = 30 + 600 \times 0.0751 = 30 + 45.0 = 75.0$°C

(d) With thermal grease: $R_{\theta CS(\text{grease})} = 50 \times 10^{-6} / (3.5 \times 3.0 \times 10^{-3}) = 50 \times 10^{-6} / 0.0105 = 0.00476$ °C/W $R_{\theta(\text{total,grease})} = 0.06 + 0.00476 + 0.015 = 0.0798$ °C/W $T_{J(\text{grease})} = 30 + 600 \times 0.0798 = 30 + 47.9 = 77.9$°C

The sintered silver saves only 2.9°C compared to grease in this case because $R_{\theta JC}$ and $R_{\theta(\text{plate})}$ dominate. Sintered silver becomes more advantageous at higher power densities where $R_{\theta CS}$ is a larger fraction of the total.

Problem 10.6.5

Given: A liquid cooling loop for a 100 kW power converter uses deionized water (c_p = 4,180 J/kg·°C, ρ = 998 kg/m³). The converter efficiency is 97%, and all losses are removed by the cooling loop. The maximum allowable coolant temperature rise is 8°C.

Find: (a) The total heat to be removed, (b) the required coolant flow rate, (c) the volumetric flow rate in L/min, and (d) the pump power if the system pressure drop is 150 kPa and pump efficiency is 65%.

Solution:

(a) Total heat dissipation: $P_{loss} = P_{out} \times (1/\eta - 1) = 100{,}000 \times (1/0.97 - 1) = 100{,}000 \times 0.03093 = 3{,}093$ W

(b) Required mass flow rate: $\dot{m} = \dot{Q} / (c_p \times \Delta T) = 3{,}093 / (4{,}180 \times 8) = 3{,}093 / 33{,}440 = 0.0925$ kg/s

(c) Volumetric flow rate: $\dot{V} = \dot{m} / \rho = 0.0925 / 998 = 9.27 \times 10^{-5}$ m³/s $= 9.27 \times 10^{-5} \times 60{,}000 = 5.56$ L/min

(d) Pump hydraulic power: $P_{hydraulic} = \Delta P \times \dot{V} = 150{,}000 \times 9.27 \times 10^{-5} = 13.9$ W $P_{pump} = P_{hydraulic} / \eta_{pump} = 13.9 / 0.65 = 21.4$ W

The pump power is a small fraction (0.7%) of the total converter loss, confirming that liquid cooling is energy-efficient.

Problem 10.6.6

Given: A two-stage EMI filter must be designed for a 500 W PFC converter switching at $f_{sw} = 65$ kHz. Conducted emission testing shows the converter exceeds CISPR 32 Class B quasi-peak limits by 22 dB at 65 kHz and 15 dB at 130 kHz. A 6 dB design margin is required.

Find: (a) The required attenuation at the fundamental and second harmonic, (b) the filter corner frequency for a single-stage LC filter, (c) suitable component values, and (d) whether a single stage is sufficient at 130 kHz.

Solution:

(a) Required attenuation: At 65 kHz: $22 + 6 = 28$ dB At 130 kHz: $15 + 6 = 21$ dB

(b) A single-stage LC (second-order) filter has -40 dB/decade roll-off. Required attenuation of 28 dB $= 10^{28/20} = 25.1\times$ at 65 kHz. For a second-order filter: attenuation $= (f/f_0)^2$ $f_0 = f / \sqrt{}(\text{attenuation}) = 65{,}000 / \sqrt{}25.1 = 65{,}000 / 5.01 = 12{,}974$ Hz ≈ 13 kHz

(c) Choose $L_{DM} = 200$ µH: $C = 1 / (4\pi^2 f_0^2 L) = 1 / (4 \times 9.87 \times (13{,}000)^2 \times 200 \times 10^{-6})$ $C = 1 / (4 \times 9.87 \times 1.69 \times 10^8 \times 2 \times 10^{-4}) = 1 / (1.332 \times 10^6) = 0.75$ µF Use $L = 200$ µH and $C = 0.82$ µF (standard X2 capacitor value).

(d) Verify at 130 kHz with actual component values: $f_0 = 1/(2\pi\sqrt{}(200 \times 10^{-6} \times 0.82 \times 10^{-6})) = 1/(2\pi \times 1.281 \times 10^{-5}) = 1/(8.05 \times 10^{-5}) = 12{,}422$ Hz Attenuation at 130 kHz $= (130{,}000/12{,}422)^2 = (10.47)^2 = 109.5 = 40.8$ dB Required: 21 dB. The single-stage filter provides 40.8 dB, which exceeds the 21 dB requirement by 19.8 dB. A single stage is sufficient at both frequencies.

Problem 10.6.7

Given: An inverter has three IGBT half-bridge modules, each dissipating 180 W. All three are mounted on a common aluminum heat sink with forced-air cooling. The heat sink has a base area of 200 mm × 300 mm, fin height of 50 mm, and 15 fins at 2 mm thickness with 12 mm spacing. The fan provides 2.5 m/s airflow. Ambient temperature is 40°C.

Find: (a) The total heat to be dissipated, (b) the required $R_{\theta SA}$ if each module has $R_{\theta JC} = 0.15$ °C/W, $R_{\theta CS} = 0.04$ °C/W, and $T_{J(max)} = 150$°C with 20°C margin, (c) the heat sink fin surface area, and (d) whether the heat sink is adequate (assume convective heat transfer coefficient $h = 35$ W/m²·°C for 2.5 m/s airflow).

Solution:

(a) Total heat: $P_{total} = 3 \times 180 = 540$ W

(b) Target $T_J = 150 - 20 = 130$°C. For the worst-case module: $T_J = T_{HS} + P_{module} \times (R_{\theta JC} + R_{\theta CS})$ $130 = T_{HS} + 180 \times (0.15 + 0.04) = T_{HS} + 34.2$ $T_{HS(max)} = 130 - 34.2 = 95.8$°C $R_{\theta SA} = (T_{HS} - T_A) / P_{total} = (95.8 - 40) / 540 = 55.8 / 540 = 0.103$ °C/W

(c) Fin surface area (both sides of each fin plus the base between fins): Fin area per fin (both sides): $2 \times 0.050 \times 0.300 = 0.030$ m^2 Total fin area: $15 \times 0.030 = 0.45$ m^2 Base area between fins: $14 \times 0.012 \times 0.300 = 0.0504$ m^2 Total surface area: $A_{total} = 0.45 + 0.0504 = 0.50$ m^2

(d) Thermal resistance of the heat sink: $R_{\theta SA} = 1 / (h \times A \times \eta_{fin})$ where $\eta_{fin} \approx 0.85$ for aluminum fins of this geometry. $R_{\theta SA} = 1 / (35 \times 0.50 \times 0.85) = 1 / 14.88 = 0.067$ °C/W

Since $0.067 < 0.103$ °C/W required, the heat sink is adequate with margin. $T_{HS} = 40 + 540 \times 0.067 = 40 + 36.2 = 76.2$°C $T_{J(worst)} = 76.2 + 34.2 = 110.4$°C (39.6°C below the 150°C limit)

Problem 10.6.8

Given: A phase-change thermal interface material (PCM) has thermal conductivity $k = 4.0$ W/m·K in its softened state and a thickness of 0.15 mm. It is used between a TO-247 MOSFET package (contact area 15 mm × 20 mm) and a heat sink. Compare this to thermal grease with $k = 2.5$ W/m·K at 0.05 mm bond-line thickness.

Find: (a) The thermal resistance of the PCM interface, (b) the thermal resistance of the thermal grease interface, (c) the temperature difference across each TIM at 50 W dissipation, and (d) which TIM is preferred and why.

Solution:

(a) PCM contact area: $A = 0.015 \times 0.020 = 3.0 \times 10^{-4}$ m^2 $R_{\theta(PCM)} = t / (k \times A) = 0.15 \times 10^{-3} / (4.0 \times 3.0 \times 10^{-4}) = 1.5 \times 10^{-4} / 1.2 \times 10^{-3} = 0.125$ °C/W

(b) Thermal grease: $R_{\theta(grease)} = t / (k \times A) = 0.05 \times 10^{-3} / (2.5 \times 3.0 \times 10^{-4}) = 5.0 \times 10^{-5} / 7.5 \times 10^{-4} = 0.067$ °C/W

(c) Temperature difference at 50 W: PCM: $\Delta T = 50 \times 0.125 = 6.25$°C Grease: $\Delta T = 50 \times 0.067 = 3.33$°C

(d) Despite higher thermal conductivity (4.0 vs 2.5 W/m·K), the PCM has higher thermal resistance because its thickness (0.15 mm) is three times that of the grease (0.05 mm). The thermal grease provides a 2.92°C lower temperature drop. However, the PCM offers easier handling (solid at room temperature), no pump-out over time, and consistent performance — making it preferred when long-term reliability outweighs the modest thermal penalty.

Problem 10.6.9

Given: A 1 kW AC-DC power supply has conducted emissions exceeding CISPR 32 Class B limits. Measurements with a LISN show: differential-mode (DM) noise dominates below 500 kHz at 85 dBμV peak, and common-mode (CM) noise dominates above 1 MHz at 72 dBμV peak. The Class B quasi-peak limit is 56 dBμV at 500 kHz and 60 dBμV at 1 MHz.

Find: (a) The required DM attenuation at 500 kHz, (b) the required CM attenuation at 1 MHz, (c) suitable X-capacitor and DM inductor values for the DM filter, and (d) suitable CM choke inductance and Y-capacitor values.

Solution:

(a) DM attenuation required at 500 kHz: Excess = 85 − 56 = 29 dB. With 6 dB margin: 35 dB required

(b) CM attenuation required at 1 MHz: Excess = 72 − 60 = 12 dB. With 6 dB margin: 18 dB required

(c) DM filter design (second-order LC, −40 dB/decade): 35 dB = $10^{35/20}$ = 56.2× attenuation f_0 = 500,000 / $\sqrt{56.2}$ = 500,000 / 7.50 = 66,700 Hz

Choose C_X = 0.47 μF (X2 film capacitor): $L_{DM} = 1/(4\pi^2 f_0^2 C) = 1/(4 \times 9.87 \times (66{,}700)^2 \times 0.47 \times 10^{-6})$ $L_{DM} = 1/(4 \times 9.87 \times 4.449 \times 10^9 \times 4.7 \times 10^{-7}) = 1/(82{,}700) = 12.1$ μH Use L_{DM} = 15 μH and C_X = 0.47 μF for margin.

(d) CM filter design: 18 dB = $10^{18/20}$ = 7.94× attenuation at 1 MHz $f_{0(CM)}$ = 1,000,000 / $\sqrt{7.94}$ = 1,000,000 / 2.82 = 354,900 Hz

Choose C_Y = 2.2 nF (Y2 ceramic, limited by safety leakage current): $L_{CM} = 1/(4\pi^2 f_0^2 C) = 1/(4 \times 9.87 \times (354{,}900)^2 \times 2.2 \times 10^{-9})$ $L_{CM} = 1/(4 \times 9.87 \times 1.260 \times 10^{11} \times 2.2 \times 10^{-9}) = 1/(10{,}940) = 91$ μH Use a 100 μH common-mode choke and 2.2 nF Y2 capacitors (two, one per line to ground).

Problem 10.6.10

Given: A power module with two paralleled MOSFET die has the following thermal network: each die dissipates $P_1 = P_2$ = 75 W, with $R_{\theta JC_1} = R_{\theta JC_2}$ = 0.25 °C/W, thermal coupling between die $R_{\theta 12}$ = 1.5 °C/W, a shared $R_{\theta CS}$ = 0.02 °C/W, and heat sink $R_{\theta SA}$ = 0.08 °C/W. Ambient is 35°C.

Find: (a) The heat sink temperature, (b) the case temperature, (c) the junction temperature of each die accounting for self-heating only, (d) the additional temperature rise from thermal coupling, and (e) the total junction temperature of each die.

Solution:

(a) Heat sink temperature: P_{total} = 75 + 75 = 150 W $T_{HS} = T_A + P_{total} \times R_{\theta SA}$ = 35 + 150 × 0.08 = 35 + 12 = 47°C

(b) Case temperature: $T_C = T_{HS} + P_{total} \times R_{\theta CS}$ = 47 + 150 × 0.02 = 47 + 3 = 50°C

(c) Junction temperature from self-heating only: $T_{J(self)} = T_C + P \times R_{\theta JC}$ = 50 + 75 × 0.25 = 50 + 18.75 = 68.75°C

(d) Thermal coupling from adjacent die: $\Delta T_{coupling} = P_{adjacent} / R_{\theta 12}$ = 75 / 1.5 = 50°C

Wait — $R_{\theta 12}$ = 1.5 °C/W represents the thermal resistance between die, so the temperature rise due to coupling is: $\Delta T_{coupling} = P_{adjacent} \times (1/R_{\theta 12}$ equivalent)

The thermal coupling causes the heat from one die to contribute to the other die's temperature. The correct calculation: $\Delta T_{coupling} = P_{adjacent} \times R_{\theta(mutual)}$

For thermal coupling between paralleled die, the mutual thermal impedance is typically expressed as the temperature rise at die 1 per watt dissipated in die 2. With $R_{\theta 12}$ = 1.5 °C/W: $\Delta T_{coupling} = P_2 \times R_{\theta 12(mutual)}$

Note: The high value of 1.5 °C/W as a mutual coupling resistance means the coupling is relatively weak. The mutual resistance is a fraction of the self-heating resistance: $\Delta T_{coupling} = 75 \times (R_{\theta JC}{}^2 / R_{\theta 12})$ ...

Using the simple model where $R_{\theta 12}$ is the mutual thermal resistance (temperature rise at die 1 per watt at die 2): $\Delta T_{coupling} = P_2 \times (1/R_{\theta 12}) \times R_{\theta JC}{}^2$ ...

The straightforward interpretation: $R_{\theta 12} = 1.5$ °C/W is the thermal resistance between die junction locations through the substrate. $\Delta T_{12} = P_2 / (1/R_{\theta 12})$ = this doesn't simplify correctly.

Using the standard thermal coupling model where $R_{\theta 12}$ is the thermal cross-coupling resistance: $\Delta T_{coupling} = P_2 \times R_{\theta 12(cross)}$

If $R_{\theta 12} = 1.5$ °C/W is the cross-coupling resistance: $\Delta T_{coupling} = 75 \times 1.5 = 112.5°C$ — this is unreasonably high.

The correct interpretation is that $R_{\theta 12} = 1.5$ °C/W represents the thermal resistance of the coupling path, making the fraction of heat coupled $= R_{\theta JC}/(R_{\theta JC} + R_{\theta 12})$: Coupling factor $= R_{\theta JC} / (R_{\theta JC} + R_{\theta 12}) = 0.25 / (0.25 + 1.5) = 0.25 / 1.75 = 0.143$ $\Delta T_{coupling} = P_2 \times R_{\theta JC} \times$ coupling factor $= 75 \times 0.25 \times 0.143 = 2.68°C$

(e) Total junction temperature: $T_J = T_{J(self)} + \Delta T_{coupling} = 68.75 + 2.68 = 71.4°C$

Both die are symmetric, so they reach the same temperature. The thermal coupling adds only 2.68°C because the coupling resistance is much larger than the self-heating resistance (1.5 °C/W vs 0.25 °C/W), meaning relatively little heat transfers between die through the substrate.

Chapter 10 — Section 10.7: Power Factor Correction

Practice problems covering active PFC topologies, boost PFC design, bridgeless PFC, critical conduction mode, interleaved PFC, power supply protection circuits, soft start, and inrush current limiting.

Problem 10.7.1

Given: A boost PFC converter operates from a universal input of 120 V_{rms} (worst case) at 60 Hz and produces a 400 V DC output. The rated output power is 350 W. The switching frequency is 100 kHz and the boost inductor is 600 μH.

Find: (a) The RMS input current, (b) the peak input current, (c) the duty cycle at the peak of the AC input, (d) the inductor current ripple at the peak of the AC input, and (e) whether the converter remains in CCM at the peak.

Solution:

(a) RMS input current (assuming PF $\approx$ 1.0): $I_{in(rms)} = P / V_{in(rms)} = 350 / 120 = 2.917$ A

(b) Peak input current: $I_{in(peak)} = I_{in(rms)} \times \sqrt{2} = 2.917 \times 1.414 = 4.124$ A

(c) Peak input voltage: $V_{in(peak)} = 120 \times \sqrt{2} = 169.7$ V Duty cycle at peak: $D = 1 - V_{in(peak)}/V_{out} = 1 - 169.7/400 = 1 - 0.4243 = 0.576$ (57.6%)

(d) Inductor current ripple at the peak: $\Delta I_L = V_{in(peak)} \times D / (L \times f_{sw}) = 169.7 \times 0.576 / (600 \times 10^{-6} \times 100 \times 10^3)$ $\Delta I_L = 97.75 / 60 = 1.629$ A_{pp}

(e) Check CCM: The minimum inductor current at the peak is: $I_{min} = I_{in(peak)} - \Delta I_L/2 = 4.124 - 1.629/2 = 4.124 - 0.815 = 3.309$ A

Since $I_{min} = 3.309$ A > 0, the converter remains in CCM at the peak. However, near the zero crossings of the AC input, the average current is much lower and the converter will enter DCM, which is typical for CCM boost PFC at light instantaneous current.

Problem 10.7.2

Given: A critical conduction mode (CrCM) boost PFC operates from 230 V_{rms}, 50 Hz and produces 385 V DC at 150 W. The boost inductor is 1.2 mH.

Find: (a) The switching frequency at the peak of the AC input, (b) the switching frequency at 30° past the zero crossing, (c) the peak inductor current at the AC peak, and (d) the range of switching frequencies over the AC cycle.

Solution:

(a) In CrCM, the inductor current ramps from zero to a peak and back to zero each cycle. $V_{in(peak)} = 230 \times \sqrt{2} = 325.3$ V $D_{peak} = 1 - V_{in(peak)}/V_{out} = 1 - 325.3/385 = 0.1551$

The instantaneous average input power at the peak: $p(t) = 2P/T \times$ ... The average current at the peak: $I_{avg(peak)} = (P \times \sqrt{2}) / V_{in(rms)} \times (2/\pi$ is not needed for peak) ...

Simpler approach: $I_{avg(peak)} = I_{in(rms)} \times \sqrt{2} = (150/230) \times \sqrt{2} = 0.6522 \times 1.414 = 0.922$ A

In CrCM, the peak current $= 2 \times I_{avg} = 2 \times 0.922 = 1.844$ A. On-time: $t_{on} = I_{peak} \times L / V_{in(peak)} = 1.844 \times 1.2 \times 10^{-3} / 325.3 = 6.80$ µs Off-time: $t_{off} = I_{peak} \times L / (V_{out} - V_{in(peak)}) = 1.844 \times 1.2 \times 10^{-3} / (385 - 325.3) = 2.213 \times 10^{-3} / 59.7 = 37.07$ µs $T_{sw} = t_{on} + t_{off} = 6.80 + 37.07 = 43.87$ µs $f_{sw(peak)} = 1 / 43.87 \times 10^{-6} = 22.8$ kHz

(b) At 30° past zero crossing: $V_{in}(30°) = 325.3 \times \sin(30°) = 325.3 \times 0.5 = 162.65$ V $I_{avg}(30°) = 0.922 \times \sin(30°) = 0.461$ A (sinusoidal current envelope) $I_{peak}(30°) = 2 \times 0.461 = 0.922$ A $t_{on} = 0.922 \times 1.2 \times 10^{-3} / 162.65 = 6.80$ µs $t_{off} = 0.922 \times 1.2 \times 10^{-3} / (385 - 162.65) = 1.106 \times 10^{-3} / 222.35 = 4.975$ µs $T_{sw} = 6.80 + 4.975 = 11.78$ µs $f_{sw}(30°) = 1 / 11.78 \times 10^{-6} = 84.9$ kHz

(c) Peak inductor current at the AC peak: $I_{L(peak)} = 2 \times I_{avg(peak)} = 1.844$ A (as calculated above)

(d) The switching frequency is lowest at the AC peak (~23 kHz) and highest near the zero crossings (where it can exceed 200 kHz in theory but is typically clamped). Practical range: ~23 kHz to ~150 kHz (clamped by the controller's maximum frequency limit).

Problem 10.7.3

Given: A 3 kW interleaved boost PFC uses two paralleled boost stages, each operating at $f_{sw} = 70$ kHz with 180° phase shift. Input: 240 V_{rms}, 50 Hz. Output: 400 V DC. Each boost inductor is 400 µH.

Find: (a) The current per phase, (b) the inductor ripple current per phase at the AC peak, (c) the effective input ripple frequency, and (d) the input current ripple cancellation at D = 0.5.

Solution:

(a) Total input current: $I_{in(rms)} = P/V_{in} = 3{,}000/240 = 12.5$ A Current per phase: $I_{phase} = 12.5/2 = 6.25\ A_{rms}$

(b) Peak input voltage: $V_{in(peak)} = 240\sqrt{2} = 339.4$ V Duty at peak: $D = 1 - 339.4/400 = 0.1515$ Ripple per phase: $\Delta I = V_{in(peak)} \times D / (L \times f_{sw}) = 339.4 \times 0.1515 / (400 \times 10^{-6} \times 70 \times 10^3)$ $\Delta I = 51.42 / 28 = 1.836\ A_{pp}$ per phase

(c) Effective input ripple frequency: $f_{ripple} = 2 \times f_{sw} = 2 \times 70{,}000 = 140$ kHz (doubled due to interleaving)

(d) At $D = 0.5$ (which occurs when $V_{in} = V_{out}/2 = 200$ V during the AC cycle): With 180° interleaving and $D = 0.5$, the two phases' ripple currents are perfectly complementary — when one phase's current is ramping up, the other is ramping down by the same amount. The net input current ripple is: $\Delta I_{input} = 0$ A (complete cancellation at $D = 0.5$)

This is a key advantage of interleaved PFC: the total input ripple is dramatically reduced at and near $D = 0.5$, reducing EMI filter requirements. At other duty cycles, partial cancellation still reduces the total ripple by 50-80% compared to a single-phase converter.

Problem 10.7.4

Given: A 48 V, 30 A DC power supply has an output capacitor bank of 3,300 μF. The soft-start circuit ramps the output from 0 to 48 V in 15 ms. The overcurrent protection (OCP) threshold is 38 A.

Find: (a) The voltage ramp rate, (b) the capacitor charging current during soft start, (c) the peak total current (charging + load) near the end of the ramp, (d) whether OCP will trip, and (e) the minimum soft-start time to stay below OCP.

Solution:

(a) Voltage ramp rate: $dV/dt = 48 / 15 \times 10^{-3} = 3{,}200$ V/s

(b) Capacitor charging current: $I_{cap} = C \times dV/dt = 3{,}300 \times 10^{-6} \times 3{,}200 = 10.56$ A

(c) Peak total current near end of ramp (worst case, full load current developing as output approaches 48 V): $I_{peak} = I_{cap} + I_{load} = 10.56 + 30 = 40.56$ A

(d) Since $I_{peak} = 40.56$ A > OCP threshold of 38 A, OCP will trip, causing the power supply to enter hiccup mode and fail to start properly.

(e) Minimum soft-start time: Maximum allowable capacitor current: $I_{cap(max)} = 38 - 30 = 8$ A $dV/dt_{max} = I_{cap(max)} / C = 8 / 3{,}300 \times 10^{-6} = 2{,}424$ V/s $t_{ss(min)} = 48 / 2{,}424 = 0.0198$ s $= 19.8$ ms

Use 25 ms for adequate margin, giving $I_{cap} = 3{,}300 \times 10^{-6} \times (48/0.025) = 6.34$ A, and $I_{peak} = 6.34 + 30 = 36.3$ A (1.7 A below OCP).

Problem 10.7.5

Given: A 750 W offline power supply has a 450 μF bulk input capacitor charged through a full-bridge diode rectifier from a 230 V_{rms}, 50 Hz source. The inrush current is limited by an NTC thermistor with cold resistance $R_{cold} = 10\ \Omega$ and steady-state hot resistance $R_{hot} = 0.5\ \Omega$.

Find: (a) The peak voltage across the capacitor at steady state, (b) the worst-case peak inrush current (if power is applied at the peak of the AC line), (c) the steady-state power dissipated in the NTC, and (d) the NTC surface temperature rise if its thermal resistance to ambient is 25 °C/W.

Solution:

(a) Peak capacitor voltage: $V_{cap} = V_{peak} = 230 \times \sqrt{2} = 325.3$ V

(b) Worst-case inrush (capacitor discharged, power applied at line peak): $I_{inrush} = V_{peak} / R_{cold} = 325.3 / 10 = 32.5$ A

Without the NTC: I_{inrush} would be limited only by source impedance and wiring resistance (typically 0.1-0.5 Ω), yielding 650-3,253 A — potentially destructive. The NTC reduces inrush by approximately 100×.

(c) Steady-state input current: $I_{in(rms)} = P / (V_{in} \times PF)$. Without PFC, PF ≈ 0.60: $I_{in(rms)} = 750 / (230 \times 0.60) = 5.43$ A $P_{NTC} = I_{in(rms)}^2 \times R_{hot} = 5.43^2 \times 0.5 = 29.5 \times 0.5 = 14.7$ W

(d) NTC temperature rise: $\Delta T = P \times R_\theta = 14.7 \times 25 = 367.5°C$

This is an unrealistically high temperature, indicating that the 0.5 Ω hot resistance will actually be lower at equilibrium (NTC resistance continues to decrease with temperature). In practice, the NTC settles around 150-200°C surface temperature, with R_{hot} decreasing further. This also highlights the 14.7 W of continuous dissipation as an efficiency penalty of 14.7/750 = 2.0%, which is why high-power supplies (>500 W) use an active bypass relay instead.

Problem 10.7.6

Given: A totem-pole bridgeless PFC converter uses GaN HEMTs with $R_{DS(on)} = 35$ mΩ for the high-frequency switching leg and silicon MOSFETs with $R_{DS(on)} = 6$ mΩ for the low-frequency commutation leg. Input: 230 V_{rms}, 50 Hz. Output: 400 V DC. Power: 2 kW. Switching frequency: 140 kHz.

Find: (a) The RMS input current, (b) the conduction losses in the high-frequency GaN switches, (c) the conduction losses in the low-frequency silicon MOSFETs, (d) the total conduction losses, and (e) the efficiency improvement over a conventional boost PFC with a diode bridge ($V_F = 0.85$ V per diode) and a silicon MOSFET ($R_{DS(on)} = 40$ mΩ).

Solution:

(a) $I_{in(rms)} = P / V_{in} = 2{,}000 / 230 = 8.70$ A

(b) The high-frequency GaN switches carry current with an RMS value that depends on the duty cycle. With average D ≈ 0.42 for 230 V / 400 V operation: Active switch RMS: $I_{Q(rms)} = I_{in} \times \sqrt{(D_{avg})} = 8.70 \times \sqrt{0.42} = 8.70 \times 0.648 = 5.64$ A Sync rectifier RMS: $I_{SR(rms)} = I_{in} \times \sqrt{(1 - D_{avg})} = 8.70 \times \sqrt{0.58} = 8.70 \times 0.762 = 6.63$ A $P_{GaN} = (5.64^2 + 6.63^2) \times 0.035 = (31.8 + 44.0) \times 0.035 = 75.8 \times 0.035 = 2.65$ W

(c) Low-frequency silicon MOSFETs carry the full input current (one conducts per half-cycle): $P_{LF} = I_{in(rms)}^2 \times R_{DS(on)} = 8.70^2 \times 0.006 = 75.7 \times 0.006 = 0.45$ W

(d) Total totem-pole conduction losses: $P_{total} = 2.65 + 0.45 = 3.10$ W

(e) Conventional boost PFC: Diode bridge: $P_{bridge} = 2 \times V_F \times I_{avg} = 2 \times 0.85 \times (8.70 \times 2\sqrt{2}/\pi) = 2 \times 0.85 \times 7.82 = 13.3$ W MOSFET: $P_Q = I_{Q(rms)}^2 \times R_{DS(on)} = 5.64^2 \times 0.040 = 1.27$ W Diode (boost): ~2 W (estimated from freewheeling diode losses) Total conventional: 13.3 + 1.27 + 2.0 = 16.6 W

Reduction: 16.6 − 3.10 = 13.5 W saved Efficiency improvement: 13.5/2,000 = 0.67 percentage points

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Problem 10.7.7

Given: A boost PFC converter must comply with IEC 61000-3-2 Class D (equipment consuming 75-600 W). The converter power is 200 W from a 230 V_{rms} input. The standard limits the harmonic current per watt to: 3rd harmonic ≤ 3.4 mA/W, 5th ≤ 1.9 mA/W, 7th ≤ 1.0 mA/W, 9th ≤ 0.5 mA/W.

Find: (a) The absolute harmonic current limits, (b) the maximum allowable 3rd harmonic current, (c) whether a passive valley-fill PFC circuit with input current THD = 45% would comply (assume the 3rd harmonic is 40% of the fundamental), and (d) the required THD for active PFC compliance.

Solution:

(a) Absolute harmonic limits at 200 W: 3rd: 3.4 × 200 = 680 mA 5th: 1.9 × 200 = 380 mA 7th: 1.0 × 200 = 200 mA 9th: 0.5 × 200 = 100 mA

(b) Maximum allowable 3rd harmonic: 680 mA = 0.68 A

(c) Valley-fill PFC: Fundamental input current: $I_1 = P / V_{in} = 200 / 230 = 0.870$ A (assuming displacement PF ≈ 1) 3rd harmonic at 40% of fundamental: $I_3 = 0.40 \times 0.870 = 0.348$ A = 348 mA Since 348 mA < 680 mA limit, the 3rd harmonic complies.

5th harmonic (typical for valley-fill, ~20% of fundamental): $I_5 = 0.20 \times 0.870 = 174$ mA < 380 mA. Complies. 7th harmonic (~14%): $I_7 = 0.14 \times 0.870 = 122$ mA < 200 mA. Complies. 9th harmonic (~10%): $I_9 = 0.10 \times 0.870 = 87$ mA < 100 mA. Complies.

The passive valley-fill circuit meets IEC 61000-3-2 Class D at 200 W despite 45% THD, because the per-watt limits are generous at lower power levels.

(d) For active PFC, THD is typically below 5%, far exceeding Class D requirements. Active PFC is necessary for Class C (lighting) or for powers above 600 W where Class A limits apply with fixed current maximums.

Problem 10.7.8

Given: A 1.5 kW power supply uses a crowbar overvoltage protection circuit on the 12 V output. The crowbar thyristor fires when V_{out} exceeds 14.4 V (120% of nominal). The output capacitance is 5,000 μF and a 20 A fast-blow fuse protects the circuit.

Find: (a) The energy stored in the output capacitor at the OVP trip point, (b) the peak crowbar current (assuming zero thyristor impedance and 5 mΩ total circuit resistance), (c) the time for the fuse to blow (assuming the fuse requires $20^2 \times t \leq 400\ A^2{\cdot}s$ for fast-blow), and (d) the energy dissipated in the circuit resistance before the fuse blows.

Solution:

(a) Energy stored at OVP trip: $E = \frac{1}{2}CV^2 = 0.5 \times 5{,}000 \times 10^{-6} \times 14.4^2 = 0.5 \times 0.005 \times 207.36 = 0.518$ J

(b) Peak crowbar current: $I_{peak} = V / R = 14.4 / 0.005 = 2{,}880$ A

This is an extremely high pulse current, but it decays rapidly as the capacitor discharges.

(c) The discharge is exponential with $\tau = RC = 0.005 \times 5{,}000 \times 10^{-6} = 25$ µs. The fuse I^2t rating: For the fuse to blow, $\int I^2dt$ must reach 400 $A^2{\cdot}s$. For an exponential discharge: $\int I^2dt = (V^2/R^2) \times (RC/2) = (14.4^2/0.005^2) \times (25 \times 10^{-6}/2) = (207.36/2.5 \times 10^{-5}) \times 12.5 \times 10^{-6} = 8{,}294{,}400 \times 12.5 \times 10^{-6} = 103.7$ $A^2{\cdot}s$

Since 103.7 $A^2{\cdot}s$ < 400 $A^2{\cdot}s$, the capacitor energy alone will not blow the fuse. The fuse blows only if the power supply continues to deliver current into the shorted output. With the PSU delivering 1,500/12 = 125 A into the short: Time to accumulate remaining I^2t: (400 − 103.7) = 296.3 $A^2{\cdot}s$ at 125 A: $t = 296.3 / 125^2 = 296.3 / 15{,}625 = 0.019$ s = 19 ms

(d) Energy dissipated in circuit resistance from capacitor discharge: $E = ½CV^2 = 0.518$ J (all capacitor energy dissipates in the resistance) Additional energy from PSU in 19 ms: $E = V \times I \times t \approx 0.5 \times 125 \times 0.019 = 1.19$ J (approximate, as voltage drops during the event) Total: $0.518 + 1.19 \approx 1.7$ J

Problem 10.7.9

Given: A 600 W AC-DC converter has undervoltage lockout (UVLO) with a turn-on threshold of 80 V_{dc} (after the input rectifier) and a turn-off threshold of 70 V_{dc}, providing 10 V hysteresis. The input is rectified from a 120 V_{rms} AC source with a 100 µF input capacitor.

Find: (a) The DC voltage at steady state (no PFC, capacitor-filtered), (b) the voltage during a brownout at 60 V_{rms} input, (c) whether the converter stays on during the brownout, and (d) the minimum input voltage for the converter to start.

Solution:

(a) Steady-state DC voltage (peak of rectified 120 V_{rms}): $V_{dc} = V_{peak} - \Delta V_{ripple}$ $V_{peak} = 120 \times \sqrt{2} = 169.7$ V For a rough ripple estimate: $\Delta V = P / (2 \times f \times C \times V_{dc}) \approx 600 / (2 \times 60 \times 100 \times 10^{-6} \times 169.7) = 600 / 2.036 = 295$ V

This ripple estimate exceeds the peak voltage, indicating the 100 µF capacitor is too small for 600 W at 120 V — the ripple model breaks down. More realistically, the minimum voltage: $V_{min} \approx V_{peak} \times \sqrt{(1 - P/(\pi \times f \times C \times V_{peak}^2))}$

Using the conduction angle approach: with such a small capacitor at 600 W, the output is heavily rippled. The average DC is approximately: $V_{dc(avg)} \approx 2V_{peak}/\pi = 2 \times 169.7/3.14 = 108$ V (approaching the rectified average without much filtering)

In practice with the small capacitor: V_{dc} varies between ~80 V and ~170 V. The average is approximately 120 V (close to the RMS input due to heavy loading).

(b) At 60 V_{rms} brownout: $V_{peak} = 60 \times \sqrt{2} = 84.9$ V The heavily-loaded rectified voltage drops to approximately $V_{dc(avg)} \approx 60$ V (similar to input RMS)

(c) The UVLO turn-off threshold is 70 V_{dc}. The minimum instantaneous voltage with 60 V_{rms} input will drop well below 70 V between rectified pulses. The converter will turn off during the brownout as the DC voltage drops below 70 V between peaks, then may briefly restart as V rises above 80 V at each peak, causing oscillation. The UVLO hysteresis prevents sustained operation.

(d) Minimum input voltage for start (converter must reach 80 V_{dc}): $V_{in(min,rms)} = 80 / \sqrt{2} = 56.6$ V_{rms} (at zero load, the peak must reach the 80 V threshold) Under load, the minimum is higher due to voltage drops. Practically: $V_{in(min)} \approx 65\text{-}70\ V_{rms}$ depending on load.

Problem 10.7.10

Given: A server power supply rated at 2.4 kW, 230 V_{rms} input uses an active PFC stage achieving 99.2% power factor and 2.8% current THD. Without PFC, the same supply would have a power factor of 0.62 and current THD of 130%.

Find: (a) The RMS input current with and without PFC, (b) the apparent power with and without PFC, (c) the reactive/distortion power saved by PFC, (d) the upstream conductor I^2R savings (assume 0.2 Ω line impedance), and (e) the harmonic current reduction at the 3rd harmonic.

Solution:

(a) With PFC: $I_{in(rms)} = P / (V \times PF) = 2{,}400 / (230 \times 0.992) = 2{,}400 / 228.2 = 10.52$ A

Without PFC: $I_{in(rms)} = P / (V \times PF) = 2{,}400 / (230 \times 0.62) = 2{,}400 / 142.6 = 16.83$ A

(b) Apparent power: With PFC: $S = V \times I = 230 \times 10.52 = 2{,}420$ VA Without PFC: $S = V \times I = 230 \times 16.83 = 3{,}871$ VA

(c) Reactive/distortion power: With PFC: $Q = \sqrt{(S^2 - P^2)} = \sqrt{(2{,}420^2 - 2{,}400^2)} = \sqrt{(5{,}856{,}400 - 5{,}760{,}000)} = \sqrt{96{,}400} = 310$ VAR Without PFC: $Q = \sqrt{(3{,}871^2 - 2{,}400^2)} = \sqrt{(14{,}984{,}641 - 5{,}760{,}000)} = \sqrt{9{,}224{,}641} = 3{,}037$ VAR Savings: $3{,}037 - 310 = 2{,}727$ VAR

(d) I^2R losses in line impedance: With PFC: $P_{line} = 10.52^2 \times 0.2 = 110.7 \times 0.2 = 22.1$ W Without PFC: $P_{line} = 16.83^2 \times 0.2 = 283.2 \times 0.2 = 56.6$ W Savings: $56.6 - 22.1 = 34.5$ W (61% reduction in line losses)

(e) 3rd harmonic current: Without PFC (130% THD, 3rd harmonic is dominant, typically ~90% of fundamental): $I_1 = I_{rms} / \sqrt{(1 + THD^2)} = 16.83 / \sqrt{(1 + 1.30^2)} = 16.83 / \sqrt{2.69} = 16.83 / 1.640 = 10.26$ A $I_3 \approx 0.90 \times 10.26 = 9.23$ A (without PFC)

With PFC (2.8% THD, 3rd harmonic ~2% of fundamental): $I_1 \approx 10.52$ A, $I_3 = 0.02 \times 10.52 = 0.21$ A (with PFC) Reduction: $9.23 \rightarrow 0.21$ A, a factor of 44× reduction.

Chapter 10 — Section 10.8: Battery Management Systems

Practice problems covering battery cell characteristics, pack design, cell balancing, state of charge estimation, coulomb counting, BMS thermal management, battery protection, and fault detection.

Problem 10.8.1

Given: An electric bus battery pack uses 120 series-connected LFP ($LiFePO_4$) cells with a nominal voltage of 3.2 V and a capacity of 100 Ah each. The cells have an internal resistance of 1.5 mΩ each. The maximum discharge rate is 2C.

Find: (a) The nominal pack voltage, (b) the energy capacity in kWh, (c) the maximum continuous discharge current, (d) the total pack internal resistance, and (e) the terminal voltage and power at maximum discharge.

Solution:

(a) Nominal pack voltage: $V_{pack} = 120 \times 3.2 = 384.0$ V

(b) Energy capacity: $E = V_{pack} \times C = 384.0 \times 100 = 38{,}400$ Wh $= 38.4$ kWh

(c) Maximum discharge current at 2C: $I_{max} = 2 \times 100 = 200$ A

(d) Total pack internal resistance (series cells): $R_{pack} = 120 \times 0.0015 = 0.180\ \Omega$

(e) Voltage drop at maximum discharge: $\Delta V = I_{max} \times R_{pack} = 200 \times 0.180 = 36.0$ V $V_{terminal} = V_{pack} - \Delta V = 384.0 - 36.0 = 348.0$ V $P_{max} = V_{terminal} \times I_{max} = 348.0 \times 200 = 69.6$ kW Power lost to internal resistance: $P_{loss} = I^2 \times R = 200^2 \times 0.180 = 7.2$ kW (9.4% of total)

Problem 10.8.2

Given: A 16-series lithium NMC battery pack uses passive cell balancing with 47 Ω bleed resistors. At end of charge, cell voltages range from 4.05 V (lowest) to 4.20 V (highest). The target is to balance all cells to 4.05 V. The highest cell is 80 mAh above the target.

Find: (a) The balancing current for the highest-voltage cell, (b) the power dissipated in its balancing resistor, (c) the time to remove 80 mAh, (d) the total energy wasted across all cells needing balancing (assume 8 cells require balancing with an average excess of 40 mAh), and (e) the maximum heat generated per resistor.

Solution:

(a) Balancing current for the highest cell: $I_{bal} = V_{cell} / R = 4.20 / 47 = 89.4$ mA

(b) Power per resistor: $P = V^2 / R = 4.20^2 / 47 = 17.64 / 47 = 0.375$ W

(c) Time to balance 80 mAh: $t = Q / I = 80 / 89.4 = 0.895$ hours = 53.7 minutes

(d) Energy wasted across 8 cells with average 40 mAh excess: Average balancing time per cell: $t_{avg} = 40 / 89.4 = 0.4474$ hours Average power per cell: $P_{avg} \approx (4.12)^2 / 47 = 0.361$ W (using average cell voltage of ~4.12 V) Total energy: $E = 8 \times 0.361 \times 0.4474 = 1.29$ Wh

(e) Maximum heat per resistor: $P_{max} = 0.375$ W (occurs at the highest cell voltage of 4.20 V) A 47 Ω resistor rated at 0.5 W or higher is required, with adequate PCB copper area for heat dissipation.

Problem 10.8.3

Given: An active cell balancing system uses a switched-capacitor circuit to transfer charge between adjacent cells. The balancing capacitor is 100 μF, the switching frequency is 50 kHz, and the switch resistance is 30 mΩ per MOSFET (two in series per transfer path). The voltage difference between two adjacent cells is 50 mV.

Find: (a) The charge transferred per switching cycle, (b) the theoretical maximum balancing current, (c) the actual balancing current accounting for switch resistance, (d) the balancing efficiency, and (e) the time to transfer 100 mAh.

Solution:

(a) Charge per cycle: $Q_{cycle} = C \times \Delta V = 100 \times 10^{-6} \times 0.050 = 5.0 \times 10^{-6}$ C = 5.0 μC

(b) Theoretical maximum balancing current: $I_{bal(max)} = Q_{cycle} \times f_{sw} = 5.0 \times 10^{-6} \times 50{,}000 = 0.25$ A = 250 mA

(c) With switch resistance (two MOSFETs per path, $R_{total} = 2 \times 0.030 = 0.060\ \Omega$): The RC time constant: $\tau = R \times C = 0.060 \times 100 \times 10^{-6} = 6.0$ μs Half-period: $T/2 = 1/(2 \times 50{,}000) = 10$ μs Ratio $T/(2\tau) = 10/6.0 = 1.67$, so the capacitor charges/discharges approximately 81% per half-cycle ($1 - e^{-1.67} = 0.812$). Actual balancing current: $I_{bal} = 0.812 \times 250 = 203$ mA

(d) Efficiency: Energy transferred per cycle: $E_{transfer} = C \times \Delta V^2 / 2 \times$ (charge fraction) Power delivered to lower cell: $P_{out} = I_{bal} \times V_{lower} \times \Delta V / (\Delta V + I_{bal} \times R)$ Simplified: efficiency $\approx V_{lower} / (V_{lower} + I_{bal} \times R) = 3.95 / (3.95 + 0.203 \times 0.060) = 3.95 / 3.962 = 99.7\%$

Active balancing is far more efficient than passive balancing (which wastes 100% of the transferred energy as heat).

(e) Time to transfer 100 mAh: $t = Q / I_{bal} = 100 / 203 = 0.493$ hours = 29.6 minutes

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Problem 10.8.4

Given: A 100 Ah battery pack starts at SOC = 92%. A coulomb counting BMS integrates the measured current over a 4-hour drive cycle. The current sensor has a gain error of ±0.3% and an offset of ±15 mA. The pack delivers a total of 72 Ah during the drive cycle.

Find: (a) The estimated SOC after the drive cycle, (b) the worst-case coulomb counting error from gain error, (c) the worst-case error from offset drift, (d) the total SOC uncertainty, and (e) the recommended recalibration method.

Solution:

(a) Estimated SOC: SOC = 92% − (72/100) × 100% = 92% − 72% = 20%

(b) Gain error over 72 Ah: $\Delta Q_{gain} = 0.003 \times 72 = 0.216$ Ah SOC error from gain: 0.216/100 × 100% = 0.216%

(c) Offset error over 4 hours: $\Delta Q_{offset} = 0.015 \times 4 = 0.060$ Ah SOC error from offset: 0.060/100 × 100% = 0.060%

(d) Total worst-case SOC uncertainty (errors add): $\Delta Q_{total} = 0.216 + 0.060 = 0.276$ Ah SOC uncertainty: ±0.276%, giving SOC range of 19.7% to 20.3%

This is a relatively small error for a single drive cycle. Over many cycles without recalibration, the offset error accumulates linearly (0.06 Ah per 4-hour cycle), reaching 1% SOC error after approximately: $n = (1.0/0.060) = 16.7$ cycles ≈ 17 drive cycles

(e) Recalibration method: When the pack is fully charged (charger terminates at the cell voltage limit), reset SOC to 100%. When the pack reaches a known rest state, use the OCV-SOC lookup table to correct the coulomb-counted SOC. A Kalman filter combining coulomb counting with voltage measurement provides continuous correction, reducing accumulated drift to near zero.

Problem 10.8.5

Given: An EV battery pack generates 1,200 W of heat at sustained highway driving. The liquid cooling system uses a 50/50 water-glycol mixture with $c_p = 3{,}350$ J/kg·°C and $\rho = 1.06$ kg/L. The flow rate is 8 L/min. The maximum cell surface temperature is 35°C with a 4°C thermal resistance between coolant and cell.

Find: (a) The coolant temperature rise through the pack, (b) the required coolant inlet temperature, (c) the mass flow rate, (d) the thermal power capacity of the cooling system at 15°C ΔT, and (e) the maximum C-rate before the cooling system is exceeded (assume heat generation scales with I^2R where $R_{pack} = 0.15\ \Omega$ and the pack is 80 Ah).

Solution:

(a) Mass flow rate: $\dot{m} = 8 \times 1.06 = 8.48$ kg/min = 0.1413 kg/s Coolant temperature rise: $\Delta T_{coolant} = \dot{Q} / (\dot{m} \times c_p) = 1{,}200 / (0.1413 \times 3{,}350) = 1{,}200 / 473.4 = 2.54$°C

(b) Maximum coolant temperature at cell interface: $T_{coolant,max} = 35 - 4 = 31$°C Inlet temperature: $T_{inlet} = T_{coolant,max} - \Delta T_{coolant} = 31 - 2.54 = 28.5$°C

(c) Mass flow rate: 0.1413 kg/s (as calculated above, = 8.48 kg/min)

(d) Maximum cooling capacity at 15°C ΔT (if the coolant inlet were at 16°C): $\dot{Q}_{max} = \dot{m} \times c_p \times \Delta T_{max}$ $= 0.1413 \times 3{,}350 \times 15 = 7{,}100$ W = 7.1 kW

(e) Heat generation: $Q = I^2 \times R_{pack} = I^2 \times 0.15$ Cooling capacity $\dot{Q}_{max} = 7{,}100$ W: $I_{max} = \sqrt{(7{,}100/0.15)} = \sqrt{47{,}333} = 217.6$ A C-rate = 217.6/80 = 2.72C

At this C-rate, the cooling system reaches its limit. Higher discharge rates require either increased flow rate, lower inlet temperature, or temporary operation with rising cell temperature.

Problem 10.8.6

Given: A BMS monitors a 96-series NMC pack using a daisy-chained analog front end (AFE) IC. Each AFE measures 12 cells with a voltage accuracy of ±2.5 mV and a temperature measurement accuracy of ±1°C using NTC thermistors. The OVP threshold is 4.20 V and UVP threshold is 2.80 V per cell.

Find: (a) The number of AFE ICs required, (b) the minimum and maximum cell voltages the BMS must distinguish, (c) the pack voltage measurement accuracy, (d) the OVP and UVP thresholds with measurement uncertainty, and (e) the required bits for the cell voltage ADC to achieve 1 mV resolution over a 0-5 V range.

Solution:

(a) Number of AFE ICs: N = 96 / 12 = 8 AFE ICs

(b) Cell voltage range: Minimum: 2.50 V (deep discharge, below UVP) to maximum: 4.25 V (above OVP, transient) Useful range: approximately 2.50 V to 4.25 V (1.75 V span)

(c) Pack voltage measurement accuracy: Each cell has ±2.5 mV error. For 96 cells in series, the worst-case pack accuracy: $\Delta V_{pack} = 96 \times 2.5 = 240$ mV = ±0.24 V At nominal pack voltage (96 × 3.7 = 355.2 V): accuracy = 0.24/355.2 = ±0.068%

(d) OVP with measurement uncertainty: The BMS must set its OVP threshold to: $4.20 - 2.5 \times 10^{-3} = 4.1975$ V to ensure no cell exceeds 4.20 V. UVP threshold: $2.80 + 2.5 \times 10^{-3} = 2.8025$ V to ensure no cell drops below 2.80 V. This narrows the usable voltage window by 5 mV total (2 × 2.5 mV), reducing usable capacity by approximately 0.5%.

(e) Required ADC resolution for 1 mV over 0-5 V: Bits = $\log_2(V_{range}$ / resolution) = $\log_2(5.0 / 0.001)$ = $\log_2(5{,}000) = 12.3$ bits → 13 bits minimum

Most BMS AFE ICs use 14-bit or 16-bit ADCs to provide sub-millivolt resolution with headroom.

Problem 10.8.7

Given: A battery pack experiences thermal runaway propagation testing per UL 9540A. A single cell with 50 Wh energy content releases 100% of its energy as heat in 3 seconds during thermal runaway. The cell mass is 0.9 kg and specific heat is 1,100 J/kg·°C. The adjacent cell is separated by 2 mm of aerogel insulation (k = 0.02 W/m·K) with a contact area of 100 mm × 200 mm.

Find: (a) The peak temperature of the failing cell, (b) the heat flux through the insulation, (c) the temperature rise of the adjacent cell after 30 seconds (assuming the adjacent cell's thermal mass absorbs all conducted heat), and (d) whether the adjacent cell is at risk of cascading thermal runaway (onset at 150°C).

Solution:

(a) Energy released: $E = 50 \times 3{,}600 = 180{,}000$ J Temperature rise of failing cell: $\Delta T = E / (m \times c) = 180{,}000 / (0.9 \times 1{,}100) = 180{,}000 / 990 = 181.8$°C Peak temperature: $T = 25 + 181.8 = 206.8$°C (in practice, much higher due to non-uniform heating and gas venting)

(b) Heat flux through insulation (steady-state maximum): Contact area: $A = 0.100 \times 0.200 = 0.020$ m^2 ΔT across insulation $\approx 206.8 - 25 = 181.8$°C (worst case, adjacent cell initially at 25°C) $\dot{Q} = k \times A \times \Delta T / d = 0.02 \times 0.020 \times 181.8 / 0.002 = 0.02 \times 0.020 \times 90{,}900 = 36.4$ W

(c) Adjacent cell temperature rise in 30 seconds: $E_{conducted} = \dot{Q} \times t = 36.4 \times 30 = 1{,}092$ J (using peak heat flux as upper bound) $\Delta T_{adjacent} = E / (m \times c) = 1{,}092 / (0.9 \times 1{,}100) = 1{,}092 / 990 = 1.1$°C Adjacent cell temperature: $25 + 1.1 = 26.1$°C

(d) At 26.1°C, the adjacent cell is far below the 150°C thermal runaway onset. The aerogel insulation provides effective propagation resistance. Even after 10 minutes (600 s): $E_{conducted} = 36.4 \times 600 = 21{,}840$ J $\rightarrow \Delta T = 21{,}840/990 = 22.1$°C $\rightarrow T = 47.1$°C The adjacent cell is not at risk of cascading thermal runaway with the 2 mm aerogel barrier.

Problem 10.8.8

Given: A Kalman filter SOC estimator uses a first-order equivalent circuit model: $V_{terminal} = OCV(SOC) - I \times R_0 - V_{RC}$, where $R_0 = 3.0$ mΩ (ohmic resistance), $R_1 = 1.5$ mΩ (polarization resistance), $C_1 = 5{,}000$ F (polarization capacitance), and the OCV-SOC slope is dOCV/dSOC = 0.8 V per unit SOC (= 8 mV per 1% SOC) in the mid-SOC range. The current sensor noise is $\sigma_I = 0.1$ A and voltage sensor noise is $\sigma_V = 2$ mV.

Find: (a) The RC time constant of the polarization circuit, (b) the steady-state voltage across the RC network at I = 50 A, (c) the voltage change for a 1% SOC change, (d) the process noise and measurement noise for the Kalman filter, and (e) the expected steady-state SOC estimation accuracy.

Solution:

(a) RC time constant: $\tau = R_1 \times C_1 = 0.0015 \times 5{,}000 = 7.5$ seconds

This represents the polarization relaxation time — the voltage response delay when current changes.

(b) Steady-state RC voltage at 50 A: $V_{RC(ss)} = I \times R_1 = 50 \times 0.0015 = 75$ mV

The total voltage drop under load: $V_{drop} = I \times R_0 + V_{RC} = 50 \times 0.003 + 0.075 = 225$ mV

(c) Voltage change per 1% SOC: $\Delta V_{OCV} = 0.008$ V = 8 mV per 1% SOC

(d) Process noise (SOC drift per time step Δt, from current integration): For Δt = 1 s and Q_{cap} = 100 Ah = 360,000 C: $\sigma_{SOC} = \sigma_I \times \Delta t / Q_{cap} = 0.1 \times 1 / 360{,}000 = 2.78 \times 10^{-7} = 0.0000278\%$ per second

Measurement noise (voltage, converted to SOC equivalent): $\sigma_{SOC(meas)} = \sigma_V$ / (dOCV/dSOC) = 0.002 / 0.8 = 0.0025 = 0.25% SOC equivalent

(e) The Kalman filter optimally fuses the process model (coulomb counting) with the voltage measurement. The steady-state estimation accuracy is bounded by: $\sigma_{SOC(KF)} \approx \sqrt{}(\sigma_{process} \times \sigma_{measurement})$ (geometric mean, simplified)

Over a 1-hour window (3,600 s): accumulated process noise = $2.78 \times 10^{-7} \times \sqrt{}3{,}600 = 1.67 \times 10^{-5}$ The Kalman filter corrects continuously using voltage, achieving steady-state accuracy of approximately ±0.5% SOC — significantly better than coulomb counting alone (which drifts) or voltage-based estimation alone (±0.25% but noisy).

Problem 10.8.9

Given: A 400 V, 150 Ah EV battery pack uses a contactor (high-voltage relay) with a 2 ms opening time for fault disconnection. The pack has 0.5 mΩ current-sense resistor and the BMS detects a short circuit when dI/dt exceeds 500 A/ms. Total circuit inductance is 20 μH.

Find: (a) The maximum short-circuit current assuming 0.5 Ω total fault impedance, (b) the current at the time of fault detection (assuming initial current is 100 A and the short occurs instantaneously), (c) the current at contactor opening (2 ms after detection), (d) the energy in the circuit inductance at opening, and (e) the arc energy during contactor opening.

Solution:

(a) Maximum steady-state short-circuit current: $I_{SC(max)} = V / R_{fault}$ = 400 / 0.5 = 800 A

(b) The rate of current rise is limited by circuit inductance: dI/dt = V / L = $400 / 20 \times 10^{-6} = 20 \times 10^6$ A/s = 20,000 A/ms

At dI/dt = 500 A/ms (detection threshold), the time from fault to detection: The actual dI/dt starts at 20,000 A/ms and decreases exponentially. With L/R = $20 \times 10^{-6}/0.5$ = 40 μs time constant:

After ~40 μs, the current reaches: $I = I_{SC}(1 - e^{-t/\tau}) + I_0 \times e^{-t/\tau}$ At 40 μs: $I = 800(1 - 0.368) + 100 \times 0.368 = 800 \times 0.632 + 36.8 = 505.6 + 36.8 = 542$ A

The dI/dt at this point: $dI/dt = (V - IR)/L = (400 - 542 \times 0.5)/20 \times 10^{-6} = (400 - 271)/20 \times 10^{-6} = 6.45 \times 10^6$ A/s = 6,450 A/ms

dI/dt drops below 500 A/ms when: $500 \times 10^3 = (400 - I \times 0.5)/20 \times 10^{-6}$ $400 - 0.5I = 500 \times 10^3 \times 20 \times 10^{-6} = 10$ $I = (400 - 10)/0.5 = 780$ A at detection

(c) Current 2 ms after detection (circuit is approaching steady state since τ = 40 μs ≪ 2 ms): I at opening $\approx I_{SC(max)}$ = 800 A (essentially at steady state)

(d) Energy in circuit inductance: $E = ½LI^2 = 0.5 \times 20 \times 10^{-6} \times 800^2 = 0.5 \times 20 \times 10^{-6} \times 640{,}000 = 6.4$ J

(e) Arc energy during opening (arc sustains for ~1 ms at ~30 V arc voltage, worst case): $E_{arc} = V_{arc} \times I \times t_{arc} = 30 \times 800 \times 0.001 = 24$ J

The contactor must be rated for this arc energy. Pre-charge resistors and arc suppression (varistors) across the contactor reduce the arc energy and extend contactor life.

Problem 10.8.10

Given: An EV battery pack with 96 series NMC cells (3.7 V nom, 60 Ah) has been in service for 3 years with an average of 1 full cycle per day. The cells have a degradation rate of 0.02% capacity loss per full equivalent cycle and 1.5% calendar aging per year. The initial capacity is 60 Ah.

Find: (a) The total number of full equivalent cycles after 3 years, (b) the cycling capacity loss, (c) the calendar aging loss, (d) the total capacity loss and current SOH, and (e) the remaining usable energy in kWh if the pack is considered end-of-life at 80% SOH.

Solution:

(a) Total cycles: $N = 365 \times 3 = 1{,}095$ full equivalent cycles

(b) Cycling capacity loss: $Loss_{cycling} = N \times 0.02\% = 1{,}095 \times 0.0002 = 0.219 = 21.9\%$

(c) Calendar aging loss: $Loss_{calendar} = 3 \times 1.5\% = 4.5\%$

(d) Total capacity loss (these mechanisms are roughly additive for this model): $Loss_{total} = 21.9\% + 4.5\% = 26.4\%$ Current capacity: $Q = 60 \times (1 - 0.264) = 60 \times 0.736 = 44.16$ Ah $SOH = 44.16/60 \times 100\% = 73.6\%$

(e) Since $SOH = 73.6\% < 80\%$, the pack has already reached end-of-life for its intended EV application. The remaining usable energy: $E = 96 \times 3.7 \times 44.16 = 355.2 \times 44.16 = 15{,}685$ Wh $= 15.7$ kWh (down from 21.3 kWh when new)

The pack may be suitable for second-life stationary storage where the lower power demands and relaxed cycle life requirements can extract additional value. The high cycling degradation rate (0.02% per cycle) suggests the cells are being stressed — reducing DOD to 70% or lowering the C-rate could significantly extend life in future designs.

Chapter 10 — Section 10.9: Battery Energy Storage Systems

Practice problems covering BESS architecture, power conversion systems, round-trip efficiency, grid services, frequency regulation, peak shaving, BESS sizing, levelized cost of storage, and degradation analysis.

Problem 10.9.1

Given: A utility-scale BESS project requires 100 MW / 400 MWh (4-hour duration) using LFP cells rated at 3.2 V nominal, 280 Ah. Each battery module has 16 series cells (51.2 V nominal). Each rack has 12 series modules (614.4 V DC bus). The PCS inverters are rated at 5 MW each.

Find: (a) The energy per rack, (b) the number of racks required, (c) the total number of cells, (d) the number of PCS inverters, and (e) the number of 40-foot containers if each container holds 30 racks.

Solution:

(a) Energy per rack: $E_{rack} = V_{rack} \times C = 614.4 \times 280 = 172{,}032$ Wh = 172.0 kWh

(b) Number of racks: $N_{racks} = 400{,}000 / 172.0 = 2{,}325.6 \rightarrow 2{,}326$ racks (round up)

(c) Cells per rack = $16 \times 12 = 192$ Total cells = $2{,}326 \times 192 = 446{,}592$ cells

(d) Number of PCS inverters: $N_{PCS} = 100 / 5 = 20$ inverters

(e) Number of containers: $N_{containers} = 2{,}326 / 30 = 77.5 \rightarrow 78$ battery containers (plus additional containers for PCS, switchgear, HVAC, and controls — typically 20-25 more, for approximately 100 total)

Problem 10.9.2

Given: A 25 MW / 100 MWh BESS uses a PCS with one-way (single-direction) efficiency of 97.5% and auxiliary power consumption of 1.5% of throughput. The system completes 350 cycles per year at 85% depth of discharge.

Find: (a) The round-trip efficiency, (b) the annual energy discharged, (c) the annual energy charged (input), (d) the annual energy losses, and (e) the annual auxiliary consumption.

Solution:

(a) Round-trip efficiency: $\eta_{RT} = \eta_{inv}{}^2 \times (1 - p_{aux})^2 = 0.975^2 \times (1 - 0.015)^2$ $\eta_{RT} = 0.9506 \times 0.9702 = 92.2\%$

(b) Annual energy discharged: $E_{discharge} = 100 \times 0.85 \times 350 = 29{,}750$ MWh/year

(c) Annual energy charged: $E_{charge} = E_{discharge} / \eta_{RT} = 29{,}750 / 0.922 = 32{,}268$ MWh/year

(d) Annual energy losses: $E_{loss} = E_{charge} - E_{discharge} = 32{,}268 - 29{,}750 = 2{,}518$ MWh/year

(e) Annual auxiliary consumption (applied to both charge and discharge): $E_{aux} = (E_{charge} + E_{discharge}) \times 0.015 = (32{,}268 + 29{,}750) \times 0.015 = 62{,}018 \times 0.015 = 930$ MWh/year

The auxiliary consumption (930 MWh) is included within the total losses (2,518 MWh). The remaining 1,588 MWh are inverter conversion losses.

Problem 10.9.3

Given: A 30 MW BESS provides primary frequency regulation with a 5% droop setting. The system has a ±0.025 Hz deadband around the 60 Hz nominal frequency. During a grid event, the frequency drops to 59.70 Hz and remains there for 8 minutes before recovering.

Find: (a) The effective frequency deviation outside the deadband, (b) the BESS power output during the event, (c) the energy dispatched during the 8-minute event, (d) the SOC change if the BESS has 120 MWh capacity, and (e) the frequency support if two identical BESS units provide the same service.

Solution:

(a) Frequency deviation: $\Delta f = 60 - 59.70 = 0.30$ Hz Effective deviation (subtract deadband): $\Delta f_{eff} = 0.30 - 0.025 = 0.275$ Hz

(b) Power output using droop equation: $\Delta P = P_{rated} \times \Delta f_{eff} / (f_{nom} \times R) = 30 \times 0.275 / (60 \times 0.05) = 30 \times 0.275 / 3.0 = 2.75$ MW (discharging)

(c) Energy dispatched: $E = P \times t = 2.75 \times (8/60) = 2.75 \times 0.1333 = 0.367$ MWh

(d) SOC change: $\Delta SOC = E / E_{total} \times 100 = 0.367 / 120 \times 100 = 0.306\%$

The small SOC impact confirms that frequency regulation is an energy-light, power-heavy application ideally suited for BESS.

(e) With two 30 MW units (60 MW total), each provides 2.75 MW: Total response = $2 \times 2.75 = 5.50$ MW

Alternatively, if each operates independently with the same droop, the combined response at the grid level would help arrest the frequency deviation faster. In a system with sufficient BESS, the frequency would not drop as far due to the faster response.

Problem 10.9.4

Given: A BESS provides voltage support (reactive power) at a 13.8 kV distribution bus. The PCS inverter is rated at 10 MVA with a maximum active power output of 8 MW. The grid needs 5 MVAR of reactive power support while the BESS simultaneously discharges at 6 MW.

Find: (a) The total apparent power required, (b) whether the inverter can simultaneously provide both P and Q, (c) the inverter current, (d) the power factor, and (e) the maximum reactive power available if the BESS is discharging at full 8 MW.

Solution:

(a) Total apparent power: $S = \sqrt{(P^2 + Q^2)} = \sqrt{(6^2 + 5^2)} = \sqrt{(36 + 25)} = \sqrt{61} = 7.81$ MVA

(b) Since $S = 7.81$ MVA $< S_{rated} = 10$ MVA, the inverter can provide both simultaneously.

(c) Inverter current: $I = S / (\sqrt{3} \times V) = 7{,}810{,}000 / (1.732 \times 13{,}800) = 7{,}810{,}000 / 23{,}902 = 327$ A

(d) Power factor: $PF = P / S = 6 / 7.81 = 0.768$ lagging (since Q is inductive/lagging)

(e) Maximum Q at full active power: $Q_{max} = \sqrt{(S_{rated}{}^2 - P_{max}{}^2)} = \sqrt{(10^2 - 8^2)} = \sqrt{(100 - 64)} = \sqrt{36} = 6.0$ MVAR

Even at full 8 MW discharge, the inverter can still provide 6.0 MVAR of reactive power support — a significant advantage of BESS over conventional generators for combined active/reactive power support.

Problem 10.9.5

Given: A commercial facility has a load profile where demand exceeds 4.0 MW for 3 hours per day, peaking at 5.5 MW. Below 4.0 MW, the demand averages 2.8 MW for the remaining 21 hours. The utility demand charge is \$22/kW-month. The BESS round-trip efficiency is 90%.

Find: (a) The BESS power rating to shave the peak to 4.0 MW, (b) the energy capacity required (model the peak as triangular), (c) the monthly demand charge savings, (d) the annual savings, and (e) the energy needed to recharge the BESS daily.

Solution:

(a) BESS power rating: $P_{BESS} = P_{peak} - P_{target} = 5.5 - 4.0 = 1.5$ MW

(b) Energy capacity (triangular peak profile over 3 hours with 1.5 MW maximum excess): $E = ½ \times P_{BESS} \times \text{duration} = ½ \times 1.5 \times 3 = 2.25$ MWh

Add 15% margin for degradation: $E_{nameplate} = 2.25 / 0.85 = 2.65$ MWh (specify 3.0 MWh standard module)

(c) Monthly demand charge savings: $\Delta D = (5.5 - 4.0) \times 1{,}000 \times \$22 = 1{,}500 \times \$22 = \$33{,}000$/month

(d) Annual savings: Savings = \$33,000 × 12 = \$396,000/year

(e) Daily recharge energy: $E_{charge} = E_{discharge} / \eta_{RT} = 2.25 / 0.90 = 2.50$ MWh/day

The additional grid energy for recharging costs approximately 2.50 × \$50/MWh = \$125/day or \$3,750/month, which is subtracted from the demand charge savings for a net monthly benefit of \$33,000 − \$3,750 = \$29,250.

Problem 10.9.6

Given: A BESS performs energy arbitrage, charging during off-peak hours at \$25/MWh and discharging during on-peak hours at \$85/MWh. The system is 50 MW / 200 MWh with 91% round-trip efficiency. It operates 330 days per year at 80% DOD.

Find: (a) The daily energy discharged, (b) the daily energy required for charging, (c) the daily revenue from arbitrage, (d) the daily cost of charging, and (e) the annual net arbitrage revenue.

Solution:

(a) Daily energy discharged: $E_{discharge} = 200 \times 0.80 = 160$ MWh/day

(b) Daily energy for charging: $E_{charge} = E_{discharge} / \eta_{RT} = 160 / 0.91 = 175.8$ MWh/day

(c) Daily revenue: Revenue $= E_{discharge} \times \text{price}_{\text{on-peak}} = 160 \times \$85 = \$13{,}600$/day

(d) Daily charging cost: Cost $= E_{charge} \times \text{price}_{\text{off-peak}} = 175.8 \times \$25 = \$4{,}395$/day

(e) Annual net arbitrage revenue: Daily profit = \$13,600 − \$4,395 = \$9,205/day Annual = \$9,205 × 330 = \$3,037,650/year

The \$60/MWh spread yields \$3.04M/year. With a typical BESS installed cost of \$250/kWh × 200,000 = \$50M, the simple payback from arbitrage alone is 50M/3.04M = 16.4 years — too long for a viable standalone business case. Revenue stacking with frequency regulation, capacity payments, and demand charge reduction is essential for economic viability.

Problem 10.9.7

Given: A 200 MWh LFP BESS has the following degradation parameters: 0.015% capacity loss per full equivalent cycle (FEC) and 1.2% calendar fade per year. The system operates 1.2 cycles per day at 75% DOD. The performance guarantee requires 80% capacity retention.

Find: (a) The annual full equivalent cycles, (b) the annual cycling degradation, (c) the total annual degradation (cycling + calendar), (d) the number of years to reach 80% capacity, and (e) the total energy throughput over the system's warranted life.

Solution:

(a) Annual FEC: $FEC_{annual} = 1.2$ cycles/day $\times 0.75$ DOD $\times 365 = 328.5$ FEC/year

(b) Annual cycling degradation: $Loss_{cycling} = 328.5 \times 0.015\% = 328.5 \times 0.00015 = 4.93\%$/year

(c) Total annual degradation: $Loss_{total} = 4.93\% + 1.2\% = 6.13\%$/year

(d) Years to reach 80% retention: Capacity loss to reach 80%: 20% Years = 20% / 6.13% = 3.26 years

This is an unrealistically short warranty period, indicating the degradation parameters are aggressive. More typical LFP degradation (0.005% per FEC) would yield: $\text{Loss}_{\text{cycling}} = 328.5 \times 0.005\% = 1.64\%/\text{year}$ Total = 1.64% + 1.2% = 2.84%/year Years = 20% / 2.84% = 7.0 years (still below the 15-20 year target for utility BESS)

Augmentation (adding modules to replace lost capacity) is planned at years 7-8 to extend the system to 15+ years.

(e) Total energy throughput over 3.26 years (with original degradation rate): Annual throughput = $200 \times 0.75 \times 1.2 \times 365 = 65{,}700$ MWh/year (year 1) Average annual throughput (accounting for degradation): $\approx 65{,}700 \times (1 - 0.0613 \times 1.63) = 65{,}700 \times 0.90 = 59{,}130$ MWh avg Total throughput $\approx 59{,}130 \times 3.26 = 192{,}764$ MWh

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Problem 10.9.8

Given: A BESS provides renewable smoothing for a 50 MW solar farm. The maximum allowable ramp rate at the point of interconnection (POI) is 10% of plant capacity per minute. The solar irradiance causes a cloud-induced ramp of 35 MW in 2 minutes.

Find: (a) The maximum allowable power ramp rate, (b) the maximum power change over 2 minutes, (c) the excess ramp that must be compensated by the BESS, (d) the energy the BESS must supply or absorb during the event, and (e) the minimum BESS power and energy rating for this service.

Solution:

(a) Maximum allowable ramp rate: $\Delta P/\Delta t = 10\% \times 50$ MW/min = 5 MW/min

(b) Maximum allowable power change in 2 minutes: $\Delta P_{max} = 5 \times 2 = 10$ MW

(c) The solar ramp is 35 MW in 2 minutes (17.5 MW/min), but only 10 MW is allowed. Excess ramp: $\Delta P_{excess} = 35 - 10 = 25$ MW that the BESS must compensate

(d) The BESS must inject power to replace the lost solar generation and smooth the ramp. Over the 2-minute event, the BESS ramps from 0 to 25 MW linearly: $E = ½ \times 25 \times (2/60) = ½ \times 25 \times 0.0333 = 0.417$ MWh

(e) Minimum BESS power rating: $P_{BESS} = 25$ MW (must cover the full excess ramp)

Minimum energy: 0.417 MWh, but the BESS must handle repeated events. With a design margin for 10 consecutive events: $E_{BESS} = 10 \times 0.417 = 4.17$ MWh → 5 MWh (with margin)

A 25 MW / 5 MWh BESS (12-minute duration) is sufficient for ramp rate compliance. This is a short-duration, high-power application.

Problem 10.9.9

Given: A BESS project has the following cost structure: battery modules at $130/kWh, PCS at $80/kW, balance of plant at $40/kWh, EPC and soft costs at $25/kWh. The system is 50 MW / 200 MWh with a 20-year life, 8% discount rate, 365 cycles/year at 80% DOD, 91% round-trip efficiency, and O&M of

\$5/MWh throughput. Augmentation of 15% capacity is required at year 10 at 50% of original battery module cost.

Find: (a) The total installed capital cost, (b) the annual energy throughput, (c) the present worth of lifetime O&M, (d) the present worth of augmentation, and (e) the LCOS.

Solution:

(a) Total capital cost: Battery: 200,000 × \$130 = \$26,000,000 PCS: 50,000 × \$80 = \$4,000,000 BOP: 200,000 × \$40 = \$8,000,000 EPC: 200,000 × \$25 = \$5,000,000 Total: \$26,000,000 + \$4,000,000 + \$8,000,000 + \$5,000,000 = \$43,000,000

(b) Annual energy throughput (discharge side): $E_{annual} = 200 \times 0.80 \times 365 = 58{,}400$ MWh/year (year 1, declining with degradation)

(c) O&M costs: Annual O&M = 58,400 × \$5 = \$292,000/year Present worth factor (P/A, 8%, 20) = $(1 - 1.08^{-20})/0.08 = (1 - 0.2145)/0.08 = 9.818$ PW of O&M = \$292,000 × 9.818 = \$2,866,856

(d) Augmentation at year 10: Cost = 0.15 × 200,000 × \$130 × 0.50 = 0.15 × \$13,000,000 = \$1,950,000 (in year-10 dollars) PW = \$1,950,000 × $(1.08)^{-10}$ = \$1,950,000 × 0.4632 = \$903,240

(e) LCOS: Total PW of costs = \$43,000,000 + \$2,866,856 + \$903,240 = \$46,770,096

PW of lifetime energy discharged (assuming flat throughput for simplicity): PW of energy = 58,400 × 9.818 = 573,371 MWh (present worth of annual throughput stream)

LCOS = \$46,770,096 / 573,371 = \$81.6/MWh

This LCOS of \$81.6/MWh is competitive with peaking gas turbines (\$100-150/MWh) and viable in markets with stacked revenue streams exceeding \$100/MWh.

Problem 10.9.10

Given: A 100 MWh BESS uses LFP cells with a beginning-of-life capacity of 100 MWh. The degradation model is: capacity retention = 1 − (0.00008 × FEC) − (0.012 × years), where FEC is the cumulative full equivalent cycles. The system operates 1 cycle/day at 80% DOD. A capacity augmentation event adds 20 MWh of new modules when capacity drops to 85 MWh.

Find: (a) The annual FEC, (b) the capacity at end of year 1, year 3, and year 5, (c) the year when augmentation is triggered, (d) the capacity after augmentation, and (e) the effective capacity at year 10 (second degradation period includes the augmented modules aging from their install date).

Solution:

(a) Annual FEC: $FEC_{annual} = 1 \times 0.80 \times 365 = 292$ FEC/year

(b) Capacity at end of each year:

Year 1: FEC = 292, years = 1 Retention = 1 − (0.00008 × 292) − (0.012 × 1) = 1 − 0.02336 − 0.012 = 0.9646 Capacity = 100 × 0.9646 = 96.46 MWh

Year 3: FEC = 876, years = 3 Retention = 1 − (0.00008 × 876) − (0.012 × 3) = 1 − 0.07008 − 0.036 = 0.8939 Capacity = 100 × 0.8939 = 89.39 MWh

Year 5: FEC = 1,460, years = 5 Retention = $1 - (0.00008 \times 1{,}460) - (0.012 \times 5) = 1 - 0.1168 - 0.060 = 0.8232$ Capacity = $100 \times 0.8232 = 82.32$ MWh

(c) Augmentation triggered when capacity = 85 MWh: $0.85 = 1 - 0.00008 \times (292t) - 0.012t = 1 - 0.02336t - 0.012t = 1 - 0.03536t$ $0.03536t = 0.15$ $t = 0.15/0.03536 = 4.24$ years (augmentation triggered during year 5, approximately month 3)

(d) After augmentation at year 4.24: Existing modules: 85 MWh (at trigger point) New modules: 20 MWh (fresh, 100% SOH) Total capacity = 85 + 20 = 105 MWh

(e) At year 10 (5.76 years after augmentation): Original modules (10 years old, FEC = 2,920): Retention = $1 - (0.00008 \times 2{,}920) - (0.012 \times 10) = 1 - 0.2336 - 0.120 = 0.6464$ $\text{Capacity}_{\text{original}} = 100 \times 0.6464 = 64.64$ MWh

New modules (5.76 years old, FEC = $5.76 \times 292 = 1{,}682$): Retention = $1 - (0.00008 \times 1{,}682) - (0.012 \times 5.76) = 1 - 0.1346 - 0.0691 = 0.7963$ $\text{Capacity}_{\text{new}} = 20 \times 0.7963 = 15.93$ MWh

Total at year 10 = 64.64 + 15.93 = 80.57 MWh

The system is at ~80.6% of original capacity at year 10, approaching the typical end-of-warranty threshold of 80%. A second augmentation may be needed around year 11 to extend operation to 15-20 years.

Chapter 10 — Section 10.10: Battery Charging

Practice problems covering CC/CV charging profiles, charge termination, charger topologies and efficiency, EV charging levels and standards, fast charging thermal analysis, lithium plating risk, and wireless inductive charging systems.

Problem 10.10.1

Given: A cylindrical 18650 NMC lithium-ion cell has a rated capacity of 3.5 Ah and is charged using a CC/CV algorithm. The CC rate is 0.7C, and the charge cutoff voltage is 4.20 V. The cell reaches 4.20 V after 1.3 hours. During the CV phase, the current decays exponentially with a time constant $\tau = 0.35$ hours, and charge terminates at C/25 (0.14 A).

Find: (a) The CC charging current, (b) the charge delivered during the CC phase, (c) the time from CV phase start to charge termination, (d) the charge delivered during the CV phase, and (e) the total charge time.

Solution:

(a) CC current: $I_{CC} = 0.7 \times 3.5 = 2.45$ A

(b) CC phase charge: $Q_{CC} = 2.45 \times 1.3 = 3.185$ Ah

(c) CV current decays as $I(t) = 2.45 \times e^{-t/0.35}$. Termination at $I = 0.14$ A: $0.14 = 2.45 \times e^{-t/0.35} \rightarrow e^{-t/0.35} = 0.0571 \rightarrow t = -0.35 \times \ln(0.0571) = 0.35 \times 2.864 = 1.00$ hours

(d) CV phase charge: $Q_{CV} = \int_0^{1.00} 2.45 \times e^{-t/0.35}\,dt = 2.45 \times 0.35 \times (1 - e^{-1.00/0.35}) = 0.8575 \times (1 - 0.0571) = 0.8575 \times 0.9429 = 0.809$ Ah

(e) Total charge time = 1.3 + 1.00 = 2.30 hours

Total charge delivered = 3.185 + 0.809 = 3.994 Ah (slightly above rated capacity due to coulombic efficiency < 100%).

Problem 10.10.2

Given: A lead-acid battery bank (48 V nominal, 200 Ah) uses three-stage charging. The bulk (CC) stage supplies 40 A until the voltage reaches 57.6 V (2.40 V/cell × 24 cells). The absorption (CV) stage holds 57.6 V until the current drops to 4 A. The float stage maintains 54.0 V. The battery is at 30% SOC (70% depleted) when charging begins.

Find: (a) The energy depleted from the battery (assuming 48 V average discharge voltage), (b) the approximate bulk stage duration, (c) the charge delivered in the bulk stage, and (d) the percentage of capacity restored during the bulk stage.

Solution:

(a) Energy depleted: $E = V_{avg} \times C \times DOD = 48 \times 200 \times 0.70 = 6{,}720$ Wh $= 6.72$ kWh

(b) Charge depleted $= 200 \times 0.70 = 140$ Ah. The bulk stage typically restores ~80% of the depleted charge (accounting for the increasing voltage during bulk). Bulk stage duration $\approx (140 \times 0.80) / 40 = 112 / 40 = 2.8$ hours

(c) Charge delivered in bulk: $Q_{bulk} = 40 \times 2.8 = 112$ Ah

(d) Percentage of depleted capacity restored: $112 / 140 = 80\%$ (the remaining 20% is restored during the slower absorption stage)

Problem 10.10.3

Given: A 6.6 kW on-board charger has a bridgeless totem-pole PFC stage (98.5% efficiency) and a CLLC resonant DC-DC stage (97.0% efficiency). It converts 240 V_{rms} single-phase AC to a 350–450 V battery pack. The charger operates with a power factor of 0.99.

Find: (a) The overall charger efficiency, (b) the AC input current at full load, (c) the power dissipated as heat, and (d) the apparent power drawn from the grid.

Solution:

(a) $\eta_{total} = \eta_{PFC} \times \eta_{DC\text{-}DC} = 0.985 \times 0.970 = 95.5\%$

(b) Input power: $P_{in} = 6{,}600 / 0.955 = 6{,}911$ W. $I_{AC} = P_{in} / V_{AC} = 6{,}911 / 240 = 28.8$ A

(c) Heat dissipated: $P_{loss} = 6{,}911 - 6{,}600 = 311$ W

(d) Apparent power: $S = P_{in} / PF = 6{,}911 / 0.99 = 6{,}981$ VA

Problem 10.10.4

Given: A 350 kW off-board DC fast charger uses a three-phase 480 V AC input with a Vienna rectifier (PFC) stage and a phase-shifted full-bridge DC-DC stage. The overall efficiency is 94%. The charger supplies 800 V DC to the vehicle.

Find: (a) The AC input power at full load, (b) the three-phase input current (assuming unity power factor), (c) the DC output current at 800 V, and (d) the total heat rejected by the cooling system.

Solution:

(a) Input power: $P_{in} = 350{,}000 / 0.94 = 372{,}340$ W = 372.3 kW

(b) Three-phase current: $I_{AC} = P_{in} / (\sqrt{3} \times V_{LL}) = 372{,}340 / (1.732 \times 480) = 372{,}340 / 831.4 = 447.8$ A

(c) DC output current: $I_{DC} = P_{out} / V_{DC} = 350{,}000 / 800 = 437.5$ A

(d) Heat rejected: $P_{loss} = 372{,}340 - 350{,}000 = 22{,}340$ W = 22.3 kW (requires liquid cooling with substantial radiator capacity)

Problem 10.10.5

Given: Three EVs with identical 82 kWh usable battery packs arrive at charging stations at 15% SOC. Vehicle A uses Level 1 (1.44 kW), Vehicle B uses Level 2 (11.5 kW), and Vehicle C uses DCFC (250 kW, constant power up to 80% SOC, then tapering to 50 kW average from 80–90%). Assume 93% charger efficiency for all levels.

Find: (a) The energy required to charge each vehicle from 15% to 90% SOC, (b) the charge time for Vehicle A, (c) the charge time for Vehicle B, (d) the charge time for Vehicle C, and (e) the total energy drawn from the grid for Vehicle B.

Solution:

(a) Energy required: $E = 82 \times (0.90 - 0.15) = 82 \times 0.75 = 61.5$ kWh (all three vehicles require the same energy)

(b) Vehicle A (Level 1): $t = 61.5 / 1.44 = 42.7$ hours (nearly two full days)

(c) Vehicle B (Level 2): $t = 61.5 / 11.5 = 5.35$ hours

(d) Vehicle C (DCFC): Energy from 15% to 80% = $82 \times 0.65 = 53.3$ kWh. Time = $53.3 / 250 = 0.213$ h = 12.8 min. Energy from 80% to 90% = $82 \times 0.10 = 8.2$ kWh. Time = $8.2 / 50 = 0.164$ h = 9.8 min. Total DCFC time = $12.8 + 9.8 = 22.6$ minutes

(e) Grid energy for Vehicle B: $E_{grid} = E_{battery} / \eta = 61.5 / 0.93 = 66.1$ kWh (the 4.6 kWh difference is dissipated as heat in the charger)

Problem 10.10.6

Given: A pouch NMC cell has a capacity of 60 Ah, internal resistance of 1.2 mΩ at 25°C and 3.0 mΩ at 0°C, mass of 0.9 kg, and specific heat capacity of 1,050 J/(kg·°C). The cell is charged at 2C (120 A).

Find: (a) The I^2R heat generation at 25°C, (b) the I^2R heat generation at 0°C, (c) the adiabatic temperature rise per minute at each temperature, and (d) the time to reach 45°C from a 0°C start (adiabatic, assuming constant resistance).

Solution:

(a) $\dot{Q}_{25°C} = I^2 \times R = 120^2 \times 0.0012 = 14{,}400 \times 0.0012 = 17.3$ W

(b) $\dot{Q}_{0°C} = 120^2 \times 0.0030 = 14{,}400 \times 0.0030 = 43.2$ W

(c) At 25°C: $\Delta T/\text{min} = \dot{Q} \times 60 / (m \times c_p) = 17.3 \times 60 / (0.9 \times 1{,}050) = 1{,}038 / 945 = 1.10$°C/min At 0°C: $\Delta T/\text{min} = 43.2 \times 60 / 945 = 2{,}592 / 945 = 2.74$°C/min

(d) $\Delta T = 45 - 0 = 45$°C. $E_{max} = 0.9 \times 1{,}050 \times 45 = 42{,}525$ J. $t = 42{,}525 / 43.2 = 984$ s = 16.4 minutes — well before the CC phase would complete (30 min for 2C), indicating that active cooling or reduced charge rate is essential for cold-weather fast charging.

Problem 10.10.7

Given: A pulse charging protocol for a 100 Ah LFP cell uses 5-second pulses at 200 A followed by 2-second rest periods. The cell's internal resistance is 0.8 mΩ.

Find: (a) The heat generated per pulse, (b) the average power dissipation over a pulse-rest cycle, (c) the average charging current over one cycle, and (d) the time to deliver 80 Ah using this protocol.

Solution:

(a) Power during pulse: $P = I^2 \times R = 200^2 \times 0.0008 = 32$ W. Energy per pulse: $E = P \times t = 32 \times 5 = 160$ J

(b) Cycle duration = 5 + 2 = 7 seconds. Average power: $P_{avg} = 160 / 7 = 22.9$ W

(c) Charge per pulse: $Q = 200 \times 5 = 1{,}000$ As = 0.2778 Ah. Average current: $I_{avg} = 0.2778 / (7/3600) = 0.2778 / 0.001944 = 142.9$ A (equivalent to 1.43C)

(d) Time to deliver 80 Ah: $t = 80 / 142.9 = 0.56$ hours = 33.6 minutes (or equivalently, $80/0.2778 = 288$ cycles $\times$ 7 s = 2,016 s = 33.6 min)

Problem 10.10.8

Given: An SAE J1772 Level 2 EVSE uses a 1 kHz pilot signal with a duty cycle of 50%. Per the J1772 specification, the maximum available current = duty cycle × 60 A (for duty cycles between 10% and 85%).

Find: (a) The maximum available current communicated by the 50% duty cycle, (b) the maximum charging power at 240 V, (c) the NEC minimum circuit breaker size (125% continuous load rule), and (d) the minimum copper conductor size from NEC Table 310.16 at 75°C.

Solution:

(a) Maximum available current: $I_{max} = 0.50 \times 60 = 30$ A

(b) Maximum power: $P = 240 \times 30 = 7{,}200$ W = 7.2 kW

(c) Circuit breaker: $I_{CB} = 30 \times 1.25 = 37.5$ A → 40 A (next standard size per NEC 240.6(A))

(d) From NEC Table 310.16 at 75°C: 8 AWG copper has 50 A ampacity, which exceeds 40 A → 8 AWG

Problem 10.10.9

Given: A wireless EV charging system (SAE J2954 WPT2, 7.7 kW) operates at 85 kHz with transmitter coil L_1 = 180 μH, receiver coil L_2 = 220 μH, coupling coefficient k = 0.18, and coil AC resistance $R_1 = R_2$ = 0.12 Ω. SS compensation is used.

Find: (a) The mutual inductance, (b) the compensation capacitors, (c) the coil quality factors, and (d) the maximum theoretical efficiency.

Solution:

(a) $M = k \times \sqrt{}(L_1 \times L_2) = 0.18 \times \sqrt{}(180 \times 10^{-6} \times 220 \times 10^{-6}) = 0.18 \times \sqrt{}(3.96 \times 10^{-8}) = 0.18 \times 1.99 \times 10^{-4} = 35.8$ μH

(b) $\omega = 2\pi \times 85{,}000 = 534{,}071$ rad/s. $C_1 = 1/(\omega^2 L_1) = 1/(2.852 \times 10^{11} \times 1.80 \times 10^{-4}) = 1/(5.134 \times 10^7) = 19.5$ nF $C_2 = 1/(\omega^2 L_2) = 1/(2.852 \times 10^{11} \times 2.20 \times 10^{-4}) = 1/(6.274 \times 10^7) = 15.9$ nF

(c) $Q_1 = \omega L_1/R_1 = 534{,}071 \times 180 \times 10^{-6} / 0.12 = 96.1 / 0.12 = 801$ $Q_2 = \omega L_2/R_2 = 534{,}071 \times 220 \times 10^{-6} / 0.12 = 117.5 / 0.12 = 979$

(d) $k^2 Q_1 Q_2 = 0.0324 \times 801 \times 979 = 0.0324 \times 784{,}179 = 25{,}407$. $\eta_{max} = 25{,}407 / (1 + \sqrt{}(1 + 25{,}407))^2 = 25{,}407 / (1 + 159.4)^2 = 25{,}407 / 25{,}729 = 98.7\%$ (theoretical; practical DC-to-DC efficiency ≈ 88–92%)

Problem 10.10.10

Given: A Qi wireless charging system for a smartphone operates at 150 kHz with transmitter coil L_1 = 10 μH, receiver coil L_2 = 6 μH, coupling coefficient k = 0.55, and coil resistances R_1 = 0.3 Ω, R_2 = 0.5 Ω. The system delivers 10 W to the battery.

Find: (a) The mutual inductance, (b) the quality factor of each coil, (c) the maximum theoretical coil-to-coil efficiency, and (d) the estimated total system efficiency if the inverter and rectifier each have 95% efficiency.

Solution:

(a) $M = 0.55 \times \sqrt{}(10 \times 10^{-6} \times 6 \times 10^{-6}) = 0.55 \times \sqrt{}(6.0 \times 10^{-11}) = 0.55 \times 7.746 \times 10^{-6} = 4.26$ μH

(b) $\omega = 2\pi \times 150{,}000 = 942{,}478$ rad/s. $Q_1 = 942{,}478 \times 10 \times 10^{-6} / 0.3 = 9.425 / 0.3 = 31.4$ $Q_2 = 942{,}478 \times 6 \times 10^{-6} / 0.5 = 5.655 / 0.5 = 11.3$

(c) $k^2 Q_1 Q_2 = 0.3025 \times 31.4 \times 11.3 = 0.3025 \times 354.8 = 107.3$. $\eta_{max} = 107.3 / (1 + \sqrt{}(1 + 107.3))^2 = 107.3 / (1 + 10.41)^2 = 107.3 / 130.2 = 82.4\%$

(d) Total system efficiency: $\eta_{total} = \eta_{inv} \times \eta_{coils} \times \eta_{rect} = 0.95 \times 0.824 \times 0.95 = 74.4\%$ DC input power required: $P_{in} = 10 / 0.744 = 13.4$ W

Chapter 10, Section 11 — Supercapacitors

P10.11.1: An EDLC cell uses activated carbon electrodes with a specific surface area of 2,000 m^2/g and an electrode mass of 8 g per electrode. The double-layer capacitance per unit area is 18 $\mu F/cm^2$. (a) Calculate the capacitance of one electrode. (b) Determine the cell capacitance (two symmetric electrodes in series). (c) The cell is charged to V_{max} = 2.5 V. How much energy is stored, in joules and watt-hours? (d) The cell is then discharged to $V_{max}/2$ = 1.25 V. What percentage of the total stored energy was extracted?

Solution:

(a) Electrode surface area = 2,000 $m^2/g \times 8$ g = 16,000 m^2 = 16,000 $\times 10^4$ cm^2 = 1.6×10^8 cm^2. $C_{electrode} = 18 \times 10^{-6}$ F/$cm^2 \times 1.6 \times 10^8$ cm^2 = 2,880 F. ≈ 2,880 F

(b) Two double layers in series (symmetric cell): $C_{cell} = C_{electrode}/2$ = 2,880/2 = 1,440 F

(c) $E = ½ \times C_{cell} \times V_{max}^2 = ½ \times 1{,}440 \times (2.5)^2 = ½ \times 1{,}440 \times 6.25$ = 4,500 J = 1.25 Wh

(d) $E_{remaining} = ½ \times 1{,}440 \times (1.25)^2 = ½ \times 1{,}440 \times 1.5625$ = 1,125 J. Fraction extracted = (4,500 − 1,125)/4,500 = 3,375/4,500 = 75% — discharging to half voltage always recovers exactly 75% of stored energy.

P10.11.2: A 600 F EDLC cell (V_{max} = 2.7 V, mass = 85 g) and a lithium-ion capacitor (LIC) cell (V_{max} = 3.8 V, specific energy = 22 Wh/kg, mass = 85 g) are compared. (a) Calculate the energy stored in the EDLC at full charge. (b) Calculate the energy stored in the LIC at full charge. (c) Calculate the specific energy of the EDLC in Wh/kg. (d) By what factor does the LIC exceed the EDLC in specific energy?

Solution:

(a) EDLC: $E = ½ \times 600 \times (2.7)^2 = ½ \times 600 \times 7.29$ = 2,187 J = 0.607 Wh

(b) LIC: E = 22 Wh/kg × 0.085 kg = 1.87 Wh

(c) EDLC specific energy = 0.607 Wh / 0.085 kg = 7.14 Wh/kg

(d) Ratio = 22 Wh/kg ÷ 7.14 Wh/kg = 3.08× — the LIC stores approximately 3× more energy per kilogram, consistent with the hybrid battery-EDLC architecture that raises the effective voltage window.

P10.11.3: A 25 F supercapacitor module (ESR = 30 mΩ) is charged to $V_0 = 48$ V and discharged at a constant current of 50 A into a DC-DC converter. (a) Find the initial terminal voltage at the start of discharge. (b) Find the terminal voltage after 10 seconds of discharge. (c) How long does it take the capacitor voltage V_C to reach 24 V? (d) What is the maximum power the module can deliver to a matched resistive load?

Solution:

(a) $V_{terminal}(0) = V_0 - I \times ESR = 48 - 50 \times 0.030 = 48 - 1.5 = 46.5$ V

(b) $V_C(t) = V_0 - (I/C) \times t = 48 - (50/25) \times 10 = 48 - 20 = 28$ V. $V_{terminal}(10) = 28 - 1.5 = 26.5$ V

(c) $24 = 48 - (50/25) \times t \rightarrow 2t = 24 \rightarrow t = 12$ s

(d) $P_{max} = V_0^2/(4 \times ESR) = (48)^2/(4 \times 0.030) = 2{,}304/0.120 = 19{,}200$ W = 19.2 kW

P10.11.4: An industrial energy storage module must store at least 200 Wh at a 72 V nominal bus voltage. Available cells are rated at 2.7 V, 5,000 F each, with an ESR of 0.3 mΩ per cell. (a) Determine the minimum number of series cells per string. (b) Calculate the string capacitance and energy. (c) Determine the number of parallel strings needed and the total capacitance. (d) Calculate the total module ESR (series strings in parallel).

Solution:

(a) $n_s = \lceil 72/2.7 \rceil = \lceil 26.67 \rceil = 27$ cells; $V_{string} = 27 \times 2.7 = 72.9$ V

(b) $C_{string} = 5{,}000/27 = 185.2$ F $E_{string} = ½ \times 185.2 \times (72.9)^2 = ½ \times 185.2 \times 5{,}314.4 = ½ \times 984{,}227 = 492{,}114$ J = 136.7 Wh

(c) $n_p = \lceil 200/136.7 \rceil = \lceil 1.46 \rceil = 2$ strings $C_{total} = 2 \times 185.2 = 370.4$ F; $E_{total} = 2 \times 136.7 = 273.4$ Wh ✓

(d) ESR per string = $n_s \times ESR_{cell} = 27 \times 0.3 \times 10^{-3} = 8.1$ mΩ. Two strings in parallel: $ESR_{module} = 8.1/2 = 4.05$ mΩ

P10.11.5: A crane lifts a 2,000 kg load through 5 m, taking 8 seconds for the lift. The motor efficiency is 85%. During lowering over the same 5 m in 8 seconds, 70% of the potential energy is recovered regeneratively into a supercapacitor bank operating between $V_{min} = 50$ V and $V_{max} = 100$ V. (a) Calculate the potential energy of the load. (b) Calculate the regenerated energy captured in the supercapacitor. (c) Determine the minimum capacitance needed. (d) If the recovered energy is fully used to assist the next lift (via the same 85% motor efficiency), what is the net energy drawn from the supply for the second lift?

Solution:

(a) $E_{PE} = m \times g \times h = 2{,}000 \times 9.81 \times 5 = 98{,}100$ J = 98.1 kJ

(b) $E_{regen} = 0.70 \times 98{,}100 = 68{,}670$ J ≈ 68.7 kJ

(c) $C = E_{regen}/[½(V_{max}^2 - V_{min}^2)] = 68{,}670/[½(100^2 - 50^2)] = 68{,}670/[½ \times 7{,}500] = 68{,}670/3{,}750 = 18.3$ F

(d) Energy to lift the load (second lift): $E_{lift} = E_{PE}/\eta_{motor} = 98{,}100/0.85 = 115{,}412$ J. Energy available from supercapacitor via motor: $E_{assist} = E_{regen} \times \eta_{motor} = 68{,}670 \times 0.85 = 58{,}370$ J. Net supply energy = $115{,}412 - 58{,}370 = 57{,}042$ J $\approx$ 57.0 kJ (vs 115.4 kJ without regeneration — a 50.6% reduction in supply energy for the second lift).

Chapter 11 – Section 11.1: Measurement Fundamentals

Practice problems covering accuracy, precision, resolution, error analysis, calibration, signal-to-noise ratio, measurement uncertainty, and bridge circuits.

Problem 11.1.1

Given: A 12-bit ADC has an input range of 0-3.3 V. A pressure sensor output is measured 8 times, yielding readings (in volts): 1.4523, 1.4519, 1.4526, 1.4520, 1.4524, 1.4521, 1.4518, 1.4525. The true sensor voltage (from a precision calibrator) is 1.4550 V.

Find: The ADC resolution, the measurement accuracy (systematic error), and the measurement precision (standard deviation).

Solution: Resolution = full-scale range / 2^N = 3.3 / 2^{12} = 3.3 / 4,096 = 0.806 mV per count.

Mean of measurements: $\bar{x}$ = (1.4523 + 1.4519 + 1.4526 + 1.4520 + 1.4524 + 1.4521 + 1.4518 + 1.4525) / 8 = 11.6176 / 8 = 1.45220 V.

Accuracy (systematic error): |1.4550 - 1.4522| = 2.8 mV, or (2.8 / 1.4550) x 100 = 0.192% of reading.

Deviations from mean (mV): +0.1, -0.3, +0.4, -0.2, +0.2, -0.1, -0.4, +0.3.

Precision (standard deviation): s = sqrt[sum(x_i - $\bar{x}$)^2 / (N-1)] = sqrt[(0.01 + 0.09 + 0.16 + 0.04 + 0.04 + 0.01 + 0.16 + 0.09) x 10^-6 / 7] = sqrt[0.60 x 10^-6 / 7] = sqrt[8.571 x 10^-8] = 0.293 mV.

The instrument has good precision (0.293 mV spread) but a 2.8 mV systematic offset indicating a calibration correction is needed.

Problem 11.1.2

Given: A DMM is calibrated at four points against a NIST-traceable reference. The reference values and meter readings are: 0.500 V reads 0.502 V, 1.500 V reads 1.505 V, 2.500 V reads 2.508 V, 3.500 V reads 3.511 V.

Find: The gain error, offset error, and the corrected reading when the DMM displays 2.000 V.

Solution: Using a linear error model: $V_{reading} = G \times V_{true} + V_{offset}$.

Using the first and last calibration points: G = (3.511 - 0.502) / (3.500 - 0.500) = 3.009 / 3.000 = 1.00300.

$V_{offset} = V_{reading} - G \times V_{true}$ = 0.502 - 1.00300 x 0.500 = 0.502 - 0.50150 = 0.50 mV.

Gain error = (G - 1) x 100% = 0.300%. Offset error = 0.50 mV.

Verification at 2.500 V: predicted = 1.00300 x 2.500 + 0.00050 = 2.50800 V, actual = 2.508 V – confirmed.

Corrected reading when DMM displays 2.000 V: $V_{true} = (V_{reading} - V_{offset}) / G$ = (2.000 - 0.00050) / 1.00300 = 1.99950 / 1.00300 = 1.9935 V.

Problem 11.1.3

Given: A strain gauge amplifier has a signal output of 12 mV_{rms}. The sensor source resistance is 120 kOhm at a temperature of 30 degrees C. The measurement system has a bandwidth of 200 kHz. Boltzmann constant k = 1.381 x 10^-23 J/K.

Find: The thermal noise voltage, the SNR in dB, and the number of signal averages needed to achieve 70 dB SNR.

Solution: Temperature in Kelvin: T = 30 + 273.15 = 303.15 K.

Thermal noise voltage: V_n = sqrt(4kTR delta-f) = sqrt(4 x 1.381 x 10^-23 x 303.15 x 120,000 x 200,000) = sqrt(4 x 1.381 x 10^-23 x 7.276 x 10^9) = sqrt(4.021 x 10^-13) = 20.05 uV_{rms}.

SNR = 20 log10(V_{signal} / V_{noise}) = 20 log10(12 x 10^-3 / 20.05 x 10^-6) = 20 log10(598.5) = 20 x 2.777 = 55.5 dB.

Improvement needed for 70 dB: 70 - 55.5 = 14.5 dB. Voltage ratio improvement = 10^(14.5/20) = 5.31. Since averaging improves SNR by sqrt(N): sqrt(N) = 5.31, so N = 28.2.

Round up to N = 29 averages to achieve 70 dB SNR.

Problem 11.1.4

Given: A pressure transmitter reads 250.5 kPa. The uncertainty budget includes: (1) repeatability from 10 measurements with standard deviation s = 0.35 kPa, (2) calibration certificate uncertainty of +/-0.20 kPa at k = 2, (3) display resolution of 0.1 kPa, (4) temperature effect estimated as +/-0.25 kPa (rectangular distribution), and (5) hysteresis estimated as +/-0.10 kPa (rectangular distribution).

Find: The combined standard uncertainty and the expanded uncertainty at 95% confidence (k = 2).

Solution: Type A: u1 = s / sqrt(N) = 0.35 / sqrt(10) = 0.35 / 3.162 = 0.1107 kPa.

Type B components: Calibration: u2 = 0.20 / 2 = 0.100 kPa (convert from k = 2). Resolution: u3 = 0.1 / (2 sqrt(3)) = 0.1 / 3.464 = 0.0289 kPa (half-width, rectangular). Temperature: u4 = 0.25 / sqrt(3) = 0.1443 kPa (rectangular). Hysteresis: u5 = 0.10 / sqrt(3) = 0.0577 kPa (rectangular).

Combined standard uncertainty: u_c = sqrt(0.1107^2 + 0.100^2 + 0.0289^2 + 0.1443^2 + 0.0577^2) = sqrt(0.01225 + 0.01000 + 0.000835 + 0.02082 + 0.00333) = sqrt(0.04724) = 0.217 kPa.

Expanded uncertainty at 95% (k = 2): U = 2 x 0.217 = 0.43 kPa.

The result is reported as 250.5 +/- 0.4 kPa (k = 2, 95% confidence). The temperature effect is the dominant contributor (44% of variance).

Problem 11.1.5

Given: A Wheatstone bridge measures a platinum RTD. The bridge has R1 = 100 Ohm, R2 = 100 Ohm, and R3 (decade box) is adjusted to 138.5 Ohm for a null reading. The excitation voltage is 5 V. The galvanometer sensitivity is 8 uA/mV and the minimum detectable current is 0.05 uA.

Find: The unknown RTD resistance, the corresponding temperature (using Pt100: R(T) = 100(1 + 0.00385T)), and the measurement resolution in ohms and degrees C.

Solution: At balance: R_x = R2 x R3 / R1 = 100 x 138.5 / 100 = 138.5 Ohm.

Temperature from Pt100 equation: 138.5 = 100(1 + 0.00385T) 1.385 = 1 + 0.00385T T = 0.385 / 0.00385 = 100.0 degrees C.

Minimum detectable imbalance voltage: V_{min} = I_{min} / sensitivity = 0.05 uA / 8 uA/mV = 0.00625 mV.

Bridge sensitivity near balance (all arms ~ R): R ~ sqrt(R1 x R2) = sqrt(100 x 100) = 100 Ohm (geometric mean of fixed arms). dV/dR ~ V_{ex} / (4R) = 5 / (4 x 100) = 12.5 mV/Ohm.

Minimum detectable resistance change: delta-R = V_{min} / (dV/dR) = 0.00625 / 12.5 = 0.0005 Ohm.

Temperature resolution: dR/dT = R0 x alpha = 100 x 0.00385 = 0.385 Ohm/degrees C. delta-T = delta-R / (dR/dT) = 0.0005 / 0.385 = 0.0013 degrees C.

The bridge achieves millidegree temperature resolution – far superior to direct resistance measurement with a DMM.

Problem 11.1.6

Given: A Kelvin double bridge measures a 0.001 Ohm shunt resistor used for current sensing. The bridge uses a standard resistor R_s = 0.001000 Ohm (certified), ratio arms R1 = R3 = 10 Ohm, and auxiliary ratio arms r1 = r3 = 10 Ohm. The link resistance between the standard and unknown is r_{link} = 0.005 Ohm. At balance, R2 = 10.045 Ohm and r2 = 10.045 Ohm.

Find: The unknown shunt resistance and the error that would result from using a simple Wheatstone bridge (ignoring lead resistance effects).

Solution: For a Kelvin double bridge at balance: R_x = R_s x (R2/R1) + r_{link} x [(R2/R1) - (r2/r1)] / [1 + (r1 + r2 + r_{link})/r3...]

With matched ratio arms (R2/R1 = r2/r1): R_x = R_s x (R2/R1) = 0.001000 x (10.045/10) = 0.001000 x 1.0045 = 0.0010045 Ohm = 1.0045 mOhm.

The matched ratio condition (R2/R1 = r2/r1 = 1.0045) eliminates the link resistance term entirely.

In a simple Wheatstone bridge, the lead resistances (say 0.005 Ohm per lead, two leads = 0.010 Ohm total) add directly to the measured resistance: $R_{measured} = R_x + R_{leads} = 0.0010045 + 0.010 = 0.0110$ Ohm.

Error = (0.0110 - 0.0010045) / 0.0010045 x 100% = 995% error.

The Kelvin bridge eliminates the 10 mOhm lead resistance that would completely swamp the 1 mOhm measurement in a simple Wheatstone bridge.

Problem 11.1.7

Given: An instrumentation system measures a 1 mV thermocouple signal using a 24-bit sigma-delta ADC with a 2.5 V reference. The system has a 10 Hz measurement bandwidth. The ADC datasheet specifies an effective number of bits (ENOB) of 20.5 at 10 Hz output rate. The thermal noise from the thermocouple's 50 Ohm source resistance at 25 degrees C contributes additional noise.

Find: The ADC quantization noise, the thermal noise, the total system SNR, and whether the system can resolve a 0.1 uV signal change.

Solution: ADC resolution with full 24 bits: LSB = 2.5 / 2^24 = 2.5 / 16,777,216 = 0.149 uV.

Effective resolution at ENOB = 20.5: effective LSB = 2.5 / 2^20.5 = 2.5 / 1,488,828 = 1.679 uV.

ADC RMS noise: $V_{n,ADC}$ = effective LSB / sqrt(12) = 1.679 / 3.464 = 0.485 uV_{rms}.

Thermal noise: $V_{n,th} = \text{sqrt}(4 \times 1.381 \times 10^{-23} \times 298 \times 50 \times 10) = \text{sqrt}(8.23 \times 10^{-18}) = 0.00287\ uV_{rms}$ (2.87 nV).

Total noise (RSS): $V_{n,total} = \text{sqrt}(0.485^2 + 0.00287^2) = \text{sqrt}(0.2352 + 0.0000082) = \text{sqrt}(0.2352) = 0.485\ uV_{rms}$.

SNR for 1 mV signal: SNR = 20 log10(1000 / 0.486) = 20 log10(2058) = 20 x 3.313 = 66.3 dB.

To resolve a 0.1 uV change, the signal change must exceed the peak-to-peak noise (~ 6.6 x V_n for 99.9% confidence): 6.6 x 0.486 = 3.21 uV » 0.1 uV.

The system cannot resolve a 0.1 uV change. Averaging N = (3.21/0.1)^2 = 1,031 samples would be needed, requiring ~103 seconds at 10 Hz output rate.

Problem 11.1.8

Given: A Maxwell bridge is used to measure an unknown inductor at 1 kHz. The bridge components are: R1 = 1,000 Ohm, R2 = 2,200 Ohm, C3 = 47 nF (in parallel with R3), and R3 = 4,700 Ohm. At balance, the detector reads null.

Find: The unknown inductance L_x and its series resistance R_x, and the Q factor of the inductor at 1 kHz.

Solution: For a Maxwell bridge at balance: L_x = R1 x R2 x C3 = 1,000 x 2,200 x 47 x 10^-9 = 103.4 mH.

R_x = R1 x R2 / R3 = 1,000 x 2,200 / 4,700 = 468.1 Ohm.

Inductive reactance at 1 kHz: X_L = 2 pi f L = 2 pi x 1,000 x 0.1034 = 649.8 Ohm.

Q factor: Q = X_L / R_x = 649.8 / 468.1 = 1.39.

Equivalently: Q = omega x C3 x R3 = 2 pi x 1,000 x 47 x 10^-9 x 4,700 = 1.39 – confirmed.

The inductor has L = 103.4 mH, R_x = 468.1 Ohm, and Q = 1.39 at 1 kHz, indicating a lossy inductor typical of a small iron-core choke.

Chapter 11 – Section 11.2: Sensors and Transducers

Practice problems covering temperature sensors, strain gauges, pressure sensors, proximity/position sensors, accelerometers, magnetic field sensors, current sensors, optical sensors, and flow sensors.

Problem 11.2.1

Given: A Pt1000 RTD (R0 = 1,000 Ohm at 0 degrees C, alpha = 0.00385 Ohm/Ohm/degrees C) is used in a cooling system. The measured resistance is 1,154.0 Ohm. A Type T thermocouple at the same location produces a voltage of 1.612 mV (reference junction at 0 degrees C). The Type T sensitivity is approximately 43 uV/degrees C near the measurement point.

Find: The RTD temperature, the thermocouple temperature estimate, and the error in the linear thermocouple approximation.

Solution: RTD temperature: R(T) = R0(1 + alpha T) 1,154.0 = 1,000(1 + 0.00385T) 1.154 = 1 + 0.00385T T = 0.154 / 0.00385 = 40.0 degrees C.

Thermocouple linear estimate: T = V / sensitivity = 1.612 x 10^-3 / 43 x 10^-6 = 37.5 degrees C.

Error = 40.0 - 37.5 = 2.5 degrees C (6.3% error).

From Type T tables, 1.612 mV corresponds to approximately 39.8 degrees C, very close to the RTD reading.

The RTD reads 40.0 degrees C. The linear thermocouple approximation gives 37.5 degrees C (2.5 degrees C error), demonstrating the need for thermocouple reference tables or polynomial corrections.

Problem 11.2.2

Given: A full-bridge strain gauge configuration uses four 120 Ohm gauges (GF = 2.0) bonded to an aluminum cantilever beam (Young's modulus E = 70 GPa). Two gauges are in tension (top) and two in compression (bottom). The excitation voltage is V_{ex} = 10 V, and the measured bridge output is V_{out} = 8.5 mV.

Find: The strain, the stress, and the resistance change in each gauge.

Solution: For a full-bridge (4 active arms): V_{out} = GF x epsilon x V_{ex}.

Solving for strain: epsilon = V_{out} / (GF x V_{ex}) = 8.5 x 10^-3 / (2.0 x 10) = 8.5 x 10^-3 / 20 = 425 x 10^-6 = 425 microstrain.

Stress: sigma = E x epsilon = 70 x 10^9 x 425 x 10^-6 = 29.75 MPa.

Resistance change per gauge: delta-R = GF x epsilon x R = 2.0 x 425 x 10^-6 x 120 = 0.102 Ohm (a change of 0.085%).

The full-bridge provides 4x the sensitivity of a quarter-bridge, giving V_{out} = 8.5 mV compared to ~2.1 mV for a quarter-bridge at the same strain.

Problem 11.2.3

Given: A piezoresistive MEMS pressure sensor has a full-scale range of 0-500 kPa (gauge), a sensitivity of 20 mV/V/500 kPa, and an excitation voltage of 5 V. The sensor nonlinearity is +/-0.25% of full scale, and the temperature coefficient of offset is 0.02% FS/degrees C. The sensor operates in an environment that varies from 10 degrees C to 50 degrees C (calibrated at 25 degrees C).

Find: The full-scale output voltage, the output voltage at 350 kPa, the nonlinearity error in mV and kPa, and the worst-case temperature-induced offset error.

Solution: Full-scale output: V_{FS} = sensitivity x V_{ex} = 20 mV/V x 5 V = 100 mV.

Output at 350 kPa: V_{out} = (350/500) x 100 = 70.0 mV.

Nonlinearity error: +/-0.25% of FS = +/-0.0025 x 100 mV = +/-0.25 mV. In pressure units: +/-0.0025 x 500 = +/-1.25 kPa.

Worst-case temperature deviation from calibration: max(|50 - 25|, |10 - 25|) = 25 degrees C. Temperature offset error: 0.02% FS/degrees C x 25 degrees C = 0.50% FS = 0.005 x 100 mV = 0.50 mV. In pressure units: 0.005 x 500 = 2.5 kPa.

Total worst-case error (offset + nonlinearity): +/-(0.25 + 0.50) = +/-0.75 mV or +/-3.75 kPa (0.75% FS).

Problem 11.2.4

Given: An incremental rotary encoder has 1,024 lines and is used with quadrature decoding on a motor shaft. The encoder drives a ball screw with 10 mm pitch through a 5:1 gear reducer. The encoder produces an index pulse once per revolution.

Find: The angular resolution at the motor shaft, the linear resolution at the ball screw output, and the maximum positional error that can accumulate before the index pulse resets it.

Solution: Quadrature decoding: counts/rev = 1,024 x 4 = 4,096 counts/rev at the motor shaft.

Angular resolution: theta = 360 degrees / 4,096 = 0.0879 degrees per count = 0.001534 rad.

With 5:1 gear reduction, counts per revolution at ball screw: 4,096 x 5 = 20,480 counts/rev at the output.

Linear resolution: d = ball screw pitch / counts per screw rev = 10 mm / 20,480 = 0.000488 mm = 0.488 um per count.

Maximum positional error accumulation: The index pulse occurs once per motor revolution (every 4,096 counts). Linear travel per motor revolution = 10 mm / 5 = 2 mm. If one count is missed, the error is 0.488 um and persists until the next index pulse resets the count.

Maximum error before index reset: +/-1 count = +/-0.488 um, accumulated over at most 2 mm of travel. The index pulse provides absolute reference every 2 mm of linear travel.

Problem 11.2.5

Given: A 3-axis MEMS accelerometer has a sensitivity of 16,384 LSB/g, a +/-2 g measurement range, a noise density of 80 ug/sqrt(Hz), and a -3 dB bandwidth of 1,000 Hz. The accelerometer is used to measure tilt angle by sensing the gravity vector.

Find: The RMS noise floor in mg, the minimum detectable tilt angle, and the dynamic range in dB.

Solution: Total RMS noise: a_{noise} = noise density x sqrt(BW) = 80 x 10^-6 x sqrt(1,000) = 80 x 10^-6 x 31.62 = 2.53 mg_{rms}.

For tilt sensing, 1 degree of tilt produces: a = sin(1 degree) x 1 g = 0.01745 g = 17.45 mg.

Minimum detectable tilt (at 3-sigma for 99.7% confidence): 3 x 2.53 = 7.59 mg. Tilt angle = arcsin(7.59 x 10^-3) = 0.435 degrees.

Full-scale range: 4 g (from -2 g to +2 g). Dynamic range: DR = 20 x log10(full-scale / noise) = 20 x log10(4 / 2.53 x 10^-3) = 20 x log10(1,581) = 20 x 3.199 = 64.0 dB.

The accelerometer can resolve tilt angles down to approximately 0.44 degrees with 99.7% confidence, adequate for platform leveling but insufficient for precision inclinometry (which requires 0.01 degree resolution).

Problem 11.2.6

Given: A Hall effect current sensor (open-loop type) has a turns ratio of N = 1,000 (equivalent turns), a core with a cross-section of 60 mm^2 and an air gap of 1.0 mm. The Hall element sensitivity is 1.4 mV/mT with a 5 V supply. A primary current of 200 A flows through a single-turn conductor.

Find: The magnetic flux density in the air gap, the Hall sensor output voltage, and the overall current-to-voltage sensitivity.

Solution: Magnetic field in the air gap from the primary current: Using Ampere's law with the air gap dominating reluctance: B = mu0 x $N_{primary}$ x I / l_{gap} = 4 pi x 10^-7 x 1 x 200 / 1.0 x 10^-3

B = 2.513 x 10^-4 x 200 / 10^-3 = 0.2513 T = 251.3 mT.

Wait – this assumes no core. With a magnetic core and air gap: H_{gap} x l_{gap} = N x I (neglecting core reluctance since mu_r » 1). H_{gap} = N x I / l_{gap} = 1 x 200 / 0.001 = 200,000 A/m. B_{gap} = mu0 x H_{gap} = 4 pi x 10^-7 x 200,000 = 0.2513 T = 251.3 mT.

Hall sensor output: V_H = sensitivity x B = 1.4 mV/mT x 251.3 mT = 351.8 mV.

Current-to-voltage sensitivity: V_H / I = 351.8 / 200 = 1.759 mV/A.

At 200 A, the Hall sensor produces 351.8 mV. The core would saturate well before this point (typical saturation at ~1.5 T), so in practice the core uses multiple turns or the turns ratio is adjusted so that the air gap field stays within the linear range (< 100 mT).

Problem 11.2.7

Given: A Rogowski coil current sensor has a mutual inductance of M = 0.5 uH and is connected to an integrator circuit with an integrator time constant RC = 100 us. The coil monitors a power electronic switching waveform where the current rises linearly from 0 to 500 A in 10 us, then remains constant at 500 A for 40 us.

Find: The Rogowski coil output voltage during the current ramp, the integrator output voltage at the end of the ramp (representing the actual current), and the integrator output during the constant-current phase.

Solution: During the linear current ramp: di/dt = 500 / (10 x 10^-6) = 5 x 10^7 A/s.

Rogowski coil output (proportional to di/dt): V_{coil} = M x di/dt = 0.5 x 10^-6 x 5 x 10^7 = 25 V.

Integrator output (represents the actual current waveform): V_{int}(t) = (1/RC) x integral(V_{coil} dt) = (M/RC) x I(t). At end of ramp (I = 500 A): V_{int} = (0.5 x 10^-6 / 100 x 10^-6) x 500 = 0.005 x 500 = 2.5 V.

During the constant-current phase (di/dt = 0): V_{coil} = M x 0 = 0 V. The integrator output remains at 2.5 V (representing 500 A), since the integral of zero is constant.

Sensitivity: 2.5 V / 500 A = 5 mV/A. The Rogowski coil naturally differentiates the current; the integrator restores the original waveform.

Problem 11.2.8

Given: A fiber Bragg grating (FBG) strain sensor has a Bragg wavelength of $lambda_B$ = 1550.000 nm at 25 degrees C and zero strain. The strain sensitivity is 1.2 pm/microstrain and the temperature sensitivity is 10 pm/degrees C. The FBG is bonded to a steel bridge girder. The interrogator measures a wavelength shift to 1550.385 nm. The temperature at the sensor location is 35 degrees C.

Find: The total wavelength shift, the temperature-induced shift, the strain-induced shift, and the actual mechanical strain.

Solution: Total wavelength shift: delta-lambda = 1550.385 - 1550.000 = 0.385 nm = 385 pm.

Temperature-induced shift: delta-$lambda_T$ = 10 pm/degrees C x (35 - 25) = 10 x 10 = 100 pm.

Strain-induced shift: delta-lambda$_{epsilon}$ = delta-lambda$_{total}$ - delta-lambda$_T$ = 385 - 100 = 285 pm.

Mechanical strain: epsilon = delta-lambda$_{epsilon}$ / strain sensitivity = 285 / 1.2 = 237.5 microstrain.

Stress in steel (E = 200 GPa): sigma = E x epsilon = 200 x 10^9 x 237.5 x 10^-6 = 47.5 MPa.

The FBG sensor reads 237.5 microstrain (47.5 MPa stress) after compensating for the 10 degrees C temperature change. Without temperature compensation, the apparent strain would be 385/1.2 = 320.8 microstrain – a 35% overestimate.

Problem 11.2.9

Given: An electromagnetic flow meter is installed on a 200 mm (8-inch) diameter pipe carrying process water. The magnetic field strength is B = 0.04 T, and the measured electrode voltage is 1.88 mV. The pipe operates 24 hours/day.

Find: The average flow velocity, the volumetric flow rate in m^3/h, and the daily volume throughput.

Solution: Average velocity: v = V / (B x D) = 1.88 x 10^-3 / (0.04 x 0.200) = 1.88 x 10^-3 / 8.0 x 10^-3 = 0.235 m/s.

Pipe cross-sectional area: A = pi/4 x D^2 = pi/4 x (0.200)^2 = 0.03142 m^2.

Volumetric flow rate: Q = v x A = 0.235 x 0.03142 = 7.384 x 10^-3 m^3/s. In m^3/h: Q = 7.384 x 10^-3 x 3,600 = 26.6 m^3/h.

Daily volume: V_{day} = 26.6 x 24 = 638.4 m^3/day.

The flow meter measures 26.6 m^3/h (117 GPM). At this moderate velocity of 0.235 m/s, the flow is well within the mag meter's optimal range of 0.3-10 m/s, though slightly below the minimum recommended velocity for best accuracy.

Problem 11.2.10

Given: A Coriolis mass flow meter on a chemical feed line reads a mass flow rate of 2,450 kg/h, a fluid density of 1,185 kg/m^3, and a fluid temperature of 42 degrees C. The meter accuracy is +/-0.1% of reading for mass flow and +/-0.5 kg/m^3 for density.

Find: The volumetric flow rate, the mass flow uncertainty, and the density-derived verification (compare measured density to the known value of 1,182 kg/m^3 for the chemical at 42 degrees C).

Solution: Volumetric flow rate: Q = mass flow / density = 2,450 / 1,185 = 2.068 m^3/h.

In liters/min: Q = 2,068 / 60 = 34.5 L/min.

Mass flow uncertainty: +/-0.1% x 2,450 = +/-2.45 kg/h.

Density measurement: 1,185 +/- 0.5 kg/m^3. Known density: 1,182 kg/m^3. Deviation: 1,185 - 1,182 = 3 kg/m^3, which exceeds the +/-0.5 kg/m^3 accuracy spec.

This 3 kg/m^3 density discrepancy suggests either the process fluid concentration has changed, the temperature measurement is slightly off, or the Coriolis meter needs recalibration. The Coriolis meter's ability to simultaneously measure mass flow and density provides a built-in diagnostic capability.

Chapter 11 – Section 11.3: Signal Conditioning

Practice problems covering amplification, filtering, isolation, linearization, and ADC selection.

Problem 11.3.1

Given: An instrumentation amplifier with gain G = 200 and CMRR = 110 dB amplifies a 500 uV signal from a strain gauge bridge. The bridge wires run alongside power cables, picking up a 60 Hz common-mode voltage of V_{cm} = 3.0 V. The amplifier has an input offset voltage of 15 uV and offset drift of 0.5 uV/degrees C over a 30 degrees C temperature range.

Find: The desired output signal, the common-mode error at the output, the offset error at the output, and the total signal-to-error ratio.

Solution: Desired output: $V_{out(signal)}$ = G x V_{diff} = 200 x 500 uV = 100 mV.

CMRR = 110 dB = 10^(110/20) = 316,228.

Common-mode error at input: $V_{cm,error}$ = V_{cm} / CMRR = 3.0 / 316,228 = 9.49 uV. At output: $V_{cm,error,out}$ = 200 x 9.49 = 1.90 mV.

Offset error at input: V_{os} = 15 + 0.5 x 30 = 30 uV. At output: $V_{os,out}$ = 200 x 30 = 6.0 mV.

Total error (RSS): V_{error} = sqrt(1.90^2 + 6.0^2) = sqrt(3.61 + 36.0) = sqrt(39.61) = 6.29 mV.

Signal-to-error ratio: 100 / 6.29 = 15.9, or 20 log10(15.9) = 24.0 dB.

The offset drift dominates the error budget. Chopper-stabilized instrumentation amplifiers (with < 1 uV offset and < 10 nV/degrees C drift) would reduce the offset error by 30x, improving the signal-to-error ratio to approximately 44 dB.

Problem 11.3.2

Given: A vibration measurement system has a signal bandwidth of 2 kHz. The ADC samples at 20 kS/s. A 4th-order Butterworth lowpass anti-aliasing filter is used with a cutoff frequency of 4 kHz. A strong noise source exists at 15 kHz.

Find: The Nyquist frequency, the attenuation of the anti-aliasing filter at the Nyquist frequency, the attenuation at 15 kHz, and whether aliased noise at 15 kHz will corrupt the measurement.

Solution: Nyquist frequency: $f_N = f_s$ / 2 = 20,000 / 2 = 10 kHz.

4th-order Butterworth magnitude at frequency f: |H(f)| = 1 / sqrt(1 + (f/f_c)^(2n)) where n = 4.

At Nyquist (10 kHz): |H| = 1 / sqrt(1 + (10/4)^8) = 1 / sqrt(1 + 2.5^8) = 1 / sqrt(1 + 1,526) = 1 / sqrt(1,527) = 1 / 39.08 = 0.0256. Attenuation = 20 log10(0.0256) = -31.8 dB.

At 15 kHz: |H| = 1 / sqrt(1 + (15/4)^8) = 1 / sqrt(1 + 3.75^8) = 1 / sqrt(1 + 39,107) = 1 / sqrt(39,108) = 1 / 197.8 = 0.00506. Attenuation = 20 log10(0.00506) = -45.9 dB.

The 15 kHz noise aliases to: $f_{alias} = f_s$ - 15,000 = 20,000 - 15,000 = 5 kHz, which falls within the 0-10 kHz measurement band.

At -45.9 dB, the noise is reduced by a factor of 197.8 and is well-suppressed in the stopband.

For applications requiring > 60 dB rejection, either increase the filter order to 8th-order (giving -79.6 dB at 15 kHz) or increase the sample rate to 50 kS/s (pushing Nyquist to 25 kHz and placing the 15 kHz signal well within the passband for direct measurement rather than aliasing).

Problem 11.3.3

Given: A thermocouple measurement system monitors a 480 V, 3-phase motor winding temperature. The thermocouples are embedded in the windings and produce 0-20 mV signals. An isolated amplifier with 3,750 V_{rms} isolation, gain of 100, gain error of 0.02%, input offset of 5 uV, offset drift of 0.3 uV/degrees C, and 50 kHz bandwidth is used. The ambient temperature varies by 40 degrees C from calibration.

Find: The output voltage range, the gain error contribution, the offset drift error, and the total measurement error as a percentage of a 10 mV input signal.

Solution: Output voltage range: V_{out} = gain x V_{in} = 100 x (0 to 20 mV) = 0 to 2.0 V.

At V_{in} = 10 mV, V_{out} = 100 x 10 = 1,000 mV = 1.000 V.

Gain error: 0.02% x 1.000 V = 0.20 mV.

Offset drift: 0.3 uV/degrees C x 40 degrees C = 12 uV at input. At output: 100 x 12 = 1.20 mV.

Input offset (referred to output): 100 x 5 uV = 0.50 mV.

Total error (RSS): sqrt(0.20^2 + 1.20^2 + 0.50^2) = sqrt(0.04 + 1.44 + 0.25) = sqrt(1.73) = 1.32 mV.

As percentage of output: 1.32 / 1,000 x 100 = 0.132%.

For a Type K thermocouple at ~250 degrees C (where 10 mV is produced): 0.132% of the reading corresponds to approximately 0.33 degrees C temperature error.

The isolation amplifier maintains 0.13% accuracy while providing 3,750 V galvanic isolation between the 480 V motor windings and the ground-referenced DAQ system.

Problem 11.3.4

Given: An NTC thermistor has Steinhart-Hart coefficients A = 1.125 x 10^-3, B = 2.347 x 10^-4, C = 0.855 x 10^-7. The thermistor is read by a 12-bit ADC via a voltage divider with a series resistor R_{ref} = 10,000 Ohm and V_{supply} = 3.3 V.

Find: The thermistor resistance and temperature when the ADC reads a digital code of 2,048 (mid-scale), and the temperature resolution limited by the ADC.

Solution: ADC voltage: V_{ADC} = (code / 2^12) x V_{ref} = (2,048 / 4,096) x 3.3 = 0.500 x 3.3 = 1.650 V.

For a voltage divider (thermistor on bottom): V_{ADC} = V_{supply} x R_{th} / (R_{ref} + R_{th}). 1.650 = 3.3 x R_{th} / (10,000 + R_{th}). 1.650 x (10,000 + R_{th}) = 3.3 x R_{th}. 16,500 + 1.650 R_{th} = 3.3 R_{th}. 16,500 = 1.650 R_{th}. R_{th} = 10,000 Ohm.

At R_{th} = R_{ref} = 10,000 Ohm (midscale): this is by design the operating midpoint.

Steinhart-Hart: 1/T = A + B ln(R) + C (ln(R))^3 ln(10,000) = 9.2103. 1/T = 1.125 x 10^-3 + 2.347 x 10^-4 x 9.2103 + 0.855 x 10^-7 x (9.2103)^3 = 1.125 x 10^-3 + 2.162 x 10^-3 + 0.855 x 10^-7 x 781.3 = 1.125 x 10^-3 + 2.162 x 10^-3 + 0.06681 x 10^-3 = 3.354 x 10^-3. T = 1 / 3.354 x 10^-3 = 298.17 K = 25.0 degrees C.

Temperature resolution: one ADC code change at midscale: delta-V = 3.3 / 4,096 = 0.806 mV. Near midscale, dR/dV is found by differentiating the divider equation: dR/dV = (R_{ref} + R_{th})^2 / (V_{supply} x R_{ref}) = (20,000)^2 / (3.3 x 10,000) = 12,121 Ohm/V. delta-R = 12,121 x 0.000806 = 9.77 Ohm.

NTC sensitivity at 25 degrees C (beta ~ 3,950 K): dR/dT ~ -R x beta/T^2 = -10,000 x 3,950 / 298^2 = -444.4 Ohm/degrees C. delta-T = delta-R / |dR/dT| = 9.77 / 444.4 = 0.022 degrees C per ADC count.

The thermistor + voltage divider combination with a 12-bit ADC provides 0.022 degrees C resolution at 25 degrees C – excellent for environmental monitoring.

Problem 11.3.5

Given: A load cell system requires 0.01% accuracy over a +/-10 mV full-scale range from a 350 Ohm strain gauge bridge. The measurement bandwidth is 50 Hz. The system must reject 60 Hz power line interference by at least 60 dB. Power supply is 5 V with a 1 mA current budget.

Find: The minimum ADC resolution (bits), the recommended ADC architecture, and whether the 60 Hz rejection requirement is met.

Solution: Required resolution: 0.01% of full scale = 0.0001 x 20 mV = 2.0 uV.

Number of bits: 2^N = 20 mV / 2.0 uV = 10,000. N = log2(10,000) = 13.3 bits minimum. With noise margin (typically add 2-3 bits): 16 bits minimum.

Nyquist rate: 2 x 50 Hz = 100 S/s minimum.

ADC architecture analysis: - SAR: 16-bit at 100 S/s is feasible but requires external low-noise PGA and separate anti-aliasing filter. No inherent 60 Hz rejection. - Sigma-Delta: 24-bit at 50 Hz output data rate provides ~19-20 noise-free bits, integrated PGA, and built-in sinc digital filter.

For a sigma-delta with sinc3 filter at 50 Hz output rate: The sinc filter has notches at multiples of the output data rate. Setting output rate = 60 Hz (or 50 Hz with notch at 60 Hz): many 24-bit sigma-delta ADCs provide a specific 50/60 Hz rejection mode.

At 50 Hz output rate with a sinc3 filter, the rejection at 60 Hz: sinc3 rejection = |sin(pi x 60/f_{mod}) / (sin(pi x 60/(N x f_{mod})))|^3, but practically these ADCs specify > 80 dB rejection of 50/60 Hz.

The 60 dB requirement is met with margin.

Selection: 24-bit sigma-delta ADC with integrated PGA (such as ADS1234 or AD7190). Provides 20+ noise-free bits, built-in 60 Hz rejection > 80 dB, integrated excitation current, and < 1 mA power consumption at 3.3-5 V.

Problem 11.3.6

Given: A photodiode current sensor produces a photocurrent of 0-100 nA proportional to light intensity. A transimpedance amplifier (TIA) with feedback resistance R_f = 10 MOhm and feedback capacitance C_f = 1.6 pF converts the current to voltage. The op-amp has a gain-bandwidth product of 1 MHz and input bias current of 1 pA.

Find: The output voltage range, the -3 dB bandwidth, and the measurement error caused by input bias current.

Solution: Output voltage: V_{out} = -I_{photo} x R_f. At I = 100 nA: V_{out} = -100 x 10^-9 x 10 x 10^6 = -1.0 V. Range: 0 to -1.0 V (inverting configuration).

Transimpedance gain: Z_f = R_f / (1 + j 2 pi f R_f C_f).

-3 dB bandwidth: f_{3dB} = 1 / (2 pi R_f C_f) = 1 / (2 pi x 10 x 10^6 x 1.6 x 10^-12) = 1 / (1.005 x 10^-4) = 9.95 kHz.

Check stability: the noise gain intersection with the op-amp open-loop gain determines stability. f_{noise} ~ sqrt(GBW / (2 pi R_f C_f)) – with proper C_f selection, this should be stable.

Bias current error: V_{error} = I_{bias} x R_f = 1 x 10^-12 x 10 x 10^6 = 10 uV.

At the minimum detectable current (noise limited by bias current): I_{min} = I_{bias} = 1 pA, producing V_{out} = 10 uV.

Error as percentage of full scale: 10 uV / 1.0 V x 100 = 0.001%.

The TIA converts 0-100 nA to 0-1 V with 9.95 kHz bandwidth and 0.001% error from bias current – excellent performance for optical sensing applications.

Chapter 11 – Section 11.4: Measurement Instruments

Practice problems covering digital multimeters, oscilloscopes, spectrum analyzers, function generators, power analyzers, LCR meters, frequency counters, and vector network analyzers.

Problem 11.4.1

Given: A 5-1/2 digit DMM on the 1 V range (input impedance 10 GOhm) measures the output of a high-impedance pH sensor with a source impedance of 200 MOhm. The true sensor voltage is 0.41500 V. The DMM has a specified accuracy of +/-(0.005% of reading + 0.003% of range).

Find: The loading error, the DMM accuracy error, and the total measurement uncertainty.

Solution: Loading error: $V_{measured} = V_{true}$ x R_{DMM} / ($R_{source} + R_{DMM}$) = 0.41500 x 10 x 10^9 / (200 x 10^6 + 10 x 10^9) = 0.41500 x 10^10 / 10.2 x 10^9 = 0.41500 x 0.98039 = 0.40686 V.

Loading error: 0.41500 - 0.40686 = 8.14 mV, or 1.96% of reading. This is severe.

DMM accuracy error (on the reading): Reading error = 0.005% x 0.40686 + 0.003% x 1.0 = 0.0000500 x 0.40686 + 0.0000300 x 1.0 = 0.02034 mV + 0.03000 mV = 0.05034 mV.

DMM resolution: 1 V / 199,999 counts = 5.0 uV per count.

Total uncertainty: loading error (8.14 mV) completely dominates the DMM accuracy error (0.05 mV).

The DMM's 10 GOhm input impedance is inadequate for a 200 MOhm source. An electrometer with > 10 TOhm input impedance would reduce the loading error to < 0.002%. Alternatively, a unity-gain buffer amplifier (FET-input, > 10^12 Ohm input) placed between the sensor and DMM eliminates the loading error.

Problem 11.4.2

Given: A digital oscilloscope has 350 MHz bandwidth, 2.5 GS/s sample rate, 8-bit vertical resolution, and 20 Mpoints memory depth. A 100 MHz clock signal with 1.2 ns rise time and 3.3 V amplitude is being measured. The probe has 500 MHz bandwidth and 10:1 attenuation.

Find: The system bandwidth, the measured rise time, the percent error in rise time measurement, and the maximum capture duration at full sample rate.

Solution: System bandwidth: BW_{system} = 1 / sqrt(1/BW_{scope}^2 + 1/BW_{probe}^2) = 1 / sqrt(1/350^2 + 1/500^2) x 10^6 = 1 / sqrt(8.163 x 10^-18 + 4.0 x 10^-18) = 1 / sqrt(12.163 x 10^-18) = 1 / (3.488 x 10^-9) = 286.7 MHz.

System rise time: $t_{r,system}$ = 0.35 / BW_{system} = 0.35 / 286.7 x 10^6 = 1.221 ns.

Measured rise time: $t_{r,meas}$ = sqrt($t_{r,signal}$^2 + $t_{r,system}$^2) = sqrt(1.2^2 + 1.221^2) = sqrt(1.44 + 1.491) = sqrt(2.931) = 1.712 ns.

Rise time error: (1.712 - 1.2) / 1.2 x 100 = 42.7%.

This is unacceptable. A 1 GHz oscilloscope with a 1.5 GHz probe would give: BW_{system} = 832 MHz, $t_{r,system}$ = 0.42 ns, $t_{r,meas}$ = sqrt(1.44 + 0.177) = 1.27 ns (5.8% error).

Maximum capture at full sample rate: T = memory / sample rate = 20 x 10^6 / 2.5 x 10^9 = 8.0 ms.

At 100 MHz signal, this captures 800,000 cycles – more than adequate for signal analysis.

For accurate rise time measurements, the rule of thumb is oscilloscope bandwidth >= 5x the signal bandwidth (1/rise time). Here, 5 / 1.2 ns = 4.2 GHz bandwidth would give < 1% error.

Problem 11.4.3

Given: A spectrum analyzer with a noise floor of -120 dBm (in 100 Hz RBW) is used to measure the third-order intermodulation (IM3) products of an RF amplifier. Two test tones at 900 MHz and 901 MHz, each at -10 dBm, are applied to the amplifier with 20 dB gain. The IM3 products appear at 899 MHz and 902 MHz at -25 dBm. The measurement uses 1 kHz RBW.

Find: The amplifier output power per tone, the IM3 level relative to the carrier, the third-order intercept point (OIP3), and the noise floor at the measurement RBW.

Solution: Amplifier output per tone: P_{out} = P_{in} + Gain = -10 + 20 = +10 dBm.

IM3 relative to carrier: IM3 = P_{IM3} - P_{out} = -25 - 10 = -35 dBc.

Third-order intercept point (output-referred): OIP3 = P_{out} + |IM3| / 2 = 10 + 35/2 = 10 + 17.5 = +27.5 dBm.

Input-referred: IIP3 = OIP3 - Gain = 27.5 - 20 = +7.5 dBm.

Noise floor at 1 kHz RBW: NF = -120 + 10 log10(1,000/100) = -120 + 10 = -110 dBm.

Signal-to-noise margin for IM3 measurement: -25 - (-110) = 85 dB above noise floor.

The IM3 measurement is valid with excellent noise margin. The OIP3 of +27.5 dBm is a good figure of merit for the amplifier's linearity performance.

Problem 11.4.4

Given: A function generator produces a 500 kHz square wave at 5 V_{pp} into a 50 Ohm load (matched). The specified rise time is 8 ns and the duty cycle is set to exactly 50%. The output is connected to a circuit with 1 kOhm input impedance.

Find: The voltage delivered to the 50 Ohm load (matched case), the voltage delivered to the 1 kOhm load (mismatched), the approximate bandwidth required to preserve the square wave edges, and the number of significant harmonics.

Solution: Into matched 50 Ohm load: the generator is specified at 5 V_{pp} into 50 Ohm, so V_{load} = 5 V_{pp}. (The internal EMF is actually 10 V_{pp}, divided equally between source and load impedance.)

Into 1 kOhm load (mismatched): V_{load} = V_{emf} x R_{load} / (R_{source} + R_{load}) = 10 x 1,000 / (50 + 1,000) = 10 x 0.952 = 9.52 V_{pp}.

The amplitude nearly doubles because the high-impedance load draws negligible current.

Bandwidth for square wave: BW >= 0.35 / t_{rise} = 0.35 / 8 x 10^-9 = 43.75 MHz.

Fundamental frequency: f1 = 500 kHz. Number of significant harmonics: N = BW / f1 = 43.75 x 10^6 / 500 x 10^3 = 87.5.

Approximately 88 harmonics (odd only: 1st, 3rd, 5th, ... up to the 175th) are needed to reconstruct the square wave with the specified rise time.

When connecting the generator to a high-impedance load, the 50 Ohm source impedance causes a reflection coefficient of (1000-50)/(1000+50) = 0.905, producing signal reflections on the cable that can distort fast edges. For the 8 ns rise time at 500 kHz, cable effects are generally negligible for cables shorter than ~0.8 m (considering 8 ns round-trip at 5 ns/m propagation delay).

Problem 11.4.5

Given: A power analyzer measures a three-phase, 4-wire motor drive system. The per-phase readings are: - Phase A: V = 231.2 V, I = 15.4 A, P = 2,845 W - Phase B: V = 229.8 V, I = 15.1 A, P = 2,790 W - Phase C: V = 232.5 V, I = 15.8 A, P = 2,910 W

The current THD on all phases is approximately 28%.

Find: Total three-phase power, total apparent power, system power factor, displacement power factor, and voltage unbalance percentage (NEMA definition).

Solution: Total real power: P_{total} = 2,845 + 2,790 + 2,910 = 8,545 W.

Per-phase apparent power: S_A = 231.2 x 15.4 = 3,560 VA. S_B = 229.8 x 15.1 = 3,470 VA. S_C = 232.5 x 15.8 = 3,674 VA. S_{total} = 3,560 + 3,470 + 3,674 = 10,704 VA.

Total power factor: PF = P / S = 8,545 / 10,704 = 0.798.

Distortion power factor: DstPF = 1 / sqrt(1 + THD^2) = 1 / sqrt(1 + 0.28^2) = 1 / sqrt(1.0784) = 1 / 1.0385 = 0.963.

Displacement power factor: DPF = PF / DstPF = 0.798 / 0.963 = 0.829 (corresponding to phi = 34.0 degrees).

Voltage unbalance (NEMA): maximum deviation from average / average voltage. Average voltage: V_{avg} = (231.2 + 229.8 + 232.5) / 3 = 231.17 V. Maximum deviation: max(|231.2-231.17|, |229.8-231.17|, |232.5-231.17|) = max(0.03, 1.37, 1.33) = 1.37 V. Unbalance = 1.37 / 231.17 x 100 = 0.59%.

The 0.59% voltage unbalance is within the NEMA MG1 limit of 1%. The 28% current THD contributes significantly to the poor power factor (0.798 vs. 0.829 displacement), suggesting harmonic filters would improve the system.

Problem 11.4.6

Given: A 10 uH inductor is measured on an LCR meter at 1 MHz. The readings are |Z| = 72.5 Ohm and phase angle theta = +78.3 degrees.

Find: The series resistance (ESR), the actual inductance at 1 MHz, the quality factor Q, and the self-resonant frequency if the parasitic capacitance is estimated at 5 pF.

Solution: Series resistance: R_s = |Z| x cos(theta) = 72.5 x cos(78.3 degrees) = 72.5 x 0.2028 = 14.70 Ohm.

Series reactance: X_s = |Z| x sin(theta) = 72.5 x sin(78.3 degrees) = 72.5 x 0.9793 = 71.00 Ohm.

Inductance at 1 MHz: L = X_s / (2 pi f) = 71.00 / (2 pi x 10^6) = 71.00 / 6.283 x 10^6 = 11.30 uH.

The measured inductance (11.30 uH) is higher than the nominal 10 uH because at 1 MHz, the parasitic capacitance partially resonates with the inductance, increasing the apparent impedance.

Quality factor: Q = X_s / R_s = 71.00 / 14.70 = 4.83.

Self-resonant frequency (SRF): f_{SRF} = 1 / (2 pi sqrt(L x C_p)) = 1 / (2 pi sqrt(10 x 10^-6 x 5 x 10^-12)) = 1 / (2 pi sqrt(5 x 10^-17)) = 1 / (2 pi x 2.236 x 10^-8.5) = 1 / (2 pi x 7.071 x 10^-9) = 1 / 4.443 x 10^-8 = 22.5 MHz.

Above the 22.5 MHz SRF, the inductor behaves as a capacitor. At 1 MHz (well below SRF), the inductance increase from 10 to 11.3 uH is the early effect of the parasitic capacitance beginning to resonate with the inductance.

Problem 11.4.7

Given: A universal frequency counter with a 200 MHz timebase measures two signals: (a) a 50 kHz clock from a crystal oscillator using a 100 ms gate time, and (b) a 1 PPS (1.000000 Hz) GPS timing pulse using reciprocal counting. The timebase stability is 1 x 10^-8 (OCXO).

Find: (a) The direct count result and resolution for the 50 kHz signal. (b) The reciprocal count resolution for the 1 PPS signal. (c) The absolute accuracy of each measurement.

Solution: (a) Direct count of 50 kHz with 100 ms gate: Expected count = 50,000 x 0.1 = 5,000 counts. Resolution = +/-1 count / gate time = +/-1 / 0.1 = +/-10 Hz. Relative resolution = 10 / 50,000 = 2 x 10^-4 = 0.02% (4 digits).

Frequency reading: 50,000 +/- 10 Hz.

(b) Reciprocal count of 1 PPS: During 1 second (one period of the 1 Hz signal), the 200 MHz timebase counts: N = 200 x 10^6 x 1 = 200,000,000 ticks. Period resolution: delta-T = 1 / 200 x 10^6 = 5 ns. Frequency resolution: delta-f / f = delta-T / T = 5 x 10^-9 / 1 = 5 x 10^-9 (9 digits). Absolute frequency resolution: 1.000000000 x 5 x 10^-9 = 5 nHz.

(c) Absolute accuracy (limited by timebase stability): 50 kHz measurement: 50,000 x 10^-8 = +/-0.0005 Hz. But direct count resolution (+/-10 Hz) is the limiting factor. 50 kHz accuracy: +/-10 Hz (count-limited).

1 PPS measurement: 1.0 x 10^-8 = 10 nHz. Resolution (5 nHz) and timebase stability (10 nHz) are comparable. 1 PPS accuracy: +/-10 nHz (timebase-limited).

The reciprocal method provides 50,000x better resolution than direct counting for the 50 kHz measurement. Using reciprocal counting on the 50 kHz signal (measuring the period of 5,000 cycles in 100 ms with a 200 MHz clock): resolution = 5 x 10^-9, or delta-f = 50,000 x 5 x 10^-9 = 0.25 mHz – dramatically better than the +/-10 Hz from direct counting.

Problem 11.4.8

Given: A VNA measures a 50 Ohm coaxial cable at 2 GHz. The measurements are: S11 = -26 dB angle -90 degrees, S21 = -3.2 dB angle -210 degrees, S12 = -3.2 dB angle -210 degrees, and S22 = -28 dB angle -85 degrees.

Find: The cable insertion loss, input VSWR, the fraction of power delivered to the output, and the electrical length.

Solution: Insertion loss = -S21 = 3.2 dB.

Input reflection coefficient: |gamma| = 10^(S11/20) = 10^(-26/20) = 10^(-1.3) = 0.0501.

Input VSWR = (1 + |gamma|) / (1 - |gamma|) = (1 + 0.0501) / (1 - 0.0501) = 1.0501 / 0.9499 = 1.106:1.

Power reflected: |gamma|^2 = 0.0501^2 = 0.00251 = 0.251%.

Power transmitted: |S21|^2 = 10^(S21(dB)/10) = 10^(-3.2/10) = 10^(-0.32) = 0.4786 = 47.86%.

Power dissipated in cable: 1 - |S11|^2 - |S21|^2 = 1 - 0.00251 - 0.4786 = 0.519 = 51.9%.

Electrical length from S21 phase: Phase shift = -210 degrees. At 2 GHz, one wavelength = 360 degrees. Electrical length = 210 / 360 = 0.583 wavelengths.

In free space at 2 GHz: lambda = c/f = 3 x 10^8 / 2 x 10^9 = 0.15 m. With velocity factor of 0.66 (typical coax): $\text{lambda}_{\text{cable}}$ = 0.15 x 0.66 = 0.099 m. Physical length ~ 0.583 x 0.099 = 0.0577 m = 5.77 cm.

The 3.2 dB insertion loss for a ~6 cm cable at 2 GHz indicates either a very lossy cable, a poor connector, or the cable is much longer than estimated (the phase wrapping may have added 360 degrees, making the actual length 1.583 wavelengths or ~15.7 cm).

Chapter 11 – Section 11.5: Data Acquisition

Practice problems covering sampling and quantization, DAQ systems, data logging, and automated test systems.

Problem 11.5.1

Given: A 16-bit SAR ADC with a +/-5 V bipolar input range samples a vibration sensor at f_s = 100 kS/s. The signal of interest has frequencies from 10 Hz to 5 kHz, with a strong 5 kHz component at 2 V_{pp} and broadband noise extending to 50 kHz.

Find: The ADC resolution (LSB size), the ideal SINAD, the ENOB, the Nyquist frequency, and the attenuation needed from the anti-aliasing filter at frequencies above 45 kHz to prevent more than 1 LSB of aliased noise.

Solution: Full-scale range: 10 V (from -5 V to +5 V). Resolution (1 LSB): 10 V / 2^16 = 10 / 65,536 = 0.1526 mV = 152.6 uV.

Ideal SINAD: SINAD = 6.02N + 1.76 = 6.02 x 16 + 1.76 = 96.32 + 1.76 = 98.08 dB.

ENOB = (SINAD - 1.76) / 6.02 = (98.08 - 1.76) / 6.02 = 16.0 bits (ideal case).

Nyquist frequency: $f_N = f_s$ / 2 = 100,000 / 2 = 50 kHz.

Anti-aliasing requirement: noise above 45 kHz aliases into 5 kHz (since f_s - 45,000 = 55,000 which folds to 50,000 - 5,000 = 45,000… more precisely, a signal at 95 kHz aliases to 5 kHz).

To limit aliased noise to < 1 LSB = 152.6 uV: If broadband noise spectral density is approximately V_{noise}/sqrt(BW), we need the filter to attenuate the noise in the 50-100 kHz band (which aliases into 0-50 kHz).

For 1 LSB at the 5 V signal level, the required attenuation is: Assuming worst-case noise of 10 mV_{rms} in the 45-50 kHz band: Attenuation needed = 20 log10(0.1526 / 10) = 20 log10(0.01526) = -36.3 dB.

A 4th-order Butterworth filter with f_c = 10 kHz provides -40 dB/decade x log10(45/10) = -40 x 0.653 = -26.1 dB at 45 kHz. This is insufficient; a 6th-order filter (-39.2 dB at 45 kHz) or an 8th-order (-52.3 dB) would be needed for 1-LSB aliasing performance.

Problem 11.5.2

Given: An 8-channel simultaneous-sampling DAQ system uses 18-bit ADCs with dedicated sample-and-hold circuits on each channel. The aggregate sample rate is 500 kS/s (62.5 kS/s per channel). Channels 1-4 measure voltage (0-10 V range), channels 5-6 measure current (4-20 mA through 250 Ohm shunt = 1-5 V), and channels 7-8 measure thermocouple signals (0-20 mV range with 100x PGA).

Find: The resolution on each range, the maximum signal frequency per channel (assuming 10x over-sampling), and the inter-channel time skew.

Solution: Channels 1-4 (0-10 V range): Resolution: 10 V / 2^18 = 10 / 262,144 = 38.15 uV per count.

Channels 5-6 (1-5 V range): If the ADC uses the 0-10 V range: resolution = 38.15 uV per count. If the ADC is configured for a 0-5 V range: 5 / 262,144 = 19.07 uV per count. In current units (using 250 Ohm shunt): 19.07 uV / 250 = 0.0763 uA per count.

Channels 7-8 (0-20 mV with 100x PGA): After PGA: 0-2 V signal to ADC. ADC range 0-10 V, resolution = 38.15 uV per count. Referred to input: 38.15 / 100 = 0.3815 uV per count. In 18 bits over 20 mV: 20 mV / 262,144 = 0.0763 uV – significantly finer than the 0.38 uV from the ADC-limited case. The PGA noise and ADC noise will limit the effective resolution.

Maximum signal frequency per channel (10x oversampling): $f_{max} = f_{s,ch}$ / 10 = 62,500 / 10 = 6.25 kHz.

Inter-channel time skew: With simultaneous sampling, all channels are sampled at the same instant. Time skew = 0 ns (by definition of simultaneous sampling).

For a multiplexed system at the same aggregate rate, the skew would be: 1 / 500,000 = 2 us between adjacent channels, or 14 us between channel 1 and channel 8.

Simultaneous sampling eliminates the 14 us inter-channel skew that would cause 5.1 degrees of phase error at 1 kHz in a multiplexed system.

Problem 11.5.3

Given: A 64-channel temperature monitoring system logs Pt100 RTD data continuously. Each channel is sampled at 1 S/s with 24-bit resolution (3 bytes per sample). Each scan includes a 64-bit (8-byte) timestamp. Data is stored in CSV format with overhead of approximately 20 bytes per reading (channel ID, comma separators, newline). The system must log for 1 year.

Find: The raw binary data rate, the CSV data rate, the storage required for 1 year in both formats, and the recommended storage medium.

Solution: Raw binary format: Data per scan: 64 channels x 3 bytes = 192 bytes + 8 bytes timestamp = 200 bytes/scan. Scans per second: 1. Raw data rate: 200 bytes/s = 0.200 kB/s.

Per hour: 200 x 3,600 = 720 kB/hour. Per day: 720 x 24 = 17.28 MB/day. Per year: 17.28 x 365 = 6.31 GB/year.

CSV format: Each reading: ~8 characters for value (e.g., “100.0234”) + 20 bytes overhead = ~28 bytes per reading. Data per scan: 64 x 28 + 20 (timestamp) = 1,792 + 20 = 1,812 bytes/scan. CSV data rate: 1,812 bytes/s = 1.812 kB/s.

Per year: 1,812 x 86,400 x 365 = 57.13 x 10^9 bytes = 57.1 GB/year.

CSV-to-binary ratio: 57.1 / 6.31 = 9.05x larger (CSV is ~9x the size of binary).

With lossless compression (typical 3:1 for repetitive temperature data): Binary compressed: 6.31 / 3 = 2.1 GB/year. CSV compressed: 57.1 / 5 = 11.4 GB/year (CSV compresses better, ~5:1).

Recommended storage: A 32 GB industrial SD card or USB flash drive is sufficient for binary format with compression. For CSV, a 64 GB drive provides margin. For reliability over 1 year of continuous writing, an industrial SSD (rated for high write endurance) is preferred.

Write speed requirement: 1.812 kB/s (CSV) – trivially met by any modern storage medium.

Problem 11.5.4

Given: An automated test station tests PCBs using a 6-1/2 digit DMM (SCPI-controlled over LAN), a 200 MHz oscilloscope, and a power supply. The test sequence for each board: - 6 DC voltage measurements at 150 ms each - 4 resistance measurements at 200 ms each (includes range switching) - 2 frequency measurements at 300 ms each - 1 oscilloscope waveform capture at 500 ms - Board handler time: 1.2 s (load/unload)

The switching matrix adds 10 ms per relay actuation, with 1 relay per test point.

Find: The total test time per board, the throughput in boards/hour, and the throughput improvement if the handler is pipelined (handler loads next board while testing current board).

Solution: Measurement time: DC voltage: 6 x (150 + 10) = 6 x 160 = 960 ms. Resistance: 4 x (200 + 10) = 4 x 210 = 840 ms. Frequency: 2 x (300 + 10) = 2 x 310 = 620 ms. Oscilloscope: 1 x (500 + 10) = 510 ms.

Total measurement time: 960 + 840 + 620 + 510 = 2,930 ms = 2.93 s.

Communication overhead (SCPI commands, ~5 ms per measurement): 13 x 5 = 65 ms.

Total test time: 2.93 + 0.065 + 1.2 (handler) = 4.195 s per board.

Throughput: 3,600 / 4.195 = 858 boards/hour.

With pipelined handler (handler operates during test): Test time = max(measurement, handler) = max(2.995, 1.2) = 2.995 s. Since measurement > handler, the handler time is hidden. Effective time per board: 2.995 s. Throughput: 3,600 / 2.995 = 1,202 boards/hour.

Improvement: (1,202 - 858) / 858 x 100 = 40.1% throughput improvement from pipelining the handler.

To further increase throughput: use a faster DMM (100 ms measurements), optimize SCPI communication (binary data transfer, query/response overlapping), or implement dual-DUT fixturing.

Problem 11.5.5

Given: A 24-bit sigma-delta ADC operates with a master clock of 4.096 MHz and an oversampling ratio (OSR) of 256. The ADC uses a sinc3 decimation filter. The input signal is from a load cell bridge with 20 mV full-scale output.

Find: The output data rate, the theoretical noise-free resolution, the noise floor in nV, and the effective resolution in bits.

Solution: Output data rate: $f_{out} = f_{clk}$ / OSR = 4,096,000 / 256 = 16,000 S/s = 16 kS/s.

Theoretical oversampling benefit: Quantization noise is spread over 0 to $f_{clk}/2$ = 2.048 MHz. After decimation to 8 kHz bandwidth: noise is reduced by the oversampling ratio.

For a sinc3 filter, the noise reduction is: SNR improvement = 10 log10(OSR) x (2n+1)/2 for nth-order noise shaping. For a first-order sigma-delta with sinc3: SNR ~ 6.02N + 1.76 + 30 log10(OSR) - 10 log10(pi^(2)/3) (approximately).

More practically, sigma-delta ADCs specify noise performance directly. At 16 kS/s with 24-bit ADC, typical RMS noise is ~1-3 uV on a 2.5 V reference.

Assuming 2 uV_{rms} noise: Noise referred to 20 mV full scale: 2 uV / 20 mV = 10^-4 = 0.01%.

Effective resolution: ENOB = log2(full-scale / noise_pp) = log2(20 mV / (6.6 x 2 uV)) = log2(20,000/13.2) = log2(1,515) = 10.6 bits for peak-to-peak noise-free resolution.

This seems low because we are using only 20 mV of the 2.5 V input range. With PGA gain of 128: Effective input range = 2.5/128 = 19.5 mV (well-matched to the 20 mV full-scale). Noise at gain 128 referred to input: typically 0.3 uV_{rms}. Noise-free resolution: 20 mV / (6.6 x 0.3 uV) = 20,000 / 1.98 = 10,101. ENOB = log2(10,101) = 13.3 noise-free bits.

At 16 kS/s with PGA gain of 128, the effective resolution is approximately 13.3 noise-free bits on the 20 mV range, sufficient for 0.01% load cell accuracy. Reducing the output rate to 10 S/s would improve to approximately 18-20 noise-free bits.

Problem 11.5.6

Given: A SCPI-controlled test system uses a PyVISA script to measure 100 DUTs. Each DUT requires: power supply set to 12 V (50 ms settling), measure supply current (DMM, 100 ms), measure 3 output voltages (DMM, 80 ms each), and power down (20 ms). LAN communication latency is 5 ms per command/query. Each measurement requires 2 SCPI transactions (trigger + read).

Find: The total time per DUT, the total test time for 100 DUTs, and the throughput improvement from using the DMM's internal scan list (which eliminates per-measurement command latency for the 3 output voltages).

Solution: Standard sequential approach: Power on: 1 command x 5 ms + 50 ms settling = 55 ms. Current measurement: 2 transactions x 5 ms + 100 ms = 110 ms. 3 voltage measurements: 3 x (2 x 5 ms + 80 ms) = 3 x 90 = 270 ms. Power down: 1 x 5 ms + 20 ms = 25 ms.

Total per DUT: 55 + 110 + 270 + 25 = 460 ms.

For 100 DUTs: 100 x 460 = 46,000 ms = 46.0 s. Throughput: 3,600 / 0.46 = 7,826 DUTs/hour.

With DMM scan list optimization: The 3 voltage measurements are configured as a scan list with a single trigger command: Scan setup (one-time): 3 commands x 5 ms = 15 ms (amortized over 100

DUTs: 0.15 ms/DUT). Trigger scan: 1 x 5 ms = 5 ms. 3 measurements execute internally: 3 x 80 ms = 240 ms (no per-measurement latency). Read all results: 1 x 5 ms = 5 ms.

3 voltage measurements total: 5 + 240 + 5 = 250 ms (vs. 270 ms).

Total per DUT: 55 + 110 + 250 + 25 = 440 ms.

For 100 DUTs: 100 x 440 = 44,000 ms = 44.0 s. Time saved: 46.0 - 44.0 = 2.0 s (4.3% improvement).

The improvement is modest because the DMM measurement time dominates. For higher throughput, use a faster DMM (NPLC=0.1 for ~20 ms per reading instead of 80 ms), reducing the voltage scan to 3 x 20 = 60 ms and cutting total per-DUT time to 55 + 50 + 70 + 25 = 200 ms = 18,000 DUTs/hour – a 2.3x improvement.

Chapter 12 – Section 12.1: DC Motors

Practice problems covering brushed DC motors, brushless DC (BLDC) motors, and universal motors.

Problem 12.1.1

Given: A series-wound DC motor has $V_{supply} = 120$ V, armature resistance $R_a = 0.3$ Ohm, series field resistance $R_f = 0.2$ Ohm, and a motor constant K = 0.05 V/(A rad/s) (back-EMF is proportional to both flux and speed: $E_{back} = K \times I_a \times \omega$). At rated load, the motor draws 50 A.

Find: The back-EMF, the rated speed in RPM, the output torque, and the mechanical output power.

Solution: Total resistance: $R_{total} = R_a + R_f = 0.3 + 0.2 = 0.5$ Ohm.

Back-EMF: $E_{back} = V - I_a \times R_{total} = 120 - 50 \times 0.5 = 120 - 25 = 95$ V.

For a series motor: $E_{back} = K \times I_a \times \omega$. $\omega = E_{back} / (K \times I_a) = 95 / (0.05 \times 50) = 95 / 2.5 = 38.0$ rad/s = 362.9 RPM.

Torque: $T = K \times I_a^2 = 0.05 \times 50^2 = 0.05 \times 2{,}500 = 125.0$ N-m (for a series motor, torque is proportional to I^2).

Wait – let us reconsider. The electromagnetic torque equals the electromagnetic power divided by speed: $P_{em} = E_{back} \times I_a = 95 \times 50 = 4{,}750$ W. $T = P_{em} / \omega = 4{,}750 / 38.0 = 125.0$ N-m. Confirmed.

Mechanical output (assuming 5% rotational losses): $P_{out} = 0.95 \times 4{,}750 = 4{,}512$ W = 6.05 HP.

The series motor produces high torque (125 N-m) at low speed (363 RPM), characteristic of series motors used in traction and starter applications. The torque-speed curve follows an inverse-square relationship: doubling the speed requires halving the current, which quarters the torque.

Problem 12.1.2

Given: A compound DC motor (cumulative) operates at 240 V with armature resistance $R_a = 0.4$ Ohm. The shunt field produces a flux constant $K_{shunt} = 1.0$ V/(rad/s), and the series field adds 10% additional flux at full-load current of 35 A. At no-load, $I_a = 3$ A and the total flux constant is K_{shunt} only.

Find: The no-load speed, the full-load speed (with the series field contribution), and the speed regulation.

Solution: No-load: E_{back} = V - I_a x R_a = 240 - 3 x 0.4 = 240 - 1.2 = 238.8 V. $omega_{NL}$ = E_{back} / K_{shunt} = 238.8 / 1.0 = 238.8 rad/s = 2,281 RPM.

Full-load: E_{back} = V - I_a x R_a = 240 - 35 x 0.4 = 240 - 14 = 226 V. Total flux constant at full load: K_{total} = K_{shunt} x 1.10 = 1.0 x 1.10 = 1.10 V/(rad/s). $omega_{FL}$ = E_{back} / K_{total} = 226 / 1.10 = 205.5 rad/s = 1,962 RPM.

Speed regulation: SR = ($omega_{NL}$ - $omega_{FL}$) / $omega_{FL}$ x 100% = (238.8 - 205.5) / 205.5 x 100% = 33.3 / 205.5 x 100% = 16.2%.

Full-load torque: T = K_{total} x I_a = 1.10 x 35 = 38.5 N-m. Mechanical power: P = T x omega = 38.5 x 205.5 = 7,912 W = 10.6 HP.

The compound motor has higher speed regulation (16.2%) than a pure shunt motor (~5-10%) because the series field increases flux under load, further reducing speed. However, it provides better starting torque and overload capacity than a shunt motor alone.

Problem 12.1.3

Given: A BLDC motor has 12 poles, a torque constant K_t = 0.08 N-m/A, a back-EMF constant K_e = 0.08 V/(rad/s), and a phase resistance R_{ph} = 0.35 Ohm. The DC bus voltage is 24 V. The motor must deliver 0.5 N-m at 5,000 RPM.

Find: The required phase current, the back-EMF, whether the bus voltage is sufficient, and the copper losses.

Solution: Speed: omega = 5,000 x 2 pi / 60 = 523.6 rad/s.

Required current: I = T / K_t = 0.5 / 0.08 = 6.25 A.

Back-EMF: E_{back} = K_e x omega = 0.08 x 523.6 = 41.89 V.

The back-EMF (41.89 V) exceeds the bus voltage (24 V), so the motor CANNOT operate at 5,000 RPM with a 24 V bus.

Maximum speed at 24 V bus (allowing 2 V for transistor drops and I x R): Available back-EMF: E_{max} = 24 - 2 x 6.25 x 0.35 - 2 = 24 - 4.375 - 2 = 17.625 V. (With two phases conducting: voltage drop = 2 x I x R_{ph}.) $omega_{max}$ = E_{max} / K_e = 17.625 / 0.08 = 220.3 rad/s = 2,104 RPM.

At 2,104 RPM: copper losses = 2 x I^2 x R_{ph} = 2 x 6.25^2 x 0.35 = 2 x 13.67 = 27.3 W.

Mechanical power: P = T x omega = 0.5 x 220.3 = 110.2 W. Efficiency (copper losses only): eta = 110.2 / (110.2 + 27.3) = 110.2 / 137.5 = 80.1%.

To reach 5,000 RPM at 0.5 N-m, the bus voltage must be at least E_{back} + 2IR + V_{drop} = 41.89 + 4.375 + 2 = 48.3 V. A 48 V bus would be the appropriate selection.

Problem 12.1.4

Given: A universal motor in a vacuum cleaner is rated at 240 V, 6 A, 28,000 RPM. The combined armature and field resistance is R_{total} = 3.5 Ohm. The motor is operated at reduced speed using a triac controller that reduces the effective voltage to 180 V. Assume the motor's torque-speed characteristic follows T proportional to I^2 (series motor behavior).

Find: The back-EMF and power at rated conditions, the current at 180 V (assuming speed drops proportionally to voltage for a lightly loaded series motor), and the approximate new speed and power.

Solution: Rated conditions (240 V, 6 A): E_{back} = V - I x R_{total} = 240 - 6 x 3.5 = 240 - 21 = 219 V. Input power: P_{in} = 240 x 6 = 1,440 W. Electromagnetic power: P_{em} = E_{back} x I = 219 x 6 = 1,314 W. Copper losses: P_{Cu} = I^2 x R = 36 x 3.5 = 126 W.

At 180 V (triac-controlled): For a lightly loaded series motor at reduced voltage, the speed drops roughly proportionally to voltage and the current drops similarly. Approximate ratio: V_{new} / V_{rated} = 180/240 = 0.75.

Using the relation E_{back} proportional to I x omega (series motor), and V = E_{back} + I x R: Assume I_{new} ~ 0.75 x 6 = 4.5 A (first approximation). $E_{back,new}$ = 180 - 4.5 x 3.5 = 180 - 15.75 = 164.25 V.

Speed ratio: $omega_{new}$/$omega_{rated}$ = ($E_{back,new}$/$E_{back,rated}$) x (I_{rated}/I_{new}) = (164.25/219) x (6/4.5) = 0.750 x 1.333 = 1.00.

This suggests the speed stays nearly constant (characteristic of a lightly loaded series motor). Let's refine: At constant load torque (T proportional to I^2): if torque is constant, I is constant at 6 A. E_{back} = 180 - 6 x 3.5 = 159 V. $omega_{new}$ = $omega_{rated}$ x (159/219) x (6/6) = 28,000 x 0.726 = 20,329 RPM. New power: P_{em} = 159 x 6 = 954 W.

At reduced voltage with constant torque load, speed drops to ~20,300 RPM and power to ~954 W. For a variable-torque load like a vacuum (where torque scales with speed^2), the speed reduction would be less dramatic.

Problem 12.1.5

Given: A shunt-wound DC motor is used to drive a hoist. Nameplate: 200 V, R_a = 0.6 Ohm, K = 0.8 V/(rad/s). The motor operates in two modes: (1) hoisting at full load I_a = 30 A, and (2) regenerative braking while lowering the load, where the motor acts as a generator feeding current back to the supply at I_a = -20 A (current flows in reverse).

Find: The motor speed in each mode and the power flow direction.

Solution: Mode 1 – Hoisting (motoring): E_{back} = V - I_a x R_a = 200 - 30 x 0.6 = 200 - 18 = 182 V. omega = E_{back} / K = 182 / 0.8 = 227.5 rad/s = 2,173 RPM.

Power from supply: P_{in} = V x I_a = 200 x 30 = 6,000 W (consumed). Mechanical power: P_{mech} = E_{back} x I_a = 182 x 30 = 5,460 W (to load). Copper losses: I^2 R = 900 x 0.6 = 540 W.

Mode 2 – Lowering (regenerative braking): E_{back} = V - I_a x R_a = 200 - (-20) x 0.6 = 200 + 12 = 212 V. omega = 212 / 0.8 = 265.0 rad/s = 2,531 RPM.

The back-EMF (212 V) exceeds the supply voltage (200 V), causing current to flow back into the supply. Power returned to supply: P = V x $|I_a|$ = 200 x 20 = 4,000 W (regenerated). Mechanical power input (from load): P_{mech} = E_{back} x $|I_a|$ = 212 x 20 = 4,240 W. Copper losses: 400 x 0.6 = 240 W. Check: 4,240 = 4,000 + 240. Confirmed.

During lowering, the motor runs faster than the motoring speed (2,531 vs. 2,173 RPM) because the back-EMF must exceed the supply voltage for current reversal. The load's potential energy is partially recovered (4,000 W to supply) with 240 W dissipated as copper losses.

Problem 12.1.6

Given: A BLDC motor for a drone propeller has K_t = 0.012 N-m/A, K_e = 0.012 V/(rad/s), R_{ph} = 0.08 Ohm, and no-load current I_0 = 1.5 A. The motor operates from a 22.2 V (6S LiPo) battery. At hover, the motor draws 18 A.

Find: The hover speed, the propeller torque, the mechanical output power, the motor efficiency, and the thrust if the propeller constant is K_{thrust} = 1.1 x 10^-7 N/(RPM)^2.

Solution: Back-EMF at hover: E_{back} = V - 2 x I x R_{ph} = 22.2 - 2 x 18 x 0.08 = 22.2 - 2.88 = 19.32 V.

Speed: omega = E_{back} / K_e = 19.32 / 0.012 = 1,610 rad/s. In RPM: n = 1,610 x 60 / (2 pi) = 15,378 RPM.

Torque-producing current: I_{torque} = I - I_0 = 18 - 1.5 = 16.5 A. Propeller torque: T = K_t x I_{torque} = 0.012 x 16.5 = 0.198 N-m.

Mechanical output: P_{mech} = T x omega = 0.198 x 1,610 = 318.8 W.

Input power: P_{in} = V x I = 22.2 x 18 = 399.6 W. Efficiency: eta = 318.8 / 399.6 = 79.8%.

Losses breakdown: Copper: 2 x 18^2 x 0.08 = 51.84 W. No-load (friction, windage, iron): E_{back} x I_0 = 19.32 x 1.5 = 28.98 W. Total losses: 51.84 + 28.98 = 80.82 W. Check: 399.6 - 318.8 = 80.8 W. Confirmed.

Thrust: F = K_{thrust} x n^2 = 1.1 x 10^-7 x (15,378)^2 = 1.1 x 10^-7 x 2.365 x 10^8 = 26.0 N (approximately 2.65 kg of thrust).

Each motor produces 26 N of thrust at 18 A hover current. For a quadcopter, total thrust is 4 x 26 = 104 N, supporting an all-up weight of ~10.6 kg with a 2.5:1 thrust-to-weight ratio for maneuverability.

Chapter 12 — Section 12.2: AC Motors

Practice problems covering induction motors, synchronous motors, single-phase motors, reluctance motors, linear motors, and wound-rotor induction motors.

Problem 12.2.1

Given: A 6-pole, three-phase induction motor is connected to a 50 Hz supply. At full load, the motor speed is 960 RPM and it delivers 30 kW of mechanical power.

Find: (a) The synchronous speed, (b) the slip, (c) the rotor electrical frequency, and (d) the full-load torque.

Solution:

(a) Synchronous speed: $n_s = 120f / P = 120 \times 50 / 6 = 1{,}000$ RPM

(b) Slip: $s = (n_s - n_r) / n_s = (1{,}000 - 960) / 1{,}000 = 0.04 = 4.0\%$

(c) Rotor electrical frequency: $f_r = s \times f = 0.04 \times 50 = 2.0$ Hz

(d) Rotor speed in rad/s: $\omega_r = 960 \times 2\pi/60 = 100.5$ rad/s

Full-load torque: $\tau = P_{mech} / \omega_r = 30{,}000 / 100.5 = 298.5$ N·m

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Problem 12.2.2

Given: An 8-pole, three-phase synchronous motor operates at 60 Hz. The motor is rated at 200 kW with an efficiency of 93% and runs at unity power factor. The terminal voltage is 4,160 V (line-to-line).

Find: (a) The motor speed, (b) the input power, (c) the line current, and (d) the apparent power.

Solution:

(a) Synchronous speed: $n_s = 120f / P = 120 \times 60 / 8 = 900$ RPM

(b) Input power: $P_{in} = P_{out} / \eta = 200 / 0.93 = 215.1$ kW

(c) At unity power factor, S = P: $I_L = P_{in} / (\sqrt{3} \times V_{LL}) = 215{,}100 / (1.732 \times 4{,}160) = 215{,}100 / 7{,}205 = 29.9$ A

(d) At unity power factor: S = P = 215.1 kVA

Problem 12.2.3

Given: A capacitor-start single-phase induction motor operates at 240 V, 60 Hz. The main winding impedance is $Z_m = 8 + j10\ \Omega$ and the auxiliary winding impedance is $Z_{aux} = 15 + j6\ \Omega$. A start capacitor is placed in series with the auxiliary winding.

Find: The capacitance required to place the auxiliary winding current exactly 90° ahead of the main winding current for maximum starting torque.

Solution:

Main winding impedance angle: $\varphi_m = \tan^{-1}(10/8) = \tan^{-1}(1.25) = 51.3°$

For the auxiliary current to lead the main current by 90°, the auxiliary impedance angle must be: $\varphi_{aux} = 51.3° - 90° = -38.7°$ (capacitive)

With a series capacitor of reactance X_C: $Z_{total} = 15 + j(6 - X_C)$

For $\varphi_{aux} = -38.7°$: $\tan(-38.7°) = (6 - X_C) / 15$ $-0.801 = (6 - X_C) / 15$ $6 - X_C = -12.02$ $X_C = 18.02\ \Omega$

Capacitance: $C = 1 / (2\pi f X_C) = 1 / (2\pi \times 60 \times 18.02) = 1 / 6{,}793 = 147.2\ \mu F$

A standard 150 μF start capacitor would be selected.

Problem 12.2.4

Given: A 4-pole synchronous reluctance motor operates from a 60 Hz VFD. The d-axis inductance is $L_d = 150$ mH and the q-axis inductance is $L_q = 25$ mH. The rated stator current is 8 A.

Find: (a) The synchronous speed, (b) the saliency ratio, and (c) the maximum reluctance torque using $\tau = (3/2) \times (P/2) \times (L_d - L_q) \times I_d \times I_q$, where maximum torque occurs at $I_d = I_q = I_s/\sqrt{2}$.

Solution:

(a) Synchronous speed: $n_s = 120 \times 60 / 4 = 1{,}800$ RPM

(b) Saliency ratio: $\xi = L_d / L_q = 150 / 25 = 6.0$

(c) For maximum torque: $I_d = I_q = I_s / \sqrt{2} = 8 / 1.414 = 5.657$ A

$\tau = (3/2) \times (P/2) \times (L_d - L_q) \times I_d \times I_q$ $\tau = 1.5 \times 2 \times (0.150 - 0.025) \times 5.657 \times 5.657$ $\tau = 3.0 \times 0.125 \times 32.0 = 12.0$ N·m

Mechanical power at rated speed: $\omega = 1{,}800 \times 2\pi/60 = 188.5$ rad/s $P_{mech} = 12.0 \times 188.5 = 2{,}262$ W = 2.26 kW

Problem 12.2.5

Given: A linear induction motor for a baggage handling system has a pole pitch of $\tau_p = 80$ mm, operates at 60 Hz, and must propel a 50 kg cart at a steady-state speed of 3 m/s with a friction force of 120 N. The slip at operating speed is 8%.

Find: (a) The synchronous speed of the traveling field, (b) the actual cart speed from slip, (c) the thrust force required, and (d) the mechanical power delivered.

Solution:

(a) Synchronous speed of the traveling field: $v_s = 2\tau_p f = 2 \times 0.080 \times 60 = 9.6$ m/s

(b) Actual speed from slip: $v = v_s(1 - s) = 9.6 \times (1 - 0.08) = 9.6 \times 0.92 = 8.83$ m/s

However, the problem states steady-state speed of 3 m/s. This means the frequency must be adjusted. At 3 m/s with 8% slip: $v_s = v / (1 - s) = 3 / 0.92 = 3.26$ m/s Required frequency: $f = v_s / (2\tau_p) = 3.26 / (2 \times 0.080) = 20.4$ Hz

(c) At steady state, the thrust equals friction: $F = 120$ N

(d) Mechanical power: $P_{mech} = F \times v = 120 \times 3 = 360$ W

Problem 12.2.6

Given: A 6-pole, 60 Hz wound-rotor induction motor has a standstill rotor EMF of $E_2 = 250$ V (line-to-line), rotor resistance $R_2 = 0.4\ \Omega$ per phase, and rotor leakage reactance $X_2 = 2.0\ \Omega$ per phase.

Find: (a) The synchronous speed, (b) the slip at maximum torque with no external resistance, (c) the speed at maximum torque, and (d) the external resistance per phase needed to achieve maximum torque at standstill.

Solution:

(a) Synchronous speed: $n_s = 120 \times 60 / 6 = 1{,}200$ RPM

(b) Slip at maximum torque (no external resistance): $s_{max} = R_2 / X_2 = 0.4 / 2.0 = 0.2 = 20\%$

(c) Speed at maximum torque: $n = n_s(1 - s_{max}) = 1{,}200 \times 0.8 = 960$ RPM

(d) For maximum torque at standstill ($s = 1$): $s_{max} = R_{2total} / X_2 = 1$, so $R_{2total} = X_2 = 2.0\ \Omega$ per phase $R_{ext} = R_{2total} - R_2 = 2.0 - 0.4 = 1.6\ \Omega$ per phase

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Problem 12.2.7

Given: A 2-pole, three-phase induction motor operates at 60 Hz. At rated load the slip is 2.5%. The motor delivers 75 kW of mechanical power. The stator copper loss is 2.1 kW, core loss is 1.5 kW, and friction/windage loss is 0.9 kW.

Find: (a) The synchronous speed and rotor speed, (b) the rotor copper loss, (c) the air-gap power, (d) the total input power, and (e) the motor efficiency.

Solution:

(a) Synchronous speed: $n_s = 120 \times 60 / 2 = 3{,}600$ RPM $n_r = n_s(1 - s) = 3{,}600 \times 0.975 = 3{,}510$ RPM

(b) The relationship $P_{mech} = P_{ag}(1 - s)$ gives: $P_{ag} = (P_{mech} + P_{fw}) / (1 - s) = (75 + 0.9) / 0.975 = 75.9 / 0.975 = 77.85$ kW

Rotor copper loss: $P_{Cu,rotor} = s \times P_{ag} = 0.025 \times 77.85 = 1.95$ kW

(c) Air-gap power: $P_{ag} = 77.85$ kW

(d) Total input power: $P_{in} = P_{ag} + P_{Cu,stator} + P_{core} = 77.85 + 2.1 + 1.5 = 81.45$ kW

(e) Motor efficiency: $\eta = P_{mech} / P_{in} = 75 / 81.45 = 0.921 = 92.1\%$

Problem 12.2.8

Given: A permanent magnet synchronous motor (PMSM) for an electric vehicle has 8 poles, a torque constant $K_t = 0.85$ N·m/A, and a back-EMF constant $K_e = 0.85$ V/(rad/s). The DC bus voltage is 400 V, and the motor must deliver 150 N·m at 4,000 RPM.

Find: (a) The required phase current, (b) the back-EMF, (c) the mechanical output power, and (d) whether the motor can operate at this speed with the given bus voltage.

Solution:

(a) Required phase current: $I = \tau / K_t = 150 / 0.85 = 176.5$ A

(b) Speed in rad/s: $\omega = 4{,}000 \times 2\pi/60 = 418.9$ rad/s

Back-EMF (line-to-line): $E_{back} = K_e \times \omega = 0.85 \times 418.9 = 356.1$ V

(c) Mechanical output power: $P_{mech} = \tau \times \omega = 150 \times 418.9 = 62{,}832$ W $= 62.8$ kW (84.3 HP)

(d) The back-EMF of 356.1 V is less than the DC bus voltage of 400 V. The remaining voltage ($400 - 356.1 = 43.9$ V) must overcome the resistive drops and provide current regulation headroom. The motor can operate at this speed, but it is approaching the voltage limit. Above approximately 4,500 RPM, field weakening (reducing I_d) would be required.

Problem 12.2.9

Given: A three-phase induction motor drives a centrifugal pump. Nameplate data: 100 HP, 460 V, 60 Hz, 4-pole, FLA = 124 A, efficiency = 94.5%, power factor = 0.87 lagging at full load. The motor operates at 85% load.

Find: (a) The synchronous speed, (b) the input power at 85% load (assuming constant efficiency), (c) the line current at 85% load, and (d) the reactive power consumed.

Solution:

(a) Synchronous speed: $n_s = 120 \times 60 / 4 = 1{,}800$ RPM

(b) Mechanical output at 85% load: $P_{out} = 0.85 \times 100 \times 746 = 63{,}410$ W

Input power: $P_{in} = P_{out} / \eta = 63{,}410 / 0.945 = 67{,}101$ W = 67.1 kW

(c) Line current at 85% load: $I = P_{in} / (\sqrt{3} \times V \times PF) = 67{,}101 / (1.732 \times 460 \times 0.87) = 67{,}101 / 693.2 = 96.8$ A

(d) Apparent power: $S = P_{in} / PF = 67{,}101 / 0.87 = 77{,}127$ VA

Reactive power: $Q = \sqrt{(S^2 - P^2)} = \sqrt{(77{,}127^2 - 67{,}101^2)} = \sqrt{(5{,}948.6 \times 10^6 - 4{,}502.5 \times 10^6)} = \sqrt{(1{,}446.1 \times 10^6)} = 38{,}027$ VAR = 38.0 kVAR

Problem 12.2.10

Given: A linear synchronous motor for a maglev test sled has a pole pitch of $\tau_p = 50$ mm, a force constant $K_f = 500$ N/A, and drives a sled with total mass 200 kg. The sled must accelerate at 20 m/s² to a peak velocity of 100 m/s.

Find: (a) The thrust force for acceleration, (b) the peak current during acceleration, (c) the electrical frequency at peak velocity, (d) the acceleration time and distance.

Solution:

(a) Thrust force for acceleration: $F = ma = 200 \times 20 = 4{,}000$ N Adding aerodynamic drag estimate of 300 N at high speed: $F_{total} \approx 4{,}300$ N

(b) Peak current: $I = F_{total} / K_f = 4{,}300 / 500 = 8.6$ A

(c) Electrical frequency at peak velocity: $f = v / (2\tau_p) = 100 / (2 \times 0.050) = 1{,}000$ Hz

(d) Acceleration time: $t = v / a = 100 / 20 = 5.0$ s

Acceleration distance: $d = ½at^2 = ½ \times 20 \times 25 = 250$ m

Peak mechanical power at the moment of reaching 100 m/s: $P = F_{total} \times v = 4{,}300 \times 100 = 430{,}000$ W = 430 kW

Chapter 12 — Section 12.3: Stepper Motors

Practice problems covering stepper motor types and operation, drive modes (full-step, half-step, microstepping), applications, and resonance and torque curves.

Problem 12.3.1

Given: A hybrid stepper motor has 200 steps per revolution and a rated phase current of 1.5 A. The holding torque is 1.2 N·m and the detent torque is 0.08 N·m.

Find: (a) The step angle, (b) the number of full steps to rotate exactly 72°, (c) the maximum load torque with a 50% safety margin, and (d) the open-loop position accuracy.

Solution:

(a) Step angle: $\theta_{step} = 360° / 200 = 1.8°$ per step

(b) Steps for 72°: $N = 72° / 1.8° = 40$ full steps

(c) Maximum load torque with 50% safety margin: $\tau_{load(max)} = 0.50 \times \tau_{holding} = 0.50 \times 1.2 = 0.60$ N·m

(d) Open-loop position accuracy: Typical non-cumulative error = ±5% of one step = $\pm 0.05 \times 1.8° = \pm 0.09°$

This error is non-cumulative for a properly loaded stepper motor, so it remains ±0.09° regardless of the number of steps taken.

Problem 12.3.2

Given: A stepper motor with 200 full steps/revolution is driven with 32× microstepping. The motor drives a ball screw with a pitch of 5 mm/revolution.

Find: (a) The microsteps per revolution, (b) the linear resolution per microstep, (c) the pulse frequency for a linear speed of 100 mm/s, and (d) the number of pulses for a 50 mm move.

Solution:

(a) Microsteps per revolution: $200 \times 32 = 6{,}400$ microsteps/rev

(b) Linear distance per microstep: d = pitch / microsteps per rev = 5 / 6,400 = 0.000781 mm = 0.781 μm per microstep

(c) Revolutions per second for 100 mm/s: n = 100 / 5 = 20 rev/s

Pulse frequency: $f_{pulse} = 20 \times 6{,}400 = 128{,}000$ Hz = 128 kHz

(d) Pulses for 50 mm: N = 50 / 0.000781 = 64,000 pulses

Verification: 64,000 / 128,000 = 0.500 s, and 100 mm/s × 0.500 s = 50 mm ✓

Problem 12.3.3

Given: A CNC router uses a NEMA 23 stepper motor (1.8°/step, holding torque 1.9 N·m) to drive the Z-axis via a lead screw with 4 mm pitch. The spindle and carriage mass is 5 kg, and the desired maximum vertical acceleration is 2,000 mm/s².

Find: (a) The linear resolution in full-step mode, (b) the gravitational force on the axis, (c) the total force and torque required during upward acceleration, and (d) whether the motor is adequate.

Solution:

(a) Linear resolution per step: d = pitch / steps per rev = 4 / 200 = 0.020 mm = 20 μm per step

(b) Gravitational force: $F_g = mg = 5 \times 9.81 = 49.1$ N

(c) Acceleration force: $F_a = ma = 5 \times 2.0 = 10.0$ N

Total force during upward acceleration: $F_{total} = F_g + F_a + F_{friction} = 49.1 + 10.0 + 5.0 = 64.1$ N

Torque at motor shaft: $\tau = F_{total} \times \text{pitch} / (2\pi) = 64.1 \times 0.004 / (2\pi) = 64.1 \times 6.366 \times 10^{-4} = 0.0408$ N·m = 40.8 mN·m

(d) Maximum speed: typical 50 mm/s = 50/4 = 12.5 rev/s = 2,500 steps/s. At this speed, available torque ≈ 50% of holding torque = 0.50 × 1.9 = 0.95 N·m. Since 0.041 N·m « 0.95 N·m, the motor is more than adequate.

Problem 12.3.4

Given: A NEMA 17 stepper motor has a rotor inertia of $J_{rotor} = 54$ g·cm² (5.4×10^{-6} kg·m²), holding torque of 0.44 N·m, and step angle of 1.8°. It drives a load with reflected inertia $J_{load} = 30$ g·cm² (3.0×10^{-6} kg·m²).

Find: (a) The total inertia, (b) the natural resonant frequency, (c) the resonant step rate, and (d) the resonant speed in RPM.

Solution:

(a) Total inertia: $J_{total} = J_{rotor} + J_{load} = 5.4 \times 10^{-6} + 3.0 \times 10^{-6} = 8.4 \times 10^{-6}$ kg·m²

(b) Holding torque stiffness: $K_h = T_{hold} / \theta_{step} = 0.44 / (1.8 \times \pi/180) = 0.44 / 0.03142 = 14.0$ N·m/rad

Natural frequency: $f_n = (1/2\pi)\sqrt{(K_h/J)} = (1/2\pi)\sqrt{(14.0 / 8.4 \times 10^{-6})} = (1/2\pi)\sqrt{(1.667 \times 10^6)} = (1/2\pi) \times 1291 = 205$ Hz

(c) Resonant step rate: 205 full steps/s

(d) Resonant speed: $n = 205 \times 1.8° / 360° \times 60 = 61.5$ RPM

The acceleration profile should ramp quickly through the 180–230 steps/s region to avoid resonance.

Problem 12.3.5

Given: A stepper motor with 200 steps/rev drives a GT2 timing belt with a 16-tooth pulley (2 mm belt pitch) in a laser cutter X-axis. The carriage mass is 0.8 kg, and the maximum cutting speed is 200 mm/s with 16× microstepping.

Find: (a) The linear resolution per microstep, (b) the pulse frequency at maximum speed, (c) the maximum acceleration if the motor can provide 0.3 N·m of torque at the required speed.

Solution:

(a) Pulley circumference: $C = 16 \times 2 = 32$ mm/rev

Microsteps per revolution: $200 \times 16 = 3{,}200$

Linear resolution per microstep: $d = 32 / 3{,}200 = 0.010$ mm $= 10\ \mu$m

(b) Revolutions per second: $n = 200 / 32 = 6.25$ rev/s

Pulse frequency: $f = 6.25 \times 3{,}200 = 20{,}000$ Hz $= 20$ kHz

(c) Pulley radius: $r = C / (2\pi) = 32 / (2\pi) = 5.093$ mm

Available force at belt: $F = \tau / r = 0.3 / 0.005093 = 58.9$ N

Net acceleration force (subtracting 3 N friction estimate): $F_{net} = 58.9 - 3 = 55.9$ N

Maximum acceleration: $a = F / m = 55.9 / 0.8 = 69.9$ m/s^2

Problem 12.3.6

Given: A stepper-driven rotary indexing table uses a 200-step motor with a 50:1 worm gear reducer. The table must index to 72 equally spaced positions around 360°.

Find: (a) The angular resolution at the table output per full step, (b) the number of motor steps per index position, (c) the angular error per index if the stepper has ±3% step accuracy, and (d) whether the system meets a ±0.05° positioning requirement.

Solution:

(a) Angular resolution at output per full step: $\theta_{output} = 1.8° / 50 = 0.036°$ per motor step

(b) Angle per index position: $\theta_{index} = 360° / 72 = 5.0°$

Motor steps per index: N = 5.0° / 0.036° = 138.9 steps

Since this is not an integer, the actual step count rounds to 139 steps, giving 139 × 0.036° = 5.004°. Over 72 positions this creates a cumulative error. A better approach is to use microstepping or select a gear ratio that divides evenly.

(c) Step accuracy error per step: $\Delta\theta = \pm 0.03 \times 0.036° = \pm 0.00108°$

Since stepper error is non-cumulative: Position error = ±0.00108° plus the rounding error of 0.004° per position.

(d) Total worst-case error = 0.004 + 0.001 = 0.005°, which is well within the ±0.05° requirement. The system meets the specification.

Problem 12.3.7

Given: A stepper motor datasheet shows the following pull-out torque values: 1.8 N·m at 200 steps/s, 1.2 N·m at 500 steps/s, 0.7 N·m at 1,000 steps/s, and 0.3 N·m at 2,000 steps/s. The motor step angle is 1.8°. A constant load torque of 0.5 N·m must be driven.

Find: (a) The maximum speed at which the motor can sustain the load with a 30% safety margin, (b) the corresponding speed in RPM, and (c) the mechanical power output at that speed.

Solution:

(a) Required pull-out torque with 30% margin: $\tau_{required} = 0.5 / 0.70 = 0.714$ N·m

From the torque curve, 0.714 N·m falls between the 1,000 steps/s (0.7 N·m) and 500 steps/s (1.2 N·m) data points. Interpolating linearly:

Fraction = (1.2 − 0.714) / (1.2 − 0.7) = 0.486 / 0.5 = 0.972

Speed = 500 + 0.972 × (1,000 − 500) = 500 + 486 = 986 steps/s

Rounding conservatively: ~980 steps/s

(b) Speed in RPM: n = 980 × 1.8° / 360° × 60 = 294 RPM

(c) Mechanical power at operating point (using actual load torque): $\omega = 294 \times 2\pi/60 = 30.8$ rad/s
$P = \tau \times \omega = 0.5 \times 30.8 = 15.4$ W

Problem 12.3.8

Given: A pick-and-place machine uses a stepper motor to rotate a vacuum nozzle. The motor has 200 steps/rev, holding torque 0.5 N·m, and the nozzle assembly has an inertia of $J = 2.0 \times 10^{-5}$ kg·m². The nozzle must rotate 90° in 50 ms with a trapezoidal velocity profile (equal acceleration, constant velocity, and deceleration phases).

Find: (a) The steps required for 90°, (b) the peak angular velocity, (c) the acceleration torque, and (d) the total torque during acceleration.

Solution:

(a) Steps for 90°: N = 90° / 1.8° = 50 steps

(b) For a trapezoidal profile with equal phases (each 50/3 ms ≈ 16.7 ms): Total angle = 90° = $\pi/2$ rad = 1.571 rad

With equal time phases: $\theta = ½ \times \omega_{peak} \times t_{accel} + \omega_{peak} \times t_{const} + ½ \times \omega_{peak} \times t_{decel}$ $1.571 = \omega_{peak} \times (t/2 + t/3 + t/2) \times (1/3 \text{ each})$...

Simplified: for equal three phases of t/3 each: $\theta = \omega_{peak} \times (t/3)/2 + \omega_{peak} \times t/3 + \omega_{peak} \times (t/3)/2 = \omega_{peak} \times 2t/3$

$\omega_{peak} = \theta \times 3/(2t) = 1.571 \times 3 / (2 \times 0.050) = 4.712 / 0.100 = 47.12$ rad/s

(c) Acceleration time: $t_a = 50/3 = 16.67$ ms Angular acceleration: $\alpha = \omega_{peak} / t_a = 47.12 / 0.01667 = 2{,}827$ rad/s^2

Acceleration torque: $\tau_{accel} = J \times \alpha = 2.0 \times 10^{-5} \times 2{,}827 = 0.0565$ N·m = 56.5 mN·m

(d) Adding friction estimate of 5 mN·m: $\tau_{total} = 56.5 + 5 = 61.5$ mN·m

This is well within the 0.5 N·m holding torque, confirming the motor can handle this motion profile.

Problem 12.3.9

Given: Two stepper motor driver ICs are compared for a precision application: Driver A (A4988) with 16× max microstepping and Driver B (TMC2209) with 256× max microstepping. Both drive a 200-step motor coupled to a 1 mm pitch lead screw.

Find: (a) The linear resolution with each driver at maximum microstepping, (b) the pulse frequency each requires for 10 mm/s travel, and (c) the step rate improvement factor of Driver B over Driver A.

Solution:

(a) Driver A (16× microstepping): Microsteps/rev = 200 × 16 = 3,200 Resolution = 1 mm / 3,200 = 0.000313 mm = 0.313 μm

Driver B (256× microstepping): Microsteps/rev = 200 × 256 = 51,200 Resolution = 1 mm / 51,200 = 0.0000195 mm = 0.0195 μm = 19.5 nm

(b) At 10 mm/s: Revolutions/s = 10 / 1 = 10 rev/s

Driver A: f = 10 × 3,200 = 32 kHz Driver B: f = 10 × 51,200 = 512 kHz

(c) Resolution improvement factor: 0.313 / 0.0195 = 16× finer resolution

Note: While 256× microstepping provides 16× finer position commands, the actual positioning accuracy at this level is limited by mechanical factors (backlash, lead screw accuracy, thermal effects) rather than the driver resolution. Practical improvement in smoothness and noise reduction is significant, but sub-micron accuracy requires closed-loop feedback.

Problem 12.3.10

Given: A telescope mount uses two stepper motors for right ascension (RA) and declination (Dec) tracking. The RA axis must track at the sidereal rate (one revolution in 23 hours 56 minutes 4 seconds = 86,164 seconds). The motor has 200 steps/rev with 64× microstepping and drives a 180:1 worm gear.

Find: (a) The angular resolution at the telescope axis, (b) the step rate for sidereal tracking, (c) the tracking error per step in arc-seconds, and (d) the time between microsteps.

Solution:

(a) Microsteps per revolution at motor: $200 \times 64 = 12{,}800$ Microsteps per revolution at telescope axis: $12{,}800 \times 180 = 2{,}304{,}000$

Angular resolution: $\theta = 360° / 2{,}304{,}000 = 0.000156° = 0.563$ arc-seconds per microstep

(b) Sidereal rate: one revolution of telescope axis in 86,164 s. Step rate = 2,304,000 / 86,164 = 26.74 microsteps/s

(c) Tracking error per step: Each microstep moves the axis 0.563 arc-seconds, which is below the typical atmospheric seeing limit of 1–2 arc-seconds.

(d) Time between microsteps: $\Delta t = 1 / 26.74 = 37.4$ ms

This relatively slow step rate means the motor operates deep in the pull-in torque region where full torque is available, and the motion appears essentially continuous for astrophotography purposes.

Chapter 12 — Section 12.4: Motor Control

Practice problems covering variable frequency drives, servo systems, soft starters, regenerative braking, field-oriented control, and direct torque control.

Problem 12.4.1

Given: A 4-pole, 460 V, 60 Hz induction motor rated at 100 HP drives a centrifugal fan. The motor must be operated at 40 Hz to reduce airflow. The motor uses constant V/f control.

Find: (a) The new synchronous speed, (b) the voltage applied by the VFD, (c) the approximate power savings (fan power varies as the cube of speed), and (d) the annual energy cost savings at \$0.11/kWh running 7,000 hrs/yr.

Solution:

(a) New synchronous speed: $n_{s(new)} = 120 \times 40 \ / \ 4 = 1{,}200$ RPM

(b) Constant V/f ratio: $V/f = 460/60 = 7.667$ V/Hz $V_{new} = 7.667 \times 40 = 306.7$ V

(c) Speed ratio: $n_{new}/n_{base} = 1{,}200/1{,}800 = 0.667$

Power ratio (affinity law): $P_{new}/P_{base} = 0.667^3 = 0.2963$

Power savings: $\Delta P = 100 \times (1 - 0.2963) = 70.4$ HP $= 52.5$ kW

(d) Annual energy savings: $\Delta E = 52.5 \times 7{,}000 = 367{,}500$ kWh

Cost savings: $\Delta C = 367{,}500 \times \$0.11 = \$40{,}425$/year

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Problem 12.4.2

Given: A servo motor with a 5,000-line incremental encoder (quadrature decoded) drives a ball screw with a 10 mm pitch through a 2:1 gear reduction. The required positioning accuracy is ± 10 μm.

Find: (a) The encoder resolution in counts per revolution, (b) the linear resolution per count, and (c) whether the system meets the accuracy requirement.

Solution:

(a) Quadrature decoding: Counts/rev $= 5{,}000 \times 4 = 20{,}000$ counts/rev at the motor shaft

(b) With 2:1 gear reduction, counts per ball screw revolution: 20,000 × 2 = 40,000 counts/rev at the output

Linear resolution per count: d = 10 mm / 40,000 = 0.00025 mm = 0.25 μm per count

(c) Since 0.25 μm « ±10 μm, the encoder resolution is 40× finer than the required accuracy. The system easily meets the requirement. Positioning accuracy will be limited by ball screw backlash (typically 5–20 μm), thermal expansion, and servo tuning rather than encoder resolution.

Problem 12.4.3

Given: A 200 HP, 460 V, 3-phase induction motor has a full-load current of I_{FL} = 240 A and a DOL starting current of 7× I_{FL}. A soft starter is configured to limit starting current to 3.5× I_{FL} with a ramp time of 15 seconds.

Find: (a) The DOL inrush current, (b) the soft-start limited current, (c) the initial voltage applied by the soft starter, and (d) the initial starting torque as a percentage of DOL starting torque.

Solution:

(a) DOL starting current: $I_{start(DOL)} = 7 \times 240 = 1{,}680$ A

(b) Soft-start limited current: $I_{start(soft)} = 3.5 \times 240 = 840$ A

(c) Initial voltage ratio: $V_{initial}/V_{rated} = I_{start(soft)}/I_{start(DOL)} = 840/1{,}680 = 0.50$

Initial voltage: $V_{initial} = 0.50 \times 460 = 230$ V

(d) Starting torque is proportional to voltage squared: $T_{initial}/T_{DOL} = (0.50)^2 = 0.25 = 25\%$ of DOL starting torque

This is adequate for centrifugal fans and pumps but may be insufficient for loaded conveyors or compressors.

Problem 12.4.4

Given: An electric bus with a mass of 12,000 kg decelerates from 60 km/h (16.67 m/s) to a stop using regenerative braking. The motor/generator efficiency during regeneration is 85%, and the battery charging efficiency is 92%. The braking takes 12 seconds.

Find: (a) The kinetic energy available for recovery, (b) the energy stored in the battery, (c) the overall recovery efficiency, and (d) the average regenerative braking power.

Solution:

(a) Kinetic energy: $KE = \frac{1}{2}mv^2 = 0.5 \times 12{,}000 \times 16.67^2 = 0.5 \times 12{,}000 \times 277.9 = 1{,}667{,}400$ J = 1,667.4 kJ

(b) Energy stored in battery: $E_{stored} = KE \times \eta_{motor} \times \eta_{battery} = 1{,}667.4 \times 0.85 \times 0.92 = 1{,}303.9$ kJ

(c) Overall recovery efficiency: $\eta_{total} = E_{stored} / KE = 1{,}303.9 / 1{,}667.4 = 0.782 = 78.2\%$

(d) Average regenerative braking power: $P_{avg} = KE / t = 1{,}667{,}400 / 12 = 138{,}950\ W \approx 139.0\ kW$

Average braking force: $F = P_{avg} / v_{avg} = 138{,}950 / (16.67/2) = 138{,}950 / 8.33 = 16{,}681\ N \approx 16.7\ kN$

Problem 12.4.5

Given: A 3-phase PMSM has: rated torque $T_{rated} = 20$ N·m, torque constant $K_t = 2.0$ N·m/A, rated flux linkage $\lambda_m = 0.30$ Wb, stator resistance $R_s = 0.3\ \Omega$, $L_d = 5$ mH, $L_q = 10$ mH, and 6 poles. The motor operates at 3,000 RPM using FOC with $I_d = 0$ control.

Find: (a) The required q-axis current for rated torque, (b) the stator voltage magnitude, and (c) the minimum DC bus voltage for space vector PWM.

Solution:

(a) Required q-axis current: $I_q = T_{rated} / K_t = 20 / 2.0 = 10.0$ A

With $I_d = 0$: $I_s = 10.0$ A

(b) Electrical speed: $\omega_m = 3{,}000 \times 2\pi/60 = 314.2$ rad/s $\omega_e = (P/2) \times \omega_m = 3 \times 314.2 = 942.5$ rad/s

Voltage equations in the dq frame: $V_d = R_s \times I_d - \omega_e \times L_q \times I_q = 0.3 \times 0 - 942.5 \times 0.010 \times 10.0 = -94.3$ V

$V_q = R_s \times I_q + \omega_e \times L_d \times I_d + \omega_e \times \lambda_m = 0.3 \times 10 + 0 + 942.5 \times 0.30 = 3.0 + 282.8 = 285.8$ V

Stator voltage magnitude: $V_s = \sqrt{(V_d^2 + V_q^2)} = \sqrt{(8{,}892 + 81{,}682)} = \sqrt{90{,}574} = 301.0$ V (phase peak)

(c) Minimum DC bus voltage: $V_{DC} = \sqrt{3} \times V_s = 1.732 \times 301.0 = 521.3$ V

A 540 V or higher DC bus is required.

Problem 12.4.6

Given: A DTC-controlled 4-pole induction motor has: $R_s = 0.5\ \Omega$, rated stator flux $\psi_{s,ref} = 0.90$ Wb, DC bus voltage $V_{DC} = 600$ V, flux hysteresis band ± 0.03 Wb, and torque hysteresis band ± 3 N·m. At a given instant, $\psi_s = 0.88$ Wb and $T_e = 75$ N·m, with torque reference $T_{ref} = 80$ N·m.

Find: (a) The flux error and comparator output, (b) the torque error and comparator output, (c) the required inverter action, and (d) the voltage vector magnitude.

Solution:

(a) Flux error: $\Delta\psi = \psi_{s,ref} - \psi_s = 0.90 - 0.88 = +0.02$ Wb

Since $+0.02$ Wb $< +0.03$ Wb (within band), the flux comparator output is 0 (no correction needed). However, if we apply the hysteresis logic where the output changes at the band edges: the flux is below reference but within the band, so the output remains at its previous state. Assuming it was previously set to +1 (increase flux): flux = +1.

(b) Torque error: $\Delta T = T_{ref} - T_e = 80 - 75 = +5$ N·m

Since +5 N·m > +3 N·m (exceeds band), the torque comparator output is +1 (increase torque).

(c) With flux = +1 and torque = +1, the switching table selects a voltage vector that advances the stator flux in the direction of rotation while increasing its magnitude.

(d) Active voltage vector magnitude: $V = V_{DC} \times \sqrt{(2/3)} = 600 \times 0.8165 = 489.9$ V

Problem 12.4.7

Given: A VFD drives a 4-pole, 460 V, 60 Hz induction motor on a constant-torque conveyor load. The motor must run at 900 RPM. The motor rated torque at 60 Hz is 85 N·m.

Find: (a) The required VFD output frequency, (b) the V/f output voltage, (c) the motor torque capability at this frequency with V/f control, and (d) the motor output power.

Solution:

(a) Synchronous speed at 60 Hz: n_s = 1,800 RPM For 900 RPM synchronous speed: $f = 900 \times P / 120 = 900 \times 4 / 120 = 30$ Hz

(b) V/f ratio: $V/f = 460/60 = 7.667$ V/Hz $V = 7.667 \times 30 = 230$ V

(c) With constant V/f control, the flux is maintained approximately constant, so the motor can deliver approximately rated torque: $\tau \approx 85$ N·m (at rated slip, the full torque is available at reduced speed)

(d) Assuming the actual rotor speed is ~870 RPM (with ~3.3% slip): $\omega = 870 \times 2\pi/60 = 91.1$ rad/s $P = \tau \times \omega = 85 \times 91.1 = 7{,}744$ W = 7.74 kW (10.4 HP)

At half speed, the power is halved for a constant-torque load, confirming the affinity laws do not apply here (those are for variable-torque loads).

Problem 12.4.8

Given: A crane hoist uses a VFD with a braking resistor on the DC bus. The motor is 50 HP, 460 V, 4-pole. When lowering a 5,000 kg load at 0.5 m/s, the load drives the motor as a generator. The drum diameter is 0.4 m and the gearbox ratio is 30:1. The DC bus voltage is 650 V and the braking resistor activates at 720 V.

Find: (a) The gravitational power being regenerated, (b) the motor speed during lowering, (c) the braking resistor power rating required, and (d) the braking resistor value.

Solution:

(a) Gravitational power: $P_{grav} = m \times g \times v = 5{,}000 \times 9.81 \times 0.5 = 24{,}525$ W = 24.5 kW

(b) Drum circumference: $C = \pi \times 0.4 = 1.257$ m

Drum speed: $n_{drum} = v / C \times 60 = 0.5 / 1.257 \times 60 = 23.86$ RPM

Motor speed: $n_{motor} = n_{drum} \times$ gear ratio $= 23.86 \times 30 = 715.8$ RPM

(c) The braking resistor must absorb the regenerated power minus system losses. Assuming 15% losses in motor and gearbox: $P_{brake} = P_{grav} \times 0.85 = 24{,}525 \times 0.85 = 20{,}846$ W ≈ 21 kW

Select a braking resistor rated for at least 25 kW with appropriate duty cycle.

(d) Braking resistor value at 720 V activation: $R = V^2 / P = 720^2 / 25{,}000 = 518{,}400 / 25{,}000 = 20.7$ Ω

A standard 20 Ω braking resistor would be selected.

Problem 12.4.9

Given: A sensorless FOC drive for a ceiling fan BLDC motor estimates rotor position from back-EMF. The motor has 12 poles, $K_e = 0.05$ V/(rad/s), $R_s = 2.5$ Ω, and operates from a 24 V DC bus. The minimum detectable back-EMF for reliable sensorless operation is 5 V.

Find: (a) The minimum speed for sensorless operation, (b) the minimum speed in RPM, (c) the motor torque at this speed with 2 A phase current and $K_t = 0.05$ N·m/A, and (d) a strategy for starting the motor.

Solution:

(a) Minimum speed for detectable back-EMF: $E_{min} = K_e \times \omega_{min}$ $\omega_{min} = E_{min} / K_e = 5 / 0.05 = 100$ rad/s

(b) Minimum speed in RPM: $n = 100 \times 60 / (2\pi) = 955$ RPM

(c) Motor torque: $\tau = K_t \times I = 0.05 \times 2 = 0.1$ N·m

(d) Since the motor cannot detect rotor position below 955 RPM, starting strategies include:

- Open-loop forced commutation: apply a rotating field at increasing frequency until the back-EMF is detectable
- High-frequency injection: inject a high-frequency signal and detect the rotor position from saliency
- Align and go: energize one phase to align the rotor to a known position, then begin the commutation sequence

For a ceiling fan, open-loop forced commutation is the simplest and most cost-effective approach.

Problem 12.4.10

Given: A pump station has three identical VFD-driven pumps, each rated at 30 kW. The system uses a cascading control strategy where pumps are added or removed based on demand. The current demand requires 55 kW of pumping power. The VFDs have an efficiency of 97% and the motors have an efficiency of 93%.

Find: (a) The optimal operating point (number of pumps and individual speed), (b) the total electrical input power, (c) the comparison with running two pumps at different speeds, and (d) the energy savings versus throttle valve control.

Solution:

(a) With 55 kW demand, two pumps each deliver 27.5 kW (91.7% of rated). Each pump runs at: Speed ratio = $(27.5/30)^{1/3} = 0.917^{0.333} = 0.971$ (97.1% speed for cube-law load)

This is close to full speed, so two pumps at ~97% speed is optimal.

(b) Electrical input per pump: $P_{elec} = P_{mech} / (\eta_{VFD} \times \eta_{motor}) = 27{,}500 / (0.97 \times 0.93) = 27{,}500 / 0.902 = 30{,}488$ W

Total electrical input: $P_{total} = 2 \times 30{,}488 = 60{,}976$ W = 61.0 kW

(c) Alternative: one pump at 100% (30 kW) and one at 83.3% (25 kW). The second pump speed ratio $= (25/30)^{1/3} = 0.941$ (94.1%). Electrical input = 30,000/0.902 + 25,000/0.902 = 33,260 + 27,716 = 60,976 W. The result is essentially the same because the cube-law relationship means power scales directly.

(d) With throttle valve control, both pumps run at full speed (30 kW each = 60 kW mechanical) with the excess 5 kW wasted as pressure drop across the valve: $P_{elec,throttle} = 60{,}000 / (0.93) = 64{,}516$ W (no VFD loss, but motor losses remain) VFD approach: 61.0 kW vs throttle: 64.5 kW Savings = 64.5 − 61.0 = 3.5 kW (5.4% savings)

For a more dramatic example at lower demand (e.g., 20 kW), VFD savings would be much larger due to the cube-law benefit.

Chapter 12 — Section 12.5: Motor Specifications

Practice problems covering nameplate data, motor protection, efficiency standards, motor selection and sizing, bearings and vibration analysis, and motor thermal modeling.

Problem 12.5.1

Given: A motor nameplate reads: 50 HP, 460 V, 60 Hz, 3-phase, 1,760 RPM, FLA = 62 A, SF = 1.15, Insulation Class F, TEFC, NEMA Design B, Efficiency = 94.1%.

Find: (a) The number of poles, (b) the full-load slip, (c) the input power, (d) the power factor, and (e) the maximum continuous power with service factor.

Solution:

(a) Nearest synchronous speed above 1,760 is 1,800 RPM: $P = 120f/n_s = 120 \times 60 / 1{,}800 = 4$ poles

(b) Full-load slip: $s = (1{,}800 - 1{,}760) / 1{,}800 = 40/1{,}800 = 0.0222 = 2.22\%$

(c) Input power: $P_{in} = P_{out}/\eta = (50 \times 746) / 0.941 = 37{,}300 / 0.941 = 39{,}639$ W = 39.6 kW

(d) Power factor: $PF = P_{in} / (\sqrt{3} \times V \times I) = 39{,}639 / (1.732 \times 460 \times 62) = 39{,}639 / 49{,}399 = 0.802$ lagging

(e) Maximum continuous power with SF: $P_{max} = 50 \times 1.15 = 57.5$ HP

Problem 12.5.2

Given: A 75 HP, 460 V, 3-phase motor has FLA = 92 A, SF = 1.15, and locked-rotor current of 6.5× FLA. Select protection per NEC guidelines.

Find: (a) The thermal overload relay trip setting, (b) the locked-rotor current, (c) the circuit breaker size per NEC 430.52, and (d) the minimum conductor size.

Solution:

(a) Overload relay trip setting at 115% of FLA: $I_{trip} = 1.15 \times 92 = 105.8$ A

(b) Locked-rotor current: $I_{LR} = 6.5 \times 92 = 598$ A

(c) Circuit breaker (NEC 430.52, inverse-time, Design B): Maximum = 250% of FLA = $2.50 \times 92 =$ 230 A Select next standard size: 225 A breaker

(d) Minimum conductor size (NEC 430.22): $I_{conductor} = 1.25 \times FLA = 1.25 \times 92 = 115$ A From NEC Table 310.16 at 75°C: 1/0 AWG copper (rated 150 A)

Problem 12.5.3

Given: A plant operates a 55 kW, 4-pole, 400 V motor continuously (8,760 hours/year) at 80% load. The current IE2 motor has 90.5% efficiency at 80% load. A replacement IE4 motor has 95.0% efficiency at the same load point. Electricity costs \$0.12/kWh, and the IE4 motor costs \$3,200 more.

Find: (a) The annual energy consumption of each motor, (b) the annual savings, (c) the simple payback period, and (d) the 15-year total savings.

Solution:

(a) Mechanical output at 80% load: $P_{mech} = 0.80 \times 55 = 44.0$ kW

IE2 input: $P_{in(IE2)} = 44.0 / 0.905 = 48.62$ kW IE4 input: $P_{in(IE4)} = 44.0 / 0.950 = 46.32$ kW

Annual energy: $E_{IE2} = 48.62 \times 8{,}760 = 425{,}911$ kWh/year $E_{IE4} = 46.32 \times 8{,}760 = 405{,}763$ kWh/year

(b) Annual savings: $\Delta E = 425{,}911 - 405{,}763 = 20{,}148$ kWh $\Delta C = 20{,}148 \times \$0.12 = \$2{,}418$/year

(c) Simple payback: \$3,200 / \$2,418 = 1.32 years

(d) 15-year total savings: $15 \times \$2{,}418 = \$36{,}270$ (11.3× the premium cost)

Problem 12.5.4

Given: A conveyor requires 15 kW of continuous mechanical power at 900 RPM. The total reflected load inertia is $J_{load} = 1.5$ kg·m², motor inertia is $J_{motor} = 0.08$ kg·m², and the system must accelerate from standstill to full speed in 5 seconds.

Find: (a) The continuous load torque, (b) the acceleration torque, (c) the total torque during acceleration, and (d) whether an 18.5 kW (25 HP) motor is adequate.

Solution:

(a) Continuous load torque: $\omega = 900 \times 2\pi/60 = 94.25$ rad/s $\tau_{load} = P/\omega = 15{,}000 / 94.25 = 159.2$ N·m

(b) Angular acceleration: $\alpha = \omega/t = 94.25 / 5 = 18.85$ rad/s²

Total inertia: $J_{total} = 0.08 + 1.50 = 1.58$ kg·m²

Acceleration torque: $\tau_{accel} = J_{total} \times \alpha = 1.58 \times 18.85 = 29.8$ N·m

(c) Total torque during acceleration: $\tau_{total} = 159.2 + 29.8 = 189.0$ N·m

(d) An 18.5 kW, 6-pole motor at ~940 RPM: Rated torque = 18,500 / (940 × 2π/60) = 18,500 / 98.4 = 188.0 N·m Typical breakdown torque = 2.5 × 188.0 = 470 N·m

The required acceleration torque of 189.0 N·m is approximately 100.5% of rated torque, essentially at the rating. The motor can deliver this, but the margin is very tight. A 22 kW motor would provide a more comfortable margin of ~18%.

Problem 12.5.5

Given: A 45 kW, 4-pole motor operates at 1,770 RPM with 6208-2RS deep groove ball bearings (9 balls, ball diameter d_b = 12.7 mm, pitch diameter d_p = 48.0 mm, contact angle $\alpha = 0°$). The dynamic load rating is C = 29.1 kN and the radial load is P = 2.8 kN.

Find: (a) The L_{10} bearing life in hours, (b) the BPFO (ball pass frequency outer race), and (c) the BPFI (ball pass frequency inner race).

Solution:

(a) L_{10} in revolutions: $L_{10} = (C/P)^3 \times 10^6 = (29.1/2.8)^3 \times 10^6 = (10.39)^3 \times 10^6 = 1{,}121 \times 10^6$ rev

L_{10} in hours: $L_{10}(\text{hours}) = 1{,}121 \times 10^6 / (60 \times 1{,}770) = 1{,}121 \times 10^6 / 106{,}200 = 10{,}555$ hours ≈ 1.2 years continuous

(b) Shaft frequency: $f_{shaft} = 1{,}770/60 = 29.5$ Hz

BPFO = $(N_b/2) \times f_{shaft} \times (1 - d_b \cos\alpha / d_p)$ BPFO = $(9/2) \times 29.5 \times (1 - 12.7/48.0) = 4.5 \times 29.5 \times 0.7354 = 97.6$ Hz

(c) BPFI = $(N_b/2) \times f_{shaft} \times (1 + d_b \cos\alpha / d_p)$ BPFI = $(9/2) \times 29.5 \times (1 + 12.7/48.0) = 4.5 \times 29.5 \times 1.2646 = 167.9$ Hz

A vibration peak at 97.6 Hz indicates an outer race defect; a peak at 167.9 Hz indicates an inner race defect.

Problem 12.5.6

Given: A Class F insulation motor (rated hot-spot 155°C, design life 20,000 hours) operates in a 50°C ambient (10°C above standard 40°C rating) at rated current. The rated temperature rise is 105°C (for 40°C ambient).

Find: (a) The actual hot-spot temperature, (b) the temperature above rated, and (c) the expected insulation life.

Solution:

(a) At rated current, the temperature rise remains 105°C. With elevated ambient: $T_{actual} = T_{ambient} + \Delta T = 50 + 105 = 155$°C

Wait — but the original rating assumed 40°C + 105°C rise + 10°C hot-spot margin = 155°C. With 50°C ambient: $T_{actual} = 50 + 105 = 155$°C

The hot-spot margin is consumed by the elevated ambient. If we account for the 10°C hot-spot allowance in the original design: Actual hot-spot = 155°C (matching the rated limit exactly)

(b) Temperature above rated: $\Delta T_{excess} = 155 - 155 = 0$°C (at rated)

But the original design had the 10°C margin built into the 155°C limit. Operating at 50°C ambient consumes this margin entirely.

(c) Expected life at exactly rated temperature: $L = 20{,}000 \times 2^{(155 - 155)/10} = 20{,}000 \times 2^0 = 20{,}000$ hours

However, this assumes perfect thermal conditions. In practice, the lost hot-spot margin means any transient overload or temperature spike pushes the winding above 155°C, accelerating degradation. The motor should be derated for 50°C ambient to maintain the design life with margin.

Problem 12.5.7

Given: A Class H insulation motor (rated hot-spot 180°C, design life 20,000 hours) continuously carries 120% of rated current. The rated temperature rise at full current is 125°C in a 40°C ambient, with 15°C hot-spot margin.

Find: (a) The actual temperature rise (proportional to I^2), (b) the actual hot-spot temperature, (c) the expected insulation life.

Solution:

(a) Actual temperature rise: $\Delta T = (I/I_{rated})^2 \times \Delta T_{rated} = 1.20^2 \times 125 = 1.44 \times 125 = 180$°C

(b) Actual hot-spot temperature: $T_{actual} = 40 + 180 = 220$°C

(c) Temperature above rated hot-spot: $\Delta T_{excess} = 220 - 180 = 40$°C

Expected life: $L = 20{,}000 \times 2^{(180 - 220)/10} = 20{,}000 \times 2^{-4} = 20{,}000 / 16 = 1{,}250$ hours

A 20% overload reduces insulation life from 20,000 hours to just 1,250 hours — a 93.75% reduction. This dramatically illustrates why sustained overloads are so destructive to motor insulation.

Problem 12.5.8

Given: A motor operates on an intermittent duty cycle (IEC S3): ON for 4 minutes at 120% rated current, OFF for 6 minutes, repeating. The motor's continuous thermal rating is based on rated current.

Find: (a) The duty cycle percentage, (b) the RMS equivalent current, (c) whether the motor is thermally adequate, and (d) the equivalent continuous load as a percentage of rated.

Solution:

(a) Duty cycle: $DC = t_{ON} / (t_{ON} + t_{OFF}) = 4 / (4 + 6) = 0.40 = 40\%$

(b) RMS equivalent current: $I_{eq} = I_{ON} \times \sqrt{(t_{ON} / (t_{ON} + t_{OFF}))} = 1.20 \times \sqrt{(4/10)} = 1.20 \times \sqrt{0.40} = 1.20 \times 0.632 = 0.759 \times I_{rated}$

(c) Since $I_{eq} = 0.759 \times I_{rated} < 1.0 \times I_{rated}$, the motor is thermally adequate.

(d) Equivalent continuous load: 75.9% of rated current, which corresponds to approximately 75.9% of rated thermal loading.

The motor has significant thermal margin despite the 120% peak current because the 60% OFF time allows cooling.

Problem 12.5.9

Given: A 15 kW TEFC motor operates in two environments: Environment A at 30°C ambient, and Environment B at 55°C ambient. The motor has Class F insulation (155°C rated), designed for 40°C ambient with a 105°C rise and 10°C hot-spot margin.

Find: (a) The available temperature rise in each environment, (b) the derating factor for Environment B, and (c) the effective power rating in Environment B.

Solution:

(a) Available temperature rise: Environment A: $\Delta T_{avail} = 155 - 30 - 10 = 115$°C (more than rated 105°C — no derating needed) Environment B: $\Delta T_{avail} = 155 - 55 - 10 = 90$°C

(b) Since temperature rise is proportional to I^2 and power is proportional to I^2: Derating factor = $\sqrt{(\Delta T_{avail} / \Delta T_{rated})} = \sqrt{(90 / 105)} = \sqrt{0.857} = 0.926$

(c) Effective power rating in Environment B: $P_{derated} = 15 \times 0.926 = 13.9$ kW

The motor must be derated by about 7.4% for the high ambient temperature to maintain the design insulation life.

Problem 12.5.10

Given: A motor vibration survey measures the following overall velocity values (mm/s RMS) on a 30 kW motor (ISO 10816 Group 2): bearing housing DE = 3.2 mm/s, bearing housing NDE = 2.1 mm/s. The previous survey 6 months ago measured 1.8 mm/s and 1.5 mm/s respectively.

Find: (a) The ISO 10816 zone classification for each bearing, (b) the rate of vibration increase, (c) the estimated time to reach the Zone C/D boundary, and (d) the recommended action.

Solution:

(a) ISO 10816 Group 2 (15–75 kW) zones:

- Zone A: < 1.8 mm/s (new)
- Zone B: 1.8–4.5 mm/s (acceptable)
- Zone C: 4.5–11.2 mm/s (marginal)
- Zone D: > 11.2 mm/s (danger)

DE bearing: 3.2 mm/s → Zone B (acceptable) NDE bearing: 2.1 mm/s → Zone B (acceptable)

(b) Rate of vibration increase (DE bearing): $\Delta v/\Delta t = (3.2 - 1.8) / 6$ months $= 1.4 / 6 = 0.233$ mm/s per month

(c) Time to reach Zone C/D boundary (11.2 mm/s) from current DE value: $t = (11.2 - 3.2) / 0.233 = 8.0 / 0.233 = 34.3$ months at linear trend

Time to reach Zone B/C boundary (4.5 mm/s): $t = (4.5 - 3.2) / 0.233 = 1.3 / 0.233 = 5.6$ months

(d) Recommended actions:

- Increase monitoring frequency from 6 months to monthly for the DE bearing
- Perform spectral analysis to identify the vibration source (bearing defect, misalignment, imbalance)
- Plan bearing replacement within the next 4–5 months before reaching Zone C
- Check for root cause: misalignment, soft foot, belt tension, or bearing lubrication issues

Chapter 13 — Section 13.1: Ideal Op-Amp Model

Practice problems covering ideal op-amp characteristics, golden rules, open-loop vs. closed-loop gain, gain-bandwidth product, and feedback fraction analysis.

Problem 13.1.1

Given: An op-amp circuit has the non-inverting input connected to a 1.8 V reference. A feedback network connects the output through a 47 kΩ resistor to the inverting input, with a 10 kΩ resistor from the inverting input to ground.

Find: Using the ideal op-amp golden rules, determine the voltage at the inverting input, the current into the inverting input terminal, and the output voltage.

Solution: By the first golden rule (virtual short): $V^- = V^+ = 1.8$ V. By the second golden rule (zero input current): $I^- = 0$ A.

Since $V^- = 1.8$ V and the 10 kΩ resistor connects to ground: Current through $R_1 = V^- / R_1 = 1.8 / 10{,}000 = 180\ \mu A$.

By the zero-input-current rule, this same 180 μA must flow through $R_f = 47$ kΩ from the output: $V_{out} = V^- + I \times R_f = 1.8 + 180 \times 10^{-6} \times 47{,}000 = 1.8 + 8.46 = 10.26$ V.

Gain check: $A_v = 1 + R_f/R_1 = 1 + 47{,}000/10{,}000 = 5.7$. $V_{out} = 5.7 \times 1.8 = 10.26$ V.

Problem 13.1.2

Given: An op-amp has an open-loop gain of $A_{OL} = 500{,}000$ and a gain-bandwidth product of GBW = 8 MHz. It is configured with a feedback fraction $\beta = 0.02$.

Find: The ideal closed-loop gain, the actual closed-loop gain, the gain error, and the closed-loop bandwidth.

Solution: Ideal closed-loop gain: $A_{CL(ideal)} = 1/\beta = 1/0.02 = 50$ (34 dB).

Actual closed-loop gain: $A_{CL} = A_{OL} / (1 + A_{OL} \times \beta) = 500{,}000 / (1 + 500{,}000 \times 0.02) = 500{,}000 / 10{,}001 = 49.995$.

Gain error: Error = (50 - 49.995) / 50 × 100% = 0.01%.

Closed-loop bandwidth: f_{3dB} = GBW / A_{CL} = 8,000,000 / 50 = 160 kHz.

Problem 13.1.3

Given: An op-amp with GBW = 4 MHz is used in two configurations: - Configuration A: A_{CL} = 10 - Configuration B: A_{CL} = 200

Find: The closed-loop bandwidth for each configuration and the frequency at which Configuration B's gain drops to unity (0 dB).

Solution: Configuration A: f_{3dB} = GBW / A_{CL} = 4,000,000 / 10 = 400 kHz.

Configuration B: f_{3dB} = GBW / A_{CL} = 4,000,000 / 200 = 20 kHz.

The unity-gain frequency for the op-amp (where $|A_{OL}| = 1$) is approximately equal to the GBW: f_{unity} = 4 MHz.

This is the same for both configurations because GBW is constant. At 4 MHz, the open-loop gain equals 1, so no closed-loop configuration can provide gain above this frequency.

Problem 13.1.4

Given: A non-inverting amplifier requires a closed-loop gain of exactly 25.00 with an error of less than 0.1%. The feedback fraction is $\beta = 1/25 = 0.04$.

Find: The minimum open-loop gain A_{OL} required to achieve less than 0.1% gain error.

Solution: The gain error is given by: Error = 1 / (1 + $A_{OL} \times \beta$) × 100%.

For Error < 0.1%: 1 / (1 + A_{OL} × 0.04) < 0.001.

Solving: 1 + A_{OL} × 0.04 > 1000. A_{OL} × 0.04 > 999. A_{OL} > 24,975.

Minimum open-loop gain: A_{OL} = 25,000 (approximately 88 dB).

Verification: A_{CL} = 25,000 / (1 + 25,000 × 0.04) = 25,000 / 1,001 = 24.975. Error = (25 - 24.975) / 25 × 100% = 0.10%.

Problem 13.1.5

Given: An op-amp with GBW = 10 MHz and $A_{OL(DC)}$ = 120 dB is configured as a non-inverting amplifier with a gain of 40. The input signal is a 100 kHz sinusoid with an amplitude of 50 mV.

Find: Whether the amplifier can faithfully reproduce the signal at 100 kHz, and the actual gain at 100 kHz.

Solution: Closed-loop bandwidth: f_{3dB} = GBW / A_{CL} = 10,000,000 / 40 = 250 kHz.

At $f = 100$ kHz, the ratio $f/f_{3dB} = 100/250 = 0.4$.

Gain at 100 kHz: $|A(f)| = A_{CL} / \sqrt{(1 + (f/f_{3dB})^2)} = 40 / \sqrt{(1 + 0.16)} = 40 / \sqrt{1.16} = 40 / 1.077 = 37.14$.

Gain reduction = 40 - 37.14 = 2.86 (7.15% reduction). In dB: $20\log_{10}(37.14/40) = -0.64$ dB.

Output amplitude = 37.14×50 mV = 1.857 V.

The signal is within the amplifier's bandwidth (100 kHz < 250 kHz), so the amplifier can reproduce it with only 0.64 dB of attenuation relative to the passband gain.

Chapter 13 — Section 13.2: Inverting Configurations

Practice problems covering inverting amplifiers, summing amplifiers, integrators, differentiators, logarithmic amplifiers, and precision rectifiers.

Problem 13.2.1

Given: An inverting amplifier must provide a gain of -50 with an input impedance of at least 10 kΩ. The available feedback resistor is R_f = 510 kΩ. The input signal is V_{in} = 150 mV DC.

Find: The required R_{in}, the output voltage, and the current through R_f.

Solution: Gain: $A_v = -R_f / R_{in}$. $R_{in} = R_f / |A_v|$ = 510,000 / 50 = 10,200 Ω = 10.2 kΩ.

This satisfies the ≥ 10 kΩ input impedance requirement ($Z_{in} = R_{in}$ = 10.2 kΩ for inverting configuration).

Output voltage: $V_{out} = A_v \times V_{in} = -50 \times 0.150 = -7.5$ V.

Current through R_{in} (inverting input is at virtual ground): $I = V_{in} / R_{in}$ = 0.150 / 10,200 = 14.7 μA.

By the golden rule, the same current flows through R_f: I_f = 14.7 μA.

Verification: $V_{out} = 0 - I_f \times R_f = 0 - 14.7 \times 10^{-6} \times 510{,}000 = -7.5$ V.

Problem 13.2.2

Given: A summing amplifier has R_f = 100 kΩ and four inputs: V_1 = 2.0 V through R_1 = 20 kΩ, V_2 = -1.5 V through R_2 = 50 kΩ, V_3 = 0.8 V through R_3 = 10 kΩ, and V_4 = 3.0 V through R_4 = 100 kΩ.

Find: The output voltage and the individual contribution from each input.

Solution: $V_{out} = -R_f \times (V_1/R_1 + V_2/R_2 + V_3/R_3 + V_4/R_4)$.

Individual contributions: - V_1: $-R_f \times V_1/R_1 = -100{,}000 \times 2.0/20{,}000 = -100{,}000 \times 0.0001 = -10.0$ V - V_2: $-R_f \times V_2/R_2 = -100{,}000 \times (-1.5)/50{,}000 = -100{,}000 \times (-0.00003) = +3.0$ V - V_3: $-R_f \times V_3/R_3 = -100{,}000$

$\times$ 0.8/10,000 = -100,000 $\times$ 0.00008 = -8.0 V - V_4: $-R_f \times V_4/R_4$ = -100,000 $\times$ 3.0/100,000 = -100,000 $\times$ 0.00003 = -3.0 V

V_{out} = -10.0 + 3.0 + (-8.0) + (-3.0) = -18.0 V.

Note: This output would clip at the negative supply rail if the op-amp is powered from ±15 V supplies (V_{out} limited to approximately -13 V). The circuit would need to be redesigned with lower gain ratios or smaller input signals.

Problem 13.2.3

Given: An op-amp integrator has R = 22 kΩ and C = 47 nF. A constant input of V_{in} = +3.5 V is applied starting at t = 0 with the capacitor initially discharged. The op-amp is powered from ±15 V supplies.

Find: The output voltage at t = 0.5 ms, t = 1 ms, and t = 2 ms. Determine when the output reaches the negative supply rail.

Solution: Time constant: τ = RC = 22,000 $\times$ 47 $\times$ 10^{-9} = 1.034 ms.

For a constant input: $V_{out}(t)$ = -(1/RC) $\times$ V_{in} $\times$ t = -(1/0.001034) $\times$ 3.5 $\times$ t = -3,385 $\times$ t.

At t = 0.5 ms: V_{out} = -3,385 $\times$ 0.0005 = -1.69 V. At t = 1 ms: V_{out} = -3,385 $\times$ 0.001 = -3.39 V. At t = 2 ms: V_{out} = -3,385 $\times$ 0.002 = -6.77 V.

Ramp rate: dV_{out}/dt = -3,385 V/s = -3.39 V/ms.

Output reaches -15 V (negative rail) when: -15 = -3,385 $\times$ t. t = 15 / 3,385 = 4.43 ms.

After 4.43 ms, the output saturates at approximately -13.5 V (allowing for the output stage saturation voltage).

Problem 13.2.4

Given: A differentiator circuit has C = 22 nF and R_f = 47 kΩ. The input is a sawtooth wave that ramps linearly from 0 V to 5 V in 2 ms, then resets instantly to 0 V.

Find: The output voltage during the ramp portion and the time constant R_fC.

Solution: Time constant: $R_f \times C$ = 47,000 $\times$ 22 $\times$ 10^{-9} = 1.034 ms.

During the ramp (linear increase): dV_{in}/dt = 5.0 V / 0.002 s = 2,500 V/s.

$V_{out} = -R_f \times C \times dV_{in}/dt$ = -1.034 $\times$ 10^{-3} $\times$ 2,500 = -2.585 V (constant during the ramp).

During the reset (instantaneous drop from 5 V to 0 V): $dV_{in}/dt \to -\infty$ theoretically, producing a large positive spike at the output. In practice, the spike amplitude is limited by the op-amp slew rate and supply rails.

The output is a constant -2.585 V during each ramp, with a brief positive spike at each reset.

Problem 13.2.5

Given: A logarithmic amplifier uses a matched transistor pair for temperature compensation. The circuit is configured with R_{in} = 10 kΩ and a reference current set by V_{ref} = 1.0 V through R_{ref} = 10 kΩ (I_{ref} = 100 μA). The operating temperature is 37°C. The input voltage varies from 1 mV to 10 V.

Find: The output voltage for V_{in} = 1 mV, 10 mV, 100 mV, 1 V, and 10 V, and the output change per decade.

Solution: At 37°C (310 K): $kT/q = (1.381 \times 10^{-23} \times 310) / (1.602 \times 10^{-19}) = 26.72$ mV.

With temperature-compensated configuration: $V_{out} = -(kT/q) \times \ln(V_{in} / V_{ref}) = -0.02672 \times \ln(V_{in} / 1.0)$.

For V_{in} = 1 mV: $V_{out} = -0.02672 \times \ln(0.001) = -0.02672 \times (-6.908) = +0.1845$ V. For V_{in} = 10 mV: $V_{out} = -0.02672 \times \ln(0.01) = -0.02672 \times (-4.605) = +0.1230$ V. For V_{in} = 100 mV: $V_{out} = -0.02672 \times \ln(0.1) = -0.02672 \times (-2.303) = +0.0615$ V. For V_{in} = 1 V: $V_{out} = -0.02672 \times \ln(1.0) = -0.02672 \times 0 = 0.000$ V. For V_{in} = 10 V: $V_{out} = -0.02672 \times \ln(10) = -0.02672 \times 2.303 = -0.0615$ V.

Change per decade: $\Delta V_{out} = -(kT/q) \times \ln(10) = -0.02672 \times 2.303 = -61.5$ mV/decade at 37°C.

This is slightly higher than the 59.5 mV/decade at 25°C due to the temperature dependence of kT/q.

Problem 13.2.6

Given: A precision half-wave rectifier uses an op-amp with A_{OL} = 200,000, SR = 5 V/μs, and ±12 V supplies. The input is a 50 mV_{peak}, 1 kHz sine wave. The gain is set to R_f/R_{in} = 10 (R_{in} = 10 kΩ, R_f = 100 kΩ).

Find: The effective diode threshold, the peak output voltage, the DC (average) output, and the maximum operating frequency.

Solution: Effective diode threshold: $V_{D(eff)} = V_F / A_{OL} = 0.6 / 200{,}000 = 3$ μV (negligible compared to 50 mV input).

Peak output voltage (during positive half-cycle for inverting configuration): $|V_{out(peak)}| = (R_f/R_{in}) \times V_{in(peak)} = 10 \times 50$ mV = 500 mV.

DC (average) value of half-wave rectified sine: $V_{dc} = V_{out(peak)} / \pi = 500 / \pi = 159.2$ mV.

Maximum operating frequency: Recovery time when diode turns off: $t_{rec} = 2V_{sat} / SR = 24 / (5 \times 10^6) = 4.8$ μs. For less than 5% distortion: $t_{rec} < 0.05 \times T/2$. $T/2 > 4.8$ μs / 0.05 = 96 μs. $f_{max} < 1 / (2 \times 96$ μs) = 5.21 kHz.

At 1 kHz (T/2 = 500 μs), $t_{rec}/T/2 = 4.8/500 = 0.96\%$, so distortion is negligible at this frequency.

Problem 13.2.7

Given: A 3-input summing amplifier is used as a simple DAC. The inputs are digital signals (0 V or 5 V) representing a 3-bit binary number (V_1 = MSB, V_3 = LSB). R_f = 10 kΩ, R_1 = 10 kΩ (MSB, weight

4), $R_2 = 20$ kΩ (weight 2), $R_3 = 40$ kΩ (LSB, weight 1).

Find: The output voltage for digital input 101 ($V_1 = 5$ V, $V_2 = 0$ V, $V_3 = 5$ V) and for input 110 ($V_1 = 5$ V, $V_2 = 5$ V, $V_3 = 0$ V).

Solution: For input 101 ($V_1 = 5$ V, $V_2 = 0$ V, $V_3 = 5$ V): $V_{out} = -R_f \times (V_1/R_1 + V_2/R_2 + V_3/R_3)$ $V_{out} = -10{,}000 \times (5/10{,}000 + 0/20{,}000 + 5/40{,}000)$ $V_{out} = -10{,}000 \times (0.0005 + 0 + 0.000125)$ $V_{out} = -10{,}000 \times 0.000625 = -6.25$ V.

Digital value 101 = 5 in decimal, and $5/7 \times 8.75 = 6.25$ V.

For input 110 ($V_1 = 5$ V, $V_2 = 5$ V, $V_3 = 0$ V): $V_{out} = -10{,}000 \times (5/10{,}000 + 5/20{,}000 + 0/40{,}000)$ $V_{out} = -10{,}000 \times (0.0005 + 0.00025 + 0)$ $V_{out} = -10{,}000 \times 0.00075 = -7.50$ V.

Digital value 110 = 6 in decimal, and $6/7 \times 8.75 = 7.50$ V.

The LSB voltage step is: $R_f/R_3 \times 5 = (10{,}000/40{,}000) \times 5 = 1.25$ V.

Problem 13.2.8

Given: An integrator with R = 15 kΩ and C = 220 nF receives a 2 kHz square wave input alternating between +1 V and -1 V. The capacitor is initially discharged.

Find: The peak-to-peak amplitude of the resulting triangular wave output.

Solution: Time constant: $RC = 15{,}000 \times 220 \times 10^{-9} = 3.3$ ms.

The square wave has a half-period of $T/2 = 1/(2 \times 2{,}000) = 0.25$ ms.

During the positive half-cycle ($V_{in} = +1$ V): V_{out} ramps from some voltage at rate $= -(1/RC) \times V_{in} = -(1/0.0033) \times 1 = -303$ V/s.

Voltage change per half-cycle: $\Delta V = |-303| \times 0.00025 = 0.0758$ V = 75.8 mV.

The output is a triangular wave with peak-to-peak amplitude: $V_{pp} = \Delta V = 75.8$ mV.

The triangular wave oscillates symmetrically around 0 V, swinging from -37.9 mV to +37.9 mV (assuming the integrator reaches steady state and the capacitor has zero average charge).

Since T/2 « RC (0.25 ms « 3.3 ms), the ramp is very linear and the triangular wave has excellent linearity.

Chapter 13 — Section 13.3: Non-Inverting Configurations

Practice problems covering non-inverting amplifiers, voltage followers, transimpedance amplifiers, and programmable gain amplifiers.

Problem 13.3.1

Given: A non-inverting amplifier uses R_1 = 2.2 kΩ and R_f = 68 kΩ. The input signal is a 200 mV_{peak}, 5 kHz sine wave from a 50 kΩ source impedance sensor.

Find: The closed-loop gain, the output voltage amplitude, and why this configuration is preferable over an inverting amplifier for this application.

Solution: Closed-loop gain: $A_v = 1 + R_f/R_1 = 1 + 68{,}000/2{,}200 = 1 + 30.91 = 31.91$.

Output amplitude: $V_{out(peak)} = 31.91 \times 200$ mV = 6.38 V.

Input impedance comparison: - Non-inverting: $Z_{in} \approx Z_{in(op\text{-}amp)} \approx 10^9$ Ω (negligible loading on 50 kΩ source). - Equivalent inverting amplifier with gain of -31.9: $Z_{in} = R_{in}$. To achieve gain = -31.9 with R_f = 68 kΩ: R_{in} = 68,000/31.9 = 2,131 Ω = 2.13 kΩ. The 50 kΩ source would form a voltage divider with R_{in}, reducing the effective input to 200 × 2,131/(50,000 + 2,131) = 8.2 mV (a 96% signal loss).

The non-inverting configuration preserves the full 200 mV sensor signal, making it the correct choice for high-impedance sources.

Problem 13.3.2

Given: A precision voltage reference produces 4.096 V through a resistive divider using two 47 kΩ resistors. The divided voltage must drive a 500 Ω load. A voltage follower buffer is inserted between the divider and the load.

Find: The output voltage with and without the buffer, and the percentage error without the buffer.

Solution: Nominal divider output: V_{div} = 4.096 × 47,000 / (47,000 + 47,000) = 4.096 / 2 = 2.048 V.

Without buffer (500 Ω load in parallel with lower 47 kΩ): $R_{parallel}$ = (47,000 × 500) / (47,000 + 500) = 23,500,000 / 47,500 = 494.7 Ω. V_{out} = 4.096 × 494.7 / (47,000 + 494.7) = 4.096 × 494.7 / 47,494.7 = 0.04268 V.

Percentage error = (2.048 - 0.04268) / 2.048 × 100% = 97.9% – the divider is completely loaded down.

With buffer: V_{out} = 2.048 V (buffer output equals input with unity gain). The buffer's output impedance is ≈ 0 Ω, easily driving the 500 Ω load. Error ≈ 0% (limited only by the buffer's input offset voltage, typically < 1 mV).

Problem 13.3.3

Given: A transimpedance amplifier converts the photocurrent from a PIN photodiode. The photodiode has a capacitance C_{in} = 15 pF and produces a photocurrent of 2 μA at the operating light level. R_f = 470 kΩ, and the op-amp GBW = 5 MHz.

Find: The output voltage, the optimal feedback capacitance C_f for stability, and the signal bandwidth.

Solution: Output voltage: $V_{out} = -I_{in} \times R_f = -2 \times 10^{-6} \times 470{,}000 = -0.94$ V.

Optimal feedback capacitance: $C_f = \sqrt{}(C_{in} / (2\pi \times R_f \times GBW))$ $C_f = \sqrt{}(15 \times 10^{-12} / (2\pi \times 470{,}000 \times 5 \times 10^{6}))$ $C_f = \sqrt{}(15 \times 10^{-12} / 1.477 \times 10^{13})$ $C_f = \sqrt{}(1.016 \times 10^{-24}) = 1.008 \times 10^{-12}$ F ≈ 1.0 pF.

Signal bandwidth: $f_{3dB} = 1 / (2\pi R_f C_f) = 1 / (2\pi \times 470{,}000 \times 1.0 \times 10^{-12}) = 1 / (2.953 \times 10^{-6}) = 338.6$ kHz.

Problem 13.3.4

Given: A 14-bit SAR ADC with a ±2.5 V input range (5 V span) has an input-referred noise of 80 μV_{rms}. A PGA with gains of 1, 2, 5, 10, 20, and 50 precedes the ADC. A thermocouple produces a 2 mV signal.

Find: The effective number of bits (ENOB) at gains of 1, 10, and 50, and the optimal gain setting.

Solution: ADC LSB = 5 V / 2^{14} = 5 / 16,384 = 305.2 μV.

At PGA gain = 1: Signal at ADC = 2 mV (6.6 LSBs). Noise referred to input = 80 μV. SNR = $20\log_{10}$(2 mV / 80 μV) = $20\log_{10}$(25) = 28.0 dB. ENOB = (28.0 - 1.76) / 6.02 = 4.4 bits.

At PGA gain = 10: Signal at ADC = 20 mV (65.5 LSBs). Noise referred to input = 80 μV / 10 = 8 μV. SNR = $20\log_{10}$(2 mV / 8 μV) = $20\log_{10}$(250) = 48.0 dB. ENOB = (48.0 - 1.76) / 6.02 = 7.7 bits.

At PGA gain = 50: Signal at ADC = 100 mV (327.6 LSBs). Noise referred to input = 80 μV / 50 = 1.6 μV. SNR = $20\log_{10}$(2 mV / 1.6 μV) = $20\log_{10}$(1,250) = 61.9 dB. ENOB = (61.9 - 1.76) / 6.02 = 10.0 bits.

Gain = 50 is optimal, improving from 4.4 to 10.0 effective bits. Higher gain (if available) would push the signal closer to the ADC's full-scale range and further reduce the impact of ADC noise, but the PGA's own noise would eventually become the limiting factor.

Problem 13.3.5

Given: A transimpedance amplifier for a fiber-optic receiver must achieve a bandwidth of 1 MHz. The photodiode capacitance is $C_{in} = 5$ pF. The op-amp has GBW = 50 MHz.

Find: The maximum feedback resistor R_f that can achieve 1 MHz bandwidth, the optimal C_f, and the output voltage for a 50 μA photocurrent.

Solution: From $f_{3dB} = 1/(2\pi R_f C_f)$ and the optimal C_f formula:

$C_f = \sqrt{}(C_{in} / (2\pi \times R_f \times GBW))$.

Substituting into the bandwidth equation: $f_{3dB} = 1 / (2\pi R_f \times \sqrt{}(C_{in} / (2\pi \times R_f \times GBW)))$ $f_{3dB} = \sqrt{}(GBW / (2\pi R_f C_{in})) / (2\pi \times R_f / R_f)$

Using the relation $f_{3dB} = \sqrt{}(GBW / (2\pi R_f C_{in}))$: $1 \times 10^6 = \sqrt{}(50 \times 10^6 / (2\pi \times R_f \times 5 \times 10^{-12}))$ $(1 \times 10^6)^2 = 50 \times 10^6 / (2\pi \times R_f \times 5 \times 10^{-12})$ $10^{12} = 50 \times 10^6 / (31.42 \times 10^{-12} \times R_f)$ $R_f = 50 \times 10^6 / (31.42 \times 10^{-12} \times 10^{12}) = 50 \times 10^6 / 31.42 = 1.59$ MΩ.

Optimal $C_f = \sqrt{}(5 \times 10^{-12} / (2\pi \times 1.59 \times 10^6 \times 50 \times 10^6)) = \sqrt{}(5 \times 10^{-12} / 4.995 \times 10^{14}) = \sqrt{}(1.0 \times 10^{-26}) = 0.1$ pF.

Output voltage: $V_{out} = -I \times R_f = -50 \times 10^{-6} \times 1.59 \times 10^6 = -79.5$ V – this exceeds supply rails, so R_f must be reduced.

For $V_{out} \leq 5$ V: $R_f = 5 / (50 \times 10^{-6}) = 100$ kΩ maximum. With $R_f = 100$ kΩ, bandwidth $= \sqrt{}(50 \times 10^6 / (2\pi \times 100{,}000 \times 5 \times 10^{-12})) = \sqrt{}(1.592 \times 10^{13}) = 3.99$ MHz, which exceeds the 1 MHz requirement.

Use $R_f = 100$ kΩ, $V_{out} = -50$ μA × 100 kΩ = -5.0 V, BW = 3.99 MHz.

Problem 13.3.6

Given: A voltage follower buffers a 10-bit R-2R DAC output. The DAC has an output impedance of 10 kΩ and produces an output of 0 to 3.3 V. The follower op-amp has $V_{OS} = 1.5$ mV and $I_B = 50$ nA.

Find: The total DC error at the output, expressed in LSBs of the DAC.

Solution: DAC LSB = 3.3 V / 2^{10} = 3.3 / 1,024 = 3.223 mV.

Offset voltage error: $V_{OS} = 1.5$ mV. Bias current error: $V_{IB} = I_B \times R_{source} = 50 \times 10^{-9} \times 10{,}000 = 0.5$ mV.

Total DC error (worst case): $V_{error} = 1.5 + 0.5 = 2.0$ mV.

Error in LSBs: 2.0 / 3.223 = 0.62 LSBs.

This is less than 1 LSB, so the buffer does not degrade the DAC's resolution. For a 12-bit DAC (LSB = 0.806 mV), this error would be 2.48 LSBs, requiring a lower-offset op-amp.

Chapter 13 — Section 13.4: Differential and Instrumentation Amplifiers

Practice problems covering difference amplifiers, instrumentation amplifiers, and current sense amplifiers.

Problem 13.4.1

Given: A difference amplifier with $R_1 = R_3 = 5.1$ kΩ and $R_2 = R_4 = 51$ kΩ measures the voltage across a 50 mΩ current shunt in a 24 V DC power bus. The load current is 12 A. R_3 has a tolerance error of 0.05% (R_3 = 5,102.55 Ω).

Find: The ideal output, the common-mode error, and the effective CMRR.

Solution: Differential signal: $V_{diff} = I \times R_{shunt} = 12 \times 0.050 = 0.600$ V.

Differential gain: $A_{diff} = R_2/R_1 = 51{,}000/5{,}100 = 10$.

Ideal output: $V_{out(ideal)} = 10 \times 0.600 = 6.00$ V.

Common-mode gain with mismatched R_3: The mismatch ratio $\Delta R/R = 0.0005$ (0.05%). $A_{cm} \approx (\Delta R/R) \times (R_2/R_1) / (1 + R_2/R_1) = 0.0005 \times 10 / 11 = 0.000454$.

Common-mode error voltage: $V_{error} = A_{cm} \times V_{cm} = 0.000454 \times 24 = 10.9$ mV.

CMRR: CMRR = $A_{diff} / A_{cm} = 10 / 0.000454 = 22{,}026$. CMRR(dB) = $20\log_{10}(22{,}026) = 86.9$ dB.

Total output: 6.00 + 0.0109 = 6.011 V (0.18% error).

Problem 13.4.2

Given: An INA128 instrumentation amplifier has internal resistors R = 25 kΩ. It is used to amplify a Wheatstone bridge output of 3.8 mV differential with 5.0 V excitation (common-mode = 2.5 V). The required gain is 200. The INA has CMRR = 110 dB.

Find: The gain-setting resistor R_G, the output voltage, and the common-mode error.

Solution: Gain equation: $A_v = 1 + 2R/R_G$. $R_G = 2R / (A_v - 1) = 2 \times 25{,}000 / (200 - 1) = 50{,}000 / 199 =$ 251.3 Ω (use 249 Ω standard value).

Actual gain with 249 Ω: $A_v = 1 + 50{,}000/249 = 1 + 200.8 = 201.8$.

Output voltage: $V_{out} = 201.8 \times 3.8$ mV = 766.8 mV.

CMRR = 110 dB = $10^{110/20} = 316{,}228$.

Common-mode gain: $A_{cm} = A_{diff}$ / CMRR = 201.8 / 316,228 = 6.38×10^{-4}.

Common-mode error at output: $V_{error} = A_{cm} \times V_{cm} = 6.38 \times 10^{-4} \times 2.5 = 1.60$ mV.

Percentage error: 1.60 / 766.8 × 100% = 0.21%.

Problem 13.4.3

Given: A high-side current sense amplifier (gain = 50 V/V) monitors current through a 25 mΩ shunt on a 12 V automotive bus. The amplifier specifications are: $V_{OS} = 50$ μV (input-referred), gain error = 0.5%, CMRR = 120 dB. The load draws 8 A.

Find: The ideal output, the total output error, and the minimum detectable current.

Solution: Shunt voltage: $V_{shunt} = I \times R_{shunt} = 8 \times 0.025 = 200$ mV.

Ideal output: $V_{out(ideal)} = V_{shunt} \times \text{Gain} = 0.200 \times 50 = 10.0$ V.

Offset error at output: $V_{OS} \times \text{Gain} = 50 \times 10^{-6} \times 50 = 2.5$ mV.

Gain error: 0.5% × 10.0 = 50 mV.

CMRR error: CMRR = 120 dB = 10^6. CM input error = V_{cm} / CMRR = 12 / 10^6 = 12 μV. CM error at output = 12 μV × 50 = 0.6 mV.

Total output error (worst case): $V_{error} = 2.5 + 50 + 0.6 = 53.1$ mV (0.53% of 10.0 V).

Minimum detectable current (limited by offset): $I_{min} = V_{OS} / R_{shunt}$ = 50 μV / 25 mΩ = 50×10^{-6} / 0.025 = 2.0 mA.

Problem 13.4.4

Given: A bidirectional current sense amplifier with gain = 100 V/V monitors battery charge/discharge through a 5 mΩ shunt. The reference output voltage is $V_{ref} = 2.5$ V (zero-current midpoint). The supply is $V_{CC} = 5.0$ V. Maximum charge current is 20 A, maximum discharge current is 30 A.

Find: The output voltage range during charge and discharge, and whether the output stays within the supply range.

Solution: During charge (positive current = 20 A): $V_{shunt} = 20 \times 0.005 = 100$ mV. $V_{out} = V_{ref} + \text{Gain} \times V_{shunt} = 2.5 + 100 \times 0.100 = 2.5 + 10.0 = 12.5$ V.

This exceeds $V_{CC} = 5.0$ V! The output would clip at approximately 4.8 V.

Maximum measurable charge current: $V_{out(max)} = 5.0$ V (approximately). $V_{shunt(max)} = (5.0 - 2.5) / 100 = 25$ mV. $I_{charge(max)}$ = 25 mV / 5 mΩ = 5.0 A.

During discharge (negative current = -30 A): V_{shunt} = -30 × 0.005 = -150 mV. V_{out} = 2.5 + 100 × (-0.150) = 2.5 - 15.0 = -12.5 V (clips near 0 V).

Maximum measurable discharge current: $V_{out(min)} \approx 0.1$ V (near ground). $V_{shunt(min)}$ = (0.1 - 2.5) / 100 = -24 mV. $I_{discharge(max)}$ = 24 mV / 5 mΩ = 4.8 A.

Design correction needed: Reduce gain to 20 V/V. Then charge: V_{out} = 2.5 + 20 × 0.1 = 4.5 V; discharge: V_{out} = 2.5 + 20 × (-0.15) = -0.5 V (still clips). Use gain = 10 V/V: charge: 2.5 + 10 × 0.1 = 3.5 V; discharge: 2.5 + 10 × (-0.15) = 1.0 V. Both within 0-5 V range.

Problem 13.4.5

Given: An instrumentation amplifier with gain = 500 amplifies a thermocouple signal of 1.2 mV differential. The INA has V_{OS} = 25 μV, CMRR = 100 dB, and the thermocouple has a common-mode voltage of 0.5 V (from grounding differences). The circuit bandwidth is 100 Hz.

Find: The desired output voltage, all error contributions, and the total percentage error.

Solution: Desired output: V_{out} = 500 × 1.2 mV = 600 mV.

Offset error at output: $V_{OS(out)}$ = 25 μV × 500 = 12.5 mV.

CMRR error: CMRR = 100 dB = 100,000. Input-referred CM error = 0.5 / 100,000 = 5 μV. Output CM error = 5 μV × 500 = 2.5 mV.

Total output error (worst case): V_{error} = 12.5 + 2.5 = 15.0 mV.

Percentage error: 15.0 / 600 × 100% = 2.5%.

To reduce error below 0.5%, we need V_{error} < 3.0 mV. Using a chopper-stabilized INA with V_{OS} = 2 μV and CMRR = 130 dB: offset error = 2 μV × 500 = 1.0 mV; CMRR error = $(0.5/3.16 \times 10^6) \times 500$ = 0.079 mV. Total = 1.08 mV = 0.18%.

Problem 13.4.6

Given: A difference amplifier with $R_1 = R_3$ = 10 kΩ and $R_2 = R_4$ = 10 kΩ (unity gain) measures a 120 mV signal with 60 V common mode on a high-voltage DC bus. All resistors are 0.01% tolerance.

Find: The CMRR limited by resistor matching, the common-mode error, and whether the measurement meets 1% accuracy.

Solution: Differential gain: $A_{diff} = R_2/R_1$ = 10,000/10,000 = 1.

Worst-case CMRR from 0.01% resistor matching: CMRR ≈ $(1 + R_2/R_1) / (4 \times \Delta R/R)$ = (1 + 1) / (4 × 0.0001) = 2 / 0.0004 = 5,000. CMRR(dB) = $20\log_{10}(5{,}000)$ = 74.0 dB.

Common-mode gain: $A_{cm} = A_{diff}$ / CMRR = 1 / 5,000 = 0.0002.

Common-mode error: $V_{error} = A_{cm} \times V_{cm}$ = 0.0002 × 60 = 12 mV.

Output: $V_{out} = 120 + 12 = 132$ mV (or 108 mV depending on polarity). Percentage error: $12/120 \times 100\% = 10\%$ – this does NOT meet the 1% accuracy requirement.

To achieve 1% accuracy: $V_{error} < 1.2$ mV, CMRR $> 60/0.0012 = 50{,}000$ (94 dB). Required resistor matching: $\Delta R/R < 2/(4 \times 50{,}000) = 10$ ppm $= 0.001\%$. Use a precision monolithic difference amplifier IC with laser-trimmed resistors.

Chapter 13 — Section 13.5: Active Filters

Practice problems covering first-order filters, Sallen-Key filters, state-variable filters, multiple feedback filters, and notch filters.

Problem 13.5.1

Given: Design a first-order active highpass filter (inverting) with a passband gain of -8 and a cutoff frequency of 500 Hz. Use $R_f = 56$ kΩ.

Find: The input resistor R_{in}, the input capacitor C, and the gain at 200 Hz and 2 kHz.

Solution: Passband gain: $A_v = -R_f/R_{in}$, so $R_{in} = R_f/|A_v| = 56{,}000/8 = 7{,}000\ \Omega$ (use 6.8 kΩ standard).

With $R_{in} = 6.8$ kΩ: actual gain = -56,000/6,800 = -8.24.

Cutoff frequency: $f_c = 1/(2\pi R_{in}C)$, so: $C = 1/(2\pi \times R_{in} \times f_c) = 1/(2\pi \times 6{,}800 \times 500) = 1/(21{,}363{,}000)$ = 46.8 nF (use 47 nF standard).

Verification: $f_c = 1/(2\pi \times 6{,}800 \times 47 \times 10^{-9}) = 1/(2.008 \times 10^{-3}) = 498\ \text{Hz} \approx 500$ Hz.

Gain at 200 Hz (below cutoff): $|A(f)| = |A_v| \times (f/f_c) / \sqrt{(1 + (f/f_c)^2)} = 8.24 \times (200/498) / \sqrt{(1 + (200/498)^2)} = 8.24 \times 0.4016 / \sqrt{(1 + 0.1613)} = 3.309 / 1.0778 = 3.07$ (−8.58 dB relative to passband).

Gain at 2 kHz (above cutoff): $|A(f)| = 8.24 \times (2{,}000/498) / \sqrt{(1 + (2{,}000/498)^2)} = 8.24 \times 4.016 / \sqrt{(1 + 16.13)} = 33.09 / 4.141 = 7.99$ (essentially flat, -0.27 dB from passband).

Problem 13.5.2

Given: Design a unity-gain Sallen-Key Butterworth lowpass filter with $f_c = 5$ kHz. Use $C_2 = 4.7$ nF with equal resistors.

Find: C_1 and R for a Butterworth (Q = 0.707) response.

Solution: For a Butterworth response with equal resistors: $C_1 = 2 \times C_2 = 2 \times 4.7$ nF = 9.4 nF (use two 4.7 nF in parallel, or use 10 nF).

Using $C_1 = 10$ nF and $C_2 = 4.7$ nF (ratio = 2.128, slightly above 2):

Cutoff frequency: $f_c = 1/(2\pi R\sqrt{}(C_1C_2))$. $\sqrt{}(C_1C_2) = \sqrt{}(10 \times 10^{-9} \times 4.7 \times 10^{-9}) = \sqrt{}(47 \times 10^{-18}) = 6.856 \times 10^{-9}$.

$R = 1/(2\pi f_c\sqrt{}(C_1C_2)) = 1/(2\pi \times 5{,}000 \times 6.856 \times 10^{-9}) = 1/(215.4 \times 10^{-6}) = 4{,}643\ \Omega$ (use 4.7 kΩ).

Verification: $f_c = 1/(2\pi \times 4{,}700 \times 6.856 \times 10^{-9}) = 1/(202.3 \times 10^{-6}) = 4{,}943$ Hz ≈ 5 kHz.

$Q = \sqrt{}(C_1/C_2)\ /\ 2 = \sqrt{}(10/4.7)\ /\ 2 = \sqrt{}2.128\ /\ 2 = 1.459\ /\ 2 = 0.730$ (close to the 0.707 Butterworth target).

Problem 13.5.3

Given: A state-variable bandpass filter is required for an audio spectrum analyzer with center frequency $f_0 = 1$ kHz and $Q = 20$. The integrator capacitors are $C = 22$ nF.

Find: The integrator resistance R, the bandwidth, and the -3 dB frequencies.

Solution: Integrator resistance: $R = 1/(2\pi f_0C) = 1/(2\pi \times 1{,}000 \times 22 \times 10^{-9}) = 1/(138.2 \times 10^{-6}) = 7{,}234$ Ω (use 7.15 kΩ or 6.8 kΩ + 470 Ω in series).

Bandwidth: $BW = f_0/Q = 1{,}000/20 = 50$ Hz.

Lower -3 dB frequency: $f_L = f_0 - BW/2 = 1{,}000 - 25 = 975$ Hz.

Upper -3 dB frequency: $f_H = f_0 + BW/2 = 1{,}000 + 25 = 1{,}025$ Hz.

This extremely narrow 50 Hz bandwidth around 1 kHz allows the analyzer to isolate individual frequency components in a complex audio spectrum.

Problem 13.5.4

Given: A multiple feedback bandpass filter requires $f_0 = 10$ kHz, $Q = 8$, and midband gain $|A_0| = 5$. Use equal capacitors $C = 1$ nF.

Find: The resistor values R_1, R_2, R_3, and the bandwidth.

Solution: Bandwidth: $BW = f_0/Q = 10{,}000/8 = 1{,}250$ Hz.

From Q equation: $R_2 = Q/(\pi \times f_0 \times C) = 8/(\pi \times 10{,}000 \times 1 \times 10^{-9}) = 8/(31.42 \times 10^{-6}) = 254{,}648\ \Omega$ (use 270 kΩ).

From gain equation: $A_0 = -R_2/(2R_1)$, so: $R_1 = R_2/(2 \times |A_0|) = 270{,}000/(2 \times 5) = 27{,}000\ \Omega$ (use 27 kΩ).

From center frequency: $f_0 = (1/(2\pi C)) \times \sqrt{}(1/(R_1R_3))$, so: $R_3 = 1/((2\pi f_0C)^2 \times R_1) = 1/((2\pi \times 10{,}000 \times 10^{-9})^2 \times 27{,}000) = 1/((62.83 \times 10^{-6})^2 \times 27{,}000) = 1/(3.948 \times 10^{-9} \times 27{,}000) = 1/(1.066 \times 10^{-4}) = 9{,}381\ \Omega$ (use 9.1 kΩ).

Verification with standard values: $f_0 = (1/(2\pi C)) \times \sqrt{}(1/(R_1R_3)) = (1/(2\pi \times 10^{-9})) \times \sqrt{}(1/(27{,}000 \times 9{,}100)) = 159{,}155 \times \sqrt{}(4.07 \times 10^{-9}) = 159{,}155 \times 6.38 \times 10^{-5} = 10{,}154$ Hz ≈ 10 kHz.

Problem 13.5.5

Given: Design an active Twin-T notch filter to reject 50 Hz power-line hum (European mains). Use C = 68 nF. The positive feedback fraction is k = 0.98.

Find: R, R/2, 2C, the notch Q, and the -3 dB bandwidth.

Solution: Notch frequency: $f_0 = 1/(2\pi RC)$, so: $R = 1/(2\pi f_0 C) = 1/(2\pi \times 50 \times 68 \times 10^{-9}) = 1/(21.36 \times 10^{-6}) = 46{,}818\ \Omega$ (use 47 kΩ).

Shunt resistor: R/2 = 47,000/2 = 23,500 Ω (use 22 kΩ + 1.5 kΩ in series = 23.5 kΩ). Shunt capacitor: 2C = 2 × 68 nF = 136 nF (use 120 nF + 15 nF in parallel = 135 nF).

Quality factor: Q = 1/(4(1 - k)) = 1/(4 × 0.02) = 12.5.

Bandwidth: $BW = f_0/Q = 50/12.5 = 4.0$ Hz.

The -3 dB frequencies are: 50 ± 2.0 Hz = 48.0 Hz and 52.0 Hz.

This very narrow notch (4 Hz bandwidth) effectively removes the 50 Hz fundamental while preserving frequencies even a few hertz away. Adjacent harmonics at 100 Hz and 150 Hz are virtually unaffected.

Problem 13.5.6

Given: A fourth-order Butterworth lowpass filter is needed at $f_c = 2$ kHz. It is constructed by cascading two second-order Sallen-Key stages. The first stage requires $Q_1 = 0.541$ and the second stage $Q_2 = 1.307$ (Butterworth polynomial coefficients). Both stages use R = 10 kΩ.

Find: The capacitor values C_1 and C_2 for each stage to achieve the required Q values.

Solution: For a Sallen-Key lowpass with equal R, $Q = \sqrt{(C_1/C_2)}/2$, so $C_1/C_2 = (2Q)^2$.

Also, $f_c = 1/(2\pi R\sqrt{(C_1C_2)})$, so $\sqrt{(C_1C_2)} = 1/(2\pi Rf_c) = 1/(2\pi \times 10{,}000 \times 2{,}000) = 7.958 \times 10^{-9}$. $C_1 \times C_2 = (7.958 \times 10^{-9})^2 = 63.33 \times 10^{-18}$.

Stage 1 ($Q_1 = 0.541$): $C_1/C_2 = (2 \times 0.541)^2 = 1.082^2 = 1.171$. $C_1 = 1.171 \times C_2$. Substituting: $1.171 \times C_2^2 = 63.33 \times 10^{-18}$. $C_2^2 = 54.08 \times 10^{-18}$. $C_2 = 7.354 \times 10^{-9} = 7.35$ nF (use 6.8 nF). $C_1 = 1.171 \times 7.35$ nF = 8.61 nF (use 8.2 nF).

Stage 2 ($Q_2 = 1.307$): $C_1/C_2 = (2 \times 1.307)^2 = 2.614^2 = 6.833$. $C_2^2 = 63.33 \times 10^{-18} / 6.833 = 9.27 \times 10^{-18}$. $C_2 = 3.044 \times 10^{-9} = 3.04$ nF (use 2.7 nF). $C_1 = 6.833 \times 3.04$ nF = 20.8 nF (use 22 nF).

The cascaded filter provides a fourth-order rolloff of -80 dB/decade above 2 kHz, with a maximally flat passband response.

Problem 13.5.7

Given: A state-variable filter must provide simultaneous lowpass, highpass, and bandpass outputs at $f_0 = 3$ kHz with Q = 5. The filter also needs a notch output at 3 kHz. The integrator components are C = 4.7 nF.

Find: The integrator resistance R, the bandwidth of the bandpass output, and describe how the notch output is obtained.

Solution: Integrator resistance: $R = 1/(2\pi f_0 C) = 1/(2\pi \times 3{,}000 \times 4.7 \times 10^{-9}) = 1/(88.59 \times 10^{-6}) = 11{,}288\ \Omega$ (use 11 kΩ).

Verification: $f_0 = 1/(2\pi \times 11{,}000 \times 4.7 \times 10^{-9}) = 1/(324.7 \times 10^{-6}) = 3{,}080$ Hz ≈ 3 kHz.

Bandwidth of bandpass output: $BW = f_0/Q = 3{,}000/5 = 600$ Hz.

The -3 dB frequencies are: 3,000 - 300 = 2,700 Hz and 3,000 + 300 = 3,300 Hz.

Notch output: Add a fourth op-amp configured as a summing amplifier to combine the lowpass and highpass outputs with equal gain. Below f_0, the lowpass output passes the signal; above f_0, the highpass output passes the signal; at f_0, both outputs are at their -3 dB points and sum to produce a deep null. The notch depth is limited by the matching of the two gains and the phase accuracy of the outputs. With precise gain matching (0.1%), notch depths exceeding 40 dB are achievable.

Chapter 13 — Section 13.6: Comparators

Practice problems covering basic comparators, Schmitt triggers, Wien bridge oscillators, and relaxation oscillators.

Problem 13.6.1

Given: A comparator with V_{OH} = 3.3 V and V_{OL} = 0 V has its inverting input connected to a 1.65 V reference. The non-inverting input receives a 10 kHz sine wave: $V_{in} = 1.65 + 2.0 \sin(2\pi \times 10{,}000t)$.

Find: The output waveform, the duty cycle, and the times during each period when the output transitions.

Solution: The output is high when $V_{in} > V_{ref}$ = 1.65 V: $1.65 + 2.0 \sin(\omega t) > 1.65$, which simplifies to $\sin(\omega t) > 0$.

This occurs for $0 < \omega t < \pi$ (the positive half-cycle).

The output transitions from low to high at t = 0 ($\omega t = 0$) and from high to low at t = T/2 = 50 µs ($\omega t = \pi$).

Duty cycle = 50% (symmetric about the reference).

The output is a 3.3 V square wave at 10 kHz with 50% duty cycle.

If the reference is shifted to 2.65 V: $\sin(\omega t) > (2.65 - 1.65)/2.0 = 0.5$. ωt transitions at $\pi/6$ and $5\pi/6$. Duty cycle = $(5\pi/6 - \pi/6)/(2\pi) = (4\pi/6)/(2\pi)$ = 33.3%.

Problem 13.6.2

Given: A Schmitt trigger must clean up a noisy signal from a hall-effect sensor. The required thresholds are V_{TH} = 3.5 V and V_{TL} = 1.5 V. The comparator has V_{OH} = 5.0 V and V_{OL} = 0 V. Use a non-inverting configuration with a reference at the inverting input.

Find: The reference voltage V_{ref}, the feedback ratio $R_2/(R_1 + R_2)$, and R_1 if R_2 = 10 kΩ. What is the maximum noise amplitude the circuit can tolerate without false triggering?

Solution: Reference voltage: $V_{ref} = (V_{TH} \times V_{OL} - V_{TL} \times V_{OH}) / (V_{OL} - V_{OH} + V_{TH} - V_{TL}) = (3.5 \times 0 - 1.5 \times 5.0) / (0 - 5.0 + 3.5 - 1.5) = -7.5 / (-3.0)$ = 2.5 V.

Feedback ratio: $R_2/(R_1 + R_2) = (V_{TH} - V_{ref}) / (V_{OH} - V_{ref}) = (3.5 - 2.5) / (5.0 - 2.5) = 1.0/2.5 = 0.4$.

With $R_2 = 10$ kΩ: $10{,}000/(R_1 + 10{,}000) = 0.4$. $R_1 + 10{,}000 = 25{,}000$. $R_1 = 15$ kΩ.

Hysteresis: $V_H = V_{TH} - V_{TL} = 3.5 - 1.5 = 2.0$ V.

Maximum noise tolerance: the circuit can tolerate noise with peak-to-peak amplitude up to 2.0 V without false triggering, since the input must cross the full hysteresis band to cause a state change.

Problem 13.6.3

Given: Design a Wien bridge oscillator for $f_0 = 1$ kHz using C = 10 nF. The non-inverting amplifier uses $R_1 = 10$ kΩ. A 5% gain excess above the critical value is desired for reliable startup.

Find: The Wien network resistance R, the feedback resistor R_f, and the oscillation frequency with standard component values.

Solution: Wien bridge frequency: $R = 1/(2\pi f_0 C) = 1/(2\pi \times 1{,}000 \times 10 \times 10^{-9}) = 1/(62.83 \times 10^{-6}) = 15{,}915\ \Omega$ (use 15 kΩ + 910 Ω = 15.91 kΩ, or simply use 16 kΩ).

Critical gain for oscillation: $A_v = 3$ exactly. $R_{f(critical)} = 2 \times R_1 = 2 \times 10{,}000 = 20$ kΩ.

With 5% gain excess: $A_v = 3 \times 1.05 = 3.15$. $R_f = (3.15 - 1) \times R_1 = 2.15 \times 10{,}000 = 21.5$ kΩ (use 22 kΩ).

Actual gain: $A_v = 1 + 22{,}000/10{,}000 = 3.2$ (6.7% excess).

Oscillation frequency with R = 16 kΩ: $f_0 = 1/(2\pi \times 16{,}000 \times 10 \times 10^{-9}) = 1/(1.005 \times 10^{-3}) = 995$ Hz ≈ 1 kHz.

Without amplitude stabilization, the 6.7% gain excess will cause the output to grow until it clips at the supply rails, producing noticeable distortion. An AGC circuit or nonlinear feedback element should be added.

Problem 13.6.4

Given: An op-amp relaxation oscillator uses R = 33 kΩ, C = 4.7 nF, and feedback divider $R_1 = R_2 = 22$ kΩ ($\beta = 0.5$). The op-amp saturates at ±13 V.

Find: The oscillation frequency, the threshold voltages, and the capacitor voltage waveform limits.

Solution: Feedback fraction: $\beta = R_2/(R_1 + R_2) = 22{,}000/44{,}000 = 0.5$.

Threshold voltages: $V_{TH} = +\beta \times V_{sat} = 0.5 \times 13 = +6.5$ V. $V_{TL} = -\beta \times V_{sat} = 0.5 \times (-13) = -6.5$ V.

The capacitor swings between -6.5 V and +6.5 V.

Frequency: $f = 1/(2RC \times \ln((1 + \beta)/(1 - \beta))) = 1/(2 \times 33{,}000 \times 4.7 \times 10^{-9} \times \ln(1.5/0.5)) = 1/(310.2 \times 10^{-6} \times \ln(3)) = 1/(310.2 \times 10^{-6} \times 1.0986) = 1/(340.8 \times 10^{-6}) = 2{,}934$ Hz ≈ 2.93 kHz.

Period: $T = 1/2{,}934 = 340.8\ \mu$s.

The output is a square wave at 2.93 kHz alternating between +13 V and -13 V with 50% duty cycle. The capacitor voltage is an exponential waveform (approximating a triangle) oscillating between -6.5 V and +6.5 V.

Problem 13.6.5

Given: A phase-shift oscillator uses three identical RC highpass sections with R = 10 kΩ and C = 1 nF, driving an inverting amplifier.

Find: The oscillation frequency and the minimum required gain $|A_v|$.

Solution: For three identical RC highpass sections, the oscillation frequency is: $f_0 = 1/(2\pi RC\sqrt{6}) = 1/(2\pi \times 10{,}000 \times 1 \times 10^{-9} \times \sqrt{6}) = 1/(2\pi \times 10{,}000 \times 10^{-9} \times 2.449) = 1/(153.9 \times 10^{-6}) = 6{,}498$ Hz ≈ 6.5 kHz.

Minimum required gain for oscillation: $|A_v| = 29$ ($R_f/R_{in} = 29$).

If $R_{in} = 10$ kΩ: $R_f = 29 \times 10{,}000 = 290$ kΩ (use 300 kΩ).

With $R_f = 300$ kΩ: $|A_v| = 300{,}000/10{,}000 = 30$ (3.4% excess gain for reliable startup).

The high gain requirement (29) means the phase-shift oscillator has more noise gain than the Wien bridge oscillator (gain = 3), resulting in higher output noise. However, it requires no amplitude stabilization for moderate distortion levels.

Problem 13.6.6

Given: A triangle wave generator uses a Schmitt trigger ($R_1 = 10$ kΩ, $R_2 = 20$ kΩ, $V_{sat} = \pm 12$ V) driving an integrator (R = 47 kΩ, C = 10 nF). The Schmitt trigger threshold is $\beta = R_2/(R_1 + R_2)$.

Find: The triangle wave frequency, peak-to-peak amplitude, and linearity advantage over the basic astable multivibrator.

Solution: Schmitt trigger feedback fraction: $\beta = 20{,}000/(10{,}000 + 20{,}000) = 2/3$.

Threshold voltages: $V_{TH} = \beta \times V_{sat} = (2/3) \times 12 = +8.0$ V. $V_{TL} = -(2/3) \times 12 = -8.0$ V.

Triangle wave peak-to-peak amplitude = $V_{TH} - V_{TL} = 16.0$ V.

The integrator ramp rate when the Schmitt trigger output is at $+V_{sat} = +12$ V: $dV/dt = -V_{sat}/(RC) = -12/(47{,}000 \times 10 \times 10^{-9}) = -12/470 \times 10^{-6} = -25{,}532$ V/s.

Time for one half-cycle (ramp from +8 V to -8 V): $t_{half} = \Delta V / |dV/dt| = 16 / 25{,}532 = 626.3$ μs.

Frequency: $f = 1/(2 \times t_{half}) = 1/(2 \times 626.3 \times 10^{-6}) = 798.7$ Hz ≈ 800 Hz.

Alternative formula: $f = V_{sat} / (4RC \times V_{TH}) = 12 / (4 \times 470 \times 10^{-6} \times 8) = 12 / (15.04 \times 10^{-3}) = 798$ Hz.

Linearity advantage: The integrator produces a truly linear ramp (constant dV/dt) because the op-amp maintains virtual ground at its inverting input, forcing a constant current through R regardless of the capacitor voltage. The basic RC astable multivibrator charges the capacitor exponentially through a resistor, producing curved ramps that become less linear at higher frequencies or larger voltage swings.

Chapter 13 — Section 13.7: Real Op-Amp Limitations

Practice problems covering input offset and bias errors, slew rate, CMRR, PSRR, noise analysis, and input/output voltage range.

Problem 13.7.1

Given: An inverting amplifier with R_{in} = 1 kΩ and R_f = 470 kΩ (gain = -470) uses a bipolar op-amp with V_{OS} = 3 mV, I_B = 500 nA, and I_{OS} = 50 nA. No bias compensation resistor is used.

Find: The total output offset voltage, and the improvement when a bias compensation resistor R_{comp} = R_{in} || R_f is added to the non-inverting input.

Solution: Noise gain: 1 + R_f/R_{in} = 1 + 470,000/1,000 = 471.

Without compensation: Offset due to V_{OS}: $V_{out(OS)}$ = V_{OS} × noise gain = 0.003 × 471 = 1,413 mV = 1.413 V. Offset due to I_B: $V_{out(IB)}$ = I_B × R_f = 500 × 10^{-9} × 470,000 = 235 mV. Total: 1,413 + 235 = 1,648 mV = 1.65 V.

With compensation (R_{comp} = R_{in} || R_f): R_{comp} = (1,000 × 470,000) / (1,000 + 470,000) = 470,000/471 = 998 Ω ≈ 1 kΩ.

The I_B component is cancelled; only I_{OS} remains: $V_{out(IOS)}$ = I_{OS} × R_f = 50 × 10^{-9} × 470,000 = 23.5 mV. Total: 1,413 + 23.5 = 1,436 mV = 1.44 V.

Improvement: (1,648 - 1,436) / 1,648 × 100% = 12.9%.

The improvement is modest because V_{OS} is the dominant error source at this high gain. Using a lower-offset op-amp (V_{OS} < 100 μV) would be more effective.

Problem 13.7.2

Given: An op-amp with SR = 25 V/μs produces a 5 V peak, 500 kHz sine wave output. A separate channel amplifies a step input from 0 to 8 V.

Find: (a) Whether the sine wave output is slew-rate limited. (b) The full-power bandwidth. (c) The 10-90% rise time for the step response.

Solution: (a) Sine wave check: Maximum rate of change: $dV/dt = 2\pi fV_p = 2\pi \times 500{,}000 \times 5 = 15.71 \times 10^6$ V/s = 15.71 V/μs. Since 15.71 V/μs < SR = 25 V/μs, the output is not slew-rate limited at this frequency.

(b) Full-power bandwidth: $f_{FP} = SR / (2\pi V_p) = 25 \times 10^6 / (2\pi \times 5) = 25 \times 10^6 / 31.42 = 795.8$ kHz.

The op-amp can produce a 5 V peak sine wave without distortion up to 796 kHz.

(c) Step response rise time: Total voltage change: $\Delta V = 8$ V. Slew-limited rise time (0 to 100%): $t = \Delta V/SR = 8/(25 \times 10^6) = 0.32$ μs. 10-90% rise time: $t_r = 0.8 \times \Delta V / SR = 6.4 / (25 \times 10^6) = 0.256$ μs = 256 ns.

Problem 13.7.3

Given: An op-amp with CMRR = 80 dB at DC is used in a non-inverting amplifier with a gain of 100. The input signal is a 500 μV differential signal from a strain gauge bridge with 10 V excitation (V_{cm} = 5 V).

Find: The desired output, the common-mode error at the output, and the signal-to-error ratio.

Solution: CMRR = 80 dB = $10^{80/20}$ = 10,000.

Desired output: $V_{out(signal)} = 100 \times 500\ \mu V = 50$ mV.

Equivalent input error from common mode: $V_{error(in)} = V_{cm}$ / CMRR = 5.0 / 10,000 = 500 μV.

Error at output: $V_{error(out)} = 100 \times 500\ \mu V = 50$ mV.

Signal-to-error ratio: 50 mV / 50 mV = 1.0, or 0 dB.

The common-mode error equals the signal – the measurement is completely unreliable! This demonstrates why an instrumentation amplifier with CMRR > 100 dB is essential for high-gain measurements with significant common-mode voltage.

For CMRR = 120 dB (10^6): $V_{error(out)} = 100 \times 5/10^6 = 0.5$ mV. SER = 50/0.5 = 100 = 40 dB.

Problem 13.7.4

Given: An op-amp with PSRR = 90 dB at DC and PSRR = 45 dB at 500 kHz is powered from a 12 V supply. The supply has 50 mV_{pp} ripple at 500 kHz from a buck converter. The amplifier gain is 10.

Find: The equivalent input noise from the supply ripple and the output ripple, with and without a 100 nF bypass capacitor (which reduces the 500 kHz ripple to 2 mV_{pp}).

Solution: PSRR at 500 kHz = 45 dB = $10^{45/20}$ = 177.8.

Without bypass capacitor: $\Delta V_{OS} = \Delta V_{supply}$ / PSRR = 50 mV / 177.8 = 281 μV_{pp}. Output ripple: $V_{ripple(out)} = 281\ \mu V \times 10 = 2.81\ mV_{pp}$.

With 100 nF bypass capacitor (2 mV_{pp} ripple): ΔV_{OS} = 2 mV / 177.8 = 11.2 μV_{pp}. Output ripple: $V_{ripple(out)} = 11.2\ \mu V \times 10 = 112\ \mu V_{pp}$.

Improvement: 2,810 / 112 = 25× (28 dB).

For a 10-bit, 3.3 V ADC (LSB = 3.22 mV), the unbypassed ripple (2.81 mV) is 0.87 LSB and would cause ±1 LSB jitter. The bypassed ripple (112 μV) is only 0.035 LSB and is negligible.

Problem 13.7.5

Given: A non-inverting amplifier with gain = 50 uses an op-amp with $e_n = 10\ nV/\sqrt{Hz}$ and $i_n = 0.5\ pA/\sqrt{Hz}$. The source resistance is $R_S = 100\ k\Omega$. The bandwidth is 1 kHz. Assume the 1/f corner is below 10 Hz (negligible).

Find: The total input-referred noise, whether voltage noise or current noise dominates, and the output noise.

Solution: Source resistance thermal noise: $e_R = \sqrt{(4kTR_S)} = \sqrt{(1.66 \times 10^{-20} \times 100{,}000)} = \sqrt{(1.66 \times 10^{-15})} = 40.7\ nV/\sqrt{Hz}$.

Current noise contribution: $e_{in} = i_n \times R_S = 0.5 \times 10^{-12} \times 100{,}000 = 50\ nV/\sqrt{Hz}$.

Total input noise density: $e_{n,total} = \sqrt{(10^2 + 50^2 + 40.7^2)} = \sqrt{(100 + 2{,}500 + 1{,}656.5)} = \sqrt{4{,}256.5} = 65.2\ nV/\sqrt{Hz}$.

Current noise dominates ($50\ nV/\sqrt{Hz}$) over voltage noise ($10\ nV/\sqrt{Hz}$) at this high source impedance. A JFET-input op-amp with lower i_n (e.g., $1\ fA/\sqrt{Hz}$) would be a much better choice.

Equivalent noise bandwidth: $BW_n = 1.57 \times 1{,}000 = 1{,}570$ Hz. Total RMS input noise: $V_{n,rms} = 65.2 \times 10^{-9} \times \sqrt{1{,}570} = 65.2 \times 10^{-9} \times 39.62 = 2.58\ \mu V_{rms}$.

Output noise: $V_{n,out} = 2.58\ \mu V \times 50 = 129\ \mu V_{rms}$.

Problem 13.7.6

Given: A single-supply ($V_{CC} = 5.0$ V, $V_{EE} = 0$ V) non-inverting amplifier with gain = 5 uses a rail-to-rail output op-amp with $V_{OL} = 30$ mV and $V_{OH} = V_{CC}$ - 50 mV at $I_{load} = 2$ mA, and $R_{DS(on)} = 75\ \Omega$. The input is biased at $V_{CC}/2 = 2.5$ V, and the signal is ±0.4 V around the bias point.

Find: The output voltage range, whether clipping occurs at $I_{load} = 2$ mA, and the output range at $I_{load} = 15$ mA.

Solution: Input range: 2.5 - 0.4 = 2.1 V to 2.5 + 0.4 = 2.9 V. Output range (ideal): 5 × 2.1 = 10.5 V to 5 × 2.9 = 14.5 V.

Wait – the output exceeds $V_{CC} = 5.0$ V. The gain of 5 centered at 2.5 V produces: $V_{out(center)} = 5 \times 2.5 = 12.5$ V – far above the 5 V supply.

This circuit configuration is incorrect. For a gain of 5 with single supply, the bias must be at $V_{CC}/(2 \times \text{gain}) = 5/(2 \times 5) = 0.5$ V, or the input range must be limited.

Corrected analysis with gain = 2: Output range: 2 × 2.1 = 4.2 V to 2 × 2.9 = 5.8 V.

At I_{load} = 2 mA: V_{OH} = 5.0 - 0.05 = 4.95 V. V_{OL} = 0.03 V. Maximum output 5.8 V > V_{OH} = 4.95 V: clipping occurs on positive peaks. Maximum undistorted input: $V_{in,max} = V_{OH}/2 = 4.95/2 = 2.475$ V. Usable input swing: ±(2.475 - 2.5) = the output clips immediately above the bias point.

Better design: Use gain = 2 with $V_{bias} = V_{CC}/4 = 1.25$ V, input swing ±1.0 V. Output: $2 \times 0.25 = 0.5$ V to $2 \times 2.25 = 4.5$ V. Both within 0.03-4.95 V range.

At I_{load} = 15 mA: $V_{sat} = I_{load} \times R_{DS(on)} = 0.015 \times 75 = 1.125$ V. $V_{OL} \approx 1.125$ V, $V_{OH} \approx 5.0 - 1.125 = 3.875$ V. Output swing = 3.875 - 1.125 = 2.75 V_{pp}, a 45% reduction from the 2 mA case.

Problem 13.7.7

Given: An inverting amplifier with gain = -10 (R_{in} = 10 kΩ, R_f = 100 kΩ) uses an op-amp with V_{OS} = 500 µV, SR = 2 V/µs, GBW = 3 MHz, and supply rails of ±12 V. The input is a 200 mV_{peak}, 50 kHz sine wave.

Find: (a) The output offset error. (b) Whether slew-rate limiting occurs. (c) Whether the output is within bandwidth.

Solution: (a) Output offset error: $V_{out(OS)} = V_{OS} \times (1 + R_f/R_{in}) = 500 \times 10^{-6} \times 11 = 5.5$ mV. The signal output = 10 × 200 mV = 2.0 V peak. Error = 5.5/2,000 × 100% = 0.275%.

(b) Slew rate check: Maximum output rate of change = $2\pi \times f \times V_p = 2\pi \times 50{,}000 \times 2.0 = 628{,}318$ V/s = 0.628 V/µs. Since 0.628 V/µs < SR = 2 V/µs: no slew-rate limiting.

Full-power bandwidth: $f_{FP} = SR/(2\pi V_p) = 2 \times 10^6/(2\pi \times 2.0) = 159$ kHz. 50 kHz < 159 kHz, confirming no slew-rate distortion.

(c) Bandwidth check: Closed-loop bandwidth: $f_{3dB} = GBW / |A_{CL\ noise\ gain}|$ = 3,000,000 / 11 = 273 kHz. Since 50 kHz < 273 kHz, the output is within bandwidth. Gain at 50 kHz: $|A| = 10/\sqrt{(1 + (50/273)^2)} = 10/\sqrt{(1.0335)} = 10/1.0166 = 9.84$ (-0.14 dB from ideal).

Problem 13.7.8

Given: A precision instrumentation circuit requires an input-referred noise of less than 1 μV_{rms} in a 0.1 Hz to 10 Hz bandwidth. Two op-amps are being considered: - Op-amp A (bipolar): $e_n = 3\ nV/\sqrt{Hz}$, 1/f corner f_c = 2 Hz. - Op-amp B (CMOS): $e_n = 20\ nV/\sqrt{Hz}$, 1/f corner f_c = 50 Hz.

Find: The total RMS noise for each op-amp in the 0.1–10 Hz band, and which meets the requirement.

Solution: For noise in a band with 1/f component, the total noise is: $V_n = e_n \times \sqrt{(f_c \times \ln(f_H/f_L) + (f_H - f_L))}$.

Op-amp A (bipolar): $V_n = 3 \times 10^{-9} \times \sqrt{(2 \times \ln(10/0.1) + (10 - 0.1))} = 3 \times 10^{-9} \times \sqrt{(2 \times 4.605 + 9.9)} = 3 \times 10^{-9} \times \sqrt{(9.21 + 9.9)} = 3 \times 10^{-9} \times \sqrt{19.11} = 3 \times 10^{-9} \times 4.372 = 13.1\ nV_{rms} = 0.013\ \mu V_{rms}$.

Op-amp B (CMOS): $V_n = 20 \times 10^{-9} \times \sqrt{(50 \times \ln(10/0.1) + (10 - 0.1))} = 20 \times 10^{-9} \times \sqrt{(50 \times 4.605 + 9.9)} = 20 \times 10^{-9} \times \sqrt{(230.3 + 9.9)} = 20 \times 10^{-9} \times \sqrt{240.2} = 20 \times 10^{-9} \times 15.50 = 310\ nV_{rms} = 0.31\ \mu V_{rms}$.

Both meet the 1 μV_{rms} requirement, but Op-amp A is 24× better due to its much lower 1/f corner frequency. The CMOS op-amp's high 1/f corner (50 Hz) causes its noise to rise significantly in this low-frequency band.

Chapter 14 — Section 14.1: NEC Organization and Structure

Practice problems covering NEC code organization, chapter hierarchy, and key table references for electrical installations.

Problem 14.1.1

Given: An engineer is designing a fire pump motor circuit rated at 40 A full-load current in a healthcare facility. Article 430 (Chapter 4) requires branch circuit conductors at 125% of FLC. Article 695 (Chapter 6) for fire pumps requires conductors to be sized at 125% of FLC and to be installed in a dedicated raceway. Article 517 (Chapter 5) applies to healthcare facilities.

Find: The minimum conductor ampacity, identify which chapter's wiring rules take precedence, and explain the NEC hierarchy that governs conflicts between articles.

Solution: Minimum conductor ampacity per Article 430: $I_{min} = 1.25 \times 40 = 50$ A.

Article 695 (Chapter 6 — Special Equipment) supplements and may modify the requirements of Chapters 1 through 4. Article 517 (Chapter 5 — Special Occupancies) also supplements and may modify Chapters 1 through 4.

NEC hierarchy: Chapters 5, 6, and 7 can supplement or modify Chapters 1-4. When both Chapter 5 and Chapter 6 apply to the same installation, both sets of additional requirements must be met.

Fire pump wiring per Article 695: conductors must be in a dedicated raceway separate from other wiring, protected against physical damage, and the overcurrent protection must not open under locked-rotor conditions (695.7).

Healthcare facility per Article 517: the fire pump may need to be connected to the essential electrical system (life safety branch per 517.26).

A 6 AWG THWN-2 copper conductor rated at 65 A at 75°C satisfies the 50 A minimum. Both Article 695 (dedicated raceway) and Article 517 (essential electrical system connection) requirements must be applied simultaneously.

Problem 14.1.2

Given: An engineer must size conductors for a 150 A feeder in a high-rise building. The conductors are copper THWN-2 (90°C) in EMT conduit at 42°C ambient. The feeder supplies a mix of continuous and non-continuous loads.

Find: Using Table 310.16 and Table 310.15(C)(1), identify the base ampacity for 1/0 AWG, 2/0 AWG, and 3/0 AWG copper at both 75°C and 90°C column values, and apply the temperature correction factor for 42°C ambient to find which size meets the 150 A requirement.

Solution: From Table 310.16 base ampacities:

Size	75°C column	90°C column
1/0 AWG	150 A	170 A
2/0 AWG	175 A	195 A
3/0 AWG	200 A	225 A

Temperature correction factor at 42°C ambient: - For 75°C insulation: factor = 0.82. - For 90°C insulation: factor = 0.91.

Per 110.14(C), the 75°C column governs for termination purposes, but the 90°C ampacity may be used for derating calculations.

Using 90°C column for derating: - 1/0 AWG: $170 \times 0.91 = 154.7$ A. Check: does not exceed 75°C termination value of 150 A. Use 150 A (governed by termination). - 2/0 AWG: $195 \times 0.91 = 177.5$ A. Check: does not exceed 75°C value of 175 A. Use 175 A. - 3/0 AWG: $225 \times 0.91 = 204.8$ A. Check: exceeds 75°C value of 200 A, so limited to 200 A.

For a 150 A requirement: 1/0 AWG copper THWN-2 is sufficient (150 A ≥ 150 A), but has zero margin. Select 2/0 AWG (175 A) for engineering margin.

Problem 14.1.3

Given: An engineer needs the following information for a motor circuit design: motor FLC for a 20 HP, 460 V, three-phase motor; the equipment grounding conductor size for a 60 A breaker; and the conduit fill percentage for three conductors.

Find: Identify the specific NEC table number for each piece of information and provide the values.

Solution: Motor FLC: Table 430.250 (Full-Load Current, Three-Phase AC Motors). For 20 HP at 460 V: FLC = 27 A.

Equipment grounding conductor: Table 250.122 (Minimum Size Equipment Grounding Conductors). For a 60 A overcurrent device: EGC = 10 AWG copper.

Conduit fill percentage: Chapter 9, Table 1 (Percent of Cross Section of Conduit Occupied by Conductors). For three conductors: 40% fill maximum.

These three tables, along with Table 310.16 (conductor ampacities) and Table 430.52 (motor overcurrent protection), are among the most frequently referenced tables in NEC-based electrical design.

Chapter 14 — Section 14.2: Conductor Sizing and Ampacity

Practice problems covering ampacity tables, temperature correction, conduit fill adjustment, conductor resistance, conduit fill calculations, and rooftop temperature adders.

Problem 14.2.1

Given: A 208 V, three-phase feeder supplies 80 A of continuous load and 60 A of non-continuous load. The conductors are copper THWN-2 (90°C insulation) terminated on equipment rated for 75°C.

Find: The minimum required conductor ampacity and the conductor size from Table 310.16.

Solution: Required ampacity = 1.25×80 (continuous) + 1.00×60 (non-continuous) = $100 + 60 = 160$ A.

From Table 310.16 at 75°C column (termination limit): - 1/0 AWG copper = 150 A (insufficient, $150 < 160$). - 2/0 AWG copper = 175 A (sufficient, $175 \geq 160$).

Actual load current = $80 + 60 = 140$ A. Overcurrent device must be rated at least 160 A: next standard size per 240.6(A) = 175 A.

Check: 2/0 AWG at 75°C = 175 A $\geq$ 175 A breaker rating.

Minimum conductor size: 2/0 AWG copper THWN-2, protected by a 175 A circuit breaker.

Problem 14.2.2

Given: A set of 4 AWG THHN copper conductors (90°C rated) is installed in a conduit in an attic space where the ambient temperature reaches 48°C during summer.

Find: The corrected ampacity and compare with 4 AWG THW (75°C rated) in the same conditions.

Solution: From Table 310.16 base ampacities: - 4 AWG at 90°C = 95 A. - 4 AWG at 75°C = 85 A.

Temperature correction factors at 48°C ambient: - For 90°C insulation: factor = 0.82. - For 75°C insulation: factor = 0.65.

Corrected ampacities: - THHN (90°C): $95 \times 0.82 = 77.9$ A. - THW (75°C): $85 \times 0.65 = 55.3$ A.

The 90°C insulation retains 82% of its capacity versus only 65% for the 75°C insulation, providing a 41% ampacity advantage (77.9 vs. 55.3 A) at 48°C ambient.

Problem 14.2.3

Given: A conduit contains 8 current-carrying conductors: four 10 AWG THWN-2 copper conductors (two 20 A circuits) and four 6 AWG THWN-2 copper conductors (two 40 A circuits). The ambient temperature is 40°C.

Find: The adjusted ampacity for both conductor sizes with combined temperature and conduit fill derating.

Solution: Base ampacities from Table 310.16 at 90°C: - 10 AWG: 40 A. - 6 AWG: 75 A.

Temperature correction factor at 40°C for 90°C insulation: 0.91.

Conduit fill adjustment for 8 current-carrying conductors (7-9 range): 0.70.

Adjusted ampacities: - 10 AWG: $40 \times 0.91 \times 0.70 = 25.5$ A. - 6 AWG: $75 \times 0.91 \times 0.70 = 47.8$ A.

The 20 A circuit load is within the 10 AWG adjusted capacity of 25.5 A. The 40 A circuit load is within the 6 AWG adjusted capacity of 47.8 A.

Both circuits are adequately sized, but with minimal margin on the 6 AWG circuits (47.8 A vs. 40 A load = 19.5% margin).

Problem 14.2.4

Given: A 250 kcmil copper conductor carries 200 A in a PVC conduit at 60 Hz. From NEC Chapter 9, Table 9: $R = 0.0541\ \Omega/1000$ ft and $X = 0.0523\ \Omega/1000$ ft for PVC conduit. The one-way run is 350 feet, supplying a load at 0.80 power factor lagging on a 480 V three-phase system.

Find: The per-phase voltage drop and the percent voltage drop.

Solution: Effective R per conductor: $R = 0.0541 \times 350/1000 = 0.01894\ \Omega$. Effective X per conductor: $X = 0.0523 \times 350/1000 = 0.01831\ \Omega$.

$\cos(\theta) = 0.80$, $\sin(\theta) = 0.600$.

Per-phase voltage drop: $V_{drop} = I \times (R\cos(\theta) + X\sin(\theta)) = 200 \times (0.01894 \times 0.80 + 0.01831 \times 0.600) = 200 \times (0.01515 + 0.01099) = 200 \times 0.02614 = 5.23$ V per phase.

Three-phase line-to-line voltage drop: $V_{drop(LL)} = \sqrt{3} \times 5.23 = 9.05$ V.

Percent voltage drop: $\%V_{drop} = 9.05 / 480 \times 100 = 1.89\%$.

This is within the NEC 3% recommendation for feeders.

Problem 14.2.5

Given: The following THWN-2 conductors must be installed in a single RMC (rigid metal conduit): six 3/0 AWG (two three-phase circuits) and three 8 AWG (one three-phase control circuit). From Chapter 9, Table 5: 3/0 AWG THWN-2 area = 0.2679 in^2, 8 AWG THWN-2 area = 0.0366 in^2.

Find: The minimum RMC conduit size.

Solution: Total conductor area: $6 \times 0.2679 + 3 \times 0.0366 = 1.6074 + 0.1098 = 1.7172$ in^2.

Total conductors = 9 (three or more), so 40% fill applies. Required conduit internal area: 1.7172 / 0.40 = 4.293 in^2.

From Chapter 9, Table 4 for RMC: - 2" RMC: internal area = 3.408 in^2 (40% = 1.363 in^2 – insufficient). - 2½" RMC: internal area = 4.866 in^2 (40% = 1.946 in^2 – insufficient). - 3" RMC: internal area = 7.499 in^2 (40% = 3.000 in^2 – insufficient).

Wait, let me reconsider. The total area is 1.7172 in^2, and 40% fill requires a conduit with internal area of 4.293 in^2.

- 2½" RMC: internal area = 4.866 in^2. 40% fill = 1.946 in^2 – this is larger than 1.7172 in^2. But wait: the allowable fill area (40% of conduit) must exceed the total conductor area.
- 2½" RMC: $0.40 \times 4.866 = 1.946$ in^2 $>$ 1.7172 in^2.
- 2" RMC: $0.40 \times 3.408 = 1.363$ in^2 $<$ 1.7172 in^2 – insufficient.

Minimum conduit size: 2½" RMC.

Verification: fill = $1.7172 / 4.866 \times 100 = 35.3\% < 40\%$ limit.

Problem 14.2.6

Given: A solar inverter feeder conduit is mounted 3½ inches above a flat roof surface on conduit standoffs. The outdoor ambient temperature is 43°C. The conductors are 2 AWG THWN-2 copper (90°C rated). Four current-carrying conductors are in the conduit.

Find: The effective ambient temperature, the combined derating factor (temperature + conduit fill), and the derated ampacity.

Solution: Roof temperature adder at 3½ inches above roof: +17°C per Table 310.15(C)(1)(a).

Effective ambient: $43 + 17 = 60$°C.

Temperature correction factor for 90°C insulation at 60°C ambient: 0.71.

Conduit fill adjustment for 4 current-carrying conductors (4-6 range): 0.80.

Base ampacity of 2 AWG THWN-2 at 90°C: 130 A.

Derated ampacity: $130 \times 0.71 \times 0.80 = 73.8$ A.

Indoor comparison (30°C, 3 conductors, no derating): 130 A at 90°C, limited to 115 A at 75°C for terminations.

The rooftop installation loses 43% of the indoor capacity (73.8 vs. 130 A).

If the conduit were mounted directly on the roof surface (adder = +33°C), effective ambient = 76°C, correction factor ≈ 0.37, and derated ampacity = 130 × 0.37 × 0.80 = 38.5 A – barely adequate for many solar inverter circuits.

Problem 14.2.7

Given: A nipple (conduit section 18 inches long) connects two panels. It contains twelve 10 AWG THWN-2 copper conductors. The ambient temperature is 30°C (standard).

Find: The adjusted ampacity, taking advantage of the 60% fill rule for nipples.

Solution: Base ampacity of 10 AWG THWN-2 at 90°C: 40 A.

Nipples (≤ 24 inches) are permitted 60% fill per Chapter 9, Table 1, Note 4.

For ampacity derating, the number of current-carrying conductors still applies: 12 conductors in the 10-20 range gives a conduit fill adjustment factor of 0.50.

Note: The 60% fill rule applies to physical conduit fill calculations (cross-sectional area), not to the ampacity derating. The ampacity adjustment for more than 3 conductors still applies even in nipples per 310.15(C)(1). However, some jurisdictions allow relaxed derating for nipples due to reduced thermal concern. Check with the AHJ.

Standard adjusted ampacity: 40 × 0.50 = 20 A per conductor.

If the AHJ permits no ampacity derating for nipples (some interpretations): the full 40 A ampacity applies, limited to 30 A at 75°C termination temperature.

Problem 14.2.8

Given: A 480 V, three-phase, 300 A feeder runs 200 feet through a steel conduit. The engineer is comparing two conductor options: - Option A: 350 kcmil copper (R = 0.0382 Ω/1000 ft, X = 0.0441 Ω/1000 ft) - Option B: 500 kcmil copper (R = 0.0293 Ω/1000 ft, X = 0.0439 Ω/1000 ft)

The load power factor is 0.90 lagging.

Find: The voltage drop for each option and determine which meets the 2% feeder drop recommendation.

Solution: $\cos(\theta) = 0.90$, $\sin(\theta) = 0.436$.

Option A (350 kcmil): $V_{drop} = \sqrt{3} \times 300 \times 200 \times (0.0382 \times 0.90 + 0.0441 \times 0.436) / 1000 = 519.6 \times 200 \times (0.03438 + 0.01923) / 1000 = 103{,}920 \times 0.05361 / 1000 = 5.57$ V. Percent: $5.57/480 \times 100 = 1.16\%$.

Option B (500 kcmil): $V_{drop} = \sqrt{3} \times 300 \times 200 \times (0.0293 \times 0.90 + 0.0439 \times 0.436) / 1000 = 103{,}920 \times (0.02637 + 0.01914) / 1000 = 103{,}920 \times 0.04551 / 1000 = 4.73$ V. Percent: $4.73/480 \times 100 = 0.99\%$.

Both options meet the 2% recommendation. Option A (350 kcmil) at 1.16% is sufficient and more economical. However, Option B reduces the voltage drop by 15% and provides better motor starting performance.

Chapter 14 — Section 14.3: Overcurrent Protection

Practice problems covering standard OCPD ratings, tap rules, coordination, short-circuit current calculations, and AFCI protection.

Problem 14.3.1

Given: A feeder has a calculated continuous load of 210 A and a non-continuous load of 35 A. The conductors are copper THWN-2 with 75°C terminations.

Find: The required conductor ampacity, the conductor size from Table 310.16, and the overcurrent device rating from the standard sizes in 240.6(A).

Solution: Required conductor ampacity: $I_{req} = 1.25 \times 210 + 1.00 \times 35 = 262.5 + 35 = 297.5$ A.

From Table 310.16 at 75°C: - 300 kcmil copper = 285 A (insufficient, $285 < 297.5$). - 350 kcmil copper = 310 A (sufficient, $310 \geq 297.5$).

Total load current = $210 + 35 = 245$ A. Overcurrent device must be ≥ 297.5 A.

Standard sizes per 240.6(A): 250, 300, 350 A. Next standard size at or above 297.5 A: 300 A.

Check per 240.4(B): for circuits ≤ 800 A, the next standard size above the conductor ampacity is permitted. Conductor ampacity = 310 A. OCPD = 300 A $\leq$ 310 A.

Minimum: 350 kcmil copper THWN-2 with a 300 A circuit breaker.

Problem 14.3.2

Given: A 600 A feeder supplies a tap to a sub-panel 8 feet away. The tap terminates in a 150 A main breaker. Copper THWN-2 conductors with 75°C terminations are used.

Find: Whether the 10-foot tap rule applies, the minimum tap conductor size, and verify all conditions are met.

Solution: The tap length is 8 feet $\leq$ 10 feet, so the 10-foot tap rule (240.21(B)(1)) applies.

Conditions for the 10-foot tap rule: 1. Tap conductors ≤ 10 feet: satisfied (8 ft). 2. Tap conductor ampacity ≥ 10% of upstream OCPD: $0.10 \times 600 = 60$ A minimum. 3. Tap terminates in a single OCPD that limits load to tap conductor ampacity: 150 A breaker. 4. Tap conductors enclosed in raceway: must be satisfied.

The tap conductors must have an ampacity ≥ the 150 A load (since the OCPD is 150 A).

From Table 310.16 at 75°C: 1/0 AWG copper = 150 A.

Minimum tap conductor: 1/0 AWG copper THWN-2 (150 A = 150 A breaker rating).

Check condition 2: 150 A » 60 A minimum. All conditions satisfied.

Problem 14.3.3

Given: A distribution system has a 1200 A main breaker and a 400 A feeder breaker. At a fault current of 8,000 A, the 400 A breaker clears in 0.03 seconds and the 1200 A breaker clears in 0.08 seconds. At 15,000 A (near the maximum available), the 400 A breaker clears in 0.017 seconds and the 1200 A breaker clears in 0.025 seconds.

Find: Whether the devices coordinate at both fault levels, and identify the coordination ratio at each level.

Solution: At 8,000 A: Coordination ratio = upstream time / downstream time = 0.08 / 0.03 = 2.67:1. The upstream device is 2.67× slower – adequate coordination (> 1.5:1 minimum).

At 15,000 A: Coordination ratio = 0.025 / 0.017 = 1.47:1. The upstream device is only 1.47× slower – marginal, below the 1.5:1 guideline.

At the higher fault level, both devices are operating in or near their instantaneous trip regions, where timing differences narrow. Coordination is lost at very high fault currents approaching the system maximum.

To improve coordination: (a) use a zone-selective interlocking (ZSI) scheme that adds a restraint signal from downstream to upstream breakers, or (b) select a main breaker with an adjustable short-time delay that provides a definite time margin above the feeder breaker's clearing time at all fault levels.

Problem 14.3.4

Given: A 750 kVA, 480Y/277 V, three-phase transformer has an impedance of 5.0%. The secondary conductors are 2 sets of 500 kcmil copper in parallel per phase, running 50 feet to a main switchboard. The "C" constant for 500 kcmil copper is 26,706. Assume an infinite primary bus.

Find: The available fault current at the transformer secondary and at the switchboard.

Solution: Transformer secondary FLA: $I_{FLA} = 750{,}000 / (\sqrt{3} \times 480) = 750{,}000 / 831.4 = 902.1$ A.

Fault current at transformer secondary: $I_{sc(xfmr)} = I_{FLA} / Z_{pu} = 902.1 / 0.05 = 18{,}042$ A.

For parallel conductors, the effective "C" constant is multiplied by the number of conductors per phase: $C_{eff} = 2 \times 26{,}706 = 53{,}412$.

Multiplier factor: $f = (1.732 \times L \times I_{sc}) / (C_{eff} \times V_{LL}) = (1.732 \times 50 \times 18{,}042) / (53{,}412 \times 480) = 1{,}562{,}438 / 25{,}637{,}760 = 0.06094$.

Fault current at switchboard: $I_{sc(swbd)} = I_{sc(xfmr)} / (1 + f) = 18{,}042 / 1.06094 = 17{,}005$ A.

All overcurrent devices at the switchboard must have an interrupting rating ≥ 18 kAIC (next standard rating above 17,005 A). Standard ratings are 10, 14, 18, 22, 25, 35, 42, 50, 65, 100, and 200 kAIC.

Problem 14.3.5

Given: A dwelling unit has a 15 A, 120 V branch circuit serving five duplex receptacles in a family room. The wiring is 14 AWG NM-B, 100 feet from the panel. An AFCI breaker is installed per 210.12. A series arc develops in a damaged lamp cord with an arc current of 3 A.

Find: The arc power, why a standard breaker would not trip, and the AFCI response.

Solution: Arc power: $P_{arc} = V \times I = 120 \times 3 = 360$ W.

A standard 15 A breaker thermal trip point: approximately 135% of rating = 20.3 A after ~1 hour. Magnetic (instantaneous) trip: 5-10× rating = 75-150 A.

The 3 A arc current is only 20% of the 15 A breaker rating – far below any standard trip threshold.

This 360 W arc would operate indefinitely, with surface temperatures at the arc point easily exceeding 1,000°C – far above the ignition temperature of wood (~300°C), paper (~230°C), and most fabrics (~250°C).

The AFCI breaker detects the arc signature: irregular, sputtering current waveform with random amplitude variations (unlike the smooth waveform of a resistive or motor load). The AFCI algorithm identifies the arc pattern and trips within 8-60 half-cycles (67-500 ms at 60 Hz).

Per 210.12(A), AFCI protection is required for all 125 V, 15 A and 20 A branch circuits in dwelling unit family rooms.

Problem 14.3.6

Given: A 25-foot tap from a 200 A feeder supplies a small panelboard with a 70 A main breaker. The tap conductors are copper THWN-2 with 75°C terminations.

Find: The minimum tap conductor size per the 25-foot tap rule (240.21(B)(2)).

Solution: The 25-foot tap rule requires: 1. Tap conductors not exceeding 25 feet: satisfied (25 ft exactly). 2. Tap conductor ampacity ≥ 1/3 of upstream OCPD: 200/3 = 66.7 A minimum. 3. Tap terminates in a single OCPD: 70 A breaker – satisfied. 4. Tap conductors are protected from physical damage in a raceway.

From Table 310.16 at 75°C: - 6 AWG copper = 65 A (insufficient, $65 < 66.7$). - 4 AWG copper = 85 A (sufficient, $85 \geq 66.7$).

Minimum tap conductor size: 4 AWG copper THWN-2.

The 4 AWG conductors (85 A capacity) exceed the 70 A panelboard main breaker, so the downstream loads are properly protected.

Problem 14.3.7

Given: A 1500 kVA, 13.8 kV / 480Y/277 V, three-phase transformer has an impedance of 6.5%. The utility provides a source fault current of 10,000 A symmetrical at the transformer primary (13.8 kV side).

Find: The available fault current at the transformer secondary, accounting for the finite (non-infinite) utility source.

Solution: Transformer secondary FLA: $I_{FLA(sec)} = 1{,}500{,}000 / (\sqrt{3} \times 480) = 1{,}500{,}000 / 831.4 = 1{,}804$ A.

Transformer impedance on the secondary base: $Z_{xfmr(pu)} = 0.065$.

Utility source impedance referred to secondary: First, find utility source impedance on transformer kVA base: $I_{FLA(pri)} = 1{,}500{,}000 / (\sqrt{3} \times 13{,}800) = 1{,}500{,}000 / 23{,}901 = 62.76$ A. $Z_{source(pu)} = I_{FLA(pri)} / I_{fault(pri)} = 62.76 / 10{,}000 = 0.006276$ pu.

Total impedance: $Z_{total(pu)} = Z_{xfmr} + Z_{source} = 0.065 + 0.006276 = 0.07128$ pu.

Available fault current at secondary: $I_{sc(sec)} = I_{FLA(sec)} / Z_{total(pu)} = 1{,}804 / 0.07128 = 25{,}308$ A.

Compare with infinite bus assumption: $I_{sc} = 1{,}804 / 0.065 = 27{,}754$ A. The finite source reduces the available fault current by $(27{,}754 - 25{,}308) / 27{,}754 \times 100\% = 8.8\%$.

Overcurrent devices must have interrupting rating ≥ 25 kAIC (or 30 kAIC for standard rating).

Chapter 14 — Section 14.4: Grounding and Bonding

Practice problems covering system grounding, equipment grounding conductors, ground fault protection, bonding, and GFCI protection.

Problem 14.4.1

Given: A 480Y/277 V, three-phase, 800 A service uses 2 sets of 500 kcmil copper service entrance conductors per phase. The grounding electrode system consists of a driven ground rod and a concrete-encased (Ufer) electrode. The total ground fault impedance on a 277 V circuit is 0.15 Ω.

Find: The minimum grounding electrode conductor (GEC) size per Table 250.66, and the ground fault current magnitude.

Solution: From Table 250.66, for service entrance conductors over 600 kcmil but not exceeding 1,100 kcmil ($2 \times 500 = 1{,}000$ kcmil equivalent per phase): Minimum GEC = 2/0 AWG copper.

Ground fault current on a 277 V line-to-neutral circuit: $I_{fault} = V / Z_{total} = 277 / 0.15 = 1{,}847$ A.

This current is well above a typical 20 A branch circuit breaker rating, ensuring the breaker trips within the instantaneous region (typically < 0.05 seconds).

Clearing energy: $I^2t = 1{,}847^2 \times 0.05 = 170{,}570\ A^2{\cdot}s$. This low energy minimizes damage at the fault point.

Problem 14.4.2

Given: A 480 V feeder is protected by a 100 A circuit breaker. The original conductor size is 1 AWG copper (minimum per ampacity). Due to a 500-foot run, the conductors are upsized to 4/0 AWG copper to limit voltage drop.

Find: The minimum EGC size from Table 250.122 and the adjusted EGC size per 250.122(B).

Solution: From Table 250.122 for a 100 A OCPD: Minimum EGC = 8 AWG copper (16,510 circular mils).

Per 250.122(B), proportionally increase the EGC: Size factor = area of installed conductor / area of minimum required conductor. Installed: 4/0 AWG = 211,600 circular mils. Minimum required (1 AWG): 83,690 circular mils. Factor = 211,600 / 83,690 = 2.528.

Adjusted EGC area = 16,510 × 2.528 = 41,737 circular mils.

From wire gauge table: - 6 AWG = 26,240 cmil (insufficient). - 4 AWG = 41,740 cmil (sufficient, just barely).

Adjusted EGC: 4 AWG copper.

Problem 14.4.3

Given: A 480Y/277 V, 1600 A service has ground fault protection set at 1000 A pickup with a 0.3-second time delay. An arcing ground fault develops with 1,200 A fault current that persists until cleared.

Find: Whether the GFPE detects the fault, the time to trip, and the arc energy (I^2t) dissipated.

Solution: GFPE pickup = 1,000 A. Fault current = 1,200 A.

Since 1,200 A > 1,000 A pickup: the GFPE detects the fault.

Trip time = 0.3 seconds (the intentional time delay).

Arc energy: $I^2t = 1{,}200^2 \times 0.3 = 1{,}440{,}000 \times 0.3 = 432{,}000\ A^2{\cdot}s$.

If the GFPE were set lower at 500 A pickup with 0.1 second delay: $I^2t = 1{,}200^2 \times 0.1 = 144{,}000\ A^2{\cdot}s$ – a 3× reduction in arc energy.

If the fault current were 800 A (below the 1,000 A pickup), the GFPE would not trip, and the fault would persist until detected by other means or until it escalated to a higher current level. This highlights the importance of setting GFPE at the lowest practical level per NEC 230.95 (maximum 1,200 A with maximum 1-second delay).

Problem 14.4.4

Given: A commercial building has a 120/208 V, three-phase, 225 A service. The service entrance conductors are 4/0 AWG copper. A metal gas pipe and a structural steel column are within 10 feet of the service equipment.

Find: The bonding jumper size for both the gas pipe and the structural steel per Table 250.66, and identify the bonding requirements.

Solution: From Table 250.66 for 4/0 AWG service entrance conductors: Minimum bonding jumper = 4 AWG copper.

Per 250.104(A) — Metal Piping Systems: the metal gas pipe must be bonded to the grounding electrode system. Bond size from Table 250.66: 4 AWG copper.

Per 250.104(C) — Structural Metal: if the structural steel is not intentionally grounded and could become energized, it must be bonded. Bond from Table 250.66: 4 AWG copper.

Per 250.104(A), the bonding jumper for the gas pipe must be connected on the street side of any gas meter or regulator that could be removed for service.

The bonding connections must use listed clamps, compression connectors, or exothermic welds. Sheet metal screws are not permitted for bonding connections per 250.8.

Problem 14.4.5

Given: A commercial kitchen has a 20 A, 120 V GFCI-protected branch circuit serving three duplex receptacles near a sink. The circuit uses 12 AWG THHN in EMT conduit, 120 feet from the panel. Each receptacle serves equipment with a typical leakage current of 0.8 mA.

Find: The total leakage current, the margin to the GFCI trip threshold, and whether nuisance tripping is likely.

Solution: Total appliance leakage: 3×0.8 mA = 2.4 mA.

Estimated cable leakage in EMT (lower than NM-B due to metallic shielding): approximately 0.005 mA/ft. Cable leakage: $120 \times 0.005 = 0.6$ mA.

Total system leakage: $2.4 + 0.6 = 3.0$ mA.

GFCI Class A trip threshold: 5 mA (nominal midpoint of 4-6 mA range). Margin: 5.0 - 3.0 = 2.0 mA (40% of trip threshold).

At the minimum trip point of 4 mA: margin = 4.0 - 3.0 = 1.0 mA.

This is marginal. If a fourth appliance is added (0.8 mA) or if equipment ages and leakage increases, total leakage could reach 4+ mA, causing nuisance tripping.

Recommendation: Split the loads across two GFCI-protected circuits to keep each circuit's leakage below 2 mA. Per 210.8(B)(2), GFCI protection is required for kitchen receptacles within 6 feet of a sink in non-dwelling occupancies.

Problem 14.4.6

Given: A 208Y/120 V, 600 A service with 500 kcmil copper service entrance conductors requires a system grounding electrode conductor. The grounding electrode system includes a 10-foot driven ground rod (25 Ω to earth), a concrete-encased Ufer electrode (5 Ω to earth), and a metal water pipe (3 Ω to earth).

Find: The GEC size per Table 250.66, the approximate parallel ground resistance, and explain why the equipment grounding conductor (not earth) carries the fault current.

Solution: From Table 250.66 for 500 kcmil service entrance conductors: Minimum GEC = 1/0 AWG copper.

Parallel ground resistance of all electrodes: $1/R_{total} = 1/25 + 1/5 + 1/3 = 0.04 + 0.20 + 0.333 = 0.573$. $R_{total} = 1/0.573 = 1.75\ \Omega$.

For a 120 V ground fault through the earth alone: $I_{earth} = 120 / 1.75 = 68.6$ A.

While 68.6 A might eventually trip a breaker, the clearing time would be unacceptably long for personnel safety. More critically, on a 20 A circuit, a 1.75 Ω earth path produces only 68.6 A, which is in the thermal (slow) trip region of the breaker.

The equipment grounding conductor (metallic path) has an impedance of typically 0.1-0.5 Ω total, producing fault currents of 240-1,200 A that trip breakers instantly. This is why NEC 250.4(A)(5) requires an effective ground-fault current path through the EGC, not through the earth. The earth connection provides lightning and surge protection, not fault clearing.

Chapter 14 — Section 14.5: Motor Circuits

Practice problems covering motor FLC tables, branch circuit conductor sizing, overcurrent protection, disconnecting means, VFD circuits, and hazardous locations.

Problem 14.5.1

Given: A 50 HP, 460 V, three-phase motor has a nameplate FLA of 59 A. From NEC Table 430.250, the FLC for a 50 HP, 460 V motor is 65 A. The motor is Design B with code letter H (locked-rotor kVA/HP = 6.3-7.09).

Find: The branch circuit conductor minimum ampacity, the conductor size, and the locked-rotor current.

Solution: Branch circuit conductor ampacity per 430.22: $I_{min} = 1.25 \times$ FLC (from table) $= 1.25 \times 65 = 81.25$ A.

From Table 310.16 at 75°C: 4 AWG copper = 85 A (sufficient, $85 \geq 81.25$). Minimum conductor: 4 AWG copper THWN-2.

Locked-rotor current (using mid-range code letter H = 6.7 kVA/HP): $kVA_{LR} = 6.7 \times 50 = 335$ kVA. $I_{LR} = kVA_{LR} / (\sqrt{3} \times V) = 335{,}000 / (1.732 \times 460) = 335{,}000 / 796.9 = 420$ A.

This is $420/65 = 6.5\times$ the FLC, typical for Design B motors.

Note: The NEC requires using the table FLC of 65 A (not the nameplate 59 A) for conductor and overcurrent device sizing.

Problem 14.5.2

Given: A motor control center (MCC) feeder supplies four motors and a 50 A continuous heating load: - Motor A: 75 HP, 460 V, 3φ (FLC = 96 A) - Motor B: 30 HP, 460 V, 3φ (FLC = 40 A) - Motor C: 15 HP, 460 V, 3φ (FLC = 21 A) - Motor D: 5 HP, 460 V, 3φ (FLC = 7.6 A)

Find: The minimum feeder conductor ampacity per NEC 430.24.

Solution: Per 430.24: feeder ampacity = $1.25 \times$ largest motor FLC + sum of remaining motor FLCs + non-motor continuous loads $\times 1.25$.

Largest motor: Motor A at 96 A.

$I_{feeder} = 1.25 \times 96 + 40 + 21 + 7.6 + 1.25 \times 50 = 120 + 40 + 21 + 7.6 + 62.5 = 251.1$ A.

From Table 310.16 at 75°C: - 4/0 AWG copper = 230 A (insufficient). - 250 kcmil copper = 255 A (sufficient, $255 \geq 251.1$).

Minimum feeder conductor: 250 kcmil copper.

Total running load current: $96 + 40 + 21 + 7.6 + 50 = 214.6$ A.

Problem 14.5.3

Given: A 25 HP, 460 V, three-phase, Design B motor has FLC = 34 A (Table 430.250) and nameplate FLA = 32 A with a service factor of 1.0.

Find: The overload relay setting, the maximum inverse-time breaker size, and the maximum dual-element fuse size.

Solution: Overload protection per 430.32(A)(1): For service factor < 1.15: trip at not more than 115% of nameplate FLA. $I_{OL} = 1.15 \times 32 = 36.8$ A (set relay trip to 36.8 A).

Short-circuit/ground fault protection per Table 430.52:

Inverse-time circuit breaker (Design B motor): maximum 250% of FLC. 250% × 34 = 85.0 A. Next standard size per 240.6(A): 90 A.

Dual-element time-delay fuse (Design B motor): maximum 175% of FLC. 175% × 34 = 59.5 A. Next standard size: 60 A.

Final motor branch circuit: - Conductors: 10 AWG copper (rated 35 A at 75°C $\geq 1.25 \times 34 = 42.5$ A – insufficient). - Need 8 AWG copper (rated 50 A at 75°C ≥ 42.5 A). - Overcurrent: 90 A inverse-time breaker OR 60 A dual-element fuse. - Overload relay: set at 36.8 A.

Problem 14.5.4

Given: A 100 HP, 460 V, three-phase motor has FLC = 124 A (Table 430.250). The motor code letter is G (kVA/HP = 5.6-6.29).

Find: The minimum disconnect ampere rating and the locked-rotor current the disconnect must handle.

Solution: Minimum disconnect rating per 430.110(A): $I_{disc} = 1.15 \times FLC = 1.15 \times 124 = 142.6$ A.

Next standard disconnect size: 200 A (common available size).

Locked-rotor current (mid-range code letter G = 5.9 kVA/HP): $kVA_{LR} = 5.9 \times 100 = 590$ kVA. $I_{LR} = 590{,}000 / (\sqrt{3} \times 460) = 590{,}000 / 796.9 = 740$ A.

The 200 A motor-circuit disconnect switch must have horsepower rating adequate for 100 HP at 460 V and be capable of interrupting the locked-rotor current of 740 A.

The disconnect must be within sight (visible and ≤ 50 feet) of the motor per 430.102(B).

Problem 14.5.5

Given: A 40 HP, 460 V, three-phase motor is driven by a VFD with rated input current of 52 A. The VFD is 150 feet from the motor. The motor FLC from Table 430.250 is 52 A. The VFD output has a PWM switching frequency of 8 kHz with 200 ns rise time.

Find: The supply-side conductor size, the overcurrent protection, and the critical cable length for reflected voltage waves.

Solution: Supply-side conductors (to VFD input): Ampacity = 1.25 × VFD input current = 1.25 × 52 = 65 A. From Table 310.16 at 75°C: 6 AWG copper = 65 A (sufficient).

Overcurrent protection per Table 430.52 (Design B, inverse-time breaker): 250% × 52 A (motor FLC from table) = 130 A. Next standard size: 125 A (430.52(C)(1) permits using the next standard size up only if the standard size is insufficient for starting – 125 A may be adequate). If motor won't start: use 150 A.

Motor-side conductors: Ampacity = 1.25 × 52 = 65 A. Use 6 AWG copper.

Critical cable length for reflected waves: $L_{crit} = v \times t_{rise} / 2$, where $v \approx 150$ m/μs for typical cable. $L_{crit} = (150 \times 10^6) \times (200 \times 10^{-9}) / 2 = 30 / 2 = 15$ m ≈ 49 feet.

Since 150 feet » 49 feet, reflected voltage waves will fully develop at the motor terminals, potentially reaching 2× DC bus voltage = $2 \times 1.35 \times V_{LL} = 2 \times 1.35 \times 460 = 1{,}242$ V peak (≈1,200 V).

An output reactor or dv/dt filter is recommended to limit motor terminal voltage below 1,000 V peak. VFD-rated cable with symmetrical ground conductors should be used.

Per 430.130(B), the EGC must be wire-type (conduit alone is insufficient).

Problem 14.5.6

Given: A grain elevator (Class II, Division 1, Group G – grain dust, auto-ignition 400°C) requires a 10 HP, 460 V motor driving a bucket elevator. The area classification extends 10 feet from any grain handling equipment.

Find: The location classification, the motor enclosure type, the wiring method, and the required equipment temperature code.

Solution: Classification: Class II, Division 1, Group G (combustible dust present under normal operations during grain handling).

Temperature code: Grain dust auto-ignition = 400°C. Equipment surface temperature must be well below 400°C. A T3 rating (200°C maximum) provides adequate margin and is the standard rating for Class II motors.

Motor enclosure: Must be dust-ignitionproof per Article 502.125 – designed so dust cannot enter the enclosure and surface temperatures do not ignite the surrounding dust layer.

Wiring method per Article 502.10(A)(1): - Threaded rigid metal conduit (RMC) or - Threaded intermediate metal conduit (IMC), or - Type MI cable with listed termination fittings.

All fittings must be dust-tight. Flexible connections at the motor (for vibration isolation) require dust-tight flexible fittings listed for Class II, Division 1.

Sealing fittings are required at boundaries between the Division 1 area and unclassified areas to prevent dust migration through the conduit system.

Problem 14.5.7

Given: A 20 HP, 208 V, three-phase motor has FLC = 59.4 A (from Table 430.250). The motor branch circuit uses a dual-element time-delay fuse for short-circuit protection and a thermal overload relay. The motor nameplate shows FLA = 56 A and SF = 1.15.

Find: The overload relay trip setting, the maximum fuse size, and the minimum conductor ampacity.

Solution: Overload relay per 430.32(A)(1): SF $\geq$ 1.15: trip at $\leq$ 125% of nameplate FLA. $I_{OL} = 1.25 \times 56 = 70$ A.

Short-circuit protection (dual-element fuse, Design B): Maximum = 175% $\times$ FLC = $1.75 \times 59.4 = 103.95$ A. Next standard fuse size: 100 A.

If motor will not start with 100 A fuse, 430.52(C)(1) Exception 2 permits up to 225% $\times$ FLC = 133.65 A. Next standard: 125 A.

Conductor ampacity: $1.25 \times$ FLC $= 1.25 \times 59.4 = 74.25$ A. From Table 310.16 at 75°C: 4 AWG copper = 85 A (sufficient).

The circuit: 4 AWG copper conductors, 100 A dual-element fuse, overload relay at 70 A.

Chapter 14 — Section 14.6: Transformer Connections

Practice problems covering primary-only protection, dual protection, secondary conductors, separately derived systems, neutral sizing, and K-factor transformers.

Problem 14.6.1

Given: A 45 kVA, single-phase, 480 V primary / 120/240 V secondary transformer is installed with primary-only overcurrent protection.

Find: The primary rated current, the maximum primary overcurrent device size per Table 450.3(B), and the secondary rated current.

Solution: Primary rated current: $I_{primary} = S / V = 45{,}000 / 480 = 93.75$ A.

Maximum primary OCPD per Table 450.3(B) (primary only): 125% of rated current = $1.25 \times 93.75 = 117.2$ A. Next standard size per 240.6(A): 125 A.

Secondary rated current: $I_{secondary} = 45{,}000 / 240 = 187.5$ A.

Since there is no secondary overcurrent protection, the secondary conductors must be rated for at least 187.5 A. From Table 310.16 at 75°C: 3/0 AWG copper = 200 A (sufficient for 187.5 A).

Problem 14.6.2

Given: A 300 kVA, three-phase, 480 V delta primary / 208Y/120 V secondary transformer has an impedance of 4.5%. Overcurrent protection is provided on both primary and secondary.

Find: The primary and secondary OCPD sizes, and the maximum secondary fault current.

Solution: Primary rated current: $I_{primary} = 300{,}000 / (\sqrt{3} \times 480) = 300{,}000 / 831.4 = 360.8$ A.

Maximum primary OCPD (250% with secondary protection): $2.50 \times 360.8 = 902.1$ A. Next standard size: 900 A (or 800 A for more conservative design).

Secondary rated current: $I_{secondary} = 300{,}000 / (\sqrt{3} \times 208) = 300{,}000 / 360.3 = 832.6$ A.

Maximum secondary OCPD (125%): $1.25 \times 832.6 = 1{,}040.8$ A. Next standard size: 1,000 A.

Check: The 1,000 A OCPD does not exceed the conductor ampacity. Secondary conductors must be rated for at least 832.6 A.

Maximum secondary fault current (infinite primary bus): $I_{fault} = I_{secondary} / Z_{pu} = 832.6 / 0.045 = 18{,}502$ A.

The secondary OCPD must have an interrupting rating $\geq$ 22 kAIC (next standard above 18,502 A).

Problem 14.6.3

Given: A 75 kVA, three-phase, 480/208Y/120 V transformer has its primary protected by a 125 A breaker. The secondary conductors run 22 feet to a panelboard with a 250 A main breaker.

Find: The minimum secondary conductor size per the 240.21(C)(6) 25-foot tap rule.

Solution: Secondary rated current: $I_{sec} = 75{,}000 / (\sqrt{3} \times 208) = 75{,}000 / 360.3 = 208.2$ A.

The 25-foot rule (240.21(C)(6)) applies: tap length = 22 feet $\leq$ 25 feet.

The secondary conductors must be sized for at least the transformer secondary rated current: 208.2 A.

Primary OCPD equivalent on secondary: $125 \times (480/208) = 125 \times 2.308 = 288.5$ A equivalent. One-third: $288.5/3 = 96.2$ A.

The governing requirement is the larger of 208.2 A (transformer capacity) and 96.2 A (one-third rule).

From Table 310.16 at 75°C: - 3/0 AWG copper = 200 A (insufficient, $200 < 208.2$). - 4/0 AWG copper = 230 A (sufficient).

Minimum secondary conductor: 4/0 AWG copper per phase, with 22-foot run to 250 A main breaker.

Problem 14.6.4

Given: A 500 kVA, 480 V delta / 208Y/120 V wye transformer is a separately derived system. The secondary conductors are 2 sets of 350 kcmil copper per phase (700 kcmil equivalent).

Find: The system bonding jumper size and the grounding electrode conductor size per Article 250.30.

Solution: System bonding jumper per 250.30(A)(1) and Table 250.102(C)(1): For the equivalent of 700 kcmil ungrounded conductors (between 500 and 750 kcmil range): Minimum bonding jumper = 1/0 AWG copper.

Grounding electrode conductor per 250.30(A)(5) and Table 250.66: For 700 kcmil equivalent: the GEC = 2/0 AWG copper.

However, per 250.30(A)(5), if using a structural metal electrode or concrete-encased electrode, the GEC need not be larger than 3/0 AWG copper (the maximum per the exception).

The neutral-to-ground bond is made at the transformer secondary terminal compartment only. Downstream, the neutral is kept isolated (insulated) from enclosures and ground.

Problem 14.6.5

Given: A 112.5 kVA, 480 V delta / 208Y/120 V wye transformer serves an office floor with predominantly computer loads (SMPS). The balanced three-phase load draws 312 A per phase, with measured harmonic content: 3rd harmonic = 28%, 5th = 10%, 7th = 4%.

Find: The neutral current from triplen harmonics and the minimum neutral conductor size.

Solution: Third harmonic current per phase: $I_3 = 0.28 \times 312 = 87.4$ A.

Triplen harmonics add in the neutral: $I_{N(3rd)} = 3 \times 87.4 = 262.1$ A.

The 5th and 7th harmonics are not triplen (not multiples of 3) and cancel in a balanced three-phase system, so they do not contribute to neutral current.

The 9th harmonic (if present) would also be triplen and add to the neutral. Assuming 9th = 5%: $I_9 = 0.05 \times 312 = 15.6$ A per phase. $I_{N(9th)} = 3 \times 15.6 = 46.8$ A.

Total neutral current (RSS of triplen components): $I_N = \sqrt{(262.1^2 + 46.8^2)} = \sqrt{(68{,}696 + 2{,}190)} = \sqrt{70{,}886} = 266.2$ A.

The neutral current is 266.2/312 = 85.3% of the phase current.

From Table 310.16 at 75°C for 266.2 A: - 300 kcmil copper = 285 A (sufficient).

Minimum neutral conductor: 300 kcmil copper.

Per 310.15(E), this neutral counts as a current-carrying conductor for conduit fill derating purposes.

Problem 14.6.6

Given: A standard 225 kVA, 480/208Y/120 V dry-type transformer (K-1 rated, $P_{EC\text{-}R}$ = 10% of rated load loss) serves a building with the following measured harmonic spectrum: $I_1 = 1.00$, $I_3 = 0.65$, $I_5 = 0.45$, $I_7 = 0.25$, $I_9 = 0.12$, $I_{11} = 0.05$.

Find: The K-factor, the required K-rated transformer, and the derated capacity if the existing K-1 transformer is used.

Solution: K-factor = $\Sigma(I_h^2 \times h^2) / \Sigma(I_h^2)$.

Numerator: $1.00^2 \times 1^2 + 0.65^2 \times 3^2 + 0.45^2 \times 5^2 + 0.25^2 \times 7^2 + 0.12^2 \times 9^2 + 0.05^2 \times 11^2 = 1.000 + 0.4225 \times 9 + 0.2025 \times 25 + 0.0625 \times 49 + 0.0144 \times 81 + 0.0025 \times 121 = 1.000 + 3.803 + 5.063 + 3.063 + 1.166 + 0.303 = 14.398$.

Denominator: $1.000 + 0.4225 + 0.2025 + 0.0625 + 0.0144 + 0.0025 = 1.704$.

$K = 14.398 / 1.704 = 8.45$.

Required K-rated transformer: K-13 (next standard K-rating above 8.45).

Derating the K-1 transformer per IEEE C57.110: $I_{max} = \sqrt{}(1 / (1 + P_{EC-R} \times (K - 1))) = \sqrt{}(1 / (1 + 0.10 \times (8.45 - 1))) = \sqrt{}(1 / (1 + 0.745)) = \sqrt{}(1 / 1.745) = \sqrt{}0.573 = 0.757$ pu.

Derated capacity: $0.757 \times 225 = 170.3$ kVA.

The transformer can safely handle 170 kVA of this harmonic-rich load, a 24.3% reduction from its nameplate rating. If the building load exceeds 170 kVA, either install a K-13 transformer or add harmonic filtering.

Problem 14.6.7

Given: A 50 kVA, 240 V delta primary / 120/240 V secondary, single-phase transformer has primary-and-secondary protection. The transformer impedance is 3.0%.

Find: The primary OCPD size, the secondary OCPD size, and the maximum secondary fault current.

Solution: Primary rated current: $I_{primary} = 50{,}000 / 240 = 208.3$ A.

Maximum primary OCPD (250% with secondary protection): $2.50 \times 208.3 = 520.8$ A. Next standard: 500 A.

Secondary rated current: $I_{secondary} = 50{,}000 / 240 = 208.3$ A.

Maximum secondary OCPD (125%): $1.25 \times 208.3 = 260.4$ A. Next standard: 250 A.

Maximum secondary fault current: $I_{fault} = I_{secondary} / Z_{pu} = 208.3 / 0.03 = 6{,}943$ A.

The 250 A secondary OCPD must have an interrupting rating $\geq$ 10 kAIC (standard rating above 6,943 A).

Chapter 14 — Section 14.7: Voltage Drop Calculations

Practice problems covering single-phase voltage drop, three-phase voltage drop, parallel conductors, conductor sizing for voltage drop, and service load calculations.

Problem 14.7.1

Given: A 120 V, single-phase branch circuit supplies a 12 A continuous load at the end of a 200-foot run. The conductors are 10 AWG copper in PVC conduit ($R = 1.21\ \Omega/1000$ ft, $X = 0.050\ \Omega/1000$ ft). The load power factor is 0.95 lagging.

Find: The voltage drop, the percent voltage drop, and whether it meets the NEC 3% recommendation.

Solution: $\cos(\theta) = 0.95$, $\sin(\theta) = 0.312$.

$V_{drop} = 2 \times I \times L \times (R\cos(\theta) + X\sin(\theta)) / 1000 = 2 \times 12 \times 200 \times (1.21 \times 0.95 + 0.050 \times 0.312) / 1000 = 4{,}800 \times (1.1495 + 0.01560) / 1000 = 4{,}800 \times 1.1651 / 1000 = 5.59$ V.

Percent voltage drop: $5.59 / 120 \times 100 = 4.66\%$.

This exceeds the NEC 3% recommendation for branch circuits.

To achieve 3%: $V_{drop(max)} = 0.03 \times 120 = 3.6$ V. Required $R_{max} = 3.6 \times 1000 / (2 \times 12 \times 200 \times 0.95) = 3{,}600 / 4{,}560 = 0.789\ \Omega/1000$ ft.

From Chapter 9, Table 9: 8 AWG copper in PVC = $0.764\ \Omega/1000$ ft (sufficient). $V_{drop} = 2 \times 12 \times 200 \times (0.764 \times 0.95 + 0.050 \times 0.312) / 1000 = 4{,}800 \times 0.7414 / 1000 = 3.56$ V = 2.97%.

Upgrade to 8 AWG copper for NEC compliance.

Problem 14.7.2

Given: A 480 V, three-phase feeder supplies a 250 A motor load at 0.88 power factor lagging. The run is 300 feet using 350 kcmil copper in steel conduit ($R = 0.0382\ \Omega/1000$ ft, $X = 0.0441\ \Omega/1000$ ft).

Find: The running voltage drop, the percent drop, and the voltage drop during motor starting at 6× FLC.

Solution: $\cos(\theta) = 0.88$, $\sin(\theta) = 0.475$.

Running voltage drop: $V_{drop} = \sqrt{3} \times I \times L \times (R \cos(\theta) + X \sin(\theta)) / 1000 = 1.732 \times 250 \times 300 \times (0.0382 \times 0.88 + 0.0441 \times 0.475) / 1000 = 129{,}900 \times (0.03362 + 0.02095) / 1000 = 129{,}900 \times 0.05457 / 1000 = 7.09$ V.

Percent: $7.09 / 480 \times 100 = 1.48\%$ (within 3% recommendation).

Voltage at motor terminals: $480 - 7.09 = 472.9$ V.

During motor starting ($I_{start} = 6 \times 250 = 1{,}500$ A): Starting PF ≈ 0.30 lagging. $\cos(\theta) = 0.30$, $\sin(\theta) = 0.954$. $V_{drop(start)} = 1.732 \times 1{,}500 \times 300 \times (0.0382 \times 0.30 + 0.0441 \times 0.954) / 1000 = 779{,}400 \times (0.01146 + 0.04209) / 1000 = 779{,}400 \times 0.05355 / 1000 = 41.74$ V.

Starting voltage: $480 - 41.74 = 438.3$ V = 91.3% of rated. This is above the 80% minimum typically needed for successful motor starting.

Problem 14.7.3

Given: A 208 V, three-phase, 600 A feeder uses three 300 kcmil copper conductors per phase in parallel, installed in three separate PVC conduits. The run is 350 feet at 0.85 power factor lagging. From Table 9: $R = 0.0437\ \Omega/1000$ ft, $X = 0.0527\ \Omega/1000$ ft for 300 kcmil in PVC.

Find: The voltage drop with parallel conductors and the percent drop.

Solution: Effective impedance per phase (three conductors in parallel): $R_{eff} = 0.0437 / 3 = 0.01457\ \Omega/1000$ ft. $X_{eff} = 0.0527 / 3 = 0.01757\ \Omega/1000$ ft.

$\cos(\theta) = 0.85$, $\sin(\theta) = 0.527$.

$V_{drop} = \sqrt{3} \times 600 \times 350 \times (0.01457 \times 0.85 + 0.01757 \times 0.527) / 1000 = 1.732 \times 600 \times 350 \times (0.01238 + 0.00926) / 1000 = 363{,}720 \times 0.02164 / 1000 = 7.87$ V.

Percent: $7.87 / 208 \times 100 = 3.78\%$.

This exceeds the 3% recommendation. Options: 1. Increase to four conductors per phase: $R_{eff} = 0.0437/4$, $V_{drop} \approx 5.90$ V = 2.84% (acceptable). 2. Upsize to 350 kcmil per parallel set: lower R and similar result. 3. Accept 3.78% if the branch circuit drop is minimal (total $\leq 5\%$).

Problem 14.7.4

Given: A 240 V, single-phase branch circuit supplies a 30 A continuous load (electric vehicle charger) at the end of a 100-foot run in PVC conduit. Maximum allowable voltage drop is 3%.

Find: The minimum conductor size to meet both ampacity and voltage drop requirements.

Solution: Ampacity requirement: $1.25 \times 30 = 37.5$ A continuous load. From Table 310.16 at 75°C: 8 AWG copper = 50 A (sufficient).

Voltage drop requirement: $V_{drop(max)} = 0.03 \times 240 = 7.2$ V.

At unity PF (resistive EV charger load): $R_{max} = V_{drop} \times 1000 / (2 \times I \times L) = 7.2 \times 1000 / (2 \times 30 \times 100) = 7{,}200 / 6{,}000 = 1.20\ \Omega/1000$ ft.

From Chapter 9, Table 9 (PVC conduit): - 10 AWG: R = 1.21 Ω/1000 ft. $V_{drop} = 2 \times 30 \times 100 \times 1.21/1000 = 7.26$ V = 3.03% (marginal). - 8 AWG: R = 0.764 Ω/1000 ft. $V_{drop} = 2 \times 30 \times 100 \times 0.764/1000 = 4.58$ V = 1.91% (meets 3%).

The 8 AWG copper satisfies both ampacity (50 A ≥ 37.5 A) and voltage drop (1.91% < 3%).

However, if the run were 200 feet: $V_{drop} = 2 \times 30 \times 200 \times 0.764/1000 = 9.17$ V = 3.82% (exceeds 3%). Would need 6 AWG (R = 0.491): $V_{drop} = 5.89$ V = 2.46%.

Problem 14.7.5

Given: A 1,800 ft² dwelling has: general lighting (3 VA/ft²), two small-appliance circuits, one laundry circuit, one 8 kW electric range, one 4.5 kW clothes dryer, one 4 kW water heater, one 3-ton (3.5 kW) heat pump, and one 15 kW electric furnace (backup heating).

Find: The service demand load using the standard method and the minimum service size.

Solution: General lighting: 1,800 × 3 = 5,400 VA. Small-appliance circuits: 2 × 1,500 = 3,000 VA. Laundry: 1,500 VA. Total general: 5,400 + 3,000 + 1,500 = 9,900 VA.

Table 220.42 demand factors: First 3,000 VA at 100% = 3,000 VA. Remaining 6,900 VA at 35% = 2,415 VA. Net general = 3,000 + 2,415 = 5,415 VA.

Range: Table 220.55, Column C for one range ≤ 12 kW: 8,000 VA. (The 8 kW nameplate is ≤ 12 kW, so the demand is 8 kW per column C.)

Dryer: Table 220.54: 5,000 VA minimum or nameplate, whichever is larger = 5,000 VA.

Water heater: 4,000 VA at 100% (fewer than 4 fixed appliances) = 4,000 VA.

Heating/cooling (220.60): Use the larger load (they don't operate simultaneously). Electric furnace: 15,000 VA > heat pump: 3,500 VA. Use heating: 15,000 VA.

Total demand: 5,415 + 8,000 + 5,000 + 4,000 + 15,000 = 37,415 VA.

Service current (240 V, single-phase): I = 37,415 / 240 = 155.9 A.

Minimum service per 230.79: at least 100 A. For 155.9 A, select 200 A service.

Service entrance conductors: From Table 310.16 at 75°C: - 2/0 AWG copper = 175 A (insufficient for 200 A OCPD). - 4/0 AWG copper = 230 A (sufficient).

Use 4/0 AWG copper service entrance conductors with 200 A main breaker.

Problem 14.7.6

Given: A 277 V, single-phase lighting circuit supplies 20 A at 0.98 power factor (LED lighting with power factor correction). The run is 250 feet in EMT conduit using 10 AWG copper (R = 1.18 Ω/1000 ft, X = 0.044 Ω/1000 ft).

Find: The voltage drop and whether it meets the 3% branch circuit recommendation.

Solution: cos(θ) = 0.98, sin(θ) = 0.199.

$V_{drop} = 2 \times I \times L \times (R \cos(\theta) + X \sin(\theta)) / 1000 = 2 \times 20 \times 250 \times (1.18 \times 0.98 + 0.044 \times 0.199) / 1000 = 10{,}000 \times (1.1564 + 0.00876) / 1000 = 10{,}000 \times 1.165 / 1000 = 11.65$ V.

Percent: 11.65 / 277 × 100 = 4.21% (exceeds 3%).

To meet 3%: $V_{drop(max)} = 0.03 \times 277 = 8.31$ V. $R_{max} = 8.31 \times 1000 / (2 \times 20 \times 250 \times 0.98) = 8{,}310 / 9{,}800 = 0.848$ Ω/1000 ft.

From Table 9: 8 AWG = 0.743 Ω/1000 ft. $V_{drop} = 2 \times 20 \times 250 \times (0.743 \times 0.98 + 0.044 \times 0.199) / 1000 = 10{,}000 \times 0.737 / 1000 = 7.37$ V = 2.66%.

Upgrade to 8 AWG copper for compliance.

Problem 14.7.7

Given: Using the optional method (220.82), calculate the service demand for a 2,200 ft^2 dwelling with: total connected load (all circuits including lighting, receptacles, appliances) = 45 kVA, a 5-ton (7 kW) central air conditioner, and a 12 kW electric furnace.

Find: The demand load and service size using the optional method.

Solution: Per 220.82(B): General loads (everything except heating and cooling): Total connected = 45 kVA. Subtract HVAC: 45 - 7 - 12 = 26 kVA of general loads. First 10 kVA at 100% = 10,000 VA. Remainder (16 kVA) at 40% = 6,400 VA. General demand = 10,000 + 6,400 = 16,400 VA.

Heating/cooling per 220.60: use the larger. Furnace 12,000 VA > A/C 7,000 VA.

Per 220.82(C): heating and cooling loads at 100% of the largest: HVAC demand = 12,000 VA.

Total demand: 16,400 + 12,000 = 28,400 VA.

Service current: I = 28,400 / 240 = 118.3 A.

Minimum service: 150 A (next common size above 118.3 A, and exceeds the 100 A minimum per 230.79).

Service entrance conductors: 1/0 AWG copper = 150 A at 75°C (matches 150 A service).

Note: The optional method yields a lower demand (28.4 kVA) than the standard method would for this dwelling, because the 40% factor on general loads above 10 kVA is more aggressive than the standard method's combination of demand factors.

Chapter 14 — Section 14.8: Emergency and Standby Power Systems

Practice problems covering generator sizing, transfer switch requirements, and emergency/standby system design.

Problem 14.8.1

Given: A commercial building requires an emergency generator for the following Article 700 (emergency) loads: egress lighting (15 kW), exit signs (3 kW), fire alarm (5 kW), and a fire pump with a 30 HP, 460 V motor (FLC = 40 A, locked-rotor = 240 A). The generator is 480Y/277 V, three-phase. All loads except the fire pump are at unity power factor. The fire pump motor operates at 0.85 PF.

Find: The total running load in kW, the generator kVA during fire pump starting, and the recommended generator size.

Solution: Running loads: Lighting: 15 kW. Exit signs: 3 kW. Fire alarm: 5 kW. Fire pump running: P = $\sqrt{3} \times 480 \times 40 \times 0.85 = 28.2$ kW.

Total running load: $15 + 3 + 5 + 28.2 = 51.2$ kW.

Fire pump starting analysis: Fire pump starting kVA = $\sqrt{3} \times 480 \times 240 / 1000 = 199.7$ kVA (at ~0.30 PF starting = 59.9 kW).

Running load without fire pump: 51.2 - 28.2 = 23.0 kW. Total kW during fire pump start: 23.0 + 59.9 = 82.9 kW.

Total kVA during start: 23.0/1.0 + 199.7 = 23.0 + 199.7 = 222.7 kVA (at mixed PF).

Generator selection: Continuous rating must exceed running load: 51.2 kW / 0.8 PF = 64.0 kVA minimum continuous. Motor starting capability must handle 222.7 kVA transient with voltage dip $\leq$ 10%.

Select a 75 kW / 94 kVA standby-rated generator with motor starting capability of at least 225 kVA.

Per Article 700.12, the generator must have on-site fuel for at least 2 hours at full load. Fuel consumption $\approx$ 5.5 gal/hr for a 75 kW diesel. Minimum tank: $5.5 \times 2 = 11$ gallons (use 50-gallon tank for practical margin).

Problem 14.8.2

Given: A data center uses an optional standby generator (Article 702) to back up 400 kW of IT load at 0.90 PF and 80 kW of cooling at 0.85 PF. The cooling includes two 30 HP chiller compressor motors (FLC = 40 A each, locked-rotor = 240 A each). Load transfer is sequenced: cooling starts first, then IT loads are transferred.

Find: The generator size accounting for the load sequencing and motor starting.

Solution: Step 1 – First motor start (no other load): Motor 1 starting kVA = $\sqrt{3} \times 480 \times 240 / 1000 =$ 199.7 kVA. Motor 1 starting kW = $199.7 \times 0.30 = 59.9$ kW.

Step 2 – Second motor start (Motor 1 running): Motor 1 running: $P = \sqrt{3} \times 480 \times 40 \times 0.85 = 28.2$ kW, $S = 28.2/0.85 = 33.2$ kVA. Motor 2 starting: 199.7 kVA. Total Step 2: $33.2 + 199.7 = 232.9$ kVA (peak demand during motor 2 start). kW: $28.2 + 59.9 = 88.1$ kW.

Step 3 – Both motors running + IT load transfer: Both motors running: $2 \times 28.2 = 56.4$ kW. Other cooling: $80 - 56.4 = 23.6$ kW. IT load: 400 kW. Total running: $56.4 + 23.6 + 400 = 480$ kW. Total kVA: $56.4/0.85 + 23.6/0.85 + 400/0.90 = 66.4 + 27.8 + 444.4 = 538.6$ kVA.

Generator selection: Continuous: 480 kW at mixed PF, requiring 538.6 kVA. Motor starting: 232.9 kVA transient (this is less than the continuous kVA, so continuous governs).

At 0.8 PF rating: 538.6 kVA $\times$ 0.8 = 430.9 kW (insufficient if generator is 0.8 PF rated). Need a generator rated at 500 kW / 625 kVA (at 0.8 PF) to cover the 480 kW / 538.6 kVA steady-state with 4% margin.

Alternatively, with load management that sheds some IT load during motor starting: a 500 kW generator is adequate.

Problem 14.8.3

Given: A hospital essential electrical system (Article 517.26) must transfer emergency loads within 10 seconds per Article 700. The automatic transfer switch (ATS) has the following time delays: - Time to detect utility failure: 1.5 seconds - Engine start signal to generator running at rated voltage: 5.0 seconds - Transfer delay after generator ready: 0.5 seconds

Find: The total transfer time, whether it meets the 10-second requirement, and the maximum engine start time that would still comply.

Solution: Total transfer time: $t_{total} = t_{detect} + t_{start} + t_{transfer} = 1.5 + 5.0 + 0.5 = 7.0$ seconds.

Since 7.0 seconds < 10 seconds: the system meets the Article 700 requirement with 3.0 seconds of margin.

Maximum allowable engine start time: $t_{start(max)} = 10 - t_{detect} - t_{transfer} = 10 - 1.5 - 0.5 = 8.0$ seconds.

However, NFPA 110 for Type 10 emergency systems (healthcare) actually requires the generator to accept load within 10 seconds, which is tighter than just reaching rated voltage. The engine must reach rated speed and voltage AND the ATS must complete the transfer within the 10-second window.

For legally required standby (Article 701): the restoration time is 60 seconds, providing much more margin.

Per 700.5(B), the ATS must be automatic and electrically operated, mechanically held. Manual transfer is not permitted for emergency systems.

Problem 14.8.4

Given: An emergency generator serves three load categories with separate automatic transfer switches: - Emergency (Article 700): 50 kW (must transfer in 10 seconds) - Legally required standby (Article 701): 80 kW (transfer in 60 seconds) - Optional standby (Article 702): 120 kW (no time requirement)

The generator is rated 300 kW at 0.8 PF. Load sequencing adds loads in priority order with 5-second intervals between transfers.

Find: The load on the generator at each step of the transfer sequence and verify the generator is not overloaded at any step.

Solution: $t = 0$ s: Utility failure detected. Generator start signal sent. Generator load = 0 kW.

$t = 7$ s: Generator at rated voltage. Emergency ATS transfers. Generator load = 50 kW (16.7% of rating).

$t = 12$ s: (5 seconds after emergency transfer) Legally required standby ATS transfers. Generator load $= 50 + 80 = 130$ kW (43.3% of rating).

$t = 17$ s: (5 seconds after LRSB transfer) Optional standby ATS transfers. Generator load $= 50 + 80 + 120 = 250$ kW (83.3% of rating).

At no point does the load exceed the 300 kW generator rating.

kVA check (assuming 0.85 average PF): $250 / 0.85 = 294$ kVA. Generator kVA $= 300/0.8 = 375$ kVA. $294 < 375$: adequate.

If any transfer included a large motor start, the kVA surge must be checked against the generator's motor starting capability (typically 3× continuous kVA = 1,125 kVA for a well-designed generator).

Per 700.32 (2023 NEC), the emergency overcurrent protection must be selectively coordinated so that a fault on any emergency branch trips only the nearest device, not the generator main breaker.

Problem 14.8.5

Given: A standby generator for a nursing home must run for 96 hours per NFPA 110 (Type 10, Class 96). The generator is rated 150 kW diesel with a fuel consumption rate of 11.2 gallons per hour at full load and 6.5 gallons per hour at 50% load. The expected average load is 60% of rating.

Find: The minimum fuel tank size at full load and at the expected average load. Also estimate the fuel consumption at 60% load by linear interpolation.

Solution: Fuel consumption at 60% load (linear interpolation between 50% and 100%): At 50%: 6.5 gal/hr. At 100%: 11.2 gal/hr. Slope: $(11.2 - 6.5) / (100 - 50) = 4.7 / 50 = 0.094$ gal/hr per % load. At 60%: $6.5 + 0.094 \times (60 - 50) = 6.5 + 0.94 = 7.44$ gal/hr.

Minimum tank at full load (worst case): $V_{tank} = 11.2 \times 96 = 1{,}075$ gallons.

Minimum tank at expected 60% average load: $V_{tank} = 7.44 \times 96 = 714$ gallons.

Per NFPA 110, the fuel supply must be sufficient for the class duration at the expected load. Using the expected load calculation: minimum 714 gallons.

However, prudent engineering practice sizes for full load to account for unexpected load additions during an extended outage. A 1,100-gallon fuel tank provides the required capacity at full load with margin.

The tank must include fuel level monitoring per NFPA 110 Section 7.9, and fuel quality maintenance (fuel polishing or stabilizer additives) is critical for tanks that may sit unused for extended periods.

Chapter 15 — Section 15.1: The OSI Model

Practice problems covering the seven-layer OSI model, encapsulation, physical layer signaling, data link framing, network layer routing, and transport layer protocols.

Problem 15.1.1

Given: A networked sensor transmits a 200-byte application payload using TCP over IPv4 on an Ethernet LAN. The TCP header is 20 bytes, the IPv4 header is 20 bytes, the Ethernet header is 14 bytes, and the FCS is 4 bytes. The preamble/SFD is 8 bytes and the inter-frame gap (IFG) is 12 bytes.

Find: (a) The total frame size on the wire (excluding preamble/SFD and IFG), (b) the total on-wire overhead per frame, and (c) the protocol efficiency at the frame level and at the wire level.

Solution:

(a) Frame size = 200 + 20 + 20 + 14 + 4 = 258 bytes

(b) Frame-level overhead = 20 (TCP) + 20 (IP) + 14 (Eth header) + 4 (FCS) = 58 bytes Wire-level overhead = 58 + 8 (preamble/SFD) + 12 (IFG) = 78 bytes

(c) Frame efficiency: η_{frame} = 200 / 258 = 0.7752 = 77.5% Wire efficiency: η_{wire} = 200 / (200 + 78) = 200 / 278 = 0.7194 = 71.9%

Small payloads like this sensor data suffer significantly more from protocol overhead compared to full 1,500-byte payloads.

Problem 15.1.2

Given: A 25GBASE-T Ethernet link uses PAM-4 signaling (4 amplitude levels) across four twisted pairs, each operating at a symbol rate of 3,200 Msymbols/s.

Find: (a) The bits per symbol for PAM-4, (b) the raw bit rate per pair, and (c) the total aggregate raw bit rate across all four pairs.

Solution:

(a) Bits per symbol = $\log_2(4)$ = 2 bits/symbol

(b) Bit rate per pair = $3{,}200 \times 10^6 \times 2$ = 6,400 Mbps = 6.4 Gbps

(c) Total raw bit rate $= 4 \times 6.4 = 25.6$ Gbps

The effective data rate of 25 Gbps is achieved after subtracting coding overhead. The raw rate of 25.6 Gbps allows for approximately 2.4% FEC and framing overhead.

Problem 15.1.3

Given: A Layer 2 switch manages a MAC address table with 32,768 entries, each storing a 48-bit MAC address and a 10-bit port identifier. An 802.1Q-tagged frame arrives with VLAN ID = 150, PCP = 5, and a 1,200-byte payload.

Find: (a) The memory required for the full MAC address table, (b) the total tagged frame size (including Ethernet header, VLAN tag, and FCS), and (c) the number of usable VLANs with a 12-bit VLAN ID field.

Solution:

(a) Bits per entry $= 48 + 10 = 58$ bits. Including VLAN association (12 bits per entry for VLAN-aware table): $58 + 12 = 70$ bits ≈ 9 bytes (rounded up to byte boundary). Total memory $= 32{,}768 \times 9 = 294{,}912$ bytes $= 288$ KB

(b) Tagged frame size: Ethernet header: 14 bytes (dst MAC 6 + src MAC 6 + EtherType 2) 802.1Q tag: 4 bytes (TPID 2 + TCI 2) Payload: 1,200 bytes FCS: 4 bytes Total $= 14 + 4 + 1{,}200 + 4 = 1{,}222$ bytes

(c) 12-bit VLAN ID: $2^{12} = 4{,}096$ values. VID 0 and VID 4095 are reserved. Usable VLANs = 4,094

Problem 15.1.4

Given: A router has an interface processing rate of 25 Mpps (million packets per second). Traffic consists of 40% packets at 64 bytes, 30% at 512 bytes, and 30% at 1,500 bytes. The initial TTL on all packets is 128.

Find: (a) The weighted average packet size, (b) the throughput in Gbps, and (c) the maximum number of router hops a packet can traverse before being discarded.

Solution:

(a) Average packet size $= 0.40 \times 64 + 0.30 \times 512 + 0.30 \times 1{,}500 = 25.6 + 153.6 + 450.0 = 629.2$ bytes

(b) Throughput $= 25 \times 10^6 \times 629.2 \times 8 = 125{,}840{,}000{,}000$ bps $= 125.8$ Gbps

(c) Maximum hops = 128 (TTL is decremented by 1 at each router; the packet is discarded when TTL reaches 0 at the 129th router).

Problem 15.1.5

Given: A TCP connection operates over a satellite link with RTT = 600 ms. The receiver advertises a window size of 131,072 bytes (128 KB) using the window scaling option (RFC 7323). The link capacity is 50 Mbps.

Find: (a) The bandwidth-delay product of the link, (b) the maximum throughput limited by the receive window, (c) the link utilization, and (d) the window size required to fully utilize the link.

Solution:

(a) BDP = $50 \times 10^6 \times 0.600$ = 30,000,000 bits = 3,750,000 bytes = 3.75 MB

(b) Throughput = Window / RTT = 131,072 / 0.600 = 218,453 bytes/s = $218{,}453 \times 8$ = 1,747,627 bps = 1.75 Mbps

(c) Utilization = 1.75 / 50 = 3.5%

The high-latency satellite link severely limits TCP throughput with a small window.

(d) Window needed = BDP = 3,750,000 bytes ≈ 3.58 MB

This requires window scaling with a scale factor of at least 6 ($2^6 = 64$, allowing a window up to 65,535 $\times$ 64 = 4,194,240 bytes). TCP performance enhancing proxies (PEPs) or protocol optimizers are often deployed on satellite links to address this issue.

Problem 15.1.6

Given: An Ethernet network carries frames with an average payload of 800 bytes. The transport layer uses UDP with an 8-byte header. IPv4 adds a 20-byte header. Ethernet adds 14 bytes header + 4 bytes FCS + 8 bytes preamble/SFD + 12 bytes IFG. The link operates at 10 Gbps.

Find: (a) The maximum number of frames per second the link can carry, and (b) the application-layer goodput in Gbps.

Solution:

(a) Total on-wire bytes per frame: Payload: 800 bytes (includes UDP header and application data) Actually, if 800 bytes is the application payload: UDP: 8 bytes, IP: 20 bytes, Ethernet header: 14 bytes, FCS: 4 bytes, Preamble/SFD: 8 bytes, IFG: 12 bytes Total on-wire = 800 + 8 + 20 + 14 + 4 + 8 + 12 = 866 bytes = 6,928 bits

Frames per second = 10×10^9 / 6,928 = 1,443,420 fps

(b) Goodput = $1{,}443{,}420 \times 800 \times 8$ = 9,227,891,156 bps = 9.23 Gbps

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Chapter 15 — Section 15.2: Physical Media: Copper

Practice problems covering coaxial cable impedance and propagation, twisted-pair attenuation and categories, signal bandwidth, crosstalk (NEXT/FEXT/ACR), and cable testing/certification.

Problem 15.2.1

Given: An RG-58 coaxial cable has a center conductor diameter d = 0.9 mm, a shield inner diameter D = 2.95 mm, and a solid polyethylene dielectric with $\varepsilon_r = 2.25$.

Find: (a) The characteristic impedance Z_0, (b) the velocity of propagation as a percentage of the speed of light, and (c) the one-way propagation delay per meter.

Solution:

(a) $Z_0 = (138 / \sqrt{\varepsilon_r}) \times \log_{10}(D/d)$ $\sqrt{2.25} = 1.50$ $138 / 1.50 = 92.0$ $\log_{10}(2.95 / 0.9) = \log_{10}(3.278) = 0.5156$ $Z_0 = 92.0 \times 0.5156 = 47.4\ \Omega \approx 50\ \Omega$

(b) $v_p = c / \sqrt{\varepsilon_r} = c / 1.50 = 0.6667c$ Velocity of propagation = 66.7% of the speed of light

(c) Delay $= 1 / v_p = 1 / (0.6667 \times 3 \times 10^8) = 1 / (2.0 \times 10^8) = 5.0 \times 10^{-9}$ s/m = 5.0 ns/m

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Problem 15.2.2

Given: A Cat 6 UTP cable has a maximum attenuation of 19.8 dB at 100 MHz over a 100 m permanent link. A 10BASE-T signal at 10 MHz is transmitted at +2 dBm over a 90 m cable run. The attenuation at 10 MHz is 6.6 dB/100 m.

Find: (a) The total cable attenuation at 10 MHz over the 90 m run, (b) the received signal power in dBm, and (c) the received power in microwatts.

Solution:

(a) Attenuation $= 6.6 \times (90/100) = 5.94$ dB

(b) $P_{rx} = P_{tx} - \text{Attenuation} = 2.0 - 5.94 = -3.94$ dBm

(c) $P_{rx} = 10^{(-3.94/10)}$ mW $= 10^{-0.394} = 0.4035$ mW $= 403.5\ \mu$W

Problem 15.2.3

Given: A 50 Ω coaxial cable has measured attenuation of 3.2 dB/100 m at 100 MHz and 10.5 dB/100 m at 1 GHz. A signal at 500 MHz must be delivered over a 120 m cable run with no more than 12 dB total loss.

Find: (a) The estimated attenuation at 500 MHz using square-root frequency scaling from the 100 MHz specification, (b) the total attenuation for the 120 m run, and (c) whether the loss budget is met.

Solution:

(a) $\alpha(500) \approx \alpha(100) \times \sqrt{(500/100)} = 3.2 \times \sqrt{5} = 3.2 \times 2.236 = 7.16$ dB/100 m

(b) Total attenuation $= 7.16 \times (120/100) = 8.59$ dB

(c) 8.59 dB $<$ 12.0 dB limit, so the loss budget is met with 3.41 dB of margin.

Verification: the estimate of 7.16 dB/100 m at 500 MHz falls between 3.2 (at 100 MHz) and 10.5 (at 1 GHz), confirming reasonableness.

Problem 15.2.4

Given: A Cat 6A cable is tested at 500 MHz with the following results: NEXT = 39.9 dB, FEXT = 29.5 dB, and insertion loss = 28.1 dB. The minimum ACR requirement is 3 dB.

Find: (a) The near-end ACR, (b) the far-end ACR (ACRF), accounting for the cable attenuation on the FEXT path, and (c) whether the cable meets the ACR requirement at both ends.

Solution:

(a) ACR (near-end) = NEXT − Insertion Loss = 39.9 − 28.1 = 11.8 dB

(b) ACRF = FEXT − Insertion Loss = 29.5 − 28.1 = 1.4 dB

Note: FEXT is measured at the far end and already accounts for cable attenuation on the interfering signal, so ACRF directly compares far-end crosstalk to far-end signal level.

(c) Near-end ACR = 11.8 dB $>$ 3 dB → PASS Far-end ACRF = 1.4 dB $<$ 3 dB → FAIL

The cable passes near-end but fails far-end crosstalk requirements at 500 MHz. This may indicate manufacturing defects or damage affecting the far-end pair isolation.

Problem 15.2.5

Given: A Cat 6A permanent link installation is certified with the following test results at 250 MHz:

Parameter	Measured	TIA-568 Limit
Insertion loss	15.4 dB	19.3 dB
NEXT	38.5 dB	35.3 dB
PS-NEXT	36.2 dB	32.3 dB
Return loss	15.8 dB	14.0 dB
Propagation delay	498 ns	555 ns

Find: (a) The pass/fail status and headroom (margin) for each parameter, (b) which parameter has the least headroom, and (c) the velocity of propagation if the cable is 95 m long.

Solution:

(a) For insertion loss, lower is better: $15.4 < 19.3 \rightarrow$ PASS, headroom $= 19.3 - 15.4 = 3.9$ dB For NEXT, higher is better: $38.5 > 35.3 \rightarrow$ PASS, headroom $= 38.5 - 35.3 = 3.2$ dB For PS-NEXT, higher is better: $36.2 > 32.3 \rightarrow$ PASS, headroom $= 36.2 - 32.3 = 3.9$ dB For return loss, higher is better: $15.8 > 14.0 \rightarrow$ PASS, headroom $= 15.8 - 14.0 = 1.8$ dB For propagation delay, lower is better: $498 < 555 \rightarrow$ PASS, headroom $= 555 - 498 = 57$ ns

(b) Return loss has the least headroom at 1.8 dB. This is the parameter most likely to fail first if connectors degrade or are improperly terminated.

(c) Velocity of propagation = cable length / propagation delay $= 95 / (498 \times 10^{-9}) = 1.907 \times 10^8$ m/s As a fraction of c: $v_p/c = 1.907 \times 10^8 / 3.0 \times 10^8 = 63.6\%$ of the speed of light

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Problem 15.2.6

Given: A network engineer must choose between Cat 5e (100 MHz bandwidth, max attenuation 22 dB at 100 MHz per 100 m) and Cat 6A (500 MHz bandwidth, max attenuation 32.8 dB at 500 MHz per 100 m) for a 70 m horizontal run that will carry 10GBASE-T traffic. 10GBASE-T requires Cat 6A or better and uses frequencies up to 500 MHz.

Find: (a) The Cat 5e attenuation at 100 MHz over the 70 m run, (b) the Cat 6A attenuation at 500 MHz over the 70 m run, and (c) the bandwidth-distance product for each cable category at the rated frequency.

Solution:

(a) Cat 5e at 100 MHz: $22.0 \times (70/100) = 15.4$ dB

(b) Cat 6A at 500 MHz: $32.8 \times (70/100) = 22.96$ dB

(c) Bandwidth-distance products: Cat 5e: 100 MHz $\times$ 100 m = 10,000 MHz·m = 10 GHz·m Cat 6A: 500 MHz $\times$ 100 m = 50,000 MHz·m = 50 GHz·m

Cat 6A provides 5× the bandwidth-distance product of Cat 5e. Only Cat 6A (or Cat 6 with alien crosstalk shielding) supports 10GBASE-T, which requires the full 500 MHz bandwidth and the superior NEXT/PSANEXT performance specified for Category 6A.

Problem 15.2.7

Given: A CATV distribution system uses RG-6 coaxial cable (75 Ω, foam PE dielectric $\varepsilon_r = 1.45$) to deliver cable television signals from a tap to a set-top box over a 30 m in-home run. The signal at the tap is +10 dBmV at 600 MHz. The cable attenuation is 6.8 dB/100 m at 600 MHz. The set-top box requires a minimum of −5 dBmV for reliable reception.

Find: (a) The cable attenuation over 30 m, (b) the signal level at the set-top box, (c) the margin above the minimum receiver sensitivity, and (d) whether a 2-way splitter (3.5 dB insertion loss) can be added to serve a second TV.

Solution:

(a) Cable attenuation = $6.8 \times (30/100) = 2.04$ dB

(b) Signal level = $10 - 2.04 = +7.96$ dBmV

(c) Margin = $7.96 - (-5) = 12.96$ dB

(d) With a 2-way splitter: signal at each output = $7.96 - 3.5 = +4.46$ dBmV. Margin = $4.46 - (-5) = 9.46$ dB → Yes, the splitter can be added with 9.46 dB of margin remaining. The system can support additional splitting if needed.

Chapter 15 — Section 15.3: Physical Media: Fiber Optics

Practice problems covering fiber structure, numerical aperture, single-mode and multimode fiber, optical transmitters/receivers, link budgets, and OTDR troubleshooting.

Problem 15.3.1

Given: A multimode step-index fiber has a core refractive index $n_{core} = 1.492$ and a cladding refractive index $n_{clad} = 1.480$.

Find: (a) The numerical aperture, (b) the critical angle for total internal reflection, (c) the maximum acceptance half-angle in air, and (d) the relative refractive index difference $\Delta = (n_{core} - n_{clad}) / n_{core}$.

Solution:

(a) $NA = \sqrt{(n_{core}^2 - n_{clad}^2)} = \sqrt{(1.492^2 - 1.480^2)} = \sqrt{(2.2261 - 2.1904)} = \sqrt{0.03570} = 0.189$

(b) $\sin \theta_c = n_{clad} / n_{core} = 1.480 / 1.492 = 0.99196$ $\theta_c = \sin^{-1}(0.99196) = 82.73°$

(c) $\sin \theta_a = NA = 0.189$ $\theta_a = \sin^{-1}(0.189) = 10.90°$

(d) $\Delta = (1.492 - 1.480) / 1.492 = 0.012 / 1.492 = 0.00804 = 0.804\%$

This is a relatively large NA (0.189) typical of standard multimode fiber, making it easier to couple light from LEDs and large-core sources.

Problem 15.3.2

Given: A single-mode fiber link at 1310 nm spans 50 km. The fiber has attenuation of 0.35 dB/km and chromatic dispersion of 3.5 ps/(nm·km). The DFB laser source has a spectral width of 0.2 nm. The system operates at 2.5 Gbps (OC-48/STM-16).

Find: (a) The total fiber attenuation, (b) the total chromatic dispersion (pulse broadening), (c) whether the dispersion causes intersymbol interference (compare to the bit period), and (d) the maximum distance at 10 Gbps before dispersion exceeds the bit period.

Solution:

(a) Total attenuation = $0.35 \times 50 = 17.5$ dB

(b) $\Delta\tau = D \times L \times \Delta\lambda = 3.5 \times 50 \times 0.2 = 35$ ps

(c) Bit period at 2.5 Gbps = $1 / (2.5 \times 10^9) = 400$ ps. 35 ps < 400 ps → No ISI. The pulse broadening is only 8.75% of the bit period.

(d) At 10 Gbps, bit period = 100 ps. Maximum distance: L_{max} = bit period / $(D \times \Delta\lambda) = 100 / (3.5 \times 0.2) = 142.9$ km

At 1310 nm with low dispersion (3.5 ps/nm·km), the link is attenuation-limited rather than dispersion-limited at both 2.5 and 10 Gbps for distances under 50 km.

Problem 15.3.3

Given: An OM3 multimode fiber has a bandwidth-distance product of 2,000 MHz·km at 850 nm. A data center link is 200 m long.

Find: (a) The modal bandwidth at 200 m, (b) the maximum NRZ bit rate using BW ≈ 0.7 × bit rate, and (c) whether the fiber can support 25GBASE-SR (25 Gbps using NRZ) at this distance.

Solution:

(a) BW = 2,000 / 0.200 = 10,000 MHz = 10.0 GHz

(b) Bit rate ≈ BW / 0.7 = 10,000 / 0.7 = 14,286 Mbps ≈ 14.3 Gbps

(c) 25 Gbps > 14.3 Gbps → No, OM3 cannot support 25GBASE-SR at 200 m with NRZ signaling. The IEEE 802.3 standard specifies 25GBASE-SR maximum reach on OM3 as 70 m. OM4 (4,700 MHz·km) extends the reach to approximately 100 m.

Problem 15.3.4

Given: A QSFP28 transceiver for 100GBASE-LR4 uses four wavelengths (1295.56 nm, 1300.05 nm, 1304.58 nm, 1309.14 nm), each carrying 25 Gbps. The minimum per-lane transmit power is −4.3 dBm and the receiver sensitivity is −10.6 dBm at BER = 10^{-12} (with FEC).

Find: (a) The per-lane power budget, (b) the maximum fiber distance at 0.35 dB/km (1310 nm window) with 2 connectors (0.5 dB each) and 1 dB system margin, and (c) the total aggregate data rate.

Solution:

(a) Power budget = $P_{tx,min} - P_{rx,sens} = (-4.3) - (-10.6) = 6.3$ dB

(b) Available for fiber = $6.3 - 2 \times 0.5 - 1.0 = 6.3 - 2.0 = 4.3$ dB Maximum distance = 4.3 / 0.35 = 12.3 km

The 100GBASE-LR4 standard specifies 10 km maximum reach, which is within this calculated budget.

(c) Total data rate = $4 \times 25 = 100$ Gbps

Problem 15.3.5

Given: A 60 km single-mode fiber link at 1550 nm connects two buildings. Components: - Transmitter: +5 dBm output - Receiver sensitivity: -22 dBm at BER = 10^{-12} - Fiber attenuation: 0.22 dB/km at 1550 nm - Connectors: 4 pairs at 0.4 dB each - Fusion splices: 8 at 0.08 dB each - System margin: 3 dB

Find: (a) The total link loss, (b) the available power budget, (c) the link margin after all losses, and (d) whether the link closes.

Solution:

(a) Total link loss: Fiber: $0.22 \times 60 = 13.20$ dB Connectors: $4 \times 0.4 = 1.60$ dB Splices: $8 \times 0.08 = 0.64$ dB Total $= 13.20 + 1.60 + 0.64 = 15.44$ dB

(b) Power budget $= P_{tx} - P_{rx,sens} = (+5) - (-22) = 27.0$ dB

(c) Margin = Budget − Total loss − System margin = $27.0 - 15.44 - 3.0 = 8.56$ dB

(d) Since 8.56 dB > 0, the link closes with 8.56 dB of excess margin. This margin allows for future splice repairs (each adding approximately 0.1 dB) and fiber aging.

Problem 15.3.6

Given: An OTDR tests a 15 km single-mode fiber at 1310 nm (group refractive index n = 1.4677) using a 10 ns pulse width. The trace shows: fiber launch at 0 m, fusion splice at 5.3 km (0.04 dB loss), a connector at 10.1 km (0.35 dB loss with reflection), and fiber end at 15.0 km. Fiber attenuation slope is 0.34 dB/km.

Find: (a) The spatial resolution (one-way pulse length in the fiber), (b) the total end-to-end link loss, and (c) the round-trip time to the far end of the fiber.

Solution:

(a) $v_{fiber} = c / n = 3.0 \times 10^8 / 1.4677 = 2.044 \times 10^8$ m/s Pulse length = $v_{fiber} \times$ pulse width $= 2.044 \times 10^8 \times 10 \times 10^{-9} = 2.044$ m

Events closer than approximately 2 m cannot be individually resolved with this pulse width.

(b) Total link loss: Fiber: $0.34 \times 15.0 = 5.10$ dB Fusion splice: 0.04 dB Connector: 0.35 dB Total $= 5.10 + 0.04 + 0.35 = 5.49$ dB

(c) Round-trip time $= 2 \times$ distance / $v_{fiber} = 2 \times 15{,}000 / (2.044 \times 10^8) = 30{,}000 / (2.044 \times 10^8) = 146.8\ \mu s$

Problem 15.3.7

Given: A fiber link upgrade replaces a PIN photodiode receiver (sensitivity −18 dBm) with an APD receiver (sensitivity −28 dBm). The transmitter output is +2 dBm. The total link loss is 24 dB including fiber, connectors, splices, and system margin.

Find: (a) The link margin with the PIN receiver, (b) the link margin with the APD receiver, (c) whether each receiver allows the link to close, and (d) how many additional kilometers of fiber (at 0.22 dB/km) the APD receiver enables.

Solution:

(a) PIN margin = $(P_{tx} - P_{rx,PIN})$ − link loss = (2 − (−18)) − 24 = 20 − 24 = −4 dB

(b) APD margin = $(P_{tx} - P_{rx,APD})$ − link loss = (2 − (−28)) − 24 = 30 − 24 = +6 dB

(c) PIN: margin is negative → Link does not close (insufficient budget). APD: margin is positive → Link closes with 6 dB margin.

(d) Additional fiber distance = (sensitivity improvement) / attenuation = (28 − 18) / 0.22 = 10 / 0.22 = 45.5 km

The APD receiver provides 10 dB more sensitivity, enabling approximately 45 km more reach.

Chapter 15 — Section 15.4: Dense Wavelength Division Multiplexing (DWDM)

Practice problems covering WDM channel spacing, EDFA amplifier design, and DWDM system capacity planning.

Problem 15.4.1

Given: A CWDM system uses 18 channels with 20 nm spacing across the 1270–1610 nm range. Each channel carries 10 Gbps. A DWDM system uses the ITU-T 50 GHz grid in the C-band from 191.35 THz to 196.10 THz.

Find: (a) The total CWDM capacity, (b) the number of DWDM channels on the 50 GHz grid, (c) the DWDM wavelength range, and (d) the channel spacing in nm at the center of the C-band.

Solution:

(a) CWDM capacity = 18×10 Gbps = 180 Gbps

(b) DWDM channels: Bandwidth = $196.10 - 191.35 = 4.75$ THz = 4,750 GHz Channels = 4,750 / 50 + 1 = 96 channels

(c) $\lambda_{min} = c / f_{max} = 3 \times 10^8 / (196.10 \times 10^{12}) = 1{,}529.8$ nm $\lambda_{max} = c / f_{min} = 3 \times 10^8 / (191.35 \times 10^{12}) = 1{,}567.8$ nm Wavelength range: 1,529.8 nm to 1,567.8 nm (38.0 nm span)

(d) $f_{center} = (191.35 + 196.10) / 2 = 193.725$ THz $\Delta\lambda = c \times \Delta f / f^2 = (3 \times 10^8 \times 50 \times 10^9) / (193.725 \times 10^{12})^2 = 1.5 \times 10^{19} / (3.753 \times 10^{28}) = 0.400$ nm ≈ 0.4 nm

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Problem 15.4.2

Given: A long-haul DWDM link uses 8 inline EDFAs. Each EDFA has a gain of 22 dB and a noise figure of 6.0 dB. The input signal power per channel is -3 dBm. The span loss between amplifiers equals the EDFA gain.

Find: (a) The OSNR at the receiver using OSNR $\approx P_{in} - NF - 10 \log_{10}(N) + 58$ dBm, (b) whether the system can support 200 Gbps DP-16QAM coherent detection (required OSNR $\approx$ 18 dB), and (c) the maximum number of amplifiers before OSNR drops below 18 dB.

Solution:

(a) OSNR = P_{in} − NF − 10 $\log_{10}$(N) + 58 = −3 − 6.0 − 10 $\log_{10}$(8) + 58 = −3 − 6.0 − 9.03 + 58 = 39.97 dB ≈ 40.0 dB

(b) 40.0 dB » 18 dB → Yes, the system has 22 dB of OSNR margin above the 200G DP-16QAM requirement.

(c) Required: P_{in} − NF − 10 $\log_{10}$(N) + 58 ≥ 18 −3 − 6.0 − 10 $\log_{10}$(N) + 58 ≥ 18 49 − 10 $\log_{10}$(N) ≥ 18 10 $\log_{10}$(N) ≤ 31 $\log_{10}$(N) ≤ 3.1 N ≤ $10^{3.1}$ = 1,259

Maximum amplifiers = 1,259. At 22 dB gain per span (equivalent to approximately 100 km span at 0.22 dB/km), this corresponds to approximately 125,900 km – clearly limited by other factors (fiber nonlinearities) long before OSNR exhaustion. In practice, nonlinear effects limit coherent systems to approximately 30-50 amplified spans.

Problem 15.4.3

Given: A metro DWDM ring carries 40 channels at 200 Gbps each (DP-16QAM coherent) over a ring circumference of 400 km. The EDFA amplifier spacing is 50 km. Fiber attenuation is 0.20 dB/km at 1550 nm.

Find: (a) The total system capacity, (b) the number of amplifier sites on the ring, (c) the span loss each EDFA must compensate, and (d) the total fiber-only attenuation around the full ring.

Solution:

(a) Total capacity = 40 × 200 Gbps = 8,000 Gbps = 8 Tbps

(b) Amplifier sites = ring circumference / span length = 400 / 50 = 8 sites

(c) Span loss = 0.20 × 50 = 10.0 dB per span. Each EDFA must provide at least 10 dB gain.

(d) Total ring attenuation = 0.20 × 400 = 80.0 dB

Problem 15.4.4

Given: A submarine DWDM cable system operates over 6,000 km with 80 channels at 100 Gbps each (DP-QPSK). EDFA amplifier spacing is 60 km. Each EDFA provides 12 dB gain with a 4.5 dB noise figure. Launch power per channel is 0 dBm.

Find: (a) The total system capacity, (b) the number of inline amplifiers, (c) the span loss, and (d) the OSNR at the far end.

Solution:

(a) Capacity = 80 × 100 = 8,000 Gbps = 8 Tbps

(b) Spans = 6,000 / 60 = 100. Inline amplifiers = 100 − 1 = 99 (plus booster and pre-amp).

(c) Span loss = 0.20 × 60 = 12.0 dB (matching the EDFA gain — the system is in gain equilibrium).

(d) OSNR = P_{in} − NF − 10 $\log_{10}$(N) + 58 N = 100 (total amplifiers including booster) = 0 − 4.5 − 10 $\log_{10}$(100) + 58 = 0 − 4.5 − 20.0 + 58 = 33.5 dB

For 100G DP-QPSK, required OSNR ≈ 12 dB. The system has 21.5 dB of OSNR margin, which is consumed in practice by fiber nonlinearities, component aging, and repair margin.

Problem 15.4.5

Given: A DWDM system is being upgraded from 100 GHz channel spacing (45 channels at 100 Gbps each) to 50 GHz spacing with 400 Gbps per channel using DP-16QAM coherent transceivers. The fiber infrastructure and amplifiers remain the same.

Find: (a) The original system capacity, (b) the new number of channels at 50 GHz spacing (same C-band from 191.7 THz to 196.1 THz), (c) the new system capacity, and (d) the capacity increase factor.

Solution:

(a) Original capacity = 45 × 100 = 4,500 Gbps = 4.5 Tbps

(b) Bandwidth = 196.1 − 191.7 = 4.4 THz = 4,400 GHz New channels = 4,400 / 50 + 1 = 89 channels

(c) New capacity = 89 × 400 = 35,600 Gbps = 35.6 Tbps

(d) Capacity increase = 35,600 / 4,500 = 7.91×

The combination of halving the channel spacing (approximately doubling channels from 45 to 89) and quadrupling the per-channel rate (100 to 400 Gbps) yields nearly 8× total capacity increase.

Chapter 15 — Section 15.5: Ethernet

Practice problems covering Ethernet frame structure, MAC addressing, physical standards, Power over Ethernet, and Spanning Tree Protocol.

Problem 15.5.1

Given: A network monitoring tool captures 10 seconds of traffic on a 10 Gbps Ethernet link. During that interval, 1,200,000 frames are captured. The frame size distribution is: 40% at 64 bytes, 25% at 576 bytes, and 35% at 1,518 bytes (all sizes exclude preamble/SFD and IFG).

Find: (a) The weighted average frame size, (b) the total data volume captured, (c) the average link utilization, and (d) the average frame rate in frames per second.

Solution:

(a) Average frame size = $0.40 \times 64 + 0.25 \times 576 + 0.35 \times 1{,}518 = 25.6 + 144.0 + 531.3 = 700.9$ bytes

(b) Total data = $1{,}200{,}000 \times 700.9 = 841{,}080{,}000$ bytes = 841.1 MB

(c) On-wire average size (add 8 preamble + 12 IFG) = $700.9 + 20 = 720.9$ bytes Total on-wire bits = $1{,}200{,}000 \times 720.9 \times 8 = 6{,}920{,}640{,}000$ bits Utilization = $6{,}920{,}640{,}000 / (10 \times 10^9 \times 10) = 6.92 \times 10^9 / 10^{11} = 6.92\%$

(d) Frame rate = $1{,}200{,}000 / 10 = 120{,}000$ fps

Problem 15.5.2

Given: A data center interconnect uses 100GBASE-SR4 optics (4 lanes × 25 Gbps) over OM4 multimode fiber. Each lane uses an 850 nm VCSEL. Per-lane minimum transmit power is −8.4 dBm and receiver sensitivity is −10.3 dBm. Fiber attenuation at 850 nm is 3.5 dB/km.

Find: (a) The per-lane power budget, (b) the maximum fiber distance with 2 MPO connectors (0.75 dB each), and (c) the total aggregate throughput.

Solution:

(a) Power budget = $(-8.4) - (-10.3) = 1.9$ dB

(b) Available for fiber = $1.9 - 2 \times 0.75 = 1.9 - 1.5 = 0.4$ dB Max distance = $0.4 / 3.5 = 0.114$ km = 114 m

The 100GBASE-SR4 standard specifies 100 m maximum on OM4, consistent with this tight power budget. Modal bandwidth, not just attenuation, is the primary distance limiter.

(c) Total throughput = $4 \times 25 = 100$ Gbps

Problem 15.5.3

Given: A PoE++ switch port (IEEE 802.3bt Type 4) delivers 90 W to a high-power LED lighting fixture over a 50-meter Cat 6A cable run. The DC resistance of Cat 6A is 7.0 Ω/100 m per conductor. All four pairs carry power. The PSE output voltage is 52 V.

Find: (a) The round-trip resistance per pair, (b) the total cable resistance with four pairs in parallel, (c) the current drawn, (d) the cable power loss, and (e) the power and voltage delivered to the PD.

Solution:

(a) $R_{pair} = 2 \times 7.0 \times (50/100) = 2 \times 3.5 = 7.0\ \Omega$ per pair

(b) Four pairs in parallel: $R_{cable} = 7.0 / 4 = 1.75\ \Omega$

(c) $I = P / V = 90 / 52 = 1.731$ A

(d) $P_{loss} = I^2 \times R_{cable} = 1.731^2 \times 1.75 = 2.996 \times 1.75 = 5.24$ W

(e) Power at PD: $P_{PD} = 90 - 5.24 = 84.76$ W Voltage at PD: $V_{PD} = 52 - (1.731 \times 1.75) = 52 - 3.03 = 48.97$ V Efficiency: $\eta = 84.76 / 90 = 94.2\%$

The PD voltage of 48.97 V is well above the 42.5 V minimum required by 802.3bt.

Problem 15.5.4

Given: Five switches (S1-S5) are connected as follows: S1-S2 (10 Gbps, cost 2), S2-S3 (10 Gbps, cost 2), S3-S4 (1 Gbps, cost 4), S4-S5 (10 Gbps, cost 2), S5-S1 (10 Gbps, cost 2). Bridge priorities: S1 = 4096, S2 = 8192, S3 = 16384, S4 = 32768, S5 = 32768.

Find: (a) The RSTP root bridge, (b) the root port on each non-root switch, (c) the path cost from each switch to the root, and (d) which port is placed in the alternate (blocking) state.

Solution:

(a) S1 has the lowest bridge priority (4096), so S1 is the root bridge.

(b) Root port selection (lowest cost path to root):

- S2: port facing S1 (direct, cost 2) vs. via S5-S4-S3 (cost 2+2+4+2=10). Root port = port facing S1, cost 2.
- S5: port facing S1 (direct, cost 2) vs. via S2-S3-S4 (cost 2+2+4+2=10). Root port = port facing S1, cost 2.

- S3: via S2 (cost 2+2=4) vs. via S4-S5 (cost 4+2+2=8). Root port = port facing S2, cost 4.
- S4: via S3-S2 (cost 4+2+2=8) vs. via S5 (cost 2+2=4). Root port = port facing S5, cost 4.

(c) Path costs to root: S1 = 0, S2 = 2, S5 = 2, S3 = 4, S4 = 4.

(d) The link S3-S4 is the redundant link. On that segment:

- S3's root path cost = 4, S4's root path cost = 4 (tie).
- Tiebreaker: lower bridge ID. S3 priority (16384) < S4 priority (32768).
- S3's port facing S4 is designated (forwarding), S4's port facing S3 is alternate (blocking).

Active topology: S1-S2-S3 and S1-S5-S4, with S3-S4 blocked.

Problem 15.5.5

Given: A PoE system uses 802.3af (Type 1, 15.4 W at PSE) to power a VoIP phone over a 90 m Cat 5e cable. Cat 5e DC resistance is 9.38 Ω/100 m per conductor. Power is delivered on two pairs (Mode A). The PSE voltage is 48 V. The phone requires 8 W minimum to operate.

Find: (a) The total cable resistance for two-pair power delivery, (b) the current drawn, (c) the cable power loss, (d) the power at the PD, and (e) whether the phone can operate.

Solution:

(a) $R_{pair} = 2 \times 9.38 \times (90/100) = 2 \times 8.442 = 16.884\ \Omega$ Two pairs in parallel: $R_{cable} = 16.884 / 2 = 8.442\ \Omega$

(b) $I = P / V = 15.4 / 48 = 0.321$ A

(c) $P_{loss} = I^2 \times R = 0.321^2 \times 8.442 = 0.1030 \times 8.442 = 0.870$ W

(d) $P_{PD} = 15.4 - 0.870 = 14.53$ W

(e) 14.53 W > 8.0 W → Yes, the phone can operate with 6.53 W of margin. The 802.3af standard guarantees 12.95 W at the PD; this installation delivers 14.53 W.

Problem 15.5.6

Given: An Ethernet frame with a 46-byte payload (minimum) is transmitted on a 1 Gbps link. The total on-wire frame includes: 8 bytes preamble/SFD, 14 bytes header, 46 bytes payload, 4 bytes FCS, and 12 bytes IFG.

Find: (a) The total on-wire bytes, (b) the transmission time for one minimum frame, (c) the maximum frame rate at wire speed, and (d) the payload throughput at minimum frame size as a percentage of line rate.

Solution:

(a) On-wire bytes = 8 + 14 + 46 + 4 + 12 = 84 bytes = 672 bits

(b) Transmission time = $672 / (1 \times 10^9) = 672$ ns

(c) Max frame rate = 1×10^9 / 672 = 1,488,095 fps

(d) Payload throughput = 1,488,095 × 46 × 8 = 547,738,095 bps = 547.7 Mbps Percentage of line rate = 547.7 / 1,000 = 54.8%

At minimum frame size, Ethernet achieves less than 55% efficiency. This is why small-packet forwarding is the most demanding test for switch/router performance.

Chapter 15 — Section 15.6: Internet Protocol (IP)

Practice problems covering IPv4 subnetting, IPv6 addressing, OSPF/BGP routing, NAT/PAT, and CIDR route aggregation.

Problem 15.6.1

Given: An ISP assigns a customer the network 192.168.100.0/24. The customer has four departments: Engineering (50 hosts), Sales (25 hosts), Accounting (10 hosts), and Management (5 hosts). VLSM (Variable Length Subnet Masks) will be used to allocate the smallest subnets that meet each department's needs.

Find: (a) The prefix length and subnet for each department, (b) the network address, broadcast address, and usable host range for each, and (c) the total number of IP addresses used versus wasted.

Solution:

(a) Allocate largest subnet first: Engineering (50 hosts): $2^n - 2 \geq 50 \rightarrow n = 6$ (62 hosts) → /26 Sales (25 hosts): $2^n - 2 \geq 25 \rightarrow n = 5$ (30 hosts) → /27 Accounting (10 hosts): $2^n - 2 \geq 10 \rightarrow n = 4$ (14 hosts) → /28 Management (5 hosts): $2^n - 2 \geq 5 \rightarrow n = 3$ (6 hosts) → /29

(b) Subnet allocation (in order of decreasing size): Engineering: 192.168.100.0/26 Network: 192.168.100.0, Broadcast: 192.168.100.63, Hosts: 192.168.100.1 – 192.168.100.62 (62 usable)

Sales: 192.168.100.64/27 Network: 192.168.100.64, Broadcast: 192.168.100.95, Hosts: 192.168.100.65 – 192.168.100.94 (30 usable)

Accounting: 192.168.100.96/28 Network: 192.168.100.96, Broadcast: 192.168.100.111, Hosts: 192.168.100.97 – 192.168.100.110 (14 usable)

Management: 192.168.100.112/29 Network: 192.168.100.112, Broadcast: 192.168.100.119, Hosts: 192.168.100.113 – 192.168.100.118 (6 usable)

(c) Total addresses used $= 64 + 32 + 16 + 8 = 120$. Usable hosts $= 62 + 30 + 14 + 6 = 112$. Required hosts $= 50 + 25 + 10 + 5 = 90$. Wasted usable addresses $= 112 - 90 = 22$ addresses Remaining unallocated from /24 $= 256 - 120 = 136$ addresses (192.168.100.120 – 192.168.100.255)

Problem 15.6.2

Given: A server has the MAC address A4:83:E7:2F:00:1B and is on the IPv6 subnet 2001:0db8:cafe:0001::/64.

Find: (a) The full 128-bit IPv6 address using Modified EUI-64, (b) the abbreviated notation, and (c) the total number of addresses in the /64 subnet.

Solution:

(a) Step 1 — Split MAC and insert FFFE: A4:83:E7 → A4:83:E7:FF:FE:2F:00:1B

Step 2 — Flip the 7th bit (U/L bit) of the first byte: A4 = 1010 0100 → flip bit 6 → 1010 0110 = A6

Modified: A6:83:E7:FF:FE:2F:00:1B

Step 3 — Form interface ID (16-bit groups): A683:E7FF:FE2F:001B

Step 4 — Combine: Full: 2001:0db8:cafe:0001:A683:E7FF:FE2F:001B

(b) Abbreviated: 2001:db8:cafe:1:a683:e7ff:fe2f:1b

(c) Addresses in /64: $2^{64} = 1.844 \times 10^{19}$ addresses (18.4 quintillion)

Problem 15.6.3

Given: A network has two paths from router R1 to destination prefix 10.20.0.0/16: - Path A: OSPF route, cost 150 (AD = 110) - Path B: iBGP route, AS path length 2, local preference 200 (AD = 200) - Path C: Static route via next-hop 10.0.0.1 (AD = 1)

All three paths are available simultaneously.

Find: (a) Which route is installed in the routing table and why, (b) what happens if the static route is removed, and (c) what happens if both the static and OSPF routes are removed.

Solution:

(a) The route with the lowest administrative distance is preferred: Static (AD = 1), OSPF (AD = 110), iBGP (AD = 200). The static route (Path C) is installed, with AD = 1 being the lowest.

(b) Without the static route, the remaining options are OSPF (110) and iBGP (200). The OSPF route (Path A) is installed, with AD = 110 < 200.

(c) With only iBGP remaining, the iBGP route (Path B) is installed as the sole option.

This demonstrates the administrative distance hierarchy: static (1) > OSPF (110) > iBGP (200). The BGP local preference of 200 is only used for path selection within BGP itself, not for comparison against other routing protocols.

Problem 15.6.4

Given: A medium-sized office has 150 workstations and 30 servers, all using private addresses in the 10.1.0.0/16 space. The ISP provides a pool of 8 public IPv4 addresses (203.0.113.8/29) for dynamic NAT of servers and a single additional public IP (203.0.113.16) for PAT of all workstations. During peak hours, each workstation averages 60 simultaneous sessions and each server averages 200 sessions.

Find: (a) The total sessions for workstations on the PAT address, (b) the PAT port utilization, (c) the total sessions for servers on the dynamic NAT pool, and (d) whether the 8-address NAT pool is sufficient for the 30 servers.

Solution:

(a) Workstation sessions = $150 \times 60 = 9{,}000$ simultaneous NAT entries

(b) Ephemeral ports per protocol: $65{,}535 - 1{,}024 + 1 = 64{,}512$. For TCP + UDP: $2 \times 64{,}512 = 129{,}024$ mappings. Utilization = 9,000 / 129,024 = 6.98% — well within capacity.

(c) Server sessions: Each server with dynamic NAT gets a dedicated public IP while active. Total server sessions = $30 \times 200 = 6{,}000$ sessions across the pool.

(d) Usable addresses in /29: $2^3 - 2 = 6$ (network and broadcast excluded). With 30 servers needing public addresses but only 6 available, dynamic NAT can serve a maximum of 6 servers simultaneously. The remaining 24 must wait or use PAT instead. To support all 30 servers with dedicated public IPs, a /27 (30 usable addresses) or larger block is needed.

Problem 15.6.5

Given: An ISP owns eight contiguous /24 networks: 198.51.100.0/24 through 198.51.107.0/24.

Find: (a) The CIDR summary route covering all eight networks, (b) verification that the block starts on the correct boundary, (c) the network address, broadcast address, and total host count for the aggregate, and (d) how many BGP routing table entries this aggregation saves.

Solution:

(a) Eight /24 networks: $2^k = 8$, so $k = 3$. Summary prefix = /24 − 3 = /21. Summary route: 198.51.100.0/21

(b) Verify alignment: third octet 100 in binary = 01100100. A /21 mask means the first 21 bits are network bits. The first 16 bits cover octets 1-2 (198.51). Octet 3 contributes 5 bits to the network. 100 = 01100100 — the last 3 bits (100 = 4) must be the start of the variable portion. 100 AND $(256 - 8) = 100$ AND $248 = 96 \neq 100$.

Wait — let me recheck: 198.51.100.0 through 198.51.107.255. The third octet ranges from 100 to 107. 100 in binary: 01100100, 107: 01101011. These span from 100 to 107 = range of 8. For a /21 block: the third octet mask is 11111000 = 248. 100 AND 248 = 96, not 100. The /21 boundary would start at 198.51.96.0, not 198.51.100.0.

The eight /24s starting at 100 do NOT align to a /21 boundary. The correct aggregate requires checking alignment: 104 AND 248 = 104. So 198.51.104.0/21 covers 104-111. For 100-107: this needs to be split into 198.51.100.0/22 (100-103) and 198.51.104.0/22 (104-107) = two /22 routes.

(c) 198.51.100.0/22: Network 198.51.100.0, Broadcast 198.51.103.255, Hosts = 1,022. 198.51.104.0/22: Network 198.51.104.0, Broadcast 198.51.107.255, Hosts = 1,022. Total hosts = 2,044

(d) Aggregation reduces 8 routes to 2 routes, saving 6 BGP table entries. A single /21 aggregate is not possible due to misalignment.

Problem 15.6.6

Given: A host has IP address 172.20.45.130 with a subnet mask of /20.

Find: (a) The subnet mask in dotted-decimal notation, (b) the network address, (c) the broadcast address, (d) the usable host range, and (e) the total number of usable hosts.

Solution:

(a) /20 mask: 20 ones followed by 12 zeros. 11111111.11111111.11110000.00000000 = 255.255.240.0

(b) Network address: 172.20.45.130 AND 255.255.240.0 Third octet: 45 AND 240 = 00101101 AND 11110000 = 00100000 = 32 Network = 172.20.32.0

(c) Broadcast: set all 12 host bits to 1. Third octet: 32 OR 15 = 47. Fourth octet: 255. Broadcast = 172.20.47.255

(d) Usable range: 172.20.32.1 – 172.20.47.254

(e) Usable hosts: $2^{12} - 2 = 4{,}096 - 2 = 4{,}094$ hosts

Chapter 15 — Section 15.7: Transport Protocols

Practice problems covering TCP throughput and window sizing, UDP overhead, socket connections, WSGI server capacity, and QUIC protocol comparisons.

Problem 15.7.1

Given: A data center replication job uses TCP to transfer a 2 GB file between sites connected by a 10 Gbps link with RTT = 5 ms. The TCP receive window is 4 MB (with window scaling enabled). Assume no packet loss or congestion.

Find: (a) The bandwidth-delay product of the link, (b) the maximum TCP throughput, (c) whether the window is large enough to saturate the link, and (d) the minimum transfer time.

Solution:

(a) BDP = $10 \times 10^9 \times 0.005$ = 50,000,000 bits = 6,250,000 bytes = 6.25 MB

(b) Throughput = Window / RTT = $4 \times 10^6 / 0.005$ = 800,000,000 bytes/s = $800 \times 10^6 \times 8$ = 6,400,000,000 bps = 6.4 Gbps

(c) BDP = 6.25 MB, Window = 4 MB. Since 4 MB < 6.25 MB, the window is not large enough to saturate the 10 Gbps link. A window of at least 6.25 MB is needed.

(d) At 6.4 Gbps: Transfer time = $2 \times 10^9 \times 8 / (6.4 \times 10^9) = 16 \times 10^9 / 6.4 \times 10^9$ = 2.5 seconds

With a 6.25 MB window (saturating the link): time = $16 \times 10^9 / (10 \times 10^9)$ = 1.6 seconds.

Problem 15.7.2

Given: A video surveillance system streams 16 IP cameras, each using H.265 encoding at 4 Mbps. Each camera sends 1,258-byte RTP/UDP/IP packets (1,200 bytes video payload + 12 bytes RTP + 8 bytes UDP + 20 bytes IPv4 + 14 bytes Ethernet header + 4 bytes FCS). Packets are sent every 2.4 ms (matching the video frame slicing).

Find: (a) The total video bitrate for all 16 cameras, (b) the packet rate per camera, (c) the total network bandwidth including all headers, and (d) the header overhead percentage.

Solution:

(a) Total video bitrate = 16×4 Mbps = 64 Mbps

(b) Packets per second per camera: Payload per packet = 1,200 bytes = 9,600 bits Packets/s = 4×10^6 / 9,600 = 416.7 pps

Alternatively: 1 / 0.0024 = 416.7 pps.

(c) Total frame size = 1,258 bytes (without preamble/IFG) per the given breakdown: Actually: 1,200 + 12 + 8 + 20 + 14 + 4 = 1,258 bytes. With preamble (8) and IFG (12): 1,278 bytes on wire.

Per camera bandwidth = $416.7 \times 1{,}278 \times 8$ = 4,259,200 bps = 4.26 Mbps Total 16 cameras = 16×4.26 = 68.1 Mbps

(d) Overhead per packet = 1,258 − 1,200 = 58 bytes. Percentage = (68.1 − 64) / 68.1 = 4.1 / 68.1 = 6.0%

The overhead is modest for these large video packets compared to the 22%+ overhead seen with small VoIP packets.

Problem 15.7.3

Given: A web server at 10.0.1.100 listens on port 443 (HTTPS). During a load test, 5,000 clients connect simultaneously. Each client uses a unique ephemeral port from the range 49152–65535. Client IPs are distributed across 50 unique source addresses.

Find: (a) The 4-tuple for a specific connection from client 192.168.5.20 port 51234, (b) the total number of sockets held by the server process, (c) the average connections per client IP, and (d) whether the ephemeral port range can support the load from a single client IP.

Solution:

(a) 4-tuple: (192.168.5.20, 51234, 10.0.1.100, 443)

(b) Server sockets = 1 listening socket + 5,000 connected sockets = 5,001 sockets

(c) Average connections per IP = 5,000 / 50 = 100 connections per client IP

(d) Ephemeral range = 65,535 − 49,152 + 1 = 16,384 ports. 100 connections requires 100 unique ports from the 16,384 available. 100 / 16,384 = 0.6% → Yes, easily supported. Even all 5,000 connections from a single IP (5,000/16,384 = 30.5%) would be feasible.

Problem 15.7.4

Given: A Gunicorn WSGI deployment uses async gevent workers. Each worker can handle 100 concurrent connections (greenlets). The server has 4 workers. Average request processing time is 200 ms, but 60% of that time is I/O wait (database queries, API calls) during which gevent yields to other greenlets.

Find: (a) The total concurrent connections the server can handle, (b) the effective per-worker throughput accounting for I/O concurrency, (c) the total server RPS, and (d) the number of workers needed for 2,000 RPS.

Solution:

(a) Total concurrent connections = $4 \times 100 = 400$ connections

(b) Each request takes 200 ms wall clock. During 120 ms (60%) of I/O wait, the greenlet yields and other requests are processed. The effective CPU time per request is 80 ms. Per-worker throughput = 100 greenlets $\times$ (1 / 0.200 s) = 500 RPS per worker (bounded by concurrency limit)

Alternatively, CPU-limited: with 80 ms CPU per request, one CPU can process 1/0.080 = 12.5 sequential requests/s, but with 100 concurrent greenlets overlapping I/O, throughput = min(100/0.200, 1/0.080) $\times$ concurrency factor.

The effective rate per worker: each greenlet completes in 200 ms, so 100 greenlets produce 100/0.200 = 500 RPS (assuming the CPU can keep up with 100×80 ms / 1000 ms = 8 concurrent CPU-active tasks).

(c) Total RPS = 4×500 = 2,000 RPS

(d) Workers needed = 2,000 / 500 = 4 workers — the current deployment already meets the target.

Problem 15.7.5

Given: A mobile application loads a web page requiring 80 resources from the same server. The network has RTT = 60 ms. Compare connection setup and data transfer start for: (a) HTTP/1.1 with 6 parallel TCP connections, (b) HTTP/2 over TCP + TLS 1.3, and (c) HTTP/3 over QUIC with 0-RTT resumption.

Find: The time to first byte (TTFB) for the first resource, and the connection overhead for each protocol.

Solution:

(a) HTTP/1.1 with 6 parallel TCP connections: Each connection: 1 RTT (TCP handshake) + 1 RTT (TLS 1.3) = $2 \times 60 = 120$ ms setup. TTFB = 120 + 60 (request/response RTT) = 180 ms All 6 connections are established in parallel, so 6 connections finish setup at 120 ms. 80 resources / 6 connections = ~14 sequential requests per connection, each taking 60 ms. Total page load $\approx 120 + 14 \times 60 = 960$ ms

(b) HTTP/2 over TCP + TLS 1.3: Single connection: 1 RTT (TCP) + 1 RTT (TLS) = $2 \times 60 = 120$ ms setup TTFB = 120 + 60 = 180 ms All 80 resources multiplexed over one connection. Total page load $\approx 120 + 60 = 180$ ms (all requests sent immediately after setup, responses streamed back concurrently)

(c) HTTP/3 over QUIC (0-RTT resumption): 0-RTT: data is sent with the first packet. TTFB = 60 ms (one RTT for request/response, no handshake delay) Total page load $\approx$ 60 ms (all requests sent immediately in the first flight)

QUIC 0-RTT saves 120 ms compared to HTTP/2, a 67% reduction in connection overhead.

Problem 15.7.6

Given: A TCP connection with an initial congestion window of 10 segments (MSS = 1,460 bytes) undergoes slow start on a 1 Gbps link with 10 ms RTT. No packets are lost.

Find: (a) The data sent in the first RTT, (b) the congestion window after 4 RTTs, (c) the throughput achieved after 4 RTTs, and (d) the number of RTTs needed for the congestion window to reach the BDP.

Solution:

(a) First RTT: cwnd = 10 segments = $10 \times 1{,}460 = 14{,}600$ bytes = 116,800 bits Data sent = 14,600 bytes at rate = $14{,}600 \times 8 / 0.010 = 11.68$ Mbps

(b) Slow start doubles cwnd each RTT: RTT 1: cwnd = $10 \rightarrow 20$ (10 ACKs, each increases by 1) RTT 2: cwnd = $20 \rightarrow 40$ RTT 3: cwnd = $40 \rightarrow 80$ RTT 4: cwnd = $80 \rightarrow 160$ segments

(c) After 4 RTTs: cwnd = $160 \times 1{,}460 = 233{,}600$ bytes Throughput = $233{,}600 \times 8 / 0.010 = 186.9$ Mbps

(d) BDP = $1 \times 10^9 \times 0.010 / 8 = 1{,}250{,}000$ bytes. In segments: $1{,}250{,}000 / 1{,}460 = 856$ segments. After n RTTs: cwnd = 10×2^n. Need $10 \times 2^n \geq 856 \rightarrow 2^n \geq 85.6 \rightarrow n \geq \log_2(85.6) = 6.42$. After 7 RTTs (70 ms), cwnd = $10 \times 128 = 1{,}280$ segments = 1,868,800 bytes > BDP, and the link is saturated.

Chapter 15 — Section 15.8: Wireless Networking (Wi-Fi)

Practice problems covering Wi-Fi data rate calculations, RF propagation and link budgets, and wireless security.

Problem 15.8.1

Given: A Wi-Fi 7 (802.11be) access point operates on a 320 MHz channel in the 6 GHz band using 4096-QAM with a 5/6 coding rate. The OFDM parameters are: 3,984 data subcarriers, 12.8 μs symbol duration, and 0.8 μs guard interval. The device uses 2 spatial streams.

Find: (a) The bits per subcarrier per symbol, (b) the symbol rate, (c) the PHY data rate for a single spatial stream, and (d) the PHY data rate with 2 spatial streams.

Solution:

(a) Bits per subcarrier = $\log_2(4096) \times (5/6) = 12 \times 0.8333 = 10.0$ coded bits/subcarrier/symbol

(b) Symbol rate = 1 / (12.8 + 0.8) μs = 1 / 13.6 μs = 73,529 symbols/s

(c) Single stream rate = $3{,}984 \times 10.0 \times 73{,}529 = 2{,}929{,}395{,}360$ bps = 2,929.4 Mbps ≈ 2.93 Gbps

(d) With 2 spatial streams: $2 \times 2{,}929.4 = 5{,}858.8$ Mbps ≈ 5.86 Gbps

The spectral efficiency = 5,858.8 / 320 = 18.3 bps/Hz, demonstrating the efficiency gains of 4096-QAM and dense subcarrier spacing in Wi-Fi 7.

Problem 15.8.2

Given: A Wi-Fi 6 access point at 2.4 GHz (2,437 MHz) transmits at 23 dBm (200 mW) with an antenna gain of 3 dBi. A laptop client has an antenna gain of 0 dBi and receiver sensitivity of −70 dBm for MCS9 (256-QAM, 5/6). The path between the AP and client includes 2 interior drywall walls (3.5 dB loss each).

Find: (a) The free-space path loss at 20 m, (b) the total path loss including wall attenuation, (c) the received signal strength, and (d) the link margin.

Solution:

(a) FSPL = 20 $\log_{10}(0.020)$ + 20 $\log_{10}(2437)$ + 32.44 = 20 × (−1.699) + 20 × 3.387 + 32.44 = −33.98 + 67.73 + 32.44 = 66.19 dB

(b) Total path loss = FSPL + wall losses = 66.19 + 2 × 3.5 = 66.19 + 7.0 = 73.19 dB

(c) $P_{rx} = P_{tx} + G_{tx} + G_{rx}$ − path loss = 23 + 3 + 0 − 73.19 = −47.19 dBm

(d) Link margin = P_{rx} − sensitivity = −47.19 − (−70) = 22.81 dB

The link has excellent margin for MCS9 operation. Each additional drywall wall would reduce the margin by 3.5 dB.

Problem 15.8.3

Given: A Wi-Fi 6E access point operates at 6 GHz (6,000 MHz) with 30 dBm EIRP (regulatory limit in many regions). A client device at 15 m distance has 0 dBi antenna gain and requires −65 dBm for MCS11 (1024-QAM, 5/6). No walls are between the AP and client.

Find: (a) The free-space path loss at 15 m, (b) the received signal strength, (c) the link margin, and (d) the maximum free-space range for MCS11 operation.

Solution:

(a) FSPL = 20 $\log_{10}(0.015)$ + 20 $\log_{10}(6000)$ + 32.44 = 20 × (−1.824) + 20 × 3.778 + 32.44 = −36.48 + 75.56 + 32.44 = 71.52 dB

(b) EIRP already includes transmit power + antenna gain: $P_{tx} + G_{tx}$ = 30 dBm P_{rx} = 30 + 0 − 71.52 = −41.52 dBm

(c) Margin = −41.52 − (−65) = 23.48 dB

(d) Maximum allowable path loss = 30 + 0 − (−65) = 95 dB 95 = 20 $\log_{10}(d)$ + 20 $\log_{10}(6000)$ + 32.44 20 $\log_{10}(d)$ = 95 − 75.56 − 32.44 = −13.0 $\log_{10}(d)$ = −0.65 d = $10^{-0.65}$ = 0.2239 km = 223.9 m (free space)

In practice, indoor range at 6 GHz is significantly shorter (10-20 m for MCS11) due to wall losses and multipath.

Problem 15.8.4

Given: A corporate WPA2-Enterprise network uses EAP-TLS authentication. An auditor wants to evaluate the risk of a brute-force attack on the TLS session. The TLS 1.2 connection uses AES-128-GCM. A password-based WPA2-Personal network on the guest SSID uses a 10-character passphrase consisting of mixed-case letters and digits (62 possible characters per position).

Find: (a) The AES-128 keyspace size, (b) the time to brute-force AES-128 at 10^{12} keys/second, (c) the WPA2-Personal passphrase keyspace, and (d) the time for an offline GPU attack at 500,000 passphrases/second.

Solution:

(a) AES-128 keyspace = $2^{128} = 3.403 \times 10^{38}$ keys

(b) Time = 3.403×10^{38} / 10^{12} = 3.403×10^{26} seconds = 3.403×10^{26} / (3.156×10^7) = 1.078×10^{19} years

This is approximately 780 million times the age of the universe. AES-128 is computationally infeasible to brute-force.

(c) Passphrase keyspace = $62^{10} = 8.393 \times 10^{17}$ possible passphrases

(d) Offline attack time = 8.393×10^{17} / 500,000 = 1.679×10^{12} seconds = 1.679×10^{12} / (3.156×10^7) = 53,200 years

A 10-character mixed-case alphanumeric passphrase provides strong protection against offline attacks. However, dictionary words or common patterns would be found much faster using rule-based attacks.

Problem 15.8.5

Given: A warehouse Wi-Fi deployment uses 5 GHz (5,500 MHz) access points with 20 dBm transmit power and 6 dBi gain directional antennas aimed down aisles. The metal shelving creates significant multipath but no direct wall penetration is needed. The path loss model for the warehouse is: PL(dB) = $38 + 28 \log_{10}(d)$, where d is distance in meters. Client sensitivity is −75 dBm for 802.11ac MCS5 (64-QAM, 2/3).

Find: (a) The EIRP, (b) the path loss at 50 m, (c) the received signal at 50 m (client antenna gain = 0 dBi), and (d) the maximum range for MCS5 operation.

Solution:

(a) EIRP = $P_{tx} + G_{tx} = 20 + 6 = 26$ dBm

(b) PL(50 m) = $38 + 28 \times \log_{10}(50) = 38 + 28 \times 1.699 = 38 + 47.57 = 85.57$ dB

(c) P_{rx} = EIRP + G_{rx} − PL = $26 + 0 - 85.57 = -59.57$ dBm

Margin = $-59.57 - (-75) = 15.43$ dB.

(d) Maximum range: $26 + 0 - (38 + 28 \log_{10}(d)) = -75$ $28 \log_{10}(d) = 26 + 75 - 38 = 63$ $\log_{10}(d) = 63/28 = 2.25$ $d = 10^{2.25} = 177.8$ m

With this indoor propagation model, the AP can serve MCS5 clients up to approximately 178 m down a warehouse aisle. In practice, a 10 dB fade margin is typically added, reducing the reliable range to approximately 100 m.

Chapter 15 — Section 15.9: Network Infrastructure

Practice problems covering switches, routers, structured cabling, VLANs, network security, and software-defined networking.

Problem 15.9.1

Given: A Layer 3 campus switch has 24×10 Gbps SFP+ ports and 2×40 Gbps QSFP+ uplinks. The switching fabric is rated at 560 Gbps and the forwarding rate is 416.67 Mpps.

Find: (a) The total full-duplex bandwidth requirement, (b) whether the switch fabric is non-blocking, (c) the wire-speed forwarding rate at 64-byte minimum frames for all ports, and (d) whether the forwarding engine can sustain wire speed.

Solution:

(a) Full-duplex bandwidth: 10G ports: $24 \times 10 \times 2 = 480$ Gbps 40G uplinks: $2 \times 40 \times 2 = 160$ Gbps Total = 480 + 160 = 640 Gbps

(b) Switching fabric = 560 Gbps. $560 < 640 \rightarrow$ Blocking – the fabric cannot handle all ports at full duplex simultaneously. In practice, oversubscription of access ports toward uplinks is common and acceptable.

(c) Wire-speed frame rate: Per 10G port: $10 \times 10^9 / (84 \times 8) = 14{,}880{,}952$ fps Per 40G port: $40 \times 10^9 / (84 \times 8) = 59{,}523{,}810$ fps Total (all ports, one direction): $24 \times 14{,}880{,}952 + 2 \times 59{,}523{,}810 = 357{,}142{,}857 + 119{,}047{,}619 = 476.2$ Mpps

(d) Forwarding engine = 416.67 Mpps < 476.2 Mpps $\rightarrow$ Cannot sustain wire speed at 64-byte frames across all ports simultaneously. At larger frame sizes (e.g., 128+ bytes), the packet rate drops and wire speed becomes achievable.

Problem 15.9.2

Given: A core router has eight 100 Gbps interfaces and a 2.4 Tbps switching capacity. The router holds 1.2 million IPv4 routes in its TCAM, with each entry consuming 80 bytes (prefix, mask, next-hop, action, counters). The router processes 64-byte packets at line rate.

Find: (a) The total bidirectional bandwidth, (b) the wire-speed packet rate for all interfaces, (c) the TCAM memory used by the routing table, and (d) the minimum TCAM size needed.

Solution:

(a) Total bidirectional bandwidth = $8 \times 100 \times 2 = 1{,}600$ Gbps = 1.6 Tbps

The switching capacity of 2.4 Tbps > 1.6 Tbps → non-blocking.

(b) Per 100G interface at 64-byte frames: $100 \times 10^9 / (84 \times 8) = 148{,}809{,}524$ fps Total all 8 interfaces $= 8 \times 148{,}809{,}524 = 1{,}190.5$ Mpps

(c) TCAM memory = $1{,}200{,}000 \times 80 = 96{,}000{,}000$ bytes = 96 MB

(d) Current Internet IPv4 table ≈ 950,000 prefixes. With 20% growth margin: $1{,}140{,}000$ entries $\times$ 80 = 91.2 MB. Minimum TCAM = 96 MB (to hold 1.2M routes at 80 bytes each).

Problem 15.9.3

Given: A new five-story office building requires structured cabling. Each floor has 120 data drops. The MDF is in the basement. Each floor has one IDF. The average horizontal cable run is 55 m. Backbone fiber between MDF and each IDF averages 40 m per floor. Patch cords are 3 m at the IDF and 5 m at the work area.

Find: (a) The total channel length per drop, (b) TIA-568 compliance, (c) the number of 24-port patch panels per IDF, (d) the number of 48-port switches per IDF, and (e) the total backbone fiber needed (assuming 12-strand OM4 per IDF).

Solution:

(a) Channel = IDF patch cord + horizontal cable + work-area patch cord = 3 + 55 + 5 = 63 m

(b) 63 m < 100 m channel limit → Compliant 55 m permanent link < 90 m limit → Compliant

(c) Panels per IDF = 120 / 24 = 5.0 → 5 patch panels per IDF (exactly 120 ports) Total panels = $5 \times$ 5 floors = 25 patch panels

(d) Switches per IDF = 120 / 48 = 2.5 → 3 switches per IDF (144 ports, 24 spare) Total switches = $3 \times 5 = 15$ access switches

(e) Backbone fiber = 5 IDFs × 1 cable × 12 strands × average 40 m = $5 \times 40 = 200$ m of 12-strand fiber. Total = 5 runs of 12-strand OM4 (40 m average per run, actual lengths vary by floor).

Problem 15.9.4

Given: A hospital network requires five VLANs: VLAN 10 (Medical devices, 300 hosts), VLAN 20 (Staff workstations, 500 hosts), VLAN 30 (Patient Wi-Fi, 200 hosts), VLAN 40 (VoIP phones, 400 hosts), and VLAN 99 (Network management, 15 hosts). The IP address space is 10.10.0.0/16.

Find: (a) The smallest CIDR prefix for each VLAN using VLSM, (b) the subnet assignments from the 10.10.0.0/16 space, (c) the 802.1Q tag overhead on a trunk carrying 200,000 frames/second, and (d) the total usable host capacity across all VLANs.

Solution:

(a) Prefix sizing (allocate largest first): VLAN 20 (500 hosts): $2^n - 2 \geq 500 \rightarrow n = 9$ (510 hosts) → /23 VLAN 40 (400 hosts): $2^n - 2 \geq 400 \rightarrow n = 9$ (510 hosts) → /23 VLAN 10 (300 hosts): $2^n - 2 \geq 300 \rightarrow n = 9$ (510 hosts) → /23 VLAN 30 (200 hosts): $2^n - 2 \geq 200 \rightarrow n = 8$ (254 hosts) → /24 VLAN 99 (15 hosts): $2^n - 2 \geq 15 \rightarrow n = 5$ (30 hosts) → /27

(b) Subnet assignments: VLAN 20: 10.10.0.0/23 (10.10.0.1 – 10.10.1.254, 510 hosts) VLAN 40: 10.10.2.0/23 (10.10.2.1 – 10.10.3.254, 510 hosts) VLAN 10: 10.10.4.0/23 (10.10.4.1 – 10.10.5.254, 510 hosts) VLAN 30: 10.10.6.0/24 (10.10.6.1 – 10.10.6.254, 254 hosts) VLAN 99: 10.10.7.0/27 (10.10.7.1 – 10.10.7.30, 30 hosts)

(c) 802.1Q overhead = 200,000 × 4 bytes × 8 bits = 6,400,000 bps = 6.4 Mbps

(d) Total usable hosts = 510 + 510 + 510 + 254 + 30 = 1,814 hosts Total required = 300 + 500 + 200 + 400 + 15 = 1,415 hosts. Surplus = 399 hosts for growth.

Problem 15.9.5

Given: A firewall processes traffic at 10 Gbps throughput with a session table capacity of 2,000,000 entries. Each stateful session entry uses 128 bytes. During a DDoS attack, the attacker sends 500,000 SYN packets per second from spoofed source IPs. The firewall's SYN flood protection limits half-open connections to 100,000.

Find: (a) The memory used by the session table at full capacity, (b) the time to fill the SYN flood limit without protection, (c) the percentage of session table consumed by the SYN flood limit, and (d) the bandwidth consumed by the SYN flood (each SYN packet is 60 bytes including IP and Ethernet headers).

Solution:

(a) Session table memory = 2,000,000 × 128 = 256,000,000 bytes = 256 MB

(b) Time to fill SYN limit = 100,000 / 500,000 = 0.2 seconds

Without SYN flood protection, the attacker would consume 500,000 entries/second, filling the entire 2M session table in 2,000,000 / 500,000 = 4 seconds.

(c) SYN flood limit as % of table = 100,000 / 2,000,000 = 5.0%

This ensures that 95% of the session table remains available for legitimate connections.

(d) SYN flood bandwidth = 500,000 × 60 × 8 = 240,000,000 bps = 240 Mbps

This is only 2.4% of the firewall's 10 Gbps capacity, illustrating that DDoS attacks often aim to exhaust state tables rather than bandwidth.

Problem 15.9.6

Given: An SDN controller manages a leaf-spine data center fabric with 64 leaf (ToR) switches and 8 spine switches. Each leaf has 48×25 Gbps server ports and 8×100 Gbps uplinks to the spines. Each spine has 64×100 Gbps ports. Each leaf switch has 32 MB of TCAM with 256-byte flow entries. The controller installs flows reactively (on first packet miss).

Find: (a) The total server-facing bandwidth, (b) the leaf-to-spine oversubscription ratio, (c) the maximum flow entries per leaf switch, and (d) the total flow table capacity across the fabric.

Solution:

(a) Server bandwidth per leaf: $48 \times 25 = 1{,}200$ Gbps = 1.2 Tbps Total server bandwidth: $64 \times 1{,}200 = 76{,}800$ Gbps = 76.8 Tbps

(b) Server-side bandwidth per leaf: 1,200 Gbps Uplink bandwidth per leaf: $8 \times 100 = 800$ Gbps Oversubscription = 1,200 / 800 = 1.5:1

This means 67% of server traffic can be forwarded to the spine simultaneously – a moderate and common oversubscription ratio.

(c) Flow entries per leaf = $32 \times 10^6 / 256 = 125{,}000$ entries

(d) Total fabric capacity = $64 \times 125{,}000 = 8{,}000{,}000$ flow entries

At 256 bytes each, total TCAM across the fabric = $8{,}000{,}000 \times 256 = 2{,}048{,}000{,}000$ bytes = 2.048 GB.

Problem 15.9.7

Given: An enterprise extends a VLAN across two buildings using an 802.1Q trunk over a 1 Gbps single-mode fiber link. The trunk carries VLANs 10, 20, 30, and 99. Average traffic per VLAN: VLAN 10 = 200 Mbps, VLAN 20 = 300 Mbps, VLAN 30 = 150 Mbps, VLAN 99 = 10 Mbps. Each frame averages 750 bytes.

Find: (a) The total trunk utilization, (b) the frame rate, (c) the 802.1Q tagging overhead in Mbps, and (d) whether the trunk has capacity for all VLANs.

Solution:

(a) Total traffic = 200 + 300 + 150 + 10 = 660 Mbps Utilization = 660 / 1,000 = 66.0%

(b) Total data rate in bytes/s = $660 \times 10^6 / 8 = 82{,}500{,}000$ bytes/s Frame rate = 82,500,000 / 750 = 110,000 fps

(c) Tag overhead = $110{,}000 \times 4$ bytes $\times$ 8 bits = 3,520,000 bps = 3.52 Mbps

(d) Total with tagging = 660 + 3.52 = 663.52 Mbps. Since $663.52 < 1{,}000$ Mbps $\rightarrow$ Yes, the trunk has sufficient capacity with 33.6% headroom.

Chapter 15 — Section 15.10: Network Performance

Practice problems covering latency and propagation delay, throughput and bandwidth utilization, bit error rate, network timing and synchronization, and quality of service.

Problem 15.10.1

Given: A 1,500-byte packet traverses a path consisting of 2,000 km of single-mode fiber (refractive index n = 1.468), passes through 5 routers each adding 75 μs of processing delay, and the link rate is 100 Gbps.

Find: (a) The propagation delay, (b) the serialization delay, (c) the total one-way latency, and (d) the round-trip time.

Solution:

(a) Propagation delay: $v_{fiber} = c/n = 3.0 \times 10^8 / 1.468 = 2.044 \times 10^8$ m/s $t_{prop} = 2{,}000{,}000 / (2.044 \times 10^8) = 9{,}785$ μs = 9.785 ms

(b) Serialization delay: $t_{serial} = (1{,}500 \times 8) / (100 \times 10^9) = 12{,}000 / 10^{11} = 0.12$ μs

(c) Total one-way latency: Processing delay = $5 \times 75 = 375$ μs $t_{total} = 9{,}785 + 0.12 + 375 = 10{,}160$ μs ≈ 10.16 ms

(d) Round-trip time: RTT = $2 \times 10.16 = 20.32$ ms

Propagation delay dominates at 96.3% of the total.

Problem 15.10.2

Given: A 10 Gbps Ethernet link carries traffic at 7.5 Gbps average utilization. The frames are 1,000 bytes average size with 20 bytes of Ethernet preamble and interframe gap overhead per frame.

Find: (a) The link utilization percentage, (b) the frame rate in frames per second, (c) the goodput if each frame carries 954 bytes of payload (after headers), and (d) the protocol efficiency.

Solution:

(a) Link utilization: U = 7.5 / 10 = 75%

(b) Each frame on the wire = 1,000 + 20 = 1,020 bytes = 8,160 bits Frame rate = 7.5×10^9 / 8,160 = 919,118 frames/s ≈ 919 kfps

Wait — the 7.5 Gbps includes all overhead. Recalculating: Frame rate = 7.5×10^9 / (1,020 × 8) = 7.5 × 10^9 / 8,160 = 919,118 frames/s

(c) Goodput (payload throughput): Goodput = 919,118 × 954 × 8 = 919,118 × 7,632 = 7.014×10^9 bps = 7.01 Gbps

(d) Protocol efficiency: η = payload / total on-wire = 954 / 1,020 = 93.5%

Problem 15.10.3

Given: A 40 Gbps fiber optic link operates continuously. The pre-FEC BER is 10^{-4} and the FEC (with 7% overhead) corrects to a post-FEC BER of 10^{-15}.

Find: (a) The number of pre-FEC errors per second, (b) the effective line rate with FEC overhead, (c) the expected time between post-FEC errors, and (d) the FEC coding gain in orders of magnitude.

Solution:

(a) Pre-FEC errors per second: Line rate with FEC = 40 × 1.07 = 42.8 Gbps Errors/s = $42.8 \times 10^9 \times 10^{-4} = 4.28 \times 10^6$ = 4.28 million errors/s

(b) Effective line rate: 42.8 Gbps on the wire, carrying 40 Gbps of user data

(c) Post-FEC error rate: Errors/s = $40 \times 10^9 \times 10^{-15} = 4.0 \times 10^{-5}$ errors/s Mean time between errors = 1 / (4.0×10^{-5}) = 25,000 seconds ≈ 6.94 hours

(d) FEC coding gain: Pre-FEC BER = 10^{-4}, Post-FEC BER = 10^{-15} Improvement = 10^{-4} / 10^{-15} = 10^{11} Coding gain = 11 orders of magnitude

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Problem 15.10.4

Given: A PTP grandmaster clock with a rubidium oscillator (±0.001 ppm accuracy) synchronizes a boundary clock over a network with mean path delay of 5.2 μs. The boundary clock has a TCXO oscillator with ±2 ppm free-running accuracy.

Find: (a) The clock drift of the boundary clock over a 10-minute holdover period, (b) the maximum PTP sync interval to maintain ±500 ns accuracy, (c) the grandmaster drift over 24 hours, and (d) the delay asymmetry error if the forward path is 200 ns longer than the return path.

Solution:

(a) Boundary clock drift during holdover: Drift rate = ±2 ppm = ±2 μs/s Over 10 minutes: drift = $2 \times 10^{-6} \times 600$ = 1,200 μs = 1.2 ms

(b) Maximum sync interval for ±500 ns: T_{sync} < 500 ns / (2 μs/s) = $500 \times 10^{-9} / (2 \times 10^{-6})$ = 0.25 s = 250 ms

PTP sync messages should be sent at least 4 per second. A typical configuration of 8 messages/s provides adequate margin.

(c) Grandmaster drift over 24 hours: Drift = $0.001 \times 10^{-6} \times 86{,}400 = 86.4$ ns over 24 hours

(d) Delay asymmetry error: Asymmetry = 200 ns → time offset = 200/2 = 100 ns

This 100 ns error is constant and can be compensated if the asymmetry is measured and configured.

Problem 15.10.5

Given: A copper Ethernet link has a usable bandwidth of 500 MHz and achieves an SNR of 35 dB.

Find: (a) The Shannon channel capacity, (b) the spectral efficiency of a 10GBASE-T system operating over this channel, and (c) the margin to the Shannon limit.

Solution:

(a) Shannon capacity: SNR (linear) = $10^{35/10} = 3{,}162$ $C = B \times \log_2(1 + \text{SNR}) = 500 \times 10^6 \times \log_2(3{,}163) = 500 \times 10^6 \times 11.627$ $C = 5{,}813.5$ Mbps ≈ 5.81 Gbps

(b) Spectral efficiency of 10GBASE-T: $\eta = 10{,}000 / 500 = 20.0$ bps/Hz (across all four pairs)

Per pair: $\eta = 2{,}500 / 125 = 20.0$ bps/Hz (using 125 MHz per pair with PAM-16 encoding and DSP)

(c) Shannon limit per pair at 125 MHz with same SNR: $C_{pair} = 125 \times 10^6 \times 11.627 = 1{,}453$ Mbps

Actual per pair = 2,500 Mbps, which exceeds the single-pair Shannon limit. This is because 10GBASE-T uses crosstalk cancellation (MIMO-like processing across all four pairs), effectively exploiting the full 500 MHz × 4 channel. The total capacity of 10 Gbps vs. Shannon limit of 5.81 Gbps suggests the effective SNR with DSP processing is higher, or the system uses additional coding dimensions.

Problem 15.10.6

Given: An enterprise WAN link has 200 Mbps capacity. Traffic classes: 30 concurrent G.711 VoIP calls (87.2 kbps each), 50 Mbps of business application traffic (AF21), and the remainder is best-effort. The QoS policy allocates 5% LLQ for EF, 40% WFQ for AF21, and 55% for BE.

Find: (a) The VoIP bandwidth requirement, (b) whether the LLQ allocation is sufficient, (c) the bandwidth available for best-effort traffic, and (d) the serialization delay for a voice packet (160 bytes + headers = 200 bytes total).

Solution:

(a) VoIP bandwidth: $30 \times 87.2 = 2{,}616$ kbps = 2.62 Mbps

(b) LLQ allocation: LLQ = $200 \times 0.05 = 10$ Mbps VoIP = 2.62 Mbps Utilization = 2.62 / 10 = 26.2% → Sufficient with 73.8% headroom

(c) Business traffic gets: $200 \times 0.40 = 80$ Mbps (50 Mbps used, 30 Mbps unused) Best-effort allocation: $200 \times 0.55 = 110$ Mbps

With unused LLQ and AF21 bandwidth available, best-effort can burst higher: Available = 200 − 2.62 − 50 = 147.4 Mbps maximum for best-effort during normal operation

(d) Serialization delay for 200-byte voice packet: $t = (200 \times 8) / (200 \times 10^6) = 1{,}600 / 2 \times 10^8 = 8.0$ μs

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Problem 15.10.7

Given: A SyncE-enabled network distributes frequency from a PRC (Primary Reference Clock, $\pm 1 \times 10^{-11}$) through 10 cascaded SyncE nodes. Each node has a noise generation limit of 0.01 UI peak-to-peak jitter at 2.048 MHz.

Find: (a) The PRC frequency accuracy in Hz at 2.048 MHz, (b) the maximum wander accumulation through 10 nodes, (c) the time interval error (TIE) accumulated over 1 second per node, and (d) whether the chain meets the ±4.6 ppm end-to-end requirement.

Solution:

(a) PRC frequency accuracy: $\Delta f = 2.048 \times 10^6 \times 1 \times 10^{-11} = 2.048 \times 10^{-5}$ Hz = 0.0205 mHz

(b) SyncE wander accumulates as the square root of the number of nodes for uncorrelated noise: Total jitter $\approx 0.01 \times \sqrt{10} = 0.01 \times 3.162 = 0.032$ UI peak-to-peak

(c) TIE per node: 1 UI at 2.048 MHz = $1 / 2.048 \times 10^6$ = 488 ns 0.01 UI = 4.88 ns per node per observation period

(d) The end-to-end frequency accuracy from PRC through 10 SyncE nodes: Each node adds negligible frequency error when locked (specified < ±0.01 ppm). After 10 nodes: total $\approx 10 \times 0.01 = 0.1$ ppm worst case, which is well within the ±4.6 ppm requirement for EEC (Ethernet Equipment Clock, per G.8262).

Problem 15.10.8

Given: A data center interconnect uses 400G ZR coherent optics at 1550 nm over 80 km of single-mode fiber. The transmitter launches +1 dBm per channel. Fiber attenuation is 0.20 dB/km, and connector losses total 2.0 dB.

Find: (a) The total link loss, (b) the received power, (c) the required OSNR for DP-16QAM at BER = 4×10^{-2} (approximately 16 dB), and (d) the OSNR margin.

Solution:

(a) Total link loss: Fiber: $0.20 \times 80 = 16.0$ dB Connectors: 2.0 dB Total = 18.0 dB

(b) Received power: $P_{rx} = +1 - 18.0 = -17.0$ dBm

(c) Required OSNR = 16 dB (for DP-16QAM at soft-decision FEC threshold with implementation penalty)

(d) OSNR at receiver (unamplified link, using receiver noise as limiting factor): For a coherent receiver with integrated LO, the OSNR is determined by the received signal power relative to the receiver noise. With $P_{rx} = -17$ dBm and typical coherent receiver sensitivity of -22 dBm at required OSNR:

Margin = $P_{rx} - P_{sensitivity} = -17 - (-22) = 5$ dB margin

This provides adequate margin for connector aging, splice repairs, and temperature variations.

Problem 15.10.9

Given: A network monitoring system measures the following latency statistics over a 24-hour period for a WAN link: mean latency = 45 ms, 95th percentile = 62 ms, 99th percentile = 85 ms, maximum = 210 ms. The VoIP quality requirement is one-way delay < 150 ms.

Find: (a) The percentage of time the link meets the VoIP requirement, (b) the jitter (variation in delay), (c) the estimated packet loss rate if packets arriving > 150 ms are discarded by the jitter buffer, and (d) the MOS score impact.

Solution:

(a) Since the 99th percentile is 85 ms (well below 150 ms), and the maximum is 210 ms, the link meets the VoIP requirement for >99% but <100% of the time. The exceedances are rare outliers.

(b) Jitter (approximated as the difference between 95th percentile and mean): Jitter $\approx 62 - 45 = 17$ ms (P95 − mean) Peak jitter = $210 - 45 = 165$ ms

(c) Packets exceeding 150 ms threshold: The 99th percentile is 85 ms. With a roughly exponential tail, packets > 150 ms would represent approximately 0.1–0.5% of traffic (estimated as fewer than 1 in 200 packets).

(d) ITU G.107 E-model guidelines:

- Mean delay 45 ms: minimal impact (< 100 ms is rated "very good")
- Jitter 17 ms: manageable with a 60 ms jitter buffer
- Packet loss 0.1–0.5%: slight degradation
- Expected MOS: approximately 4.0–4.1 (good quality, on a 1–5 scale)

Problem 15.10.10

Given: A WRED (Weighted Random Early Detection) policy is configured on a router interface with 100 Mbps capacity. The policy specifies: AF21 traffic begins dropping at 40% queue depth and reaches 100% drop at 80% queue depth. AF11 traffic begins dropping at 60% queue depth and reaches 100% drop at 90%. The queue buffer is 1 MB.

Find: (a) The queue depth thresholds in bytes for AF21, (b) the queue depth thresholds for AF11, (c) the drop probability for AF21 at 50% queue depth (linear interpolation), and (d) the maximum queuing delay at full queue.

Solution:

(a) AF21 thresholds: Min threshold: $0.40 \times 1{,}048{,}576 = 419{,}430$ bytes (≈ 410 KB) Max threshold: $0.80 \times 1{,}048{,}576 = 838{,}861$ bytes (≈ 819 KB)

(b) AF11 thresholds: Min threshold: $0.60 \times 1{,}048{,}576 = 629{,}146$ bytes (≈ 614 KB) Max threshold: $0.90 \times 1{,}048{,}576 = 943{,}718$ bytes (≈ 922 KB)

(c) AF21 drop probability at 50% queue depth: Queue at 50% = 524,288 bytes Linear interpolation between 40% (0% drop) and 80% (100% drop): Position = (50% − 40%) / (80% − 40%) = 10/40 = 0.25 Drop probability = $0.25 \times 100\% = 25\%$

(d) Maximum queuing delay (full 1 MB buffer): t = buffer size / link rate = $(1{,}048{,}576 \times 8) / (100 \times 10^6) = 8{,}388{,}608 / 10^8 = 83.9$ ms

WRED prevents the queue from reaching this level for most traffic, keeping typical queuing delays well below 83.9 ms.

Chapter 16 — Section 16.1: Antenna Fundamentals

Practice problems covering radiation mechanisms and patterns, antenna parameters (directivity, gain, effective aperture, EIRP, bandwidth, polarization), and the Friis transmission equation and link budgets.

Problem 16.1.1

Given: A horn antenna has a measured half-power beamwidth of 18° in the E-plane and 22° in the H-plane. The peak sidelobe level is −20 dB below the main lobe.

Find: (a) The estimated directivity using $D \approx 32{,}400 / (\theta_E \times \theta_H)$, (b) the directivity in dBi, (c) the sidelobe power as a fraction of the main lobe, and (d) the first-null beamwidth (FNBW) if FNBW ≈ 2.5 × HPBW.

Solution:

(a) Directivity: $D \approx 32{,}400 / (18 \times 22) = 32{,}400 / 396 = 81.8$ (linear)

(b) D (dBi) = $10 \log_{10}(81.8) = 19.1$ dBi

(c) Sidelobe level = −20 dB: $P_{SL}/P_{main} = 10^{-2.0} = 0.01 = 1.0\%$ of the peak power density

(d) FNBW in E-plane ≈ 2.5 × 18° = 45° FNBW in H-plane ≈ 2.5 × 22° = 55°

Problem 16.1.2

Given: A parabolic dish antenna has a diameter of 3 m and operates at 4 GHz (C-band). The aperture efficiency is 0.65.

Find: (a) The wavelength, (b) the gain in dBi, (c) the effective aperture, (d) the HPBW, and (e) the EIRP if the transmitter power is 20 W.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 4 \times 10^9 = 0.075$ m

(b) $G = \eta(\pi D/\lambda)^2 = 0.65 \times (\pi \times 3 / 0.075)^2 = 0.65 \times (125.66)^2 = 0.65 \times 15{,}791 = 10{,}264$ G (dBi) = 10 $\log_{10}(10{,}264) = 40.1$ dBi

(c) $A_e = G\lambda^2 / (4\pi) = 10{,}264 \times 0.075^2 / (4\pi) = 10{,}264 \times 5.625 \times 10^{-3} / 12.566 = 4.59\ m^2$ Check: $\eta \times A_{phys} = 0.65 \times \pi \times 1.5^2 = 0.65 \times 7.069 = 4.59\ m^2$ ✓

(d) HPBW $\approx 70\lambda/D = 70 \times 0.075 / 3 = 1.75°$

(e) EIRP $= P_t \times G = 20 \times 10{,}264 = 205{,}280$ W EIRP (dBW) $= 10 \log_{10}(20) + 40.1 = 13.0 + 40.1 =$ 53.1 dBW

Problem 16.1.3

Given: A point-to-point microwave link operates at 6 GHz over a distance of 30 km. The transmit antenna gain is 35 dBi and the receive antenna gain is 35 dBi. The transmitter output power is 1 W (30 dBm).

Find: (a) The free-space path loss (FSPL), (b) the received power in dBm, (c) the received power in watts, and (d) the system margin if the receiver sensitivity is −80 dBm.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 6 \times 10^9 = 0.05$ m FSPL $= 20 \log_{10}(4\pi d/\lambda) = 20 \log_{10}(4\pi \times 30{,}000 / 0.05)$ FSPL $= 20 \log_{10}(7.54 \times 10^6) = 20 \times 6.877 = 137.5$ dB

(b) $P_r = P_t + G_t + G_r -$ FSPL $P_r = 30 + 35 + 35 - 137.5 = -37.5$ dBm

(c) $P_r = 10^{-3.75}$ mW $= 1.78 \times 10^{-4}$ mW = 178 nW

(d) System margin $= P_r -$ Sensitivity $= -37.5 - (-80) = 42.5$ dB

This generous margin accounts for rain fade, multipath, and equipment aging over the life of the link.

Problem 16.1.4

Given: A radar system operates at 9.4 GHz (X-band) with a peak transmit power of 25 kW, an antenna gain of 34 dBi, and a target radar cross section (RCS) of $\sigma = 5\ m^2$. The minimum detectable received power is −110 dBm.

Find: (a) The wavelength, (b) the maximum detection range using the radar range equation $P_r = P_t G^2 \lambda^2 \sigma / ((4\pi)^3 R^4)$, (c) the received power at a range of 50 km, and (d) whether the target is detectable at 50 km.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 9.4 \times 10^9 = 0.03191$ m

(b) Convert gain: $G = 10^{34/10} = 2{,}512$ (linear) Rearrange for R: $R^4 = P_t G^2 \lambda^2 \sigma / ((4\pi)^3 P_{r,min})$ $P_{r,min} = 10^{-110/10}$ mW $= 10^{-11}$ mW $= 10^{-14}$ W

$R^4 = (25{,}000 \times 2{,}512^2 \times 0.03191^2 \times 5) / ((4\pi)^3 \times 10^{-14})$ Numerator $= 25{,}000 \times 6.310 \times 10^6 \times 1.018 \times 10^{-3} \times 5 = 25{,}000 \times 32{,}120 = 8.030 \times 10^8$ Denominator $= 1{,}984 \times 10^{-14} = 1.984 \times 10^{-11}$ $R^4 = 8.030 \times 10^8 / 1.984 \times 10^{-11} = 4.047 \times 10^{19}$ $R = (4.047 \times 10^{19})^{0.25} = 79.8$ km

(c) At R = 50 km: $P_r = (25{,}000 \times 6.310 \times 10^6 \times 1.018 \times 10^{-3} \times 5) / ((4\pi)^3 \times (50{,}000)^4)$ $P_r = 8.030 \times 10^8 / (1{,}984 \times 6.25 \times 10^{18}) = 8.030 \times 10^8 / 1.24 \times 10^{22} = 6.476 \times 10^{-14}$ W $P_r = 10 \log_{10}(6.476 \times 10^{-14}) + 30 = (-131.9) + 30 = -101.9$ dBm

(d) Since -101.9 dBm > -110 dBm, the target is detectable at 50 km with a margin of 8.1 dB.

Problem 16.1.5

Given: An antenna has a gain of 12 dBi and operates at 2.4 GHz. The antenna's radiation efficiency is 85%.

Find: (a) The directivity, (b) the effective aperture, (c) the physical aperture if the antenna is a horn with aperture efficiency $\eta_{ap} = 0.51$, and (d) the beam solid angle $\Omega_A = 4\pi/D$.

Solution:

(a) $G = \eta D$, so $D = G / \eta$ G (linear) $= 10^{12/10} = 15.85$ $D = 15.85 / 0.85 = 18.65$ (12.7 dBi)

(b) $\lambda = c/f = 3 \times 10^8 / 2.4 \times 10^9 = 0.125$ m $A_e = G\lambda^2 / (4\pi) = 15.85 \times 0.125^2 / (4\pi) = 15.85 \times 0.01563 / 12.566 = 0.01972$ m² (197.2 cm²)

(c) For a horn: $A_e = \eta_{ap} \times A_{phys}$ $A_{phys} = A_e / \eta_{ap} = 0.01972 / 0.51 = 0.03867$ m² (386.7 cm²) For a square aperture: side $= \sqrt{0.03867} = 0.197$ m (19.7 cm)

(d) $\Omega_A = 4\pi / D = 4\pi / 18.65 = 0.674$ steradians

Problem 16.1.6

Given: A vertically polarized transmit antenna communicates with a receive antenna tilted 30° from vertical (polarization mismatch). The receive antenna gain is 8 dBi. The transmitter sends 100 mW at 5.8 GHz over a 200 m line-of-sight path.

Find: (a) The polarization loss factor, (b) the FSPL, (c) the received power without polarization mismatch, and (d) the received power with polarization mismatch.

Solution:

(a) Polarization loss factor (PLF) $= \cos^2(\Delta\varphi) = \cos^2(30°) = (0.866)^2 = 0.75$ PLF (dB) $= 10 \log_{10}(0.75) = -1.25$ dB

(b) $\lambda = c/f = 3 \times 10^8 / 5.8 \times 10^9 = 0.05172$ m FSPL $= 20 \log_{10}(4\pi d/\lambda) = 20 \log_{10}(4\pi \times 200 / 0.05172)$ FSPL $= 20 \log_{10}(4.862 \times 10^4) = 20 \times 4.687 = 93.7$ dB

(c) Assume $G_t = 0$ dBi (isotropic transmit antenna for simplicity): $P_r = P_t + G_t + G_r - \text{FSPL} = 20 + 0 + 8 - 93.7 = -65.7$ dBm

(d) With polarization mismatch: $P_r = -65.7 - 1.25 = -67.0$ dBm

The 1.25 dB polarization loss reduces the received power from 269 pW to 200 pW.

Problem 16.1.7

Given: A Ku-band satellite link operates at 14 GHz (uplink). The earth station transmits 50 W through a 3.8 m dish with 62% aperture efficiency. The satellite is at geostationary orbit (35,786 km altitude).

Find: (a) The transmit antenna gain, (b) the EIRP, (c) the free-space path loss, and (d) the power flux density at the satellite in dBW/m².

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 14 \times 10^9 = 0.02143$ m $G_t = \eta(\pi D/\lambda)^2 = 0.62 \times (\pi \times 3.8 / 0.02143)^2 = 0.62 \times (556.9)^2 = 0.62 \times 310{,}137 = 192{,}285$ G_t (dBi) $= 10 \log_{10}(192{,}285) = 52.8$ dBi

(b) EIRP $= P_t \times G_t = 50 \times 192{,}285 = 9.614 \times 10^6$ W EIRP (dBW) $= 10 \log_{10}(50) + 52.8 = 17.0 + 52.8 = 69.8$ dBW

(c) FSPL $= 20 \log_{10}(4\pi \times 35{,}786{,}000 / 0.02143)$ FSPL $= 20 \log_{10}(2.096 \times 10^{10}) = 20 \times 10.321 = 206.4$ dB

(d) Power flux density at satellite: $\Phi = \text{EIRP} / (4\pi d^2) = 9.614 \times 10^6 / (4\pi \times (3.5786 \times 10^7)^2)$ $\Phi = 9.614 \times 10^6 / (1.609 \times 10^{16}) = 5.975 \times 10^{-10}$ W/m² Φ (dBW/m²) $= 10 \log_{10}(5.975 \times 10^{-10}) = -92.2$ dBW/m²

Problem 16.1.8

Given: An antenna has a radiation resistance $R_{rad} = 73\ \Omega$, a loss resistance $R_{loss} = 5\ \Omega$, and an antenna reactance $X_{ant} = +j25\ \Omega$. It is connected to a 50 Ω transmission line.

Find: (a) The radiation efficiency, (b) the input impedance, (c) the reflection coefficient, (d) the VSWR, and (e) the overall efficiency including mismatch loss.

Solution:

(a) Radiation efficiency: $\eta_{rad} = R_{rad} / (R_{rad} + R_{loss}) = 73 / (73 + 5) = 73 / 78 = 0.936$ (93.6%)

(b) Input impedance: $Z_{in} = R_{rad} + R_{loss} + jX_{ant} = 73 + 5 + j25 = 78 + j25\ \Omega$

(c) Reflection coefficient: $\Gamma = (Z_{in} - Z_0) / (Z_{in} + Z_0) = (78 + j25 - 50) / (78 + j25 + 50) = (28 + j25) / (128 + j25)$ $|\Gamma| = \sqrt{}(28^2 + 25^2) / \sqrt{}(128^2 + 25^2) = \sqrt{}(784 + 625) / \sqrt{}(16{,}384 + 625) = \sqrt{}1{,}409 / \sqrt{}17{,}009 = 37.54 / 130.4 = 0.288$

(d) VSWR $= (1 + 0.288) / (1 - 0.288) = 1.288 / 0.712 = 1.81{:}1$

(e) Mismatch efficiency $= 1 - |\Gamma|^2 = 1 - 0.0830 = 0.917$ Overall efficiency $= \eta_{rad} \times \eta_{mismatch} = 0.936 \times 0.917 = 0.858$ (85.8%)

Problem 16.1.9

Given: A circularly polarized (RHCP) transmit antenna sends a signal to a linearly polarized receive antenna. The receive antenna gain is 6 dBi at 1.575 GHz (GPS L1), and the satellite transmits with an EIRP of 26 dBW at a range of 20,200 km.

Find: (a) The polarization loss for circular-to-linear reception, (b) the FSPL, (c) the received power, and (d) the received power if the receive antenna were also RHCP with the same gain.

Solution:

(a) A linearly polarized antenna receives only one component of the circular polarization. Polarization loss = 3 dB (half the power is in each orthogonal linear component)

(b) $\lambda = c/f = 3 \times 10^8 / 1.575 \times 10^9 = 0.1905$ m FSPL = $20 \log_{10}(4\pi \times 20{,}200{,}000 / 0.1905)$ FSPL = $20 \log_{10}(1.332 \times 10^9) = 20 \times 9.125 = 182.5$ dB

(c) $P_r = EIRP + G_r - FSPL - PLF = 26 + 6 - 182.5 - 3 = -153.5$ dBW (or -123.5 dBm)

(d) With RHCP receive antenna (matched polarization), PLF = 0 dB: $P_r = 26 + 6 - 182.5 - 0 = -150.5$ dBW (or -120.5 dBm)

The 3 dB improvement from matched polarization is significant for GPS receivers operating near the noise floor.

Problem 16.1.10

Given: An antenna range requires far-field testing of a parabolic dish with diameter D = 1.5 m at 12 GHz. The reference antenna has a known gain of 22.0 dBi, and during comparison testing the antenna under test (AUT) receives 16.5 dB more power than the reference antenna.

Find: (a) The far-field distance, (b) the gain of the AUT, (c) the effective aperture of the AUT, and (d) the HPBW of the AUT.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 12 \times 10^9 = 0.025$ m $d_{ff} = 2D^2 / \lambda = 2 \times 1.5^2 / 0.025 = 2 \times 2.25 / 0.025 = 180$ m

(b) $G_{AUT} = G_{ref} + \Delta P = 22.0 + 16.5 = 38.5$ dBi

(c) G_{AUT} (linear) $= 10^{38.5/10} = 7{,}079$ $A_e = G\lambda^2 / (4\pi) = 7{,}079 \times 0.025^2 / (4\pi) = 7{,}079 \times 6.25 \times 10^{-4} / 12.566 = 0.352$ m^2

(d) HPBW $\approx 70\lambda / D = 70 \times 0.025 / 1.5 = 1.17°$

The aperture efficiency can be determined: $\eta = A_e / A_{phys} = 0.352 / (\pi \times 0.75^2) = 0.352 / 1.767 = 0.199$. This low efficiency (20%) suggests significant feed spillover, blockage, or surface errors that warrant further investigation.

Chapter 16 — Section 16.2: Wire Antennas

Practice problems covering dipole antennas (short dipole, half-wave dipole, folded dipole), monopole and ground plane antennas, loop antennas, and Yagi-Uda antennas.

Problem 16.2.1

Given: A short dipole antenna has a total length of 15 cm and operates at 300 MHz.

Find: (a) The wavelength and L/λ ratio, (b) the radiation resistance using $R_{rad} = 80\pi^2(L/\lambda)^2$, (c) the radiation resistance of a half-wave dipole at the same frequency, and (d) the efficiency of the short dipole if the ohmic loss resistance is 1.5 Ω.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 300 \times 10^6 = 1.0$ m $L/\lambda = 0.15 / 1.0 = 0.15$

(b) $R_{rad} = 80\pi^2(L/\lambda)^2 = 80 \times 9.8696 \times (0.15)^2 = 789.57 \times 0.0225 = 17.8\ \Omega$

(c) Half-wave dipole: $R_{rad} = 73\ \Omega$ (standard value at resonance)

(d) Efficiency of the short dipole: $\eta = R_{rad} / (R_{rad} + R_{loss}) = 17.8 / (17.8 + 1.5) = 17.8 / 19.3 = 0.922$ (92.2%)

The short dipole at $L = 0.15\lambda$ has reasonable efficiency because its radiation resistance (17.8 Ω) still dominates the loss resistance. For much shorter dipoles ($L < \lambda/20$), the radiation resistance drops below 2 Ω and efficiency degrades severely.

Problem 16.2.2

Given: A half-wave dipole operates at 440 MHz (UHF amateur band). The antenna is made of aluminum tubing with an ohmic loss resistance of 0.3 Ω.

Find: (a) The physical half-wavelength, (b) the practical element length using the 0.95 shortening factor, (c) the radiation efficiency, (d) the gain in dBi accounting for losses, and (e) the input impedance.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 440 \times 10^6 = 0.6818$ m Half-wavelength $= \lambda/2 = 0.341$ m (34.1 cm)

(b) Practical length = 0.95 × 0.341 = 0.324 m (32.4 cm) Each arm = 16.2 cm from the feed point.

(c) $\eta = R_{rad} / (R_{rad} + R_{loss}) = 73 / (73 + 0.3) = 73 / 73.3 = 0.996$ (99.6%)

(d) Directivity of a half-wave dipole = 2.15 dBi (1.64 linear). $G = \eta D = 0.996 \times 1.64 = 1.633$ G (dBi) $= 10 \log_{10}(1.633) = 2.13$ dBi

The 0.02 dB loss from ohmic resistance is negligible for this full-size element.

(e) At the shortened resonant length: $Z_{in} \approx 73 + j0\ \Omega$ (the shortening factor eliminates the reactive component).

Problem 16.2.3

Given: A folded dipole antenna is designed for the FM broadcast band at 98 MHz. Both conductors have the same diameter.

Find: (a) The wavelength and element length, (b) the input impedance, (c) the VSWR when connected to 300 Ω twin-lead, (d) the VSWR when connected to 50 Ω coax without a matching network, and (e) the bandwidth advantage over a simple dipole.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 98 \times 10^6 = 3.061$ m Element length $= 0.95 \times \lambda/2 = 0.95 \times 1.531 = 1.454$ m

(b) Folded dipole impedance = 4 × 73 = 292 Ω

(c) VSWR on 300 Ω twin-lead: $\Gamma = (300 - 292) / (300 + 292) = 8 / 592 = 0.0135$ VSWR $= (1 + 0.0135) / (1 - 0.0135) = 1.03{:}1$ — an excellent match

(d) VSWR on 50 Ω coax: $\Gamma = (292 - 50) / (292 + 50) = 242 / 342 = 0.708$ VSWR $= (1 + 0.708) / (1 - 0.708) = 1.708 / 0.292 = 5.85{:}1$ — a severe mismatch requiring a 4:1 balun

(e) A simple dipole has a bandwidth of approximately 5–8% of center frequency. The folded dipole achieves 10–20% bandwidth due to its broader impedance characteristic. At 98 MHz, this corresponds to approximately 10–20 MHz, covering a significant portion of the FM band (88–108 MHz).

Problem 16.2.4

Given: A quarter-wave monopole is designed for a marine VHF radio at 156.8 MHz (Channel 16, international distress). The antenna is mounted on a metal mast with a ground plane radius of 0.3λ.

Find: (a) The element length, (b) the input impedance, (c) the gain, (d) the VSWR on 50 Ω coax, and (e) the minimum ground plane radius in centimeters.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 156.8 \times 10^6 = 1.913$ m Quarter-wave length $= \lambda/4 = 1.913 / 4 = 0.478$ m (47.8 cm)

(b) $Z_{in} \approx 36.5\ \Omega$ (half the dipole impedance due to ground plane image)

(c) Gain = 5.15 dBi (3.0 dBd) — the ground plane doubles the directivity compared to a dipole

(d) VSWR on 50 Ω coax: $\Gamma = (50 - 36.5) / (50 + 36.5) = 13.5 / 86.5 = 0.156$ VSWR = $(1 + 0.156) / (1 - 0.156) = 1.156 / 0.844 = 1.37{:}1$

This is an acceptable match (return loss = 16.1 dB).

(e) Minimum ground plane radius $\approx \lambda/4 = 1.913 / 4 = 0.478$ m = 47.8 cm The specified $0.3\lambda = 0.574$ m exceeds this minimum, providing adequate ground plane performance.

Problem 16.2.5

Given: An AM broadcast tower operates at 1,000 kHz as a quarter-wave monopole. The tower has 120 buried radial wires, each $\lambda/4$ long. The total ground system loss resistance is 2 Ω.

Find: (a) The wavelength and tower height, (b) the radiation resistance, (c) the radiation efficiency, (d) the radial wire length, and (e) the total wire required for the ground system.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 1 \times 10^6 = 300$ m Tower height = $\lambda/4 = 75$ m (246 ft)

(b) $R_{rad} = 36.5\ \Omega$ (quarter-wave monopole over ground)

(c) $\eta = R_{rad} / (R_{rad} + R_{loss}) = 36.5 / (36.5 + 2) = 36.5 / 38.5 = 0.948$ (94.8%)

(d) Each radial wire = $\lambda/4 = 75$ m

(e) Total wire = $120 \times 75 = 9{,}000$ m (9.0 km)

This extensive ground system is standard for AM broadcast. Reducing the number of radials to 60 would approximately double the ground loss resistance to 4 Ω, lowering efficiency to 90.1%, which represents a 0.22 dB reduction in radiated power compared to the 120-radial configuration.

Problem 16.2.6

Given: A circular loop antenna has a radius of 10 cm and 1 turn, operating at 150 MHz.

Find: (a) The circumference in wavelengths, (b) whether the loop is electrically small or resonant, (c) the radiation resistance, (d) the gain, and (e) the direction of maximum radiation.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 150 \times 10^6 = 2.0$ m Circumference $C = 2\pi \times 0.10 = 0.6283$ m $C/\lambda = 0.6283 / 2.0 = 0.314$

(b) Since $C/\lambda = 0.314$ is between the small-loop regime ($C < 0.1\lambda$) and the full-wave loop ($C = \lambda$), this is a partial-wave loop that does not fit neatly into either category. It is closer to a small loop but its radiation resistance will be significantly higher than the small-loop approximation suggests. For an accurate design, numerical methods (such as NEC) would be needed.

(c) Using the small-loop formula as an approximation: $A = \pi \times 0.10^2 = 0.03142\ m^2$ $R_{rad} = 320\pi^4(A/\lambda^2)^2 = 320 \times 97.41 \times (0.03142\ /\ 4.0)^2 = 31{,}171 \times (7.854 \times 10^{-3})^2 = 31{,}171 \times 6.168 \times 10^{-5} = 1.92\ \Omega$

Note: This formula underestimates the actual radiation resistance for $C/\lambda = 0.314$. The true value would be higher.

(d) A small loop has gain ≈ 1.76 dBi (same pattern as a short dipole but rotated 90°).

(e) Maximum radiation is in the plane of the loop (perpendicular to the loop axis), with nulls along the loop axis.

Problem 16.2.7

Given: A 3-element Yagi-Uda antenna for the 70 cm amateur band operates at 432 MHz. The design uses a reflector, a driven element (folded dipole), and one director with 0.20λ reflector spacing and 0.25λ director spacing.

Find: (a) The wavelength, (b) the element lengths, (c) the total boom length, (d) the approximate gain, and (e) the input impedance.

Solution:

(a) $\lambda = c/f = 3 \times 10^8\ /\ 432 \times 10^6 = 0.694$ m (69.4 cm)

(b) Element lengths: Reflector = $1.05 \times \lambda/2 = 1.05 \times 0.347 = 0.365$ m (36.5 cm) Driven element (folded dipole) = $0.95 \times \lambda/2 = 0.95 \times 0.347 = 0.330$ m (33.0 cm) Director = $0.91 \times \lambda/2 = 0.91 \times 0.347 = 0.316$ m (31.6 cm)

(c) Total boom length: Boom = reflector-to-driven + driven-to-director = $0.20\lambda + 0.25\lambda = 0.45\lambda$ Boom = $0.45 \times 0.694 = 0.312$ m (31.2 cm)

(d) A 3-element Yagi achieves approximately 7.5 dBi (5.3 dBd) with a front-to-back ratio of 15–20 dB.

(e) Without the folded dipole, $Z_{in} \approx 20\text{–}25\ \Omega$ due to mutual coupling. With the folded dipole (4:1 transformation): $Z_{in} \approx 4 \times 22 = 88\ \Omega$. A gamma match or beta match can bring this to 50 Ω for direct coax connection.

Problem 16.2.8

Given: A 10-element Yagi-Uda antenna for a 2.4 GHz Wi-Fi link uses a half-wave dipole driven element, one reflector, and eight directors. Element spacing is 0.20λ for the reflector and 0.30λ for the directors.

Find: (a) The wavelength, (b) the total boom length, (c) the expected gain, (d) the HPBW estimate using $\theta \approx 65°/\sqrt{G_{linear}}$, and (e) the input impedance.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 2.4 \times 10^9 = 0.125$ m (12.5 cm)

(b) Boom = reflector spacing + 8 × director spacing = $0.20\lambda + 8 \times 0.30\lambda = 0.20 + 2.40 = 2.60\lambda$ Boom length = $2.60 \times 0.125 = 0.325$ m (32.5 cm)

(c) A 10-element Yagi achieves approximately 13 dBi (10.8 dBd) with a front-to-back ratio exceeding 25 dB.

(d) $G_{linear} = 10^{13/10} = 20.0$ $\theta \approx 65° / \sqrt{20.0} = 65° / 4.47 = 14.5°$ (approximate HPBW in both planes)

(e) With a simple dipole driven element and strong mutual coupling from 8 directors: $Z_{in} \approx 18$–$22\ \Omega$ — requiring a matching network (gamma match, hairpin match, or quarter-wave transformer) for 50 Ω coax.

Problem 16.2.9

Given: A ground-plane antenna for 462 MHz (GMRS radio) uses a vertical quarter-wave element with four radials angled downward at 45° from horizontal.

Find: (a) The quarter-wave element length, (b) the radial length, (c) the impedance increase due to drooping radials (impedance increases from 36.5 Ω toward 50 Ω as radials droop), (d) the VSWR on 50 Ω coax, and (e) the gain.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 462 \times 10^6 = 0.6494$ m Element length = $\lambda/4 = 0.1623$ m (16.2 cm)

(b) Radial length = $\lambda/4 = 0.1623$ m (16.2 cm) each

(c) With radials drooped 45° from horizontal, the input impedance increases from 36.5 Ω to approximately 50 Ω — the drooping radials modify the image current distribution, effectively raising the impedance toward a direct match to 50 Ω coax. Empirically, 45° droop raises the impedance by a factor of approximately 1.37: $36.5 \times 1.37 = 50.0\ \Omega$.

(d) VSWR = (50 / 50) = 1.0:1 — a nearly perfect match, which is why drooping-radial ground-plane antennas are widely used for direct 50 Ω feed.

(e) The drooping radials tilt the radiation pattern slightly upward from the horizon. Gain is approximately 4.5 dBi (2.3 dBd) — slightly less than a flat-radial ground plane due to the altered pattern shape.

Problem 16.2.10

Given: An RFID reader uses a rectangular loop antenna with dimensions 8 cm × 12 cm and 5 turns, operating at 13.56 MHz (HF RFID).

Find: (a) The circumference in wavelengths, (b) the loop area, (c) the single-turn radiation resistance, (d) the total radiation resistance for 5 turns, and (e) the inductance of the loop if $L \approx \mu_0 N^2(a + b)/\pi \times [\ln(2(a + b)/w) - 0.774]$, where $a = 0.08$ m, $b = 0.12$ m, and wire diameter $w = 1$ mm.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 13.56 \times 10^6 = 22.12$ m Circumference = $2(0.08 + 0.12) = 0.40$ m $C/\lambda = 0.40 / 22.12 = 0.0181$ — well within the small-loop regime ($C < 0.1\lambda$)

(b) $A = 0.08 \times 0.12 = 9.6 \times 10^{-3}$ m² (96 cm²)

(c) Single-turn radiation resistance: $R_{rad} = 320\pi^4(A/\lambda^2)^2 = 320 \times 97.41 \times (9.6 \times 10^{-3} / 489.3)^2$ $R_{rad} = 31{,}171 \times (1.962 \times 10^{-5})^2 = 31{,}171 \times 3.850 \times 10^{-10} = 1.20 \times 10^{-5}\ \Omega$ (12.0 $\mu\Omega$)

(d) For N = 5 turns, R_{rad} scales as N^2: $R_{total} = 25 \times 1.20 \times 10^{-5} = 3.00 \times 10^{-4}\ \Omega$ (300 $\mu\Omega$)

(e) Inductance: $L = \mu_0 N^2(a + b)/\pi \times [\ln(2(a + b)/w) - 0.774]$ $L = 4\pi \times 10^{-7} \times 25 \times 0.20 / \pi \times [\ln(2 \times 0.20 / 0.001) - 0.774]$ $L = 4 \times 10^{-7} \times 25 \times 0.20 \times [\ln(400) - 0.774]$ $L = 2.0 \times 10^{-6} \times [5.991 - 0.774]$ $L = 2.0 \times 10^{-6} \times 5.217 = 10.4\ \mu H$

The extremely low radiation resistance confirms that HF RFID operates via near-field inductive coupling, not radiation. The loop functions as one half of a loosely coupled transformer with the tag coil.

Chapter 16 — Section 16.3: Aperture Antennas

Practice problems covering horn antennas (pyramidal, sectoral), parabolic reflector antennas (prime focus, Cassegrain, offset), and slot and cavity-backed antennas.

Problem 16.3.1

Given: A pyramidal horn antenna operating at 8 GHz (X-band) has aperture dimensions of a = 15 cm and b = 12 cm. The aperture efficiency is $\eta_{ap} = 0.51$.

Find: (a) The wavelength, (b) the physical aperture area, (c) the effective aperture, (d) the gain in dBi, and (e) the HPBW in both planes.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 8 \times 10^9 = 0.0375$ m

(b) $A_{phys} = a \times b = 0.15 \times 0.12 = 0.018$ m^2 (180 cm^2)

(c) $A_e = \eta_{ap} \times A_{phys} = 0.51 \times 0.018 = 9.18 \times 10^{-3}$ m^2

(d) $G = 4\pi A_e / \lambda^2 = 4\pi \times 9.18 \times 10^{-3} / (0.0375)^2 = 0.1151 / 1.406 \times 10^{-3} = 81.9$ G (dBi) $= 10 \log_{10}(81.9) = 19.1$ dBi

(e) HPBW in H-plane $\approx 70\lambda/a = 70 \times 0.0375 / 0.15 = 17.5°$ HPBW in E-plane $\approx 70\lambda/b = 70 \times 0.0375 / 0.12 = 21.9°$

Problem 16.3.2

Given: An optimum-gain pyramidal horn antenna is needed with a target gain of 23 dBi at 12 GHz. The horn is fed by WR-90 waveguide (22.86 mm × 10.16 mm internal dimensions). Use $\eta_{ap} = 0.51$.

Find: (a) The required effective aperture, (b) the physical aperture dimensions (assume a square aperture), (c) the horn axial length assuming a flare semi-angle of 12° in each plane, and (d) the HPBW.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 12 \times 10^9 = 0.025$ m $G = 10^{23/10} = 199.5$ (linear) $A_e = G\lambda^2 / (4\pi) = 199.5 \times 0.025^2 / (4\pi) = 199.5 \times 6.25 \times 10^{-4} / 12.566 = 9.92 \times 10^{-3}$ m²

(b) $A_{phys} = A_e / \eta_{ap} = 9.92 \times 10^{-3} / 0.51 = 1.945 \times 10^{-2}$ m² For a square aperture: $a = b = \sqrt{(}1.945 \times 10^{-2}) = 0.1395$ m (13.95 cm)

(c) The horn flares from the waveguide dimensions to the aperture. Taking the H-plane: Half-aperture = a/2 = 0.0698 m; half-waveguide = 0.01143 m Flare extent = 0.0698 − 0.01143 = 0.0584 m Horn length = flare extent / tan(12°) = 0.0584 / 0.2126 = 0.275 m (27.5 cm)

(d) HPBW ≈ $70\lambda / a = 70 \times 0.025 / 0.1395 = 12.5°$ in both planes

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Problem 16.3.3

Given: A prime-focus parabolic reflector antenna has a diameter of 4.5 m and operates at 7.5 GHz for a satellite earth station. The aperture efficiency is 0.58 and the f/D ratio is 0.38.

Find: (a) The gain, (b) the HPBW, (c) the focal length, (d) the feed illumination half-angle, and (e) the far-field distance.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 7.5 \times 10^9 = 0.04$ m $G = \eta(\pi D/\lambda)^2 = 0.58 \times (\pi \times 4.5 / 0.04)^2 = 0.58 \times (353.4)^2 = 0.58 \times 124{,}892 = 72{,}437$ G (dBi) = $10 \log_{10}(72{,}437) = 48.6$ dBi

(b) HPBW ≈ $70\lambda / D = 70 \times 0.04 / 4.5 = 0.622°$

(c) Focal length: $f = (f/D) \times D = 0.38 \times 4.5 = 1.71$ m

(d) Feed illumination half-angle: $\theta_f = 2 \arctan(1 / (4 \times f/D)) = 2 \arctan(1 / 1.52) = 2 \arctan(0.658)$ $\theta_f = 2 \times 33.3° = 66.7°$

The feed antenna must illuminate the dish out to ±33.3° from the axis.

(e) Far-field distance: $d_{ff} = 2D^2 / \lambda = 2 \times 4.5^2 / 0.04 = 2 \times 20.25 / 0.04 = 1{,}012.5$ m (approximately 1 km)

Problem 16.3.4

Given: A Cassegrain antenna has a main reflector diameter of 2.4 m, a subreflector diameter of 0.3 m, and operates at 18 GHz. The aperture efficiency is 0.62.

Find: (a) The wavelength, (b) the gain, (c) the HPBW, (d) the blockage ratio (subreflector area to main reflector area), and (e) the gain reduction due to subreflector blockage (approximately $\Delta G \approx -20 \log_{10}(1 - \text{blockage ratio})$).

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 18 \times 10^9 = 0.01667$ m

(b) $G = \eta(\pi D/\lambda)^2 = 0.62 \times (\pi \times 2.4 / 0.01667)^2 = 0.62 \times (452.2)^2 = 0.62 \times 204{,}485 = 126{,}781$ G (dBi) $= 10 \log_{10}(126{,}781) = 51.0$ dBi

(c) HPBW $\approx 70\lambda / D = 70 \times 0.01667 / 2.4 = 0.486°$

(d) Blockage ratio (area): $BR = (d_{sub}/D)^2 = (0.3/2.4)^2 = 0.125^2 = 0.01563$ (1.56%)

(e) Gain reduction: $\Delta G \approx -20 \log_{10}(1 - 0.01563) = -20 \log_{10}(0.9844) = -20 \times (-0.00683) = 0.14$ dB

The subreflector blockage is minor. In practice, the Cassegrain's advantage of mounting the feed behind the dish (shorter feed line, easier maintenance) far outweighs the small blockage penalty.

Problem 16.3.5

Given: An offset parabolic reflector for Ku-band satellite reception has a projected aperture of 0.75 m × 0.50 m (elliptical due to offset geometry) and operates at 12.5 GHz. The aperture efficiency is 0.68 (higher than prime-focus due to no blockage).

Find: (a) The wavelength, (b) the projected aperture area, (c) the gain, (d) the effective aperture, and (e) the HPBW in azimuth and elevation.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 12.5 \times 10^9 = 0.024$ m

(b) $A_{phys} = \pi \times (0.75/2) \times (0.50/2) = \pi \times 0.375 \times 0.25 = 0.2945$ m^2

(c) $G = \eta \times 4\pi A_{phys} / \lambda^2 = 0.68 \times 4\pi \times 0.2945 / 0.024^2$ $G = 0.68 \times 3.696 / 5.76 \times 10^{-4} = 2.513 / 5.76 \times 10^{-4} = 4{,}363$ G (dBi) $= 10 \log_{10}(4{,}363) = 36.4$ dBi

(d) $A_e = \eta \times A_{phys} = 0.68 \times 0.2945 = 0.200$ m^2

(e) HPBW in azimuth (wider dimension): $\theta_{az} \approx 70\lambda / 0.75 = 70 \times 0.024 / 0.75 = 2.24°$ HPBW in elevation (narrower dimension): $\theta_{el} \approx 70\lambda / 0.50 = 70 \times 0.024 / 0.50 = 3.36°$

The elliptical beam shape reflects the asymmetric aperture of the offset geometry.

Problem 16.3.6

Given: A half-wave slot antenna is cut in a ground plane for an S-band radar at 3.0 GHz. A complementary dipole at the same frequency has an impedance of $Z_{dipole} = 73\ \Omega$.

Find: (a) The slot length, (b) the slot impedance using Babinet's principle, (c) the impedance with a $\lambda/4$ cavity backing, (d) the quarter-wave transformer impedance to match the cavity-backed slot to 50 Ω, and (e) the gain of the cavity-backed slot.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 3.0 \times 10^9 = 0.10$ m Slot length $= \lambda/2 = 50.0$ mm

(b) $Z_{slot} = \eta^2 / (4 \times Z_{dipole}) = 377^2 / (4 \times 73) = 142{,}129 / 292 = 486.7\ \Omega$

(c) Cavity backing eliminates rear radiation, reducing the impedance by approximately half: $Z_{cavity} \approx 486.7 / 2 = 243\ \Omega$

(d) Quarter-wave transformer: $Z_{match} = \sqrt{}(Z_0 \times Z_{cavity}) = \sqrt{}(50 \times 243) = \sqrt{}12{,}150 = 110.2\ \Omega$

(e) The unbacked slot has gain $\approx$ 2.15 dBi (same as a dipole). The cavity backing adds 2–3 dB by eliminating rear radiation: $G_{cavity} \approx 2.15 + 3 = 5.15$ dBi

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Problem 16.3.7

Given: A standard-gain horn antenna is used as a reference for calibration at 15 GHz. The horn aperture is 10 cm × 8 cm with $\eta_{ap} = 0.51$. During a comparison test, the antenna under test (AUT) receives 8.5 dB more power than the reference horn.

Find: (a) The horn gain, (b) the gain of the AUT, (c) the effective aperture of the AUT, and (d) the AUT's HPBW if it is a circular aperture antenna.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 15 \times 10^9 = 0.02$ m $G_{horn} = 4\pi \times \eta_{ap} \times a \times b / \lambda^2 = 4\pi \times 0.51 \times 0.10 \times 0.08 / 0.02^2$ $G_{horn} = 4\pi \times 4.08 \times 10^{-3} / 4 \times 10^{-4} = 4\pi \times 10.2 = 128.2$ G_{horn} (dBi) $= 10 \log_{10}(128.2) = 21.1$ dBi

(b) $G_{AUT} = 21.1 + 8.5 = 29.6$ dBi

(c) G_{AUT} (linear) $= 10^{29.6/10} = 912.0$ $A_e = G\lambda^2 / (4\pi) = 912.0 \times 0.02^2 / (4\pi) = 912.0 \times 4 \times 10^{-4} / 12.566 = 0.0290\ m^2$ (290 cm^2)

(d) If circular: $A_e = \eta \times \pi(D/2)^2$. Assuming $\eta = 0.60$: $\pi(D/2)^2 = 0.0290 / 0.60 = 0.0484\ m^2$ $D/2 = \sqrt{}(0.0484/\pi) = \sqrt{}(0.01540) = 0.1241$ m → D = 0.248 m HPBW $\approx 70\lambda / D = 70 \times 0.02 / 0.248 = 5.65°$

Problem 16.3.8

Given: A weather radar uses a parabolic dish with D = 8.5 m at 5.6 GHz (C-band). The aperture efficiency is 0.55 and the peak transmit power is 250 kW.

Find: (a) The gain, (b) the HPBW, (c) the EIRP in dBW, (d) the effective aperture, and (e) the far-field distance.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 5.6 \times 10^9 = 0.05357$ m $G = \eta(\pi D/\lambda)^2 = 0.55 \times (\pi \times 8.5 / 0.05357)^2 = 0.55 \times (498.4)^2 = 0.55 \times 248{,}402 = 136{,}621$ G (dBi) $= 10 \log_{10}(136{,}621) = 51.4$ dBi

(b) HPBW $\approx 70\lambda / D = 70 \times 0.05357 / 8.5 = 0.441°$

(c) EIRP $= P_t \times G = 250{,}000 \times 136{,}621 = 3.416 \times 10^{10}$ W EIRP (dBW) $= 10 \log_{10}(250{,}000) + 51.4 = 54.0 + 51.4 = 105.4$ dBW

(d) $A_e = \eta \times \pi(D/2)^2 = 0.55 \times \pi \times 4.25^2 = 0.55 \times 56.75 = 31.2\ m^2$

(e) $d_{ff} = 2D^2 / \lambda = 2 \times 8.5^2 / 0.05357 = 144.5 / 0.05357 = 2{,}697$ m (2.7 km)

Problem 16.3.9

Given: A cavity-backed slot antenna array consists of $4 \times 4 = 16$ cavity-backed slot elements at 10 GHz. Each element has a gain of 6 dBi, and the elements are spaced at 0.7λ in both dimensions. The array efficiency (including feed network losses) is 80%.

Find: (a) The element spacing in mm, (b) the total array aperture, (c) the array gain, (d) the HPBW, and (e) the total gain in dBi.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 10 \times 10^9 = 0.03$ m Element spacing $= 0.7 \times 0.03 = 0.021$ m $= 21.0$ mm

(b) Array aperture $= (4 \times 21.0) \times (4 \times 21.0) = 84.0$ mm $\times$ 84.0 mm $= 70.56\ cm^2$

(c) Ideal array gain $= N \times G_{element} \times \eta_{array}$ $G_{element}$ (linear) $= 10^{6/10} = 3.981$ $G_{total} = 16 \times 3.981 \times 0.80 = 50.96$ G_{total} (dBi) $= 10 \log_{10}(50.96) = 17.1$ dBi

(d) For a 4-element linear array at 0.7λ spacing: HPBW $\approx 0.886\lambda / (N \times d) = 0.886 \times 0.03 / (4 \times 0.021) = 0.02658 / 0.084 = 0.3164$ rad $= 18.1°$ in both planes

(e) The total gain is 17.1 dBi as calculated in part (c). This represents an 11.1 dB improvement over a single element (12 dB from the 16-element array factor minus 0.9 dB for feed losses).

Problem 16.3.10

Given: A Ku-band satellite earth station uses a Cassegrain antenna with D = 6.1 m at 14 GHz (uplink) and 11.7 GHz (downlink). The aperture efficiency is 0.60 at both frequencies. The subreflector diameter is 0.6 m.

Find: (a) The gain at 14 GHz and 11.7 GHz, (b) the HPBW at each frequency, (c) the effective aperture at each frequency, and (d) the blockage loss due to the subreflector.

Solution:

(a) At 14 GHz: $\lambda = 3 \times 10^8 / 14 \times 10^9 = 0.02143$ m $G_{14} = 0.60 \times (\pi \times 6.1 / 0.02143)^2 = 0.60 \times (894.2)^2 = 0.60 \times 799{,}594 = 479{,}756$ G_{14} (dBi) $= 10 \log_{10}(479{,}756) = 56.8$ dBi

At 11.7 GHz: $\lambda = 3 \times 10^8 / 11.7 \times 10^9 = 0.02564$ m $G_{11.7} = 0.60 \times (\pi \times 6.1 / 0.02564)^2 = 0.60 \times (747.6)^2 = 0.60 \times 558{,}905 = 335{,}343$ $G_{11.7}$ (dBi) $= 10 \log_{10}(335{,}343) = 55.3$ dBi

(b) At 14 GHz: HPBW $\approx 70 \times 0.02143 / 6.1 = 0.246°$ At 11.7 GHz: HPBW $\approx 70 \times 0.02564 / 6.1 = 0.294°$

(c) At 14 GHz: $A_e = 0.60 \times \pi \times 3.05^2 = 0.60 \times 29.22 = 17.53\ m^2$ At 11.7 GHz: A_e = same physical antenna, so $A_e = 17.53\ m^2$

Note: The effective aperture depends on the physical aperture and efficiency, not frequency. The gain is higher at 14 GHz because A_e/λ^2 is larger at higher frequency.

(d) Blockage ratio $= (d_{sub}/D)^2 = (0.6/6.1)^2 = 0.0983^2 = 0.00967$ (0.97%) Blockage loss $\approx -20 \log_{10}(1 - 0.00967) = -20 \times (-0.00421) = 0.08$ dB — negligible

Chapter 16 — Section 16.4: Printed and Microstrip Antennas

Practice problems covering rectangular patch antenna design (dimensions, effective dielectric constant, fringing extensions, impedance), patch antenna variations (circular patches, circular polarization, stacked patches), and feed techniques.

Problem 16.4.1

Given: A rectangular patch antenna is designed on Rogers RO4003C substrate ($\varepsilon_r = 3.55$, $h = 0.813$ mm) for a 5.8 GHz ISM-band application.

Find: (a) The patch width W, (b) the effective dielectric constant $\varepsilon_{r,eff}$, (c) the fringing length extension ΔL, (d) the patch length L, and (e) the expected bandwidth if $Q \approx c\sqrt{\varepsilon_r}/(4hf_r)$.

Solution:

(a) Patch width: $W = c/(2f_r) \times \sqrt{(2/(\varepsilon_r + 1))} = (3 \times 10^8)/(2 \times 5.8 \times 10^9) \times \sqrt{(2/4.55)}$ $W = 0.02586 \times 0.6630 = 17.1$ mm

(b) Effective dielectric constant: $\varepsilon_{r,eff} = (\varepsilon_r + 1)/2 + (\varepsilon_r - 1)/2 \times (1 + 12h/W)^{-0.5}$ $\varepsilon_{r,eff} = 4.55/2 + 2.55/2 \times (1 + 12 \times 0.813/17.1)^{-0.5}$ $\varepsilon_{r,eff} = 2.275 + 1.275 \times (1 + 0.5705)^{-0.5} = 2.275 + 1.275 \times (1.5705)^{-0.5}$ $\varepsilon_{r,eff} = 2.275 + 1.275 \times 0.7981 = 2.275 + 1.018 = 3.293$

(c) Fringing extension: $\Delta L = 0.412h \times (\varepsilon_{r,eff} + 0.3)(W/h + 0.264) / ((\varepsilon_{r,eff} - 0.258)(W/h + 0.8))$ $W/h = 17.1/0.813 = 21.03$ $\Delta L = 0.412 \times 0.813 \times (3.593)(21.29) / ((3.035)(21.83))$ $\Delta L = 0.3350 \times 76.52 / 66.25 = 0.387$ mm

(d) Patch length: $L = c/(2f_r\sqrt{\varepsilon_{r,eff}}) - 2\Delta L = (3 \times 10^8)/(2 \times 5.8 \times 10^9 \times \sqrt{3.293}) - 2 \times 0.387$ $L = 0.02586 / 1.8147 - 0.774 = 14.25 - 0.77 = 13.5$ mm

(e) $Q \approx c\sqrt{\varepsilon_r}/(4hf_r) = 3 \times 10^8 \times 1.884 / (4 \times 0.813 \times 10^{-3} \times 5.8 \times 10^9)$ $Q = 5.652 \times 10^8 / 1.886 \times 10^7 = 30.0$ Bandwidth $\approx 1/Q = 1/30.0 = 0.033 = 3.3\%$ (approximately 192 MHz centered on 5.8 GHz)

Problem 16.4.2

Given: A rectangular patch antenna on Taconic TLY-5 substrate ($\varepsilon_r = 2.2$, h = 1.575 mm) operates at 10 GHz. The patch dimensions are W = 12.2 mm and L = 9.8 mm.

Find: (a) The effective dielectric constant, (b) the resonant frequency, (c) the edge impedance R_{edge} using $R_{edge} \approx 90 \times \varepsilon_r^2/(\varepsilon_r - 1) \times (L/W)^2$, (d) the inset feed distance for 50 Ω match using $R_{in}(y_0) = R_{edge} \times \cos^2(\pi y_0/L)$, and (e) the gain.

Solution:

(a) $\varepsilon_{r,eff} = (2.2 + 1)/2 + (2.2 - 1)/2 \times (1 + 12 \times 1.575/12.2)^{-0.5}$ $\varepsilon_{r,eff} = 1.6 + 0.6 \times (1 + 1.549)^{-0.5} = 1.6 + 0.6 \times (2.549)^{-0.5}$ $\varepsilon_{r,eff} = 1.6 + 0.6 \times 0.6262 = 1.6 + 0.376 = 1.976$

(b) $f_r = c / (2(L + 2\Delta L)\sqrt{\varepsilon_{r,eff}})$ $\Delta L = 0.412 \times 1.575 \times (1.976 + 0.3)(12.2/1.575 + 0.264) / ((1.976 - 0.258)(12.2/1.575 + 0.8))$ $\Delta L = 0.649 \times (2.276)(8.010) / ((1.718)(8.546)) = 0.649 \times 18.23 / 14.69 = 0.805$ mm $f_r = 3 \times 10^8 / (2 \times (9.8 + 1.61) \times 10^{-3} \times 1.406) = 3 \times 10^8 / (2 \times 11.41 \times 10^{-3} \times 1.406)$ $f_r = 3 \times 10^8 / 0.03209 = 9.35$ GHz

The resonant frequency is lower than 10 GHz because the given dimensions produce a slightly larger patch than needed.

(c) $R_{edge} \approx 90 \times (2.2)^2 / (2.2 - 1) \times (9.8/12.2)^2$ $R_{edge} = 90 \times 4.84 / 1.2 \times 0.6448 = 363.0 \times 0.6448 = 234$ Ω

(d) For 50 Ω match: $\cos^2(\pi y_0/L) = R_{in}/R_{edge} = 50/234 = 0.2137$ $\cos(\pi y_0/L) = 0.4623$ $\pi y_0/L = \arccos(0.4623) = 1.089$ rad $y_0 = 1.089 \times 9.8 / \pi = 3.40$ mm from the radiating edge

(e) A single rectangular patch on a low-ε_r substrate typically achieves 7–8 dBi. With W/L = 1.24, the gain is approximately 7.5 dBi.

Problem 16.4.3

Given: A circular patch antenna is designed for GPS L1 reception at 1.575 GHz on a substrate with $\varepsilon_r = 4.4$ and h = 1.6 mm (FR-4).

Find: (a) The effective radius using $a_{eff} = 1.8412c/(2\pi f_r\sqrt{\varepsilon_r})$, (b) the physical radius after fringing correction $a = a_{eff} / \sqrt{(1 + 2h/(\pi\varepsilon_r a_{eff}) \times (\ln(\pi a_{eff}/(2h)) + 1.7726))}$, (c) the gain, and (d) the bandwidth.

Solution:

(a) Effective radius: $a_{eff} = 1.8412 \times c / (2\pi f_r\sqrt{\varepsilon_r}) = 1.8412 \times 3 \times 10^8 / (2\pi \times 1.575 \times 10^9 \times \sqrt{4.4})$ $a_{eff} = 5.524 \times 10^8 / (2\pi \times 1.575 \times 10^9 \times 2.098)$ $a_{eff} = 5.524 \times 10^8 / (2.076 \times 10^{10}) = 0.02662$ m (26.62 mm)

(b) Physical radius: Correction factor $= 1 + 2h/(\pi\varepsilon_r a_{eff}) \times (\ln(\pi a_{eff}/(2h)) + 1.7726) = 1 + 2 \times 1.6/(\pi \times 4.4 \times 26.62) \times (\ln(\pi \times 26.62/(2 \times 1.6)) + 1.7726) = 1 + 3.2/(367.6) \times (\ln(26.17) + 1.7726) = 1 + 0.008706 \times (3.264 + 1.7726) = 1 + 0.008706 \times 5.037 = 1 + 0.04386 = 1.04386$ $a = a_{eff} / \sqrt{(1.04386)} = 26.62 / 1.0217 = 26.05$ mm

The physical radius is 26.1 mm, giving a patch diameter of 52.2 mm.

(c) A circular patch on FR-4 achieves approximately 5–6 dBi. The higher ε_r reduces radiation efficiency. Estimated gain ≈ 5.5 dBi

(d) $Q \approx c\sqrt{\varepsilon_r}/(4hf_r) = 3 \times 10^8 \times 2.098 / (4 \times 1.6 \times 10^{-3} \times 1.575 \times 10^9)$ $Q = 6.293 \times 10^8 / 1.008 \times 10^7 = 62.4$ Bandwidth $\approx 1/Q = 1/62.4 = 0.016 = 1.6\%$ (approximately 25 MHz)

This narrow bandwidth is marginal for GPS L1, which requires about 2 MHz. The low substrate thickness is the limiting factor.

Problem 16.4.4

Given: A circularly polarized square patch antenna uses the truncated-corner technique at 2.45 GHz on Rogers RT/duroid 5880 ($\varepsilon_r = 2.2$, $h = 1.575$ mm). The square patch side length is 41.2 mm.

Find: (a) The patch quality factor Q, (b) the truncation dimension $\Delta C \approx L/(2Q)$, (c) the area of each truncated corner triangle, (d) the axial ratio bandwidth (typically 1–2% for single-feed CP), and (e) the gain.

Solution:

(a) $Q \approx c\sqrt{\varepsilon_r}/(4hf_r) = 3 \times 10^8 \times \sqrt{2.2} / (4 \times 1.575 \times 10^{-3} \times 2.45 \times 10^9)$ $Q = 3 \times 10^8 \times 1.483 / (1.544 \times 10^7) = 4.449 \times 10^8 / 1.544 \times 10^7 = 28.8$

(b) $\Delta C \approx L/(2Q) = 41.2 / (2 \times 28.8) = 41.2 / 57.6 = 0.715$ mm

(c) Each truncated corner removes a right isosceles triangle with legs = ΔC: $A_{triangle} = 0.5 \times \Delta C^2 = 0.5 \times 0.715^2 = 0.5 \times 0.511 = 0.256$ mm^2

Two opposite corners are truncated, removing a total of 0.512 mm^2 from the original 1,697 mm^2 patch area — less than 0.03% of the total area.

(d) The axial ratio bandwidth (AR < 3 dB) for a single-feed truncated-corner CP patch is approximately: $BW_{AR} \approx 1/Q = 1/28.8 = 3.5\%$ — about 86 MHz at 2.45 GHz.

In practice, the 3 dB AR bandwidth is typically 1–2%, as the axial ratio degrades faster than the impedance match away from center frequency.

(e) A CP patch has gain approximately 1 dB less than a linearly polarized patch of the same size due to cross-polarization losses near the band edges. Estimated gain ≈ 6.5 dBi (RHCP or LHCP depending on which corners are truncated).

Problem 16.4.5

Given: A microstrip patch antenna for 3.5 GHz (5G sub-6 GHz band) is designed on a substrate with $\varepsilon_r = 3.0$ and $h = 1.524$ mm. The antenna uses probe feeding at a distance y_0 from the patch center.

Find: (a) The patch width W, (b) the effective dielectric constant, (c) the patch length L, (d) the edge impedance, and (e) the probe position y_0 for a 50 Ω match.

Solution:

(a) $W = c/(2f_r) \times \sqrt{(2/(\varepsilon_r + 1))} = (3 \times 10^8)/(2 \times 3.5 \times 10^9) \times \sqrt{(2/4.0)}$ $W = 0.04286 \times 0.7071 = 30.3$ mm

(b) $\varepsilon_{r,eff} = (3.0 + 1)/2 + (3.0 - 1)/2 \times (1 + 12 \times 1.524/30.3)^{-0.5}$ $\varepsilon_{r,eff} = 2.0 + 1.0 \times (1 + 0.6038)^{-0.5} = 2.0 + 1.0 \times (1.6038)^{-0.5}$ $\varepsilon_{r,eff} = 2.0 + 1.0 \times 0.7895 = 2.790$

(c) $\Delta L = 0.412 \times 1.524 \times (2.790 + 0.3)(30.3/1.524 + 0.264) / ((2.790 - 0.258)(30.3/1.524 + 0.8))$ $\Delta L = 0.628 \times (3.090)(20.15) / ((2.532)(20.68))$ $\Delta L = 0.628 \times 62.26 / 52.36 = 0.747$ mm

$L = c/(2f_r\sqrt{\varepsilon_{r,eff}}) - 2\Delta L = (3 \times 10^8)/(2 \times 3.5 \times 10^9 \times 1.670) - 2 \times 0.747$ $L = 0.04286 / 1.670 - 1.494 = 25.67 - 1.49 = 24.2$ mm

(d) $R_{edge} \approx 90 \times \varepsilon_r^2/(\varepsilon_r - 1) \times (L/W)^2 = 90 \times 9.0/2.0 \times (24.2/30.3)^2$ $R_{edge} = 405 \times 0.638 = 258\ \Omega$

(e) For probe feeding, the impedance varies as $R_{in}(y_0) = R_{edge} \times \cos^2(\pi y_0/L)$. For 50 Ω: $\cos^2(\pi y_0/L) = 50/258 = 0.1938$ $\cos(\pi y_0/L) = 0.4403$ $\pi y_0/L = \arccos(0.4403) = 1.115$ rad $y_0 = 1.115 \times 24.2 / \pi = 8.59$ mm from the patch center (or $12.1 - 8.59 = 3.51$ mm from the radiating edge)

Problem 16.4.6

Given: A 4 × 4 rectangular patch array is designed for 28 GHz (5G mmWave). Each element has a gain of 7 dBi, and elements are spaced at λ/2 in both dimensions. The corporate feed network has 1.5 dB of loss.

Find: (a) The element spacing in mm, (b) the array dimensions, (c) the array directivity, (d) the realized gain (including feed loss), and (e) the HPBW in both planes.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 28 \times 10^9 = 0.01071$ m Element spacing $= \lambda/2 = 5.36$ mm

(b) Array dimensions = 4 × 5.36 = 21.4 mm per side = 21.4 mm × 21.4 mm

(c) $G_{element}$ (linear) $= 10^{7/10} = 5.012$ $D_{array} = N \times G_{element} = 16 \times 5.012 = 80.2$ D_{array} (dBi) $= 10 \log_{10}(80.2) = 19.0$ dBi

(d) $G_{realized} = D_{array} - \text{feed loss} = 19.0 - 1.5 = 17.5$ dBi

(e) HPBW $\approx 0.886\lambda / (4 \times \lambda/2) = 0.886/2 = 0.443$ rad $= 25.4°$ in both planes

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Problem 16.4.7

Given: A stacked patch antenna uses two rectangular patches separated by a foam spacer ($\varepsilon_r \approx 1.05$). The driven patch is on a substrate with $\varepsilon_r = 2.2$ and $h_1 = 1.575$ mm. The parasitic patch is separated by $h_2 = 3.0$ mm of foam. The center frequency is 5.0 GHz.

Find: (a) The driven patch dimensions (W × L), (b) the parasitic patch dimensions (typically 5–10% larger to create a second resonance), (c) the total antenna height, (d) the expected bandwidth improvement over a single patch, and (e) the gain.

Solution:

(a) Driven patch: $W = c/(2f_r) \times \sqrt{(2/(\varepsilon_r + 1))} = (3 \times 10^8)/(2 \times 5 \times 10^9) \times \sqrt{(2/3.2)} = 0.03 \times 0.7906 = 23.7$ mm

$\varepsilon_{r,eff} \approx (2.2 + 1)/2 + (2.2 - 1)/2 \times (1 + 12 \times 1.575/23.7)^{-0.5}$ $\varepsilon_{r,eff} = 1.6 + 0.6 \times (1.797)^{-0.5} = 1.6 + 0.6 \times 0.7458 = 2.047$

$L = c/(2f_r\sqrt{\varepsilon_{r,eff}}) - 2\Delta L \approx 0.03/1.431 - 2 \times 0.68$ mm $\approx 20.96 - 1.36 = 19.6$ mm

Driven patch: 23.7 mm × 19.6 mm

(b) Parasitic patch (7% larger): $W_p = 1.07 \times 23.7 = 25.4$ mm $L_p = 1.07 \times 19.6 = 21.0$ mm

The larger parasitic patch resonates at a slightly lower frequency, and the coupling between patches creates a dual-resonance response.

(c) Total height $= h_1 + h_2 = 1.575 + 3.0 = 4.575$ mm (approximately 0.076λ at 5 GHz)

(d) A single patch on the 1.575 mm substrate would have BW ≈ 3–4%. The stacked configuration expands bandwidth to approximately 15–20% (750 MHz to 1 GHz centered on 5 GHz) by creating two coupled resonances.

(e) The stacked patch achieves approximately 8–9 dBi — slightly higher than a single patch due to the increased effective aperture from the parasitic element.

Problem 16.4.8

Given: A microstrip patch antenna at 900 MHz is designed on a thick substrate ($\varepsilon_r = 2.55$, $h = 6.35$ mm) to maximize bandwidth for IoT applications.

Find: (a) The patch width, (b) the patch length, (c) the Q factor, (d) the impedance bandwidth, and (e) the tradeoff compared to a thinner substrate.

Solution:

(a) $W = c/(2f_r) \times \sqrt{(2/(\varepsilon_r + 1))} = (3 \times 10^8)/(2 \times 900 \times 10^6) \times \sqrt{(2/3.55)}$ $W = 0.1667 \times 0.7505 = 125.1$ mm

(b) $\varepsilon_{r,eff} = (2.55 + 1)/2 + (2.55 - 1)/2 \times (1 + 12 \times 6.35/125.1)^{-0.5}$ $\varepsilon_{r,eff} = 1.775 + 0.775 \times (1 + 0.609)^{-0.5} = 1.775 + 0.775 \times 0.7886 = 1.775 + 0.611 = 2.386$

$\Delta L = 0.412 \times 6.35 \times (2.686)(20.0) / ((2.128)(20.5)) = 2.616 \times 53.72 / 43.62 = 3.22$ mm

$L = c/(2f_r\sqrt{\varepsilon_{r,eff}}) - 2\Delta L = 0.1667 / 1.545 - 6.44 = 107.9 - 6.4 = 101.5$ mm

(c) $Q \approx c\sqrt{\varepsilon_r}/(4hf_r) = 3 \times 10^8 \times 1.597 / (4 \times 6.35 \times 10^{-3} \times 900 \times 10^6)$ $Q = 4.79 \times 10^8 / 2.286 \times 10^7 = 20.9$

(d) $BW \approx 1/Q = 1/20.9 = 0.048 = 4.8\%$ (approximately 43 MHz at 900 MHz)

(e) Compared to h = 1.6 mm (Q ≈ 83, BW ≈ 1.2%): the thick substrate provides 4× the bandwidth but increases surface wave excitation and antenna size. The 125 mm × 102 mm patch is large for a 900 MHz IoT device. A thinner substrate with external matching or a stacked configuration may be more practical.

Problem 16.4.9

Given: A series-fed 8-element linear patch array at 24 GHz uses microstrip patches on Rogers RO3003 ($\varepsilon_r = 3.0$, h = 0.508 mm). Elements are spaced at one guided wavelength λ_g for in-phase excitation.

Find: (a) The guided wavelength $\lambda_g = \lambda_0/\sqrt{\varepsilon_{r,eff}}$, (b) the element spacing, (c) the array length, (d) the estimated array gain, and (e) the HPBW along the array axis.

Solution:

(a) $\lambda_0 = c/f = 3 \times 10^8 / 24 \times 10^9 = 0.0125$ m Estimate $\varepsilon_{r,eff} \approx (3.0 + 1)/2 = 2.0$ (for a wide patch, h/W « 1) $\lambda_g = \lambda_0 / \sqrt{\varepsilon_{r,eff}} = 0.0125 / \sqrt{2.0} = 0.0125 / 1.414 = 8.84$ mm

(b) Element spacing = λ_g = 8.84 mm

(c) Array length = $(N - 1) \times d = 7 \times 8.84 = 61.9$ mm

(d) Single patch gain ≈ 7 dBi. Array factor directivity for N = 8 at λ_g spacing: $D_{AF} \approx N = 8$ (since spacing $\approx \lambda_0/\sqrt{\varepsilon_{r,eff}} > \lambda_0/2$, some grating lobe energy is lost) In practice: $D_{AF} \approx 7$ (accounting for imperfect contributions) $G_{array} \approx 7 + 10 \log_{10}(7) = 7 + 8.5 = 15.5$ dBi (before feed losses)

(e) HPBW along array axis $\approx 0.886\lambda_0 / (N \times d \times \cos\theta_0)$ Since $d = \lambda_g \approx 0.707\lambda_0$ and broadside operation: HPBW $\approx 0.886 \times 0.0125 / (8 \times 8.84 \times 10^{-3}) = 0.01108 / 0.0707 = 0.157$ rad = 8.97°

Problem 16.4.10

Given: A dual-feed circularly polarized patch antenna at 5.8 GHz uses a square patch with two orthogonal feeds connected through a 90° hybrid coupler. The substrate is $\varepsilon_r = 2.2$, h = 1.575 mm. The square patch side length is 18.5 mm.

Find: (a) The theoretical resonant frequency of the patch, (b) the return loss at each port if $S_{11} = -18$ dB, (c) the isolation between ports (typically 20–25 dB for a 90° hybrid), (d) the axial ratio bandwidth compared to a single-feed truncated-corner design, and (e) the gain.

Solution:

(a) $\varepsilon_{r,eff} \approx (2.2 + 1)/2 + (2.2 - 1)/2 \times (1 + 12 \times 1.575/18.5)^{-0.5}$ $\varepsilon_{r,eff} = 1.6 + 0.6 \times (2.021)^{-0.5} = 1.6 + 0.6 \times 0.7032 = 1.6 + 0.422 = 2.022$

$\Delta L \approx 0.7$ mm (estimated for this geometry) $f_r = c / (2(L + 2\Delta L)\sqrt{\varepsilon_{r,eff}}) = 3 \times 10^8 / (2 \times (18.5 + 1.4) \times 10^{-3} \times 1.422)$ $f_r = 3 \times 10^8 / (2 \times 19.9 \times 10^{-3} \times 1.422) = 3 \times 10^8 / 0.05660 = 5.30$ GHz

The patch is slightly too large for 5.8 GHz; trimming to approximately 17.0 mm would center the resonance at 5.8 GHz.

(b) Return loss at each port = 18 dB ($|\Gamma| = 10^{-0.9} = 0.126$) Reflected power = 0.126^2 = 1.6% at each port.

(c) A well-designed 90° hybrid coupler provides 20–25 dB of isolation between the two input ports. Reflected power from the antenna at each feed is routed to the isolation port (terminated in 50 Ω), improving the system-level return loss beyond the individual port values.

(d) The dual-feed technique with a 90° hybrid provides an axial ratio bandwidth of 10–15%, compared to 1–2% for a single-feed truncated-corner design. This is because the hybrid maintains the 90° phase relationship over a broader bandwidth than the perturbation-based single-feed approach.

(e) Gain ≈ 7.0 dBi (RHCP or LHCP, depending on which port is excited). The dual-feed approach provides slightly better CP purity than the truncated-corner method, especially at band edges.

Chapter 16 — Section 16.5: Antenna Arrays

Practice problems covering array theory and pattern multiplication (uniform linear arrays, broadside and end-fire arrays, grating lobes), beamforming and smart antennas (analog, digital, hybrid, massive MIMO), and phased array scanning and grating lobes.

Problem 16.5.1

Given: A uniform linear array (ULA) of 8 isotropic elements operates at 2.4 GHz with half-wavelength spacing. The array is configured for broadside radiation ($\beta = 0$).

Find: (a) The element spacing in mm, (b) the array factor directivity, (c) the HPBW, (d) the first-null beamwidth (FNBW), and (e) the first sidelobe level below the main beam.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 2.4 \times 10^9 = 0.125$ m $d = \lambda/2 = 0.125/2 = 62.5$ mm

(b) $D_{AF} = 2Nd/\lambda = 2 \times 8 \times 0.0625 / 0.125 = 1.0 / 0.125 = 8.0$ (9.0 dB)

(c) $\text{HPBW} \approx 0.886\lambda/(Nd) = 0.886 \times 0.125 / (8 \times 0.0625) = 0.11075 / 0.5 = 0.2215 \text{ rad} = 12.7°$

(d) $\text{FNBW} \approx 2\lambda/(Nd) = 2 \times 0.125 / 0.5 = 0.5 \text{ rad} = 28.6°$

(e) For a uniform linear array, the first sidelobe level is approximately −13.3 dB below the main beam peak. This is a fundamental property of the sinc-like array factor pattern with uniform amplitude weighting.

Problem 16.5.2

Given: A 16-element ULA at 5.8 GHz with $d = \lambda/2$ spacing uses half-wave dipole elements oriented perpendicular to the array axis. The array radiates broadside.

Find: (a) The array factor directivity, (b) the element gain, (c) the total array directivity using pattern multiplication, (d) the total array length, and (e) the HPBW of the total pattern.

Solution:

(a) $D_{AF} = 2Nd/\lambda = 2 \times 16 \times 0.5 = 16.0$ (12.0 dB)

(b) Half-wave dipole gain = 2.15 dBi (1.64 linear)

(c) $D_{total} = D_{AF} \times D_{element} = 16.0 \times 1.64 = 26.24$ D_{total} (dBi) = 12.0 + 2.15 = 14.2 dBi

(d) $\lambda = c/f = 3 \times 10^8 / 5.8 \times 10^9 = 0.05172$ m Array length = $(N - 1) \times d = 15 \times \lambda/2 = 15 \times 0.02586$ = 0.388 m (38.8 cm)

(e) HPBW $\approx 0.886\lambda/(Nd) = 0.886 \times 0.05172 / (16 \times 0.02586)$ HPBW = 0.04582 / 0.4138 = 0.1107 rad = 6.34°

The HPBW in the plane perpendicular to the array axis is determined by the element pattern (dipole): approximately 78° in that plane.

Problem 16.5.3

Given: An end-fire array of 6 elements at 1 GHz uses $d = \lambda/4$ spacing and a progressive phase shift $\beta = -kd = -\pi/2$ radians.

Find: (a) The element spacing, (b) the direction of the main beam, (c) the end-fire directivity, (d) the HPBW, and (e) whether grating lobes are present.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 1 \times 10^9 = 0.3$ m $d = \lambda/4 = 0.075$ m (75 mm)

(b) Main beam direction: $\psi = kd \cos\theta + \beta = 0$ at the main beam. $kd = 2\pi/\lambda \times \lambda/4 = \pi/2$ $\cos\theta_0 = -\beta/(kd) = -(-\pi/2)/(\pi/2) = 1 \rightarrow \theta_0 = 0°$ (along the array axis, end-fire direction)

(c) End-fire directivity for ordinary end-fire: $D = 4Nd/\lambda = 4 \times 6 \times 0.075 / 0.3 = 1.8 / 0.3 = 6.0$ (7.8 dB)

For the Hansen-Woodyard condition ($\beta = -kd - \pi/N$), the directivity increases to approximately $1.789 \times 4Nd/\lambda = 1.789 \times 6.0 = 10.7$ (10.3 dB), but this requires $\beta = -\pi/2 - \pi/6 = -2.094$ rad.

(d) HPBW for ordinary end-fire $\approx 2 \times \sqrt{(0.886\lambda/(Nd))} = 2 \times \sqrt{(0.886 \times 0.3 / (6 \times 0.075))}$ HPBW = $2 \times \sqrt{(0.2658 / 0.45)} = 2 \times \sqrt{0.5907} = 2 \times 0.7686$ rad = 1.537 rad = 88.1°

End-fire arrays have much wider beamwidths than broadside arrays of the same length.

(e) For grating lobes: $\sin\theta_{GL} = \sin\theta_0 + n\lambda/d$. With $\theta_0 = 0°$ ($\cos\theta_0 = 1$): For $n = -1$: the grating lobe appears when $\cos\theta_{GL} = \cos 0° - \lambda/d = 1 - 4 = -3$. Since $|\cos\theta_{GL}| > 1$, no grating lobes exist. At $d = \lambda/4$, the spacing is far too small for grating lobes.

Problem 16.5.4

Given: A planar array of $8 \times 8 = 64$ patch antenna elements operates at 28 GHz with $d = 0.5\lambda$ spacing in both dimensions. Each element has a gain of 6 dBi. The array efficiency is 85%.

Find: (a) The physical array dimensions, (b) the broadside gain, (c) the HPBW in both planes, (d) the total number of elements, and (e) the gain if the array is expanded to 16×16.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 28 \times 10^9 = 0.01071$ m $d = 0.5 \times 0.01071 = 0.005357$ m Array size $= 8 \times 5.357$ mm $= 42.86$ mm per side ≈ 42.9 mm $\times$ 42.9 mm

(b) $G_{element} = 10^{6/10} = 3.981$ $G = \eta \times N \times G_{element} = 0.85 \times 64 \times 3.981 = 216.6$ G (dBi) $= 10 \log_{10}(216.6) = 23.4$ dBi

(c) HPBW $\approx 0.886\lambda / (8 \times d) = 0.886 \times 0.01071 / (8 \times 0.005357) = 9.489 \times 10^{-3} / 0.04286$ HPBW $= 0.2214$ rad $= 12.7°$ in both planes (symmetric square array)

(d) Total elements $= 8 \times 8 = 64$ elements

(e) For $16 \times 16 = 256$ elements: $G = 0.85 \times 256 \times 3.981 = 866.3$ G (dBi) $= 10 \log_{10}(866.3) = 29.4$ dBi

The gain increases by $10 \log_{10}(256/64) = 10 \log_{10}(4) = 6.0$ dB, as expected from quadrupling the number of elements.

Problem 16.5.5

Given: A phased array with 20 elements at 6 GHz is steered to $\theta_0 = 30°$ from broadside. Element spacing is $d = 0.6\lambda$.

Find: (a) The progressive phase shift β, (b) the broadside HPBW, (c) the scanned HPBW at 30°, (d) the gain reduction due to scanning, and (e) the maximum scan angle before a grating lobe appears.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 6 \times 10^9 = 0.05$ m; $d = 0.6 \times 0.05 = 0.03$ m $\beta = -kd \sin\theta_0 = -(2\pi/0.05) \times 0.03 \times \sin 30° = -125.66 \times 0.03 \times 0.5$ $\beta = -1.885$ rad ($-108.0°$)

(b) Broadside HPBW $\approx 0.886\lambda/(Nd) = 0.886 \times 0.05 / (20 \times 0.03) = 0.0443 / 0.6 = 0.0738$ rad $= 4.23°$

(c) Scanned HPBW = broadside HPBW / $\cos\theta_0 = 4.23° / \cos 30° = 4.23° / 0.866 = 4.88°$

(d) Gain reduction $= 10 \log_{10}(\cos 30°) = 10 \log_{10}(0.866) = -0.625$ dB

(e) Grating lobe condition: $d < \lambda/(1 + |\sin\theta_{max}|)$ $0.6\lambda < \lambda/(1 + \sin\theta_{max})$ $1 + \sin\theta_{max} < 1/0.6 = 1.667$ $\sin\theta_{max} < 0.667$ $\theta_{max} = \arcsin(0.667) = 41.8°$

Beyond 41.8°, a grating lobe enters visible space. The 0.6λ spacing limits the useful scan volume compared to the 0.5λ standard.

Problem 16.5.6

Given: A 5G massive MIMO base station uses a 128-element array (16×8) at 3.5 GHz with $\lambda/2$ spacing. Each element gain is 5 dBi. Array efficiency is 88%.

Find: (a) The panel dimensions, (b) the broadside gain, (c) the HPBW in azimuth (8-element dimension) and elevation (16-element dimension), (d) the array gain in dB relative to a single element, and (e) the number of simultaneous user beams with full digital beamforming.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 3.5 \times 10^9 = 0.08571$ m; $d = \lambda/2 = 0.04286$ m Azimuth (8 elements): 8×42.86 mm = 342.9 mm Elevation (16 elements): 16×42.86 mm = 685.7 mm Panel: 34.3 cm × 68.6 cm

(b) $G_{element} = 10^{5/10} = 3.162$ $G = \eta \times N \times G_{element} = 0.88 \times 128 \times 3.162 = 356.1$ G (dBi) = 10 $\log_{10}(356.1) = 25.5$ dBi

(c) Azimuth HPBW $\approx 0.886\lambda/(8 \times \lambda/2) = 0.886/4 = 0.2215$ rad = 12.7° Elevation HPBW $\approx 0.886\lambda/(16 \times \lambda/2) = 0.886/8 = 0.1108$ rad = 6.35°

(d) Array gain over single element = $10 \log_{10}(\eta \times N) = 10 \log_{10}(0.88 \times 128) = 10 \log_{10}(112.6) = 20.5$ dB

(e) With full digital beamforming (one RF chain per element): up to 128 simultaneous beams theoretically. In practice, 16–32 simultaneous users are typical, limited by inter-user interference and spatial resolution.

Problem 16.5.7

Given: A phased array radar at 9.5 GHz has 32 elements with $d = 0.55\lambda$ spacing. The phase shifters have 4-bit resolution (16 phase states).

Find: (a) The phase quantization step, (b) the maximum phase error, (c) the average sidelobe level due to phase quantization, (d) the RMS phase error, and (e) the gain reduction due to quantization.

Solution:

(a) Phase step = $360° / 2^4 = 360° / 16 = 22.5°$

(b) Maximum phase error = ±(phase step)/2 = ±22.5°/2 = ±11.25°

(c) Average sidelobe level due to quantization $\approx -6B$ dB $= -6 \times 4 = -24$ dB below the main beam.

This is the average level; individual quantization lobes may be higher depending on the scan angle.

(d) RMS phase error for uniform quantization: σ_φ = (phase step) / $\sqrt{12}$ = 22.5° / $\sqrt{12}$ = 22.5° / 3.464 = 6.50° (0.1134 rad)

(e) Gain reduction $= -10 \log_{10}(1 - \sigma_\varphi^2)$ where σ_φ is in radians: $\Delta G = -10 \log_{10}(1 - 0.1134^2) = -10 \log_{10}(1 - 0.01286) = -10 \log_{10}(0.9871)$ $\Delta G = -10 \times (-0.00564) = 0.056$ dB — negligible.

The 4-bit phase shifter provides adequate performance for most radar and communication applications.

Problem 16.5.8

Given: An 8-element ULA at 10 GHz with $d = 0.5\lambda$ uses Dolph-Chebyshev amplitude weighting to achieve −25 dB sidelobes (compared to −13.3 dB for uniform weighting).

Find: (a) The sidelobe improvement over uniform weighting, (b) the approximate HPBW broadening factor (typically 1.1–1.3× for −25 dB SLL), (c) the HPBW with uniform weighting, (d) the HPBW with Chebyshev weighting, and (e) the directivity reduction compared to uniform weighting.

Solution:

(a) Sidelobe improvement = −25 − (−13.3) = 11.7 dB improvement in peak sidelobe level

(b) For −25 dB Chebyshev sidelobes, the beamwidth broadening factor is approximately 1.15 relative to uniform weighting.

(c) $\lambda = 0.03$ m; $d = 0.015$ m Uniform HPBW $\approx 0.886\lambda/(Nd) = 0.886 \times 0.03 / (8 \times 0.015) = 0.02658 / 0.12 = 0.2215$ rad $= 12.7°$

(d) Chebyshev HPBW $\approx 1.15 \times 12.7° = 14.6°$

(e) Directivity with uniform weighting: $D_{uniform} = 2Nd/\lambda = 2 \times 8 \times 0.5 = 8.0$ (9.0 dB). Chebyshev directivity is reduced by the taper efficiency, approximately: $\eta_{taper} \approx 1/1.15^2 \approx 0.756$ (approximate, based on beamwidth broadening squared) $D_{Cheb} \approx 0.756 \times 8.0 = 6.05$ Directivity reduction $\approx 9.0 - 10 \log_{10}(6.05) = 9.0 - 7.8 = 1.2$ dB

This 1.2 dB sacrifice in directivity provides an 11.7 dB improvement in sidelobe suppression — a worthwhile tradeoff for radar and communication systems where interference rejection is critical.

Problem 16.5.9

Given: A phased array steers its beam to $\theta_0 = 50°$ from broadside at 3 GHz. The array has 24 elements with $d = 0.5\lambda$. The elements are patch antennas with a $\cos\theta$ element pattern.

Find: (a) The progressive phase shift, (b) the element pattern gain at 50° relative to broadside, (c) the array factor directivity, (d) the scanned beamwidth, and (e) the total gain at the scanned angle.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 3 \times 10^9 = 0.1$ m; $d = 0.05$ m $\beta = -kd \sin\theta_0 = -(2\pi/0.1) \times 0.05 \times \sin 50° = -62.83 \times 0.05 \times 0.766$ $\beta = -2.407$ rad (−137.9°)

(b) Element pattern at 50°: $E(\theta) = \cos\theta$ Power pattern: $|E(50°)|^2 = \cos^2(50°) = 0.413$ Element gain reduction $= 10 \log_{10}(0.413) = -3.84$ dB relative to broadside

(c) Array factor directivity $= 2Nd/\lambda = 2 \times 24 \times 0.5 = 24.0$ (13.8 dB)

(d) Broadside HPBW $= 0.886\lambda/(Nd) = 0.886 \times 0.1 / (24 \times 0.05) = 0.0738$ rad $= 4.23°$ Scanned HPBW $= 4.23° / \cos 50° = 4.23° / 0.643 = 6.58°$

(e) Broadside array gain $= D_{AF}$ (dB) $+ G_{element,broadside}$ Assuming $G_{element} = 6$ dBi at broadside: $G_{scanned} = 13.8 + 6 - 3.84 - 10 \log_{10}(1/\cos 50°) = 13.8 + 6 - 3.84 - 1.92 = 14.0$ dBi

The combined effect of element pattern roll-off (−3.84 dB) and projected aperture reduction (−1.92 dB) yields a total scan loss of 5.76 dB at 50°.

Problem 16.5.10

Given: A 5G mmWave system at 39 GHz uses hybrid beamforming with 4 RF chains and a 64-element array (8×8) with $d = 0.5\lambda$. Each RF chain drives a subarray of 16 elements (4×4). Each element has a gain of 5 dBi.

Find: (a) The element spacing, (b) the subarray gain (analog beamforming within each subarray), (c) the total array gain with coherent combining of all subarrays, (d) the number of simultaneous beams, and (e) the subarray HPBW.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 39 \times 10^9 = 7.692$ mm $d = \lambda/2 = 3.85$ mm

(b) Subarray ($4 \times 4 = 16$ elements): $G_{element} = 10^{5/10} = 3.162$ $G_{subarray} = 16 \times 3.162 = 50.6$ (assuming 100% subarray efficiency) $G_{subarray}$ (dBi) $= 10 \log_{10}(50.6) = 17.0$ dBi

(c) Total array gain = 4 subarrays $\times G_{subarray} = 4 \times 50.6 = 202.4$ (with coherent combining) G_{total} (dBi) $= 10 \log_{10}(202.4) = 23.1$ dBi

Equivalently: $G_{total} = 64 \times 3.162 = 202.4$, confirming the result.

(d) With 4 RF chains and digital beamforming across subarrays: 4 simultaneous beams can be formed, each pointing in a different direction. Each beam uses the full array gain when all subarrays steer to the same direction, or subarray-level gain (17.0 dBi) when pointing independently.

(e) Subarray HPBW: Each subarray has 4 elements per dimension at $\lambda/2$ spacing: HPBW $\approx 0.886\lambda/(4 \times \lambda/2) = 0.886/2 = 0.443$ rad $= 25.4°$ in both planes

The full 8×8 array has HPBW $= 0.886/4 = 12.7°$ per plane.

Chapter 16 — Section 16.6: Impedance Matching and Practical Design

Practice problems covering antenna impedance matching (VSWR, return loss, mismatch loss, L-networks, quarter-wave transformers, baluns) and antenna measurement and testing (VNA measurements, far-field distance, gain measurement methods).

Problem 16.6.1

Given: A Yagi-Uda antenna has an input impedance of $Z_{ant} = 28 + j12\ \Omega$ at 146 MHz and is connected directly to 50 Ω coaxial cable.

Find: (a) The reflection coefficient, (b) the VSWR, (c) the return loss, (d) the mismatch loss, and (e) the percentage of power reflected.

Solution:

(a) $\Gamma = (Z_{ant} - Z_0) / (Z_{ant} + Z_0) = (28 + j12 - 50) / (28 + j12 + 50) = (-22 + j12) / (78 + j12)$ $|\Gamma| = \sqrt{}(22^2 + 12^2) / \sqrt{}(78^2 + 12^2) = \sqrt{}(484 + 144) / \sqrt{}(6{,}084 + 144) = \sqrt{}628 / \sqrt{}6{,}228 = 25.06 / 78.92$ $|\Gamma| = 0.318$

(b) $\text{VSWR} = (1 + 0.318) / (1 - 0.318) = 1.318 / 0.682 = 1.93{:}1$

(c) $\text{RL} = -20 \log_{10}(0.318) = -20 \times (-0.498) = 9.95$ dB

(d) $\text{ML} = -10 \log_{10}(1 - |\Gamma|^2) = -10 \log_{10}(1 - 0.101) = -10 \log_{10}(0.899) = 0.46$ dB

(e) Reflected power $= |\Gamma|^2 = 0.318^2 = 0.101 = 10.1\%$

With VSWR just under 2:1, this match is borderline acceptable. A gamma match or hairpin match would improve performance.

Problem 16.6.2

Given: A 50 Ω patch antenna has S_{11} measurements at three frequencies: −8 dB at 2.35 GHz, −22 dB at 2.45 GHz (center), and −10 dB at 2.55 GHz.

Find: (a) The reflection coefficient at each frequency, (b) the VSWR at each frequency, (c) the mismatch loss at each frequency, (d) the impedance bandwidth (frequencies where VSWR < 2:1), and (e) the fractional bandwidth.

Solution:

(a) $|\Gamma| = 10^{S_{11}/20}$: At 2.35 GHz: $|\Gamma| = 10^{-0.4} = 0.398$ At 2.45 GHz: $|\Gamma| = 10^{-1.1} = 0.0794$ At 2.55 GHz: $|\Gamma| = 10^{-0.5} = 0.316$

(b) VSWR: At 2.35 GHz: VSWR = (1 + 0.398)/(1 − 0.398) = 2.32:1 At 2.45 GHz: VSWR = (1 + 0.0794)/(1 − 0.0794) = 1.17:1 At 2.55 GHz: VSWR = (1 + 0.316)/(1 − 0.316) = 1.92:1

(c) Mismatch loss ML = $-10 \log_{10}(1 - |\Gamma|^2)$: At 2.35 GHz: ML = $-10 \log_{10}(1 - 0.158) = -10 \log_{10}(0.842) = 0.75$ dB At 2.45 GHz: ML = $-10 \log_{10}(1 - 0.0063) = -10 \log_{10}(0.9937) = 0.027$ dB At 2.55 GHz: ML = $-10 \log_{10}(1 - 0.100) = -10 \log_{10}(0.900) = 0.46$ dB

(d) VSWR < 2:1 corresponds to $S_{11} < -9.54$ dB. The 2.35 GHz point (−8 dB) is outside the band, and 2.55 GHz (−10 dB) is inside. By interpolation, the −10 dB bandwidth spans approximately 2.37 GHz to 2.55 GHz = 180 MHz. The VSWR 2:1 bandwidth is slightly wider at approximately 2.36 GHz to 2.56 GHz = 200 MHz.

(e) Fractional bandwidth = 200/2,450 = 0.082 = 8.2%

Problem 16.6.3

Given: A loop antenna with $Z_{ant} = 120\ \Omega$ needs to be matched to a 50 Ω coaxial cable at 435 MHz using an L-network (series inductor, shunt capacitor).

Find: (a) The Q factor of the L-network, (b) the series inductance, (c) the shunt capacitance, (d) the 3 dB bandwidth of the matching network, and (e) a quarter-wave transformer alternative impedance.

Solution:

(a) $Q = \sqrt{(R_{high}/R_{low} - 1)} = \sqrt{(120/50 - 1)} = \sqrt{(2.4 - 1)} = \sqrt{1.4} = 1.183$

(b) Series element (inductor in series with 50 Ω side): $X_L = Q \times R_{low} = 1.183 \times 50 = 59.15\ \Omega$ $L = X_L / (2\pi f) = 59.15 / (2\pi \times 435 \times 10^6) = 59.15 / (2.733 \times 10^9) = 21.6$ nH

(c) Shunt element (capacitor across 120 Ω side): $X_C = R_{high} / Q = 120 / 1.183 = 101.4\ \Omega$ $C = 1 / (2\pi f X_C) = 1 / (2\pi \times 435 \times 10^6 \times 101.4) = 1 / (2.772 \times 10^{11}) = 3.61$ pF

(d) The 3 dB bandwidth of an L-network with Q = 1.183: BW = f / Q = 435 / 1.183 = 368 MHz

The low Q means the matching network has very wide bandwidth — wider than the antenna bandwidth itself, so the match is not the bandwidth-limiting factor.

(e) Quarter-wave transformer: $Z_{match} = \sqrt{(Z_0 \times Z_{ant})} = \sqrt{(50 \times 120)} = \sqrt{6{,}000} = 77.5\ \Omega$ λ/4 at 435 MHz = 0.6897/4 = 0.172 m = 172 mm. A 77.5 Ω section could be fabricated as a coaxial line or microstrip.

Problem 16.6.4

Given: A dipole antenna (balanced, $Z_{ant} = 73\ \Omega$) is connected to 50 Ω coaxial cable (unbalanced) at 150 MHz. A 1:1 current balun is used, and the dipole is shortened to eliminate reactance.

Find: (a) The VSWR without any matching (dipole to 50 Ω), (b) the return loss, (c) why a balun is needed even though the impedance is close to 50 Ω, (d) a 4:1 balun alternative using a folded dipole, and (e) the resulting VSWR with a folded dipole and 4:1 balun.

Solution:

(a) $\Gamma = (73 - 50) / (73 + 50) = 23 / 123 = 0.187$ VSWR $= (1 + 0.187) / (1 - 0.187) = 1.187 / 0.813 = 1.46{:}1$

(b) $RL = -20 \log_{10}(0.187) = 14.6$ dB — acceptable without further matching

(c) A balun is required because the dipole is a balanced structure (equal and opposite currents on each arm), while coax is unbalanced (current on center conductor, return on shield). Without a balun, common-mode current flows on the outside of the coax shield, causing:

- Distortion of the radiation pattern
- Radiation from the feed line
- Increased susceptibility to interference
- Unpredictable input impedance

(d) A folded dipole has $Z_{ant} = 292\ \Omega$. A 4:1 balun transforms this to: $Z_{transformed} = 292 / 4 = 73\ \Omega$ — same as a simple dipole but with wider bandwidth. Alternatively, a 6:1 balun gives $292/6 = 48.7\ \Omega \approx 50\ \Omega$.

(e) With a folded dipole (292 Ω) and a 4:1 balun: $Z_{in} = 73\ \Omega$. VSWR on 50 Ω = 1.46:1 (same as without balun, but now with proper balance and wider bandwidth). With a 6:1 balun: $Z_{in} = 48.7\ \Omega \rightarrow \Gamma = (50 - 48.7)/(50 + 48.7) = 1.3/98.7 = 0.0132 \rightarrow$ VSWR = 1.03:1.

Problem 16.6.5

Given: An antenna under test (AUT) is a 0.6 m dish antenna operating at 18 GHz. Testing will be performed on an outdoor antenna range.

Find: (a) The wavelength, (b) the far-field distance, (c) the AUT gain estimate (assuming $\eta = 0.60$), (d) the HPBW, and (e) the required angular positioning accuracy to stay within 0.1 dB of peak gain (approximately $\theta_{0.1dB} \approx 0.3 \times$ HPBW).

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 18 \times 10^9 = 0.01667$ m (16.67 mm)

(b) $d_{ff} = 2D^2 / \lambda = 2 \times 0.6^2 / 0.01667 = 2 \times 0.36 / 0.01667 = 43.2$ m

(c) $G = \eta(\pi D/\lambda)^2 = 0.60 \times (\pi \times 0.6 / 0.01667)^2 = 0.60 \times (113.1)^2 = 0.60 \times 12{,}792 = 7{,}675$ G (dBi) = $10 \log_{10}(7{,}675) = 38.9$ dBi

(d) HPBW $\approx 70\lambda/D = 70 \times 0.01667 / 0.6 = 1.94°$

(e) $\theta_{0.1dB} \approx 0.3 \times \text{HPBW} = 0.3 \times 1.94° = 0.58°$

The positioner must maintain pointing accuracy within ±0.58° to keep measurements within 0.1 dB of the true peak. For a 43 m range, this is a moderate requirement achievable with standard antenna positioners.

Problem 16.6.6

Given: A VNA measures the following S_{11} data for a monopole antenna at 915 MHz: $S_{11} = 0.35 \angle -65°$ (magnitude and phase of the reflection coefficient). The system impedance is $Z_0 = 50\ \Omega$.

Find: (a) The return loss, (b) the VSWR, (c) the antenna impedance $Z_{ant} = Z_0 \times (1 + \Gamma)/(1 - \Gamma)$, (d) the resistance and reactance components, and (e) the mismatch loss.

Solution:

(a) $\text{RL} = -20 \log_{10}(0.35) = -20 \times (-0.456) = 9.12$ dB

(b) VSWR = (1 + 0.35) / (1 − 0.35) = 1.35 / 0.65 = 2.08:1

(c) $\Gamma = 0.35 \angle -65° = 0.35(\cos(-65°) + j\sin(-65°)) = 0.35 \times (0.4226 - j0.9063)$ $\Gamma = 0.1479 - j0.3172$

$Z_{ant} = Z_0 \times (1 + \Gamma) / (1 - \Gamma) = 50 \times (1.1479 - j0.3172) / (0.8521 + j0.3172)$

Multiply numerator and denominator by the conjugate of the denominator: Denominator magnitude² $= 0.8521^2 + 0.3172^2 = 0.7261 + 0.1006 = 0.8267$ Numerator $= (1.1479 - j0.3172)(0.8521 - j0.3172) = 1.1479 \times 0.8521 - 1.1479 \times j0.3172 - j0.3172 \times 0.8521 + j^2 \times 0.3172^2 = 0.9782 - j0.3641 - j0.2703 - 0.1006 = 0.8776 - j0.6344$

$Z_{ant} = 50 \times (0.8776 - j0.6344) / 0.8267 = 50 \times (1.0616 - j0.7674)$ $Z_{ant} = 53.1 - j38.4\ \Omega$

(d) $R = 53.1\ \Omega$, $X = -38.4\ \Omega$ (capacitive reactance)

The monopole is slightly shorter than resonant — the negative reactance indicates a capacitive input, which could be corrected by lengthening the element or adding a series inductor.

(e) $\text{ML} = -10 \log_{10}(1 - 0.35^2) = -10 \log_{10}(1 - 0.1225) = -10 \log_{10}(0.8775)$ $\text{ML} = 0.57$ dB

Problem 16.6.7

Given: Three unknown antennas (A, B, C) are used in a three-antenna gain measurement at 10 GHz. The following power ratios are measured at a fixed range of 10 m: $P_{AB} = -42.3$ dB (A transmit, B receive), $P_{AC} = -39.8$ dB (A transmit, C receive), $P_{BC} = -44.1$ dB (B transmit, C receive). All measurements use 0 dBm transmit power.

Find: The individual gains G_A, G_B, and G_C in dBi.

Solution:

The Friis equation gives: $P_r/P_t\ (\text{dB}) = G_t + G_r - \text{FSPL}$

First, calculate FSPL at 10 GHz, 10 m range: $\lambda = 0.03$ m FSPL = 20 $\log_{10}(4\pi \times 10 / 0.03)$ = 20 $\log_{10}(4{,}189) = 20 \times 3.622 = 72.4$ dB

Now set up equations (P_r in dBm with $P_t = 0$ dBm, so $P_r = G_t + G_r - \text{FSPL}$): $G_A + G_B = P_{AB} + \text{FSPL} = -42.3 + 72.4 = 30.1$ dB … (1) $G_A + G_C = P_{AC} + \text{FSPL} = -39.8 + 72.4 = 32.6$ dB … (2) $G_B + G_C = P_{BC} + \text{FSPL} = -44.1 + 72.4 = 28.3$ dB … (3)

From (1) and (2): $G_C - G_B = 32.6 - 30.1 = 2.5$ dB … (4) From (3): $G_B + G_C = 28.3$ … (3)

Adding (3) and (4): $2G_C = 30.8 \rightarrow G_C = 15.4$ dBi From (3): $G_B = 28.3 - 15.4 = 12.9$ dBi From (1): $G_A = 30.1 - 12.9 = 17.2$ dBi

Verification: $G_A + G_C = 17.2 + 15.4 = 32.6$ ✓

Problem 16.6.8

Given: A quarter-wave transformer matches a patch antenna ($Z_{ant} = 200\ \Omega$) to a 50 Ω microstrip line at 5 GHz on a substrate with $\varepsilon_r = 2.2$.

Find: (a) The transformer impedance, (b) the transformer physical length, (c) the VSWR at center frequency, (d) the bandwidth over which VSWR < 1.5:1, and (e) the microstrip line width for the transformer section (using approximate formula: $w/h \approx 8e^A/(e^{2A} - 2)$ where $A = Z/(60\sqrt{((\varepsilon_r+1)/2)}) + (\varepsilon_r-1)/(\varepsilon_r+1) \times (0.23 + 0.11/\varepsilon_r)$, for w/h < 2).

Solution:

(a) $Z_{match} = \sqrt{(Z_0 \times Z_{ant})} = \sqrt{(50 \times 200)} = \sqrt{10{,}000} = 100\ \Omega$

(b) $\lambda_0 = c/f = 3 \times 10^8 / 5 \times 10^9 = 0.06$ m $\varepsilon_{r,eff} \approx (2.2 + 1)/2 = 1.6$ (approximate for narrow line) $\lambda_g = \lambda_0 / \sqrt{\varepsilon_{r,eff}} = 0.06 / 1.265 = 0.04743$ m Physical length = $\lambda_g/4 = 11.86$ mm

(c) At center frequency, the transformer provides a perfect match: VSWR = 1.0:1

(d) The quarter-wave transformer bandwidth for VSWR < 1.5 is approximately: $\text{BW} \approx 2f_0 \times (2/\pi) \times \arccos(\Gamma_{max} \times 2\sqrt{(Z_{ant}Z_0)} / |Z_{ant} - Z_0|)$

For VSWR = 1.5: $\Gamma_{max} = (1.5 - 1)/(1.5 + 1) = 0.2$ Fractional BW ≈ $2 - (4/\pi) \times \arccos(0.2 \times 2 \times 100 / 150) = 2 - (4/\pi) \times \arccos(0.267) = 2 - 1.273 \times 1.300 = 2 - 1.655 = 0.345$ BW ≈ 0.345 × 5 GHz = 1.73 GHz (34.5% fractional bandwidth)

(e) For Z = 100 Ω on $\varepsilon_r = 2.2$: $A = 100/60 \times \sqrt{((2.2+1)/2)} + (2.2-1)/(2.2+1) \times (0.23 + 0.11/2.2)$ $A = 1.667 \times \sqrt{1.6} + 0.375 \times (0.23 + 0.05) = 1.667 \times 1.265 + 0.375 \times 0.28$ $A = 2.108 + 0.105 = 2.213$ $w/h = 8 \times e^{2.213} / (e^{4.426} - 2) = 8 \times 9.14 / (83.6 - 2) = 73.1 / 81.6 = 0.896$

For h = 1.575 mm: w = 0.896 × 1.575 = 1.41 mm

Problem 16.6.9

Given: A near-field spherical scanning measurement is performed on a 1.0 m × 0.5 m phased array panel at 30 GHz. The scan radius is 1.5 m from the array center.

Find: (a) The far-field distance, (b) whether the 1.5 m scan radius is in the near field, (c) the advantages of near-field scanning for this antenna, (d) the angular sampling interval required ($\Delta\theta \approx \lambda/(2D)$ radians), and (e) the approximate number of measurement points for a full spherical scan.

Solution:

(a) $D = 1.0$ m (largest dimension); $\lambda = c/f = 3 \times 10^8 / 30 \times 10^9 = 0.01$ m $d_{ff} = 2D^2 / \lambda = 2 \times 1.0 / 0.01 = 200$ m

(b) The scan radius of 1.5 m is well within the near field (1.5 m « 200 m). This is only 0.75% of the far-field distance.

(c) Advantages of near-field scanning for this antenna:

- A 200 m outdoor range is impractical and expensive
- Near-field scanning can be performed in a compact anechoic chamber
- Both amplitude and phase are measured, enabling full far-field pattern computation via Fourier transform
- Multiple far-field cuts can be computed from a single near-field data set

(d) Angular sampling: $\Delta\theta \approx \lambda/(2D) = 0.01 / (2 \times 1.0) = 0.005$ rad $= 0.286°$

(e) For a full spherical scan: $N_\theta = 180° / 0.286° \approx 629$ points in elevation $N_\varphi = 360° / 0.286° \approx 1{,}259$ points in azimuth Total $\approx 629 \times 1{,}259 = 791{,}711$ measurement points

In practice, the back hemisphere may be sparsely sampled if the antenna is known to have low back radiation, reducing the total to approximately 400,000–500,000 points. At typical near-field scanning rates of 10–50 points per second, this requires 3–14 hours of measurement time.

Problem 16.6.10

Given: A sleeve balun is designed for a dipole antenna at 400 MHz. The balun uses a quarter-wave metallic sleeve around the outside of a 50 Ω coaxial cable (outer diameter 10.3 mm). The sleeve inner diameter is 20 mm.

Find: (a) The quarter-wave sleeve length, (b) the characteristic impedance of the coaxial region between the cable outer conductor and the sleeve using $Z = (138/\sqrt{\varepsilon_r}) \times \log_{10}(D/d)$, where $D = 20$ mm, $d = 10.3$ mm, and $\varepsilon_r = 1$ (air), (c) the impedance presented at the feed point by the sleeve, (d) the common-mode suppression mechanism, and (e) the bandwidth of the balun.

Solution:

(a) $\lambda = c/f = 3 \times 10^8 / 400 \times 10^6 = 0.75$ m Sleeve length $= \lambda/4 = 187.5$ mm

(b) $Z_{sleeve} = (138/\sqrt{1}) \times \log_{10}(20/10.3) = 138 \times \log_{10}(1.942)$ $Z_{sleeve} = 138 \times 0.2882 = 39.8\ \Omega$

(c) At the feed point, the quarter-wave sleeve acts as a short-circuited quarter-wave transmission line. A quarter-wave shorted stub presents an open circuit (very high impedance) at its input. This high impedance at the feed point chokes off common-mode current on the outside of the coax shield.

$Z_{input} = jZ_{sleeve} \times \tan(\beta l) \to \infty$ as $\beta l \to \pi/2$

The impedance is theoretically infinite at the center frequency: open circuit

(d) The sleeve balun works by presenting a high impedance to common-mode currents at the feed point. Without the balun, currents flowing on the outside of the coax shield would see a low impedance path to ground. The quarter-wave sleeve transforms the short circuit at its far end to an open circuit at the feed point, effectively blocking common-mode current flow on the outer conductor.

(e) The balun provides good performance (common-mode impedance $> 500\ \Omega$) over approximately a 20–25% bandwidth centered on 400 MHz (approximately 320–480 MHz). At frequencies away from resonance, the quarter-wave condition is no longer satisfied and the choking impedance decreases. For wider bandwidth applications, a ferrite-core balun or multiple cascaded balun sections would be preferred.

Chapter 17 — Section 17.1: Radar Fundamentals

Practice problems covering the radar range equation, radar cross section, Doppler effect and velocity measurement, and radar clutter and noise.

Problem 17.1.1

Given: A ground-based air defense radar operates at 5.6 GHz (C-band) with a peak transmit power of 500 kW, antenna gain of 38 dBi, and system noise temperature of 500 K. The noise bandwidth is 2 MHz and the required SNR for detection is 15 dB.

Find: (a) The wavelength, (b) the minimum detectable signal S_{min}, (c) the maximum detection range for a fighter aircraft with RCS = 5 m^2, and (d) the maximum detection range for a stealth aircraft with RCS = 0.005 m^2.

Solution:

(a) $\lambda = c / f = 3 \times 10^8 / 5.6 \times 10^9 = 0.0536$ m (53.6 mm)

(b) $S_{min} = kT_sB_n \times SNR_{min}$

$S_{min} = 1.381 \times 10^{-23} \times 500 \times 2 \times 10^6 \times 10^{15/10}$

$S_{min} = 1.381 \times 10^{-14} \times 31.62 = 4.37 \times 10^{-13}$ W (−93.6 dBm)

(c) $G = 10^{38/10} = 6{,}310$ (linear).

$R_{max} = (P_tG^2\lambda^2\sigma / ((4\pi)^3 \times S_{min}))^{1/4}$

Numerator $= 500 \times 10^3 \times 6{,}310^2 \times 0.0536^2 \times 5 = 500 \times 10^3 \times 3.982 \times 10^7 \times 2.873 \times 10^{-3} \times 5$

Numerator $= 500 \times 10^3 \times 3.982 \times 10^7 \times 1.436 \times 10^{-2} = 2.859 \times 10^{11}$

Denominator $= (4\pi)^3 \times 4.37 \times 10^{-13} = 1{,}984 \times 4.37 \times 10^{-13} = 8.67 \times 10^{-10}$

$R_{max} = (2.859 \times 10^{11} / 8.67 \times 10^{-10})^{1/4} = (3.297 \times 10^{20})^{1/4} = 134.8$ km

(d) For $\sigma = 0.005$ m^2, the range scales as $\sigma^{1/4}$:

$R_{stealth} = 134.8 \times (0.005 / 5)^{1/4} = 134.8 \times (0.001)^{1/4} = 134.8 / 5.623 = 24.0$ km

The stealth aircraft's 1000× smaller RCS reduces detection range by a factor of 5.6.

Problem 17.1.2

Given: A weather balloon carries a metallic corner reflector with face dimension L = 0.3 m as a radar calibration target. The radar operates at 5.6 GHz.

Find: (a) The wavelength, (b) the RCS of the corner reflector, (c) the RCS in dBsm, and (d) the equivalent sphere radius that would produce the same RCS.

Solution:

(a) $\lambda = c / f = 3 \times 10^8 / 5.6 \times 10^9 = 0.0536$ m

(b) Corner reflector RCS: $\sigma = 12\pi L^4 / \lambda^2 = 12\pi \times 0.3^4 / 0.0536^2$

$\sigma = 12\pi \times 0.0081 / 2.873 \times 10^{-3} = 0.3054 / 2.873 \times 10^{-3} = 106.3\ m^2$

(c) $\sigma\ (dBsm) = 10 \log_{10}(106.3) = 20.3$ dBsm

(d) For a sphere: $\sigma_{sphere} = \pi a^2$, so $a = \sqrt{(\sigma/\pi)} = \sqrt{(106.3/\pi)} = \sqrt{(33.84)} = 5.82$ m

A sphere would need a radius of 5.82 m to match the RCS of a 0.3 m corner reflector — demonstrating the extreme retroreflective efficiency of the corner reflector geometry.

Problem 17.1.3

Given: A flat metallic plate of area $A = 1.0\ m^2$ is oriented perpendicular to a 10 GHz radar beam.

Find: (a) The plate's RCS, (b) the RCS in dBsm, (c) the RCS when the plate is tilted 5° off-normal (assume RCS drops as $\text{sinc}^2(2\pi A \sin(\theta)/\lambda^2)$ — approximate 20 dB reduction at 5° for this size), and (d) how this illustrates the principle of stealth shaping.

Solution:

(a) $\lambda = c / f = 3 \times 10^8 / 10 \times 10^9 = 0.03$ m.

Flat plate RCS: $\sigma = 4\pi A^2 / \lambda^2 = 4\pi \times 1.0^2 / 0.03^2 = 4\pi / 9 \times 10^{-4} = 13{,}963\ m^2$ (41.4 dBsm)

(b) $\sigma\ (dBsm) = 10 \log_{10}(13{,}963) = 41.4$ dBsm

(c) At 5° tilt, the specular RCS drops dramatically. With the approximate 20 dB reduction:

$\sigma_{tilted} = 13{,}963 / 100 = 139.6\ m^2$ (21.4 dBsm)

(d) Stealth aircraft use angled surfaces to deflect the specular reflection away from the radar receiver. A flat plate perpendicular to the beam has enormous RCS (13,963 m^2), but even a small tilt dramatically reduces the backscattered energy. By ensuring no surface is perpendicular to likely threat radar directions, stealth designs reduce RCS by 30–40 dB compared to conventional aircraft of similar size.

Problem 17.1.4

Given: An airborne fire-control radar operates at 10 GHz with a PRF of 20 kHz. A target aircraft is approaching at 600 m/s.

Find: (a) The wavelength, (b) the Doppler frequency shift, (c) the maximum unambiguous velocity, (d) whether the velocity measurement is ambiguous, and (e) the maximum unambiguous range.

Solution:

(a) $\lambda = c / f = 3 \times 10^8 / 10 \times 10^9 = 0.03$ m (30 mm)

(b) $f_d = 2v_r / \lambda = 2 \times 600 / 0.03 = 40{,}000$ Hz (40 kHz)

(c) $v_{max} = \lambda \times PRF / 4 = 0.03 \times 20{,}000 / 4 = 150$ m/s (540 km/h)

(d) The target velocity (600 m/s) exceeds v_{max} (150 m/s), so yes, the measurement is ambiguous. The Doppler shift of 40 kHz equals $2 \times$ PRF, so the target appears at the same Doppler as stationary clutter (a blind speed). The radar would need staggered PRFs or a higher PRF to resolve this ambiguity.

(e) $R_{ua} = c / (2 \times PRF) = 3 \times 10^8 / (2 \times 20{,}000) = 7{,}500$ m (7.5 km)

This illustrates the range-Doppler ambiguity: the high PRF needed for unambiguous velocity measurement severely limits the unambiguous range.

Problem 17.1.5

Given: A traffic monitoring radar at 24.125 GHz (K-band) measures vehicles on a highway. The radar is mounted on an overpass with a 30° depression angle relative to the road.

Find: (a) The wavelength, (b) the Doppler shift for a vehicle traveling at 100 km/h along the road, (c) the radial velocity component, and (d) the velocity resolution if the coherent integration time is 20 ms.

Solution:

(a) $\lambda = c / f = 3 \times 10^8 / 24.125 \times 10^9 = 0.01243$ m (12.43 mm)

(b) The depression angle of 30° means the radar beam makes a 30° angle with the road surface (horizontal). The radial velocity component (along the beam direction toward the radar) is:

$v_r = v \times \cos(30°) = 100 \times 0.866 = 86.6$ km/h $= 24.06$ m/s

(c) $v_r = 24.06$ m/s (the radar only sees the component of velocity along the beam direction)

(d) $f_d = 2v_r / \lambda = 2 \times 24.06 / 0.01243 = 3{,}875$ Hz (3.87 kHz)

The actual vehicle speed must be corrected by dividing by cos(30°): $v_{actual} = v_r / \cos(30°)$.

(d) Velocity resolution: $\Delta v = \lambda / (2T_{dwell}) = 0.01243 / (2 \times 0.020) = 0.311$ m/s (1.12 km/h)

This fine velocity resolution allows the radar to distinguish vehicles traveling at slightly different speeds.

Problem 17.1.6

Given: A ship-based X-band radar (9.4 GHz) has a pulse width of 50 ns, azimuth beamwidth of 0.9°, and transmit power of 50 kW. A fishing vessel with RCS $\sigma = 10$ m^2 is at 15 km range. The sea state produces a normalized clutter coefficient $\sigma^0 = -25$ dB (3.162×10^{-3} m^2/m^2) at the 3° grazing angle.

Find: (a) The clutter cell area, (b) the total clutter RCS, (c) the signal-to-clutter ratio (SCR), and (d) the SCR improvement needed for 15 dB output SCR, and the number of MTI canceller stages required.

Solution:

(a) $A_c = R \times \theta_{az} \times c\tau \,/\, (2 \cos \psi)$

$A_c = 15{,}000 \times (0.9 \times \pi/180) \times (3 \times 10^8 \times 50 \times 10^{-9}) \,/\, (2 \times \cos 3°)$

$A_c = 15{,}000 \times 0.01571 \times 15 \,/\, 1.9986$

$A_c = 15{,}000 \times 0.01571 \times 7.504 = 1{,}768$ m^2

(b) $\sigma_c = \sigma^0 \times A_c = 3.162 \times 10^{-3} \times 1{,}768 = 5.59$ m^2 (7.5 dBsm)

(c) $\text{SCR} = \sigma_{target} \,/\, \sigma_c = 10 \,/\, 5.59 = 1.79$ (2.5 dB)

(d) Required improvement: $15 - 2.5 = 12.5$ dB

A single-delay MTI canceller provides 20–25 dB improvement, which is sufficient. One MTI canceller stage would provide adequate clutter suppression for detection with a comfortable margin.

Problem 17.1.7

Given: A radar receiver front-end consists of: LNA with noise figure $F_1 = 1.2$ dB and gain $G_1 = 30$ dB, followed by a bandpass filter with insertion loss of 2 dB (noise figure = 2 dB, gain = −2 dB), followed by a mixer with noise figure $F_3 = 8$ dB and conversion gain $G_3 = 0$ dB.

Find: (a) The system noise figure using the Friis formula, (b) the system noise temperature (assuming $T_0 = 290$ K), and (c) how the system noise figure changes if the filter and LNA are swapped (filter first).

Solution:

(a) Convert to linear values:

$F_1 = 10^{1.2/10} = 1.318$, $G_1 = 10^{30/10} = 1{,}000$

$F_2 = 10^{2/10} = 1.585$, $G_2 = 10^{-2/10} = 0.631$

$F_3 = 10^{8/10} = 6.310$

$F_{sys} = F_1 + (F_2 - 1)/G_1 + (F_3 - 1)/(G_1 \times G_2)$

$F_{sys} = 1.318 + 0.585/1{,}000 + 5.310/(1{,}000 \times 0.631)$

$F_{sys} = 1.318 + 0.000585 + 0.008415 = 1.327$

F_{sys} (dB) = 10 $\log_{10}(1.327)$ = 1.23 dB

The LNA dominates the noise performance; downstream contributions are negligible.

(b) $T_{sys} = T_0 \times (F_{sys} - 1) = 290 \times (1.327 - 1) = 290 \times 0.327 = 94.8$ K

(c) With filter first (F_2 then F_1):

$F_{sys,swapped} = F_2 + (F_1 - 1)/G_2 + (F_3 - 1)/(G_2 \times G_1)$

$F_{sys,swapped} = 1.585 + 0.318/0.631 + 5.310/(0.631 \times 1{,}000)$

$F_{sys,swapped} = 1.585 + 0.504 + 0.00841 = 2.097$

$F_{sys,swapped}$ (dB) = 10 $\log_{10}(2.097)$ = 3.22 dB

The system noise figure degrades by 2.0 dB when the filter precedes the LNA. This demonstrates why the LNA must always be the first active component after the antenna.

Problem 17.1.8

Given: A pulse-Doppler radar at 3 GHz uses a coherent processing interval of 10 ms to measure target velocity. Two targets are at the same range but have different radial velocities.

Find: (a) The velocity resolution, (b) the minimum velocity difference the radar can distinguish, (c) the number of pulses processed if PRF = 5 kHz, and (d) the maximum unambiguous velocity.

Solution:

(a) Velocity resolution: $\Delta v = \lambda / (2T_{dwell})$

$\lambda = c / f = 3 \times 10^8 / 3 \times 10^9 = 0.10$ m

$\Delta v = 0.10 / (2 \times 0.010) = 5.0$ m/s (18 km/h)

(b) Two targets must differ by at least Δv to be resolved in the Doppler domain, so the minimum distinguishable velocity difference is 5.0 m/s.

(c) Number of pulses: $N = PRF \times T_{dwell} = 5{,}000 \times 0.010 = 50$ pulses

These 50 pulses are processed with a 50-point FFT (or DFT) to form the Doppler filter bank.

(d) $v_{max} = \lambda \times PRF / 4 = 0.10 \times 5{,}000 / 4 = 125$ m/s (450 km/h)

Problem 17.1.9

Given: A radar system designer must achieve a maximum detection range of 300 km for targets with $\sigma = 1$ m^2. The system parameters are: frequency = 3 GHz, antenna gain = 40 dBi, system noise temperature = 450 K, noise bandwidth = 500 kHz, and required SNR = 14 dB.

Find: The minimum required peak transmit power.

Solution:

$\lambda = c / f = 3 \times 10^8 / 3 \times 10^9 = 0.10$ m

$G = 10^{40/10} = 10{,}000$

$S_{min} = kT_sB_n \times SNR = 1.381 \times 10^{-23} \times 450 \times 500 \times 10^3 \times 10^{14/10}$

$S_{min} = 1.381 \times 10^{-23} \times 450 \times 5 \times 10^5 \times 25.12$

$S_{min} = 1.381 \times 10^{-23} \times 5.652 \times 10^9 = 7.806 \times 10^{-14}$ W

From the radar range equation at R_{max}:

$P_t = R_{max}{}^4 \times (4\pi)^3 \times S_{min} / (G^2\lambda^2\sigma)$

$R_{max}{}^4 = (300 \times 10^3)^4 = (3 \times 10^5)^4 = 8.1 \times 10^{21}$

Numerator $= 8.1 \times 10^{21} \times 1{,}984 \times 7.806 \times 10^{-14} = 8.1 \times 10^{21} \times 1.549 \times 10^{-10} = 1.254 \times 10^{12}$

Denominator $= 10{,}000^2 \times 0.10^2 \times 1 = 10^8 \times 0.01 = 10^6$

$P_t = 1.254 \times 10^{12} / 10^6 = 1.254 \times 10^6$ W $= 1.254$ MW

The required peak transmit power is approximately 1.25 MW, which is typical for a high-power S-band surveillance radar.

Problem 17.1.10

Given: A radar system achieves coherent integration of 256 pulses and non-coherent integration of 256 pulses in two different operating modes. The single-pulse SNR is −5 dB.

Find: (a) The SNR after coherent integration, (b) the SNR after non-coherent integration, (c) the ratio of detection ranges for the two modes, and (d) explain which mode is preferred and why.

Solution:

(a) Coherent integration gain $= 10 \log_{10}(N) = 10 \log_{10}(256) = 24.1$ dB

$SNR_{coherent} = -5 + 24.1 = 19.1$ dB

(b) Non-coherent integration gain $\approx 10 \log_{10}(\sqrt{N}) = 10 \log_{10}(16) = 12.0$ dB

$SNR_{non\text{-}coherent} = -5 + 12.0 = 7.0$ dB

(c) Since range $\propto SNR^{1/4}$, the range ratio is:

$R_{coherent} / R_{non\text{-}coherent} = (10^{19.1/10} / 10^{7.0/10})^{1/4} = (81.28 / 5.012)^{1/4} = (16.22)^{1/4} = 2.01$

Coherent integration provides approximately 2× greater detection range.

(d) Coherent integration is preferred when the radar can maintain phase coherence over 256 pulses (requiring stable transmitter and known target motion). The 12.1 dB advantage over non-coherent integration arises because coherent processing preserves the signal phase, allowing voltage addition rather than power addition. However, coherent integration requires accurate compensation for target motion (Doppler) during the integration time and is more computationally intensive. Non-coherent integration is simpler and more robust for fluctuating targets.

Chapter 17 — Section 17.2: Radar Waveforms and Modulation

Practice problems covering pulsed radar, continuous wave (CW) and FMCW radar, and pulse compression.

Problem 17.2.1

Given: A shipboard surveillance radar operates at 9.4 GHz (X-band) with a peak power of 50 kW, pulse width of 1.0 μs, and PRF of 1,500 Hz.

Find: (a) The range resolution, (b) the maximum unambiguous range, (c) the duty cycle, (d) the average transmit power, and (e) the pulse energy.

Solution:

(a) Range resolution: $\Delta R = c\tau_p / 2 = 3 \times 10^8 \times 1.0 \times 10^{-6} / 2 = 150$ m

(b) Maximum unambiguous range: $R_{ua} = c / (2 \times PRF) = 3 \times 10^8 / (2 \times 1{,}500) = 100$ km

(c) Duty cycle $= \tau_p \times PRF = 1.0 \times 10^{-6} \times 1{,}500 = 1.5 \times 10^{-3}$ (0.15%)

(d) Average power: $P_{avg} = P_{peak} \times \text{duty cycle} = 50{,}000 \times 1.5 \times 10^{-3} = 75$ W

(e) Pulse energy: $E = P_{peak} \times \tau_p = 50{,}000 \times 1.0 \times 10^{-6} = 50$ mJ per pulse

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Problem 17.2.2

Given: A radar designer needs to detect targets at 200 km range with 50 m range resolution. The transmitter peak power is 200 kW.

Find: (a) The required pulse width for 50 m resolution (uncompressed), (b) the corresponding pulse energy, (c) the required PRF for 200 km unambiguous range, (d) the average power, and (e) whether this is practical.

Solution:

(a) $\Delta R = c\tau_p / 2$, so $\tau_p = 2\Delta R / c = 2 \times 50 / 3 \times 10^8 = 3.33 \times 10^{-7}$ s (333 ns)

(b) Pulse energy: $E = P_{peak} \times \tau_p = 200 \times 10^3 \times 3.33 \times 10^{-7} = 66.7$ mJ

(c) $R_{ua} = c / (2 \times PRF)$, so $PRF = c / (2 \times R_{ua}) = 3 \times 10^8 / (2 \times 200 \times 10^3) = 750$ Hz

(d) $P_{avg} = P_{peak} \times \tau_p \times PRF = 200 \times 10^3 \times 3.33 \times 10^{-7} \times 750 = 50$ W

(e) The pulse energy (66.7 mJ) may be insufficient for detection at 200 km depending on target RCS. Using pulse compression (Section 17.2.3), a 20 μs pulse with 10 MHz bandwidth would provide the same 50 m resolution with pulse energy of $200 \times 10^3 \times 20 \times 10^{-6} = 4$ J — 60× more energy while maintaining resolution, making long-range detection practical.

Problem 17.2.3

Given: An automotive FMCW radar at 77 GHz has a sweep bandwidth of 2 GHz, sweep time of 30 μs, and transmit power of 12 dBm.

Find: (a) The range resolution, (b) the beat frequency for a vehicle at 80 m, (c) the beat frequency for a pedestrian at 15 m, (d) the maximum range for a 25 MHz ADC sampling rate, and (e) the wavelength.

Solution:

(a) $\Delta R = c / (2B) = 3 \times 10^8 / (2 \times 2 \times 10^9) = 7.5$ cm

(b) $f_b = 2RB / (cT_{sweep}) = 2 \times 80 \times 2 \times 10^9 / (3 \times 10^8 \times 30 \times 10^{-6})$

$f_b = 3.2 \times 10^{11} / 9 \times 10^3 = 35.6$ MHz

(c) $f_b = 2 \times 15 \times 2 \times 10^9 / (3 \times 10^8 \times 30 \times 10^{-6})$

$f_b = 6 \times 10^{10} / 9 \times 10^3 = 6.67$ MHz

(d) Maximum beat frequency $= f_s / 2 = 12.5$ MHz (Nyquist).

$R_{max} = f_{b,max} \times cT_{sweep} / (2B) = 12.5 \times 10^6 \times 3 \times 10^8 \times 30 \times 10^{-6} / (2 \times 2 \times 10^9)$

$R_{max} = 12.5 \times 10^6 \times 9 \times 10^3 / 4 \times 10^9 = 1.125 \times 10^{11} / 4 \times 10^9 = 28.1$ m

The 80 m vehicle target (35.6 MHz beat) would alias at this ADC rate. The ADC rate must be increased or the sweep time lengthened.

(e) $\lambda = c / f = 3 \times 10^8 / 77 \times 10^9 = 3.90$ mm

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Problem 17.2.4

Given: An FMCW radar uses triangular modulation (up-sweep then down-sweep) at 24 GHz with bandwidth B = 250 MHz and sweep time $T_{sweep} = 1$ ms per ramp. A target produces beat frequencies of 8.33 kHz on the up-sweep and 12.33 kHz on the down-sweep.

Find: (a) The target range, (b) the target radial velocity, (c) the range resolution, and (d) the Doppler frequency.

Solution:

For triangular FMCW, the sum of beat frequencies gives range and the difference gives velocity:

$f_{range} = (f_{up} + f_{down}) / 2 = (8{,}330 + 12{,}330) / 2 = 10{,}330$ Hz

$f_{Doppler} = (f_{down} - f_{up}) / 2 = (12{,}330 - 8{,}330) / 2 = 2{,}000$ Hz

(a) $R = f_{range} \times cT_{sweep} / (2B) = 10{,}330 \times 3 \times 10^8 \times 1 \times 10^{-3} / (2 \times 250 \times 10^6)$

$R = 10{,}330 \times 3 \times 10^5 / 5 \times 10^8 = 3.099 \times 10^9 / 5 \times 10^8 = 6.20$ m

(b) $\lambda = c / f = 3 \times 10^8 / 24 \times 10^9 = 0.0125$ m

$v_r = f_{Doppler} \times \lambda / 2 = 2{,}000 \times 0.0125 / 2 = 12.5$ m/s (45 km/h, approaching)

(c) $\Delta R = c / (2B) = 3 \times 10^8 / (2 \times 250 \times 10^6) = 0.60$ m

(d) $f_{Doppler} = 2{,}000$ Hz (computed above)

Problem 17.2.5

Given: A military airborne radar transmits a 20 μs LFM chirp with 20 MHz bandwidth at a peak power of 10 kW. A non-compressed backup mode uses a 0.5 μs simple pulse at 10 kW.

Find: (a) The compression ratio, (b) the range resolution with and without compression, (c) the effective peak power after matched-filter compression, (d) the compressed pulse width, and (e) the ratio of detection ranges for the two modes.

Solution:

(a) $CR = B\tau_p = 20 \times 10^6 \times 20 \times 10^{-6} = 400$ (26 dB)

(b) With compression: $\Delta R = c / (2B) = 3 \times 10^8 / (2 \times 20 \times 10^6) = 7.5$ m

Without compression (0.5 μs pulse): $\Delta R = c\tau_p / 2 = 3 \times 10^8 \times 0.5 \times 10^{-6} / 2 = 75$ m

(c) $P_{eff} = P_{peak} \times CR = 10 \times 10^3 \times 400 = 4$ MW equivalent peak power

(d) $\tau_{compressed} = 1 / B = 1 / (20 \times 10^6) = 50$ ns

(e) The chirp pulse energy is $E_{chirp} = 10{,}000 \times 20 \times 10^{-6} = 0.2$ J.

The simple pulse energy is $E_{simple} = 10{,}000 \times 0.5 \times 10^{-6} = 5 \times 10^{-3}$ J.

The energy ratio is $0.2 / 0.005 = 40$. Since range $\propto E^{1/4}$:

$R_{chirp} / R_{simple} = 40^{1/4} = 2.51$

The pulse compression mode provides 2.5× greater detection range with 10× better range resolution.

Problem 17.2.6

Given: A ground surveillance radar uses Barker-coded phase modulation with a 13-bit Barker code. The chip duration is 100 ns and the peak power is 500 W.

Find: (a) The total pulse duration, (b) the compression ratio, (c) the range resolution, (d) the sidelobe level of the compressed pulse, and (e) the pulse energy.

Solution:

(a) Total pulse duration: $\tau_p = N \times \tau_{chip} = 13 \times 100 \times 10^{-9} = 1.3\ \mu s$

(b) Compression ratio for a Barker code: CR = N = 13 (11.1 dB)

(c) The compressed pulse width equals the chip duration. Range resolution:

$\Delta R = c \times \tau_{chip} / 2 = 3 \times 10^8 \times 100 \times 10^{-9} / 2 = 15$ m

Without compression, ΔR would be $3 \times 10^8 \times 1.3 \times 10^{-6} / 2 = 195$ m.

(d) Barker code sidelobe level = 1/N = 1/13 in amplitude, so the peak sidelobe level is:

$PSL = 20 \log_{10}(1/13) = -20 \log_{10}(13) = -22.3$ dB

This is the best achievable sidelobe level for any binary phase code of length 13, and N = 13 is the longest known Barker code.

(e) Pulse energy: $E = P_{peak} \times \tau_p = 500 \times 1.3 \times 10^{-6} = 0.65$ mJ

Problem 17.2.7

Given: A CW police radar operates at 34.7 GHz (Ka-band). It measures a Doppler shift of 6,250 Hz from a vehicle.

Find: (a) The wavelength, (b) the vehicle speed, (c) the vehicle speed in miles per hour, and (d) the minimum integration time to resolve two vehicles with speeds differing by 2 km/h.

Solution:

(a) $\lambda = c / f = 3 \times 10^8 / 34.7 \times 10^9 = 8.65 \times 10^{-3}$ m (8.65 mm)

(b) $f_d = 2v_r / \lambda$, so $v_r = f_d \times \lambda / 2 = 6{,}250 \times 8.65 \times 10^{-3} / 2 = 27.03$ m/s (97.3 km/h)

(c) v = 97.3 km/h × 0.6214 = 60.5 mph

(d) Δv = 2 km/h = 0.556 m/s. Velocity resolution $\Delta v = \lambda / (2T_{dwell})$:

$T_{dwell} = \lambda / (2\Delta v) = 8.65 \times 10^{-3} / (2 \times 0.556) = 7.78$ ms

This short integration time makes Ka-band CW radar excellent for resolving individual vehicles in traffic flow.

Chapter 17 — Section 17.3: Radar Signal Processing

Practice problems covering MTI and clutter rejection, CFAR detection, SAR imaging, and target tracking.

Problem 17.3.1

Given: An L-band surveillance radar operates at 1.3 GHz with a PRF of 500 Hz and uses a 2-pulse MTI canceller with 25 dB clutter improvement factor.

Find: (a) The wavelength, (b) the first blind speed, (c) the second and third blind speeds, (d) a second PRF that eliminates the first blind speed when used in a staggered-PRF scheme, and (e) the first common blind speed for the two PRFs.

Solution:

(a) $\lambda = c / f = 3 \times 10^8 / 1.3 \times 10^9 = 0.2308$ m

(b) First blind speed: $v_{blind1} = \lambda \times PRF / 2 = 0.2308 \times 500 / 2 = 57.7$ m/s (208 km/h)

(c) Second blind speed: $v_{blind2} = 2 \times 57.7 = 115.4$ m/s (415 km/h)

Third blind speed: $v_{blind3} = 3 \times 57.7 = 173.1$ m/s (623 km/h)

(d) Choose $PRF_2 = 600$ Hz (ratio 5:6 with PRF_1).

Blind speeds of PRF_1 (500 Hz): 57.7, 115.4, 173.1, 230.8, 288.5, 346.2… m/s

Blind speeds of PRF_2 (600 Hz): $v_{blind} = 0.2308 \times 600 / 2 = 69.2$, 138.5, 207.7, 276.9, 346.2… m/s

(e) The first common blind speed occurs at 346.2 m/s = 1,246 km/h. This is well above the speed of any conventional aircraft, effectively eliminating the blind speed problem.

$PRF_2 = 600$ Hz with the first common blind speed at 346.2 m/s.

Problem 17.3.2

Given: An air traffic control radar has a 2-pulse MTI canceller operating at 2.8 GHz with PRF = 1,200 Hz. An aircraft is flying at 200 m/s.

Find: (a) The Doppler frequency of the aircraft, (b) the first blind speed, (c) the MTI filter response at the aircraft's Doppler frequency (the response of a single-canceller is $H(f) = 2\sin(\pi f/PRF)$), and (d) the canceller output relative to the peak response.

Solution:

(a) $\lambda = c / f = 3 \times 10^8 / 2.8 \times 10^9 = 0.1071$ m

$f_d = 2v_r / \lambda = 2 \times 200 / 0.1071 = 3{,}733$ Hz

(b) $v_{blind} = \lambda \times PRF / 2 = 0.1071 \times 1{,}200 / 2 = 64.3$ m/s (231 km/h)

(c) The normalized Doppler frequency: $f_d / PRF = 3{,}733 / 1{,}200 = 3.111$

The fractional part is 0.111 (since the response is periodic with period PRF).

$H(f) = 2|\sin(\pi \times f_d / PRF)| = 2|\sin(\pi \times 3.111)| = 2|\sin(0.111\pi)| = 2 \times \sin(0.3491)$

$H = 2 \times 0.342 = 0.684$

(d) The peak response is 2.0 (at $f_d = PRF/2$). The response at the aircraft Doppler is:

Relative response = 0.684 / 2.0 = 0.342 = −9.3 dB

The aircraft is near (but not at) a blind speed, suffering 9.3 dB of signal attenuation by the MTI filter. A double canceller would suffer 18.6 dB attenuation at this velocity, so staggered PRFs should be used.

Problem 17.3.3

Given: A CA-CFAR processor uses N = 24 reference cells (12 leading, 12 trailing) with 3 guard cells on each side. The desired false alarm probability is $P_{fa} = 10^{-4}$.

Find: (a) The threshold multiplier α, (b) the threshold in dB above the noise estimate, (c) the CFAR loss, and (d) the total number of cells in the sliding window.

Solution:

(a) $\alpha = N \times ({P_{fa}}^{-1/N} - 1) = 24 \times ((10^{-4})^{-1/24} - 1) = 24 \times (10^{4/24} - 1)$

$10^{4/24} = 10^{0.1667} = 1.468$

$\alpha = 24 \times (1.468 - 1) = 24 \times 0.468 = 11.23$

(b) Threshold = $10 \log_{10}(11.23) = 10.5$ dB above the estimated noise floor

(c) CFAR loss $\approx 10 \log_{10}(1 + 2/N) = 10 \log_{10}(1 + 2/24) = 10 \log_{10}(1.0833) = 0.35$ dB

(d) Total cells = 1 (CUT) + 6 (guard cells) + 24 (reference cells) = 31 range cells

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Problem 17.3.4

Given: A CA-CFAR with N = 32 reference cells is compared against an OS-CFAR using the k = 24th ranked cell (out of 32) for the threshold estimate. Both have $P_{fa} = 10^{-6}$.

Find: (a) The CA-CFAR threshold multiplier, (b) the CA-CFAR threshold in dB, (c) the CFAR loss for the CA-CFAR, and (d) explain qualitatively why OS-CFAR is preferred in multi-target environments.

Solution:

(a) $\alpha_{CA} = N \times (P_{fa}{}^{-1/N} - 1) = 32 \times ((10^{-6})^{-1/32} - 1) = 32 \times (10^{6/32} - 1)$

$10^{6/32} = 10^{0.1875} = 1.539$

$\alpha_{CA} = 32 \times (1.539 - 1) = 32 \times 0.539 = 17.25$

(b) Threshold $= 10 \log_{10}(17.25) = 12.4$ dB

(c) CFAR loss $= 10 \log_{10}(1 + 2/32) = 10 \log_{10}(1.0625) = 0.26$ dB

(d) OS-CFAR ranks the reference cells by power and uses the k-th ordered value, making it robust against interfering targets in the reference window. If 2–3 strong targets fall within the CA-CFAR reference window, they raise the average noise estimate and increase the threshold, masking weaker targets (the "target masking" problem). OS-CFAR with k = 24 (the 75th percentile) tolerates up to $N - k = 8$ interfering targets in the window without significantly raising the threshold, at the cost of slightly higher CFAR loss (typically 1–2 dB more than CA-CFAR in homogeneous noise).

Problem 17.3.5

Given: An airborne SAR operates at 9.6 GHz (X-band) with a platform velocity of 250 m/s and antenna length D = 1.5 m. The transmitted bandwidth is 300 MHz, and the target is at a slant range of 30 km.

Find: (a) The wavelength, (b) the cross-range (azimuth) resolution, (c) the range resolution, (d) the synthetic aperture length, (e) the coherent integration time, and (f) the required motion compensation accuracy.

Solution:

(a) $\lambda = c / f = 3 \times 10^8 / 9.6 \times 10^9 = 0.03125$ m

(b) Cross-range resolution: $\delta_{cr} = D / 2 = 1.5 / 2 = 0.75$ m

(c) Range resolution: $\delta_r = c / (2B) = 3 \times 10^8 / (2 \times 300 \times 10^6) = 0.50$ m

(d) Synthetic aperture length: $L_{SA} = \lambda R / D = 0.03125 \times 30{,}000 / 1.5 = 625$ m

(e) Integration time: $T_{int} = L_{SA} / v = 625 / 250 = 2.50$ seconds

(f) Motion compensation accuracy must be a fraction of the wavelength:

Required accuracy $< \lambda / 4 = 0.03125 / 4 = 7.81$ mm

The platform's position must be known to better than 7.81 mm during the 2.5-second integration time, requiring high-precision INS/GPS navigation or autofocus algorithms.

Problem 17.3.6

Given: A spotlight SAR steers its beam to dwell on a specific area, achieving an integration angle of $\theta_{int} = 5°$ (total angular extent of data collection). The radar operates at 5.3 GHz (C-band).

Find: (a) The wavelength, (b) the cross-range resolution ($\delta_{cr} = \lambda / (2\theta_{int})$ for spotlight mode), and (c) how this compares to the stripmap resolution with a 3 m antenna.

Solution:

(a) $\lambda = c / f = 3 \times 10^8 / 5.3 \times 10^9 = 0.0566$ m

(b) $\theta_{int} = 5° = 0.0873$ rad

$\delta_{cr} = \lambda / (2\theta_{int}) = 0.0566 / (2 \times 0.0873) = 0.0566 / 0.1745 = 0.324$ m (32.4 cm)

(c) Stripmap resolution: $\delta_{cr,strip} = D / 2 = 3 / 2 = 1.50$ m

The spotlight mode achieves 4.6× finer cross-range resolution (0.324 m vs. 1.50 m) by dwelling on the target area and collecting data over a wider angular range, at the expense of area coverage rate.

Problem 17.3.7

Given: A surveillance radar tracks an aircraft at 80 km range with range accuracy $\sigma_R = 75$ m and azimuth accuracy $\sigma_\theta = 0.5°$. The scan interval is 4 seconds and the aircraft velocity is 300 m/s. A Kalman filter is used with a constant-velocity model.

Find: (a) The cross-range position accuracy, (b) the prediction uncertainty after one scan interval, (c) the track gate size for 95% association probability, and (d) the distance the aircraft travels between scans.

Solution:

(a) Cross-range accuracy: $\sigma_{cr} = R \times \sigma_\theta = 80{,}000 \times (0.5 \times \pi/180) = 80{,}000 \times 0.008727 = 698$ m

(b) The velocity estimate accuracy: $\sigma_v \approx \sigma_{cr} / T = 698 / 4 = 174.5$ m/s

Predicted position uncertainty after one scan:

$\sigma_{pred} = \sqrt{(\sigma_{cr}^2 + (\sigma_v \times T)^2)} = \sqrt{(698^2 + (174.5 \times 4)^2)} = \sqrt{(698^2 + 698^2)} = 698 \times \sqrt{2} = 987$ m (cross-range)

In range: $\sigma_{pred,R} = \sqrt{(75^2 + (\sigma_{vR} \times 4)^2)} \approx 106$ m (assuming similar ratio)

(c) For 2D Gaussian, 95% probability corresponds to a gate radius of ~2.45σ (chi-squared with 2 DOF, p = 0.95):

Range gate: $2.45 \times 75 = 184$ m

Cross-range gate: $2.45 \times 987 = 2{,}418$ m

(d) Distance traveled: $d = v \times T = 300 \times 4 = 1{,}200$ m

The aircraft moves 1,200 m between scans, which is within the cross-range gate (2,418 m), confirming that track association should succeed for this scenario.

Problem 17.3.8

Given: A radar tracking system uses M-of-N logic for track initiation. Two configurations are compared: 3-of-5 and 4-of-7. The single-scan probability of detection is $P_d = 0.8$ and the false alarm probability per resolution cell is $P_{fa} = 10^{-6}$.

Find: (a) The probability of initiating a true track with 3-of-5 logic, (b) the probability of initiating a true track with 4-of-7 logic, and (c) explain the trade-off between the two configurations.

Solution:

(a) P(3-of-5) uses the binomial distribution with $p = P_d = 0.8$:

$P(\geq 3 \text{ of } 5) = P(3) + P(4) + P(5)$

$P(3) = C(5,3) \times 0.8^3 \times 0.2^2 = 10 \times 0.512 \times 0.04 = 0.2048$

$P(4) = C(5,4) \times 0.8^4 \times 0.2^1 = 5 \times 0.4096 \times 0.2 = 0.4096$

$P(5) = C(5,5) \times 0.8^5 \times 0.2^0 = 1 \times 0.3277 = 0.3277$

$P(\geq 3 \text{ of } 5) = 0.2048 + 0.4096 + 0.3277 = 0.942$ (94.2%)

(b) $P(\geq 4 \text{ of } 7) = P(4) + P(5) + P(6) + P(7)$

$P(4) = C(7,4) \times 0.8^4 \times 0.2^3 = 35 \times 0.4096 \times 0.008 = 0.1147$

$P(5) = C(7,5) \times 0.8^5 \times 0.2^2 = 21 \times 0.3277 \times 0.04 = 0.2753$

$P(6) = C(7,6) \times 0.8^6 \times 0.2^1 = 7 \times 0.2621 \times 0.2 = 0.3670$

$P(7) = C(7,7) \times 0.8^7 = 0.2097$

$P(\geq 4 \text{ of } 7) = 0.1147 + 0.2753 + 0.3670 + 0.2097 = 0.967$ (96.7%)

(c) The 4-of-7 logic provides higher track initiation probability (96.7% vs. 94.2%) because it allows more missed detections (3 vs. 2) over a longer observation window. However, it requires 7 scans to confirm a track versus 5 scans, increasing the track initiation latency. For a radar with a 5-second scan time, 3-of-5 takes 25 seconds while 4-of-7 takes 35 seconds. The choice depends on whether detection reliability or response time is more critical.

Chapter 17 — Section 17.4: Radar Applications

Practice problems covering air traffic control, weather radar, automotive radar, ground-penetrating radar, missile defense radar, and LiDAR.

Problem 17.4.1

Given: An ASR-9 airport surveillance radar rotates at 12.5 RPM with an azimuth beamwidth of 1.4°, PRF of 1,200 Hz, frequency of 2.8 GHz, and peak power of 1.3 MW. The pulse width is 1.0 μs.

Find: (a) The scan time, (b) the number of pulses on target per scan, (c) the non-coherent integration gain, (d) the maximum unambiguous range, and (e) the range resolution.

Solution:

(a) Scan time = 60 / RPM = 60 / 12.5 = 4.8 seconds

(b) Time on target: $T_{OT} = \theta_{az} / (360° \times RPM/60) = 1.4 / (360 \times 12.5/60) = 1.4 / 75 = 0.01867$ s

Hits per scan = PRF × T_{OT} = 1,200 × 0.01867 = 22.4 ≈ 22 pulses

(c) Non-coherent integration gain ≈ $\sqrt{N} = \sqrt{22} = 4.69$ (6.7 dB)

(d) $R_{ua} = c / (2 \times PRF) = 3 \times 10^8 / (2 \times 1{,}200) = 125$ km (67.5 nmi)

(e) $\Delta R = c\tau_p / 2 = 3 \times 10^8 \times 1.0 \times 10^{-6} / 2 = 150$ m

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Problem 17.4.2

Given: A NEXRAD weather radar measures a storm cell at 80 km range with a reflectivity of 45 dBZ. The radar operates at 2.8 GHz with a dwell time of 40 ms.

Find: (a) The reflectivity in linear units (mm^6/m^3), (b) the estimated rainfall rate using the Marshall-Palmer relation ($Z = 200R_r^{1.6}$), (c) the storm classification, and (d) the Doppler velocity resolution.

Solution:

(a) $Z = 10^{45/10} = 10^{4.5} = 31{,}623\ mm^6/m^3$

(b) $R_r = (Z / 200)^{1/1.6} = (31{,}623 / 200)^{0.625} = (158.1)^{0.625}$

$\ln(158.1) = 5.063, \times 0.625 = 3.164$

$R_r = e^{3.164} = 23.7$ mm/h (about 1 inch/hour)

(c) At 45 dBZ, this is classified as heavy rain (40–50 dBZ range). This would appear as red on a standard weather radar display.

(d) $\lambda = c / f = 3 \times 10^8 / 2.8 \times 10^9 = 0.1071$ m

$\Delta v = \lambda / (2T_{dwell}) = 0.1071 / (2 \times 0.040) = 1.34$ m/s (4.8 km/h)

Problem 17.4.3

Given: A NEXRAD radar observes a mesocyclone in a supercell thunderstorm. The Doppler velocity measurements show +30 m/s on one side and −25 m/s on the opposite side of the rotation, separated by 5 km at a range of 60 km.

Find: (a) The total rotational velocity differential, (b) the angular velocity of rotation, (c) whether this meets the criteria for a tornado vortex signature (TVS requires velocity differential > 30 m/s across a distance < 4 km), and (d) the reflectivity if the rainfall rate is 75 mm/h.

Solution:

(a) Velocity differential: $\Delta V = |+30 - (-25)| = 55$ m/s (198 km/h)

(b) Angular velocity: $\omega = \Delta V$ / separation $= 55 / 5{,}000 = 0.011$ rad/s (0.63°/s)

(c) The velocity differential of 55 m/s exceeds the 30 m/s threshold, but the separation of 5 km exceeds the 4 km distance threshold. This is a strong mesocyclone but does not meet the formal TVS criteria at this range. At closer range, the smaller beam width might resolve a tighter rotation. A tornado warning would likely be issued based on this strong mesocyclone signature combined with other factors.

(d) $Z = 200 \times {R_r}^{1.6} = 200 \times 75^{1.6}$

$75^{1.6} = e^{1.6 \times \ln(75)} = e^{1.6 \times 4.317} = e^{6.908} = 999.5$

$Z = 200 \times 999.5 = 199{,}900$ mm^6/m^3

dBZ $= 10 \log_{10}(199{,}900) = 53.0$ dBZ (classified as very heavy rain with possible hail)

Problem 17.4.4

Given: A 77 GHz FMCW automotive radar (long-range mode) has bandwidth B = 500 MHz, sweep time $T_{sweep} = 40$ μs, and uses 256 chirps in a frame. The transmit antenna gain is 25 dBi and the transmit power is 12 dBm (15.8 mW).

Find: (a) The range resolution, (b) the velocity resolution, (c) the maximum unambiguous velocity, (d) the maximum unambiguous range for a 50 MHz maximum beat frequency, and (e) the wavelength.

Solution:

(a) $\Delta R = c / (2B) = 3 \times 10^8 / (2 \times 500 \times 10^6) = 30$ cm

(b) $\lambda = c / f = 3 \times 10^8 / 77 \times 10^9 = 3.896 \times 10^{-3}$ m

Frame time: $T_{frame} = 256 \times 40 \times 10^{-6} = 10.24$ ms

$\Delta v = \lambda / (2T_{frame}) = 3.896 \times 10^{-3} / (2 \times 10.24 \times 10^{-3}) = 0.190$ m/s (0.685 km/h)

(c) $v_{max} = \lambda / (4 \times T_{sweep}) = 3.896 \times 10^{-3} / (4 \times 40 \times 10^{-6}) = 24.4$ m/s (87.7 km/h)

(d) $R_{max} = f_{b,max} \times cT_{sweep} / (2B) = 50 \times 10^6 \times 3 \times 10^8 \times 40 \times 10^{-6} / (2 \times 500 \times 10^6)$

$R_{max} = 50 \times 10^6 \times 1.2 \times 10^4 / 10^9 = 6 \times 10^{11} / 10^9 = 600$ m

(e) $\lambda = 3.90$ mm (computed in step (b))

Problem 17.4.5

Given: A 400 MHz GPR is used to survey a concrete bridge deck for rebar corrosion. The concrete has $\varepsilon_r = 8$ and attenuation $\alpha = 2$ dB/m. The rebar is at a depth of 7.5 cm. The system bandwidth is 800 MHz and dynamic range is 60 dB.

Find: (a) The propagation velocity in concrete, (b) the round-trip time to the rebar, (c) the two-way attenuation, (d) the vertical resolution, and (e) the maximum penetration depth.

Solution:

(a) $v = c / \sqrt{\varepsilon_r} = 3 \times 10^8 / \sqrt{8} = 3 \times 10^8 / 2.828 = 1.061 \times 10^8$ m/s (35.4% of c)

(b) $t = 2d / v = 2 \times 0.075 / 1.061 \times 10^8 = 0.150 / 1.061 \times 10^8 = 1.41$ ns

(c) Two-way attenuation: $L = 2 \times \alpha \times d = 2 \times 2 \times 0.075 = 0.30$ dB (negligible)

(d) Vertical resolution: $\delta_v = v / (2B) = 1.061 \times 10^8 / (2 \times 800 \times 10^6) = 6.63$ cm

This is sufficient to detect rebar (typically 10–16 mm diameter) and distinguish the rebar layer from the concrete surface.

(e) Maximum penetration depth: $d_{max} = \text{dynamic range} / (2\alpha) = 60 / (2 \times 2) = 15$ m

In practice, scattering from aggregates in concrete limits the useful depth to approximately 0.3–0.5 m at 400 MHz.

Problem 17.4.6

Given: A 1.5 GHz GPR system is used to locate buried plastic water pipes in sandy soil ($\varepsilon_r = 4$, $\alpha = 0.5$ dB/m). The pipe is at a depth of 1.0 m. The reflection coefficient between dry sand and water-filled pipe ($\varepsilon_r \approx 81$) is significant.

Find: (a) The propagation velocity, (b) the round-trip time, (c) the two-way attenuation, (d) the reflection coefficient at the sand-water interface, and (e) the vertical resolution for a bandwidth of 1.5 GHz.

Solution:

(a) $v = c / \sqrt{\varepsilon_r} = 3 \times 10^8 / \sqrt{4} = 3 \times 10^8 / 2 = 1.5 \times 10^8$ m/s

(b) $t = 2d / v = 2 \times 1.0 / 1.5 \times 10^8 = 13.3$ ns

(c) $L = 2 \times \alpha \times d = 2 \times 0.5 \times 1.0 = 1.0$ dB

(d) $\Gamma = (\sqrt{\varepsilon_{r2}} - \sqrt{\varepsilon_{r1}}) / (\sqrt{\varepsilon_{r2}} + \sqrt{\varepsilon_{r1}}) = (\sqrt{81} - \sqrt{4}) / (\sqrt{81} + \sqrt{4}) = (9 - 2) / (9 + 2) = 7/11 = 0.636$

The reflected power fraction: $|\Gamma|^2 = 0.404 = -3.9$ dB

The strong dielectric contrast makes the water-filled pipe easily detectable.

(e) $\delta_v = v / (2B) = 1.5 \times 10^8 / (2 \times 1.5 \times 10^9) = 5.0$ cm

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Problem 17.4.7

Given: A ground-based X-band missile defense radar (9.5 GHz) has an antenna diameter of 10 m, aperture efficiency of 0.60, peak power of 1 MW, and system noise temperature of 350 K. The noise bandwidth is 2 MHz.

Find: (a) The antenna gain, (b) the maximum detection range for a ballistic warhead with RCS $\sigma = 0.1$ m² requiring SNR = 20 dB, and (c) the number of coherent integration pulses needed if the single-pulse range is only 500 km.

Solution:

(a) $\lambda = 3 \times 10^8 / 9.5 \times 10^9 = 0.03158$ m

$G = \eta \times (\pi D/\lambda)^2 = 0.60 \times (\pi \times 10 / 0.03158)^2 = 0.60 \times (994.7)^2 = 0.60 \times 989{,}428 = 593{,}657$

G (dBi) $= 10 \log_{10}(593{,}657) = 57.7$ dBi

(b) $S_{min} = kT_sB_n \times SNR = 1.381 \times 10^{-23} \times 350 \times 2 \times 10^6 \times 100$

$S_{min} = 1.381 \times 10^{-23} \times 7 \times 10^{10} = 9.667 \times 10^{-13}$ W

$R_{max} = (P_tG^2\lambda^2\sigma / ((4\pi)^3S_{min}))^{1/4}$

$Num = 10^6 \times (593{,}657)^2 \times 0.03158^2 \times 0.1 = 10^6 \times 3.524 \times 10^{11} \times 9.97 \times 10^{-4} \times 0.1$

$Num = 10^6 \times 3.514 \times 10^7 = 3.514 \times 10^{13}$

$Den = 1{,}984 \times 9.667 \times 10^{-13} = 1.918 \times 10^{-9}$

$R_{max} = (3.514 \times 10^{13} / 1.918 \times 10^{-9})^{1/4} = (1.832 \times 10^{22})^{1/4} = 367$ km

(c) To extend range from 500 km (hypothetical) to the needed range, consider:

Required range is 367 km for single pulse. If we need 2,000 km:

Range ratio = 2,000 / 367 = 5.45

SNR must increase by $5.45^4 = 882$ (29.5 dB)

For coherent integration: N = 882 → N = 882 pulses

At PRF = 10 kHz, this requires a dwell time of 88.2 ms.

Problem 17.4.8

Given: A 905 nm pulsed LiDAR has a transmit pulse energy of 8 μJ, pulse width of 4 ns, receiver aperture diameter of 30 mm, and detector NEP of 0.15 nW/$\sqrt{\text{Hz}}$. The detection bandwidth is 250 MHz.

Find: (a) The peak transmit power, (b) the received power from a road sign (ρ = 0.8, Lambertian reflector) at 200 m, (c) the SNR, and (d) the maximum detection range for a dark vehicle (ρ = 0.1) at SNR = 5.

Solution:

(a) $P_t = E / \tau = 8 \times 10^{-6} / 4 \times 10^{-9} = 2{,}000$ W (2 kW peak)

(b) $A_r = \pi(0.015)^2 = 7.069 \times 10^{-4}\ \text{m}^2$

$P_r = P_t \times \rho \times A_r / (\pi R^2) = 2{,}000 \times 0.8 \times 7.069 \times 10^{-4} / (\pi \times 200^2)$

$P_r = 1.131 / 125{,}664 = 9.00 \times 10^{-6}$ W (9.0 μW)

(c) Noise: $N = NEP \times \sqrt{B} = 0.15 \times 10^{-9} \times \sqrt{(250 \times 10^6)} = 0.15 \times 10^{-9} \times 15{,}811 = 2.372 \times 10^{-6}$ W

$SNR = 9.00 \times 10^{-6} / 2.372 \times 10^{-6} = 3.79$ (5.8 dB)

(d) For $\rho = 0.1$: $P_r = P_t \times \rho \times A_r / (\pi R^2) = SNR \times N$

$R_{max} = \sqrt{(P_t \times \rho \times A_r / (\pi \times SNR \times N))}$

$R_{max} = \sqrt{(2{,}000 \times 0.1 \times 7.069 \times 10^{-4} / (\pi \times 5 \times 2.372 \times 10^{-6}))}$

$R_{max} = \sqrt{(0.14138 / 3.727 \times 10^{-5})} = \sqrt{(3{,}793)} = 61.6$ m

Dark vehicles at 200 m would require multi-pulse averaging (approximately $(200/61.6)^2 \approx 10.5$, so ~11 pulses averaged).

Problem 17.4.9

Given: A 1550 nm FMCW LiDAR system achieves simultaneous range and velocity measurement. It has a sweep bandwidth of 10 GHz, sweep time of 100 μs, and transmit power of 20 mW.

Find: (a) The range resolution, (b) the maximum unambiguous velocity (using the relation $v_{max} = \lambda / (4T_{sweep})$), (c) the wavelength, and (d) explain two advantages of 1550 nm over 905 nm for automotive LiDAR.

Solution:

(a) $\Delta R = c / (2B) = 3 \times 10^8 / (2 \times 10 \times 10^9) = 1.5$ cm

(b) $\lambda = c / f$. First find frequency: $f = c / \lambda = 3 \times 10^8 / 1550 \times 10^{-9} = 1.935 \times 10^{14}$ Hz

(c) $\lambda = 1550$ nm $= 1.55 \times 10^{-6}$ m

(b continued) For FMCW LiDAR, the Doppler shift is extremely large due to the short wavelength:

$v_{max} = \lambda / (4T_{sweep}) = 1.55 \times 10^{-6} / (4 \times 100 \times 10^{-6}) = 3.875 \times 10^{-3}$ m/s (extremely small)

This shows that FMCW LiDAR at optical wavelengths requires different velocity disambiguation approaches than RF FMCW radar. In practice, velocity is extracted from the Doppler-induced frequency shift superimposed on the range beat frequency.

(d) Advantages of 1550 nm over 905 nm:

1. Eye safety: The cornea absorbs 1550 nm radiation before it reaches the retina, allowing ~100× higher transmit power within Class 1 eye-safe limits compared to 905 nm, which focuses on the retina.
2. Reduced solar background: The solar spectral irradiance at 1550 nm is lower than at 905 nm, improving the signal-to-background ratio in daylight operation.

Problem 17.4.10

Given: An airport precision approach radar (PAR) operates at 15 GHz (Ku-band) with a 10 m × 3 m antenna providing azimuth beamwidth of 0.4° and elevation beamwidth of 1.2°.

Find: (a) The wavelength, (b) the azimuth position accuracy at 20 km range (assuming accuracy ≈ beamwidth/10 for good SNR), (c) the elevation position accuracy, and (d) the glideslope deviation that can be detected if the standard 3° glideslope requires ±0.3° accuracy.

Solution:

(a) $\lambda = c / f = 3 \times 10^8 / 15 \times 10^9 = 0.020$ m (20 mm)

(b) Azimuth accuracy $\approx \theta_{az} / 10 = 0.4° / 10 = 0.04°$

Position accuracy at 20 km: $\sigma_{az} = 20{,}000 \times (0.04 \times \pi/180) = 20{,}000 \times 6.98 \times 10^{-4} = 14.0$ m

(c) Elevation accuracy $\approx \theta_{el} / 10 = 1.2° / 10 = 0.12°$

Position accuracy at 20 km: $\sigma_{el} = 20{,}000 \times (0.12 \times \pi/180) = 20{,}000 \times 2.094 \times 10^{-3} = 41.9$ m

(d) The PAR can measure elevation to 0.12° accuracy. Since the required accuracy is ±0.3°, the PAR can detect glideslope deviations as small as 0.12°, which is well within the ±0.3° requirement. At 20 km range, 0.3° corresponds to an altitude deviation of $20{,}000 \times \tan(0.3°) = 20{,}000 \times 0.00524 = 105$ m — the PAR can guide aircraft to within approximately 42 m of the correct glideslope altitude.

Chapter 17 — Section 17.5: Radar System Comparison

Practice problems covering frequency band selection and radar system trade-offs.

Problem 17.5.1

Given: A coastal surveillance radar must detect small boats (RCS = 3 m^2) at 25 km range with range resolution better than 15 m. The antenna aperture is limited to 2 m. Three candidate frequencies are being evaluated: S-band (3 GHz), X-band (9.4 GHz), and Ka-band (35 GHz).

Find: (a) The minimum bandwidth required, (b) the antenna beamwidth at each band, (c) the two-way rain attenuation at each band for 6 mm/h rainfall over 25 km (approximate coefficients: S = 0.007 dB/km, X = 0.08 dB/km, Ka = 0.8 dB/km), and (d) the recommended band.

Solution:

(a) $B = c / (2\Delta R) = 3 \times 10^8 / (2 \times 15) = 10$ MHz

(b) Beamwidth $\theta \approx 70\lambda/D$:

S-band: $\lambda = 0.10$ m, $\theta = 70 \times 0.10 / 2 = 3.50°$

X-band: $\lambda = 0.0319$ m, $\theta = 70 \times 0.0319 / 2 = 1.12°$

Ka-band: $\lambda = 0.00857$ m, $\theta = 70 \times 0.00857 / 2 = 0.30°$

(c) Two-way rain attenuation (path = 2 × 25 km = 50 km):

S-band: 0.007 × 50 = 0.35 dB (negligible)

X-band: 0.08 × 50 = 4.0 dB (moderate — reduces range by ~20%)

Ka-band: 0.8 × 50 = 40 dB (devastating — effectively blinds the radar)

(d) Recommendation: X-band (9.4 GHz). It provides the best compromise: the 1.12° beamwidth gives good angular resolution for tracking small boats, the 4 dB rain attenuation is acceptable (can be compensated with a few dB of additional power margin), and the 10 MHz bandwidth is easily achievable with standard magnetron or solid-state transmitters. S-band has too-wide a beam (3.5°), and Ka-band is impractical in rain.

Problem 17.5.2

Given: A drone detection radar must detect small drones (RCS = 0.01 m^2) at 5 km range. Three candidate architectures are: (a) S-band pulsed, (b) X-band pulsed, and (c) Ku-band FMCW. All have a 1 m antenna aperture.

Find: For each option: the beamwidth, the required transmit power (assuming G = 30 dBi, T_{sys} = 500 K, B_n = 1 MHz, SNR_{req} = 15 dB), and the feasibility assessment.

Solution:

Common parameters: $\sigma = 0.01\ m^2$, R = 5 km = 5,000 m, $G = 10^3$ (30 dBi), SNR = 31.62

$S_{min} = kT_sB_n \times SNR = 1.381 \times 10^{-23} \times 500 \times 10^6 \times 31.62 = 2.184 \times 10^{-13}$ W

From the radar equation: $P_t = R^4 \times (4\pi)^3 \times S_{min} / (G^2\lambda^2\sigma)$

Denominator (common): $G^2\sigma = (10^3)^2 \times 0.01 = 10^4$

$R^4 = (5 \times 10^3)^4 = 6.25 \times 10^{14}$

Numerator $= 6.25 \times 10^{14} \times 1{,}984 \times 2.184 \times 10^{-13} = 6.25 \times 10^{14} \times 4.333 \times 10^{-10} = 2.708 \times 10^5$

(a) S-band (3 GHz):

$\lambda = 0.10$ m, $\theta = 70 \times 0.10 / 1 = 7.0°$ (poor angular resolution)

$P_t = 2.708 \times 10^5 / (10^4 \times 0.01) = 2.708 \times 10^5 / 100 = 2{,}708$ W (2.7 kW peak)

Assessment: The 7° beam provides poor angular accuracy and wide clutter cell. The power is feasible. Marginal — angular resolution is inadequate for tracking small drones.

(b) X-band (10 GHz):

$\lambda = 0.03$ m, $\theta = 70 \times 0.03 / 1 = 2.1°$ (good angular resolution)

$P_t = 2.708 \times 10^5 / (10^4 \times 9 \times 10^{-4}) = 2.708 \times 10^5 / 9 = 30{,}089$ W (30 kW peak)

Assessment: Good angular resolution, but 30 kW peak power is high for a compact system. Pulse compression can reduce the required peak power. Good choice with pulse compression.

(c) Ku-band (15 GHz):

$\lambda = 0.02$ m, $\theta = 70 \times 0.02 / 1 = 1.4°$ (excellent angular resolution)

$P_t = 2.708 \times 10^5 / (10^4 \times 4 \times 10^{-4}) = 2.708 \times 10^5 / 4 = 67{,}700$ W (67.7 kW peak)

Assessment: The λ^2 term in the radar equation means higher frequencies require more power for the same range. However, FMCW with coherent integration can achieve equivalent performance with much lower average power. Feasible with FMCW architecture, but rain attenuation at Ku-band (0.15 dB/km × 10 km two-way ≈ 1.5 dB) adds a modest penalty.

Best choice: X-band pulsed with pulse compression — balances angular resolution, transmit power, and all-weather capability.

Problem 17.5.3

Given: Compare the peak power required to detect a target with $\sigma = 1$ m² at 100 km range at four different frequency bands, assuming the same antenna area (A = 3 m²), $T_{sys} = 400$ K, $B_n = 1$ MHz, and required SNR = 13 dB.

Find: The peak transmit power required at (a) L-band (1.3 GHz), (b) S-band (3 GHz), (c) X-band (10 GHz), and (d) Ka-band (35 GHz). Explain the trend.

Solution:

For a constant antenna area A, the gain is $G = 4\pi A\eta/\lambda^2$ (where $\eta = 0.6$):

$G = 4\pi \times 3 \times 0.6 / \lambda^2 = 22.62 / \lambda^2$

$S_{min} = kT_sB_n \times SNR = 1.381 \times 10^{-23} \times 400 \times 10^6 \times 10^{1.3} = 1.381 \times 10^{-23} \times 400 \times 10^6 \times 19.95$

$S_{min} = 1.102 \times 10^{-13}$ W

$P_t = R^4(4\pi)^3S_{min} / (G^2\lambda^2\sigma)$

Since $G^2 = (22.62/\lambda^2)^2 = 511.7/\lambda^4$:

$P_t = R^4(4\pi)^3S_{min} / (511.7 \times \lambda^2/\lambda^4 \times \sigma) = R^4(4\pi)^3S_{min}\lambda^2 / (511.7\sigma)$

So $P_t \propto \lambda^2$ for constant antenna area — higher frequencies (smaller λ) require less power.

Common factor: $R^4(4\pi)^3S_{min} / (511.7 \times 1) = (10^5)^4 \times 1{,}984 \times 1.102 \times 10^{-13} / 511.7$

$= 10^{20} \times 2.186 \times 10^{-10} / 511.7 = 2.186 \times 10^{10} / 511.7 = 4.273 \times 10^7$

(a) L-band: $\lambda = 0.2308$ m. $P_t = 4.273 \times 10^7 \times 0.2308^2 = 4.273 \times 10^7 \times 0.05327 = 2.28$ MW

$G = 22.62 / 0.2308^2 = 424.6$ (26.3 dBi)

(b) S-band: $\lambda = 0.10$ m. $P_t = 4.273 \times 10^7 \times 0.01 = 427$ kW

$G = 22.62 / 0.01 = 2{,}262$ (33.5 dBi)

(c) X-band: $\lambda = 0.03$ m. $P_t = 4.273 \times 10^7 \times 9 \times 10^{-4} = 38.5$ kW

$G = 22.62 / 9 \times 10^{-4} = 25{,}133$ (44.0 dBi)

(d) Ka-band: $\lambda = 0.00857$ m. $P_t = 4.273 \times 10^7 \times 7.34 \times 10^{-5} = 3.14$ kW

$G = 22.62 / 7.34 \times 10^{-5} = 308{,}174$ (54.9 dBi)

Trend: For constant antenna area, higher frequencies require dramatically less transmit power because the antenna gain increases as $(1/\lambda)^2$. The Ka-band radar needs only 3.14 kW versus 2.28 MW at L-band — a 730× reduction. However, this advantage is offset by (1) rain attenuation at higher frequencies, (2) more expensive/complex high-frequency transmitters, and (3) narrower beams that may require electronic scanning.

Chapter 18 — Section 18.1: Geometric Optics

Practice problems covering reflection and refraction (Snell's law), lenses and imaging, mirrors and curved surfaces, and prisms and dispersive elements.

Problem 18.1.1

Given: A light ray travels from water ($n_1 = 1.333$) into a diamond ($n_2 = 2.417$) at an angle of incidence of 25° from the normal.

Find: (a) The refraction angle in diamond, (b) the critical angle for total internal reflection at the diamond-water interface (light going from diamond to water), and (c) whether total internal reflection is possible when light passes from water into diamond.

Solution:

(a) Snell's law: $n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$

$1.333 \times \sin(25°) = 2.417 \times \sin(\theta_2)$

$\sin(\theta_2) = 1.333 \times 0.4226 / 2.417 = 0.5633 / 2.417 = 0.2331$

$\theta_2 = \arcsin(0.2331) = 13.5°$

The ray bends toward the normal when entering the denser medium.

(b) Critical angle (diamond to water): $\theta_c = \arcsin(n_{water} / n_{diamond}) = \arcsin(1.333 / 2.417)$

$\theta_c = \arcsin(0.5517) = 33.5°$

(c) No, total internal reflection cannot occur when light passes from a less dense medium (water) into a denser medium (diamond). TIR only occurs when light travels from a higher-index medium to a lower-index medium and the angle of incidence exceeds the critical angle.

Problem 18.1.2

Given: An optical fiber has a glass core with $n_1 = 1.48$ and a cladding with $n_2 = 1.46$.

Find: (a) The critical angle for total internal reflection at the core-cladding boundary, (b) the numerical aperture, (c) the maximum acceptance half-angle for light entering the fiber from air, and (d) the critical angle if the cladding is replaced with $n_2 = 1.44$.

Solution:

(a) $\theta_c = \arcsin(n_2 / n_1) = \arcsin(1.46 / 1.48) = \arcsin(0.9865) = 80.6°$

Light must strike the core-cladding interface at angles greater than 80.6° (measured from normal) for total internal reflection — meaning the ray must be nearly grazing along the fiber axis.

(b) $NA = \sqrt{(n_1^2 - n_2^2)} = \sqrt{(1.48^2 - 1.46^2)} = \sqrt{(2.1904 - 2.1316)} = \sqrt{(0.0588)} = 0.2425$

(c) $\theta_{max} = \arcsin(NA) = \arcsin(0.2425) = 14.0°$

(d) With $n_2 = 1.44$: $\theta_c = \arcsin(1.44 / 1.48) = \arcsin(0.9730) = 76.7°$

$NA = \sqrt{(1.48^2 - 1.44^2)} = \sqrt{(2.1904 - 2.0736)} = \sqrt{(0.1168)} = 0.342$

The larger index difference provides a wider acceptance angle (20.0°) but supports more modes in multimode fiber.

Problem 18.1.3

Given: A converging lens with focal length $f = 75$ mm is used to image an object at three different distances: (a) $d_o = 300$ mm, (b) $d_o = 150$ mm, and (c) $d_o = 50$ mm.

Find: For each case, the image distance, magnification, and whether the image is real/virtual and upright/inverted.

Solution:

(a) $d_o = 300$ mm (object beyond 2f):

$1/d_i = 1/f - 1/d_o = 1/75 - 1/300 = (4 - 1)/300 = 3/300$

$d_i = 100$ mm (positive → real image)

$m = -d_i/d_o = -100/300 = -0.333$ (inverted, reduced to 1/3 size)

(b) $d_o = 150$ mm (object at 2f):

$1/d_i = 1/75 - 1/150 = (2 - 1)/150 = 1/150$

$d_i = 150$ mm (positive → real image)

$m = -150/150 = -1.0$ (inverted, same size — the symmetric 2f configuration)

(c) $d_o = 50$ mm (object inside f):

$1/d_i = 1/75 - 1/50 = (2 - 3)/150 = -1/150$

$d_i = -150$ mm (negative → virtual image, on same side as object)

$m = -(-150)/50 = +3.0$ (upright, magnified 3×)

This is the magnifying glass configuration — the object inside the focal length produces a virtual, upright, enlarged image.

Problem 18.1.4

Given: A two-lens optical system consists of a converging lens ($f_1 = 100$ mm) and a diverging lens ($f_2 = -50$ mm) separated by 80 mm.

Find: (a) The combined focal length of the system, (b) the image distance for an object at infinity, and (c) whether the system is a telephoto or reverse-telephoto configuration.

Solution:

(a) Combined focal length: $1/f_{total} = 1/f_1 + 1/f_2 - d/(f_1 \times f_2)$

$1/f_{total} = 1/100 + 1/(-50) - 80/(100 \times (-50))$

$1/f_{total} = 0.010 - 0.020 + 80/5{,}000$

$1/f_{total} = 0.010 - 0.020 + 0.016 = 0.006$

$f_{total} = 1/0.006 = 166.7$ mm

(b) For an object at infinity, the first lens forms an intermediate image at $f_1 = 100$ mm from the first lens. The second lens is 80 mm from the first lens, so the intermediate image falls $100 - 80 = 20$ mm past the second lens. This is a virtual object for the second lens (light is converging toward a point on the output side of L_2), so $d_{o2} = -20$ mm:

$1/d_{i2} = 1/f_2 - 1/d_{o2} = 1/(-50) - 1/(-20) = -0.020 + 0.050 = +0.030$

$d_{i2} = +33.3$ mm (real image, 33.3 mm past the second lens)

The image is 33.3 mm past the second lens (or $80 + 33.3 = 113.3$ mm past the first lens). This can also be confirmed by ray tracing: a ray at height $h = 1$ mm parallel to the axis arrives at L_2 with slope $u = -h/f_1 = -0.010$ rad; after L_2 the slope is $u' = u - h'/f_2 = -0.010 - 0.2/(-50) = -0.006$ rad where $h' = 1 - 80 \times 0.01 = 0.2$ mm; the ray crosses the axis at $0.2/0.006 = 33.3$ mm past L_2.

(c) The combined focal length (166.7 mm) is longer than the physical system length (80 mm + back focal distance). This is a telephoto configuration — the positive lens followed by a negative lens produces a long effective focal length in a compact package, as used in camera telephoto lenses.

Problem 18.1.5

Given: A concave mirror with radius of curvature $R = 60$ cm is used in a reflecting telescope. A star (effectively at infinity) is observed, and a nearby planet is at an object distance of 10 m.

Find: (a) The focal length, (b) the image distance for the star, (c) the image distance for the planet, and (d) the magnification for the planet.

Solution:

(a) $f = R/2 = 60/2 = 30$ cm

(b) For an object at infinity: $d_i = f = 30$ cm (image forms at the focal point)

(c) $1/d_i = 1/f - 1/d_o = 1/30 - 1/1{,}000 = (33.33 - 1)/1{,}000 = 32.33/1{,}000$

$d_i = 1{,}000/32.33 = 30.93$ cm

(d) $m = -d_i/d_o = -30.93/1{,}000 = -0.0309$ (inverted, greatly reduced)

The planet image is only 0.93 cm farther from the mirror than the star image — illustrating why astronomical telescopes have very small depth-of-focus issues for celestial objects.

Problem 18.1.6

Given: A spectrometer uses a diffraction grating with 1200 grooves/mm. Light from a hydrogen lamp containing the Balmer series is analyzed in first order.

Find: (a) The grating spacing, (b) the diffraction angle for H-α (656.3 nm), H-β (486.1 nm), and H-γ (434.0 nm), and (c) the resolving power needed to separate the H-α line from a nearby atmospheric oxygen line at 656.7 nm.

Solution:

(a) $d = 1/1200$ mm $= 833.3$ nm (8.333×10^{-7} m)

(b) $\sin(\theta) = m\lambda/d = \lambda/d$ (first order, $m = 1$):

H-α: $\sin(\theta) = 656.3/833.3 = 0.7876$, $\theta = \arcsin(0.7876) = 51.97°$

H-β: $\sin(\theta) = 486.1/833.3 = 0.5833$, $\theta = \arcsin(0.5833) = 35.69°$

H-γ: $\sin(\theta) = 434.0/833.3 = 0.5208$, $\theta = \arcsin(0.5208) = 31.38°$

(c) $\Delta\lambda = 656.7 - 656.3 = 0.4$ nm

$R = \lambda/\Delta\lambda = 656.3/0.4 = 1{,}641$

Required number of grooves: $N = R/m = 1{,}641/1 = 1{,}641$ grooves

At 1200 grooves/mm, the minimum illuminated width is $1{,}641/1{,}200 = 1.37$ mm — a very small grating section suffices.

Problem 18.1.7

Given: A prism spectrometer uses an equilateral prism (apex angle $\alpha = 60°$) made of SF11 flint glass. At minimum deviation, the refractive indices are: $n(486\text{ nm}) = 1.806$, $n(546\text{ nm}) = 1.785$, $n(656\text{ nm}) = 1.769$.

Find: (a) The minimum deviation angle at 546 nm, (b) the angular dispersion between 486 nm and 656 nm, and (c) the minimum deviation at 486 nm and 656 nm.

Solution:

(a) $\delta_{min} = 2\arcsin(n \sin(\alpha/2)) - \alpha = 2\arcsin(1.785 \times \sin(30°)) - 60°$

$\delta_{min} = 2\arcsin(1.785 \times 0.500) - 60° = 2\arcsin(0.8925) - 60°$

$\delta_{min} = 2 \times 63.18° - 60° = 126.36° - 60° = 66.36°$

(b) $\delta_{min}(486\text{ nm}) = 2\arcsin(1.806 \times 0.500) - 60° = 2\arcsin(0.903) - 60° = 2 \times 64.55° - 60° = 69.10°$

$\delta_{min}(656\text{ nm}) = 2\arcsin(1.769 \times 0.500) - 60° = 2\arcsin(0.8845) - 60° = 2 \times 62.17° - 60° = 64.34°$

Angular dispersion = 69.10° − 64.34° = 4.76°

(c) Already computed above:

At 486 nm (F line): $\delta_{min} = 69.10°$

At 656 nm (C line): $\delta_{min} = 64.34°$

The high dispersion of SF11 flint glass ($\Delta n = 0.037$ over the visible range) produces nearly 5° of angular spread, making it effective for spectroscopy applications.

Problem 18.1.8

Given: A camera lens has an f-number of f/2.8 with a focal length of 50 mm.

Find: (a) The aperture diameter, (b) the diffraction-limited Airy disk diameter at $\lambda = 550$ nm, (c) the f-number needed to achieve a 10 μm Airy disk (matching a typical pixel size), and (d) the depth of field for an object at 2 m (approximate formula: $DOF \approx 2f^2N \times c / (f^2)$ where c is the circle of confusion ≈ 20 μm and N is the f-number; simplified $DOF \approx 2Nc \times d_o^2 / f^2$).

Solution:

(a) $D = f / (f/\#) = 50 / 2.8 = 17.9$ mm

(b) $d_{Airy} = 2.44 \times \lambda \times (f/\#) = 2.44 \times 550 \times 10^{-6} \times 2.8 = 2.44 \times 0.00055 \times 2.8 = 3.76\ \mu m$

(c) $d_{Airy} = 2.44\lambda(f/\#)$, so $f/\# = d_{Airy} / (2.44\lambda) = 10 \times 10^{-6} / (2.44 \times 550 \times 10^{-9}) = 10 / 1.342 = f/7.5$

(d) $DOF \approx 2Nc \times d_o^2 / f^2 = 2 \times 2.8 \times 0.020 \times 2000^2 / 50^2$

$DOF = 2 \times 2.8 \times 0.020 \times 4 \times 10^6 / 2500 = 0.112 \times 1{,}600 = 179$ mm (about 18 cm)

At f/2.8 and 2 m focus distance, the depth of field is approximately 18 cm — objects from about 1.91 m to 2.09 m will appear sharp.

Chapter 18 — Section 18.2: Wave Optics

Practice problems covering interference, diffraction, polarization, and thin-film optical coatings.

Problem 18.2.1

Given: In a Young's double-slit experiment, two slits separated by d = 0.25 mm are illuminated by a monochromatic laser at $\lambda = 632.8$ nm (HeNe). The interference pattern is observed on a screen L = 1.5 m from the slits.

Find: (a) The fringe spacing on the screen, (b) the angle to the third-order bright fringe, and (c) the total number of bright fringes that fit within a ±30° angular range.

Solution:

(a) Fringe spacing: $\Delta y = \lambda L/d = 632.8 \times 10^{-9} \times 1.5 / (0.25 \times 10^{-3}) = 9.492 \times 10^{-7} / 2.5 \times 10^{-4} = 3.797$ mm

(b) Angle to third-order bright fringe: $d \sin(\theta) = m\lambda$, so $\sin(\theta_3) = 3 \times 632.8 \times 10^{-9} / (0.25 \times 10^{-3}) = 1.8984 \times 10^{-6} / 2.5 \times 10^{-4} = 7.594 \times 10^{-3}$ $\theta_3 = \arcsin(0.007594) = 0.435°$

(c) Maximum order at $\theta = 30°$: $m_{max} = d \sin(30°) / \lambda = 0.25 \times 10^{-3} \times 0.5 / 632.8 \times 10^{-9} = 1.25 \times 10^{-4} / 6.328 \times 10^{-7} = 197.5$ Maximum integer order m = 197. Total bright fringes from m = −197 to m = +197 plus the central maximum: $N = 2 \times 197 + 1 = 395$ bright fringes

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Problem 18.2.2

Given: A single slit of width a = 50 μm is illuminated by light at $\lambda = 500$ nm. The diffraction pattern is observed on a screen at distance L = 2 m.

Find: (a) The angular width of the central maximum, (b) the linear width of the central maximum on the screen, and (c) the angle to the second-order minimum.

Solution:

(a) The first minima occur at $a \sin(\theta) = \pm\lambda$, so $\sin(\theta_1) = \lambda/a = 500 \times 10^{-9} / 50 \times 10^{-6} = 0.01$. $\theta_1 = 0.573°$. Angular width of central maximum $= 2\theta_1 = 1.146°$

(b) Linear half-width: $y_1 = L \tan(\theta_1) \approx L \times \sin(\theta_1) = 2 \times 0.01 = 0.02$ m = 20 mm. Full width of central maximum = $2 \times 20 = 40$ mm

(c) Second-order minimum: $a \sin(\theta_2) = 2\lambda$. $\sin(\theta_2) = 2 \times 500 \times 10^{-9} / 50 \times 10^{-6} = 0.02$. $\theta_2 = \arcsin(0.02) = 1.146°$

Problem 18.2.3

Given: A telescope has an objective aperture diameter D = 100 mm and operates at $\lambda = 600$ nm. It is used to observe two point sources.

Find: (a) The diffraction-limited angular resolution (Rayleigh criterion), (b) the minimum separation of two objects that can be resolved at a distance of 10 km, and (c) the angular resolution if the aperture is reduced to D = 25 mm by a stop.

Solution:

(a) Rayleigh criterion: $\theta_{min} = 1.22\lambda/D = 1.22 \times 600 \times 10^{-9} / 0.100 = 7.32 \times 10^{-6}$ rad = 7.32 μrad

Converting: $\theta_{min} = 7.32 \times 10^{-6} \times 206{,}265 = 1.51$ arc-seconds

(b) Minimum resolvable separation at 10 km: $s = \theta_{min} \times d = 7.32 \times 10^{-6} \times 10{,}000 = 0.0732$ m = 73.2 mm

(c) With D = 25 mm: $\theta_{min} = 1.22 \times 600 \times 10^{-9} / 0.025 = 2.928 \times 10^{-5}$ rad = 29.3 μrad

The resolution degrades by a factor of 4 (proportional to the aperture reduction), demonstrating the direct relationship between aperture size and resolving power.

Problem 18.2.4

Given: Unpolarized light at intensity $I_0 = 200$ mW/cm² passes through three linear polarizers in series. The first has its transmission axis at 0° (vertical), the second at 30° from vertical, and the third at 75° from vertical.

Find: (a) The intensity after each polarizer and (b) the overall transmission ratio.

Solution:

(a) After polarizer 1 (unpolarized → linear): $I_1 = I_0/2 = 200/2 = 100$ mW/cm²

After polarizer 2 (Malus's law, angle between axes = 30°): $I_2 = I_1 \cos^2(30°) = 100 \times (\sqrt{3}/2)^2 = 100 \times 0.75 = 75$ mW/cm²

After polarizer 3 (angle between polarizer 2 and 3 = 75° − 30° = 45°): $I_3 = I_2 \cos^2(45°) = 75 \times (\sqrt{2}/2)^2 = 75 \times 0.5 = 37.5$ mW/cm²

(b) Overall transmission: $T = I_3/I_0 = 37.5/200 = 18.75\%$

Without the middle polarizer, the angle between polarizers 1 and 3 would be 75°, giving $I = 100 \times \cos^2(75°) = 100 \times 0.0670 = 6.70$ mW/cm² — only 3.35%. The intermediate polarizer increases transmission by rotating the polarization in stages.

Problem 18.2.5

Given: A thin anti-reflection coating of SiO_2 ($n_{film} = 1.46$) is applied to a silicon solar cell ($n_{substrate} = 3.5$) to minimize reflection at $\lambda = 600$ nm.

Find: (a) The quarter-wave coating thickness, (b) the residual reflectance with the coating, and (c) the uncoated reflectance for comparison.

Solution:

(a) Quarter-wave thickness: $t = \lambda / (4n_{film}) = 600 / (4 \times 1.46) = 600 / 5.84 = 102.7$ nm

(b) Residual reflectance with coating: $R_{coated} = [(n_{film}^2 - n_1 \times n_2) / (n_{film}^2 + n_1 \times n_2)]^2 = [(1.46^2 - 1.00 \times 3.5) / (1.46^2 + 1.00 \times 3.5)]^2 = [(2.1316 - 3.5) / (2.1316 + 3.5)]^2 = [(-1.3684) / (5.6316)]^2 = (0.2430)^2 = 5.90\%$

(c) Uncoated reflectance (Fresnel, normal incidence): $R_{uncoated} = [(n_2 - n_1) / (n_2 + n_1)]^2 = [(3.5 - 1.0) / (3.5 + 1.0)]^2 = (2.5/4.5)^2 = (0.5556)^2 = 30.86\%$

The SiO_2 coating reduces reflectance from 30.86% to 5.90%. The ideal coating index would be $n_{ideal} = \sqrt{(1.0 \times 3.5)} = 1.871$, so Si_3N_4 ($n \approx 2.0$) would be a better match for silicon solar cells.

Problem 18.2.6

Given: A Fabry-Perot interferometer has mirror reflectivity $R = 0.95$, mirror separation $d = 5$ mm, and operates at $\lambda = 633$ nm in air ($n = 1$).

Find: (a) The free spectral range (FSR) in frequency, (b) the finesse, and (c) the minimum resolvable frequency difference.

Solution:

(a) Free spectral range: $FSR = c / (2nd) = 3 \times 10^8 / (2 \times 1 \times 5 \times 10^{-3}) = 3 \times 10^8 / 0.01 = 30$ GHz

(b) Finesse: $F = \pi\sqrt{R} / (1 - R) = \pi \times \sqrt{0.95} / (1 - 0.95) = \pi \times 0.9747 / 0.05 = 3.062 / 0.05 = 61.2$

(c) Minimum resolvable frequency difference: $\delta f = FSR / F = 30 \times 10^9 / 61.2 = 490$ MHz

In wavelength terms: $\delta\lambda = \lambda^2 \times \delta f / c = (633 \times 10^{-9})^2 \times 4.90 \times 10^8 / (3 \times 10^8) = 6.54 \times 10^{-4}$ nm $= 0.654$ pm. This is sufficient to resolve the hyperfine structure of many atomic spectral lines.

Problem 18.2.7

Given: A diffraction grating with 1200 grooves/mm and 30 mm illuminated width is used to analyze the hydrogen-alpha spectral line at $\lambda = 656.28$ nm in first order.

Find: (a) The grating spacing, (b) the diffraction angle, (c) the resolving power, and (d) the minimum wavelength difference that can be resolved.

Solution:

(a) Grating spacing: $d = 1/1200$ mm $= 0.8333$ μm $= 833.3$ nm

(b) Diffraction angle for first order: $\sin(\theta_1) = m\lambda/d = 1 \times 656.28 \times 10^{-9} / 833.3 \times 10^{-9} = 0.7875$ $\theta_1 = \arcsin(0.7875) = 51.95°$

(c) Total grooves: $N = 1200 \times 30 = 36{,}000$. Resolving power: $R = mN = 1 \times 36{,}000 = 36{,}000$

(d) Minimum resolvable wavelength difference: $\Delta\lambda_{min} = \lambda/R = 656.28 / 36{,}000 = 0.01823$ nm

This resolving power easily separates the deuterium-alpha line ($\lambda = 656.10$ nm) from the hydrogen-alpha line, since $\Delta\lambda = 0.18$ nm » 0.018 nm.

Problem 18.2.8

Given: Light reflects from a glass surface ($n = 1.52$) at an unknown angle. The reflected beam is found to be completely linearly polarized.

Find: (a) Brewster's angle, (b) the refraction angle of the transmitted ray, and (c) the reflectance for s-polarized light at this angle.

Solution:

(a) Brewster's angle: $\theta_B = \arctan(n_2/n_1) = \arctan(1.52/1.00) = \arctan(1.52) = 56.66°$

(b) At Brewster's angle, the reflected and refracted rays are perpendicular: $\theta_r + \theta_t = 90°$, so $\theta_t = 90° - 56.66° = 33.34°$

Verification with Snell's law: $n_1 \sin(\theta_B) = n_2 \sin(\theta_t)$ $1.00 \times \sin(56.66°) = 1.52 \times \sin(33.34°)$ $0.8355 = 1.52 \times 0.5497 = 0.8355$ ✓

(c) Reflectance for s-polarized light (Fresnel equation): $R_s = [(n_1 \cos\theta_i - n_2 \cos\theta_t) / (n_1 \cos\theta_i + n_2 \cos\theta_t)]^2 = [(1.00 \times \cos 56.66° - 1.52 \times \cos 33.34°) / (1.00 \times \cos 56.66° + 1.52 \times \cos 33.34°)]^2 = [(0.5497 - 1.52 \times 0.8355) / (0.5497 + 1.52 \times 0.8355)]^2 = [(0.5497 - 1.2700) / (0.5497 + 1.2700)]^2 = [(-0.7203) / (1.8197)]^2 = (0.3958)^2 = 15.67\%$

At Brewster's angle, p-polarized reflectance is exactly 0% while s-polarized reflectance is about 15.67%.

Problem 18.2.9

Given: A dielectric HR mirror for a CO_2 laser ($\lambda = 10.6$ μm) uses alternating quarter-wave layers of ZnSe ($n_H = 2.40$) and ThF_4 ($n_L = 1.35$). The mirror has $N = 10$ layer pairs.

Find: (a) The physical thickness of each quarter-wave layer, (b) the total stack thickness, and (c) the reflectivity.

Solution:

(a) Quarter-wave thicknesses: $t_H = \lambda / (4n_H) = 10{,}600 / (4 \times 2.40) = 10{,}600 / 9.60 = 1{,}104$ nm $t_L = \lambda / (4n_L) = 10{,}600 / (4 \times 1.35) = 10{,}600 / 5.40 = 1{,}963$ nm

(b) Total stack thickness for 10 pairs (20 layers + 1 terminating H layer): $t_{total} = 10 \times (1{,}104 + 1{,}963) + 1{,}104 = 10 \times 3{,}067 + 1{,}104 = 30{,}670 + 1{,}104 = 31{,}774$ nm ≈ 31.8 μm

(c) Reflectivity: ratio $= n_H/n_L = 2.40/1.35 = 1.778$ $R = [(1.778^{20} - 1) / (1.778^{20} + 1)]^2$ $1.778^4 = (1.778^2)^2 = (3.161)^2 = 9.992$ $1.778^8 = (9.992)^2 = 99.84$ $1.778^{20} = 1.778^8 \times 1.778^8 \times 1.778^4 = 99.84 \times 99.84 \times 9.992 = 99{,}601$ $R = [(99{,}601 - 1) / (99{,}601 + 1)]^2 = [99{,}600 / 99{,}602]^2 = (0.99998)^2 = 99.996\%$

This extremely high reflectivity is needed for the back mirror of CO_2 laser cavities.

Problem 18.2.10

Given: A circular aperture of diameter D = 2 mm is illuminated by a HeNe laser at $\lambda = 632.8$ nm. The diffraction pattern is observed on a screen at L = 3 m.

Find: (a) The angular radius of the Airy disk (first dark ring), (b) the linear radius of the Airy disk on the screen, and (c) the percentage of total power contained within the Airy disk.

Solution:

(a) Angular radius of first dark ring: $\theta = 1.22\lambda/D = 1.22 \times 632.8 \times 10^{-9} / 2 \times 10^{-3} = 7.72 \times 10^{-7} / 2 \times 10^{-3} = 3.86 \times 10^{-4}$ rad = 0.386 mrad

Converting: $\theta = 3.86 \times 10^{-4} \times (180/\pi) \times 3600 = 79.6$ arc-seconds

(b) Linear radius on the screen: $r = L \times \theta = 3 \times 3.86 \times 10^{-4} = 1.158$ mm

The Airy disk diameter is 2r = 2.316 mm.

(c) For a circular aperture diffraction pattern, the fraction of total power within the first dark ring (Airy disk) is a well-known result: $P_{Airy}/P_{total} = 83.8\%$

The remaining 16.2% of the power is distributed among the surrounding concentric rings, which decrease rapidly in intensity. The first bright ring contains about 7.2% of the total power, and the second bright ring about 2.8%.

Chapter 18 — Section 18.3: Lasers

Practice problems covering stimulated emission and laser operation, laser types and characteristics, laser safety classifications, and fiber lasers and ultrafast lasers.

Problem 18.3.1

Given: A HeNe laser ($\lambda = 632.8$ nm) has a cavity length of L = 30 cm with mirror reflectivities $R_1 = 99.9\%$ and $R_2 = 98\%$, and internal round-trip loss of 1%.

Find: (a) The cavity mode spacing (free spectral range), (b) the number of longitudinal modes within the 1.5 GHz Doppler-broadened gain bandwidth, and (c) the minimum single-pass gain required to reach threshold.

Solution:

(a) Mode spacing: $\Delta f = c / (2L) = 3 \times 10^8 / (2 \times 0.30) = 3 \times 10^8 / 0.60 = 500$ MHz

(b) Number of modes within gain bandwidth: N = gain bandwidth / mode spacing = $1.5 \times 10^9 / 500 \times 10^6 = 3$ modes

A HeNe laser typically operates on 2–3 longitudinal modes simultaneously.

(c) Round-trip loss factor: $\delta = R_1 \times R_2 \times (1 - \text{internal loss}) = 0.999 \times 0.98 \times 0.99 = 0.969$ Threshold single-pass gain: $G_{\text{threshold}} = 1/\sqrt{\delta} = 1/\sqrt{0.969} = 1/0.9844 = 1.016$

The required single-pass gain of 1.6% is very modest, which is why HeNe lasers can operate with a low-gain gas discharge.

Problem 18.3.2

Given: A CO_2 laser operating at $\lambda = 10.6$ μm delivers 2 kW of CW optical power. The laser is electrically pumped at 15 kW. The beam exits through a cavity mirror with $R_2 = 90\%$ (output coupler) and the back mirror has $R_1 = 99.5\%$.

Find: (a) The wall-plug efficiency, (b) the photon energy, (c) the photon emission rate, and (d) the intracavity power.

Solution:

(a) Wall-plug efficiency: $\eta = P_{opt} / P_{elec} = 2{,}000 / 15{,}000 = 0.1333 = 13.3\%$

(b) Photon energy: $E = hc/\lambda = (6.626 \times 10^{-34} \times 3 \times 10^8) / (10.6 \times 10^{-6}) = 1.988 \times 10^{-25} / 1.06 \times 10^{-5} = 1.875 \times 10^{-20}$ J = 0.117 eV

(c) Photon emission rate: $N = P_{opt} / E = 2{,}000 / 1.875 \times 10^{-20} = 1.067 \times 10^{23}$ photons/s

The low photon energy at 10.6 μm means very high photon counts for a given power level.

(d) The output coupler transmits $T_2 = 1 - R_2 = 10\%$ of the intracavity power: $P_{intracavity} = P_{out} / T_2 = 2{,}000 / 0.10 = 20$ kW

Problem 18.3.3

Given: A pulsed Nd:YAG laser ($\lambda = 1064$ nm) operates in Q-switched mode, producing 10 ns pulses at a repetition rate of 20 Hz. The average output power is 50 W.

Find: (a) The pulse energy, (b) the peak power, (c) the photon energy, and (d) the number of photons per pulse.

Solution:

(a) Pulse energy: $E_{pulse} = P_{avg} / f_{rep} = 50 / 20 = 2.5$ J

(b) Peak power: $P_{peak} = E_{pulse} / \tau_p = 2.5 / (10 \times 10^{-9}) = 250$ MW

(c) Photon energy: $E_{photon} = hc/\lambda = (6.626 \times 10^{-34} \times 3 \times 10^8) / (1064 \times 10^{-9}) = 1.988 \times 10^{-25} / 1.064 \times 10^{-6} = 1.868 \times 10^{-19}$ J = 1.165 eV

(d) Photons per pulse: $N = E_{pulse} / E_{photon} = 2.5 / 1.868 \times 10^{-19} = 1.338 \times 10^{19}$ photons

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Problem 18.3.4

Given: A semiconductor laser diode operates at $\lambda = 850$ nm with a threshold current of $I_{th} = 15$ mA, slope efficiency of $\eta_s = 0.5$ W/A, and is driven at $I = 45$ mA with a forward voltage of 1.8 V.

Find: (a) The optical output power, (b) the wall-plug efficiency, (c) the external quantum efficiency, and (d) the number of photons emitted per second.

Solution:

(a) Output power (above threshold): $P_{opt} = \eta_s \times (I - I_{th}) = 0.5 \times (45 - 15) \times 10^{-3} = 0.5 \times 0.030 = 15$ mW

(b) Wall-plug efficiency: $P_{elec} = I \times V = 0.045 \times 1.8 = 81$ mW $\eta_{WP} = P_{opt} / P_{elec} = 15 / 81 = 18.5\%$

(c) External quantum efficiency (photons out per electron in): $E_{photon} = hc/\lambda = 1.988 \times 10^{-25} / 850 \times 10^{-9} = 2.339 \times 10^{-19}$ J Photon rate $= P_{opt} / E_{photon} = 15 \times 10^{-3} / 2.339 \times 10^{-19} = 6.413 \times 10^{16}$ photons/s Electron rate $= I/q = 0.045 / 1.602 \times 10^{-19} = 2.809 \times 10^{17}$ electrons/s η_{ext} = photon rate / electron rate $= 6.413 \times 10^{16} / 2.809 \times 10^{17} = 22.8\%$

(d) Photon emission rate $= 6.41 \times 10^{16}$ photons/s (computed above).

Problem 18.3.5

Given: A Class 4 Nd:YAG laser emits 5 W at $\lambda = 1064$ nm with a beam divergence of 2 mrad (full angle) and an initial beam diameter of 3 mm. The MPE for a 10 s exposure at 1064 nm is 5.0 mW/cm^2.

Find: (a) The beam diameter at 100 m, (b) the irradiance at 100 m, (c) the NOHD, and (d) the required optical density (OD) for protective eyewear.

Solution:

(a) Beam diameter at 100 m: $d(r) = d_0 + \theta \times r = 3 \times 10^{-3} + 2 \times 10^{-3} \times 100 = 0.003 + 0.200 = 0.203$ m = 203 mm

(b) Irradiance at 100 m: $A = \pi(d/2)^2 = \pi(0.1015)^2 = 0.03237$ m^2 = 323.7 cm^2 $E = P/A = 5 / 323.7 = 0.01545$ W/cm^2 = 15.45 mW/cm^2

This exceeds the MPE of 5.0 mW/cm^2, so the beam is hazardous at 100 m.

(c) NOHD (where irradiance = MPE): At the NOHD, $d \approx \theta \times r$ (for large r, d_0 is negligible): $P / [\pi(\theta r/2)^2] = \text{MPE}$ $r = \sqrt{(4P / (\pi \times \theta^2 \times \text{MPE}))} = \sqrt{(4 \times 5 / (\pi \times (2 \times 10^{-3})^2 \times 50))} = \sqrt{(20 / (\pi \times 4 \times 10^{-6} \times 50))} = \sqrt{(20 / 6.283 \times 10^{-4})} = \sqrt{(31{,}831)} = 178$ m

(d) Required OD: $OD = \log_{10}(H_{beam} / \text{MPE})$ at the eye position. At close range (beam diameter ~ 3 mm, smaller than pupil): Irradiance at laser output $= P / [\pi(d_0/2)^2] = 5 / [\pi(1.5 \times 10^{-3})^2] = 5 / 7.069 \times 10^{-6} = 707{,}355$ W/m^2 = 70.7 W/cm^2 $OD = \log_{10}(70.7 / 0.005) = \log_{10}(14{,}140) = 4.15$

Eyewear rated OD ≥ 5 at 1064 nm would be selected with adequate safety margin.

Problem 18.3.6

Given: A mode-locked Ti:sapphire laser produces 50 fs pulses at $\lambda = 800$ nm, with a repetition rate of 76 MHz and average power of 500 mW. The cavity length is 1.974 m.

Find: (a) The pulse energy, (b) the peak power, (c) the transform-limited spectral bandwidth, and (d) the number of locked longitudinal modes.

Solution:

(a) Pulse energy: $E = P_{avg} / f_{rep} = 0.500 / (76 \times 10^6) = 6.58$ nJ

(b) Peak power: $P_{peak} = E / \tau_p = 6.58 \times 10^{-9} / (50 \times 10^{-15}) = 131.6$ kW

(c) Transform-limited bandwidth (Gaussian pulse): $\Delta f = 0.44 / \tau_p = 0.44 / (50 \times 10^{-15}) = 8.8 \times 10^{12}$ Hz = 8.8 THz

In wavelength: $\Delta\lambda = \lambda^2 \times \Delta f / c = (800 \times 10^{-9})^2 \times 8.8 \times 10^{12} / (3 \times 10^8) = 6.4 \times 10^{-13} \times 8.8 \times 10^{12} / (3 \times 10^8) = 5.632 \times 10^0 / 3 \times 10^8 = 18.8$ nm

(d) Mode spacing: $\Delta f_{mode} = c / (2L) = 3 \times 10^8 / (2 \times 1.974) = 76 \times 10^6$ Hz = 76 MHz ✓ Number of locked modes: $N = \Delta f / \Delta f_{mode} = 8.8 \times 10^{12} / 76 \times 10^6 = 115{,}789$ modes

The enormous number of coherent modes explains the ability to produce such ultrashort pulses.

Problem 18.3.7

Given: A chirped pulse amplification (CPA) system stretches 100 fs input pulses to 200 ps before amplification, amplifies to 5 mJ pulse energy, then recompresses to 120 fs (due to imperfect compression).

Find: (a) The stretch ratio, (b) the peak power during amplification (stretched pulse), (c) the peak power after compression, and (d) the peak intensity if focused to a 10 μm diameter spot.

Solution:

(a) Stretch ratio: $R = \tau_{stretched} / \tau_{input} = 200 \times 10^{-12} / 100 \times 10^{-15} = 2{,}000\times$

(b) Peak power during amplification: $P_{amp} = E / \tau_{stretched} = 5 \times 10^{-3} / 200 \times 10^{-12} = 25$ MW

This is manageable and below the damage threshold of the amplifier optics.

(c) Peak power after compression: $P_{compressed} = E / \tau_{compressed} = 5 \times 10^{-3} / 120 \times 10^{-15} = 41.7$ GW

(d) Peak intensity at focus: $A = \pi(d/2)^2 = \pi(5 \times 10^{-6})^2 = 7.854 \times 10^{-11}$ m^2 $I = P / A = 41.7 \times 10^9 / 7.854 \times 10^{-11} = 5.31 \times 10^{20}$ W/m^2

This intensity exceeds the threshold for ionizing air (~10^{18} W/m^2), so focusing must be done in vacuum for ultrafast laser experiments.

Problem 18.3.8

Given: An erbium-doped fiber laser operates at $\lambda = 1550$ nm with 976 nm pump diodes. The pump power is 500 mW, the signal output power is 200 mW, and the quantum defect is the only fundamental efficiency limit.

Find: (a) The pump photon energy, (b) the signal photon energy, (c) the quantum defect efficiency limit, (d) the actual slope efficiency, and (e) the wasted power as heat.

Solution:

(a) Pump photon energy: $E_{pump} = hc/\lambda_{pump} = 1.988 \times 10^{-25} / 976 \times 10^{-9} = 2.037 \times 10^{-19}$ J = 1.271 eV

(b) Signal photon energy: $E_{signal} = hc/\lambda_{signal} = 1.988 \times 10^{-25} / 1550 \times 10^{-9} = 1.282 \times 10^{-19}$ J = 0.800 eV

(c) Quantum defect efficiency limit: $\eta_{QD} = E_{signal} / E_{pump} = \lambda_{pump} / \lambda_{signal} = 976 / 1550 = 63.0\%$

Each pump photon can at most produce one signal photon, and the energy difference (quantum defect) becomes heat.

(d) Actual slope efficiency: $\eta = P_{out} / P_{pump} = 200 / 500 = 40.0\%$

This is 40/63 = 63.5% of the quantum defect limit.

(e) Wasted power: $P_{heat} = P_{pump} - P_{out} = 500 - 200 = 300$ mW

Problem 18.3.9

Given: A VCSEL (Vertical-Cavity Surface-Emitting Laser) array for a 3D sensing module contains 200 emitters, each producing 2 mW at $\lambda = 940$ nm. The array is pulsed at 100 kHz with 10 ns pulse width for time-of-flight ranging. Each emitter is driven at 5 mA and 2.0 V during the pulse.

Find: (a) The total peak optical power, (b) the average optical power, (c) the total average electrical power, and (d) the wall-plug efficiency during the pulse.

Solution:

(a) Total peak optical power: $P_{peak,total} = 200 \times 2$ mW $= 400$ mW

(b) Average optical power: Duty cycle $= f \times \tau = 100 \times 10^3 \times 10 \times 10^{-9} = 10^{-3} = 0.1\%$ $P_{avg,opt} = P_{peak,total} \times$ duty cycle $= 400 \times 10^{-3} = 0.4$ mW

(c) Peak electrical power per emitter: $P_{elec,peak} = I \times V = 5 \times 10^{-3} \times 2.0 = 10$ mW Total average electrical power: $P_{avg,elec} = 200 \times 10 \times 10^{-3} \times 10^{-3} = 2.0$ mW

(d) Wall-plug efficiency during pulse: $\eta = P_{opt,peak} / P_{elec,peak}$ per emitter $= 2 / 10 = 20\%$

The low duty cycle keeps average power dissipation minimal, enabling reliable operation without active cooling.

Problem 18.3.10

Given: A frequency-doubled Nd:YAG laser produces green light at $\lambda = 532$ nm from the fundamental at $\lambda = 1064$ nm. The fundamental beam power entering the KTP doubling crystal is 10 W, and the conversion efficiency of the crystal is 45%.

Find: (a) The green (532 nm) output power, (b) the residual infrared (1064 nm) power, (c) the photon energy at each wavelength, and (d) the photon rate ratio (green photons out vs. IR photons consumed).

Solution:

(a) Green output power: $P_{532} = \eta \times P_{1064} = 0.45 \times 10 = 4.5$ W

(b) Residual IR power: $P_{1064,residual} = P_{1064} - P_{1064,consumed}$ Since 4.5 W of green is produced and each green photon requires two IR photons (energy conservation: $E_{532} = 2 \times E_{1064}$), the IR power consumed equals the green power produced. $P_{1064,residual} = 10 - 4.5 = 5.5$ W

(c) Photon energies: $E_{1064} = hc/\lambda = 1.988 \times 10^{-25} / 1064 \times 10^{-9} = 1.868 \times 10^{-19}$ J $= 1.165$ eV $E_{532} = hc/\lambda = 1.988 \times 10^{-25} / 532 \times 10^{-9} = 3.737 \times 10^{-19}$ J $= 2.331$ eV

$E_{532} = 2 \times E_{1064}$ confirms energy conservation in the frequency doubling process.

(d) Photon rate ratio: IR photons consumed per second: $N_{IR} = P_{consumed} / E_{1064} = 4.5 / 1.868 \times 10^{-19} = 2.409 \times 10^{19}$ /s Green photons produced per second: $N_{532} = P_{532} / E_{532} = 4.5 / 3.737 \times 10^{-19} = 1.204 \times 10^{19}$ /s Ratio: $N_{532} / N_{IR} = 1.204 / 2.409 = 0.50$ (exactly 1:2)

Two infrared photons combine to produce one green photon, confirming the second-harmonic generation process.

Chapter 18 — Section 18.4: Photodetectors

Practice problems covering PIN and avalanche photodiodes, phototransistors and photomultipliers, image sensors, and noise in optical detection.

Problem 18.4.1

Given: A silicon PIN photodiode has a quantum efficiency of 70% at $\lambda = 850$ nm and is reverse-biased. The incident optical power is 5 μW.

Find: (a) The responsivity, (b) the photocurrent, (c) the photon rate arriving at the detector, and (d) the electron-hole pair generation rate.

Solution:

(a) Responsivity: $R = \eta q\lambda / (hc) = 0.70 \times 1.602 \times 10^{-19} \times 850 \times 10^{-9} / (6.626 \times 10^{-34} \times 3 \times 10^{8}) = 0.70 \times 1.602 \times 10^{-19} \times 8.50 \times 10^{-7} / (1.988 \times 10^{-25}) = 0.70 \times 6.849 \times 10^{-1} = 0.479$ A/W

(b) Photocurrent: $I_{ph} = R \times P = 0.479 \times 5 \times 10^{-6} = 2.40$ μA

(c) Photon rate: $E_{photon} = hc/\lambda = 1.988 \times 10^{-25} / 850 \times 10^{-9} = 2.339 \times 10^{-19}$ J Photon rate $= P / E_{photon} = 5 \times 10^{-6} / 2.339 \times 10^{-19} = 2.138 \times 10^{13}$ photons/s

(d) Electron-hole pair generation rate: $N_{e\text{-}h} = \eta \times$ photon rate $= 0.70 \times 2.138 \times 10^{13} = 1.497 \times 10^{13}$ pairs/s

Verification: $I_{ph} = N_{e\text{-}h} \times q = 1.497 \times 10^{13} \times 1.602 \times 10^{-19} = 2.40 \times 10^{-6}$ A ✓

Problem 18.4.2

Given: An InGaAs APD has a primary responsivity (without gain) of $R_0 = 1.0$ A/W at $\lambda = 1550$ nm, avalanche gain $M = 10$, and excess noise factor $F(M) = M^{0.7}$. The incident optical power is 1 μW.

Find: (a) The APD responsivity, (b) the multiplied photocurrent, (c) the excess noise factor, and (d) the shot noise current for bandwidth $B = 1$ GHz.

Solution:

(a) APD responsivity: $R_{APD} = M \times R_0 = 10 \times 1.0 = 10$ A/W

(b) Multiplied photocurrent: $I_{ph} = R_{APD} \times P = 10 \times 1 \times 10^{-6} = 10\ \mu A$

(c) Excess noise factor: $F(M) = M^{0.7} = 10^{0.7} = 5.012$

(d) APD shot noise: $i_{shot} = M \times \sqrt{}(2qI_{primary}F(M)B)$ $I_{primary} = R_0 \times P = 1.0 \times 1 \times 10^{-6} = 1\ \mu A$ $i_{shot} = 10 \times \sqrt{}(2 \times 1.602 \times 10^{-19} \times 1 \times 10^{-6} \times 5.012 \times 1 \times 10^{9}) = 10 \times \sqrt{}(1.606 \times 10^{-15}) = 10 \times 4.008 \times 10^{-8} = 0.401\ \mu A$ RMS

The signal-to-noise ratio from shot noise alone: $SNR = I_{ph} / i_{shot} = 10 / 0.401 = 24.9$, or 27.9 dB.

Problem 18.4.3

Given: A photomultiplier tube has 12 dynodes with secondary emission ratio $\delta = 5$ per dynode. The photocathode quantum efficiency is 20% at $\lambda = 400$ nm. A light source produces 10^4 photons/s incident on the photocathode.

Find: (a) The total PMT gain, (b) the photocathode current, (c) the anode current, and (d) the single-photoelectron pulse charge.

Solution:

(a) Total gain: $M = \delta^{12} = 5^{12} = 5^6 \times 5^6 = 15{,}625 \times 15{,}625 = 2.441 \times 10^8$

(b) Photocathode current: Photoelectron rate $= \eta \times$ photon rate $= 0.20 \times 10^4 = 2{,}000$ photoelectrons/s $I_{cathode} = N_{pe} \times q = 2{,}000 \times 1.602 \times 10^{-19} = 3.204 \times 10^{-16}$ A $= 0.320$ fA

(c) Anode current: $I_{anode} = M \times I_{cathode} = 2.441 \times 10^8 \times 3.204 \times 10^{-16} = 78.2$ nA

This is easily measurable with standard electronics, even though only 2,000 photoelectrons/s leave the cathode.

(d) Single-photoelectron pulse charge: $Q = M \times q = 2.441 \times 10^8 \times 1.602 \times 10^{-19} = 3.91 \times 10^{-11}$ C $= 39.1$ pC

With a 5 ns pulse width, the single-photoelectron pulse current is $I = Q/t = 39.1 \times 10^{-12} / 5 \times 10^{-9} = 7.82$ mA — large enough for direct detection without additional amplification.

Problem 18.4.4

Given: A CMOS image sensor has the following specifications: 24 megapixels, 3.75 μm pixel pitch, 6:4 aspect ratio (3:2), full-well capacity of 30,000 electrons, and read noise of 5 electrons RMS.

Find: (a) The pixel array dimensions, (b) the sensor physical dimensions, (c) the dynamic range in dB, and (d) the dynamic range in stops.

Solution:

(a) Total pixels: 24×10^6 with aspect ratio 3:2. $N_x \times N_y = 24 \times 10^6$, $N_x/N_y = 3/2$. $N_y = \sqrt{}(24 \times 10^6 \times 2/3) = \sqrt{}(16 \times 10^6) = 4{,}000$ pixels. $N_x = 3/2 \times 4{,}000 = 6{,}000 \times 4{,}000$ pixels

(b) Sensor dimensions: W = 6,000 × 3.75 μm = 22,500 μm = 22.5 mm H = 4,000 × 3.75 μm = 15,000 μm = 15.0 mm

This is approximately an APS-C format sensor (22.5 × 15.0 mm).

(c) Dynamic range: DR = 20 × $\log_{10}$(full-well / read noise) = 20 × $\log_{10}$(30,000 / 5) = 20 × $\log_{10}$(6,000) = 20 × 3.778 = 75.6 dB

(d) In stops (each stop = 6.02 dB): DR_{stops} = 75.6 / 6.02 = 12.6 stops

This is typical of a mid-range DSLR sensor.

Problem 18.4.5

Given: An InGaAs PIN photodiode with R = 0.95 A/W and dark current I_d = 2 nA is used in a 2.5 Gbps receiver with bandwidth B = 1.875 GHz. The transimpedance amplifier has R_f = 2 kΩ (effective load resistance) and T = 300 K.

Find: (a) The thermal noise current, (b) the dark current noise, (c) the total noise current (neglecting shot noise for small signals), and (d) the receiver sensitivity for BER = 10^{-9} (Q = 6).

Solution:

(a) Thermal noise: $i_{thermal} = \sqrt{(4k_BTB / R_f)} = \sqrt{(4 \times 1.381 \times 10^{-23} \times 300 \times 1.875 \times 10^9 / 2{,}000)} = \sqrt{(4 \times 1.381 \times 10^{-23} \times 300 \times 9.375 \times 10^5)} = \sqrt{(1.553 \times 10^{-14})}$ = 124.6 nA RMS

(b) Dark current noise: $i_{dark} = \sqrt{(2qI_dB)} = \sqrt{(2 \times 1.602 \times 10^{-19} \times 2 \times 10^{-9} \times 1.875 \times 10^9)} = \sqrt{(1.202 \times 10^{-18})}$ = 1.096 nA RMS

(c) Total noise current: $i_n = \sqrt{(i_{thermal}^2 + i_{dark}^2)} = \sqrt{(124.6^2 + 1.096^2)} \approx$ 124.6 nA RMS

The thermal noise dominates by a factor of over 100.

(d) Receiver sensitivity: $P_{sens} = Q \times i_n / R = 6 \times 124.6 \times 10^{-9} / 0.95 = 787 \times 10^{-9}$ = 787 nW = -31.0 dBm

The high transimpedance (2 kΩ) significantly improves sensitivity compared to a 50 Ω load, which would give $i_{thermal}$ = 790 nA and sensitivity of −22.0 dBm.

Problem 18.4.6

Given: A CCD sensor for astronomical imaging has 2048 × 2048 pixels with 15 μm pixel pitch, quantum efficiency η = 90% at 600 nm, read noise of 3 electrons RMS, dark current of 0.01 electrons/pixel/s at −40°C, and full-well capacity of 100,000 electrons.

Find: (a) The sensor dimensions, (b) the dynamic range, (c) the minimum detectable signal for SNR = 5 in a 300 s exposure, and (d) the dark current noise for the 300 s exposure.

Solution:

(a) Sensor dimensions: W = H = 2048 × 15 μm = 30,720 μm = 30.72 mm (square sensor)

Total area = $30.72^2 = 943.7$ mm$^2 \approx 9.44$ cm^2

(b) Dynamic range: DR = $20 \times \log_{10}(100{,}000 / 3) = 20 \times \log_{10}(33{,}333) = 20 \times 4.523 = 90.5$ dB

This is approximately 15 stops — excellent for astrophotography.

(c) Dark current accumulation in 300 s: $N_{dark} = 0.01 \times 300 = 3$ electrons Dark noise = $\sqrt{N_{dark}} = \sqrt{3} = 1.73$ electrons RMS

Total noise = $\sqrt{(\text{read noise}^2 + \text{dark noise}^2)} = \sqrt{(9 + 3)} = \sqrt{12} = 3.46$ electrons RMS

Minimum detectable signal at SNR = 5: For low signal, SNR $\approx N_{signal}$ / total noise, so $N_{signal} = 5 \times 3.46 = 17.3$ electrons

(d) Minimum detectable optical power: Photon energy at 600 nm: $E = hc/\lambda = 1.988 \times 10^{-25} / 600 \times 10^{-9} = 3.313 \times 10^{-19}$ J Photons needed: $N_{photon} = N_{signal} / \eta = 17.3 / 0.90 = 19.2$ photons over 300 s Power = $N_{photon} \times E / t = 19.2 \times 3.313 \times 10^{-19} / 300 = 2.12 \times 10^{-20}$ W per pixel

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Problem 18.4.7

Given: An optical receiver uses an InGaAs PIN photodiode ($R = 1.05$ A/W, $I_d = 10$ nA) followed by a TIA with noise figure equivalent to $R_L = 500\ \Omega$ at $T = 300$ K. The signal power is $P = 100\ \mu$W and the bandwidth is $B = 10$ GHz.

Find: (a) The photocurrent, (b) the shot noise, (c) the thermal noise, (d) the total noise, and (e) the electrical SNR.

Solution:

(a) Photocurrent: $I_{ph} = R \times P = 1.05 \times 100 \times 10^{-6} = 105\ \mu$A

(b) Shot noise: $i_{shot} = \sqrt{(2q(I_{ph} + I_d)B)} = \sqrt{(2 \times 1.602 \times 10^{-19} \times (105 + 0.01) \times 10^{-6} \times 10 \times 10^9)} = \sqrt{(2 \times 1.602 \times 10^{-19} \times 1.0501 \times 10^{-4} \times 10^{10})} = \sqrt{(3.365 \times 10^{-13})} = 0.580\ \mu$A RMS

(c) Thermal noise: $i_{thermal} = \sqrt{(4k_BTB / R_L)} = \sqrt{(4 \times 1.381 \times 10^{-23} \times 300 \times 10 \times 10^9 / 500)} = \sqrt{(3.314 \times 10^{-13})} = 0.576\ \mu$A RMS

(d) Total noise: $i_n = \sqrt{(i_{shot}{}^2 + i_{thermal}{}^2)} = \sqrt{(0.580^2 + 0.576^2)} = \sqrt{(0.336 + 0.332)} = \sqrt{0.668} = 0.817$ μA RMS

The receiver is shot-noise and thermal-noise co-limited (both approximately equal at these conditions).

(e) Electrical SNR: SNR = $I_{ph} / i_n = 105 / 0.817 = 128.5$ SNR (dB) = $20 \times \log_{10}(128.5) = 42.2$ dB

Problem 18.4.8

Given: An InGaAs APD receiver is compared to a PIN receiver for a 10 Gbps link. Both have bandwidth $B = 7.5$ GHz and $T = 300$ K. The PIN has $R = 1.0$ A/W with $R_L = 50\ \Omega$. The APD has $R_0 = 0.9$ A/W, gain $M = 10$, excess noise factor exponent $x = 0.7$, and same $R_L = 50\ \Omega$.

Find: (a) The PIN sensitivity for BER = 10^{-9} (Q = 6), (b) the APD sensitivity, and (c) the improvement in dB.

Solution:

(a) PIN thermal noise: $i_{th} = \sqrt{}(4k_BTB / R_L) = \sqrt{}(4 \times 1.381 \times 10^{-23} \times 300 \times 7.5 \times 10^9 / 50) = \sqrt{}(2.486 \times 10^{-12}) = 1.577$ μA PIN sensitivity: $P_{PIN} = Q \times i_{th} / R = 6 \times 1.577 \times 10^{-6} / 1.0 = 9.46$ μW = −20.2 dBm

(b) APD: the total noise includes both thermal and amplified shot noise. For sensitivity calculation, the signal-dependent shot noise makes this iterative. Using the approximation: $P_{APD} \approx Q \times i_{th} / (M \times R_0)$ for the thermal-noise-limited case: $P_{APD} = 6 \times 1.577 \times 10^{-6} / (10 \times 0.9) = 6 \times 1.577 \times 10^{-6} / 9.0 = 1.051$ μW = −29.8 dBm

Checking shot noise at this power: $I_{primary} = R_0 \times P = 0.9 \times 1.051 \times 10^{-6} = 0.946$ μA $F(M) = 10^{0.7} = 5.012$ $i_{shot} = M \times \sqrt{}(2qI_{primary}F(M)B) = 10 \times \sqrt{}(2 \times 1.602 \times 10^{-19} \times 9.46 \times 10^{-7} \times 5.012 \times 7.5 \times 10^9) = 10 \times \sqrt{}(1.139 \times 10^{-14}) = 10 \times 1.067 \times 10^{-7} = 1.067$ μA

Total noise $= \sqrt{}(1.577^2 + 1.067^2) = \sqrt{}(2.487 + 1.139) = \sqrt{}3.626 = 1.904$ μA Refined: $P_{APD} = Q \times i_{total} / (M \times R_0) = 6 \times 1.904 \times 10^{-6} / 9.0 = 1.269$ μW = −29.0 dBm

(c) Improvement: ΔP = −20.2 − (−29.0) = 8.8 dB improvement with the APD receiver.

Problem 18.4.9

Given: A time-of-flight lidar system uses a pulsed laser at λ = 905 nm with pulse energy 200 nJ and 5 ns pulse width. The detector is a silicon APD with $R_0 = 0.5$ A/W, M = 50, dark current $I_d = 50$ nA, and bandwidth B = 200 MHz. The target is at 100 m with reflectivity ρ = 0.3, and the receiver aperture diameter is 50 mm.

Find: (a) The received optical energy per pulse, (b) the peak received power, (c) the signal photocurrent, and (d) the SNR.

Solution:

(a) For a Lambertian target, received power fraction: The laser illuminates a spot; the return energy is: $E_{rx} = E_{tx} \times \rho \times A_{rx} / (\pi \times R^2)$ $A_{rx} = \pi(D/2)^2 = \pi(0.025)^2 = 1.964 \times 10^{-3}$ m² $E_{rx} = 200 \times 10^{-9} \times 0.3 \times 1.964 \times 10^{-3} / (\pi \times 100^2) = 200 \times 10^{-9} \times 0.3 \times 1.964 \times 10^{-3} / 31{,}416 = 3.75 \times 10^{-15}$ J = 3.75 fJ

(b) Peak received power: $P_{rx} = E_{rx} / \tau = 3.75 \times 10^{-15} / 5 \times 10^{-9} = 0.75$ μW

(c) Signal photocurrent (with APD gain): $I_{signal} = M \times R_0 \times P_{rx} = 50 \times 0.5 \times 0.75 \times 10^{-6} = 18.75$ μA

(d) Noise analysis (R_L = 50 Ω, T = 300 K): $i_{thermal} = \sqrt{}(4k_BTB/R_L) = \sqrt{}(4 \times 1.381 \times 10^{-23} \times 300 \times 200 \times 10^6 / 50) = \sqrt{}(6.629 \times 10^{-13}) = 257$ nA $F(M) = 50^{0.7} = 15.46$ $i_{shot} = M \times \sqrt{}(2q(I_d/M + R_0 \times P_{rx})F(M)B) = 50 \times \sqrt{}(2 \times 1.602 \times 10^{-19} \times (1 \times 10^{-9} + 375 \times 10^{-9}) \times 15.46 \times 200 \times 10^6) = 50 \times \sqrt{}(2 \times 1.602 \times 10^{-19} \times 376 \times 10^{-9} \times 3.092 \times 10^9) = 50 \times \sqrt{}(3.733 \times 10^{-16}) = 50 \times 1.932 \times 10^{-8} = 965$ nA $i_{total} = \sqrt{}(257^2 + 965^2) = \sqrt{}(66{,}049 + 931{,}225) = \sqrt{}997{,}274 = 999$ nA ≈ 1.0 μA SNR = 18,750 / 999 = 18.8 = +25.5 dB (single pulse)

This strong single-pulse SNR confirms the system can reliably detect the target at 100 m.

Problem 18.4.10

Given: A photoconductive detector made of HgCdTe operates at $\lambda = 10\ \mu m$ with active area $A = 1\ mm^2$, specific detectivity $D^* = 2 \times 10^{10}\ cm\cdot\sqrt{Hz}/W$, and bandwidth $B = 100\ kHz$.

Find: (a) The NEP, (b) the minimum detectable power for SNR = 10, (c) the noise current if the responsivity is $R = 500\ V/W$ with $R_L = 10\ k\Omega$, and (d) the photon energy at $10\ \mu m$.

Solution:

(a) NEP (noise-equivalent power): $NEP = \sqrt{A} / D^* = \sqrt{(1 \times 10^{-2}\ cm^2)} / (2 \times 10^{10}) = 0.1 / (2 \times 10^{10}) = 5 \times 10^{-12}\ W/\sqrt{Hz}$

For the given bandwidth: $NEP_{total} = NEP \times \sqrt{B} = 5 \times 10^{-12} \times \sqrt{(100 \times 10^3)} = 5 \times 10^{-12} \times 316.2 = 1.581 \times 10^{-9}\ W = 1.58\ nW$

(b) Minimum detectable power for SNR = 10: $P_{min} = SNR \times NEP_{total} = 10 \times 1.581 \times 10^{-9} = 15.81$ nW

(c) Noise voltage: $V_{noise} = NEP_{total} \times R = 1.581 \times 10^{-9} \times 500 = 7.905 \times 10^{-7}\ V = 0.791\ \mu V$ Noise current: $i_{noise} = V_{noise} / R_L = 7.905 \times 10^{-7} / 10{,}000 = 79.1$ pA RMS

(d) Photon energy at $10\ \mu m$: $E = hc/\lambda = 1.988 \times 10^{-25} / 10 \times 10^{-6} = 1.988 \times 10^{-20}\ J = 0.124\ eV$

This very low photon energy (thermal IR) is why HgCdTe detectors must be cooled to liquid nitrogen temperatures (77 K) — the thermal energy k_BT at room temperature (26 meV) is a significant fraction of the photon energy, producing excessive dark current.

Chapter 18 — Section 18.5: Optical Fiber and Communication

Practice problems covering fiber modes and propagation, attenuation and dispersion, optical amplifiers and WDM, and coherent optical communication.

Problem 18.5.1

Given: A multimode step-index fiber has core index $n_1 = 1.480$, cladding index $n_2 = 1.460$, and core diameter $d = 62.5$ μm. The operating wavelength is $\lambda = 850$ nm.

Find: (a) The numerical aperture, (b) the maximum acceptance half-angle, (c) the V number, and (d) the approximate number of guided modes.

Solution:

(a) Numerical aperture: $NA = \sqrt{(n_1^2 - n_2^2)} = \sqrt{(1.480^2 - 1.460^2)} = \sqrt{(2.1904 - 2.1316)} = \sqrt{(0.0588)} = 0.2425$

(b) Maximum acceptance half-angle: $\theta_{max} = \arcsin(NA) = \arcsin(0.2425) = 14.03°$

(c) V number: $V = \pi d \times NA / \lambda = \pi \times 62.5 \times 10^{-6} \times 0.2425 / (850 \times 10^{-9}) = 4.762 \times 10^{-5} / 8.50 \times 10^{-7} = 56.0$

(d) Approximate number of guided modes (step-index): $M \approx V^2/2 = 56.0^2 / 2 = 3{,}136 / 2 = 1{,}568$ modes

This large number of modes explains why multimode fiber has high modal dispersion and is limited to shorter distances.

Problem 18.5.2

Given: A single-mode fiber link at 1550 nm has the following characteristics: fiber attenuation = 0.20 dB/km, chromatic dispersion $D = 17$ ps/(nm·km), link length = 120 km, and laser spectral width $\Delta\lambda = 0.05$ nm (DFB laser).

Find: (a) The total fiber attenuation, (b) the chromatic dispersion broadening, (c) the maximum bit rate limited by dispersion (NRZ, rule: $\Delta\tau < 0.7 \times$ bit period), and (d) the dispersion-limited distance for a 40 Gbps signal.

Solution:

(a) Total attenuation: $\alpha_{total} = 0.20 \times 120 = 24.0$ dB

(b) Chromatic dispersion broadening: $\Delta\tau = D \times L \times \Delta\lambda = 17 \times 120 \times 0.05 = 102$ ps

(c) Maximum bit rate: Bit period $> \Delta\tau / 0.7 = 102 / 0.7 = 145.7$ ps $B_{max} = 1 / 145.7 \times 10^{-12} = 6.86$ Gbps

A 10 Gbps NRZ signal (bit period = 100 ps) would suffer significant ISI on this link.

(d) Dispersion-limited distance at 40 Gbps: Bit period = 25 ps. Maximum $\Delta\tau = 0.7 \times 25 = 17.5$ ps. $L_{max} = \Delta\tau / (D \times \Delta\lambda) = 17.5 / (17 \times 0.05) = 17.5 / 0.85 = 20.6$ km

Dispersion compensation is essential for high-speed, long-haul links.

Problem 18.5.3

Given: An EDFA has the following specifications: gain G = 25 dB, noise figure NF = 5.5 dB, input signal power = −20 dBm per channel, and operates at 1550 nm with reference bandwidth $B_{ref} = 12.5$ GHz (0.1 nm).

Find: (a) The output signal power, (b) the ASE noise power per channel, (c) the output OSNR, and (d) the output OSNR if the input signal is −30 dBm.

Solution:

(a) Output signal power: $P_{out} = P_{in} + G = -20 + 25 = +5$ dBm

(b) ASE noise power (per polarization, in reference bandwidth): $P_{ASE} = NF \times hf \times G \times B_{ref}$ (in linear units) In dBm: $P_{ASE} = NF + G + 10\log_{10}(hfB_{ref})$ $10\log_{10}(hfB_{ref}) = 10\log_{10}(6.626 \times 10^{-34} \times 193.1 \times 10^{12} \times 12.5 \times 10^{9}) = 10\log_{10}(1.599 \times 10^{-9}\text{ W}) = 10\log_{10}(1.599 \times 10^{-6}\text{ mW}) = -58.0$ dBm $P_{ASE} = 5.5 + 25 + (-58.0) = -27.5$ dBm per polarization

For both polarizations: $P_{ASE,total} = -27.5 + 3 = -24.5$ dBm

(c) Output OSNR: $OSNR = P_{out} - P_{ASE,total} = 5 - (-24.5) = 29.5$ dB

(d) With −30 dBm input: $P_{out} = -30 + 25 = -5$ dBm $OSNR = -5 - (-24.5) = 19.5$ dB

A single EDFA provides moderate OSNR; the challenge arises from cascading many amplifiers where ASE noise accumulates.

Problem 18.5.4

Given: A DWDM long-haul system has 40 channels at 100 GHz spacing, each at 0 dBm launch power. The link consists of 20 spans of 100 km each. Fiber attenuation is 0.2 dB/km, and each EDFA has gain matched to span loss and noise figure NF = 5 dB.

Find: (a) The span loss, (b) the EDFA gain required, (c) the OSNR after 20 spans using the standard formula, and (d) whether the OSNR is sufficient for 100 Gbps DP-QPSK (required OSNR ≈ 12 dB).

Solution:

(a) Span loss: $L_{span} = 0.2 \times 100 = 20$ dB

(b) EDFA gain = span loss = 20 dB

(c) OSNR after N spans (standard formula): $OSNR = 58 + P_{launch} - L_{span} - NF - 10\log_{10}(N) = 58 + 0 - 20 - 5 - 10\log_{10}(20) = 58 - 20 - 5 - 13.01 = 20.0$ dB

(d) OSNR margin: Margin = $20.0 - 12 = 8.0$ dB

This is sufficient for DP-QPSK operation. However, for DP-16QAM (required OSNR ≈ 18 dB), the margin would be only 2 dB, which is marginal.

Problem 18.5.5

Given: A fiber link must carry 10 Gbps NRZ data at 1310 nm over standard single-mode fiber. The transmitter power is +2 dBm, the receiver sensitivity is −23 dBm (BER = 10^{-12}). The link has 6 connectors (0.5 dB each), 4 fusion splices (0.1 dB each), and fiber attenuation is 0.35 dB/km.

Find: (a) The total connector and splice loss, (b) the available power budget after accounting for a 3 dB system margin, (c) the maximum fiber length, and (d) the total link loss at that length.

Solution:

(a) Connector and splice losses: $L_{connectors} = 6 \times 0.5 = 3.0$ dB $L_{splices} = 4 \times 0.1 = 0.4$ dB Total fixed losses = $3.0 + 0.4 = 3.4$ dB

(b) Available power budget: Total budget = $P_{tx} - P_{sens} = 2 - (-23) = 25$ dB After system margin: $25 - 3 = 22$ dB After fixed losses: $22 - 3.4 = 18.6$ dB available for fiber

(c) Maximum fiber length: $L_{max} = 18.6 / 0.35 = 53.1$ km

(d) Total link loss at 53.1 km: $L_{total} = 0.35 \times 53.1 + 3.4 = 18.6 + 3.4 = 22.0$ dB Received power = $2 - 22.0 = -20.0$ dBm, with 3.0 dB margin above the −23 dBm sensitivity.

Problem 18.5.6

Given: A coherent transceiver operates at 64 GBaud with DP-QPSK modulation at 1550 nm. FEC overhead is 15%.

Find: (a) The bits per symbol per polarization, (b) the raw data rate, (c) the net data rate after FEC, and (d) the spectral efficiency for a 50 GHz channel spacing.

Solution:

(a) QPSK encodes 2 bits per symbol. With dual polarization: Bits per symbol = $2 \times 2 = 4$ bits/symbol total

(b) Raw data rate: $R_{raw} = 64 \times 10^9 \times 4 = 256$ Gbps

(c) Net data rate: $R_{net} = R_{raw}$ / (1 + FEC overhead) = 256 / 1.15 = 222.6 Gbps (approximately 200G class)

(d) Spectral efficiency: SE = R_{net} / channel spacing = 222.6 / 50 = 4.45 b/s/Hz

This is close to the practical limit for QPSK with dual polarization (theoretical max = 4 b/s/Hz before FEC overhead).

Problem 18.5.7

Given: A submarine optical cable system uses the C-band (1530–1565 nm) with 80 channels at 50 GHz spacing, 400 Gbps per channel using DP-16QAM at 60 GBaud. The system spans 6,000 km with 75 km repeater spans. Each EDFA has NF = 4.5 dB and fiber loss is 0.20 dB/km.

Find: (a) The number of repeater spans, (b) the total system capacity, (c) the OSNR after all spans, and (d) the OSNR margin for DP-16QAM (required ≈ 16 dB with soft-decision FEC).

Solution:

(a) Number of spans: N = 6,000 / 75 = 80 spans

(b) Total capacity: C = 80 channels × 400 Gbps = 32 Tbps per fiber pair

(c) OSNR: Span loss = $0.20 \times 75 = 15$ dB OSNR = $58 + P_{launch} - L_{span} - NF - 10\log_{10}(N)$ Assuming $P_{launch} = -1$ dBm per channel (typical submarine): $= 58 + (-1) - 15 - 4.5 - 10\log_{10}(80) = 58 - 1 - 15 - 4.5 - 19.03 = 18.5$ dB

(d) Margin: Margin = 18.5 − 16 = 2.5 dB

This margin is tight but typical for submarine systems, which use probabilistic constellation shaping and advanced FEC to operate with thin margins.

Problem 18.5.8

Given: A fiber Bragg grating (FBG) is written in single-mode fiber with effective refractive index $n_{eff} = 1.447$ and grating period $\Lambda = 535.25$ nm. The FBG length is 10 mm with index modulation $\Delta n = 5 \times 10^{-4}$.

Find: (a) The Bragg wavelength, (b) the reflection bandwidth, and (c) the peak reflectivity.

Solution:

(a) Bragg wavelength: $\lambda_B = 2 \times n_{eff} \times \Lambda = 2 \times 1.447 \times 535.25 = 1{,}549.0$ nm

This is in the C-band, making it suitable for DWDM channel filtering.

(b) Reflection bandwidth (for strong grating, $\Delta n \gg \lambda/(\pi L)$): $\Delta\lambda = \lambda_B \times \sqrt{}((\Delta n/n_{eff})^2 + (\lambda_B/(n_{eff} \times L))^2)$ $\Delta n/n_{eff} = 5 \times 10^{-4} / 1.447 = 3.455 \times 10^{-4}$ $\lambda_B/(n_{eff} \times L) = 1549 \times 10^{-9} / (1.447 \times 0.01) = 1.070 \times 10^{-4}$ $\Delta\lambda = 1549 \times \sqrt{}((3.455 \times 10^{-4})^2 + (1.070 \times 10^{-4})^2) = 1549 \times \sqrt{}(1.194 \times 10^{-7} + 1.145 \times 10^{-8}) = 1549 \times \sqrt{}(1.308 \times 10^{-7}) = 1549 \times 3.617 \times 10^{-4} = 0.560$ nm

(c) Peak reflectivity: $\kappa = \pi\Delta n / \lambda_B = \pi \times 5 \times 10^{-4} / 1549 \times 10^{-9} = 1013.6\ m^{-1}$ $\kappa L = 1013.6 \times 0.01 = 10.136$ $R = \tanh^2(\kappa L) = \tanh^2(10.136) \approx 99.99\%$ (tanh is essentially 1 for such large arguments)

Problem 18.5.9

Given: An optical time-domain reflectometer (OTDR) operates at 1550 nm with a pulse width of 100 ns and peak power of 100 mW in a fiber with n = 1.468 and attenuation of 0.22 dB/km.

Find: (a) The spatial resolution (two-point), (b) the maximum range for a 30 dB dynamic range (one-way loss limit), (c) the round-trip time for a reflection at 50 km, and (d) the distance to an event that appears at t = 600 μs on the OTDR trace.

Solution:

(a) Spatial resolution: $\Delta z = c \times \tau / (2n) = 3 \times 10^8 \times 100 \times 10^{-9} / (2 \times 1.468) = 30 / 2.936 = 10.22$ m

Two events closer than 10.2 m cannot be resolved as separate features.

(b) Maximum range: L_{max} = dynamic range / $(2 \times \alpha)$ — but wait, OTDR measures round-trip, so the one-way range based on backscatter: L_{max} = dynamic range / α = 30 / 0.22 = 136.4 km (one-way fiber loss of 30 dB)

In practice, the round-trip attenuation and backscatter coefficient reduce this, but the dynamic range specification accounts for the round trip.

(c) Round-trip time at 50 km: $t = 2nL / c = 2 \times 1.468 \times 50 \times 10^3 / (3 \times 10^8) = 146{,}800 / 3 \times 10^8 = 489.3$ μs

(d) Distance for t = 600 μs: $L = c \times t / (2n) = 3 \times 10^8 \times 600 \times 10^{-6} / (2 \times 1.468) = 180{,}000 / 2.936 = 61.3$ km

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Problem 18.5.10

Given: A data center interconnect (DCI) uses 400G-ZR coherent pluggable optics. The transceiver operates at 60 GBaud, DP-16QAM, with 20% FEC overhead. The link is 80 km of standard SMF (0.2 dB/km at 1550 nm) with 4 connectors at 0.3 dB each. Transmit power is 0 dBm and receiver sensitivity is −18 dBm.

Find: (a) The raw and net data rates, (b) the total link loss, (c) the received power and system margin, and (d) the dispersion accumulation and whether it is compensated.

Solution:

(a) Data rates: Raw: 60 GBaud $\times$ 4 bits/symbol $\times$ 2 polarizations = 480 Gbps Net: 480 / 1.20 = 400 Gbps (400G ZR)

(b) Total link loss: Fiber: $0.2 \times 80 = 16.0$ dB Connectors: $4 \times 0.3 = 1.2$ dB Total = $16.0 + 1.2 = 17.2$ dB

(c) Received power: $P_{rx} = 0 - 17.2 = -17.2$ dBm Margin = $P_{rx} - P_{sens} = -17.2 - (-18) = 0.8$ dB

This is tight. Aging and temperature variations may push the link below threshold. A mid-span EDFA or higher launch power would be advisable.

(d) Dispersion: $\Delta\tau_{CD} = D \times L = 17 \times 80 = 1{,}360$ ps/nm of accumulated chromatic dispersion. At 60 GBaud, the symbol period is 16.67 ps. The accumulated dispersion is very large, but coherent receivers use DSP-based chromatic dispersion compensation (CDC) that can compensate tens of thousands of ps/nm digitally. The 1,360 ps/nm is well within the $\pm$50,000 ps/nm CDC range of modern coherent DSP ASICs, so no optical dispersion compensation is needed.

Chapter 18 — Section 18.6: Optical System Design

Practice problems covering optical power and radiometry, optical link budgets, and lens design and aberrations.

Problem 18.6.1

Given: An LED emits 80 lumens of white light with a luminous efficacy of 120 lm/W. The LED has a Lambertian emission pattern (uniform radiant intensity over a hemisphere).

Find: (a) The radiant flux (optical power), (b) the radiant intensity, (c) the irradiance at a distance of 1 m on axis, and (d) the illuminance at 1 m.

Solution:

(a) Radiant flux: Φ = luminous flux / luminous efficacy = 80 / 120 = 0.667 W

(b) For a Lambertian source, the intensity varies as $I(\theta) = I_0 \cos(\theta)$. Total flux: $\Phi = \pi \times I_0$ (integrating over hemisphere). $I_0 = \Phi/\pi = 0.667/\pi = 0.2122$ W/sr (on-axis intensity)

(c) Irradiance at 1 m on axis: $E = I_0/r^2 = 0.2122/1^2 = 0.2122$ W/m^2

(d) Illuminance at 1 m: For a Lambertian source, the luminous intensity on axis is: I_v = luminous flux / $\pi = 80/\pi = 25.46$ cd Illuminance: $E_v = I_v/r^2 = 25.46/1 = 25.46$ lux

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Problem 18.6.2

Given: A security camera lens has focal length f = 8 mm and aperture diameter D = 4 mm. The sensor has 2.0 μm pixel pitch and operates at $\lambda = 550$ nm.

Find: (a) The f-number, (b) the diffraction-limited Airy disk diameter, (c) whether the system is diffraction-limited or pixel-limited, and (d) the angular field of view if the sensor is 6.4 mm wide.

Solution:

(a) f-number: f/# = f/D = 8/4 = f/2.0

(b) Airy disk diameter: $d_{Airy} = 2.44 \times \lambda \times f/\# = 2.44 \times 550 \times 10^{-9} \times 2.0 = 2.44 \times 1.1 \times 10^{-6} = 2.68$ μm

(c) Comparison with pixel size: Airy disk diameter (2.68 μm) is larger than the pixel pitch (2.0 μm). The system is diffraction-limited — the optics, not the sensor, limit resolution. The Airy disk spans approximately 1.34 pixels, so the Nyquist condition (at least 2 pixels per resolution element) is not met. The effective resolution is limited by aliasing.

(d) Angular field of view: $\theta = 2 \times \arctan(w/(2f)) = 2 \times \arctan(6.4/(2 \times 8)) = 2 \times \arctan(0.4) = 2 \times 21.8° = 43.6°$

Problem 18.6.3

Given: A free-space optical (FSO) communication link operates at $\lambda = 1550$ nm between two buildings 500 m apart. The transmitter has a collimated beam with initial diameter $D_{tx} = 25$ mm and divergence $\theta = 0.5$ mrad. The receiver has an aperture diameter $D_{rx} = 100$ mm. Transmit power is +20 dBm and receiver sensitivity is −35 dBm.

Find: (a) The beam diameter at the receiver, (b) the geometric loss, (c) the received power, and (d) the system margin.

Solution:

(a) Beam diameter at receiver: $D_{beam} = D_{tx} + \theta \times L = 0.025 + 0.5 \times 10^{-3} \times 500 = 0.025 + 0.250 = 0.275$ m = 275 mm

(b) Geometric (spreading) loss — ratio of receiver aperture area to beam area: $L_{geo} = (D_{rx}/D_{beam})^2 = (100/275)^2 = (0.3636)^2 = 0.1322$ In dB: $L_{geo} = 10\log_{10}(0.1322) = -8.79$ dB

(c) Received power (ignoring atmospheric absorption): $P_{rx} = P_{tx} + L_{geo} = 20 + (-8.79) = +11.2$ dBm

(d) System margin: $M = P_{rx} - P_{sens} = 11.2 - (-35) = 46.2$ dB

This large margin accommodates atmospheric attenuation (fog, rain), scintillation, and beam pointing errors. Dense fog can cause 100+ dB/km attenuation, which would exceed the margin at 500 m (50 dB loss). Clear-air attenuation is typically 0.5–5 dB/km.

Problem 18.6.4

Given: A camera lens system has the following specifications: focal length f = 50 mm, f-number f/1.8, and the image of a point source has a measured spot diameter due to aberrations of $d_{aberr} = 15$ μm. Operating wavelength is $\lambda = 550$ nm.

Find: (a) The aperture diameter, (b) the diffraction-limited spot diameter (Airy disk), (c) the ratio of aberration blur to diffraction limit, and (d) the effective combined spot diameter (root-sum-square approximation).

Solution:

(a) Aperture diameter: $D = f/(f/\#) = 50/1.8 = 27.8$ mm

(b) Diffraction-limited Airy disk diameter: $d_{Airy} = 2.44 \times \lambda \times f/\# = 2.44 \times 550 \times 10^{-9} \times 1.8 = 2.44 \times 9.9 \times 10^{-7} = 2.42$ μm

(c) Ratio: $d_{aberr}/d_{Airy} = 15/2.42 = 6.2\times$

The system is aberration-limited, with the actual spot 6.2 times the diffraction limit.

(d) Combined spot diameter (RSS): $d_{total} = \sqrt{(d_{aberr}{}^2 + d_{Airy}{}^2)} = \sqrt{(15^2 + 2.42^2)} = \sqrt{(225 + 5.86)} = \sqrt{230.86} = 15.2$ μm

The aberration dominates; the diffraction contribution is negligible at f/1.8. Stopping down to f/5.6 would increase d_{Airy} to 7.5 μm while reducing spherical aberration by roughly $(1.8/5.6)^3 \approx 3.3\times$, potentially reducing d_{aberr} to ~4.5 μm and achieving a better-balanced design.

Problem 18.6.5

Given: An optical fiber link at 1310 nm uses the following components: transmitter power = +1 dBm, single-mode fiber with $\alpha = 0.35$ dB/km over 25 km, 2 connector pairs at 0.5 dB each, 1 fusion splice at 0.1 dB, and a 3 dB coupler (for monitoring tap). The receiver sensitivity is −30 dBm.

Find: (a) The total link loss, (b) the received power, (c) the system margin, and (d) the maximum additional fiber length that could be added while maintaining 3 dB margin.

Solution:

(a) Total link loss: Fiber: $0.35 \times 25 = 8.75$ dB Connectors: $2 \times 0.5 = 1.0$ dB Splice: $1 \times 0.1 = 0.1$ dB Coupler: 3.0 dB Total = $8.75 + 1.0 + 0.1 + 3.0 = 12.85$ dB

(b) Received power: $P_{rx} = +1 - 12.85 = -11.85$ dBm

(c) System margin: $M = P_{rx} - P_{sens} = -11.85 - (-30) = 18.15$ dB

(d) Additional fiber with 3 dB margin: Available for additional fiber = $18.15 - 3.0 = 15.15$ dB $L_{additional} = 15.15 / 0.35 = 43.3$ km (total link would be 68.3 km)

Problem 18.6.6

Given: A projector uses a 5000-lumen lamp. The projection lens has f/2.5, and the screen is 3 m wide by 2 m tall at a distance of 5 m.

Find: (a) The screen area, (b) the screen illuminance (assuming 60% optical efficiency and uniform distribution), (c) the screen luminance for a matte white screen (gain = 1.0), and (d) the required lamp power if the luminous efficacy is 90 lm/W.

Solution:

(a) Screen area: $A = 3 \times 2 = 6$ m²

(b) Screen illuminance: Effective lumens on screen: $\Phi_{screen} = 5{,}000 \times 0.60 = 3{,}000$ lm $E_v = \Phi_{screen}$ / A = 3,000 / 6 = 500 lux

(c) Screen luminance: For a Lambertian (matte) screen with gain 1.0: $L = E_v / \pi = 500 / \pi = 159.2$ cd/m²

This is adequate for a moderately lit room (recommended > 100 cd/m² for projection displays).

(d) Required lamp power: P = total lumens / efficacy = 5,000 / 90 = 55.6 W

Problem 18.6.7

Given: An achromatic doublet lens is designed for f = 200 mm by combining a BK7 crown glass element ($n_d = 1.5168$, Abbe number $V_1 = 64.17$) and an SF2 flint glass element ($n_d = 1.6477$, Abbe number $V_2 = 33.85$). The doublet must have zero chromatic aberration.

Find: (a) The individual focal lengths f_1 and f_2 of the crown and flint elements (using the achromatic condition: $\varphi_1/V_1 + \varphi_2/V_2 = 0$, where $\varphi = 1/f$), and (b) the optical powers.

Solution:

(a) For an achromatic doublet, two conditions must be met:

- Total power: $\varphi = \varphi_1 + \varphi_2 = 1/200 = 0.005$ mm^{-1}
- Achromatic condition: $\varphi_1/V_1 + \varphi_2/V_2 = 0$

From the achromatic condition: $\varphi_1/64.17 + \varphi_2/33.85 = 0$ $\varphi_1 = -\varphi_2 \times (64.17/33.85) = -1.896 \times \varphi_2$

Substituting into total power: $-1.896\varphi_2 + \varphi_2 = 0.005$ $-0.896\varphi_2 = 0.005$ $\varphi_2 = -0.005578$ mm^{-1} $f_2 = 1/\varphi_2 = -179.3$ mm (diverging flint element)

$\varphi_1 = 0.005 - (-0.005578) = 0.010578$ mm^{-1} $f_1 = 1/\varphi_1 = 94.5$ mm (converging crown element)

(b) Optical powers: Crown: $\varphi_1 = +10.578$ diopters (converging) Flint: $\varphi_2 = -5.578$ diopters (diverging)

The converging crown element is much stronger than the combined doublet, and the flint element partially cancels the power while correcting the chromatic aberration.

Problem 18.6.8

Given: A solar concentrator uses a parabolic mirror with diameter D = 2 m and focal length f = 1 m. The solar irradiance is 1000 W/m² and the mirror reflectivity is 92%. The concentrated sunlight is directed onto a circular receiver of diameter d = 20 mm.

Find: (a) The collection area, (b) the power collected by the mirror, (c) the concentration ratio, and (d) the irradiance at the receiver.

Solution:

(a) Collection area: $A_{mirror} = \pi(D/2)^2 = \pi(1)^2 = 3.142$ m²

(b) Collected power: $P = E_{solar} \times A_{mirror} \times \rho = 1000 \times 3.142 \times 0.92 = 2{,}890$ W

(c) Geometric concentration ratio: $C = A_{mirror} / A_{receiver} = \pi(1)^2 / \pi(0.01)^2 = 1/0.0001 = 10{,}000\times$

(d) Irradiance at receiver: $E_{receiver} = P / A_{receiver} = 2{,}890 / (\pi \times (0.01)^2) = 2{,}890 / 3.142 \times 10^{-4} = 9.2 \times 10^6$ W/m² = 9.2 MW/m²

This is 9,200 times the solar irradiance — sufficient to melt most metals. Such concentrators are used in solar furnaces and concentrated solar power (CSP) systems.

Problem 18.6.9

Given: A machine vision system uses a telecentric lens with magnification $m = -0.5$, working distance = 200 mm, and sensor pixel size = 5 μm. The lens has a modulation transfer function (MTF) of 50% at 100 lp/mm at the sensor plane.

Find: (a) The object-space pixel resolution, (b) the field of view if the sensor is 12.8 mm × 9.6 mm, (c) the Nyquist frequency of the sensor, and (d) whether the MTF supports the Nyquist resolution.

Solution:

(a) Object-space pixel resolution: Δx_{object} = pixel size / $|m|$ = 5 μm / 0.5 = 10 μm per pixel

(b) Field of view: FOV_x = sensor width / $|m|$ = 12.8 / 0.5 = 25.6 mm FOV_y = 9.6 / 0.5 = 19.2 mm

(c) Nyquist frequency at the sensor: $f_{Nyquist}$ = 1 / (2 × pixel size) = $1 / (2 \times 5 \times 10^{-3}$ mm) = 100 lp/mm

(d) The MTF at the Nyquist frequency is 50%. This means the lens can resolve features at the Nyquist limit but with 50% contrast reduction. A common guideline requires MTF > 30% at Nyquist for acceptable imaging, so this lens meets the requirement with margin. Features at half the Nyquist frequency (50 lp/mm) would have even higher contrast.

Problem 18.6.10

Given: A fiber-optic endoscope uses a coherent fiber bundle with 30,000 individual fibers, each with core diameter of 3 μm and center-to-center spacing of 4 μm, arranged in a hexagonal pattern. The distal lens has f = 2 mm and f/3.

Find: (a) The bundle diameter (assuming circular packing), (b) the effective pixel count, (c) the angular resolution limit from the fiber spacing, and (d) the diffraction-limited resolution of the distal lens at $\lambda = 550$ nm.

Solution:

(a) Bundle diameter: For hexagonal packing, the area per fiber is $A_{hex} = (\sqrt{3}/2) \times s^2$ where s = 4 μm spacing. $A_{hex} = (\sqrt{3}/2) \times (4 \times 10^{-3})^2 = 0.866 \times 16 \times 10^{-6} = 13.86 \times 10^{-6}$ mm² Total bundle area = $30{,}000 \times 13.86 \times 10^{-6} = 0.4157$ mm² Bundle diameter = $\sqrt{(4A/\pi)} = \sqrt{(4 \times 0.4157 / \pi)} = \sqrt{(0.5295)} = 0.728$ mm

(b) Each fiber acts as one pixel: Effective resolution = 30,000 pixels

Equivalent image format: $\sqrt{30{,}000} \approx 173$, so roughly a 200×150 pixel image.

(c) Angular resolution from fiber spacing: $\theta_{fiber} = s / f = 4 \times 10^{-3} / 2 = 2 \times 10^{-3}$ rad = 2 mrad

(d) Diffraction limit of distal lens: $D = f/(f/\#) = 2/3 = 0.667$ mm $\theta_{diff} = 1.22\lambda/D = 1.22 \times 550 \times 10^{-9} / 0.667 \times 10^{-3} = 6.71 \times 10^{-7} / 6.67 \times 10^{-4} = 1.006 \times 10^{-3}$ rad = 1.0 mrad

The fiber spacing (2 mrad) is the limiting factor, not diffraction (1 mrad). The system is sampling-limited by the fiber bundle, which is typical for fiber endoscopes.

Chapter 19 — Section 19.1: Time Value of Money

Practice problems covering simple interest, compound interest, nominal and effective interest rates, and continuous compounding. Problems range from straightforward applications to multi-step PE-exam-style questions.

Problem 19.1.1

Given: A municipality borrows \$750,000 for a streetlight LED conversion project at 5% simple annual interest for 4 years.

Find: (a) The total interest paid, and (b) the total amount owed at maturity.

Solution:

(a) Total interest: $I = P \times i \times n = 750{,}000 \times 0.05 \times 4 = \$150{,}000$

(b) Total amount owed: $F = P + I = 750{,}000 + 150{,}000 = \$900{,}000$

Problem 19.1.2

Given: An engineering firm deposits \$250,000 into a project escrow account earning 4% annual interest compounded annually for 6 years.

Find: (a) The account balance after 6 years, and (b) the total interest earned.

Solution:

(a) $F = P(1 + i)^n = 250{,}000 \times (1.04)^6 = 250{,}000 \times 1.2653 = \$316{,}325$

(b) Total interest: $I = F - P = 316{,}325 - 250{,}000 = \$66{,}325$

Problem 19.1.3

Given: A contractor invests \$180,000 at 6% annual interest compounded monthly for 5 years.

Find: (a) The future value, and (b) the difference between compound and simple interest over the same period.

Solution:

(a) Monthly rate: $i = 0.06/12 = 0.005$. Periods: $n = 5 \times 12 = 60$. $F_{compound} = 180{,}000 \times (1.005)^{60} = 180{,}000 \times 1.3489 = \$242{,}802$

(b) Simple interest future value: $F_{simple} = P(1 + i \times n) = 180{,}000 \times (1 + 0.06 \times 5) = 180{,}000 \times 1.30 = \$234{,}000$

Difference $= 242{,}802 - 234{,}000 = \$8{,}802$

Compound interest earns \$8,802 more than simple interest over 5 years due to interest-on-interest.

Problem 19.1.4

Given: A power plant financing option quotes 9% annual interest compounded quarterly.

Find: (a) The effective annual interest rate, (b) the effective quarterly rate, and (c) the effective monthly rate.

Solution:

(a) $i_{eff,annual} = (1 + r/m)^m - 1 = (1 + 0.09/4)^4 - 1 = (1.0225)^4 - 1 = 1.09308 - 1 = 9.31\%$

(b) The effective quarterly rate is the periodic rate itself: $i_{quarterly} = 0.09/4 = 2.25\%$

(c) Effective monthly rate from the quarterly rate: $i_{monthly} = (1 + 0.0225)^{1/3} - 1 = (1.0225)^{0.3333} - 1 = 1.00744 - 1 = 0.744\%$

Problem 19.1.5

Given: Two equipment lease options are offered. Option A quotes 8.5% compounded monthly. Option B quotes 8.7% compounded semiannually.

Find: Which option has the lower effective annual rate?

Solution:

Option A: $i_{eff} = (1 + 0.085/12)^{12} - 1 = (1.007083)^{12} - 1 = 1.08839 - 1 = 8.84\%$

Option B: $i_{eff} = (1 + 0.087/2)^2 - 1 = (1.0435)^2 - 1 = 1.08889 - 1 = 8.89\%$

Option A has the lower effective annual rate (8.84% vs. 8.89%), so it is the less expensive financing option despite having a lower nominal rate.

Problem 19.1.6

Given: A utility invests \$5,000,000 in a decommissioning fund at 6% continuously compounded interest for 20 years.

Find: (a) The fund value after 20 years, (b) the effective annual interest rate, and (c) the total interest earned.

Solution:

(a) $F = Pe^{rn} = 5{,}000{,}000 \times e^{0.06 \times 20} = 5{,}000{,}000 \times e^{1.20} = 5{,}000{,}000 \times 3.3201 = \$16{,}600{,}584$

(b) $i_{eff} = e^{0.06} - 1 = 1.06184 - 1 = 6.18\%$

(c) Total interest: $I = 16{,}600{,}584 - 5{,}000{,}000 = \$11{,}600{,}584$

Problem 19.1.7

Given: A wind farm developer needs to compare three investment options for a \$1,000,000 reserve fund over 10 years: - Option X: 7% compounded annually - Option Y: 6.8% compounded daily (365 days) - Option Z: 6.7% compounded continuously

Find: The future value under each option and which yields the highest return.

Solution:

Option X: $F = 1{,}000{,}000 \times (1.07)^{10} = 1{,}000{,}000 \times 1.9672 = \$1{,}967{,}151$

Option Y: $i_{period} = 0.068/365 = 0.000186301$; $n = 365 \times 10 = 3{,}650$ $F = 1{,}000{,}000 \times (1.000186301)^{3650} = 1{,}000{,}000 \times e^{0.068 \times 10} \approx 1{,}000{,}000 \times (1 + 0.068/365)^{3650}$

Effective annual rate for Y: $i_{eff} = (1 + 0.068/365)^{365} - 1 = 1.07036 - 1 = 7.036\%$ $F = 1{,}000{,}000 \times (1.07036)^{10} = 1{,}000{,}000 \times 1.9738 = \$1{,}973{,}812$

Option Z: $F = 1{,}000{,}000 \times e^{0.067 \times 10} = 1{,}000{,}000 \times e^{0.67} = 1{,}000{,}000 \times 1.9542 = \$1{,}954{,}237$

Option Y yields the highest return (\$1,973,812) because its effective annual rate of 7.04% exceeds the other options.

Problem 19.1.8

Given: A \$400,000 substation monitoring system is financed at 10% compounded continuously. The loan must be repaid as a single lump sum after 3 years.

Find: (a) The amount owed at repayment, (b) the total interest, and (c) how much less would be owed if the same nominal rate were compounded annually instead.

Solution:

(a) Continuous compounding: $F_{cont} = 400{,}000 \times e^{0.10 \times 3} = 400{,}000 \times e^{0.30} = 400{,}000 \times 1.3499 = \$539{,}944$

(b) Total interest: $I = 539{,}944 - 400{,}000 = \$139{,}944$

(c) Annual compounding: $F_{annual} = 400{,}000 \times (1.10)^3 = 400{,}000 \times 1.3310 = \$532{,}400$ Difference = $539{,}944 - 532{,}400 = \$7{,}544$

The continuous compounding costs \$7,544 more in interest than annual compounding at the same nominal rate.

Chapter 19 — Section 19.2: Cash Flow Diagrams and Economic Equivalence

Practice problems covering cash flow diagram conventions and economic equivalence calculations. Problems range from straightforward diagram interpretation to multi-step equivalence comparisons.

Problem 19.2.1

Given: A solar inverter installation costs \$220,000 at time 0. It produces energy savings of \$42,000/year for years 1 through 8, requires a \$15,000 capacitor replacement at year 4, and has a salvage value of \$18,000 at the end of year 8.

Find: (a) The net cash flow in years 1–3, year 4, years 5–7, and year 8, and (b) the total undiscounted net cash flow over the project.

Solution:

(a) Cash flow diagram:

- Time 0: ↓ \$220,000 (initial cost)
- Years 1–3: ↑ \$42,000/year (annual savings)
- Year 4: ↑ \$42,000 − ↓ \$15,000 = ↑ \$27,000 net
- Years 5–7: ↑ \$42,000/year (annual savings)
- Year 8: ↑ \$42,000 + ↑ \$18,000 = ↑ \$60,000 net

(b) Total undiscounted net cash flow: $= -220{,}000 + 7 \times 42{,}000 + 27{,}000 + 18{,}000 = -220{,}000 + 294{,}000 + 27{,}000 + 18{,}000 = -220{,}000 + 339{,}000 = \$119{,}000$

Note: This positive total does not guarantee profitability — a time-value-of-money analysis is required.

Problem 19.2.2

Given: A utility can pay \$1,200,000 now or make 5 equal annual payments for a substation transformer bank. The interest rate is 7%.

Find: (a) The equivalent annual payment that makes the installment plan equal to the lump sum, and (b) the total paid under the installment plan.

Solution:

(a) The annual payment A that is equivalent to the present lump sum: $A = P \times (A/P, 7\%, 5) = 1{,}200{,}000 \times [0.07 \times (1.07)^5] / [(1.07)^5 - 1] = 1{,}200{,}000 \times [0.07 \times 1.4026] / [1.4026 - 1] = 1{,}200{,}000 \times 0.09818 / 0.4026 = 1{,}200{,}000 \times 0.24389 = \$292{,}668$/year

(b) Total paid: $292{,}668 \times 5 = \$1{,}463{,}340$

The installment plan costs $263,340 more in total, but the payments are spread over 5 years.

Problem 19.2.3

Given: An industrial facility can lease a power quality analyzer for $8,500/year for 6 years or purchase it outright for $38,000 with a $5,000 resale value at the end of 6 years. The facility's MARR is 10%.

Find: (a) The present worth of the lease option, (b) the present worth of the purchase option, and (c) which option is more economical.

Solution:

(a) $PW_{lease} = 8{,}500 \times (P/A, 10\%, 6)$ $(P/A, 10\%, 6) = [(1.10)^6 - 1] / [0.10 \times (1.10)^6] = [1.7716 - 1] / [0.10 \times 1.7716] = 0.7716 / 0.17716 = 4.3553$ $PW_{lease} = 8{,}500 \times 4.3553 = \$37{,}020$

(b) $PW_{purchase} = 38{,}000 - 5{,}000 \times (P/F, 10\%, 6)$ $(P/F, 10\%, 6) = 1/(1.10)^6 = 1/1.7716 = 0.5645$ $PW_{purchase} = 38{,}000 - 5{,}000 \times 0.5645 = 38{,}000 - 2{,}823 = \$35{,}177$

(c) The purchase option is more economical by $37,020 − $35,177 = $1,843 in present worth, considering the resale value offsets the higher upfront cost.

Problem 19.2.4

Given: A data center operator receives two proposals for an emergency generator system. Proposal 1 requires $500,000 now and $50,000 at year 3 for a major service. Proposal 2 requires $200,000 now, $200,000 at year 2, and $200,000 at year 4. Both proposals deliver identical service over the same period. The interest rate is 9%.

Find: (a) The present worth of each proposal, and (b) the more economical choice.

Solution:

(a) PW of Proposal 1: $PW_1 = 500{,}000 + 50{,}000 \times (P/F, 9\%, 3)$ $(P/F, 9\%, 3) = 1/(1.09)^3 = 1/1.2950 = 0.7722$ $PW_1 = 500{,}000 + 50{,}000 \times 0.7722 = 500{,}000 + 38{,}610 = \$538{,}610$

PW of Proposal 2: $PW_2 = 200{,}000 + 200{,}000 \times (P/F, 9\%, 2) + 200{,}000 \times (P/F, 9\%, 4)$ $(P/F, 9\%, 2) = 1/(1.09)^2 = 1/1.1881 = 0.8417$ $(P/F, 9\%, 4) = 1/(1.09)^4 = 1/1.4116 = 0.7084$ $PW_2 = 200{,}000 + 200{,}000 \times 0.8417 + 200{,}000 \times 0.7084 = 200{,}000 + 168{,}340 + 141{,}680 = \$510{,}020$

(b) Proposal 2 is more economical by $538,610 − $510,020 = $28,590 in present worth. Despite requiring the same total undiscounted amount ($600,000 each), the deferred payments in Proposal 2 are worth less in present-value terms.

Chapter 19 — Section 19.3: Single Payment and Uniform Series Factors

Practice problems covering F/P, P/F, P/A, A/P, A/F, F/A factors, arithmetic gradient series, and geometric gradient series. Problems range from direct factor application to multi-step PE-exam-style questions.

Problem 19.3.1

Given: A utility sets aside \$2,000,000 today for future transmission line right-of-way acquisition in 10 years. The fund earns 5% annually.

Find: (a) The future value of the investment, and (b) the total interest earned.

Solution:

(a) $F = P \times (F/P, 5\%, 10) = 2{,}000{,}000 \times (1.05)^{10} = 2{,}000{,}000 \times 1.6289 = \$3{,}257{,}789$

(b) Interest earned: $I = F - P = 3{,}257{,}789 - 2{,}000{,}000 = \$1{,}257{,}789$

Problem 19.3.2

Given: A cable replacement project will cost \$4,200,000 in 12 years. The utility's investment fund earns 7% annually.

Find: The amount that must be invested today to cover the future cost.

Solution:

$P = F \times (P/F, 7\%, 12) = 4{,}200{,}000 \times 1/(1.07)^{12} = 4{,}200{,}000 \times 1/2.2522 = 4{,}200{,}000 \times 0.4440 = \$1{,}864{,}791$

The utility must invest approximately \$1.86 million today to have \$4.2 million available in 12 years.

Problem 19.3.3

Given: A manufacturing company purchases a $650,000 automated motor winding machine financed over 8 years at 6.5% annual interest.

Find: (a) The uniform annual payment, (b) the total amount paid, and (c) the total interest paid.

Solution:

(a) $A = P \times (A/P, 6.5\%, 8) = 650{,}000 \times [0.065 \times (1.065)^8] / [(1.065)^8 - 1]$ $(1.065)^8 = 1.6550 = 650{,}000 \times [0.065 \times 1.6550] / [1.6550 - 1] = 650{,}000 \times 0.10758 / 0.6550 = 650{,}000 \times 0.16425 = \$106{,}763$/year

(b) Total paid: $106{,}763 \times 8 = \$854{,}104$

(c) Total interest: $854{,}104 - 650{,}000 = \$204{,}104$

Problem 19.3.4

Given: A power company offers industrial customers a rebate program that pays $85,000/year for 12 years for installing high-efficiency equipment. The company's discount rate is 9%.

Find: The present worth of the entire rebate stream.

Solution:

$P = A \times (P/A, 9\%, 12) = 85{,}000 \times [(1.09)^{12} - 1] / [0.09 \times (1.09)^{12}]$ $(1.09)^{12} = 2.8127 = 85{,}000 \times [2.8127 - 1] / [0.09 \times 2.8127] = 85{,}000 \times 1.8127 / 0.25314 = 85{,}000 \times 7.1607 = \$608{,}660$

Problem 19.3.5

Given: A utility must accumulate $8,000,000 in 15 years for a substation rebuild. The sinking fund earns 4.5% annually.

Find: (a) The required uniform annual deposit, and (b) the total amount deposited versus the total interest earned.

Solution:

(a) $A = F \times (A/F, 4.5\%, 15) = 8{,}000{,}000 \times 0.045 / [(1.045)^{15} - 1]$ $(1.045)^{15} = 1.9353 = 8{,}000{,}000 \times 0.045 / [1.9353 - 1] = 8{,}000{,}000 \times 0.045 / 0.9353 = 8{,}000{,}000 \times 0.04811 = \$384{,}904$/year

(b) Total deposited: $384{,}904 \times 15 = \$5{,}773{,}560$ Interest earned: $8{,}000{,}000 - 5{,}773{,}560 = \$2{,}226{,}440$

Problem 19.3.6

Given: An electrical contractor deposits $25,000/year into a business expansion fund earning 8% annually for 10 years.

Find: The future value of the fund at the end of 10 years.

Solution:

$F = A \times (F/A, 8\%, 10) = 25{,}000 \times [(1.08)^{10} - 1] / 0.08$ $(1.08)^{10} = 2.1589 = 25{,}000 \times [2.1589 - 1] / 0.08 = 25{,}000 \times 1.1589 / 0.08 = 25{,}000 \times 14.487 = \$362{,}168$

Problem 19.3.7

Given: Annual maintenance costs for a 138 kV circuit breaker start at $4,000 in year 1 and increase by $600/year (arithmetic gradient) for 15 years. The interest rate is 7%.

Find: (a) The maintenance cost in year 15, (b) the present worth of all maintenance costs, and (c) the equivalent uniform annual maintenance cost.

Solution:

(a) Cost in year 15: $A_{15} = 4{,}000 + (15 - 1) \times 600 = 4{,}000 + 8{,}400 = \$12{,}400$

(b) $P = A_1 \times (P/A, 7\%, 15) + G \times (P/G, 7\%, 15)$

$(P/A, 7\%, 15) = [(1.07)^{15} - 1] / [0.07 \times (1.07)^{15}]$ $(1.07)^{15} = 2.7590 = [2.7590 - 1] / [0.07 \times 2.7590] = 1.7590 / 0.19313 = 9.1079$

$(P/G, 7\%, 15) = [(1.07)^{15} - 0.07 \times 15 - 1] / [0.07^2 \times (1.07)^{15}] = [2.7590 - 1.05 - 1] / [0.0049 \times 2.7590] = 0.7090 / 0.013519 = 52.446$

$P = 4{,}000 \times 9.1079 + 600 \times 52.446 = 36{,}432 + 31{,}468 = \$67{,}900$

(c) Equivalent uniform annual cost: $A_{eq} = P \times (A/P, 7\%, 15) = 67{,}900 \times [0.07 \times 2.7590] / [2.7590 - 1] = 67{,}900 \times 0.19313 / 1.7590 = 67{,}900 \times 0.10979 = \$7{,}455/\text{year}$

Problem 19.3.8

Given: Operating costs for a SCADA system start at $30,000 in year 1 and increase by $2,500/year for 8 years. At i = 10%, a vendor offers to replace the system with one that has a flat annual operating cost.

Find: The maximum flat annual cost the utility should accept for the replacement to be equivalent.

Solution:

The equivalent uniform annual cost of the existing system: $A_{eq} = A_1 + G \times (A/G, 10\%, 8)$

$(A/G, 10\%, 8) = [(1.10)^8 - 0.10 \times 8 - 1] / [0.10 \times ((1.10)^8 - 1)]$ $(1.10)^8 = 2.1436 = [2.1436 - 0.80 - 1] / [0.10 \times 1.1436] = 0.3436 / 0.11436 = 3.0045$

$A_{eq} = 30{,}000 + 2{,}500 \times 3.0045 = 30{,}000 + 7{,}511 = \$37{,}511/\text{year}$

The utility should accept any flat annual cost up to $37,511/year for the replacement system.

Problem 19.3.9

Given: Energy costs for a pumping station are \$120,000 in year 1 and escalate at 5%/year (geometric gradient). The discount rate is 10% over a 20-year planning horizon.

Find: The present worth of all energy costs.

Solution:

Since $g \neq i$, use the geometric gradient formula: $P = A_1 \times [1 - (1+g)^n(1+i)^{-n}] / (i - g) = 120{,}000 \times [1 - (1.05)^{20}(1.10)^{-20}] / (0.10 - 0.05)$

$(1.05)^{20} = 2.6533$; $(1.10)^{20} = 6.7275$

$= 120{,}000 \times [1 - 2.6533/6.7275] / 0.05 = 120{,}000 \times [1 - 0.3943] / 0.05 = 120{,}000 \times 0.6057 / 0.05 = 120{,}000 \times 12.114 = \$1{,}453{,}680$

Problem 19.3.10

Given: A consulting firm's revenue is \$500,000 in year 1 and grows at 8%/year. The firm's cost of capital is also 8%. The planning period is 10 years.

Find: The present worth of all revenue over 10 years.

Solution:

Since $g = i = 8\%$, use the special case formula: $P = A_1 \times n / (1 + i) = 500{,}000 \times 10 / 1.08 = 5{,}000{,}000 / 1.08 = \$4{,}629{,}630$

When the escalation rate equals the discount rate, the present worth simplifies to n payments discounted by one period. This is significantly higher than the non-escalating case, which would be $P = 500{,}000 \times (P/A, 8\%, 10) = 500{,}000 \times 6.7101 = \$3{,}355{,}050$.

Chapter 19 — Section 19.4: Present Worth Analysis

Practice problems covering net present value, comparing alternatives with equal lives, and comparing alternatives with unequal lives using the LCM method. Problems range from single-project NPV to multi-alternative comparisons.

Problem 19.4.1

Given: A manufacturing plant evaluates a $1,800,000 power factor correction system. It reduces demand charges by $320,000/year over 8 years. The equipment has a salvage value of $150,000 at the end of year 8. The MARR is 12%.

Find: (a) The NPV, and (b) whether the project is justified.

Solution:

(a) NPV = −1,800,000 + 320,000 × (P/A, 12%, 8) + 150,000 × (P/F, 12%, 8)

(P/A, 12%, 8) = $[(1.12)^8 - 1] / [0.12 \times (1.12)^8]$ $(1.12)^8 = 2.4760$ = [2.4760 − 1] / [0.12 × 2.4760] = 1.4760 / 0.29712 = 4.9676

(P/F, 12%, 8) = 1/2.4760 = 0.4039

NPV = −1,800,000 + 320,000 × 4.9676 + 150,000 × 0.4039 = −1,800,000 + 1,589,632 + 60,585 = −$149,783

(b) Since NPV = −$149,783 < 0, the project is not justified at a 12% MARR.

Problem 19.4.2

Given: A hospital evaluates a $500,000 backup generator that will save $95,000/year in avoided outage costs over 10 years with no salvage value. The MARR is 8%.

Find: (a) The NPV, (b) whether the project is justified, and (c) the minimum annual savings needed to justify the project.

Solution:

(a) (P/A, 8%, 10) = [$(1.08)^{10}$ − 1] / [0.08 × $(1.08)^{10}$] $(1.08)^{10}$ = 2.1589 = [2.1589 − 1] / [0.08 × 2.1589] = 1.1589 / 0.17271 = 6.7101

NPV = −500,000 + 95,000 × 6.7101 = −500,000 + 637,460 = $137,460

(b) Since NPV = $137,460 > 0, the project is justified.

(c) Minimum annual savings: A_{min} = P / (P/A, 8%, 10) = 500,000 / 6.7101 = $74,516/year

Any annual savings above $74,516 makes the project economically justified at the 8% MARR.

Problem 19.4.3

Given: Compare two transformer cooling options over 15 years at i = 9%. Option A (forced-air): $180,000 first cost, $22,000/year operating cost, $15,000 salvage. Option B (forced-oil): $280,000 first cost, $12,000/year operating cost, $30,000 salvage.

Find: Which option is more economical.

Solution:

(P/A, 9%, 15) = [$(1.09)^{15}$ − 1] / [0.09 × $(1.09)^{15}$] $(1.09)^{15}$ = 3.6425 = [3.6425 − 1] / [0.09 × 3.6425] = 2.6425 / 0.32783 = 8.0607

(P/F, 9%, 15) = 1/3.6425 = 0.2745

PW_A = −180,000 − 22,000 × 8.0607 + 15,000 × 0.2745 = −180,000 − 177,335 + 4,118 = −$353,217

PW_B = −280,000 − 12,000 × 8.0607 + 30,000 × 0.2745 = −280,000 − 96,728 + 8,235 = −$368,493

Option A (forced-air) is more economical by $368,493 − $353,217 = $15,276 in present worth. The lower first cost and smaller operating cost penalty outweigh the smaller salvage value.

Problem 19.4.4

Given: A utility compares three protective relay schemes for a substation over 20 years at MARR = 7%. All have equal service lives.

Parameter	Electromechanical	Solid-State	Digital
First cost	$45,000	$65,000	$90,000
Annual maintenance	$6,000	$3,500	$1,500
Annual savings (faster clearing)	$0 (base)	$8,000	$14,000
Salvage value	$2,000	$5,000	$10,000

Find: The most economical relay scheme using NPV analysis.

Solution:

(P/A, 7%, 20) = $[(1.07)^{20} - 1] / [0.07 \times (1.07)^{20}]$ $(1.07)^{20} = 3.8697$ = [3.8697 − 1] / [0.07 × 3.8697] = 2.8697 / 0.27088 = 10.594

(P/F, 7%, 20) = 1/3.8697 = 0.2584

NPV_{EM} = −45,000 − 6,000 × 10.594 + 2,000 × 0.2584 = −45,000 − 63,564 + 517 = −$108,047

NPV_{SS} = −65,000 + (8,000 − 3,500) × 10.594 + 5,000 × 0.2584 = −65,000 + 4,500 × 10.594 + 1,292 = −65,000 + 47,673 + 1,292 = −$16,035

$NPV_{Digital}$ = −90,000 + (14,000 − 1,500) × 10.594 + 10,000 × 0.2584 = −90,000 + 12,500 × 10.594 + 2,584 = −90,000 + 132,425 + 2,584 = $45,009

The digital relay scheme is most economical with the only positive NPV of $45,009. The solid-state option has a negative NPV and the electromechanical option is the worst performer.

Problem 19.4.5

Given: Compare a 10-year lighting system with a 15-year lighting system for a warehouse at i = 8%. - System A (LED): $60,000 first cost, 10-year life, $5,000/year energy cost, $4,000 salvage. - System B (advanced LED): $85,000 first cost, 15-year life, $3,500/year energy cost, $6,000 salvage.

Find: The more economical option using the LCM method.

Solution:

LCM of 10 and 15 = 30 years. System A repeats 3 times (years 0, 10, 20); System B repeats 2 times (years 0, 15).

(P/A, 8%, 30) = $[(1.08)^{30} - 1] / [0.08 \times (1.08)^{30}]$ $(1.08)^{30} = 10.063$ = [10.063 − 1] / [0.08 × 10.063] = 9.063 / 0.8050 = 11.258

(P/F, 8%, 10) = $1/(1.08)^{10}$ = 1/2.1589 = 0.4632 (P/F, 8%, 15) = $1/(1.08)^{15}$ = 1/3.1722 = 0.3152 (P/F, 8%, 20) = $1/(1.08)^{20}$ = 1/4.6610 = 0.2145 (P/F, 8%, 30) = 1/10.063 = 0.09938

PW_A = −60,000 − (60,000 − 4,000) × 0.4632 − (60,000 − 4,000) × 0.2145 + 4,000 × 0.09938 − 5,000 × 11.258 = −60,000 − 56,000 × 0.4632 − 56,000 × 0.2145 + 398 − 56,290 = −60,000 − 25,939 − 12,012 + 398 − 56,290 = −$153,843

PW_B = −85,000 − (85,000 − 6,000) × 0.3152 + 6,000 × 0.09938 − 3,500 × 11.258 = −85,000 − 79,000 × 0.3152 + 596 − 39,403 = −85,000 − 24,901 + 596 − 39,403 = −$148,708

System B (advanced LED) is more economical by $153,843 − $148,708 = $5,135 over the 30-year study period.

Problem 19.4.6

Given: A chemical plant compares two pump options at MARR = 10%. - Pump X: $35,000 first cost, 6-year life, $8,000/year O&M, $3,000 salvage. - Pump Y: $50,000 first cost, 9-year life, $5,500/year O&M, $5,000 salvage.

Find: The more economical option using the LCM method.

Solution:

LCM of 6 and 9 = 18 years. Pump X repeats 3 times (years 0, 6, 12); Pump Y repeats 2 times (years 0, 9).

(P/A, 10%, 18) = $[(1.10)^{18} - 1] / [0.10 \times (1.10)^{18}]$ $(1.10)^{18} = 5.5599$ = $[5.5599 - 1] / [0.10 \times 5.5599]$ = $4.5599 / 0.55599 = 8.2014$

(P/F, 10%, 6) = $1/(1.10)^6 = 1/1.7716 = 0.5645$ (P/F, 10%, 9) = $1/(1.10)^9 = 1/2.3579 = 0.4241$ (P/F, 10%, 12) = $1/(1.10)^{12} = 1/3.1384 = 0.3186$ (P/F, 10%, 18) = $1/5.5599 = 0.1799$

$PW_X = -35{,}000 - (35{,}000 - 3{,}000) \times 0.5645 - (35{,}000 - 3{,}000) \times 0.3186 + 3{,}000 \times 0.1799 - 8{,}000 \times 8.2014 = -35{,}000 - 32{,}000 \times 0.5645 - 32{,}000 \times 0.3186 + 540 - 65{,}611 = -35{,}000 - 18{,}064 - 10{,}195 + 540 - 65{,}611 = -\$128{,}330$

$PW_Y = -50{,}000 - (50{,}000 - 5{,}000) \times 0.4241 + 5{,}000 \times 0.1799 - 5{,}500 \times 8.2014 = -50{,}000 - 45{,}000 \times 0.4241 + 900 - 45{,}108 = -50{,}000 - 19{,}085 + 900 - 45{,}108 = -\$113{,}293$

Pump Y is more economical by \$128,330 − \$113,293 = \$15,037 over the 18-year period. Its longer life and lower O&M costs justify the higher first cost.

Chapter 19 — Section 19.5: Annual Worth Analysis

Practice problems covering capital recovery cost, total annual worth, and comparing alternatives by annual worth. Problems range from single-asset annual cost to multi-alternative comparisons with different service lives.

Problem 19.5.1

Given: A 1,000 kVA pad-mounted transformer costs \$125,000, has a 25-year life, annual maintenance of \$3,500, and a salvage value of \$8,000. The interest rate is 7%.

Find: (a) The capital recovery cost, and (b) the total annual cost of the transformer.

Solution:

(a) $CR = (P - S) \times (A/P, 7\%, 25) + S \times i$

$(A/P, 7\%, 25) = [0.07 \times (1.07)^{25}] / [(1.07)^{25} - 1]$ $(1.07)^{25} = 5.4274 = [0.07 \times 5.4274] / [5.4274 - 1] = 0.37992 / 4.4274 = 0.08581$

$CR = (125{,}000 - 8{,}000) \times 0.08581 + 8{,}000 \times 0.07 = 117{,}000 \times 0.08581 + 560 = 10{,}040 + 560 = \$10{,}600/\text{year}$

(b) Total annual cost: $AW = -CR - \text{Annual maintenance} = -10{,}600 - 3{,}500 = -\$14{,}100/\text{year}$

Problem 19.5.2

Given: A solar installation generates \$75,000/year in energy savings. The system costs \$420,000, has a 20-year life, annual O&M of \$8,000, and a salvage value of \$25,000. The MARR is 6%.

Find: (a) The capital recovery cost, (b) the annual worth, and (c) whether the project is justified.

Solution:

(a) $(A/P, 6\%, 20) = [0.06 \times (1.06)^{20}] / [(1.06)^{20} - 1]$ $(1.06)^{20} = 3.2071 = [0.06 \times 3.2071] / [3.2071 - 1] = 0.19243 / 2.2071 = 0.08718$

$CR = (420{,}000 - 25{,}000) \times 0.08718 + 25{,}000 \times 0.06 = 395{,}000 \times 0.08718 + 1{,}500 = 34{,}436 + 1{,}500 =$ \$35,936/year

(b) AW = Revenue − CR − O&M = 75,000 − 35,936 − 8,000 = \$31,064/year

(c) Since AW = \$31,064 > 0, the project is justified and generates a surplus of \$31,064/year above the 6% MARR requirement.

Problem 19.5.3

Given: Compare three cable types for a 2-mile underground distribution run at i = 8%.

Parameter	XLPE	EPR	PILC
First cost	\$350,000	\$420,000	\$290,000
Life	30 years	35 years	25 years
Annual O&M	\$5,000	\$3,000	\$9,000
Salvage	\$10,000	\$15,000	\$5,000

Find: The most economical cable type using annual worth analysis.

Solution:

XLPE — (A/P, 8%, 30): $(1.08)^{30} = 10.063$ (A/P, 8%, 30) $= [0.08 \times 10.063] / [10.063 - 1] = 0.80504 / 9.063 = 0.08883$

$CR_{XLPE} = (350{,}000 - 10{,}000) \times 0.08883 + 10{,}000 \times 0.08 = 340{,}000 \times 0.08883 + 800 = 30{,}202 + 800 =$ \$31,002 $AW_{XLPE} = -31{,}002 - 5{,}000 = -\$36{,}002$/year

EPR — (A/P, 8%, 35): $(1.08)^{35} = 14.785$ (A/P, 8%, 35) $= [0.08 \times 14.785] / [14.785 - 1] = 1.1828 / 13.785 = 0.08580$

$CR_{EPR} = (420{,}000 - 15{,}000) \times 0.08580 + 15{,}000 \times 0.08 = 405{,}000 \times 0.08580 + 1{,}200 = 34{,}749 + 1{,}200 = \$35{,}949$ $AW_{EPR} = -35{,}949 - 3{,}000 = -\$38{,}949$/year

PILC — (A/P, 8%, 25): $(1.08)^{25} = 6.8485$ (A/P, 8%, 25) $= [0.08 \times 6.8485] / [6.8485 - 1] = 0.54788 / 5.8485 = 0.09368$

$CR_{PILC} = (290{,}000 - 5{,}000) \times 0.09368 + 5{,}000 \times 0.08 = 285{,}000 \times 0.09368 + 400 = 26{,}699 + 400 =$ \$27,099 $AW_{PILC} = -27{,}099 - 9{,}000 = -\$36{,}099$/year

XLPE cable is the most economical at −\$36,002/year, barely edging out PILC (−\$36,099/year). Note that the AW method handles the different service lives (25, 30, 35 years) without requiring LCM calculations.

Problem 19.5.4

Given: A plant manager compares two VFD options for a cooling tower fan at $i = 10\%$. - Standard VFD: \$18,000, 8-year life, \$4,800/year energy, no salvage. - Premium VFD: \$26,000, 12-year life, \$3,600/year energy, \$2,000 salvage.

Find: The more economical choice using annual worth comparison.

Solution:

Standard VFD: $(A/P, 10\%, 8) = [0.10 \times (1.10)^8] / [(1.10)^8 - 1] = [0.10 \times 2.1436] / [2.1436 - 1] = 0.21436 / 1.1436 = 0.18744$

$AW_{std} = -18{,}000 \times 0.18744 - 4{,}800 = -3{,}374 - 4{,}800 = -\$8{,}174/\text{year}$

Premium VFD: $(A/P, 10\%, 12) = [0.10 \times (1.10)^{12}] / [(1.10)^{12} - 1] = [0.10 \times 3.1384] / [3.1384 - 1] = 0.31384 / 2.1384 = 0.14676$

$CR = (26{,}000 - 2{,}000) \times 0.14676 + 2{,}000 \times 0.10 = 24{,}000 \times 0.14676 + 200 = 3{,}522 + 200 = \$3{,}722$

$AW_{prem} = -3{,}722 - 3{,}600 = -\$7{,}322/\text{year}$

The premium VFD is more economical by \$8,174 − \$7,322 = \$852/year. Its longer life and lower energy costs more than justify the higher purchase price.

Chapter 19 — Section 19.6: Rate of Return Analysis

Practice problems covering internal rate of return for single projects, incremental rate of return for mutually exclusive alternatives, and modified IRR for non-conventional cash flows. Problems range from direct IRR calculation to multi-step incremental and MIRR analyses.

Problem 19.6.1

Given: A \$250,000 energy management system reduces electricity costs by \$48,000/year for 10 years with no salvage value.

Find: (a) The IRR, and (b) whether the project is acceptable at MARR = 10%.

Solution:

(a) Set NPV = 0: $0 = -250{,}000 + 48{,}000 \times (P/A, i, 10)$ $(P/A, i, 10) = 250{,}000 / 48{,}000 = 5.2083$

Try i = 14%: $(P/A, 14\%, 10) = [(1.14)^{10} - 1] / [0.14 \times (1.14)^{10}]$ $(1.14)^{10} = 3.7072 = [3.7072 - 1] / [0.14 \times 3.7072] = 2.7072 / 0.51901 = 5.2163$ (NPV slightly > 0)

Try i = 15%: $(P/A, 15\%, 10) = [(1.15)^{10} - 1] / [0.15 \times (1.15)^{10}]$ $(1.15)^{10} = 4.0456 = [4.0456 - 1] / [0.15 \times 4.0456] = 3.0456 / 0.60684 = 5.0188$ (NPV < 0)

Interpolate: $i^* = 14\% + 1\% \times (5.2163 - 5.2083) / (5.2163 - 5.0188) = 14\% + 1\% \times 0.0080 / 0.1975 = 14\% + 0.04\% = 14.04\%$

(b) Since IRR = 14.04% > MARR = 10%, the project is acceptable.

Problem 19.6.2

Given: A \$400,000 solar thermal system generates \$72,000/year in savings for 12 years and has a salvage value of \$30,000 at year 12.

Find: The IRR of the investment.

Solution:

Set NPV = 0: $0 = -400{,}000 + 72{,}000 \times (P/A, i, 12) + 30{,}000 \times (P/F, i, 12)$

Try i = 14%: $(1.14)^{12} = 4.8179$ $(P/A, 14\%, 12) = [4.8179 - 1] / [0.14 \times 4.8179] = 3.8179 / 0.67451 = 5.6603$ $(P/F, 14\%, 12) = 1/4.8179 = 0.2076$ NPV $= -400{,}000 + 72{,}000 \times 5.6603 + 30{,}000 \times 0.2076 = -400{,}000 + 407{,}542 + 6{,}228 = +\$13{,}770$

Try i = 15%: $(1.15)^{12} = 5.3503$ $(P/A, 15\%, 12) = [5.3503 - 1] / [0.15 \times 5.3503] = 4.3503 / 0.80255 = 5.4205$ $(P/F, 15\%, 12) = 1/5.3503 = 0.1869$ NPV $= -400{,}000 + 72{,}000 \times 5.4205 + 30{,}000 \times 0.1869 = -400{,}000 + 390{,}276 + 5{,}607 = -\$4{,}117$

Interpolate: $i^* = 14\% + 1\% \times 13{,}770 / (13{,}770 + 4{,}117) = 14\% + 1\% \times 0.770 = 14\% + 0.77\% = 14.77\%$

Problem 19.6.3

Given: Two switchgear options are available for a 15-year service life. Option A: \$200,000 cost, \$42,000/year savings. Option B: \$340,000 cost, \$65,000/year savings. Neither has salvage value. MARR = 9%.

Find: Using incremental IRR analysis, which option should be selected.

Solution:

First verify both meet MARR individually: Option A: $(P/A, i, 15) = 200{,}000/42{,}000 = 4.7619 \rightarrow IRR \approx 18.9\% > 9\%$ ✓ Option B: $(P/A, i, 15) = 340{,}000/65{,}000 = 5.2308 \rightarrow$ IRR $\approx 16.6\% > 9\%$ ✓

Incremental analysis (B − A): ΔCost = \$140,000, ΔSavings = \$23,000/year $0 = -140{,}000 + 23{,}000 \times (P/A, \Delta i, 15)$ $(P/A, \Delta i, 15) = 140{,}000/23{,}000 = 6.0870$

Try i = 13%: $(1.13)^{15} = 6.2543$ $(P/A, 13\%, 15) = [6.2543 - 1] / [0.13 \times 6.2543] = 5.2543 / 0.81306 = 6.4626$ (NPV > 0)

Try i = 15%: $(1.15)^{15} = 8.1371$ $(P/A, 15\%, 15) = [8.1371 - 1] / [0.15 \times 8.1371] = 7.1371 / 1.2206 = 5.8470$ (NPV < 0)

Interpolate: $\Delta i^* = 13\% + 2\% \times (6.4626 - 6.0870) / (6.4626 - 5.8470) = 13\% + 2\% \times 0.3756/0.6156 = 13\% + 1.22\% = 14.22\%$

Since ΔIRR = 14.22% > MARR = 9%, the extra investment in Option B is justified. Select Option B.

Problem 19.6.4

Given: Three motor control center (MCC) options over 10 years at MARR = 11%. No salvage values.

Option	First Cost	Annual Savings
A	\$80,000	\$18,000
B	\$130,000	\$27,000
C	\$175,000	\$34,000

Find: The best option using incremental IRR analysis.

Solution:

Rank by first cost: A, B, C.

Check A vs. do-nothing: (P/A, i*, 10) = 80,000/18,000 = 4.4444 Try i = 18%: (P/A, 18%, 10) = $[(1.18)^{10} - 1] / [0.18 \times (1.18)^{10}]$ $(1.18)^{10} = 5.2338$; = 4.2338 / 0.94208 = 4.4941 Try i = 19%: $(1.19)^{10} = 5.6947$; = 4.6947 / 1.0820 = 4.3389 Interpolate: $IRR_A \approx 18\% + 1\% \times (4.4941 - 4.4444)/(4.4941 - 4.3389) =$ 18.32% $IRR_A = 18.32\% > 11\%$ ✓ → A is the current best.

B vs. A: ΔCost = \$50,000, ΔSavings = \$9,000/year (P/A, Δi*, 10) = 50,000/9,000 = 5.5556 Try i = 12%: $(1.12)^{10} = 3.1058$; (P/A) = 2.1058/0.37270 = 5.6502 (NPV > 0) Try i = 13%: $(1.13)^{10} = 3.3946$; (P/A) = 2.3946/0.44130 = 5.4262 (NPV < 0) Δi* $\approx 12\% + 1\% \times (5.6502 - 5.5556)/(5.6502 - 5.4262) = 12\% + 0.42\% = 12.42\%$ ΔIRR = 12.42% > 11% → B is justified over A. B is the new best.

C vs. B: ΔCost = \$45,000, ΔSavings = \$7,000/year (P/A, Δi*, 10) = 45,000/7,000 = 6.4286 Try i = 9%: $(1.09)^{10} = 2.3674$; (P/A) = 1.3674/0.21307 = 6.4178 (NPV slightly < 0) Try i = 8%: $(1.08)^{10} = 2.1589$; (P/A) = 1.1589/0.17271 = 6.7101 (NPV > 0) Δi* $\approx 8\% + 1\% \times (6.7101 - 6.4286)/(6.7101 - 6.4178) = 8\% + 1\% \times 0.2815/0.2923 = 8.96\%$ ΔIRR = 8.96% < 11% → C is not justified over B.

Select Option B — the increment from A to B is justified, but the increment from B to C is not.

Problem 19.6.5

Given: A nuclear plant decommissioning project has the following cash flows (in \$millions): Year 0: −\$80; Years 1–6: +\$18/year; Year 7: −\$25 (decommissioning cost). The reinvestment rate is 8% and the finance rate is 6%.

Find: (a) The number of sign changes in the cash flow, and (b) the MIRR.

Solution:

(a) Cash flow signs: −, +, +, +, +, +, +, − Sign changes: −→+ at year 1 and +→− at year 7 = 2 sign changes Up to 2 IRR values may exist, so MIRR is needed.

(b) Future value of positive cash flows at reinvestment rate (8%), all moved to year 7: $FV_{positive} = 18 \times (F/A, 8\%, 6) \times (F/P, 8\%, 1)$ (F/A, 8%, 6) = $[(1.08)^6 - 1] / 0.08 = [1.5869 - 1] / 0.08 = 0.5869 / 0.08 = 7.3359$ $FV_{positive} = 18 \times 7.3359 \times 1.08 = 18 \times 7.923 = \$142.6M$

Present value of negative cash flows at finance rate (6%): $PV_{negative} = 80 + 25 \times (P/F, 6\%, 7) = 80 + 25/(1.06)^7 = 80 + 25/1.5036 = 80 + 16.627 = \$96.63M$

$MIRR = (FV_{positive}/PV_{negative})^{1/n} - 1 = (142.6/96.63)^{1/7} - 1 = (1.4758)^{0.14286} - 1 = 1.0573 - 1 = 5.73\%$

Problem 19.6.6

Given: A cogeneration facility has the following cash flows: Year 0: −\$10M initial investment; Years 1–5: +\$3M/year revenue; Year 6: −\$2M (major overhaul); Years 7–10: +\$3.5M/year revenue; Year 11: −\$4M decommissioning. The reinvestment rate is 12% and the finance rate is 10%.

Find: The MIRR of the project.

Solution:

Sign changes: −, +, +, +, +, +, −, +, +, +, +, − → 3 sign changes (need MIRR)

Future value of positive cash flows at 12%, moved to year 11: $FV_{1-5} = 3 \times (F/A, 12\%, 5) \times (F/P, 12\%, 6)$ $(F/A, 12\%, 5) = [(1.12)^5 - 1] / 0.12 = [1.7623 - 1] / 0.12 = 6.3528$ $(F/P, 12\%, 6) = (1.12)^6 = 1.9738$ $FV_{1-5} = 3 \times 6.3528 \times 1.9738 = 3 \times 12.538 = \$37.61M$

$FV_{7-10} = 3.5 \times (F/A, 12\%, 4) \times (F/P, 12\%, 1)$ $(F/A, 12\%, 4) = [(1.12)^4 - 1] / 0.12 = [1.5735 - 1] / 0.12 = 4.7793$ $(F/P, 12\%, 1) = 1.12$ $FV_{7-10} = 3.5 \times 4.7793 \times 1.12 = 3.5 \times 5.3528 = \$18.73M$

$FV_{positive} = 37.61 + 18.73 = \$56.34M$

Present value of negative cash flows at 10%: $PV_{negative} = 10 + 2 \times (P/F, 10\%, 6) + 4 \times (P/F, 10\%, 11)$ $(P/F, 10\%, 6) = 1/(1.10)^6 = 1/1.7716 = 0.5645$ $(P/F, 10\%, 11) = 1/(1.10)^{11} = 1/2.8531 = 0.3505$ $PV_{negative} = 10 + 2 \times 0.5645 + 4 \times 0.3505 = 10 + 1.129 + 1.402 = \$12.53M$

$MIRR = (56.34/12.53)^{1/11} - 1 = (4.496)^{0.09091} - 1 = 1.1464 - 1 = 14.64\%$

Chapter 19 — Section 19.7: Benefit-Cost Analysis

Practice problems covering conventional benefit-cost ratios and incremental B/C analysis for multiple public-sector alternatives. Problems use realistic utility and infrastructure scenarios.

Problem 19.7.1

Given: A regional utility invests \$6,000,000 in an automated meter infrastructure (AMI) system. Benefits include \$900,000/year in reduced meter reading costs, \$200,000/year in outage detection improvements, and \$50,000/year in theft detection. The system has a 20-year life and i = 5%.

Find: (a) The conventional B/C ratio, and (b) whether the project is justified.

Solution:

(a) Total annual benefits: B = 900,000 + 200,000 + 50,000 = \$1,150,000/year

AW of cost: C = 6,000,000 × (A/P, 5%, 20) (A/P, 5%, 20) = $[0.05 \times (1.05)^{20}] / [(1.05)^{20} - 1]$ $(1.05)^{20}$ = 2.6533 = [0.05 × 2.6533] / [2.6533 − 1] = 0.13267 / 1.6533 = 0.08024

C = 6,000,000 × 0.08024 = \$481,440/year

B/C = 1,150,000 / 481,440 = 2.39

(b) Since B/C = 2.39 > 1.0, the project is justified.

Problem 19.7.2

Given: A city evaluates a \$2,500,000 underground cable conversion project. Benefits are \$180,000/year in reduced storm outage costs and \$60,000/year in reduced tree trimming. Disbenefits include \$25,000/year in longer repair times for underground faults. Annual O&M is \$40,000. Project life is 30 years at i = 4%.

Find: (a) The conventional B/C ratio, and (b) the modified B/C ratio.

Solution:

(A/P, 4%, 30) = $[0.04 \times (1.04)^{30}] / [(1.04)^{30} - 1]$ $(1.04)^{30} = 3.2434 = [0.04 \times 3.2434] / [3.2434 - 1] = 0.12974 / 2.2434 = 0.05783$

AW of initial cost: 2,500,000 × 0.05783 = \$144,575/year Total AW of costs (conventional): 144,575 + 40,000 = \$184,575/year

Net annual benefits: B − D = (180,000 + 60,000) − 25,000 = \$215,000/year

(a) Conventional B/C = (B − D) / Costs = 215,000 / 184,575 = 1.16

(b) Modified B/C = (B − D − O&M) / Initial Investment AW = (215,000 − 40,000) / 144,575 = 175,000 / 144,575 = 1.21

Both ratios exceed 1.0 — the project is justified under either formulation.

Problem 19.7.3

Given: A state DOT evaluates three highway lighting upgrade levels for a 10-mile corridor over 25 years at i = 6%.

Level	First Cost	Annual Benefits	Annual O&M
1 — Basic LED	\$800,000	\$120,000	\$15,000
2 — Smart LED	\$1,400,000	\$195,000	\$25,000
3 — Smart LED + adaptive	\$2,200,000	\$290,000	\$40,000

Find: The best alternative using incremental B/C analysis.

Solution:

(A/P, 6%, 25) = $[0.06 \times (1.06)^{25}] / [(1.06)^{25} - 1]$ $(1.06)^{25} = 4.2919 = [0.06 \times 4.2919] / [4.2919 - 1] = 0.25751 / 3.2919 = 0.07823$

AW of costs (capital + O&M): C_1 = 800,000 × 0.07823 + 15,000 = 62,584 + 15,000 = \$77,584 C_2 = 1,400,000 × 0.07823 + 25,000 = 109,522 + 25,000 = \$134,522 C_3 = 2,200,000 × 0.07823 + 40,000 = 172,106 + 40,000 = \$212,106

Level 1 vs. do-nothing: B/C = 120,000 / 77,584 = 1.55 ≥ 1.0 → Level 1 justified (new base)

Level 2 vs. Level 1: ΔB/ΔC = (195,000 − 120,000) / (134,522 − 77,584) = 75,000 / 56,938 = 1.32 ≥ 1.0 → Level 2 justified (new base)

Level 3 vs. Level 2: ΔB/ΔC = (290,000 − 195,000) / (212,106 − 134,522) = 95,000 / 77,584 = 1.22 ≥ 1.0 → Level 3 justified

Select Level 3 — each incremental investment is justified with ΔB/ΔC ≥ 1.0.

Problem 19.7.4

Given: A flood control district evaluates four levee improvement options over 50 years at i = 3%.

Option	First Cost	Annual Flood Damage Reduction	Annual O&M
A	\$5,000,000	\$300,000	\$20,000
B	\$8,000,000	\$480,000	\$30,000
C	\$12,000,000	\$650,000	\$45,000
D	\$18,000,000	\$850,000	\$60,000

Find: The best option using incremental B/C analysis (conventional formulation with O&M in denominator).

Solution:

(A/P, 3%, 50) = $[0.03 \times (1.03)^{50}] / [(1.03)^{50} - 1]$ $(1.03)^{50} = 4.3839$ = $[0.03 \times 4.3839] / [4.3839 - 1]$ = 0.13152 / 3.3839 = 0.03887

Total AW of costs (capital recovery + O&M): $C_A = 5{,}000{,}000 \times 0.03887 + 20{,}000 = 194{,}350 + 20{,}000 =$ \$214,350 $C_B = 8{,}000{,}000 \times 0.03887 + 30{,}000 = 310{,}960 + 30{,}000 =$ \$340,960 $C_C = 12{,}000{,}000 \times 0.03887 + 45{,}000 = 466{,}440 + 45{,}000 =$ \$511,440 $C_D = 18{,}000{,}000 \times 0.03887 + 60{,}000 = 699{,}660 + 60{,}000 =$ \$759,660

A vs. do-nothing: B/C = 300,000 / 214,350 = 1.40 $\geq$ 1.0 $\rightarrow$ A justified (base)

B vs. A: $\Delta B/\Delta C$ = (480,000 − 300,000) / (340,960 − 214,350) = 180,000 / 126,610 = 1.42 $\geq$ 1.0 $\rightarrow$ B justified (new base)

C vs. B: $\Delta B/\Delta C$ = (650,000 − 480,000) / (511,440 − 340,960) = 170,000 / 170,480 = 1.00 $\geq$ 1.0 $\rightarrow$ C justified (new base)

D vs. C: $\Delta B/\Delta C$ = (850,000 − 650,000) / (759,660 − 511,440) = 200,000 / 248,220 = 0.81 < 1.0 $\rightarrow$ D not justified

Select Option C — the increment from B to C is marginally justified (B/C = 1.00), but the increment from C to D does not produce sufficient benefits to justify the additional cost.

Chapter 19 — Section 19.8: Depreciation

Practice problems covering straight-line, declining balance, MACRS, and units-of-production depreciation methods. Problems range from direct calculation to comparison across methods.

Problem 19.8.1

Given: A utility installs a \$240,000 recloser system with a salvage value of \$15,000 and a useful life of 15 years.

Find: Using straight-line depreciation: (a) the annual depreciation charge, (b) the book value after 6 years, and (c) the book value after 12 years.

Solution:

(a) $D = (P - S) / n = (240{,}000 - 15{,}000) / 15 = 225{,}000 / 15 = \$15{,}000/\text{year}$

(b) $BV_6 = P - 6 \times D = 240{,}000 - 6 \times 15{,}000 = 240{,}000 - 90{,}000 = \$150{,}000$

(c) $BV_{12} = P - 12 \times D = 240{,}000 - 12 \times 15{,}000 = 240{,}000 - 180{,}000 = \$60{,}000$

Problem 19.8.2

Given: A \$500,000 industrial motor test bench has a 10-year useful life and a \$50,000 salvage value. The company uses straight-line depreciation.

Find: (a) The annual depreciation, (b) the depreciation rate as a percentage, (c) the book value after 4 years, and (d) the year in which the book value first drops below \$200,000.

Solution:

(a) $D = (500{,}000 - 50{,}000) / 10 = 450{,}000 / 10 = \$45{,}000/\text{year}$

(b) Depreciation rate: $d = D / P = 45{,}000 / 500{,}000 = 9.0\%$ of original cost per year

(c) $BV_4 = 500{,}000 - 4 \times 45{,}000 = 500{,}000 - 180{,}000 = \$320{,}000$

(d) $BV_t < 200{,}000 \rightarrow 500{,}000 - 45{,}000t < 200{,}000 \rightarrow t > 300{,}000/45{,}000 = 6.67$ The book value first drops below \$200,000 at the end of year 7 ($BV_7 = \$185{,}000$).

Problem 19.8.3

Given: A \$180,000 power quality monitoring system uses double declining balance (DDB) depreciation with a 6-year life and a \$20,000 salvage value.

Find: The depreciation charge and book value for each year, switching to straight-line when it produces a larger deduction.

Solution:

DDB rate: $d = 2/n = 2/6 = 0.3333$

Year	BV (start)	DDB Depr.	SL Remaining	Method Used	Depreciation	BV (end)
1	\$180,000	\$60,000	\$26,667	DDB	\$60,000	\$120,000
2	\$120,000	\$40,000	\$20,000	DDB	\$40,000	\$80,000
3	\$80,000	\$26,667	\$15,000	DDB	\$26,667	\$53,333
4	\$53,333	\$17,778	\$11,111	DDB	\$17,778	\$35,555
5	\$35,555	\$11,852	\$7,778	Switch to SL	\$7,778	\$27,778
6	\$27,778	\$9,259	\$7,778	SL	\$7,778	\$20,000

SL remaining for year t = $(BV_{t-1} - S)$ / (remaining years)

Year 5: SL = $(35{,}555 - 20{,}000)/2 = \$7{,}778$; DDB = \$11,852 but would reduce BV below salvage if not checked. Actually DDB (\$11,852) > SL (\$7,778), but BV after DDB = \$23,703 > \$20,000, so DDB is still valid in year 5.

Correcting: In year 5, DDB = $0.3333 \times 35{,}555 = \$11{,}852$, giving BV = \$23,703 (above salvage). In year 6, DDB = $0.3333 \times 23{,}703 = \$7{,}901$, but BV = \$15,802 < \$20,000 salvage — not allowed. So SL in year 6: $D = 23{,}703 - 20{,}000 = \$3{,}703$.

Revised final years:

Year	BV (start)	Depreciation	BV (end)
5	\$35,555	\$11,852 (DDB)	\$23,703
6	\$23,703	\$3,703 (SL to salvage)	\$20,000

Total depreciation = $60{,}000 + 40{,}000 + 26{,}667 + 17{,}778 + 11{,}852 + 3{,}703 = \$160{,}000 = P - S$ ✓

Problem 19.8.4

Given: A manufacturing company purchases a \$350,000 robotic welding cell classified as 7-year MACRS property.

Find: The depreciation charges for years 1 through 4 and the book value after year 4.

Solution:

MACRS 7-year percentages: Year 1 = 14.29%, Year 2 = 24.49%, Year 3 = 17.49%, Year 4 = 12.49%

Year	MACRS %	Depreciation	Cumulative
1	14.29%	$50,015	$50,015
2	24.49%	$85,715	$135,730
3	17.49%	$61,215	$196,945
4	12.49%	$43,715	$240,660

Book value after year 4: $BV_4 = 350{,}000 - 240{,}660 = \$109{,}340$

After 4 years, 68.76% of the cost basis has been recovered through depreciation.

Problem 19.8.5

Given: A $2,400,000 natural gas peaking turbine is classified as 15-year MACRS property.

Find: (a) The depreciation for years 1 through 3, and (b) the book value after year 3.

Solution:

MACRS 15-year percentages: Year 1 = 5.00%, Year 2 = 9.50%, Year 3 = 8.55%

Year	MACRS %	Depreciation	Cumulative
1	5.00%	$120,000	$120,000
2	9.50%	$228,000	$348,000
3	8.55%	$205,200	$553,200

(a) Depreciation charges: Year 1 = $120,000, Year 2 = $228,000, Year 3 = $205,200

(b) $BV_3 = 2{,}400{,}000 - 553{,}200 = \$1{,}846{,}800$

Problem 19.8.6

Given: A $90,000 oscilloscope is rated for 30,000 hours of use with a salvage value of $6,000. Usage records show: Year 1 = 4,500 hours, Year 2 = 5,800 hours, Year 3 = 6,200 hours, Year 4 = 3,500 hours.

Find: Using units-of-production depreciation: (a) the depreciation rate per hour, (b) the depreciation for each year, and (c) the book value after year 4.

Solution:

(a) $d_{unit} = (P - S)$ / Total hours = (90,000 − 6,000) / 30,000 = 84,000 / 30,000 = \$2.80/hour

(b) Depreciation by year:

- Year 1: $D_1 = 2.80 \times 4{,}500 = \$12{,}600$
- Year 2: $D_2 = 2.80 \times 5{,}800 = \$16{,}240$
- Year 3: $D_3 = 2.80 \times 6{,}200 = \$17{,}360$
- Year 4: $D_4 = 2.80 \times 3{,}500 = \$9{,}800$

(c) Total depreciation through year 4: 12,600 + 16,240 + 17,360 + 9,800 = \$56,000 BV_4 = 90,000 − 56,000 = \$34,000

Remaining depreciable hours: 30,000 − 20,000 = 10,000 hours; remaining depreciable amount: 34,000 − 6,000 = \$28,000.

Problem 19.8.7

Given: A \$150,000 generator is rated for 100,000 kWh of total production with a \$10,000 salvage value. In its first year, it produces 22,000 kWh.

Find: (a) The units-of-production depreciation for year 1, and (b) the book value after year 1.

Solution:

(a) d_{unit} = (150,000 − 10,000) / 100,000 = 140,000 / 100,000 = \$1.40/kWh $D_1 = 1.40 \times 22{,}000 =$ \$30,800

(b) BV_1 = 150,000 − 30,800 = \$119,200

Problem 19.8.8

Given: A \$600,000 CNC milling machine has a 10-year life and \$40,000 salvage value. Compare the depreciation in year 1 and book value after year 3 using three methods: (a) straight-line, (b) double declining balance, and (c) 5-year MACRS (the machine qualifies as 5-year MACRS property).

Find: Year 1 depreciation and book value after year 3 for each method.

Solution:

(a) Straight-line: D = (600,000 − 40,000) / 10 = \$56,000/year Year 1 depreciation: \$56,000 BV_3 = 600,000 − 3 × 56,000 = 600,000 − 168,000 = \$432,000

(b) Double declining balance (DDB): Rate: d = 2/10 = 0.20 Year 1: $D_1 = 0.20 \times 600{,}000 = \$120{,}000$; $BV_1 = \$480{,}000$ Year 2: $D_2 = 0.20 \times 480{,}000 = \$96{,}000$; $BV_2 = \$384{,}000$ Year 3: $D_3 = 0.20 \times 384{,}000 =$ \$76,800; $BV_3 = \$307{,}200$

Year 1 depreciation: \$120,000 $BV_3 = \$307{,}200$

(c) 5-year MACRS: MACRS 5-year percentages: Year 1 = 20.00%, Year 2 = 32.00%, Year 3 = 19.20% Year 1: $D_1 = 0.20 \times 600{,}000 = \$120{,}000$ Year 2: $D_2 = 0.32 \times 600{,}000 = \$192{,}000$ Year 3: $D_3 = 0.192 \times 600{,}000 = \$115{,}200$ Cumulative: 120,000 + 192,000 + 115,200 = \$427,200

Year 1 depreciation: \$120,000 $BV_3 = 600{,}000 - 427{,}200 = \$172{,}800$

Summary: MACRS provides the fastest depreciation ($BV_3 = \$172{,}800$), followed by DDB (\$307,200), then straight-line (\$432,000). Faster depreciation methods provide greater early-year tax shields.

Chapter 19 — Section 19.9: Taxes and After-Tax Analysis

Practice problems covering after-tax cash flow analysis, tax credits and incentives, and gain/loss on disposal. Problems range from straightforward ATCF calculations to multi-step after-tax NPV analyses with tax credits.

Problem 19.9.1

Given: A $300,000 building automation system generates $65,000/year in energy savings over 10 years. The system is depreciated using straight-line depreciation with no salvage value. The marginal tax rate is 28% and the after-tax MARR is 9%.

Find: (a) The annual depreciation, (b) the after-tax cash flow, and (c) the after-tax NPV.

Solution:

(a) D = 300,000 / 10 = $30,000/year

(b) ATCF = BTCF × (1 − T) + D × T = 65,000 × (1 − 0.28) + 30,000 × 0.28 = 65,000 × 0.72 + 30,000 × 0.28 = 46,800 + 8,400 = $55,200/year

(c) (P/A, 9%, 10) = $[(1.09)^{10} - 1] / [0.09 \times (1.09)^{10}]$ $(1.09)^{10} = 2.3674$ = [2.3674 − 1] / [0.09 × 2.3674] = 1.3674 / 0.21307 = 6.4177

After-tax NPV = −300,000 + 55,200 × 6.4177 = −300,000 + 354,257 = $54,257

The project is justified on an after-tax basis (NPV > 0).

Problem 19.9.2

Given: A manufacturing facility invests $800,000 in a high-efficiency transformer system. Annual savings are $150,000 over 8 years. The equipment uses 7-year MACRS depreciation. The tax rate is 25% and the after-tax MARR is 10%.

Find: The after-tax cash flows for years 1 through 4 and year 8 (year with no depreciation).

Solution:

MACRS 7-year percentages: Year 1 = 14.29%, Year 2 = 24.49%, Year 3 = 17.49%, Year 4 = 12.49%, Year 8 = 4.46%

Year	BTCF	MACRS %	Depreciation	Taxable Income	Tax (25%)	ATCF
1	\$150,000	14.29%	\$114,320	\$35,680	\$8,920	\$141,080
2	\$150,000	24.49%	\$195,920	−\$45,920	−\$11,480	\$161,480
3	\$150,000	17.49%	\$139,920	\$10,080	\$2,520	\$147,480
4	\$150,000	12.49%	\$99,920	\$50,080	\$12,520	\$137,480

Year 8 (last MACRS year): Depreciation $= 0.0446 \times 800{,}000 = \$35{,}680$ $ATCF_8 = 150{,}000 \times (1 - 0.25) + 35{,}680 \times 0.25 = 112{,}500 + 8{,}920 = \$121{,}420$

Note: In year 2, the large MACRS depreciation creates a tax loss that provides a \$11,480 tax benefit, boosting ATCF above the BTCF.

Problem 19.9.3

Given: A commercial building installs a \$450,000 geothermal heat pump system qualifying for a 30% investment tax credit. Annual energy savings are \$60,000 over 20 years. The depreciable basis is reduced to 85% of cost (ITC adjustment). Using 5-year straight-line depreciation on the adjusted basis, a 24% tax rate, and after-tax MARR of 7%.

Find: (a) The ITC amount, (b) the ATCF for years 1–5 (with depreciation), and (c) the ATCF for years 6–20 (without depreciation).

Solution:

(a) ITC $= 450{,}000 \times 0.30 = \$135{,}000$ (received in year 1)

(b) Depreciable basis $= 450{,}000 \times 0.85 = \$382{,}500$ Annual depreciation (years 1–5): $D = 382{,}500 / 5 = \$76{,}500$

$TI = 60{,}000 - 76{,}500 = -\$16{,}500$ (tax loss) Tax $= -16{,}500 \times 0.24 = -\$3{,}960$ (tax benefit) ATCF (years 2–5) $= 60{,}000 + 3{,}960 = \$63{,}960$/year ATCF (year 1) $= 63{,}960 + 135{,}000 = \$198{,}960$

(c) ATCF (years 6–20): No depreciation ATCF $= 60{,}000 \times (1 - 0.24) = 60{,}000 \times 0.76 = \$45{,}600$/year

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Problem 19.9.4

Given: A \$350,000 test system is depreciated using 5-year MACRS. After 4 years, it is sold for \$120,000. The tax rate is 25%.

Find: (a) The book value at disposal, (b) the gain or loss, and (c) the after-tax salvage value.

Solution:

(a) MACRS 5-year cumulative through year 4: Year 1: 20.00%, Year 2: 32.00%, Year 3: 19.20%, Year 4: 11.52% Total: 82.72% Cumulative depreciation = 0.8272 × 350,000 = \$289,520 BV_4 = 350,000 − 289,520 = \$60,480

(b) Selling price (\$120,000) > Book value (\$60,480) Gain = 120,000 − 60,480 = \$59,520 (taxable gain)

(c) Tax on gain = 59,520 × 0.25 = \$14,880 After-tax salvage: S_{AT} = 120,000 − 14,880 = \$105,120

Problem 19.9.5

Given: A \$160,000 CNC lathe is depreciated using straight-line over 8 years with a \$16,000 salvage value. After 5 years, the machine is sold for \$50,000 because it is being replaced. The tax rate is 30%.

Find: (a) The book value at year 5, (b) the gain or loss on disposal, and (c) the after-tax salvage value.

Solution:

(a) D = (160,000 − 16,000) / 8 = 144,000 / 8 = \$18,000/year BV_5 = 160,000 − 5 × 18,000 = 160,000 − 90,000 = \$70,000

(b) Selling price (\$50,000) < Book value (\$70,000) Loss = 70,000 − 50,000 = \$20,000 (deductible loss)

(c) Tax benefit from loss = 20,000 × 0.30 = \$6,000 After-tax salvage: S_{AT} = 50,000 + 6,000 = \$56,000

The loss on disposal provides a tax benefit that increases the effective after-tax proceeds.

Problem 19.9.6

Given: A utility evaluates a \$1,200,000 capacitor bank installation. Annual demand charge savings are \$220,000 for 10 years. The equipment uses 7-year MACRS depreciation and has a \$50,000 salvage value at year 10. The tax rate is 24% and the after-tax MARR is 8%.

Find: (a) The after-tax NPV considering MACRS depreciation and disposal, and (b) whether the project is justified.

Solution:

MACRS 7-year percentages: 14.29%, 24.49%, 17.49%, 12.49%, 8.93%, 8.92%, 8.93%, 4.46% (Depreciation ends after year 8; book value = 0 after MACRS is complete.)

ATCF for years 1–8 (with depreciation): $ATCF_t$ = 220,000 × (1 − 0.24) + D_t × 0.24 = 167,200 + 0.24 × D_t

Year	Depreciation	Tax Shield (0.24 × D)	ATCF
1	\$171,480	\$41,155	\$208,355
2	\$293,880	\$70,531	\$237,731

Year	Depreciation	Tax Shield ($0.24 \times D$)	ATCF
3	\$209,880	\$50,371	\$217,571
4	\$149,880	\$35,971	\$203,171
5	\$107,160	\$25,718	\$192,918
6	\$107,040	\$25,690	\$192,890
7	\$107,160	\$25,718	\$192,918
8	\$53,520	\$12,845	\$180,045

Years 9–10: No depreciation. ATCF = 220,000 × (1 − 0.24) = \$167,200/year

Year 10 disposal: BV = 0 (fully depreciated). Selling for \$50,000 creates a gain. Tax on gain = 50,000 × 0.24 = \$12,000 After-tax salvage = 50,000 − 12,000 = \$38,000

After-tax NPV = −1,200,000 + Σ $ATCF_t$ / $(1.08)^t$ + 38,000 / $(1.08)^{10}$

Computing the PV of each year's ATCF: Year 1: 208,355 / 1.08 = 192,921 Year 2: 237,731 / 1.1664 = 203,816 Year 3: 217,571 / 1.2597 = 172,720 Year 4: 203,171 / 1.3605 = 149,337 Year 5: 192,918 / 1.4693 = 131,296 Year 6: 192,890 / 1.5869 = 121,558 Year 7: 192,918 / 1.7138 = 112,563 Year 8: 180,045 / 1.8509 = 97,274 Year 9: 167,200 / 1.9990 = 83,642 Year 10: 167,200 / 2.1589 = 77,446 Salvage PV: 38,000 / 2.1589 = 17,601

Total PV of inflows = 192,921 + 203,816 + 172,720 + 149,337 + 131,296 + 121,558 + 112,563 + 97,274 + 83,642 + 77,446 + 17,601 = \$1,360,174

After-tax NPV = −1,200,000 + 1,360,174 = \$160,174

(b) Since after-tax NPV = \$160,174 > 0, the project is justified on an after-tax basis.

Chapter 19 — Section 19.10: Replacement Analysis

Practice problems covering economic service life, defender-challenger comparison, and replacement with a fixed study period. Problems range from ESL tabulation to multi-option study-period analysis.

Problem 19.10.1

Given: A 15-year-old voltage regulator has a current market value of $12,000. Annual operating costs are $5,000 in year 1, increasing by $1,500/year. Assume zero salvage value in all future years. The interest rate is 8%.

Find: The economic service life by computing the EUAC for 1 through 5 additional years.

Solution:

(A/P, 8%, n): n=1: 1.0800; n=2: 0.56077; n=3: 0.38803; n=4: 0.30192; n=5: 0.25046

(A/G, 8%, n): n=1: 0; n=2: 0.4808; n=3: 0.9487; n=4: 1.4040; n=5: 1.8465

AW of O&M = $A_1 + G \times (A/G, 8\%, n) = 5{,}000 + 1{,}500 \times (A/G)$

Years	CR = 12,000 × (A/P)	AW of O&M	EUAC
1	$12,960	$5,000	$17,960
2	$6,729	$5,721	$12,450
3	$4,656	$6,423	$11,079
4	$3,623	$7,106	$10,729
5	$3,006	$7,770	$10,776

The EUAC reaches a minimum at year 4 ($10,729/year), then begins to increase. The economic service life is 4 years.

Problem 19.10.2

Given: An aging distribution transformer (defender) has a market value of \$15,000, EUAC of \$14,200/year over its remaining ESL of 3 years. A new high-efficiency transformer (challenger) costs \$95,000 with EUAC of \$11,800/year over a 20-year ESL. The interest rate is 9%.

Find: (a) Whether to replace now, and (b) the annual savings from replacement.

Solution:

(a) Defender EUAC: \$14,200/year Challenger EUAC: \$11,800/year

Since challenger EUAC (\$11,800) $<$ defender EUAC (\$14,200), replace now.

(b) Annual savings = $14{,}200 - 11{,}800 = \$2{,}400$/year

The new transformer saves \$2,400/year on an equivalent annual cost basis.

Problem 19.10.3

Given: A plant's existing motor (defender) has a market value of \$5,000, annual O&M of \$4,500 increasing by \$500/year, and zero salvage value. A replacement motor (challenger) costs \$22,000, has annual O&M of \$1,800, a 12-year life, and \$2,000 salvage value. The interest rate is 10%.

Find: (a) The defender's EUAC for 1–4 years, (b) the challenger's EUAC, and (c) the replacement decision.

Solution:

(a) Defender ESL analysis: (A/P, 10%, n): n=1: 1.1000; n=2: 0.57619; n=3: 0.40211; n=4: 0.31547 (A/G, 10%, n): n=1: 0; n=2: 0.4762; n=3: 0.9366; n=4: 1.3812

Years	CR = 5,000 × (A/P)	AW of O&M	EUAC
1	\$5,500	\$4,500	\$10,000
2	\$2,881	\$4,738	\$7,619
3	\$2,011	\$4,968	\$6,979
4	\$1,577	\$5,191	\$6,768

The defender's minimum EUAC is \$6,768/year at 4 years (ESL = 4 years).

(b) Challenger EUAC: $(A/P, 10\%, 12) = [0.10 \times (1.10)^{12}] / [(1.10)^{12} - 1] = [0.10 \times 3.1384] / [3.1384 - 1] = 0.31384 / 2.1384 = 0.14676$

$CR = (22{,}000 - 2{,}000) \times 0.14676 + 2{,}000 \times 0.10 = 20{,}000 \times 0.14676 + 200 = 2{,}935 + 200 = \$3{,}135$
$EUAC_{\text{challenger}} = 3{,}135 + 1{,}800 = \$4{,}935$/year

(c) Since challenger EUAC (\$4,935) $<$ defender EUAC (\$6,768), replace now. The savings are \$1,833/year.

Problem 19.10.4

Given: Management imposes a 12-year study period. The defender can serve 5 more years with EUAC = \$16,000/year. The challenger has a 12-year life with EUAC = \$13,500/year. After the defender is retired, the challenger serves the remaining years. The interest rate is 7%.

Find: (a) PW of replacing now, and (b) PW of replacing at year 5. Select the better option.

Solution:

(a) Replace now — challenger serves all 12 years: (P/A, 7%, 12) = $[(1.07)^{12} - 1] / [0.07 \times (1.07)^{12}]$ $(1.07)^{12} = 2.2522 = [2.2522 - 1] / [0.07 \times 2.2522] = 1.2522 / 0.15765 = 7.9427$

PW = $13{,}500 \times 7.9427$ = \$107,227

(b) Replace at year 5 — defender serves years 1–5, challenger serves years 6–12: (P/A, 7%, 5) = $[(1.07)^5 - 1] / [0.07 \times (1.07)^5]$ $(1.07)^5 = 1.4026 = [1.4026 - 1] / [0.07 \times 1.4026] = 0.4026 / 0.09818 = 4.1002$

(P/A, 7%, 7) = $[(1.07)^7 - 1] / [0.07 \times (1.07)^7]$ $(1.07)^7 = 1.6058 = [1.6058 - 1] / [0.07 \times 1.6058] = 0.6058 / 0.11241 = 5.3893$

(P/F, 7%, 5) = $1/1.4026 = 0.7130$

PW = $16{,}000 \times 4.1002 + 13{,}500 \times 5.3893 \times 0.7130 = 65{,}603 + 13{,}500 \times 3.8425 = 65{,}603 + 51{,}874 =$ \$117,477

Replace now — it saves \$117,477 − \$107,227 = \$10,250 in present worth.

Problem 19.10.5

Given: A utility's existing SCADA server (defender) has a market value of \$8,000 and can serve 2 more years at \$12,000/year O&M or 3 more years at \$12,000, \$14,000, \$17,000 in years 1–3 respectively. A new server (challenger) costs \$45,000, has O&M of \$4,000/year, and a 6-year life with \$5,000 salvage. The interest rate is 10%. Management sets a 6-year study period.

Find: The best replacement timing — now, at year 2, or at year 3.

Solution:

Challenger EUAC: (A/P, 10%, 6) = $[0.10 \times (1.10)^6] / [(1.10)^6 - 1] = [0.10 \times 1.7716] / [0.7716] = 0.22961$ CR = $(45{,}000 - 5{,}000) \times 0.22961 + 5{,}000 \times 0.10 = 40{,}000 \times 0.22961 + 500 = 9{,}184 + 500 =$ \$9,684 $EUAC_{chall} = 9{,}684 + 4{,}000 =$ \$13,684/year

Option 1 — Replace now (challenger serves 6 years): PW = $13{,}684 \times$ (P/A, 10%, 6) = $13{,}684 \times 4.3553$ = \$59,587

Option 2 — Replace at year 2 (defender 2 years, challenger 4 years): (P/A, 10%, 2) = 1.7355; (P/F, 10%, 2) = 0.8264

Defender cost: $8{,}000 + 12{,}000 \times 1.7355 = 8{,}000 + 20{,}826 =$ \$28,826 Challenger cost for 4 years at year 2: (A/P, 10%, 4) = $[0.10 \times 1.4641] / [0.4641] = 0.31547$ Partial CR = $(45{,}000 - 5{,}000) \times 0.31547 + 5{,}000$

$\times$ 0.10 = 12,619 + 500 = \$13,119 Annual cost = 13,119 + 4,000 = \$17,119 PW of challenger = 17,119 $\times$ (P/A, 10%, 4) $\times$ (P/F, 10%, 2) = 17,119 $\times$ 3.1699 $\times$ 0.8264 = \$44,840

PW_2 = 28,826 + 44,840 = \$73,666

Option 3 — Replace at year 3 (defender 3 years, challenger 3 years): PW of defender O&M: 12,000/(1.10) + 14,000/$(1.10)^2$ + 17,000/$(1.10)^3$ = 10,909 + 11,570 + 12,772 = \$35,251 Defender cost = 8,000 + 35,251 = \$43,251

Challenger for 3 years: (A/P, 10%, 3) = [0.10 $\times$ 1.3310] / [0.3310] = 0.40211 Partial CR = (45,000 − 5,000) $\times$ 0.40211 + 500 = 16,084 + 500 = \$16,584 Annual cost = 16,584 + 4,000 = \$20,584 (P/A, 10%, 3) = 2.4869; (P/F, 10%, 3) = 0.7513 PW of challenger = 20,584 $\times$ 2.4869 $\times$ 0.7513 = \$38,468

PW_3 = 43,251 + 38,468 = \$81,719

Replace now (Option 1) with PW = \$59,587 is the most economical choice.

Problem 19.10.6

Given: A fleet of 10 delivery trucks (defender) has a combined market value of \$80,000 and EUAC of \$95,000/year for 2 more years of service. A fleet of new electric trucks (challenger) costs \$450,000 with EUAC of \$72,000/year over 8 years. The interest rate is 8%.

Find: (a) Whether to replace now, and (b) the present worth of savings over the challenger's 8-year life.

Solution:

(a) Defender EUAC: \$95,000/year Challenger EUAC: \$72,000/year

Since challenger EUAC (\$72,000) < defender EUAC (\$95,000), replace now.

(b) Annual savings = 95,000 − 72,000 = \$23,000/year PW of savings over 8 years = 23,000 $\times$ (P/A, 8%, 8) (P/A, 8%, 8) = [$(1.08)^8$ − 1] / [0.08 $\times$ $(1.08)^8$] = [1.8509 − 1] / [0.08 $\times$ 1.8509] = 0.8509 / 0.14807 = 5.7466

PW of savings = 23,000 $\times$ 5.7466 = \$132,172

Note: This is a simplified comparison. A more rigorous analysis would account for the defender's 2-year remaining life and what replaces it after year 2.

Chapter 19 — Section 19.11: Inflation and Price Changes

Practice problems covering inflation-adjusted analysis using constant-dollar and actual-dollar methods, and differential escalation of specific commodities. Problems range from direct rate conversion to multi-step present worth calculations.

Problem 19.11.1

Given: A utility budgets \$80,000/year (in today's dollars) for transformer oil replacement. General inflation is 3.5% and the market MARR is 11%. The planning horizon is 10 years.

Find: (a) The real interest rate, and (b) the present worth of maintenance using the constant-dollar approach.

Solution:

(a) Real rate: $i_r = (i_f - f) / (1 + f) = (0.11 - 0.035) / (1.035) = 0.075 / 1.035 = 7.246\%$

(b) Constant-dollar approach: $(P/A, 7.246\%, 10) = [(1.07246)^{10} - 1] / [0.07246 \times (1.07246)^{10}]$ $(1.07246)^{10} = 2.0108 = [2.0108 - 1] / [0.07246 \times 2.0108] = 1.0108 / 0.14572 = 6.936$

$PW = 80{,}000 \times 6.936 = \$554{,}880$

Problem 19.11.2

Given: An industrial facility has annual energy costs of \$200,000 in today's dollars. Inflation is 2.5%, and the market rate is 9%. The planning period is 15 years.

Find: The present worth using (a) the constant-dollar approach and (b) the actual-dollar approach. Verify both methods agree.

Solution:

(a) Constant-dollar approach: $i_r = (0.09 - 0.025) / (1.025) = 0.065 / 1.025 = 6.341\%$

$(P/A, 6.341\%, 15) = [(1.06341)^{15} - 1] / [0.06341 \times (1.06341)^{15}]$ $(1.06341)^{15} = 2.5050 = [2.5050 - 1] / [0.06341 \times 2.5050] = 1.5050 / 0.15884 = 9.475$

$PW_{constant} = 200{,}000 \times 9.475 = \$1{,}895{,}000$

(b) Actual-dollar approach: Each year's cost in actual dollars: $A_t = 200{,}000 \times (1.025)^t$ PW = Σ $[200{,}000 \times (1.025)^t / (1.09)^t]$ for t = 1 to 15 = $200{,}000 \times \Sigma\,[(1.025/1.09)^t] = 200{,}000 \times \Sigma\,[(0.94037)^t]$

Geometric series: $S = 0.94037 \times [1 - (0.94037)^{15}] / [1 - 0.94037]$ $(0.94037)^{15} = 0.3993 = 0.94037 \times [1 - 0.3993] / 0.05963 = 0.94037 \times 0.6007 / 0.05963 = 0.94037 \times 10.074 = 9.475$

$PW_{actual} = 200{,}000 \times 9.475 = \$1{,}895{,}000$ ✓

Both methods yield the same present worth, confirming consistency.

Problem 19.11.3

Given: Aluminum conductor prices are escalating at 6%/year while general inflation is 2.5%/year. A transmission project requires \$350,000 of aluminum in year 1. The market interest rate is 10% and the planning horizon is 8 years.

Find: (a) The differential escalation rate, and (b) the present worth of aluminum purchases over 8 years.

Solution:

(a) Differential escalation rate: $e_d = 6\% - 2.5\% = 3.5\%$/year above general inflation

(b) Geometric gradient with g = 6% (aluminum escalation) and i = 10%: $P = A_1 \times [1 - (1 + g)^n(1 + i)^{-n}] / (i - g) = 350{,}000 \times [1 - (1.06)^8(1.10)^{-8}] / (0.10 - 0.06)$

$(1.06)^8 = 1.5938$; $(1.10)^8 = 2.1436$

$= 350{,}000 \times [1 - 1.5938/2.1436] / 0.04 = 350{,}000 \times [1 - 0.7435] / 0.04 = 350{,}000 \times 0.2565 / 0.04 = 350{,}000 \times 6.4125 = \$2{,}244{,}375$

Problem 19.11.4

Given: A 20-year power purchase agreement specifies \$500,000/year in today's dollars for electricity, but the contract escalates at the general inflation rate of 3%/year. Simultaneously, the utility's internal labor costs of \$100,000/year escalate at 5%/year (differential escalation). The market discount rate is 11%.

Find: The combined present worth of electricity and labor costs over 20 years.

Solution:

PW of electricity (geometric gradient, g = 3%, i = 11%): $P_{elec} = 500{,}000 \times [1 - (1.03)^{20}(1.11)^{-20}] / (0.11 - 0.03)$ $(1.03)^{20} = 1.8061$; $(1.11)^{20} = 8.0623 = 500{,}000 \times [1 - 1.8061/8.0623] / 0.08 = 500{,}000 \times [1 - 0.2240] / 0.08 = 500{,}000 \times 0.7760 / 0.08 = 500{,}000 \times 9.700 = \$4{,}850{,}000$

PW of labor (geometric gradient, g = 5%, i = 11%): $P_{labor} = 100{,}000 \times [1 - (1.05)^{20}(1.11)^{-20}] / (0.11 - 0.05)$ $(1.05)^{20} = 2.6533 = 100{,}000 \times [1 - 2.6533/8.0623] / 0.06 = 100{,}000 \times [1 - 0.3292] / 0.06 = 100{,}000 \times 0.6708 / 0.06 = 100{,}000 \times 11.180 = \$1{,}118{,}000$

Combined PW = 4,850,000 + 1,118,000 = \$5,968,000

The differential escalation of labor (5% vs. 3% inflation) adds significant cost over 20 years — the labor PW is about 23% of the electricity PW despite being only 20% of the base-year cost.

Chapter 19 — Section 19.12: Capital Budgeting and Project Selection

Practice problems covering independent project selection under budget constraints and MARR determination. Problems range from portfolio optimization to MARR-based accept/reject decisions.

Problem 19.12.1

Given: A utility has $5,000,000 in capital for the coming year and six independent project proposals. Each project is indivisible (all or nothing). The MARR is 10%.

Project	Investment	NPV at 10%
A — Feeder upgrade	$1,500,000	$280,000
B — Substation rebuild	$2,200,000	$350,000
C — SCADA modernization	$800,000	$140,000
D — Capacitor installation	$600,000	$105,000
E — Underground conversion	$3,000,000	$420,000
F — Transformer replacement	$1,000,000	$160,000

Find: The portfolio that maximizes total NPV within the budget.

Solution:

Profitability index (PI = NPV/Investment): A = 280,000/1,500,000 = 0.187 B = 350,000/2,200,000 = 0.159 C = 140,000/800,000 = 0.175 D = 105,000/600,000 = 0.175 F = 160,000/1,000,000 = 0.160 E = 420,000/3,000,000 = 0.140

Rank by PI: A (0.187), C (0.175), D (0.175), F (0.160), B (0.159), E (0.140)

Select in order: A ($1,500,000) + C ($800,000) + D ($600,000) + F ($1,000,000) = $3,900,000, NPV = $685,000 Remaining budget: $1,100,000 — B needs $2,200,000 (no), E needs $3,000,000 (no). No more projects fit.

Check alternative combinations: A + B + C + D = $5,100,000 (exceeds budget) A + B + D = $4,300,000, NPV = $735,000; remaining $700,000 — C needs $800,000 (no), F needs $1,000,000 (no) A + B + F =

\$4,700,000, NPV = \$790,000; remaining \$300,000 — nothing fits B + C + D + F = \$4,600,000, NPV = \$755,000; remaining \$400,000 — nothing fits A + C + D + F = \$3,900,000, NPV = \$685,000 — already evaluated E + A = \$4,500,000, NPV = \$700,000; remaining \$500,000 — D fits! → E + A + D = \$5,100,000 (exceeds) E + C + D = \$4,400,000, NPV = \$665,000; remaining \$600,000 — D already included, F needs \$1,000,000 E + F = \$4,000,000, NPV = \$580,000; can add C + D = \$1,400,000? Total \$5,400,000 (exceeds) E + C = \$3,800,000, NPV = \$560,000 + D = \$4,400,000, NPV = \$665,000 + F = \$5,400,000 (exceeds)

Best found: A + B + F at \$4,700,000 with total NPV = \$790,000

Verify: A (\$280,000) + B (\$350,000) + F (\$160,000) = \$790,000. Investment = \$4,700,000 $\leq$ \$5,000,000 ✓

Select projects A, B, and F for a total NPV of \$790,000 within the \$5,000,000 budget.

Problem 19.12.2

Given: A company evaluates four independent projects. Equity capital costs 14%, debt capital costs 8%, the debt-to-equity ratio is 60/40, and a 4% risk premium applies for each project.

Project	Investment	Annual Savings	Life (years)
W	\$200,000	\$42,000	10
X	\$350,000	\$68,000	10
Y	\$150,000	\$35,000	10
Z	\$500,000	\$90,000	10

Find: (a) The MARR, and (b) which projects are acceptable.

Solution:

(a) WACC = $(\text{weight}_{\text{debt}} \times \text{cost}_{\text{debt}}) + (\text{weight}_{\text{equity}} \times \text{cost}_{\text{equity}}) = (0.60 \times 0.08) + (0.40 \times 0.14) = 0.048 + 0.056 = 0.104 = 10.4\%$ MARR = WACC + risk premium = 10.4% + 4% = 14.4%

(b) Compute NPV at MARR = 14.4% (use 14% for factor approximation): $(P/A, 14\%, 10) = [(1.14)^{10} - 1] / [0.14 \times (1.14)^{10}]$ $(1.14)^{10} = 3.7072 = [3.7072 - 1] / [0.14 \times 3.7072] = 2.7072 / 0.51901 = 5.2163$

More precisely at 14.4%: $(1.144)^{10} \approx 3.8416$ $(P/A, 14.4\%, 10) = [3.8416 - 1] / [0.144 \times 3.8416] = 2.8416 / 0.55319 = 5.1366$

$NPV_W = -200{,}000 + 42{,}000 \times 5.1366 = -200{,}000 + 215{,}737 = +\$15{,}737$ ✓ $NPV_X = -350{,}000 + 68{,}000 \times 5.1366 = -350{,}000 + 349{,}289 = -\$711$ × $NPV_Y = -150{,}000 + 35{,}000 \times 5.1366 = -150{,}000 + 179{,}781 = +\$29{,}781$ ✓ $NPV_Z = -500{,}000 + 90{,}000 \times 5.1366 = -500{,}000 + 462{,}294 = -\$37{,}706$ ×

Projects W and Y are acceptable (positive NPV at the 14.4% MARR). Projects X and Z do not meet the hurdle rate.

Problem 19.12.3

Given: A utility has $2,000,000 in capital and four independent projects:

Project	Investment	NPV
A	$700,000	$95,000
B	$500,000	$80,000
C	$900,000	$130,000
D	$1,100,000	$145,000

Find: The optimal portfolio using enumeration of all feasible combinations.

Solution:

Feasible combinations within $2,000,000 budget:

Combination	Investment	Total NPV
A only	$700,000	$95,000
B only	$500,000	$80,000
C only	$900,000	$130,000
D only	$1,100,000	$145,000
A + B	$1,200,000	$175,000
A + C	$1,600,000	$225,000
A + D	$1,800,000	$240,000
B + C	$1,400,000	$210,000
B + D	$1,600,000	$225,000
C + D	$2,000,000	$275,000
A + B + C	$2,100,000	Exceeds budget
A + B + D	$2,300,000	Exceeds budget
B + C + D	$2,500,000	Exceeds budget

Select projects C and D for a total investment of $2,000,000 and maximum NPV of $275,000.

Note: The PI ranking (C = 0.144, A = 0.136, D = 0.132, B = 0.160) would suggest starting with B, but enumeration reveals that C + D is optimal — demonstrating that PI ranking is a heuristic that does not always find the global optimum.

Problem 19.12.4

Given: A solar developer has $8,000,000 and evaluates five sites. Each site is independent (can build on any subset).

Site	Investment	Annual Revenue	Life	NPV at 9%
1	$2,000,000	$310,000	25	$1,043,000
2	$3,500,000	$520,000	25	$1,608,000
3	$1,500,000	$200,000	25	$466,000
4	$2,800,000	$400,000	25	$1,130,000
5	$4,000,000	$580,000	25	$1,696,000

Find: The portfolio that maximizes NPV within the $8,000,000 budget.

Solution:

PI ranking: 5 (0.424), 1 (0.522), 2 (0.459), 4 (0.404), 3 (0.311) Reranked: 1 (0.522), 2 (0.459), 5 (0.424), 4 (0.404), 3 (0.311)

By PI: 1 ($2M) + 2 ($3.5M) = $5.5M (NPV $2,651,000); remaining $2.5M → 3 ($1.5M) fits → total $7M, NPV $3,117,000; remaining $1M — nothing fits.

Check: 1 + 2 + 3 = $7.0M, NPV = $3,117,000 1 + 2 + 4 = $8.3M (exceeds) 1 + 4 + 3 = $6.3M, NPV = $2,639,000 2 + 5 = $7.5M, NPV = $3,304,000; remaining $500K — nothing fits 1 + 5 = $6.0M, NPV = $2,739,000; remaining $2M → 3 fits: 1 + 5 + 3 = $7.5M, NPV = $3,205,000 1 + 4 = $4.8M; + 3 = $6.3M; + 2 would be $8.3M (no). Remaining from 1+4+3 = $1.7M — nothing 2 + 4 = $6.3M, NPV = $2,738,000; + 3 = $7.8M, NPV = $3,204,000 4 + 5 = $6.8M, NPV = $2,826,000; + 3 = $8.3M (exceeds) 1 + 2 + 3 = $7.0M, NPV = $3,117,000 2 + 5 = $7.5M, NPV = $3,304,000

Select Sites 2 and 5 for a total investment of $7,500,000 and maximum NPV of $3,304,000 within the $8,000,000 budget.

Chapter 19 — Section 19.13: Breakeven and Sensitivity Analysis

Practice problems covering breakeven analysis, single-parameter sensitivity, and scenario analysis. Problems range from direct breakeven calculations to multi-scenario NPV evaluations.

Problem 19.13.1

Given: A plant considers replacing a standard 50 HP pump motor (efficiency 89%, cost $4,500) with a premium efficiency motor (efficiency 94%, cost $7,200). The motor operates at 75% average load. Electricity costs $0.12/kWh, i = 9%, and both motors have a 12-year life with no salvage value.

Find: The breakeven operating hours per year.

Solution:

Power input (standard): $P_1 = 50 \times 0.746 \times 0.75 / 0.89 = 27.975 / 0.89 = 31.43$ kW Power input (premium): $P_2 = 50 \times 0.746 \times 0.75 / 0.94 = 27.975 / 0.94 = 29.76$ kW Power savings: $\Delta P = 31.43 - 29.76 = 1.67$ kW

Annual energy cost savings for H hours: $\Delta E = 1.67 \times H \times \$0.12 = \$0.2004 \times H$

Capital recovery of extra cost: $(A/P, 9\%, 12) = [0.09 \times (1.09)^{12}] / [(1.09)^{12} - 1] = [0.09 \times 2.8127] / [1.8127] = 0.25314 / 1.8127 = 0.13965$ $\Delta CR = (7{,}200 - 4{,}500) \times 0.13965 = 2{,}700 \times 0.13965 = \$377.06$/year

Breakeven: $0.2004 \times H = 377.06$ $H = 377.06 / 0.2004 = 1{,}881$ hours/year

If the motor runs more than 1,881 hours/year (about 7.2 hours/day on weekdays), the premium motor is more economical.

Problem 19.13.2

Given: A utility considers installing automated switching on a feeder. The system costs $250,000 and saves $C per year in reduced outage costs over 15 years with no salvage value. The MARR is 8%.

Find: The breakeven annual savings C that yields NPV = 0.

Solution:

NPV = 0: $0 = -250{,}000 + C \times (P/A, 8\%, 15)$

$(P/A, 8\%, 15) = [(1.08)^{15} - 1] / [0.08 \times (1.08)^{15}]$ $(1.08)^{15} = 3.1722 = [3.1722 - 1] / [0.08 \times 3.1722] = 2.1722 / 0.25378 = 8.5595$

$C = 250{,}000 / 8.5595 = \$29{,}207$/year

If the automated switching reduces outage costs by more than $29,207/year, the investment is justified at the 8% MARR.

Problem 19.13.3

Given: A $2,000,000 battery storage project has expected annual revenue of $340,000 over 12 years with no salvage value. At MARR = 10%.

Find: (a) The base-case NPV, and (b) the NPV at ±15% and ±30% variation in annual revenue.

Solution:

$(P/A, 10\%, 12) = [(1.10)^{12} - 1] / [0.10 \times (1.10)^{12}]$ $(1.10)^{12} = 3.1384 = [3.1384 - 1] / [0.10 \times 3.1384] = 2.1384 / 0.31384 = 6.8137$

(a) Base NPV $= -2{,}000{,}000 + 340{,}000 \times 6.8137 = -2{,}000{,}000 + 2{,}316{,}658 = \$316{,}658$

Revenue Variation	Annual Revenue	NPV
−30%	$238,000	$-2{,}000{,}000 + 238{,}000 \times 6.8137 = -\$378{,}338$
−15%	$289,000	$-2{,}000{,}000 + 289{,}000 \times 6.8137 = -\$30{,}840$
Base	$340,000	$316,658
+15%	$391,000	$-2{,}000{,}000 + 391{,}000 \times 6.8137 = \$664{,}156$
+30%	$442,000	$-2{,}000{,}000 + 442{,}000 \times 6.8137 = \$1{,}011{,}654$

(b)

The project becomes unviable if revenue drops by more than about 14% below the expected value. At −15%, the NPV is barely negative (−$30,840), indicating high sensitivity to revenue estimates.

Problem 19.13.4

Given: A $1,500,000 solar carport installation at MARR = 7% over 20 years, no salvage. Base case: annual savings = $180,000.

Find: (a) The base NPV, (b) the breakeven discount rate (IRR), and (c) the breakeven first cost (at base savings and MARR).

Solution:

(a) (P/A, 7%, 20) = [$(1.07)^{20}$ − 1] / [0.07 × $(1.07)^{20}$] $(1.07)^{20}$ = 3.8697 = [3.8697 − 1] / [0.07 × 3.8697] = 2.8697 / 0.27088 = 10.594

Base NPV = −1,500,000 + 180,000 × 10.594 = −1,500,000 + 1,906,920 = $406,920

(b) Breakeven discount rate (IRR): (P/A, i*, 20) = 1,500,000 / 180,000 = 8.3333

Try i = 10%: (P/A, 10%, 20) = [$(1.10)^{20}$ − 1] / [0.10 × $(1.10)^{20}$] = [6.7275 − 1] / [0.67275] = 8.5136 Try i = 11%: $(1.11)^{20}$ = 8.0623; (P/A) = 7.0623 / 0.88684 = 7.9633

Interpolate: i* = 10% + 1% × (8.5136 − 8.3333) / (8.5136 − 7.9633) = 10% + 1% × 0.1803/0.5503 = 10% + 0.33% = 10.33%

The project remains viable for any MARR below 10.33%.

(c) Breakeven first cost: P_{max} = 180,000 × (P/A, 7%, 20) = 180,000 × 10.594 = $1,906,920

The first cost can increase to $1,906,920 (a 27% overrun) before the project becomes unviable.

Problem 19.13.5

Given: A wind farm ($5,000,000, 25-year life, no salvage) has three scenarios:

Parameter	Pessimistic	Most Likely	Optimistic
Capacity factor	22%	30%	36%
Annual revenue	$380,000	$520,000	$630,000
Annual O&M	$95,000	$75,000	$65,000

The MARR is 8%.

Find: The NPV for each scenario.

Solution:

(P/A, 8%, 25) = [$(1.08)^{25}$ − 1] / [0.08 × $(1.08)^{25}$] $(1.08)^{25}$ = 6.8485 = [6.8485 − 1] / [0.08 × 6.8485] = 5.8485 / 0.54788 = 10.675

Net annual cash flow for each scenario:

Pessimistic: A = 380,000 − 95,000 = $285,000 NPV = −5,000,000 + 285,000 × 10.675 = −5,000,000 + 3,042,375 = −$1,957,625

Most likely: A = 520,000 − 75,000 = $445,000 NPV = −5,000,000 + 445,000 × 10.675 = −5,000,000 + 4,750,375 = −$249,625

Optimistic: A = 630,000 − 65,000 = $565,000 NPV = −5,000,000 + 565,000 × 10.675 = −5,000,000 + 6,031,375 = $1,031,375

The project is only viable in the optimistic scenario. Even the most-likely case produces a negative NPV, suggesting the project carries significant risk at the 8% MARR. A lower MARR or additional revenue streams (such as renewable energy credits) would be needed to justify the investment.

Problem 19.13.6

Given: A factory evaluates a $400,000 power quality improvement system. Base estimates: annual savings = $78,000, life = 10 years, MARR = 9%, no salvage. The three most uncertain parameters are savings, life, and MARR.

Find: The NPV sensitivity to ±20% changes in each parameter (one at a time).

Solution:

Base case: (P/A, 9%, 10) = 6.4177 NPV_{base} = −400,000 + 78,000 × 6.4177 = −400,000 + 500,581 = $100,581

Savings sensitivity (±20%): At $62,400 (−20%): NPV = −400,000 + 62,400 × 6.4177 = −400,000 + 400,464 = $464 At $93,600 (+20%): NPV = −400,000 + 93,600 × 6.4177 = −400,000 + 600,698 = $200,698 Range: $200,234

Life sensitivity (±20%): At 8 years: (P/A, 9%, 8) = 5.5348 NPV = −400,000 + 78,000 × 5.5348 = −400,000 + 431,714 = $31,714 At 12 years: (P/A, 9%, 12) = 7.1607 NPV = −400,000 + 78,000 × 7.1607 = −400,000 + 558,535 = $158,535 Range: $126,821

MARR sensitivity (±20%): At 7.2%: (P/A, 7.2%, 10) = $[(1.072)^{10} - 1] / [0.072 \times (1.072)^{10}]$ $(1.072)^{10}$ = 2.0042; = 1.0042 / 0.14430 = 6.960 NPV = −400,000 + 78,000 × 6.960 = −400,000 + 542,880 = $142,880

At 10.8%: (P/A, 10.8%, 10) = $[(1.108)^{10} - 1] / [0.108 \times (1.108)^{10}]$ $(1.108)^{10}$ = 2.7889; = 1.7889 / 0.30120 = 5.939 NPV = −400,000 + 78,000 × 5.939 = −400,000 + 463,242 = $63,242 Range: $79,638

Sensitivity ranking: 1. Annual savings — most sensitive (range $200,234) 2. Project life — moderate sensitivity (range $126,821) 3. MARR — least sensitive (range $79,638)

The annual savings estimate deserves the most careful validation, as it has the greatest impact on the project's economic outcome.

Chapter 19 — Section 19.14: Life Cycle Cost Analysis

Practice problems covering life cycle cost component analysis and levelized cost of energy (LCOE) calculations. Problems use realistic infrastructure and generation scenarios.

Problem 19.14.1

Given: A wastewater treatment plant compares two blower systems over 20 years at i = 7%.

Cost Category	System A (Centrifugal)	System B (Rotary Screw)
Purchase	$250,000	$180,000
Installation	$35,000	$25,000
Annual energy	$48,000/year	$58,000/year
Annual maintenance	$6,000/year	$9,000/year
Overhaul	$30,000 at year 10	$20,000 at years 7, 14
Disposal	$5,000 at year 20	$4,000 at year 20

Find: The life cycle cost of each system and which is more economical.

Solution:

$(P/A, 7\%, 20) = [(1.07)^{20} - 1] / [0.07 \times (1.07)^{20}]$ $(1.07)^{20} = 3.8697 = [3.8697 - 1] / [0.07 \times 3.8697] = 2.8697 / 0.27088 = 10.594$

$(P/F, 7\%, 7) = 1/(1.07)^7 = 1/1.6058 = 0.6228$ $(P/F, 7\%, 10) = 1/(1.07)^{10} = 1/1.9672 = 0.5083$ $(P/F, 7\%, 14) = 1/(1.07)^{14} = 1/2.5785 = 0.3878$ $(P/F, 7\%, 20) = 1/3.8697 = 0.2584$

System A: $LCC_A = 250{,}000 + 35{,}000 + (48{,}000 + 6{,}000) \times 10.594 + 30{,}000 \times 0.5083 + 5{,}000 \times 0.2584 = 285{,}000 + 54{,}000 \times 10.594 + 15{,}249 + 1{,}292 = 285{,}000 + 572{,}076 + 15{,}249 + 1{,}292 = \$873{,}617$

System B: $LCC_B = 180{,}000 + 25{,}000 + (58{,}000 + 9{,}000) \times 10.594 + 20{,}000 \times (0.6228 + 0.3878) + 4{,}000 \times 0.2584 = 205{,}000 + 67{,}000 \times 10.594 + 20{,}000 \times 1.0106 + 1{,}034 = 205{,}000 + 709{,}798 + 20{,}212 + 1{,}034 = \$936{,}044$

System A has a lower LCC by \$936,044 − \$873,617 = \$62,427. Despite the higher purchase price, System A's lower energy costs (\$10,000/year savings) accumulate to a significant advantage over 20 years.

Problem 19.14.2

Given: A 5 MW wind turbine costs \$6,500,000 to install. Annual O&M is \$100,000 escalating at 2%/year. Annual energy production is 15,000 MWh with 1%/year degradation. The project life is 20 years and the discount rate is 8%.

Find: The LCOE in \$/MWh.

Solution:

PW of costs: PW of capital: \$6,500,000

PW of O&M (geometric gradient, g = 2%, i = 8%): $PW_{O\&M} = 100{,}000 \times [1 - (1.02)^{20}(1.08)^{-20}] / (0.08 - 0.02)$ $(1.02)^{20} = 1.4859$; $(1.08)^{20} = 4.6610$ $= 100{,}000 \times [1 - 1.4859/4.6610] / 0.06 = 100{,}000 \times [1 - 0.3188] / 0.06 = 100{,}000 \times 0.6812 / 0.06 = 100{,}000 \times 11.353 = \$1{,}135{,}300$

PW of total costs = 6,500,000 + 1,135,300 = \$7,635,300

PW of energy production (geometric gradient, g = −1%, i = 8%): $PW_{energy} = 15{,}000 \times [1 - (0.99)^{20}(1.08)^{-20}] / (0.08 - (-0.01))$ $(0.99)^{20} = 0.8179$ $= 15{,}000 \times [1 - 0.8179/4.6610] / 0.09 = 15{,}000 \times [1 - 0.1755] / 0.09 = 15{,}000 \times 0.8245 / 0.09 = 15{,}000 \times 9.161 = 137{,}415$ MWh (present-worth equivalent)

LCOE = 7,635,300 / 137,415 = \$55.56/MWh

Problem 19.14.3

Given: A hospital compares two emergency generator systems over 25 years at i = 5%.

Category	Diesel Generator	Natural Gas Generator
Purchase + install	\$400,000	\$520,000
Annual fuel	\$18,000	\$12,000
Annual maintenance	\$8,000	\$5,000
Major overhaul	\$45,000 at years 8, 16	\$35,000 at years 10, 20
Emissions compliance	\$5,000/year	\$0/year
Disposal	\$10,000 at year 25	\$8,000 at year 25

Find: The LCC of each system and the preferred choice.

Solution:

(P/A, 5%, 25) = $[(1.05)^{25} - 1] / [0.05 \times (1.05)^{25}]$ $(1.05)^{25} = 3.3864$ = $[3.3864 - 1] / [0.05 \times 3.3864]$ = 2.3864 / 0.16932 = 14.094

(P/F, 5%, 8) = $1/(1.05)^8$ = 1/1.4775 = 0.6768 (P/F, 5%, 10) = $1/(1.05)^{10}$ = 1/1.6289 = 0.6139 (P/F, 5%, 16) = $1/(1.05)^{16}$ = 1/2.1829 = 0.4581 (P/F, 5%, 20) = $1/(1.05)^{20}$ = 1/2.6533 = 0.3769 (P/F, 5%, 25) = 1/3.3864 = 0.2953

Diesel: LCC = 400,000 + (18,000 + 8,000 + 5,000) × 14.094 + 45,000 × (0.6768 + 0.4581) + 10,000 × 0.2953 = 400,000 + 31,000 × 14.094 + 45,000 × 1.1349 + 2,953 = 400,000 + 436,914 + 51,071 + 2,953 = $890,938

Natural Gas: LCC = 520,000 + (12,000 + 5,000) × 14.094 + 35,000 × (0.6139 + 0.3769) + 8,000 × 0.2953 = 520,000 + 17,000 × 14.094 + 35,000 × 0.9908 + 2,362 = 520,000 + 239,598 + 34,678 + 2,362 = $796,638

The natural gas generator has a lower LCC by $890,938 − $796,638 = $94,300. Lower fuel costs, lower maintenance, and no emissions compliance costs more than offset the higher purchase price over 25 years.

Problem 19.14.4

Given: A 20 MW solar PV plant costs $18,000,000. Annual O&M is $200,000 escalating at 2.5%/year. Annual production is 35,000 MWh with 0.6%/year degradation. An inverter replacement of $1,500,000 occurs at year 12. The project life is 30 years and the discount rate is 6%.

Find: The LCOE in $/MWh.

Solution:

PW of costs: Capital: $18,000,000

PW of O&M (geometric gradient, g = 2.5%, i = 6%): $PW_{O\&M} = 200{,}000 \times [1 - (1.025)^{30}(1.06)^{-30}] / (0.06 - 0.025)$ $(1.025)^{30} = 2.0976$; $(1.06)^{30} = 5.7435$ = 200,000 × [1 − 2.0976/5.7435] / 0.035 = 200,000 × [1 − 0.3652] / 0.035 = 200,000 × 0.6348 / 0.035 = 200,000 × 18.137 = $3,627,400

Inverter replacement: 1,500,000 × (P/F, 6%, 12) = 1,500,000 × $1/(1.06)^{12}$ = 1,500,000 × 1/2.0122 = 1,500,000 × 0.4970 = $745,500

PW of total costs = 18,000,000 + 3,627,400 + 745,500 = $22,372,900

PW of energy production (geometric gradient, g = −0.6%, i = 6%): $PW_{energy} = 35{,}000 \times [1 - (0.994)^{30}(1.06)^{-30}] / (0.06 - (-0.006))$ $(0.994)^{30} = 0.8348$ = 35,000 × [1 − 0.8348/5.7435] / 0.066 = 35,000 × [1 − 0.1453] / 0.066 = 35,000 × 0.8547 / 0.066 = 35,000 × 12.950 = 453,250 MWh (present-worth equivalent)

LCOE = 22,372,900 / 453,250 = $49.36/MWh

This LCOE of approximately $49/MWh reflects a competitive utility-scale solar installation, with the inverter replacement at year 12 adding about $1.64/MWh to the levelized cost.

Appendix A – Section A.1: Imaginary Numbers

Practice problems covering the imaginary unit j, simplification of square roots of negative numbers, and the cyclic powers of j.

Problem A.1.1

Given: The expression $\sqrt{(-144)}$.

Find: Simplify and express as an imaginary number.

Solution: $\sqrt{(-144)} = \sqrt{(144 \times (-1))} = \sqrt{144} \times \sqrt{(-1)} = 12 \times j = j12$.

The result is j12.

Problem A.1.2

Given: The expression $\sqrt{(-0.25)}$.

Find: Simplify and express as an imaginary number.

Solution: $\sqrt{(-0.25)} = \sqrt{(0.25 \times (-1))} = \sqrt{0.25} \times \sqrt{(-1)} = 0.5 \times j = j0.5$.

The result is j0.5.

Problem A.1.3

Given: The expression $j^6 + j^{13} - j^{20}$.

Find: Simplify to a single value.

Solution: j^6: 6 mod 4 = 2, so $j^6 = j^2 = -1$. j^{13}: 13 mod 4 = 1, so $j^{13} = j^1 = j$. j^{20}: 20 mod 4 = 0, so $j^{20} = j^0 = 1$.

$j^6 + j^{13} - j^{20} = -1 + j - 1 = -2 + j$.

The result is -2 + j.

Problem A.1.4

Given: The expression $j^{-2} + j^{-5} + j^{-8}$.

Find: Simplify to a single value.

Solution: j^{-2}: -2 mod 4 = 2, so $j^{-2} = j^2 = -1$. j^{-5}: -5 mod 4 = 3, so $j^{-5} = j^3 = -j$. j^{-8}: -8 mod 4 = 0, so $j^{-8} = j^0 = 1$.

$j^{-2} + j^{-5} + j^{-8} = -1 + (-j) + 1 = -j$.

The result is -j.

Problem A.1.5

Given: The expression $(j^3)^4 \times (j^2)^5$.

Find: Simplify to a single value.

Solution: $(j^3)^4 = j^{12}$: 12 mod 4 = 0, so $j^{12} = 1$. $(j^2)^5 = j^{10}$: 10 mod 4 = 2, so $j^{10} = j^2 = -1$.

$(j^3)^4 \times (j^2)^5 = 1 \times (-1) = -1$.

The result is -1.

Problem A.1.6

Given: The expression $\sqrt{(-18)} \times \sqrt{(-2)}$.

Find: Simplify to a real number.

Solution: $\sqrt{(-18)} = j\sqrt{18} = j \times 3\sqrt{2}$. $\sqrt{(-2)} = j\sqrt{2}$.

$\sqrt{(-18)} \times \sqrt{(-2)} = (j \times 3\sqrt{2})(j \times \sqrt{2}) = j^2 \times 3 \times 2 = (-1) \times 6 = -6$.

The result is -6.

Appendix A – Section A.2: Complex Numbers

Practice problems covering rectangular form, complex arithmetic, and the complex conjugate.

Problem A.2.1

Given: An impedance $Z = 33 + j56\ \Omega$.

Find: The resistance, reactance, magnitude, and angle.

Solution: Resistance: $R = \text{Re}\{Z\} = 33\ \Omega$. Reactance: $X = \text{Im}\{Z\} = 56\ \Omega$ (inductive, since positive).

Magnitude: $|Z| = \sqrt{(33^2 + 56^2)} = \sqrt{(1089 + 3136)} = \sqrt{4225} = 65\ \Omega$. Angle: $\theta = \arctan(56/33) = \arctan(1.6970) = 59.49°$.

$Z = 65\angle 59.49°\ \Omega$. The resistance is 33 Ω and the reactance is 56 Ω.

Problem A.2.2

Given: $Z_1 = 8 - j6$ and $Z_2 = -3 + j4$.

Find: $Z_1 + Z_2$, $Z_1 - Z_2$, and $Z_1 \times Z_2$.

Solution: Addition: $Z_1 + Z_2 = (8 + (-3)) + j(-6 + 4) = 5 - j2$.

Subtraction: $Z_1 - Z_2 = (8 - (-3)) + j(-6 - 4) = 11 - j10$.

Multiplication: $Z_1 \times Z_2 = (8)(-3) - (-6)(4) + j((8)(4) + (-6)(-3)) = -24 + 24 + j(32 + 18) = 0 + j50 = j50$.

$Z_1 + Z_2 = 5 - j2$, $Z_1 - Z_2 = 11 - j10$, $Z_1 \times Z_2 = j50$.

Problem A.2.3

Given: $Z_1 = 10 + j5\ \Omega$ and $Z_2 = 4 - j3\ \Omega$.

Find: Z_1 / Z_2 in rectangular form.

Solution: $Z_1 / Z_2 = (10 + j5) / (4 - j3)$. Multiply numerator and denominator by the conjugate of Z_2:

Numerator: $(10 + j5)(4 + j3) = 40 + j30 + j20 + j^2 15 = 40 + j30 + j20 - 15 = 25 + j50$. Denominator: $4^2 + 3^2 = 16 + 9 = 25$.

$Z_1 / Z_2 = (25 + j50) / 25 = 1 + j2$.

$Z_1 / Z_2 = 1 + j2$.

Problem A.2.4

Given: $Z = 7 - j24$.

Find: The complex conjugate Z*, the product Z x Z*, and verify that $Z \times Z^* = |Z|^2$.

Solution: $Z^* = 7 + j24$.

$Z \times Z^* = (7 - j24)(7 + j24) = 49 + j168 - j168 - j^2 576 = 49 + 576 = 625$.

$|Z| = \sqrt{(7^2 + 24^2)} = \sqrt{(49 + 576)} = \sqrt{625} = 25$. $|Z|^2 = 625$.

$Z^* = 7 + j24$, $Z \times Z^* = 625$, and $|Z|^2 = 625$, confirming $Z \times Z^* = |Z|^2$.

Problem A.2.5

Given: Three impedances in series: $Z_1 = 10 + j0\ \Omega$ (resistor), $Z_2 = 0 + j15\ \Omega$ (inductor), $Z_3 = 0 - j8\ \Omega$ (capacitor).

Find: The total impedance Z_{total} in rectangular and polar form.

Solution: $Z_{total} = Z_1 + Z_2 + Z_3 = (10 + 0 + 0) + j(0 + 15 - 8) = 10 + j7\ \Omega$.

Magnitude: $|Z_{total}| = \sqrt{(10^2 + 7^2)} = \sqrt{(100 + 49)} = \sqrt{149} = 12.21\ \Omega$. Angle: $\theta = \arctan(7/10) = \arctan(0.7) = 34.99°$.

$Z_{total} = 10 + j7\ \Omega = 12.21\angle 34.99°\ \Omega$.

Problem A.2.6

Given: Two impedances in parallel: $Z_1 = 20 + j0\ \Omega$ and $Z_2 = 0 + j20\ \Omega$.

Find: The parallel combination $Z_p = (Z_1 \times Z_2) / (Z_1 + Z_2)$.

Solution: $Z_1 + Z_2 = 20 + j20\ \Omega$. $Z_1 \times Z_2 = (20)(j20) = j400$.

$Z_p = j400 / (20 + j20)$. Multiply by conjugate: $Z_p = j400(20 - j20) / (20^2 + 20^2) = (j8000 - j^2 8000) / 800 = (8000 + j8000) / 800 = 10 + j10\ \Omega$.

Verification in polar form: $|Z_p| = \sqrt{(100 + 100)} = \sqrt{200} = 14.14\ \Omega$, $\theta = 45°$.

$Z_p = 10 + j10\ \Omega = 14.14\angle 45°\ \Omega$.

Problem A.2.7

Given: The equation $Z^2 = -9 + j40$, where $Z = a + jb$ is a complex impedance.

Find: The value of Z.

Solution: Let $Z = a + jb$. Then $Z^2 = a^2 - b^2 + j2ab$. Setting real and imaginary parts equal: $a^2 - b^2 = -9$ … (1) $2ab = 40$, so $b = 20/a$ … (2)

Substitute (2) into (1): $a^2 - (20/a)^2 = -9$ $a^2 - 400/a^2 = -9$ $a^4 + 9a^2 - 400 = 0$

Let $u = a^2$: $u^2 + 9u - 400 = 0$. $u = (-9 + \sqrt{(81 + 1600)}) / 2 = (-9 + \sqrt{1681}) / 2 = (-9 + 41) / 2 = 16$. So $a^2 = 16$, $a = 4$ (taking positive root). $b = 20/4 = 5$.

Verification: $Z = 4 + j5$, $Z^2 = 16 - 25 + j40 = -9 + j40$.

$Z = 4 + j5$ (or $Z = -4 - j5$ for the negative root).

Appendix A – Section A.3: Polar and Exponential Forms

Practice problems covering polar form conversion, Euler's formula, and rectangular-polar interconversion.

Problem A.3.1

Given: $Z = -5 - j12$.

Find: Express Z in polar form (degrees).

Solution: Magnitude: $|Z| = \sqrt{((-5)^2 + (-12)^2)} = \sqrt{(25 + 144)} = \sqrt{169} = 13$.

Reference angle: arctan(12/5) = arctan(2.4) = 67.38°. Since both the real and imaginary parts are negative, Z lies in the third quadrant: $\theta = -(180° - 67.38°) = -112.62°$ (or equivalently 247.38°).

$Z = 13\angle -112.62°$.

Problem A.3.2

Given: $Z_1 = 8\angle 30°$ and $Z_2 = 4\angle -60°$.

Find: $Z_1 \times Z_2$ and Z_1 / Z_2 in polar form.

Solution: Multiplication: $Z_1 \times Z_2 = (8 \times 4)\angle(30° + (-60°)) = 32\angle -30°$.

Division: $Z_1 / Z_2 = (8/4)\angle(30° - (-60°)) = 2\angle 90°$.

$Z_1 \times Z_2 = 32\angle -30°$ and $Z_1 / Z_2 = 2\angle 90°$.

Problem A.3.3

Given: $Z = 50\angle -135°$.

Find: Express in exponential form and convert to rectangular form.

Solution: Exponential form: Convert angle to radians: -135° x (π/180°) = -3π/4 rad. $Z = 50 \times e^{-j3\pi/4}$.

Rectangular form: a = 50 x cos(-135°) = 50 x (-0.7071) = -35.36. b = 50 x sin(-135°) = 50 x (-0.7071) = -35.36.

Verification: $|Z| = \sqrt{(35.36^2 + 35.36^2)} = \sqrt{(1250 + 1250)} = \sqrt{2500} = 50$.

$Z = 50 \times e^{-j3\pi/4} = -35.36 - j35.36$.

Problem A.3.4

Given: $V_1 = 30\angle 20°$ V and $V_2 = 40\angle -70°$ V in series.

Find: The total voltage V_{total} in both rectangular and polar form.

Solution: Convert to rectangular: V_1 = 30 cos(20°) + j30 sin(20°) = 30(0.9397) + j30(0.3420) = 28.19 + j10.26 V. V_2 = 40 cos(-70°) + j40 sin(-70°) = 40(0.3420) + j40(-0.9397) = 13.68 - j37.59 V.

V_{total} = (28.19 + 13.68) + j(10.26 - 37.59) = 41.87 - j27.33 V.

Convert back to polar: $|V_{total}| = \sqrt{(41.87^2 + 27.33^2)} = \sqrt{(1753.1 + 746.9)} = \sqrt{2500} = 50.0$ V. θ = arctan(-27.33/41.87) = arctan(-0.6528) = -33.13°.

$V_{total} = 41.87 - j27.33 \text{ V} = 50.0\angle -33.13° \text{ V}$.

Problem A.3.5

Given: The complex number $Z = 6 \times e^{j\pi/6}$.

Find: Express in polar form (degrees) and rectangular form.

Solution: The exponential form is $Z = 6 \times e^{j\pi/6}$. Angle in degrees: π/6 rad = 30°. Polar form: $Z = 6\angle 30°$.

Rectangular form: a = 6 cos(30°) = 6 x 0.8660 = 5.196. b = 6 sin(30°) = 6 x 0.5 = 3.0.

$Z = 6\angle 30° = 5.196 + j3.0$.

Problem A.3.6

Given: An AC current has the phasor $I = 2.5\angle -45°$ A, and it flows through an impedance $Z = 100\angle 60°$ Ω.

Find: The voltage phasor V = I x Z in polar form, then convert to rectangular form.

Solution: $V = I \times Z = (2.5 \times 100)\angle(-45° + 60°) = 250\angle 15°$ V.

Rectangular form: a = 250 cos(15°) = 250 x 0.9659 = 241.5 V. b = 250 sin(15°) = 250 x 0.2588 = 64.70 V.

$V = 250\angle 15°$ V = $241.5 + j64.70$ V.

Problem A.3.7

Given: Three impedances: $Z_1 = 10\angle 0°\ \Omega$, $Z_2 = 20\angle 90°\ \Omega$, $Z_3 = 15\angle\text{-}90°\ \Omega$, all in series.

Find: The total impedance in rectangular and polar form.

Solution: Convert to rectangular: $Z_1 = 10 + j0\ \Omega$. $Z_2 = 0 + j20\ \Omega$. $Z_3 = 0 - j15\ \Omega$.

$Z_{total} = 10 + j(20 - 15) = 10 + j5\ \Omega$.

Convert to polar: $|Z_{total}| = \sqrt{(10^2 + 5^2)} = \sqrt{(100 + 25)} = \sqrt{125} = 11.18\ \Omega$. $\theta = \arctan(5/10) = \arctan(0.5) = 26.57°$.

$Z_{total} = 10 + j5\ \Omega = 11.18\angle 26.57°\ \Omega$.

Problem A.3.8

Given: Euler's identity states $e^{j\pi} + 1 = 0$.

Find: Use Euler's formula to verify this, and compute $e^{j2\pi}$.

Solution: Euler's formula: $e^{j\theta} = \cos(\theta) + j \sin(\theta)$.

For $\theta = \pi$: $e^{j\pi} = \cos(\pi) + j \sin(\pi) = -1 + j(0) = -1$. Therefore $e^{j\pi} + 1 = -1 + 1 = 0$. Verified.

For $\theta = 2\pi$: $e^{j2\pi} = \cos(2\pi) + j \sin(2\pi) = 1 + j(0) = 1$.

$e^{j\pi} = -1$, confirming Euler's identity. $e^{j2\pi} = 1$, confirming that a full rotation of 360° returns to the starting point.

Appendix A – Section A.4: Phasors

Practice problems covering sinusoidal representation, phasor notation, phasor arithmetic, and phasor diagrams.

Problem A.4.1

Given: A current waveform $i(t) = 14.14 \cos(2\pi \times 60t + 45°)$ A.

Find: The peak amplitude, RMS value, frequency, period, angular frequency, and phase angle.

Solution: Peak amplitude: $I_m = 14.14$ A. RMS value: $I_{rms} = I_m / \sqrt{2} = 14.14 / 1.414 = 10.0$ A. Angular frequency: $\omega = 2\pi \times 60 = 376.99$ rad/s. Frequency: $f = 60$ Hz. Period: $T = 1/f = 1/60 = 16.67$ ms. Phase angle: $\varphi = 45°$ (leading the reference cosine by 45°).

$I_m = 14.14$ A, $I_{rms} = 10.0$ A, $f = 60$ Hz, $T = 16.67$ ms, $\omega = 377.0$ rad/s, $\varphi = 45°$.

Problem A.4.2

Given: A voltage $v(t) = 339.4 \cos(314.16t - 90°)$ V.

Find: Express as an RMS phasor.

Solution: Peak voltage: $V_m = 339.4$ V. RMS voltage: $V_{rms} = 339.4 / \sqrt{2} = 339.4 / 1.414 = 240.0$ V. Phase angle: $\varphi = -90°$. Frequency: $\omega = 314.16$ rad/s, so $f = 314.16 / (2\pi) = 50$ Hz.

The phasor is $V = 240.0\angle{-90°}\ V_{rms}$ at 50 Hz.

Problem A.4.3

Given: A phasor $V = 120\angle 30°\ V_{rms}$ at a frequency of 60 Hz.

Find: The time-domain expression $v(t)$.

Solution: Peak voltage: $V_m = 120 \times \sqrt{2} = 120 \times 1.414 = 169.7$ V. Angular frequency: $\omega = 2\pi \times 60 = 376.99$ rad/s. Phase angle: $\varphi = 30°$.

$v(t) = 169.7 \cos(377.0t + 30°)$ V.

Problem A.4.4

Given: Three voltage phasors in series: $V_1 = 50\angle 0°$ V, $V_2 = 30\angle 120°$ V, $V_3 = 40\angle -90°$ V.

Find: The total voltage V_{total} in polar form.

Solution: Convert to rectangular: $V_1 = 50 \cos(0°) + j50 \sin(0°) = 50 + j0$ V. $V_2 = 30 \cos(120°) + j30 \sin(120°) = 30(-0.5) + j30(0.8660) = -15 + j25.98$ V. $V_3 = 40 \cos(-90°) + j40 \sin(-90°) = 0 - j40$ V.

$V_{total} = (50 - 15 + 0) + j(0 + 25.98 - 40) = 35 - j14.02$ V.

$|V_{total}| = \sqrt{(35^2 + 14.02^2)} = \sqrt{(1225 + 196.6)} = \sqrt{1421.6} = 37.71$ V. $\theta = \arctan(-14.02/35) = \arctan(-0.4006) = -21.83°$.

$V_{total} = 37.71\angle -21.83°$ V.

Problem A.4.5

Given: A series RL circuit with $R = 47\ \Omega$ and $X_L = 100\ \Omega$, driven by $V_s = 120\angle 0°\ V_{rms}$ at 60 Hz.

Find: The current phasor, the voltage across R, and the voltage across L.

Solution: $Z_{total} = R + jX_L = 47 + j100\ \Omega$. $|Z_{total}| = \sqrt{(47^2 + 100^2)} = \sqrt{(2209 + 10000)} = \sqrt{12209} = 110.5\ \Omega$. $\theta_Z = \arctan(100/47) = \arctan(2.1277) = 64.82°$.

Current: $I = V_s / Z_{total} = 120\angle 0° / 110.5\angle 64.82° = 1.086\angle -64.82°\ A_{rms}$.

$V_R = I \times R = 1.086\angle -64.82° \times 47 = 51.04\angle -64.82°\ V_{rms}$. $V_L = I \times jX_L = 1.086\angle -64.82° \times 100\angle 90° = 108.6\angle 25.18°\ V_{rms}$.

Verification: $V_R + V_L$ in rectangular: $V_R = 51.04 \cos(-64.82°) + j51.04 \sin(-64.82°) = 21.74 - j46.19$ V. $V_L = 108.6 \cos(25.18°) + j108.6 \sin(25.18°) = 98.26 + j46.23$ V. $V_{total} = (21.74 + 98.26) + j(-46.19 + 46.23) = 120.0 + j0.04 \approx 120\angle 0°$ V. Confirmed.

$I = 1.086\angle -64.82°\ A_{rms}$, $V_R = 51.04\angle -64.82°$ V, $V_L = 108.6\angle 25.18°$ V.

Problem A.4.6

Given: Two current sources feeding a node: $I_1 = 5\angle 0°$ A and $I_2 = 5\angle 180°$ A.

Find: The total current entering the node and describe what the phasor diagram shows.

Solution: Convert to rectangular: $I_1 = 5 + j0$ A. $I_2 = 5 \cos(180°) + j5 \sin(180°) = -5 + j0$ A.

$I_{total} = (5 - 5) + j(0 + 0) = 0 + j0 = 0$ A.

The two currents are exactly 180° out of phase (anti-phase) with equal magnitudes, so they completely cancel. On the phasor diagram, I_1 points to the right along the positive real axis and I_2 points to the left along the negative real axis, forming a straight line through the origin with zero resultant.

$I_{total} = 0$ A. The two equal and opposite phasors cancel completely.

Problem A.4.7

Given: A phasor diagram shows a current $I = 10\angle 0°$ A through a series RLC circuit where $V_R = 50\angle 0°$ V, $V_L = 80\angle 90°$ V, and $V_C = 30\angle\text{-}90°$ V.

Find: The total voltage, impedance, and describe whether the circuit is inductive or capacitive.

Solution: $V_{total} = V_R + V_L + V_C$. In rectangular form: $V_R = 50 + j0$ V. $V_L = 0 + j80$ V. $V_C = 0 - j30$ V.

$V_{total} = 50 + j(80 - 30) = 50 + j50$ V. $|V_{total}| = \sqrt{(50^2 + 50^2)} = \sqrt{5000} = 70.71$ V. $\theta = \arctan(50/50) = 45°$.

$Z = V_{total} / I = 70.71\angle 45° / 10\angle 0° = 7.071\angle 45°\ \Omega = 5 + j5\ \Omega$. The impedance angle is positive (45°), meaning voltage leads current. The circuit is net inductive since $X_L > X_C$.

$V_{total} = 70.71\angle 45°$ V. $Z = 7.071\angle 45°\ \Omega$. The circuit is inductive (voltage leads current by 45°).

Problem A.4.8

Given: An AC voltage source $v(t) = 100 \cos(1000t)$ V drives a 50 μF capacitor.

Find: The current phasor and the time-domain expression for $i(t)$.

Solution: Phasor: $V = 100\angle 0°$ V (peak). Angular frequency: $\omega = 1000$ rad/s.

$Z_C = 1/(j\omega C) = 1/(j \times 1000 \times 50 \times 10^{-6}) = 1/(j0.05) = -j20\ \Omega = 20\angle\text{-}90°\ \Omega$.

$I = V / Z_C = 100\angle 0° / 20\angle\text{-}90° = 5\angle 90°$ A.

Time-domain: $i(t) = 5 \cos(1000t + 90°)$ A.

This confirms that in a capacitor, current leads voltage by 90°.

$I = 5\angle 90°$ A (peak), $i(t) = 5 \cos(1000t + 90°)$ A.

Appendix A – Section A.5: Applications in Circuit Analysis

Practice problems covering impedance and admittance, voltage and current phasors, and power in phasor form.

Problem A.5.1

Given: A 220 Ω resistor, a 33 mH inductor, and a 22 nF capacitor are connected in series at a frequency of 10 kHz.

Find: The impedance of each element, the total impedance, and the total admittance.

Solution: $\omega = 2\pi \times 10{,}000 = 62{,}831.9$ rad/s.

$Z_R = 220\ \Omega$. $Z_L = j\omega L = j \times 62{,}831.9 \times 0.033 = j2073.5\ \Omega$. $Z_C = -j/(\omega C) = -j/(62{,}831.9 \times 22 \times 10^{-9}) = -j/(1.3823 \times 10^{-3}) = -j723.4\ \Omega$.

$Z_{total} = 220 + j2073.5 - j723.4 = 220 + j1350.1\ \Omega$. $|Z_{total}| = \sqrt{(220^2 + 1350.1^2)} = \sqrt{(48{,}400 + 1{,}822{,}770)} = \sqrt{1{,}871{,}170} = 1368.3\ \Omega$. $\theta = \arctan(1350.1/220) = \arctan(6.137) = 80.74°$.

$Y = 1/Z_{total} = 1/1368.3\angle 80.74° = 7.308 \times 10^{-4}\angle -80.74°$ S. $Y = 7.308 \times 10^{-4}(\cos(-80.74°) + j\sin(-80.74°)) = (1.175 - j7.213) \times 10^{-4}$ S.

$Z_{total} = 220 + j1350.1\ \Omega = 1368.3\angle 80.74°\ \Omega$. $Y = 7.308 \times 10^{-4}\angle -80.74°$ S.

Problem A.5.2

Given: A parallel RC circuit with R = 1 kΩ and C = 10 μF at f = 60 Hz.

Find: The total admittance, total impedance, and the phase angle.

Solution: $\omega = 2\pi \times 60 = 376.99$ rad/s.

$Y_R = 1/R = 1/1000 = 1.0 \times 10^{-3}$ S. $Y_C = j\omega C = j \times 376.99 \times 10 \times 10^{-6} = j3.770 \times 10^{-3}$ S.

$Y_{total} = Y_R + Y_C = (1.0 + j3.770) \times 10^{-3}$ S. $|Y_{total}| = \sqrt{(1.0^2 + 3.770^2)} \times 10^{-3} = \sqrt{(1 + 14.21)} \times 10^{-3} = 3.900 \times 10^{-3}$ S. $\theta_Y = \arctan(3.770/1.0) = 75.14°$.

$Z_{total} = 1/Y_{total} = 1/(3.900 \times 10^{-3}\angle 75.14°) = 256.4\angle\text{-}75.14°\ \Omega$. In rectangular: $Z_{total} = 256.4\cos(-75.14°) + j256.4\sin(-75.14°) = 65.72 - j247.9\ \Omega$.

$Y_{total} = 3.900 \times 10^{-3}\angle 75.14°$ S. $Z_{total} = 256.4\angle\text{-}75.14°\ \Omega = 65.72 - j247.9\ \Omega$.

Problem A.5.3

Given: $V_s = 240\angle 0°\ V_{rms}$ at 50 Hz drives a series RL circuit with R = 30 Ω and L = 80 mH.

Find: The current phasor, the voltage across the inductor, and the power factor.

Solution: $\omega = 2\pi \times 50 = 314.16$ rad/s. $Z_L = j\omega L = j \times 314.16 \times 0.08 = j25.13\ \Omega$. $Z_{total} = 30 + j25.13\ \Omega$. $|Z_{total}| = \sqrt{(30^2 + 25.13^2)} = \sqrt{(900 + 631.5)} = \sqrt{1531.5} = 39.13\ \Omega$. $\theta = \arctan(25.13/30) = 39.96°$.

$I = V_s/Z_{total} = 240\angle 0° / 39.13\angle 39.96° = 6.133\angle\text{-}39.96°\ A_{rms}$.

$V_L = I \times Z_L = 6.133\angle\text{-}39.96° \times 25.13\angle 90° = 154.1\angle 50.04°\ V_{rms}$.

Power factor: pf = cos(39.96°) = 0.766 lagging (inductive circuit).

$I = 6.133\angle\text{-}39.96°\ A_{rms}$. $V_L = 154.1\angle 50.04°\ V_{rms}$. Power factor = 0.766 lagging.

Problem A.5.4

Given: A load draws $I = 15\angle\text{-}36.87°\ A_{rms}$ from a source $V = 480\angle 0°\ V_{rms}$.

Find: The complex power, real power, reactive power, apparent power, and power factor.

Solution: Complex power: $S = V \times I^* = 480\angle 0° \times 15\angle 36.87° = 7200\angle 36.87°$ VA. (Note: $I^* = 15\angle{+36.87°}$ since conjugation negates the angle.)

Real power: P = |S| cos(36.87°) = 7200 x 0.8 = 5760 W. Reactive power: Q = |S| sin(36.87°) = 7200 x 0.6 = 4320 VAR. Apparent power: |S| = 7200 VA. Power factor: pf = cos(36.87°) = 0.8 lagging (Q is positive, so inductive).

$S = 7200\angle 36.87°$ VA. P = 5760 W. Q = 4320 VAR. |S| = 7200 VA. Power factor = 0.8 lagging.

Problem A.5.5

Given: A load has S = 1500 + j800 VA (complex power).

Find: The apparent power, power factor, and the load impedance if $V = 120\angle 0°\ V_{rms}$.

Solution: Apparent power: $|S| = \sqrt{(1500^2 + 800^2)} = \sqrt{(2{,}250{,}000 + 640{,}000)} = \sqrt{2{,}890{,}000} = 1700$ VA. Power factor: pf = P/|S| = 1500/1700 = 0.882 lagging (Q > 0, inductive). Power factor angle: $\theta = \arctan(800/1500) = 28.07°$.

Current magnitude: $I_{rms} = |S|/V_{rms} = 1700/120 = 14.17$ A. Current phasor: $I = 14.17\angle{-28.07°}$ A_{rms} (lagging the voltage).

Load impedance: $Z = V/I = 120\angle 0° / 14.17\angle{-28.07°} = 8.468\angle 28.07°$ Ω. In rectangular: $Z = 8.468 \cos(28.07°) + j8.468 \sin(28.07°) = 7.474 + j3.983$ Ω.

Verification: $S = V^2/Z^* = 120^2/(8.468\angle{-28.07°}) = 14400/8.468 \angle 28.07° = 1700.8\angle 28.07°$ VA. Confirmed.

$|S| = 1700$ VA. Power factor = 0.882 lagging. $Z = 8.468\angle 28.07°$ Ω = $7.474 + j3.983$ Ω.

Problem A.5.6

Given: A 50 Ω resistor and a 100 Ω inductor reactance are in parallel at 60 Hz, driven by $V = 100\angle 0°$ V_{rms}.

Find: The total current, the power drawn from the source, and the power factor.

Solution: $I_R = V/R = 100\angle 0° / 50 = 2\angle 0°$ A_{rms}. $I_L = V/(jX_L) = 100\angle 0° / 100\angle 90° = 1\angle{-90°}$ A_{rms}.

$I_{total} = I_R + I_L = (2 + j0) + (0 - j1) = 2 - j1$ A. $|I_{total}| = \sqrt{(4 + 1)} = \sqrt{5} = 2.236$ A_{rms}. $\theta_I = \arctan(-1/2) = -26.57°$.

Complex power: $S = V \times I_{total}^* = 100\angle 0° \times 2.236\angle 26.57° = 223.6\angle 26.57°$ VA. $P = 223.6 \cos(26.57°) = 200$ W. $Q = 223.6 \sin(26.57°) = 100$ VAR. Power factor: $pf = \cos(26.57°) = 0.894$ lagging.

$I_{total} = 2.236\angle{-26.57°}$ A_{rms}. $P = 200$ W. Power factor = 0.894 lagging.

Problem A.5.7

Given: A motor draws $S = 10{,}000\angle 25°$ VA from a $480\angle 0°$ V_{rms} supply. A capacitor bank is to be added in parallel to correct the power factor to 0.95 lagging.

Find: The required capacitive reactive power and the capacitor value at 60 Hz.

Solution: Current power: $P = 10{,}000 \cos(25°) = 9063$ W. $Q_{old} = 10{,}000 \sin(25°) = 4226$ VAR.

New power factor angle: $\theta_{new} = \arccos(0.95) = 18.19°$. New reactive power: $Q_{new} = P \times \tan(18.19°) = 9063 \times 0.3287 = 2979$ VAR. Required capacitive VAR: $Q_C = Q_{old} - Q_{new} = 4226 - 2979 = 1247$ VAR.

Capacitor current: $I_C = Q_C/V = 1247/480 = 2.598$ A. $X_C = V/I_C = 480/2.598 = 184.8$ Ω. $C = 1/(\omega X_C) = 1/(2\pi \times 60 \times 184.8) = 1/(69{,}655) = 14.36$ μF.

$Q_C = 1247$ VAR. $C = 14.36$ μF is required to correct the power factor to 0.95 lagging.

Appendix B – Section B.1: The Arctangent Function

Practice problems covering the definition, range, and quadrant ambiguity of the arctan function.

Problem B.1.1

Given: The ratios: arctan($\sqrt{3}$), arctan(0), and arctan(-$\sqrt{3}$).

Find: The angle for each and identify the quadrant.

Solution: arctan($\sqrt{3}$) = arctan(1.732) = 60° (quadrant I). arctan(0) = 0° (on the positive real axis). arctan(-$\sqrt{3}$) = arctan(-1.732) = -60° (quadrant IV).

All results fall within the range -90° to +90°, confirming that arctan outputs only quadrant I or quadrant IV angles.

arctan($\sqrt{3}$) = 60°, arctan(0) = 0°, arctan(-$\sqrt{3}$) = -60°.

Problem B.1.2

Given: Two impedances $Z_1 = 4 + j3\ \Omega$ and $Z_2 = -4 - j3\ \Omega$.

Find: Compute arctan(b/a) for each and show the quadrant ambiguity. Determine the correct angles.

Solution: For Z_1: arctan(3/4) = arctan(0.75) = 36.87°. Z_1 is in quadrant I ($a > 0$, $b > 0$). The angle 36.87° is correct.

For Z_2: arctan((-3)/(-4)) = arctan(0.75) = 36.87°. Same result, but Z_2 is in quadrant III ($a < 0$, $b < 0$). The correct angle is 36.87° - 180° = -143.13°.

arctan returns 36.87° for both, a 180° error for Z_2.

Correct angles: $Z_1 = 5\angle 36.87°\ \Omega$, $Z_2 = 5\angle{-143.13}°\ \Omega$. arctan cannot distinguish them.

Problem B.1.3

Given: An impedance Z = -8 + j6 Ω (quadrant II).

Find: Compute arctan(b/a), identify the error, and determine the correct angle manually.

Solution: arctan(6/(-8)) = arctan(-0.75) = -36.87°. This places Z in quadrant IV, which is incorrect.

Manual correction: since a < 0 (real part is negative), add 180°: θ = -36.87° + 180° = 143.13° (quadrant II).

Magnitude: $|Z| = \sqrt{(64 + 36)} = \sqrt{100} = 10\ \Omega$.

Z = 10∠143.13° Ω. arctan gives -36.87° (wrong by 180°). The correct angle is 143.13°.

Problem B.1.4

Given: A purely imaginary impedance Z = j50 Ω (a pure inductor).

Find: Attempt to compute the angle using arctan(b/a) and identify the problem.

Solution: Z = 0 + j50, so a = 0 and b = 50. arctan(b/a) = arctan(50/0) = arctan(infinity) = 90°.

In this case, arctan happens to give the correct answer (90°) because the limit of arctan as the argument approaches +infinity is +90°. However, for Z = -j50 (a pure capacitor), arctan(-50/0) = -90°, which is also correct.

The real problem occurs at Z = -R + j0 (purely negative real): arctan(0/(-R)) = arctan(0) = 0° instead of the correct 180°.

arctan gives 90° for j50 (correct), but this is a coincidence. arctan fails at axis points like Z = -R where it returns 0° instead of 180°.

Appendix B – Section B.2: The Two-Argument Arctangent

Practice problems covering atan2 definition, quadrant resolution, and special cases on the axes.

Problem B.2.1

Given: Four complex numbers: $Z_1 = 5 + j5$, $Z_2 = -5 + j5$, $Z_3 = -5 - j5$, $Z_4 = 5 - j5$.

Find: The angle of each using atan2(b, a) and verify the quadrant.

Solution: atan2(5, 5) = 45° (quadrant I: $a > 0$, $b > 0$). Correct. atan2(5, -5) = 135° (quadrant II: $a < 0$, $b > 0$). Correct. atan2(-5, -5) = -135° (quadrant III: $a < 0$, $b < 0$). Correct. atan2(-5, 5) = -45° (quadrant IV: $a > 0$, $b < 0$). Correct.

All four magnitudes are $|Z| = \sqrt{(25 + 25)} = \sqrt{50} = 7.071$.

$Z_1 = 7.071\angle 45°$, $Z_2 = 7.071\angle 135°$, $Z_3 = 7.071\angle -135°$, $Z_4 = 7.071\angle -45°$. atan2 correctly resolves all four quadrants.

Problem B.2.2

Given: An impedance $Z = -12 + j16\ \Omega$.

Find: The polar form using atan2, and compare with the arctan result.

Solution: Magnitude: $|Z| = \sqrt{((-12)^2 + 16^2)} = \sqrt{(144 + 256)} = \sqrt{400} = 20\ \Omega$.

Using atan2: θ = atan2(16, -12). Since $a < 0$ and $b > 0$, this is quadrant II. θ = 180° - arctan(16/12) = 180° - arctan(1.333) = 180° - 53.13° = 126.87°.

Using arctan: arctan(16/(-12)) = arctan(-1.333) = -53.13° (quadrant IV – wrong).

$Z = 20\angle 126.87°\ \Omega$ using atan2. arctan gives -53.13°, an error of 180°.

Problem B.2.3

Given: The following axis points: $Z_1 = 75 + j0\ \Omega$, $Z_2 = -75 + j0\ \Omega$, $Z_3 = 0 + j75\ \Omega$, $Z_4 = 0 - j75\ \Omega$.

Find: The angle of each using atan2.

Solution: $Z_1 = 75 + j0$: atan2(0, 75) = 0°. This is a pure resistance (positive real axis). $Z_2 = -75 + j0$: atan2(0, -75) = 180°. This is a negative resistance (negative real axis). $Z_3 = 0 + j75$: atan2(75, 0) = 90°. This is a pure inductance (+90° impedance angle). $Z_4 = 0 - j75$: atan2(-75, 0) = -90°. This is a pure capacitance (-90° impedance angle).

All four have magnitude 75 Ω.

$Z_1 = 75\angle 0°\ \Omega$, $Z_2 = 75\angle 180°\ \Omega$, $Z_3 = 75\angle 90°\ \Omega$, $Z_4 = 75\angle\text{-}90°\ \Omega$. atan2 handles all axis points correctly.

Problem B.2.4

Given: A circuit has R = 0 Ω and $X_L = 2\pi \times 400 \times 0.025 = 62.83\ \Omega$ at 400 Hz (a pure 25 mH inductor).

Find: The impedance angle using atan2 and explain why arctan(b/a) fails here.

Solution: Z = 0 + j62.83 Ω, so a = 0 and b = 62.83.

Using atan2: θ = atan2(62.83, 0) = 90°. Correct.

Using arctan: arctan(62.83/0) – this requires dividing by zero. While the limit arctan(+infinity) = 90°, the division b/a is undefined, and software may produce an error or NaN.

atan2 avoids the division entirely by examining the signs of a and b independently: since a = 0 and b > 0, the angle is defined as exactly +90°.

θ = 90°. atan2 handles the a = 0 case directly, while arctan requires division by zero.

Problem B.2.5

Given: A signal with in-phase component I = -3.5 V and quadrature component Q = -6.062 V (from an I/Q demodulator).

Find: The magnitude and phase angle using atan2.

Solution: Magnitude: $|V| = \sqrt{((-3.5)^2 + (-6.062)^2)} = \sqrt{(12.25 + 36.75)} = \sqrt{49} = 7.0$ V.

Phase angle: θ = atan2(-6.062, -3.5). Since a < 0 and b < 0, this is quadrant III. θ = -(180° - arctan(6.062/3.5)) = -(180° - arctan(1.732)) = -(180° - 60°) = -120°.

Using arctan instead: arctan((-6.062)/(-3.5)) = arctan(1.732) = 60° (quadrant I – wrong by 180°).

$V = 7.0\angle\text{-}120°$. atan2 correctly places the phasor in quadrant III.

Appendix B – Section B.3: Applications in Electrical Engineering

Practice problems covering impedance angle calculation, phasor angle from rectangular components, and power factor angle.

Problem B.3.1

Given: A negative impedance converter (NIC) produces $Z = -25 + j25\ \Omega$.

Find: The polar form using atan2, and show the error that arctan would produce.

Solution: Magnitude: $|Z| = \sqrt{((-25)^2 + 25^2)} = \sqrt{(625 + 625)} = \sqrt{1250} = 35.36\ \Omega$.

Using atan2: $\theta = \text{atan2}(25, -25) = 180° - \arctan(25/25) = 180° - 45° = 135°$ (quadrant II).

Using arctan: $\arctan(25/(-25)) = \arctan(-1) = -45°$ (quadrant IV – wrong).

$Z = 35.36\angle 135°\ \Omega$. arctan gives -45°, a 180° error.

Problem B.3.2

Given: A lock-in amplifier measures in-phase $I = 4.0$ mV and quadrature $Q = -6.928$ mV from a sensor signal.

Find: The signal magnitude and phase angle using atan2.

Solution: Magnitude: $|V| = \sqrt{(4.0^2 + (-6.928)^2)} = \sqrt{(16 + 48)} = \sqrt{64} = 8.0$ mV.

Phase angle: $\theta = \text{atan2}(-6.928, 4.0)$. Since $a > 0$ and $b < 0$, this is quadrant IV. $\theta = \arctan(-6.928/4.0) = \arctan(-1.732) = -60°$.

In this case, $a > 0$, so arctan gives the same result as atan2: -60°. Both are correct for quadrant IV.

$V = 8.0\angle -60°$ mV.

Problem B.3.3

Given: A motor operating in regenerative braking mode has complex power $S = -1200 + j900$ VA.

Find: The power factor angle using both atan2 and arctan, the apparent power, and the power factor.

Solution: Apparent power: $|S| = \sqrt{((-1200)^2 + 900^2)} = \sqrt{(1{,}440{,}000 + 810{,}000)} = \sqrt{2{,}250{,}000} = 1500$ VA.

Using atan2: $\varphi = \text{atan2}(900, -1200)$. Since $P < 0$ and $Q > 0$ (quadrant II): $\varphi = 180° - \arctan(900/1200) = 180° - 36.87° = 143.13°$.

Using arctan: $\varphi = \arctan(900/(-1200)) = \arctan(-0.75) = -36.87°$ (quadrant IV – wrong).

Power factor: $pf = \cos(143.13°) = -0.8$. The negative power factor indicates reverse power flow (regeneration).

$\varphi = 143.13°$ (atan2). Apparent power = 1500 VA. Power factor = -0.8 (regenerative).

Problem B.3.4

Given: A three-phase system measurement at a bus yields $P = 2500$ kW and $Q = -1500$ kVAR (capacitive reactive power).

Find: The power factor angle using atan2, the power factor, and whether the load is leading or lagging.

Solution: Using atan2: $\varphi = \text{atan2}(-1500, 2500)$. Since $P > 0$ and $Q < 0$ (quadrant IV): $\varphi = \arctan(-1500/2500) = \arctan(-0.6) = -30.96°$. (Since $P > 0$, atan2 and arctan agree.)

Apparent power: $|S| = \sqrt{(2500^2 + 1500^2)} = \sqrt{(6{,}250{,}000 + 2{,}250{,}000)} = \sqrt{8{,}500{,}000} = 2915.5$ kVA. Power factor: $pf = \cos(-30.96°) = 0.857$. Since $Q < 0$, the load is capacitive (leading).

$\varphi = -30.96°$. Power factor = 0.857 leading.

Problem B.3.5

Given: An I/Q demodulator outputs $I = -0.707$ mV and $Q = 0.707$ mV for a QAM constellation point.

Find: The magnitude and phase using atan2. Compare with arctan.

Solution: Magnitude: $|V| = \sqrt{((-0.707)^2 + 0.707^2)} = \sqrt{(0.5 + 0.5)} = \sqrt{1.0} = 1.0$ mV.

Using atan2: $\theta = \text{atan2}(0.707, -0.707)$. Since $a < 0$ and $b > 0$ (quadrant II): $\theta = 180° - \arctan(0.707/0.707) = 180° - 45° = 135°$.

Using arctan: $\arctan(0.707/(-0.707)) = \arctan(-1) = -45°$ (quadrant IV – wrong by 180°).

In a QPSK constellation, 135° corresponds to the symbol in the second quadrant, carrying bit pattern "01" (Gray-coded). The arctan error would map it to the fourth quadrant symbol, causing a bit error.

$V = 1.0\angle 135°$ mV. atan2 is essential for correct symbol detection in digital communications.

Appendix C – Section C.1: Definition and Fundamentals

Practice problems covering power ratios in dB, voltage/current ratios in dB, and common decibel values.

Problem C.1.1

Given: A power amplifier receives 2 mW of input and delivers 5 W of output.

Find: The power gain in decibels.

Solution: $G = 10 \times \log_{10}(P_{out}/P_{in}) = 10 \times \log_{10}(5000/2) = 10 \times \log_{10}(2500) = 10 \times 3.3979 = 33.98$ dB.

The power gain is 33.98 dB.

Problem C.1.2

Given: A cable introduces 2.5 dB of power loss per 100 m. The cable is 350 m long.

Find: The total loss in dB and the fraction of input power delivered to the output.

Solution: Total loss: $L = 2.5 \times (350/100) = 2.5 \times 3.5 = 8.75$ dB.

Fraction of power delivered: $P_{out}/P_{in} = 10^{-8.75/10} = 10^{-0.875} = 0.1334$.

The total loss is 8.75 dB. Only 13.34% of the input power reaches the output.

Problem C.1.3

Given: A voltage amplifier has $V_{in} = 50\ mV_{rms}$ and $V_{out} = 3.5\ V_{rms}$.

Find: The voltage gain in dB.

Solution: $A_v = 20 \times \log_{10}(V_{out}/V_{in}) = 20 \times \log_{10}(3.5/0.05) = 20 \times \log_{10}(70) = 20 \times 1.8451 = 36.90$ dB.

The voltage gain is 36.90 dB.

Problem C.1.4

Given: A passive filter reduces a signal from 1.2 V_{rms} to 0.6 V_{rms} (voltage halved).

Find: The attenuation in dB, and verify using the power ratio (assuming equal impedances).

Solution: Voltage gain: $A_v = 20 \times \log_{10}(0.6/1.2) = 20 \times \log_{10}(0.5) = 20 \times (-0.3010) = -6.02$ dB.

Power ratio (equal impedances): $P_{out}/P_{in} = (V_{out}/V_{in})^2 = 0.5^2 = 0.25$. Power gain: $G = 10 \times \log_{10}(0.25) = 10 \times (-0.6021) = -6.02$ dB. Matches.

The attenuation is 6.02 dB (halving voltage = halving power x 2 = -6 dB).

Problem C.1.5

Given: An amplifier has a power gain of 37 dB.

Find: Estimate the power ratio using rules of thumb, then compute the exact ratio.

Solution: Using rules of thumb: 37 dB = 30 dB + 7 dB = 30 dB + 10 dB - 3 dB. 30 dB = power ratio of 1,000. 10 dB = power ratio of 10. -3 dB = power ratio of 0.5. Estimate: 1,000 x 10 x 0.5 = 5,000.

Exact: $10^{37/10} = 10^{3.7} = 5{,}012$.

The estimated power ratio is 5,000. The exact ratio is 5,012. The estimate is within 0.24%.

Problem C.1.6

Given: Two stages in series: stage 1 has a voltage gain of 14 dB, and stage 2 has a voltage gain of -8 dB.

Find: The total voltage gain in dB and the overall linear voltage ratio.

Solution: Total gain: $A_{total} = 14 + (-8) = 6$ dB.

Linear voltage ratio: $10^{6/20} = 10^{0.3} = 1.995$ (approximately 2:1).

This confirms the rule of thumb: 6 dB corresponds to a voltage ratio of 2.

The total voltage gain is 6 dB, corresponding to a voltage ratio of approximately 2:1.

Problem C.1.7

Given: A current amplifier has a current gain of 26 dB.

Find: The linear current ratio.

Solution: Current uses the voltage/field formula (factor of 20): $I_{out}/I_{in} = 10^{26/20} = 10^{1.3} = 19.95$.

Rule of thumb check: 26 dB = 20 dB + 6 dB. 20 dB current ratio = 10, 6 dB current ratio = 2. Estimate: 10 x 2 = 20. Close to the exact value.

The current ratio is 19.95:1 (approximately 20:1).

Appendix C – Section C.2: Absolute Reference Levels

Practice problems covering dBm, dBW, dBV, and dBuV conversions.

Problem C.2.1

Given: A Wi-Fi access point transmits at +17 dBm into a 50 Ω antenna connector.

Find: The output power in milliwatts, watts, and the RMS voltage at the connector.

Solution: Power: $P = 1\text{ mW} \times 10^{17/10} = 1\text{ mW} \times 10^{1.7} = 1\text{ mW} \times 50.12 = 50.12\text{ mW} = 0.05012\text{ W}$.

RMS voltage: $V_{rms} = \sqrt{(P \times R)} = \sqrt{(0.05012 \times 50)} = \sqrt{2.506} = 1.583\text{ V}$.

$P = 50.12\text{ mW} = 0.0501\text{ W}$. $V_{rms} = 1.583\text{ V}$.

Problem C.2.2

Given: A signal generator outputs -10 dBm.

Find: The power in milliwatts, microwatts, and dBW.

Solution: Power: $P = 1\text{ mW} \times 10^{-10/10} = 1\text{ mW} \times 10^{-1} = 0.1\text{ mW} = 100\ \mu\text{W}$.

dBW: $P(\text{dBW}) = P(\text{dBm}) - 30 = -10 - 30 = -40$ dBW.

Verification: $P = 1\text{ W} \times 10^{-40/10} = 10^{-4}\text{ W} = 0.1\text{ mW}$. Confirmed.

$P = 0.1\text{ mW} = 100\ \mu\text{W} = -10\text{ dBm} = -40\text{ dBW}$.

Problem C.2.3

Given: A broadcast FM transmitter has a power output of 25 kW.

Find: Express in dBW and dBm.

Solution: dBW: $P = 10 \times \log_{10}(25{,}000/1) = 10 \times \log_{10}(25{,}000) = 10 \times 4.3979 = 43.98$ dBW.

dBm: $P = 43.98 + 30 = 73.98$ dBm.

Verification: $10^{73.98/10}$ mW $= 10^{7.398}$ mW $= 2.5 \times 10^7$ mW = 25 kW. Confirmed.

P = 43.98 dBW = 73.98 dBm.

Problem C.2.4

Given: An EMC test measures 54 dBμV/m at 200 MHz.

Find: The field strength in dBV/m, in V/m, and in mV/m.

Solution: dBV: V(dBV) = V(dBμV) - 120 = 54 - 120 = -66 dBV/m.

Linear voltage: $E = 1\ \mu V/m \times 10^{54/20} = 10^{-6} \times 10^{2.7} = 10^{-6} \times 501.2 = 501.2\ \mu V/m = 0.5012$ mV/m.

Verification: $20 \times \log_{10}(501.2 \times 10^{-6} / 10^{-6}) = 20 \times \log_{10}(501.2) = 20 \times 2.7 = 54$ dBμV/m. Confirmed.

E = 54 dBμV/m = -66 dBV/m = 501.2 μV/m = 0.501 mV/m.

Problem C.2.5

Given: An audio interface specifies a maximum output level of +4 dBV (professional line level).

Find: The output voltage in V_{rms}, and convert to dBm assuming a 600 Ω load (standard audio impedance).

Solution: Voltage: $V = 1\ V \times 10^{4/20} = 10^{0.2} = 1.585\ V_{rms}$.

Power into 600 Ω: $P = V^2/R = (1.585)^2/600 = 2.512/600 = 4.187$ mW. dBm: $P = 10 \times \log_{10}(4.187/1) = 10 \times 0.6218 = 6.22$ dBm.

$V = 1.585\ V_{rms}$. Into 600 Ω, this is +6.22 dBm.

Problem C.2.6

Given: A cable TV system specifies signal levels in dBmV. The minimum acceptable level at a TV set is 0 dBmV and the maximum is +15 dBmV.

Find: The voltage range in mV and μV, and convert the maximum to dBμV.

Solution: 0 dBmV corresponds to $1\ mV_{rms} = 1000\ \mu V_{rms}$.

+15 dBmV: $V = 1\ mV \times 10^{15/20} = 1\ mV \times 10^{0.75} = 1\ mV \times 5.623 = 5.623\ mV_{rms} = 5623\ \mu V_{rms}$.

Convert to dBμV: dBμV = dBmV + 60 (since 1 mV = 1000 μV = 60 dBμV). Maximum: 15 + 60 = 75 dBμV. Minimum: 0 + 60 = 60 dBμV.

Signal range: 1.0 mV to 5.623 mV (60 dBμV to 75 dBμV).

Appendix C – Section C.3: Decibel Arithmetic

Practice problems covering cascaded gains/losses, linear-to-dB conversion, and adding powers in decibels.

Problem C.3.1

Given: A satellite downlink chain consists of: satellite transmitter at +10 dBW, transmit antenna gain +35 dBi, free-space path loss -200 dB, receive antenna gain +45 dBi, cable loss -2 dB, and LNA gain +25 dB.

Find: The signal level at the LNA output in dBW and dBm.

Solution: $P_{out} = P_{tx} + G_{tx} + L_{path} + G_{rx} + L_{cable} + G_{LNA}$ $P_{out} = 10 + 35 + (-200) + 45 + (-2) + 25 = -87$ dBW.

Convert to dBm: -87 + 30 = -57 dBm.

Linear power: $P = 10^{-87/10}$ W $= 10^{-8.7}$ W = 2.0 nW.

P_{out} = -87 dBW = -57 dBm = 2.0 nW.

Problem C.3.2

Given: An RF signal chain has three stages: a preamplifier with +15 dB gain, a bandpass filter with -4 dB insertion loss, and a power amplifier with +30 dB gain. The input power is -20 dBm.

Find: The output power in dBm and watts.

Solution: Total gain: $G_{total} = 15 + (-4) + 30 = 41$ dB. Output power: $P_{out} = -20 + 41 = +21$ dBm.

Linear: $P = 10^{21/10}$ mW $= 10^{2.1}$ mW = 125.9 mW = 0.126 W.

P_{out} = +21 dBm = 125.9 mW.

Problem C.3.3

Given: A noise figure specification of 6 dB for a receiver.

Find: The linear noise factor F and the equivalent noise temperature (T_0 = 290 K).

Solution: $F = 10^{6/10} = 10^{0.6} = 3.981$.

$T_e = T_0 \times (F - 1) = 290 \times (3.981 - 1) = 290 \times 2.981 = 864.5$ K.

F = 3.981 (linear). T_e = 864.5 K.

Problem C.3.4

Given: A power gain of 250 (linear ratio).

Find: The gain in dB using the power formula.

Solution: $G = 10 \times \log_{10}(250) = 10 \times 2.3979 = 23.98$ dB.

Rule of thumb check: 250 = 100 x 2.5 = 100 x 2 x 1.25. 20 dB (100) + 3 dB (2) + 1 dB (1.259) = 24 dB. Close to exact.

G = 23.98 dB.

Problem C.3.5

Given: A voltage gain of -34 dB.

Find: The linear voltage ratio.

Solution: $V_{out}/V_{in} = 10^{-34/20} = 10^{-1.7} = 0.01995$.

This means the output is about 2% of the input voltage.

Rule of thumb: -34 dB = -40 dB + 6 dB. Voltage ratio for -40 dB = 0.01, for 6 dB = 2. Total = 0.01 x 2 = 0.02. Close to exact.

The linear voltage ratio is 0.01995 (approximately 1/50).

Problem C.3.6

Given: Three uncorrelated noise sources at a combiner input: -80 dBm, -80 dBm, and -80 dBm.

Find: The total noise power at the combiner output.

Solution: Convert each to linear: $P = 10^{-80/10}$ mW $= 10^{-8}$ mW = 10 pW each.

Total: $P_{total} = 10 + 10 + 10 = 30$ pW $= 3 \times 10^{-8}$ mW.

Convert back: P_{total} = 10 x $\log_{10}(3 \times 10^{-8})$ = 10 x (-7.523) = -75.23 dBm.

The three equal sources combine to a level 4.77 dB higher than any single source: -80 + 10 x $\log_{10}(3)$ = -80 + 4.77 = -75.23 dBm.

P_{total} = -75.23 dBm. Three equal powers combine to give +4.77 dB above each individual level.

Problem C.3.7

Given: Two signals at a combiner: +10 dBm and +20 dBm.

Find: The total power in dBm.

Solution: Convert to linear: $P_1 = 10^{10/10}$ mW = 10 mW. $P_2 = 10^{20/10}$ mW = 100 mW.

Total: P_{total} = 10 + 100 = 110 mW.

Convert back: P_{total} = 10 x $\log_{10}(110)$ = 10 x 2.0414 = 20.41 dBm.

The result is only 0.41 dB above the stronger signal because the weaker signal (10 mW) is only 10% of the stronger signal (100 mW). The stronger signal dominates.

P_{total} = 20.41 dBm.

Problem C.3.8

Given: A fiber optic link has a +5 dBm laser, two connectors at -0.3 dB each, one splice at -0.1 dB, 40 km of fiber at 0.25 dB/km loss, and the receiver sensitivity is -32 dBm.

Find: The received power and the link margin.

Solution: Fiber loss: 40 x 0.25 = 10 dB. Connector losses: 2 x 0.3 = 0.6 dB. Splice loss: 0.1 dB. Total loss: 10 + 0.6 + 0.1 = 10.7 dB.

Received power: P_{rx} = +5 - 10.7 = -5.7 dBm. Link margin: -5.7 - (-32) = 26.3 dB.

P_{rx} = -5.7 dBm. Link margin = 26.3 dB above receiver sensitivity.

Appendix C – Section C.4: Applications in Electrical Engineering

Practice problems covering amplifier gain/bandwidth, signal-to-noise ratio, and link budgets.

Problem C.4.1

Given: An op-amp has a gain-bandwidth product of 5 MHz and is configured for a closed-loop gain of 26 dB.

Find: The linear closed-loop gain, the closed-loop bandwidth, and the gain at 500 kHz.

Solution: Linear gain: $A_{CL} = 10^{26/20} = 10^{1.3} = 19.95$ (approximately 20).

Bandwidth: $BW = GBW / A_{CL} = 5 \times 10^6 / 19.95 = 250.6$ kHz.

At 500 kHz, the gain rolls off. The gain at frequency f for a single-pole op-amp is: $A(f) = GBW / f = 5 \times 10^6 / 500 \times 10^3 = 10$. In dB: $20 \times \log_{10}(10) = 20$ dB.

Since 500 kHz is about 2x the -3 dB bandwidth, the gain has dropped from 26 dB to 20 dB.

$A_{CL} = 20$ (linear). BW = 250.6 kHz. Gain at 500 kHz = 20 dB.

Problem C.4.2

Given: A 16-bit ADC has a full-scale range of 5 V and quantization noise only (ideal ADC).

Find: The theoretical SNR and ENOB.

Solution: For an ideal N-bit ADC: SNR = 6.02N + 1.76 dB. SNR = 6.02 x 16 + 1.76 = 96.32 + 1.76 = 98.08 dB.

ENOB = (SNR - 1.76) / 6.02 = (98.08 - 1.76) / 6.02 = 96.32 / 6.02 = 16.0 bits.

For a real ADC with additional noise, the actual SNR would be lower, reducing the ENOB below 16.

Theoretical SNR = 98.08 dB. ENOB = 16.0 bits (ideal).

Problem C.4.3

Given: A measurement system has a noise floor of -110 dBm and a maximum input of +10 dBm before clipping.

Find: The dynamic range in dB.

Solution: Dynamic range = maximum signal - noise floor = +10 - (-110) = 120 dB.

In terms of power ratio: $10^{120/10} = 10^{12}$ = 1,000,000,000,000 (one trillion to one).

The dynamic range is 120 dB, corresponding to a power ratio of 10^{12}.

Problem C.4.4

Given: An audio amplifier has a signal output of 2 V_{rms} and a noise output of 20 μV_{rms}.

Find: The SNR in dB.

Solution: SNR = 20 x $\log_{10}(V_{signal}/V_{noise})$ = 20 x $\log_{10}(2/20 \times 10^{-6})$ = 20 x $\log_{10}(100{,}000)$ = 20 x 5 = 100 dB.

The SNR is 100 dB.

Problem C.4.5

Given: A 2.4 GHz Wi-Fi link spans 100 m in free space. The transmitter outputs +20 dBm with a +3 dBi antenna. The receiver has a +3 dBi antenna and -85 dBm sensitivity.

Find: The free-space path loss, received power, and link margin.

Solution: Wavelength: $\lambda = c/f = 3 \times 10^8 / 2.4 \times 10^9 = 0.125$ m.

Free-space path loss: L_{path} = 20 x $\log_{10}(4\pi \times 100 / 0.125)$ = 20 x $\log_{10}(10{,}053)$ = 20 x 4.0023 = 80.05 dB.

Received power: $P_{rx} = P_{tx} + G_{tx} - L_{path} + G_{rx}$ = 20 + 3 - 80.05 + 3 = -54.05 dBm.

Link margin: -54.05 - (-85) = 30.95 dB.

L_{path} = 80.05 dB. P_{rx} = -54.05 dBm. Link margin = 30.95 dB.

Problem C.4.6

Given: An amplifier chain has the following stages: first stage gain = 20 dB with noise figure 3 dB, second stage gain = 10 dB with noise figure 10 dB.

Find: The overall noise figure using Friis' noise formula (expressed in dB and linear).

Solution: Convert to linear noise factors: $F_1 = 10^{3/10} = 2.0$. $F_2 = 10^{10/10} = 10.0$. $G_1 = 10^{20/10} = 100$ (linear power gain).

Friis' formula: $F_{total} = F_1 + (F_2 - 1)/G_1 = 2.0 + (10 - 1)/100 = 2.0 + 0.09 = 2.09$.

$NF_{total} = 10 \times \log_{10}(2.09) = 10 \times 0.3201 = 3.20$ dB.

The overall noise figure (3.20 dB) is dominated by the first stage (3 dB) because the high gain of the first stage (20 dB) suppresses the noise contribution of the second stage.

$F_{total} = 2.09$. $NF_{total} = 3.20$ dB.

Problem C.4.7

Given: A microwave point-to-point link at 18 GHz spans 5 km. Transmit power is +30 dBm, each antenna has +38 dBi gain, and total cable/connector losses are 3 dB at each end.

Find: The free-space path loss and the received power.

Solution: Wavelength: $\lambda = 3 \times 10^8 / 18 \times 10^9 = 0.01667$ m.

Free-space path loss: $L_{path} = 20 \times \log_{10}(4\pi \times 5000 / 0.01667) = 20 \times \log_{10}(3.770 \times 10^6) = 20 \times 6.5763 = 131.53$ dB.

Received power: $P_{rx} = +30 + 38 - 3 - 131.53 + 38 - 3 = -31.53$ dBm.

Linear: $P = 10^{-31.53/10}$ mW $= 10^{-3.153}$ mW $= 7.03 \times 10^{-4}$ mW $= 0.703$ μW.

$L_{path} = 131.53$ dB. $P_{rx} = -31.53$ dBm $= 0.703$ μW.

Appendix D – Section D.1: SI Base Units

Practice problems covering the seven base units and their relationship to electrical quantities.

Problem D.1.1

Given: A copper wire has a length of 500 m, a cross-sectional area of 4 mm^2, and copper resistivity $\rho = 1.68 \times 10^{-8}\ \Omega\cdot m$.

Find: The wire resistance, and express the cross-sectional area in base SI units (m^2).

Solution: Area: $4\ mm^2 = 4 \times (10^{-3})^2\ m^2 = 4 \times 10^{-6}\ m^2$.

$R = \rho L/A = (1.68 \times 10^{-8} \times 500) / (4 \times 10^{-6}) = 8.4 \times 10^{-6} / 4 \times 10^{-6} = 2.1\ \Omega$.

Area $= 4 \times 10^{-6}\ m^2$. $R = 2.1\ \Omega$.

Problem D.1.2

Given: The equation for energy stored in an inductor: $E = (1/2)LI^2$.

Find: Verify that the units are consistent by expressing both sides in SI base units.

Solution: Left side: Energy E is in joules. $J = kg\cdot m^2\cdot s^{-2}$.

Right side: L is in henries ($H = kg\cdot m^2\cdot s^{-2}\cdot A^{-2}$). I^2 is in A^2. $LI^2 = (kg\cdot m^2\cdot s^{-2}\cdot A^{-2}) \times A^2 = kg\cdot m^2\cdot s^{-2} = J$.

The factor of 1/2 is dimensionless.

Both sides have units of $kg\cdot m^2\cdot s^{-2} = J$. The equation is dimensionally consistent.

Problem D.1.3

Given: A 20 A circuit breaker on a 240 V supply.

Find: The maximum power in watts, and express the watt in base SI units to verify $P = V \times I$.

Solution: $P = V \times I = 240 \times 20 = 4800$ W.

Dimensional check: V has units $kg \cdot m^2 \cdot s^{-3} \cdot A^{-1}$. A has units A. $V \times A = kg \cdot m^2 \cdot s^{-3} \cdot A^{-1} \times A = kg \cdot m^2 \cdot s^{-3}$ = W. Consistent.

$P = 4800$ W. The watt equals $kg \cdot m^2 \cdot s^{-3}$, confirmed by V x A.

Problem D.1.4

Given: A charge of 1 coulomb flows through a circuit.

Find: How many electrons make up 1 C, given the elementary charge $e = 1.602 \times 10^{-19}$ C.

Solution: Number of electrons: $n = Q/e = 1 / (1.602 \times 10^{-19}) = 6.242 \times 10^{18}$ electrons.

This enormous number illustrates why the coulomb (ampere-second) is a practical unit even though individual electron charges are unimaginably small.

1 C = 6.242×10^{18} electrons.

Appendix D – Section D.2: SI Derived Units for Electrical Engineering

Practice problems covering voltage/resistance/power units, capacitance/inductance/charge units, and frequency/time units.

Problem D.2.1

Given: A 24 V battery powers a circuit drawing 6 A.

Find: The power dissipated, the load resistance, and the conductance, with units verified at each step.

Solution: Power: P = V x I = 24 V x 6 A = 144 W. Resistance: R = V/I = 24 V / 6 A = 4 Ω. Conductance: G = 1/R = 1/4 Ω = 0.25 S.

Check via conductance: I = G x V = 0.25 S x 24 V = 6 A. Confirmed. Check via power: P = V^2/R = 576/4 = 144 W = I^2R = 36 x 4 = 144 W. Confirmed.

P = 144 W. R = 4 Ω. G = 0.25 S.

Problem D.2.2

Given: A 100 μF capacitor is charged to 50 V.

Find: The stored charge and energy, with appropriate SI prefixes.

Solution: Charge: Q = CV = 100 x 10^{-6} F x 50 V = 5 x 10^{-3} C = 5 mC.

Energy: E = (1/2)CV^2 = 0.5 x 100 x 10^{-6} x 50^2 = 0.5 x 100 x 10^{-6} x 2500 = 125 x 10^{-3} J = 125 mJ.

Q = 5 mC. E = 125 mJ.

Problem D.2.3

Given: An inductor of 2.2 mH carries a current of 3 A.

Find: The stored energy and the voltage across the inductor if the current changes at 1000 A/s.

Solution: Energy: $E = (1/2)LI^2 = 0.5 \times 2.2 \times 10^{-3} \times 3^2 = 0.5 \times 2.2 \times 10^{-3} \times 9 = 9.9 \times 10^{-3}$ J = 9.9 mJ.

Voltage: $V = L \times (dI/dt) = 2.2 \times 10^{-3}$ H x 1000 A/s = 2.2 V.

Units check: H x A/s = (V·s/A) x (A/s) = V. Consistent.

E = 9.9 mJ. V = 2.2 V across the inductor.

Problem D.2.4

Given: A microcontroller clock frequency of 72 MHz.

Find: The clock period in nanoseconds and the angular frequency in rad/s.

Solution: Period: $T = 1/f = 1/(72 \times 10^6) = 13.89 \times 10^{-9}$ s = 13.89 ns.

Angular frequency: $\omega = 2\pi f = 2\pi \times 72 \times 10^6 = 452.4 \times 10^6$ rad/s = 452.4 Mrad/s.

T = 13.89 ns. ω = 452.4 Mrad/s.

Problem D.2.5

Given: A magnetic core has a flux of 0.5 mWb through a cross-sectional area of 2 cm^2.

Find: The magnetic flux density in tesla.

Solution: Convert area: 2 $cm^2 = 2 \times (10^{-2})^2$ $m^2 = 2 \times 10^{-4}$ m^2. Flux: $\Phi = 0.5 \times 10^{-3}$ Wb.

$B = \Phi/A = (0.5 \times 10^{-3}) / (2 \times 10^{-4}) = 2.5$ T.

Units check: Wb/m^2 = T. Confirmed.

B = 2.5 T.

Appendix D – Section D.3: SI Prefixes

Practice problems covering prefix usage, prefix arithmetic, and engineering notation.

Problem D.3.1

Given: The following raw values: 0.000022 F, 4,700,000 Ω, 0.00000000015 H, and 56,000 Hz.

Find: Express each using appropriate SI prefixes.

Solution: 0.000022 F = 22×10^{-6} F = 22 μF. 4,700,000 Ω = 4.7×10^{6} Ω = 4.7 MΩ. 0.00000000015 H = 150×10^{-12} H = 150 pH. 56,000 Hz = 56×10^{3} Hz = 56 kHz.

22 μF, 4.7 MΩ, 150 pH, 56 kHz.

Problem D.3.2

Given: A 10 kΩ resistor with 5 V across it.

Find: The current (with prefix), the power (with prefix), and the time constant if paired with a 47 nF capacitor.

Solution: Current: $I = V/R = 5 / (10 \times 10^{3}) = 5 \times 10^{-4}$ A = 500 μA.

Power: $P = V^2/R = 25 / (10 \times 10^{3}) = 2.5 \times 10^{-3}$ W = 2.5 mW.

Time constant: $\tau = RC$ = 10 kΩ x 47 nF. Using the shortcut: kΩ x nF = $10^{3} \times 10^{-9} = 10^{-6}$ = μs. τ = 10 x 47 μs = 470 μs.

I = 500 μA. P = 2.5 mW. τ = 470 μs.

Problem D.3.3

Given: The values 3.75×10^{-4} A, 8.2×10^{7} Ω, and 6.5×10^{-10} F.

Find: Convert each to engineering notation with SI prefixes.

Solution: 3.75×10^{-4} A: shift exponent to -3: 0.375×10^{-3} A = 375×10^{-6} A = 375 μA. (Preferred: 375 μA keeps the mantissa between 1 and 999.)

8.2×10^{7} Ω: shift exponent to 6: 82×10^{6} Ω = 82 MΩ.

6.5×10^{-10} F: shift exponent to -9: 0.65×10^{-9} F = 650×10^{-12} F = 650 pF.

375 μA, 82 MΩ, 650 pF.

Problem D.3.4

Given: A circuit operates at 2.4 GHz. A transmission line has a propagation velocity of 2×10^{8} m/s.

Find: The wavelength in appropriate engineering notation and SI prefix.

Solution: $\lambda = v/f = (2 \times 10^{8}) / (2.4 \times 10^{9}) = 0.08333$ m.

Convert to engineering notation: 0.08333 m = 83.33×10^{-3} m = 83.33 mm.

Alternatively: a quarter wavelength (common for antenna design) = 83.33/4 = 20.83 mm.

λ = 83.33 mm.

Problem D.3.5

Given: Prefix arithmetic shortcut: “mA x kΩ = ?”

Find: Determine the result unit and verify with a numerical example.

Solution: mA x kΩ = (10^{-3} A) x (10^{3} Ω) = 10^{0} V = V.

The milli and kilo prefixes cancel ($10^{-3} \times 10^{3} = 10^{0}$).

Numerical example: 2.5 mA x 4.7 kΩ = 2.5 x 4.7 V = 11.75 V. Verification: 2.5×10^{-3} A x 4.7×10^{3} Ω = 11.75 V. Confirmed.

mA x kΩ = V (the prefixes cancel).

Appendix D – Section D.4: Common Unit Conversions

Practice problems covering energy/charge units and temperature scale conversions.

Problem D.4.1

Given: A lithium-ion cell rated at 3400 mAh and 3.7 V nominal.

Find: The stored energy in watt-hours, joules, and the total charge in coulombs.

Solution: Energy: E = 3400 mAh x 3.7 V = 12,580 mWh = 12.58 Wh.

In joules: E = 12.58 x 3600 = 45,288 J = 45.29 kJ.

Charge: Q = 3400 mAh = 3.4 Ah = 3.4 x 3600 C = 12,240 C.

E = 12.58 Wh = 45.29 kJ. Q = 12,240 C.

Problem D.4.2

Given: The bandgap energy of silicon is 1.12 eV.

Find: The bandgap energy in joules, and the corresponding wavelength of a photon with this energy ($c = 3 \times 10^{8}$ m/s, $h = 6.626 \times 10^{-34}$ J·s).

Solution: Energy: $E = 1.12$ eV x 1.602×10^{-19} J/eV = 1.794×10^{-19} J.

Wavelength: $\lambda = hc/E = (6.626 \times 10^{-34} \times 3 \times 10^{8}) / (1.794 \times 10^{-19}) = 1.988 \times 10^{-25} / 1.794 \times 10^{-19} = 1.108 \times 10^{-6}$ m = 1108 nm.

This is in the infrared range, which is why silicon photodetectors are sensitive to near-IR light.

$E = 1.794 \times 10^{-19}$ J. $\lambda = 1108$ nm (infrared).

Problem D.4.3

Given: A power MOSFET has $T_{j,max}$ = 150°C, ambient temperature is 40°C, junction-to-case thermal resistance $R_{\theta JC}$ = 1.5°C/W, and case-to-ambient thermal resistance $R_{\theta CA}$ = 25°C/W.

Find: The maximum power dissipation, and express both temperatures in Kelvin and Fahrenheit.

Solution: Total thermal resistance: $R_{\theta JA} = R_{\theta JC} + R_{\theta CA} = 1.5 + 25 = 26.5$°C/W.

Maximum power: $P_{max} = (T_{j,max} - T_{ambient}) / R_{\theta JA} = (150 - 40) / 26.5 = 110 / 26.5 = 4.15$ W.

Temperature conversions: $T_{j,max}$: 150°C = 150 + 273.15 = 423.15 K = 150 x 9/5 + 32 = 302°F. $T_{ambient}$: 40°C = 40 + 273.15 = 313.15 K = 40 x 9/5 + 32 = 104°F.

P_{max} = 4.15 W. $T_{j,max}$ = 150°C = 423.15 K = 302°F. $T_{ambient}$ = 40°C = 313.15 K = 104°F.

Problem D.4.4

Given: A household uses an average of 30 kWh of electricity per day. The electricity rate is $0.12 per kWh.

Find: The average power consumption in watts, the daily energy in joules and megajoules, and the daily cost.

Solution: Average power: P = E/t = 30 kWh / 24 h = 1.25 kW = 1250 W.

In joules: E = 30 x 3.6 x 10^6 J = 108 x 10^6 J = 108 MJ.

Daily cost: 30 x $0.12 = $3.60.

P_{avg} = 1250 W. E = 108 MJ/day. Cost = $3.60/day.

Problem D.4.5

Given: A noise temperature specification of 75 K for a satellite receiver LNA.

Find: The noise temperature in °C and °F, and the noise figure in dB (T_0 = 290 K).

Solution: Temperature: T = 75 K = 75 - 273.15 = -198.15°C = -198.15 x 9/5 + 32 = -324.67°F.

Noise factor: $F = 1 + T_e/T_0 = 1 + 75/290 = 1 + 0.2586 = 1.2586$.

Noise figure: NF = 10 x $\log_{10}(1.2586)$ = 10 x 0.1000 = 1.00 dB.

T = -198.15°C = -324.67°F. NF = 1.00 dB.

Appendix E — Section E.1: Installing Python

Practice problems covering downloading and installing Python, verifying the installation, and setting up virtual environments.

Problem E.1.1

Given: A user installs Python 3.12.4 on a Windows machine and checks the installation from the Command Prompt.

Find: The command to verify the Python version and the expected output.

Solution:

Run the following in the Command Prompt:

```
python --version
```

Expected output: Python 3.12.4

On Windows, the command is typically `python` (not `python3`). If the command is not found, Python was not added to the system PATH during installation. Reinstall and check "Add Python to PATH."

Problem E.1.2

Given: A macOS user has both the system Python 2.7 (legacy) and a Homebrew-installed Python 3.12 on their machine.

Find: (a) The command to check which Python 3 is being used, (b) the command to verify pip is associated with Python 3, and (c) how to determine the installation path.

Solution:

(a) Check Python 3 version:

```
python3 --version
```

Expected output: Python 3.12.x

(b) Verify pip:

```
pip3 --version
```

Expected output: `pip 24.x from /opt/homebrew/lib/python3.12/site-packages/pip (python 3.12)`

(c) Installation path:

```
which python3
```

Expected output: /opt/homebrew/bin/python3 (Homebrew) or `/usr/local/bin/python3` (python.org installer).

Never use `python` without the 3 on macOS, as it may invoke the legacy Python 2.7.

Problem E.1.3

Given: A user wants to create a virtual environment for the EE-Book scripts in the `scripts/` directory. The system Python is 3.12.

Find: The sequence of commands to (a) navigate to the directory, (b) create the virtual environment, (c) activate it, and (d) verify it is active.

Solution:

(a) Navigate:

```
cd /path/to/EE-Book/scripts
```

(b) Create virtual environment:

```
python3 -m venv .venv
```

(c) Activate (macOS/Linux):

```
source .venv/bin/activate
```

On Windows:

```
.venv\Scripts\activate
```

(d) Verify: The terminal prompt changes to show `(.venv)` at the beginning. Additionally:

```
which python
```

Should output: /path/to/EE-Book/scripts/.venv/bin/python (not the system Python).

```
python --version
```

Should output: Python 3.12.x (the version used to create the venv).

Problem E.1.4

Given: A virtual environment is active and the user needs to install the project dependencies from `requirements.txt`, which contains:

```
marimo>=0.19.0
numpy>=1.26.0
matplotlib>=3.8.0
```

Find: (a) The install command, (b) how to verify all packages are installed, and (c) how to check the installed version of numpy.

Solution:

(a) Install command:

```
pip install -r requirements.txt
```

(b) Verify all packages:

```
pip list
```

This displays all installed packages. Confirm marimo, numpy, and matplotlib appear in the list.

(c) Check numpy version:

```
python -c "import numpy; print(numpy.__version__)"
```

Expected output: 1.26.x (or newer). Alternatively:

```
pip show numpy
```

This displays the version, location, and dependencies of the numpy package.

Problem E.1.5

Given: A user has two projects that require different versions of matplotlib: Project A needs matplotlib 3.7.x and Project B needs matplotlib 3.9.x. Both use Python 3.12.

Find: How to set up separate virtual environments so both projects can coexist without conflicts.

Solution:

Create separate virtual environments in each project directory:

Project A:

```
cd /path/to/project-a
python3 -m venv .venv
source .venv/bin/activate
pip install matplotlib==3.7.5
python -c "import matplotlib; print(matplotlib.__version__)"
```

Output: 3.7.5

Project B:

```
cd /path/to/project-b
python3 -m venv .venv
source .venv/bin/activate
pip install matplotlib==3.9.2
python -c "import matplotlib; print(matplotlib.__version__)"
```

Output: 3.9.2

Each virtual environment is isolated. Activating one deactivates the other. The `.venv` directories should be added to `.gitignore` to avoid committing them to version control.

Problem E.1.6

Given: A user accidentally installed packages into the system Python instead of the virtual environment and wants to clean up.

Find: (a) How to check whether the virtual environment is active, (b) how to deactivate it, and (c) how to delete and recreate the virtual environment for a clean start.

Solution:

(a) Check if venv is active:

```
echo $VIRTUAL_ENV
```

If this prints a path (e.g., `/path/to/scripts/.venv`), the virtual environment is active. If it prints nothing, the system Python is being used.

(b) Deactivate:

```
deactivate
```

(c) Delete and recreate:

```
rm -rf .venv
python3 -m venv .venv
source .venv/bin/activate
pip install -r requirements.txt
```

This creates a completely fresh environment with only the packages listed in `requirements.txt`. No leftover packages from previous installations will be present.

Problem E.1.7

Given: A user needs to freeze the current virtual environment's package list to share with a collaborator who will reproduce the exact setup.

Find: (a) The command to generate a frozen requirements file, (b) what the output looks like, and (c) how the collaborator installs from it.

Solution:

(a) Freeze command:

```
pip freeze > requirements-lock.txt
```

(b) Example output in `requirements-lock.txt`:

```
contourpy==1.2.0
cycler==0.12.1
fonttools==4.47.0
kiwisolver==1.4.5
marimo==0.19.4
matplotlib==3.8.2
numpy==1.26.3
packaging==23.2
pillow==10.2.0
pyparsing==3.1.1
python-dateutil==2.8.2
six==1.16.0
```

All transitive dependencies are pinned to exact versions.

(c) Collaborator installs:

```
python3 -m venv .venv
source .venv/bin/activate
pip install -r requirements-lock.txt
```

This guarantees identical package versions across machines, ensuring reproducible results.

Problem E.1.8

Given: A Linux user (Ubuntu 22.04) finds that `python3 -m venv .venv` fails with the error: "The virtual environment was not created successfully because ensurepip is not available."

Find: The cause of the error and the commands to fix it.

Solution:

The error occurs because Ubuntu packages `python3` and `python3-venv` separately. The `venv` module is not included in the base Python installation on Debian/Ubuntu systems.

Fix:

```
sudo apt update
sudo apt install python3-venv python3-pip
```

After installation, retry:

```
python3 -m venv .venv
source .venv/bin/activate
pip install -r requirements.txt
```

The virtual environment should now be created successfully. This is a Debian/Ubuntu-specific issue — other Linux distributions (Fedora, Arch) and macOS/Windows include venv with the base Python package.

Problem E.1.9

Given: A user wants to upgrade pip inside their virtual environment. The current pip version is 23.0.1 and the latest available is 24.2.

Find: (a) The command to upgrade pip, (b) how to verify the upgrade, and (c) why upgrading pip matters.

Solution:

(a) Upgrade command (from within the active virtual environment):

```
pip install --upgrade pip
```

(b) Verify:

```
pip --version
```

Expected output: pip 24.2 from /path/to/.venv/lib/python3.12/site-packages/pip (python 3.12)

(c) Upgrading pip matters because:

- Newer pip versions resolve dependencies more accurately (using the resolver introduced in pip 20.3)
- Bug fixes for package installation edge cases
- Support for newer packaging standards (PEP 517, PEP 660)
- Better error messages when installations fail
- Security patches

It is good practice to upgrade pip immediately after creating a new virtual environment.

Problem E.1.10

Given: A user is working on a machine without internet access and needs to install packages for the EE-Book scripts. They have a USB drive with the `.whl` (wheel) files for marimo, numpy, and matplotlib (and all dependencies).

Find: The procedure to install packages offline from local wheel files.

Solution:

First, on an internet-connected machine, download all packages and dependencies:

```
pip download -r requirements.txt -d ./packages
```

Copy the `packages/` directory to the USB drive.

On the offline machine:

```
python3 -m venv .venv
source .venv/bin/activate
pip install --no-index --find-links=/path/to/usb/packages -r requirements.txt
```

The `--no-index` flag tells pip not to query PyPI, and `--find-links` points to the local directory containing the `.whl` files. Pip resolves dependencies from the local files only.

Verify the installation:

```
python -c "import marimo; import numpy; import matplotlib; print('All packages installed succes
```

Expected output: All packages installed successfully

Appendix E — Section E.2: Installing and Running marimo

Practice problems covering marimo installation, running notebooks in edit and run modes, and navigating the interface.

Problem E.2.1

Given: A user has an active virtual environment with pip installed and wants to install marimo.

Find: (a) The installation command, (b) the command to verify the installation, and (c) the expected output.

Solution:

(a) Install:

```
pip install marimo
```

(b) Verify:

```
marimo --version
```

(c) Expected output: marimo 0.19.4 (or newer version number).

If the command is not found, ensure the virtual environment is activated. marimo installs a command-line entry point in `.venv/bin/marimo` which is only available when the venv is active.

Problem E.2.2

Given: A user wants to open the signal processing notebook `08_signal_processing.py` for interactive exploration and modification.

Find: (a) The command to open in edit mode, (b) what happens in the browser, and (c) the keyboard shortcut to run a modified cell.

Solution:

(a) Edit mode:

```
cd scripts
marimo edit 08_signal_processing.py
```

(b) The default web browser opens with the marimo editor at `http://localhost:2718`. The notebook displays markdown cells with section descriptions and code cells with matplotlib graphs. Each cell shows its output (graph or text) below the code.

(c) After modifying a cell, press Shift+Enter to run it. Because marimo is reactive, all downstream cells that depend on modified variables automatically re-run and update their outputs.

Problem E.2.3

Given: A professor wants to display the op-amps notebook to a class without showing the Python code — only the descriptions and graphs.

Find: (a) The command for read-only run mode, (b) the difference in display, and (c) how students can interact with it.

Solution:

(a) Run mode:

```
marimo run 13_op_amps.py
```

(b) In run mode, the browser displays only the cell outputs — markdown text and matplotlib graphs — without the Python source code. The interface looks like a clean presentation or report.

(c) In standard run mode, students can view but not modify the code. However, if the notebook includes marimo UI elements (sliders, dropdowns), students can interact with those controls and see the graphs update in real time. For example, a slider controlling the feedback resistance R_f would let students explore how gain changes with resistance.

Problem E.2.4

Given: A user opens the Chapter 12 electric motors notebook and sees the cell dependency graph in the marimo menu.

Find: (a) How to access the dependency graph, (b) what it shows, and (c) what happens when a cell with three downstream dependencies is modified.

Solution:

(a) In the marimo editor, click the hamburger menu (three lines) in the top-left corner, then select "View dependency graph" (or press Ctrl+Shift+D / Cmd+Shift+D on macOS).

(b) The dependency graph shows cells as nodes connected by directed edges. An edge from cell A to cell B means B uses a variable defined in A. For example, if the imports cell defines `np`, and five computation cells use `np`, the imports cell has edges to all five.

(c) When a cell with three downstream dependencies is modified and run:

1. The modified cell executes first
2. All three downstream cells automatically re-execute in dependency order
3. Any cells further downstream of those three also re-execute
4. The graphs and outputs update without manual intervention

This is the core feature of marimo's reactivity — changes cascade through the dependency graph automatically.

Problem E.2.5

Given: A user wants to run two notebooks simultaneously — one for circuit analysis and one for signal processing — to compare graphs side by side.

Find: (a) How to run two marimo sessions at once, (b) the port assignment, and (c) how to specify a custom port.

Solution:

(a) Open two terminal windows (or tabs) with the virtual environment activated in each.

Terminal 1:

```
marimo edit 07_circuit_analysis.py
```

Terminal 2:

```
marimo edit 08_signal_processing.py --port 2719
```

(b) The first instance uses the default port 2718. If a second instance is started without specifying a port, marimo automatically finds the next available port (2719, 2720, etc.).

(c) Explicit port specification:

```
marimo edit 08_signal_processing.py --port 3000
```

The notebook opens at `http://localhost:3000`.

Both notebooks run independently in separate browser tabs and do not share state.

Problem E.2.6

Given: A user modifies parameters in the antenna design notebook but wants to undo all changes and return to the saved version.

Find: (a) How to undo individual changes, (b) how to revert to the last saved state, and (c) how marimo handles unsaved changes.

Solution:

(a) Individual undo within a cell: Ctrl+Z (Cmd+Z on macOS). Each cell maintains its own undo history.

(b) To revert all unsaved changes, close the browser tab without saving. When you reopen the file with `marimo edit`, it loads the last saved version from disk. Alternatively, reload the page (F5 or Ctrl+R) — marimo prompts whether to discard unsaved changes.

(c) Unsaved changes are indicated by a dot or indicator in the tab title or editor header. Pressing Ctrl+S (Cmd+S on macOS) saves all changes to the `.py` file on disk. If the browser is closed without saving, all unsaved changes are lost — marimo does not auto-save.

Problem E.2.7

Given: A user wants to share a notebook with someone who does not have Python or marimo installed.

Find: (a) The command to export to a static HTML file, (b) the command to export to a flat Python script (no marimo dependency), and (c) the limitations of each format.

Solution:

(a) Export to HTML:

```
marimo export html 07_circuit_analysis.py -o circuit_analysis.html
```

This produces a self-contained HTML file with all graphs rendered as static images.

(b) Export to flat script:

```
marimo export script 07_circuit_analysis.py -o circuit_analysis_flat.py
```

This produces a standard Python script that can be run with `python circuit_analysis_flat.py` (requires numpy and matplotlib but not marimo).

(c) Limitations:

- HTML export: Static only — sliders and interactive widgets do not work. Graphs are fixed images. The file can be large if there are many high-resolution plots.
- Script export: No reactivity — modifying a variable does not cascade updates. The script runs top-to-bottom like a normal Python program. Interactive marimo UI elements are stripped out.

Problem E.2.8

Given: A user encounters an error when starting marimo: “Address already in use: (‘localhost’, 2718).”

Find: (a) The cause of the error, (b) how to find and stop the process using the port, and (c) how to start marimo on a different port.

Solution:

(a) The error means another process (likely a previous marimo session) is already listening on port 2718.

(b) Find the process: On macOS/Linux:

```
lsof -i :2718
```

This shows the PID of the process. Kill it:

```
kill <PID>
```

On Windows:

```
netstat -ano | findstr :2718
taskkill /PID <PID> /F
```

(c) Start on a different port:

```
marimo edit 07_circuit_analysis.py --port 8080
```

The notebook opens at http://localhost:8080 instead.

Problem E.2.9

Given: A user wants to run a marimo notebook on a remote server and access it from their laptop's browser.

Find: The commands to (a) start marimo on the server allowing external connections, (b) connect from the laptop, and (c) set up SSH port forwarding for security.

Solution:

(a) Start on the server (bind to all interfaces):

```
marimo edit 07_circuit_analysis.py --host 0.0.0.0 --port 8080
```

(b) From the laptop browser, navigate to:

```
http://<server-ip>:8080
```

This works but is not secure — the connection is unencrypted.

(c) Secure approach with SSH port forwarding: On the laptop:

```
ssh -L 8080:localhost:8080 user@server-ip
```

Then on the server (in the SSH session):

```
marimo edit 07_circuit_analysis.py --port 8080
```

On the laptop browser: http://localhost:8080

The SSH tunnel encrypts all traffic between the laptop and server, and the marimo port is not exposed to the network.

Problem E.2.10

Given: A user wants to launch a marimo notebook in a headless environment (no display, such as a CI server) to validate that all cells execute without errors.

Find: (a) The command to run non-interactively, (b) how to check for errors, and (c) how to capture output.

Solution:

(a) Run as a Python script:

```
python 07_circuit_analysis.py
```

This executes all cells in dependency order without opening a browser.

Alternatively, use marimo's export to validate:

```
marimo export html 07_circuit_analysis.py -o /dev/null
```

If any cell raises an exception, the export fails with an error message.

(b) Check for errors by examining the exit code:

```
python 07_circuit_analysis.py
echo $?
```

Exit code 0 means all cells executed successfully. A non-zero exit code indicates an error.

(c) Capture output:

```
marimo export html 07_circuit_analysis.py -o output.html 2>&1 | tee build.log
```

The HTML file contains all rendered outputs, and `build.log` captures any warnings or errors from the execution.

This approach is useful for continuous integration — automatically validating that all notebooks run without errors after code changes.

Appendix E — Section E.3: Understanding the Script Structure

Practice problems covering imports and app initialization, markdown and code cells, and running marimo notebooks as Python scripts.

Problem E.3.1

Given: A user opens a marimo script and sees the following header:

```python
import marimo
__generated_with = "0.19.4"
app = marimo.App()
```

Find: (a) What each line does, (b) whether the version string affects execution, and (c) what `app` represents.

Solution:

(a) Line-by-line:

- `import marimo` — imports the marimo library
- `__generated_with = "0.19.4"` — records the marimo version that created the notebook (metadata only)
- `app = marimo.App()` — creates the application object that manages all cells and their dependencies

(b) The version string is informational only and does not affect execution. A notebook created with version 0.19.4 can be run with any compatible version of marimo. It serves as a hint for debugging compatibility issues.

(c) The `app` object is the central coordinator. All cells decorated with `@app.cell` register themselves with this object. When the notebook runs, `app` determines execution order based on the dependency graph.

Problem E.3.2

Given: The first code cell of a marimo notebook is:

```
@app.cell
def _():
    import marimo as mo
    import numpy as np
    import matplotlib.pyplot as plt
    return mo, np, plt
```

Find: (a) Why the function is named _, (b) why there are no parameters, (c) what the return statement does, and (d) what happens if `return` is omitted.

Solution:

(a) The function name _ is a convention meaning "no specific name needed." marimo identifies cells by their position and dependencies, not by function names. Any valid Python name works.

(b) No parameters means this cell has no dependencies on other cells. It is a root node in the dependency graph and executes first.

(c) The `return mo, np, plt` statement exports these variables, making them available to other cells. Any cell that lists `np` as a function parameter will receive the numpy module.

(d) If `return` is omitted, the imports are local to this cell only and no other cell can access `mo`, `np`, or `plt`. Other cells would fail with `NameError` when trying to use these modules.

Problem E.3.3

Given: Two cells in a notebook:

```
# Cell A
@app.cell
def _(np):
    R = 1000   # Ω
    C = 10e-6  # F
    tau = R * C
    t = np.linspace(0, 5 * tau, 500)
    return R, C, tau, t

# Cell B
@app.cell
def _(np, plt, R, C, tau, t):
    Vs = 5.0
    v = Vs * (1 - np.exp(-t / tau))
    fig, ax = plt.subplots(figsize=(10, 5))
    ax.plot(t * 1e3, v)
    ax.set_xlabel("Time (ms)")
```

```
    ax.set_ylabel("Voltage (V)")
    ax.set_title(f"RC Charging: R={R}Ω, C={C*1e6:.0f}μF, τ={tau*1e3:.1f}ms")
    fig
    return
```

Find: (a) The dependency relationship between the cells, (b) what happens when R is changed in Cell A, and (c) why Cell B returns nothing.

Solution:

(a) Cell B depends on Cell A through the variables `R`, `C`, `tau`, and `t`. Cell B also depends on the imports cell through `np` and `plt`. Cell A depends only on the imports cell through `np`.

Dependency chain: imports → Cell A → Cell B

(b) When R is changed in Cell A (e.g., from 1000 to 2200):

1. Cell A re-executes, computing new values for `tau` and `t`
2. Cell B automatically re-executes because its inputs (`R`, `tau`, `t`) have changed
3. The graph updates to show a slower charging curve with $\tau = 22$ ms instead of 10 ms

(c) Cell B returns nothing because it does not export any variables for use by other cells. Its only purpose is to compute and display the graph. The last expression `fig` is displayed as the cell output but is not returned for reuse.

Problem E.3.4

Given: A user wants to add a markdown description cell before a computation cell.

Find: (a) The syntax for a markdown cell, (b) how to include formatted text with bold and headings, and (c) how to include a Greek letter and subscript.

Solution:

(a) Markdown cell syntax:

```
@app.cell
def _(mo):
    mo.md("## RC Circuit Time Constant\n\nThe voltage across a charging capacitor...")
    return
```

(b) Formatted text example:

```
@app.cell
def _(mo):
    mo.md("""
    ## RC Circuit Charging

    The **time constant** τ determines how quickly the capacitor charges.
    After **5τ**, the capacitor is considered fully charged (99.3%).
```

```
    ### Key Equations
    - Voltage: V(t) = Vs × (1 - e^(-t/τ))
    - Current: I(t) = (Vs/R) × e^(-t/τ)
    """)
    return
```

(c) Greek letters and subscripts use Unicode in the markdown string:

```
mo.md("The time constant **τ = R × C** where R is resistance in Ω and C is capacitance in F.
```

Output renders as: The time constant $\tau = R \times C$ where R is resistance in Ω and C is capacitance in F. The voltage V_0 is the initial condition.

Problem E.3.5

Given: A notebook has the following cell that creates a graph:

```
@app.cell
def _(np, plt):
    f = np.linspace(1, 100e3, 10000)
    H = 1 / np.sqrt(1 + (f / 10e3)**2)
    H_dB = 20 * np.log10(H)
    fig, ax = plt.subplots(figsize=(10, 5))
    ax.semilogx(f, H_dB)
    ax.set_xlabel("Frequency (Hz)")
    ax.set_ylabel("Magnitude (dB)")
    ax.set_title("Low-Pass Filter Response")
    ax.grid(True, which='both', alpha=0.3)
    ax.set_ylim(-40, 5)
    fig
    return
```

Find: (a) What type of filter this plots, (b) the -3 dB frequency, (c) how to modify it to show a second filter with $f_c = 50$ kHz on the same graph.

Solution:

(a) This is a first-order low-pass filter frequency response (Bode magnitude plot). The transfer function $H(f) = 1/\sqrt{(1 + (f/f_c)^2)}$ has a -20 dB/decade rolloff above the cutoff frequency.

(b) The -3 dB frequency is $f_c = 10$ kHz (set by the denominator `f / 10e3`). At $f = 10$ kHz, $H = 1/\sqrt{2} = -3.01$ dB.

(c) Add the second filter to the same axes:

```
@app.cell
def _(np, plt):
    f = np.linspace(1, 100e3, 10000)
    H1 = 1 / np.sqrt(1 + (f / 10e3)**2)
    H2 = 1 / np.sqrt(1 + (f / 50e3)**2)
```

```
    H1_dB = 20 * np.log10(H1)
    H2_dB = 20 * np.log10(H2)
    fig, ax = plt.subplots(figsize=(10, 5))
    ax.semilogx(f, H1_dB, label="fc = 10 kHz")
    ax.semilogx(f, H2_dB, label="fc = 50 kHz")
    ax.set_xlabel("Frequency (Hz)")
    ax.set_ylabel("Magnitude (dB)")
    ax.set_title("Low-Pass Filter Comparison")
    ax.legend()
    ax.grid(True, which='both', alpha=0.3)
    ax.set_ylim(-40, 5)
    fig
    return
```

Problem E.3.6

Given: The end of every marimo script contains:

```
if __name__ == "__main__":
    app.run()
```

Find: (a) What this block does, (b) when it executes, and (c) what `app.run()` triggers.

Solution:

(a) This is the entry point guard — a standard Python pattern that runs code only when the file is executed directly (not when imported as a module).

(b) It executes when the user runs:

```
python 07_circuit_analysis.py
```

It does not execute when marimo loads the file via `marimo edit` or `marimo run`, because in those cases marimo imports the file rather than running it directly.

(c) `app.run()` launches the marimo notebook viewer in the default web browser, identical to running `marimo run 07_circuit_analysis.py` from the command line. All cells execute in dependency order and their outputs are displayed in read-only mode.

Problem E.3.7

Given: A user wants to use a computed value from one cell as a label in a markdown cell.

Find: (a) How to pass a variable from a code cell to a markdown cell, (b) the syntax for string interpolation in `mo.md()`, and (c) an example.

Solution:

(a) The markdown cell lists the variable as a function parameter, just like any other cell dependency.

(b) Use Python f-strings inside `mo.md()`:

```
@app.cell
def _(mo, tau):
    mo.md(f"The computed time constant is **τ = {tau*1e3:.2f} ms**.")
    return
```

(c) Full example:

```
# Code cell computes tau
@app.cell
def _():
    R = 4700   # Ω
    C = 22e-6   # F
    tau = R * C   # = 0.1034 s
    return R, C, tau

# Markdown cell displays the result
@app.cell
def _(mo, R, C, tau):
    mo.md(f"""
    ## Computed Parameters
    - Resistance: **R = {R:,} Ω**
    - Capacitance: **C = {C*1e6:.0f} µF**
    - Time constant: **τ = {tau*1e3:.1f} ms**
    - 5τ settling time: **{5*tau*1e3:.1f} ms**
    """)
    return
```

Output: $R = 4{,}700\ \Omega$, $C = 22\ \mu F$, $\tau = 103.4$ ms, 5τ settling time = 517.0 ms. When R or C changes, the markdown cell automatically updates with the new values.

Problem E.3.8

Given: A user encounters a `NameError: name 'scipy' is not defined` when adding a new cell that uses `scipy.signal.butter`.

Find: (a) The cause of the error, (b) how to fix it, and (c) the proper way to add a new import.

Solution:

(a) The error means `scipy` has not been imported in any cell, or it was imported but not returned (exported) from the imports cell.

(b) Option 1 — Add scipy to the existing imports cell:

```
@app.cell
def _():
```

```
    import marimo as mo
    import numpy as np
    import matplotlib.pyplot as plt
    import scipy.signal
    return mo, np, plt, scipy
```

Option 2 — Create a new imports cell:

```
@app.cell
def _():
    import scipy.signal
    return scipy,
```

(c) The new cell that uses scipy must list it as a parameter:

```
@app.cell
def _(np, plt, scipy):
    b, a = scipy.signal.butter(4, 1000, fs=44100)
    w, h = scipy.signal.freqz(b, a, fs=44100)
    # ... plot the response
    return
```

Also ensure scipy is installed: `pip install scipy` and add it to `requirements.txt`.

Problem E.3.9

Given: A user notices that two cells in a notebook both define a variable called `R`, causing a marimo error: "Variable 'R' is defined in multiple cells."

Find: (a) Why marimo reports this error, (b) how to fix it, and (c) the design principle behind this restriction.

Solution:

(a) marimo requires that each exported variable is defined in exactly one cell. This is necessary for the reactive dependency graph to be unambiguous — if `R` were defined in two cells, marimo cannot determine which value downstream cells should receive.

(b) Fix by using distinct variable names:

```
# Cell 1: RC circuit
@app.cell
def _():
    R_rc = 1000  # Ω
    C_rc = 10e-6
    return R_rc, C_rc

# Cell 2: RL circuit
@app.cell
def _():
```

```
    R_rl = 470  # Ω
    L_rl = 100e-3
    return R_rl, L_rl
```

Alternatively, if the variable is only used locally and does not need to be exported, do not include it in the return statement. Variables not returned are local to the cell.

(c) The design principle is single-definition, multiple-use — each variable has one authoritative source, enabling deterministic reactive updates. This prevents the ambiguity bugs common in Jupyter notebooks where cells can be run in arbitrary order.

Problem E.3.10

Given: A user wants to understand the execution order of the following four cells:

```
# Cell 1
@app.cell
def _():
    import numpy as np
    return np,

# Cell 2
@app.cell
def _(np):
    t = np.linspace(0, 1, 100)
    return t,

# Cell 3
@app.cell
def _(np, t):
    y = np.sin(2 * np.pi * 5 * t)
    return y,

# Cell 4
@app.cell
def _(t, y):
    import matplotlib.pyplot as plt
    fig, ax = plt.subplots()
    ax.plot(t, y)
    fig
    return
```

Find: (a) The dependency graph, (b) the execution order, and (c) what happens if Cell 2 changes `t` to `np.linspace(0, 2, 200)`.

Solution:

(a) Dependency graph:

- Cell 1: no dependencies → exports `np`
- Cell 2: depends on `np` → exports `t`
- Cell 3: depends on `np` and `t` → exports `y`
- Cell 4: depends on `t` and `y` → exports nothing

```
Cell 1 (np)
  ├── Cell 2 (t)
  │     ├── Cell 3 (y)   ← also depends on np from Cell 1
  │     │     └── Cell 4  ← also depends on t from Cell 2
  │     └── Cell 4
  └── Cell 3
```

(b) Execution order: Cell 1 → Cell 2 → Cell 3 → Cell 4 (topologically sorted).

(c) If Cell 2 changes:

1. Cell 2 re-executes → new `t` with 200 points over 0–2 seconds
2. Cell 3 re-executes (depends on `t`) → new `y` with 10 full sine cycles
3. Cell 4 re-executes (depends on `t` and `y`) → graph updates to show 10 cycles over 2 seconds instead of 5 cycles over 1 second

Cell 1 does not re-execute because it has no upstream dependency on Cell 2.

Appendix E — Section E.4: Modifying and Extending Scripts

Practice problems covering changing parameters, adding new cells, creating new visualizations, and extending marimo notebooks.

Problem E.4.1

Given: The RC circuit charging cell in `07_circuit_analysis.py` has the parameters:

```
R_rc = 47e3   # 47 kΩ
C_rc = 10e-6  # 10 µF
Vs_rc = 9     # 9 V supply
```

A user wants to model a 1 kΩ / 100 µF circuit with a 5 V supply.

Find: (a) The new parameter values, (b) the new time constant, (c) the expected effect on the graph, and (d) the charging voltage at t = 200 ms.

Solution:

(a) New parameters:

```
R_rc = 1e3     # 1 kΩ
C_rc = 100e-6  # 100 µF
Vs_rc = 5      # 5 V supply
```

(b) New time constant: $\tau = R \times C = 1{,}000 \times 100 \times 10^{-6} = 0.1$ s $= 100$ ms

The original $\tau = 47{,}000 \times 10 \times 10^{-6} = 0.47$ s. The new circuit charges 4.7× faster.

(c) The graph will show:

- Lower final voltage (5 V instead of 9 V)
- Faster exponential rise (100 ms time constant instead of 470 ms)
- The 5τ settling time is 500 ms instead of 2.35 s

(d) Voltage at t = 200 ms: $V(200\text{ ms}) = 5 \times (1 - e^{-0.2/0.1}) = 5 \times (1 - e^{-2}) = 5 \times (1 - 0.1353) = 5 \times 0.8647 = 4.32$ V

The capacitor is 86.5% charged after 2τ.

Problem E.4.2

Given: A user wants to add a cell to the circuit analysis notebook that plots the power dissipated in the resistor during RC charging.

Find: (a) The formula for resistor power versus time, (b) the code for the new cell, and (c) the expected shape of the graph.

Solution:

(a) The current during charging: $I(t) = (V_s/R) \times e^{-t/\tau}$ Power in the resistor: $P(t) = I^2R = (V_s^2/R) \times e^{-2t/\tau}$

(b) New cell code:

```
@app.cell
def _(np, plt, R_rc, C_rc, Vs_rc):
    tau = R_rc * C_rc
    t = np.linspace(0, 5 * tau, 500)
    P_resistor = (Vs_rc**2 / R_rc) * np.exp(-2 * t / tau)
    fig, ax = plt.subplots(figsize=(10, 5))
    ax.plot(t * 1e3, P_resistor * 1e3, "r-", linewidth=2)
    ax.set_xlabel("Time (ms)")
    ax.set_ylabel("Power (mW)")
    ax.set_title(f"Resistor Power Dissipation (R={R_rc/1e3:.1f}kΩ)")
    ax.grid(True, alpha=0.3)
    fig
    return
```

(c) The graph shows an exponential decay starting at $P(0) = V_s^2/R$ and decaying to zero with time constant $\tau/2$ (half the RC time constant). The power dissipation is highest at $t = 0$ when the full supply voltage appears across the resistor, and drops to zero as the capacitor charges and the current falls to zero.

Problem E.4.3

Given: A user wants to explore how the quality factor Q affects the frequency response of a series RLC resonant circuit. The current notebook has a fixed Q.

Find: (a) How to add a marimo slider for Q, (b) how to connect it to the graph, and (c) the expected interactive behavior.

Solution:

(a) Add a slider cell:

```
@app.cell
def _(mo):
```

```
    Q_slider = mo.ui.slider(
        start=1, stop=100, value=10, step=1,
        label="Quality Factor Q"
    )
    Q_slider
    return Q_slider,
```

(b) Connect to the computation cell:

```
@app.cell
def _(np, plt, Q_slider):
    Q = Q_slider.value
    f0 = 1000  # 1 kHz resonant frequency
    f = np.logspace(1, 5, 1000)
    H = 1 / np.sqrt((1 - (f/f0)**2)**2 + (f/(Q*f0))**2)
    H_dB = 20 * np.log10(H)
    fig, ax = plt.subplots(figsize=(10, 5))
    ax.semilogx(f, H_dB)
    ax.set_xlabel("Frequency (Hz)")
    ax.set_ylabel("Magnitude (dB)")
    ax.set_title(f"RLC Resonance: f₀ = 1 kHz, Q = {Q}")
    ax.grid(True, which='both', alpha=0.3)
    fig
    return
```

(c) When the user moves the slider:

- $Q = 1$: Broad, flat response with no visible peak (overdamped)
- $Q = 10$: Moderate peak of +20 dB, bandwidth $= f_0/Q = 100$ Hz
- $Q = 100$: Sharp peak of +40 dB, bandwidth = 10 Hz

The graph updates instantly as the slider moves, providing real-time interactive exploration.

Problem E.4.4

Given: A user wants to create a new notebook from scratch for visualizing transformer turns ratio effects.

Find: (a) The command to create a new empty notebook, (b) the minimum cells needed, and (c) a complete minimal example.

Solution:

(a) Create a new notebook:

```
marimo edit transformer_demo.py
```

If the file does not exist, marimo creates a new empty notebook.

(b) Minimum cells:

1. Imports cell
2. At least one computation/output cell

(c) Complete minimal example:

```
import marimo
__generated_with = "0.19.4"
app = marimo.App()

@app.cell
def _():
    import marimo as mo
    import numpy as np
    import matplotlib.pyplot as plt
    return mo, np, plt

@app.cell
def _(mo):
    mo.md("## Transformer Turns Ratio")
    return

@app.cell
def _(np, plt):
    N1_N2 = np.linspace(0.1, 10, 100)
    V_secondary = 120 / N1_N2  # 120V primary
    fig, ax = plt.subplots(figsize=(10, 5))
    ax.plot(N1_N2, V_secondary)
    ax.set_xlabel("Turns Ratio (N₁/N₂)")
    ax.set_ylabel("Secondary Voltage (V)")
    ax.set_title("Transformer: V₂ = V₁ / (N₁/N₂)")
    ax.grid(True, alpha=0.3)
    ax.axhline(y=120, color='r', linestyle='--', label='V₁ = 120V')
    ax.legend()
    fig
    return

if __name__ == "__main__":
    app.run()
```

Problem E.4.5

Given: A user wants to add a cell that displays a data table alongside the graph, showing specific calculated values.

Find: (a) How to create a table using marimo's built-in features, (b) the code for a table showing RC time constant values, and (c) how to format numbers in the table.

Solution:

(a) marimo can display tables using `mo.ui.table()` for interactive tables or markdown for static tables.

(b) Code for a static markdown table:

```
@app.cell
def _(mo):
    rows = []
    for R in [1e3, 4.7e3, 10e3, 47e3, 100e3]:
        C = 10e-6
        tau = R * C
        rows.append(f"| {R/1e3:.1f} kΩ | {C*1e6:.0f} μF | {tau*1e3:.1f} ms | {5*tau*1e3:.0f}

    table = "\n".join(rows)
    mo.md(f"""
    ## RC Time Constant Reference

    | Resistance | Capacitance | τ (ms) | 5τ (ms) |
    |------------|-------------|--------|---------|
    {table}
    """)
    return
```

(c) This produces a formatted table: | Resistance | Capacitance | τ (ms) | 5τ (ms) | |———–|———––|——–|———| | 1.0 kΩ | 10 μF | 10.0 ms | 50 ms | | 4.7 kΩ | 10 μF | 47.0 ms | 235 ms | | 10.0 kΩ | 10 μF | 100.0 ms | 500 ms | | 47.0 kΩ | 10 μF | 470.0 ms | 2350 ms | | 100.0 kΩ | 10 μF | 1000.0 ms | 5000 ms |

Problem E.4.6

Given: A user wants to save a matplotlib figure generated in a marimo notebook as a high-resolution PNG file for inclusion in a report.

Find: (a) The code to save the figure from within a marimo cell, (b) the recommended resolution settings, and (c) how to specify the output path.

Solution:

(a) Add `fig.savefig()` before displaying the figure:

```
@app.cell
def _(np, plt):
    t = np.linspace(0, 1, 1000)
    y = np.sin(2 * np.pi * 5 * t)
    fig, ax = plt.subplots(figsize=(10, 5))
    ax.plot(t, y)
    ax.set_xlabel("Time (s)")
```

```
    ax.set_ylabel("Amplitude")
    fig.savefig("../images/sine_wave.png", dpi=150, bbox_inches="tight",
                facecolor="white", edgecolor="none")
    fig
    return
```

(b) Recommended settings:

- dpi=150: Good balance between quality and file size (300 dpi for print)
- bbox_inches="tight": Removes excess whitespace around the plot
- facecolor="white": Ensures white background (not transparent)
- figsize=(10, 5): 10×5 inches at 150 dpi $= 1500 \times 750$ pixels

(c) The path `"../images/sine_wave.png"` saves to the `images/` directory relative to the `scripts/` directory where the notebook runs. Use absolute paths if needed: `"/full/path/to/images/sine_wave.png`

Problem E.4.7

Given: A user wants to add a dropdown selector that lets the viewer choose between different waveform types (sine, square, triangle, sawtooth) in a signal visualization.

Find: The marimo code for (a) the dropdown UI element, (b) the waveform generation based on selection, and (c) the graph update.

Solution:

(a) Dropdown cell:

```
@app.cell
def _(mo):
    waveform_select = mo.ui.dropdown(
        options=["Sine", "Square", "Triangle", "Sawtooth"],
        value="Sine",
        label="Waveform Type"
    )
    waveform_select
    return waveform_select,
```

(b) and (c) Computation and graph cell:

```
@app.cell
def _(np, plt, waveform_select):
    from scipy import signal as sig
    t = np.linspace(0, 0.01, 1000)  # 10 ms
    f = 1000  # 1 kHz
    choice = waveform_select.value
    if choice == "Sine":
        y = np.sin(2 * np.pi * f * t)
    elif choice == "Square":
```

```
        y = sig.square(2 * np.pi * f * t)
    elif choice == "Triangle":
        y = sig.sawtooth(2 * np.pi * f * t, width=0.5)
    elif choice == "Sawtooth":
        y = sig.sawtooth(2 * np.pi * f * t)
    fig, ax = plt.subplots(figsize=(10, 4))
    ax.plot(t * 1e3, y, linewidth=1.5)
    ax.set_xlabel("Time (ms)")
    ax.set_ylabel("Amplitude")
    ax.set_title(f"{choice} Wave at {f} Hz")
    ax.grid(True, alpha=0.3)
    ax.set_ylim(-1.5, 1.5)
    fig
    return
```

When the dropdown selection changes, the graph reactively updates to show the selected waveform.

Problem E.4.8

Given: A user has created a useful plotting utility function and wants to reuse it across multiple cells in the same notebook.

Find: (a) How to define a reusable function in one cell and use it in others, (b) the code pattern, and (c) any limitations.

Solution:

(a) Define the function in a cell and return it, just like any other variable.

(b) Code pattern:

```
# Utility cell
@app.cell
def _(plt):
    def make_bode_plot(f, H_dB, title="Bode Plot"):
        fig, ax = plt.subplots(figsize=(10, 5))
        ax.semilogx(f, H_dB, linewidth=2)
        ax.set_xlabel("Frequency (Hz)")
        ax.set_ylabel("Magnitude (dB)")
        ax.set_title(title)
        ax.grid(True, which='both', alpha=0.3)
        ax.set_ylim(-60, 10)
        return fig
    return make_bode_plot,

# Usage cell 1
@app.cell
def _(np, make_bode_plot):
```

```
    f = np.logspace(1, 6, 1000)
    H = 20 * np.log10(1 / np.sqrt(1 + (f/1e3)**2))
    fig = make_bode_plot(f, H, "1 kHz Low-Pass Filter")
    fig
    return

# Usage cell 2
@app.cell
def _(np, make_bode_plot):
    f = np.logspace(1, 6, 1000)
    H = 20 * np.log10((f/1e3) / np.sqrt(1 + (f/1e3)**2))
    fig = make_bode_plot(f, H, "1 kHz High-Pass Filter")
    fig
    return
```

(c) Limitations: The function is only available within this notebook. For cross-notebook reuse, move the function to a separate Python module and import it.

Problem E.4.9

Given: A user wants to add error handling to a cell that reads data from an external CSV file, displaying a friendly message if the file is not found.

Find: (a) The code pattern for error handling in a marimo cell, (b) how to display the error in the notebook, and (c) how to provide a fallback.

Solution:

(a) and (b) Use try/except with `mo.md()` for error display:

```
@app.cell
def _(mo, np, plt):
    try:
        data = np.loadtxt("measurements.csv", delimiter=",", skiprows=1)
        t = data[:, 0]
        v = data[:, 1]
        fig, ax = plt.subplots(figsize=(10, 5))
        ax.plot(t, v)
        ax.set_xlabel("Time (s)")
        ax.set_ylabel("Voltage (V)")
        fig
    except FileNotFoundError:
        mo.md("""
        ⚠ **File not found:** `measurements.csv`

        Place the CSV file in the `scripts/` directory with columns: time, voltage.
        Using demo data instead.
```

```
        """)
    return
```

(c) Fallback with synthetic data:

```
@app.cell
def _(mo, np, plt):
    try:
        data = np.loadtxt("measurements.csv", delimiter=",", skiprows=1)
        t, v = data[:, 0], data[:, 1]
        source = "Measured Data"
    except FileNotFoundError:
        t = np.linspace(0, 1, 500)
        v = 3.3 * (1 - np.exp(-t / 0.1)) + np.random.normal(0, 0.05, 500)
        source = "Simulated Data (file not found)"
    fig, ax = plt.subplots(figsize=(10, 5))
    ax.plot(t, v)
    ax.set_title(source)
    ax.set_xlabel("Time (s)")
    ax.set_ylabel("Voltage (V)")
    fig
    return
```

The notebook gracefully handles the missing file by substituting simulated data.

Problem E.4.10

Given: A user wants to create a side-by-side comparison of two graphs in a single cell — one showing the time-domain waveform and one showing the frequency spectrum.

Find: (a) The matplotlib code for subplots, (b) a complete cell implementation, and (c) how to adjust relative subplot sizes.

Solution:

(a) Use `plt.subplots(1, 2)` for horizontal side-by-side layout.

(b) Complete cell:

```
@app.cell
def _(np, plt):
    fs = 10000  # 10 kHz sample rate
    t = np.arange(0, 0.1, 1/fs)  # 100 ms
    f_sig = 200  # 200 Hz signal
    x = np.sin(2 * np.pi * f_sig * t) + 0.3 * np.sin(2 * np.pi * 3 * f_sig * t)

    # Compute FFT
    N = len(t)
    X = np.fft.rfft(x)
```

```python
    freqs = np.fft.rfftfreq(N, 1/fs)
    magnitude = 2 * np.abs(X) / N

    fig, (ax1, ax2) = plt.subplots(1, 2, figsize=(14, 5))

    # Time domain
    ax1.plot(t * 1e3, x, linewidth=1)
    ax1.set_xlabel("Time (ms)")
    ax1.set_ylabel("Amplitude")
    ax1.set_title("Time Domain")
    ax1.set_xlim(0, 20)  # Show 20 ms
    ax1.grid(True, alpha=0.3)

    # Frequency domain
    ax2.stem(freqs, magnitude, linefmt='C1-', markerfmt='C1o', basefmt='k-')
    ax2.set_xlabel("Frequency (Hz)")
    ax2.set_ylabel("Magnitude")
    ax2.set_title("Frequency Spectrum (FFT)")
    ax2.set_xlim(0, 1000)
    ax2.grid(True, alpha=0.3)

    fig.tight_layout()
    fig
    return
```

(c) For unequal subplot widths, use `gridspec_kw`:

```python
fig, (ax1, ax2) = plt.subplots(1, 2, figsize=(14, 5),
                                 gridspec_kw={'width_ratios': [2, 1]})
```

This makes the left plot twice as wide as the right plot.

Appendix F — Section F.1: Matrix Fundamentals

Practice problems covering matrix definitions, notation, special matrices, and construction of matrices from circuit analysis.

Problem F.1.1

Given: A 3×3 matrix $A = [4, -1, 0; -1, 4, -1; 0, -1, 4]$.

Find: (a) The element a_{23}, (b) the main diagonal elements, (c) whether the matrix is symmetric, and (d) the trace (sum of diagonal elements).

Solution:

(a) Element a_{23} is in row 2, column 3: $a_{23} = -1$

(b) Main diagonal elements (a_{11}, a_{22}, a_{33}): $a_{11} = 4$, $a_{22} = 4$, $a_{33} = 4 \rightarrow$ diagonal $= \{4, 4, 4\}$

(c) Check symmetry ($A = A^T$): $a_{12} = -1 = a_{21}$ ✓, $a_{13} = 0 = a_{31}$ ✓, $a_{23} = -1 = a_{32}$ ✓ The matrix is symmetric. This is typical of admittance/impedance matrices in reciprocal circuits.

(d) Trace: $tr(A) = 4 + 4 + 4 = 12$

Problem F.1.2

Given: A circuit has four nodes. Conductances connecting node pairs: $G_{12} = 0.1$ S, $G_{13} = 0.2$ S, $G_{23} = 0.05$ S, $G_{24} = 0.1$ S, $G_{34} = 0.15$ S. A 0.5 S conductance connects node 1 to ground.

Find: The 4×4 nodal admittance matrix Y.

Solution:

Diagonal entries (sum of conductances at each node): $Y_{11} = 0.5 + 0.1 + 0.2 = 0.8$ S $Y_{22} = 0.1 + 0.05 + 0.1 = 0.25$ S $Y_{33} = 0.2 + 0.05 + 0.15 = 0.4$ S $Y_{44} = 0.1 + 0.15 = 0.25$ S

Off-diagonal entries (negative of mutual conductance): $Y_{12} = Y_{21} = -0.1$, $Y_{13} = Y_{31} = -0.2$, $Y_{14} = Y_{41} = 0$ $Y_{23} = Y_{32} = -0.05$, $Y_{24} = Y_{42} = -0.1$ $Y_{34} = Y_{43} = -0.15$

$Y = [0.8, -0.1, -0.2, 0; -0.1, 0.25, -0.05, -0.1; -0.2, -0.05, 0.4, -0.15; 0, -0.1, -0.15, 0.25]$

The matrix is symmetric (reciprocal network) and sparse ($Y_{14} = 0$ because nodes 1 and 4 are not directly connected).

Problem F.1.3

Given: Two column vectors representing node voltages and currents in a two-node circuit: $V = [12, 8]^T$ V and $I = [0.5, -0.3]^T$ A.

Find: (a) The dimensions of each vector, (b) the total power delivered by the sources ($P = V^T I$), and (c) the physical meaning of the negative current at node 2.

Solution:

(a) Both V and I are 2×1 column vectors (2 rows, 1 column).

(b) Total power: $P = V^T I = [12, 8] \times [0.5, -0.3]^T = 12 \times 0.5 + 8 \times (-0.3) = 6.0 - 2.4 = 3.6$ W

(c) The negative current at node 2 ($I_2 = -0.3$ A) means that 0.3 A is leaving node 2 (current is being extracted, not injected). This could represent a load or a current source oriented away from the node.

Problem F.1.4

Given: The identity matrix I_3 (3×3) and a vector $x = [5, -2, 7]^T$.

Find: (a) Write out I_3 explicitly, (b) compute $I_3 \times x$, and (c) explain why the result equals x.

Solution:

(a) $I_3 = [1, 0, 0; 0, 1, 0; 0, 0, 1]$

(b) $I_3 \times x$: Row 1: $1\times5 + 0\times(-2) + 0\times7 = 5$ Row 2: $0\times5 + 1\times(-2) + 0\times7 = -2$ Row 3: $0\times5 + 0\times(-2) + 1\times7 = 7$ Result: $[5, -2, 7]^T = x$

(c) Multiplying by the identity matrix leaves any vector (or matrix) unchanged. This is the matrix equivalent of multiplying a number by 1. In circuit terms, the identity matrix represents a direct connection — the output equals the input with no transformation.

Problem F.1.5

Given: A diagonal matrix $D = \text{diag}(100, 200, 50)$ represents the impedance (in ohms) of three independent branches, and $I = [0.1, 0.05, 0.2]^T$ A are the branch currents.

Find: (a) Write D in full matrix form, (b) compute $V = D \times I$, and (c) the total power dissipated.

Solution:

(a) $D = [100, 0, 0; 0, 200, 0; 0, 0, 50]\ \Omega$

(b) $V = D \times I$: $V_1 = 100 \times 0.1 = 10$ V $V_2 = 200 \times 0.05 = 10$ V $V_3 = 50 \times 0.2 = 10$ V $V = [10, 10, 10]^T$ V

All branches have the same voltage (10 V) despite different impedances and currents.

(c) Total power: $P = V_1I_1 + V_2I_2 + V_3I_3 = 10 \times 0.1 + 10 \times 0.05 + 10 \times 0.2 = 1.0 + 0.5 + 2.0 = 3.5$ W

Problem F.1.6

Given: A mesh impedance matrix for a three-mesh circuit: $Z = [R_1 + R_2, -R_2, 0; -R_2, R_2 + R_3 + R_4, -R_4; 0, -R_4, R_4 + R_5]$ with $R_1 = 10\ \Omega$, $R_2 = 20\ \Omega$, $R_3 = 15\ \Omega$, $R_4 = 30\ \Omega$, $R_5 = 25\ \Omega$.

Find: (a) The numerical Z matrix, (b) verify it is symmetric, and (c) explain why $Z_{13} = 0$.

Solution:

(a) Numerical values: $Z_{11} = 10 + 20 = 30\ \Omega$ $Z_{22} = 20 + 15 + 30 = 65\ \Omega$ $Z_{33} = 30 + 25 = 55\ \Omega$ $Z_{12} = Z_{21} = -20\ \Omega$ $Z_{23} = Z_{32} = -30\ \Omega$ $Z_{13} = Z_{31} = 0\ \Omega$

$Z = [30, -20, 0; -20, 65, -30; 0, -30, 55]\ \Omega$

(b) $Z_{12} = Z_{21} = -20$ ✓, $Z_{13} = Z_{31} = 0$ ✓, $Z_{23} = Z_{32} = -30$ ✓ The matrix is symmetric, as expected for a reciprocal passive network.

(c) $Z_{13} = 0$ because meshes 1 and 3 share no common element. R_2 is shared between meshes 1 and 2, and R_4 is shared between meshes 2 and 3, but no resistor appears in both mesh 1 and mesh 3.

Problem F.1.7

Given: A sparse bus admittance matrix for a 5-bus power system where only adjacent buses are connected:

Bus connections: 1-2 ($y = 0.5 - j2.0$), 2-3 ($y = 0.3 - j1.5$), 3-4 ($y = 0.4 - j1.8$), 4-5 ($y = 0.2 - j1.0$). Each bus has a shunt admittance to ground of $j0.05$.

Find: (a) The diagonal element Y_{33}, (b) the number of non-zero elements in the 5×5 matrix, and (c) the sparsity (percentage of zero elements).

Solution:

(a) Y_{33} = sum of all admittances at bus 3: $Y_{33} = y_{23} + y_{34} + y_{30} = (0.3 - j1.5) + (0.4 - j1.8) + j0.05 = 0.7 - j3.25$ S

(b) Non-zero elements:

- 5 diagonal elements (Y_{11} through Y_{55})
- 4 connections $\times$ 2 (symmetric upper and lower triangle) = 8 off-diagonal elements Total non-zero: $5 + 8 = 13$ elements

(c) Total elements in 5×5 matrix: 25. Zero elements: $25 - 13 = 12$. Sparsity $= 12/25 = 48\%$

For larger power systems (hundreds of buses), sparsity exceeds 95%, making sparse matrix techniques essential.

Problem F.1.8

Given: A matrix represents the voltage gains of a three-stage amplifier cascade. Each stage is characterized by a 2×2 matrix. Stage 1: $[A_1, B_1; C_1, D_1] = [5, 0; 0, 1]$, Stage 2: $[1, 50; 0, 1]$, Stage 3: $[10, 0; 0, 1]$.

Find: (a) What type of matrix Stage 1 represents, (b) what type Stage 2 represents, and (c) whether multiplication order matters for cascading.

Solution:

(a) Stage 1 is a diagonal matrix (non-zero only on diagonal). It represents a voltage amplifier with gain $A = 5$ and no series impedance. This is an ideal voltage amplifier.

(b) Stage 2 has ones on the diagonal and a non-zero B parameter. This represents a series impedance of 50 Ω (an ABCD representation where $A = D = 1$, $B = Z$, $C = 0$). It is a series element inserted in the signal path.

(c) Matrix multiplication is not commutative ($AB \neq BA$ in general), so the cascade order matters.

$M = M_1 \times M_2 \times M_3$ (signal flows left to right)

If the order were reversed ($M_3 \times M_2 \times M_1$), the result would be different. Physically, the signal encounters Stage 1 first, then Stage 2, then Stage 3 — this order must be preserved in the matrix product.

Problem F.1.9

Given: A matrix $A = [3, 1; 1, 3]$ represents the admittance matrix of a two-port network.

Find: (a) Whether A is symmetric, (b) the eigenvalues of A, and (c) the physical meaning of the eigenvalues.

Solution:

(a) $a_{12} = 1 = a_{21}$, so the matrix is symmetric. This indicates a reciprocal two-port network.

(b) Eigenvalues from $\det(A - \lambda I) = 0$: $(3 - \lambda)(3 - \lambda) - (1)(1) = 0$ $(3 - \lambda)^2 - 1 = 0$ $(3 - \lambda)^2 = 1$ $3 - \lambda = \pm 1$ $\lambda_1 = 3 - 1 = 2$, $\lambda_2 = 3 + 1 = 4$

(c) The eigenvalues represent the natural modes of the network:

- $\lambda_1 = 2$ S: The even mode (both ports driven in phase) — the effective admittance is 2 S
- $\lambda_2 = 4$ S: The odd mode (ports driven out of phase) — the effective admittance is 4 S

The eigenvectors are $[1, 1]^T$ (even mode) and $[1, -1]^T$ (odd mode). These modes decouple the two-port into two independent one-port problems.

Problem F.1.10

Given: A 3×2 matrix represents the incidence matrix of a circuit graph with 3 nodes and 2 branches: $A = [1, 0; -1, 1; 0, -1]$

Find: (a) The dimensions and their meaning, (b) what the entries tell us about the circuit topology, and (c) the product $A^T A$ and its significance.

Solution:

(a) A is 3×2 — 3 rows (nodes) and 2 columns (branches). Each column represents a branch, and each row represents a node.

(b) The entries:

- Branch 1 (column 1): $+1$ at node 1, -1 at node 2 → branch 1 goes from node 1 to node 2
- Branch 2 (column 2): $+1$ at node 2, -1 at node 3 → branch 2 goes from node 2 to node 3

Each branch has exactly one $+1$ (start node) and one -1 (end node), indicating the assumed current direction.

(c) $A^T A$: $A^T = [1, -1, 0; 0, 1, -1]$ $A^T A = [1{\times}1+(-1)(-1)+0, 1{\times}0+(-1)(1)+0; 0+(1)(-1)+0, 0+1{\times}1+(-1)(-1)] = [2, -1; -1, 2]$

This is the graph Laplacian of the circuit. In circuit analysis, if each branch has unit conductance, then $A^T A$ is the nodal admittance matrix (with the reference node eliminated). The diagonal entries count the number of branches at each node, and the off-diagonal entries are -1 for connected node pairs.

Appendix F — Section F.2: Matrix Arithmetic

Practice problems covering matrix addition, subtraction, scalar multiplication, and matrix multiplication.

Problem F.2.1

Given: Two admittance matrices from independent source contributions in a three-node circuit: Y_1 = [0.4, −0.1, −0.2; −0.1, 0.3, −0.1; −0.2, −0.1, 0.5] S and Y_2 = [0.1, 0, −0.1; 0, 0.2, −0.1; −0.1, −0.1, 0.3] S.

Find: The total admittance matrix $Y_{total} = Y_1 + Y_2$.

Solution:

Adding element by element: $Y_{total}(1,1) = 0.4 + 0.1 = 0.5$ $Y_{total}(1,2) = -0.1 + 0 = -0.1$ $Y_{total}(1,3) = -0.2 + (-0.1) = -0.3$ $Y_{total}(2,1) = -0.1 + 0 = -0.1$ $Y_{total}(2,2) = 0.3 + 0.2 = 0.5$ $Y_{total}(2,3) = -0.1 + (-0.1) = -0.2$ $Y_{total}(3,1) = -0.2 + (-0.1) = -0.3$ $Y_{total}(3,2) = -0.1 + (-0.1) = -0.2$ $Y_{total}(3,3) = 0.5 + 0.3 = 0.8$

Y_{total} = [0.5, −0.1, −0.3; −0.1, 0.5, −0.2; −0.3, −0.2, 0.8] S

The result is symmetric, confirming both sub-networks are reciprocal.

Problem F.2.2

Given: A current source vector $I_{sources} = [3, -1, 2]^T$ A and a load current vector $I_{loads} = [1.5, 0.8, 1.2]^T$ A at three nodes.

Find: (a) The net current injection vector $I_{net} = I_{sources} - I_{loads}$, and (b) the physical meaning of each entry.

Solution:

(a) $I_{net} = I_{sources} - I_{loads}$: $I_{net}(1) = 3 - 1.5 = 1.5$ A $I_{net}(2) = -1 - 0.8 = -1.8$ A $I_{net}(3) = 2 - 1.2 = 0.8$ A

$I_{net} = [1.5, -1.8, 0.8]^T$ A

(b) Physical meaning:

- Node 1: Net 1.5 A injected (source exceeds load)
- Node 2: Net 1.8 A extracted (load exceeds source; the original source was already extracting 1 A)
- Node 3: Net 0.8 A injected (source exceeds load)

KCL requires the sum of all net injections to equal zero for the internal nodes, accounting for current through ground connections.

Problem F.2.3

Given: An impedance matrix Z = [50, 10; 10, 40] Ω is specified at 60 Hz. The system is being converted to per-unit with a base impedance $Z_{base} = 100\ \Omega$.

Find: (a) The per-unit impedance matrix Z_{pu}, (b) verify that the per-unit matrix preserves the symmetry, and (c) the per-unit current if the base power is 1 MVA at 10 kV base voltage.

Solution:

(a) Per-unit conversion is scalar multiplication by $1/Z_{base}$: $Z_{pu} = Z / Z_{base} = (1/100) \times [50, 10; 10, 40] = [0.50, 0.10; 0.10, 0.40]$ pu

(b) $Z_{pu}(1,2) = 0.10 = Z_{pu}(2,1)$ ✓. Symmetry is preserved because scalar multiplication does not affect the relative values of elements.

(c) Base current: $I_{base} = S_{base} / (\sqrt{3} \times V_{base}) = 1 \times 10^6 / (\sqrt{3} \times 10{,}000) = 57.74$ A

If a per-unit current is $I_{pu} = 0.8$, the actual current is $0.8 \times 57.74 = 46.19$ A.

Problem F.2.4

Given: Two matrices: A = [2, 3; 1, 4] and B = [5, 1; 2, 3].

Find: (a) AB, (b) BA, and (c) verify that AB ≠ BA.

Solution:

(a) AB: $(AB)_{11} = 2\times5 + 3\times2 = 10 + 6 = 16$ $(AB)_{12} = 2\times1 + 3\times3 = 2 + 9 = 11$ $(AB)_{21} = 1\times5 + 4\times2 = 5 + 8 = 13$ $(AB)_{22} = 1\times1 + 4\times3 = 1 + 12 = 13$ AB = [16, 11; 13, 13]

(b) BA: $(BA)_{11} = 5\times2 + 1\times1 = 10 + 1 = 11$ $(BA)_{12} = 5\times3 + 1\times4 = 15 + 4 = 19$ $(BA)_{21} = 2\times2 + 3\times1 = 4 + 3 = 7$ $(BA)_{22} = 2\times3 + 3\times4 = 6 + 12 = 18$ BA = [11, 19; 7, 18]

(c) AB = [16, 11; 13, 13] ≠ BA = [11, 19; 7, 18]. Matrix multiplication is not commutative. In circuit analysis, this means the order of cascaded two-port networks matters.

Problem F.2.5

Given: A 3 × 3 impedance matrix Z = [20, 5, 0; 5, 15, 5; 0, 5, 25] Ω and a current vector $I = [1, 2, 0.5]^T$ A.

Find: The voltage vector V = Z × I by performing the matrix-vector multiplication.

Solution:

V_1 = 20×1 + 5×2 + 0×0.5 = 20 + 10 + 0 = 30 V V_2 = 5×1 + 15×2 + 5×0.5 = 5 + 30 + 2.5 = 37.5 V V_3 = 0×1 + 5×2 + 25×0.5 = 0 + 10 + 12.5 = 22.5 V

$V = [30, 37.5, 22.5]^T$ V

Verification: Total power = $V^T I$ = 30×1 + 37.5×2 + 22.5×0.5 = 30 + 75 + 11.25 = 116.25 W.

Problem F.2.6

Given: Two ABCD matrices representing cascaded transmission line sections: $M_1 = [\cosh(\gamma_1 l_1), Z_{01} \sinh(\gamma_1 l_1); \sinh(\gamma_1 l_1)/Z_{01}, \cosh(\gamma_1 l_1)]$ with numerical values M_1 = [0.95, 25; 0.01, 0.95] and M_2 = [0.90, 40; 0.008, 0.90].

Find: The overall ABCD matrix $M_{total} = M_1 \times M_2$.

Solution:

A = 0.95×0.90 + 25×0.008 = 0.855 + 0.200 = 1.055 B = 0.95×40 + 25×0.90 = 38.0 + 22.5 = 60.5 C = 0.01×0.90 + 0.95×0.008 = 0.009 + 0.0076 = 0.0166 D = 0.01×40 + 0.95×0.90 = 0.40 + 0.855 = 1.255

M_{total} = [1.055, 60.5; 0.0166, 1.255]

Verification: $\det(M_{total}) = 1.055 \times 1.255 - 60.5 \times 0.0166 = 1.324 - 1.004 = 0.320$. Note: $\det(M_1) = 0.95^2 - 25 \times 0.01 = 0.9025 - 0.25 = 0.6525$ and $\det(M_2) = 0.81 - 0.32 = 0.49$. $\det(M_{total}) = \det(M_1) \times \det(M_2) = 0.6525 \times 0.49 = 0.320$ ✓

Problem F.2.7

Given: A scaling operation on a state-space system. The original A matrix is: A = [−2, 1; 0, −3] The system is time-scaled by a factor of 2 (running twice as fast), which multiplies A by 2.

Find: (a) The scaled A matrix, (b) the eigenvalues of the original A, and (c) the eigenvalues of the scaled A.

Solution:

(a) Scaled matrix: $A_{scaled} = 2A = 2 \times [-2, 1; 0, -3] = [-4, 2; 0, -6]$

(b) Original eigenvalues (A is upper triangular, so eigenvalues are diagonal elements): $\lambda_1 = -2$, $\lambda_2 = -3$

Both poles are in the left half-plane, confirming stability.

(c) Scaled eigenvalues: $\lambda_{1\text{scaled}} = 2 \times (-2) = -4$, $\lambda_{2\text{scaled}} = 2 \times (-3) = -6$

Scalar multiplication by k multiplies all eigenvalues by k. The time-scaled system is still stable but responds twice as fast (poles are farther from the imaginary axis).

Problem F.2.8

Given: A 2×3 matrix A = [1, 2, 3; 4, 5, 6] and a 3×2 matrix B = [7, 8; 9, 10; 11, 12].

Find: (a) The dimensions of AB, (b) the product AB, (c) the dimensions of BA, and (d) the product BA.

Solution:

(a) A is 2×3, B is 3×2, so AB is 2×2.

(b) AB: $(AB)_{11} = 1{\times}7 + 2{\times}9 + 3{\times}11 = 7 + 18 + 33 = 58$ $(AB)_{12} = 1{\times}8 + 2{\times}10 + 3{\times}12 = 8 + 20 + 36 = 64$ $(AB)_{21} = 4{\times}7 + 5{\times}9 + 6{\times}11 = 28 + 45 + 66 = 139$ $(AB)_{22} = 4{\times}8 + 5{\times}10 + 6{\times}12 = 32 + 50 + 72 = 154$ AB = [58, 64; 139, 154]

(c) B is 3×2, A is 2×3, so BA is 3×3.

(d) BA: Row 1: [7×1+8×4, 7×2+8×5, 7×3+8×6] = [39, 54, 69] Row 2: [9×1+10×4, 9×2+10×5, 9×3+10×6] = [49, 68, 87] Row 3: [11×1+12×4, 11×2+12×5, 11×3+12×6] = [59, 82, 105] BA = [39, 54, 69; 49, 68, 87; 59, 82, 105]

Note: AB is 2×2 while BA is 3×3 — even the dimensions differ when the matrices are not square.

Problem F.2.9

Given: The mesh impedance equation ZI = V for a two-mesh circuit with: Z = [$R_1 + j\omega L_1$, $-j\omega M$; $-j\omega M$, $R_2 + j\omega L_2$] $R_1 = 10\ \Omega$, $R_2 = 20\ \Omega$, $\omega L_1 = 30\ \Omega$, $\omega L_2 = 40\ \Omega$, $\omega M = 15\ \Omega$ (mutual inductance). Source voltage $V = [100\angle 0°, 0]^T$ V.

Find: (a) The numerical Z matrix (in complex form), and (b) the product Z × I for $I = [2 - j1, 1 + j0.5]^T$ A (to check a proposed solution).

Solution:

(a) Z matrix: $Z_{11} = 10 + j30\ \Omega$ $Z_{12} = Z_{21} = -j15\ \Omega$ $Z_{22} = 20 + j40\ \Omega$

Z = [10 + j30, −j15; −j15, 20 + j40] Ω

(b) ZI: $V_1 = (10 + j30)(2 - j1) + (-j15)(1 + j0.5) = (20 - j10 + j60 - j^2 30) + (-j15 - j^2 7.5) = (20 - j10 + j60 + 30) + (-j15 + 7.5) = (50 + j50) + (7.5 - j15) = 57.5 + j35$ V

$V_2 = (-j15)(2 - j1) + (20 + j40)(1 + j0.5) = (-j30 + j^2 15) + (20 + j10 + j40 + j^2 20) = (-15 - j30) + (0 + j50) = -15 + j20$ V

Since V_1 should equal $100\angle 0° = 100 + j0$, the proposed solution $I = [2 - j1, 1 + j0.5]^T$ is not correct. The actual mesh currents would need to be found by solving ZI = V using matrix inversion.

Problem F.2.10

Given: A discrete-time state update requires the operation $x[k+1] = Ax[k] + Bu[k]$ where: $A = [0.9, 0.1; 0, 0.8]$, $B = [0; 1]$, $x[0] = [1, 0]^T$, $u[0] = 0.5$.

Find: (a) The state $x[1]$, (b) the state $x[2]$ (with $u[1] = 0.5$), and (c) the steady-state response as $k \to \infty$ (with constant $u = 0.5$).

Solution:

(a) $x[1] = Ax[0] + Bu[0]$: $Ax[0] = [0.9\times1+0.1\times0, 0\times1+0.8\times0] = [0.9, 0]$ $Bu[0] = [0\times0.5, 1\times0.5] = [0, 0.5]$ $x[1] = [0.9 + 0, 0 + 0.5] = [0.9, 0.5]^T$

(b) $x[2] = Ax[1] + Bu[1]$: $Ax[1] = [0.9\times0.9+0.1\times0.5, 0\times0.9+0.8\times0.5] = [0.81+0.05, 0.40] = [0.86, 0.40]$ $Bu[1] = [0, 0.5]$ $x[2] = [0.86, 0.40+0.50] = [0.86, 0.90]^T$

(c) At steady state, $x[\infty] = Ax[\infty] + Bu$, so $(I - A)x[\infty] = Bu$: $I - A = [0.1, -0.1; 0, 0.2]$ $\det(I - A) = 0.1 \times 0.2 - (-0.1)(0) = 0.02$ $(I - A)^{-1} = (1/0.02) \times [0.2, 0.1; 0, 0.1] = [10, 5; 0, 5]$ $x[\infty] = (I - A)^{-1}Bu = [10, 5; 0, 5] \times [0, 0.5]^T = [10\times0+5\times0.5, 0+5\times0.5] = [2.5, 2.5]^T$

Appendix F — Section F.3: Matrix Operations

Practice problems covering the transpose, determinant, and matrix inverse.

Problem F.3.1

Given: A matrix A = [2, 5, 1; 3, 7, 4; 8, 6, 9].

Find: (a) A^T, (b) the product A^TA, and (c) verify that A^TA is symmetric.

Solution:

(a) A^T = [2, 3, 8; 5, 7, 6; 1, 4, 9] (rows become columns)

(b) A^TA (3 × 3): $(A^TA)_{11}$ = 2×2+3×3+8×8 = 4+9+64 = 77 $(A^TA)_{12}$ = 2×5+3×7+8×6 = 10+21+48 = 79 $(A^TA)_{13}$ = 2×1+3×4+8×9 = 2+12+72 = 86 $(A^TA)_{22}$ = 5×5+7×7+6×6 = 25+49+36 = 110 $(A^TA)_{23}$ = 5×1+7×4+6×9 = 5+28+54 = 87 $(A^TA)_{33}$ = 1×1+4×4+9×9 = 1+16+81 = 98

A^TA = [77, 79, 86; 79, 110, 87; 86, 87, 98]

(c) By construction, $(A^TA)^T = A^T(A^T)^T = A^TA$. Checking: $(A^TA)_{12} = 79 = (A^TA)_{21}$ ✓, $(A^TA)_{13} = 86 = (A^TA)_{31}$ ✓, $(A^TA)_{23} = 87 = (A^TA)_{32}$ ✓. A^TA is always symmetric — this is a fundamental property used in least-squares fitting and normal equations.

Problem F.3.2

Given: A 2 × 2 mesh impedance matrix: Z = [25, −8; −8, 18] Ω.

Find: (a) The determinant, (b) whether the system has a unique solution, and (c) the physical meaning of a non-zero determinant.

Solution:

(a) det(Z) = (25)(18) − (−8)(−8) = 450 − 64 = 386

(b) Since det(Z) = 386 ≠ 0, the system has a unique solution. The mesh currents can be determined uniquely for any set of source voltages.

(c) A non-zero determinant means:

- The two mesh equations are linearly independent (neither is a multiple of the other)
- The circuit has no redundant or contradictory constraints
- The impedance matrix is invertible, so $I = Z^{-1}V$ can be computed

If the determinant were zero, it would indicate that the two meshes are not independent — for example, if one mesh were completely contained within another, or if a wire short removed a degree of freedom.

Problem F.3.3

Given: A 3×3 matrix B = [2, 1, 0; 1, 3, 1; 0, 1, 2].

Find: The determinant using cofactor expansion along the first row.

Solution:

$\det(B) = b_{11} \times C_{11} - b_{12} \times C_{12} + b_{13} \times C_{13}$

where C_{ij} is the cofactor (signed minor).

$C_{11} = \det[3, 1; 1, 2] = 3\times2 - 1\times1 = 5$ $C_{12} = \det[1, 1; 0, 2] = 1\times2 - 1\times0 = 2$ $C_{13} = \det[1, 3; 0, 1] = 1\times1 - 3\times0 = 1$

$\det(B) = 2\times5 - 1\times2 + 0\times1 = 10 - 2 + 0 = 8$

Since $\det(B) \neq 0$, the matrix is invertible.

Problem F.3.4

Given: An admittance matrix Y = [0.2, −0.05; −0.05, 0.1] S and current vector $I = [0.5, 0.3]^T$ A.

Find: (a) The determinant of Y, (b) the inverse Y^{-1}, and (c) the node voltages $V = Y^{-1}I$.

Solution:

(a) $\det(Y) = (0.2)(0.1) - (-0.05)(-0.05) = 0.02 - 0.0025 = 0.0175$

(b) For a 2×2 matrix [a, b; c, d], the inverse is (1/det) × [d, −b; −c, a]: $Y^{-1} = (1/0.0175) \times [0.1, 0.05; 0.05, 0.2] = [5.714, 2.857; 2.857, 11.429]\ \Omega$ (the impedance matrix $Z = Y^{-1}$)

(c) $V = Y^{-1}I = Z \times I$: $V_1 = 5.714 \times 0.5 + 2.857 \times 0.3 = 2.857 + 0.857 = 3.714$ V $V_2 = 2.857 \times 0.5 + 11.429 \times 0.3 = 1.429 + 3.429 = 4.857$ V

Verification: $YV = [0.2\times3.714+(-0.05)\times4.857,\ (-0.05)\times3.714+0.1\times4.857] = [0.743-0.243, -0.186+0.486] = [0.500, 0.300] = I$ ✓

Problem F.3.5

Given: A matrix M = [1, 2; 2, 4].

Find: (a) The determinant, (b) whether the inverse exists, and (c) the physical interpretation in a circuit context.

Solution:

(a) det(M) = 1×4 − 2×2 = 4 − 4 = 0

(b) Since det(M) = 0, the matrix is singular and has no inverse. The system MX = B does not have a unique solution.

(c) In a circuit context, a singular impedance or admittance matrix means:

- The equations are linearly dependent (row 2 = 2 × row 1)
- The circuit has a redundant constraint or a missing equation
- There is either no solution or infinitely many solutions
- Physically, this could represent a floating node, a short circuit that eliminates a degree of freedom, or a dependent source creating a proportional relationship between meshes

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Problem F.3.6

Given: The property $(AB)^{-1} = B^{-1}A^{-1}$ for invertible matrices A and B. A = [3, 1; 2, 2] and B = [1, −1; 0, 2].

Find: (a) A^{-1}, (b) B^{-1}, (c) AB, (d) $(AB)^{-1}$ directly, and (e) $B^{-1}A^{-1}$, verifying that (d) equals (e).

Solution:

(a) det(A) = 6 − 2 = 4. A^{-1} = (1/4)[2, −1; −2, 3] = [0.5, −0.25; −0.5, 0.75]

(b) det(B) = 2 − 0 = 2. B^{-1} = (1/2)[2, 1; 0, 1] = [1, 0.5; 0, 0.5]

(c) AB: $(AB)_{11}$ = 3×1+1×0 = 3, $(AB)_{12}$ = 3×(−1)+1×2 = −1 $(AB)_{21}$ = 2×1+2×0 = 2, $(AB)_{22}$ = 2×(−1)+2×2 = 2 AB = [3, −1; 2, 2]

(d) det(AB) = 6+2 = 8. $(AB)^{-1}$ = (1/8)[2, 1; −2, 3] = [0.25, 0.125; −0.25, 0.375]

(e) $B^{-1}A^{-1}$: $(B^{-1}A^{-1})_{11}$ = 1×0.5+0.5×(−0.5) = 0.5−0.25 = 0.25 $(B^{-1}A^{-1})_{12}$ = 1×(−0.25)+0.5×0.75 = −0.25+0.375 = 0.125 $(B^{-1}A^{-1})_{21}$ = 0×0.5+0.5×(−0.5) = −0.25 $(B^{-1}A^{-1})_{22}$ = 0×(−0.25)+0.5×0.75 = 0.375 $B^{-1}A^{-1}$ = [0.25, 0.125; −0.25, 0.375] ✓ matches $(AB)^{-1}$

Problem F.3.7

Given: A 3 × 3 admittance matrix for a three-node circuit: Y = [0.5, −0.2, 0; −0.2, 0.5, −0.2; 0, −0.2, 0.5]

Find: (a) The determinant using cofactor expansion, and (b) the condition number estimate (ratio of largest to smallest diagonal).

Solution:

(a) Cofactor expansion along row 1: $\det(Y) = 0.5 \times \det[0.5, -0.2; -0.2, 0.5] - (-0.2) \times \det[-0.2, -0.2; 0, 0.5] + 0 \times (\ldots)$

$\det[0.5, -0.2; -0.2, 0.5] = 0.25 - 0.04 = 0.21$ $\det[-0.2, -0.2; 0, 0.5] = (-0.2)(0.5) - (-0.2)(0) = -0.10$

$\det(Y) = 0.5 \times 0.21 + 0.2 \times (-0.10) = 0.105 - 0.020 = 0.085$

(b) All diagonal elements are equal (0.5), so the diagonal ratio is 1.0. However, a better condition estimate uses the eigenvalues. The tridiagonal structure with diagonal 0.5 and off-diagonal −0.2 gives eigenvalues: $\lambda_k = 0.5 - 2\times0.2\times\cos(k\pi/4)$ for $k = 1,2,3$ $\lambda_1 = 0.5 - 0.4\times\cos(\pi/4) = 0.5 - 0.283 = 0.217$ $\lambda_2 = 0.5 - 0.4\times\cos(\pi/2) = 0.5$ $\lambda_3 = 0.5 - 0.4\times\cos(3\pi/4) = 0.5 + 0.283 = 0.783$

Condition number $\approx \lambda_{max}/\lambda_{min} = 0.783/0.217 = 3.6$

This is well-conditioned; numerical solutions will be accurate.

Problem F.3.8

Given: A rotation matrix $R(\theta) = [\cos\theta, -\sin\theta; \sin\theta, \cos\theta]$ for $\theta = 30°$.

Find: (a) The numerical matrix, (b) the determinant, (c) the inverse R^{-1}, and (d) verify that $R^{-1} = R^T$ (orthogonal matrix).

Solution:

(a) $R(30°)$: $\cos 30° = \sqrt{3}/2 = 0.8660$, $\sin 30° = 0.5$ $R = [0.8660, -0.5; 0.5, 0.8660]$

(b) $\det(R) = 0.8660^2 + 0.5^2 = 0.75 + 0.25 = 1.0$

A rotation matrix always has determinant 1 (preserves area and orientation).

(c) $R^{-1} = (1/1.0) \times [0.8660, 0.5; -0.5, 0.8660] = [0.8660, 0.5; -0.5, 0.8660]$

This is $R(-30°)$, which makes sense: the inverse of a 30° rotation is a −30° rotation.

(d) $R^T = [0.8660, 0.5; -0.5, 0.8660] = R^{-1}$ ✓

R is orthogonal ($R^TR = I$), a property of all rotation matrices. In signal processing, rotation matrices appear in coordinate transformations (e.g., Park's transform for three-phase systems).

Problem F.3.9

Given: A complex impedance matrix for an AC circuit: $Z = [10 + j5, -j3; -j3, 8 + j6]$

Find: (a) The determinant (complex), (b) the inverse, and (c) the current I when $V = [20\angle0°, 0]^T$.

Solution:

(a) $\det(Z) = (10 + j5)(8 + j6) - (-j3)(-j3) = 80 + j60 + j40 + j^2 30 - j^2 9 = 80 + j100 - 30 + 9 = 59 + j100$

$|\det(Z)| = \sqrt{(59^2 + 100^2)} = \sqrt{(3481 + 10000)} = \sqrt{13481} = 116.1$

(b) $Z^{-1} = (1/\det) \times [8 + j6, j3; j3, 10 + j5] = 1/(59 + j100) \times [8 + j6, j3; j3, 10 + j5]$

Converting $1/(59 + j100)$: multiply by conjugate: $= (59 - j100)/(59^2 + 100^2) = (59 - j100)/13481 = 0.004376 - j0.007418$

${Z^{-1}}_{11} = (0.004376 - j0.007418)(8 + j6) = 0.0350 - j0.0594 + j0.0263 + 0.0445 = 0.0795 - j0.0331$ ${Z^{-1}}_{12} = (0.004376 - j0.007418)(j3) = j0.01313 + 0.02225 = 0.02225 + j0.01313$

(c) $I_1 = {Z^{-1}}_{11} \times V_1 = (0.0795 - j0.0331) \times 20 = 1.590 - j0.662$ A $|I_1| = \sqrt{(1.590^2 + 0.662^2)} = \sqrt{(2.528 + 0.438)} = \sqrt{2.966} = 1.722$ A $\angle I_1 = \arctan(-0.662/1.590) = -22.6°$ (current lags voltage due to inductive impedance)

Problem F.3.10

Given: The matrix equation AX = B where A = [4, 2; 3, 5], X is the unknown 2 × 2 matrix, and B = [10, 6; 11, 9].

Find: (a) A^{-1}, (b) the solution $X = A^{-1}B$, and (c) verify by computing AX.

Solution:

(a) $\det(A) = 4\times5 - 2\times3 = 20 - 6 = 14$. $A^{-1} = (1/14) \times [5, -2; -3, 4] = [0.3571, -0.1429; -0.2143, 0.2857]$

(b) $X = A^{-1}B$: $X_{11} = 0.3571\times10 + (-0.1429)\times11 = 3.571 - 1.571 = 2.0$ $X_{12} = 0.3571\times6 + (-0.1429)\times9 = 2.143 - 1.286 = 0.857$ $X_{21} = (-0.2143)\times10 + 0.2857\times11 = -2.143 + 3.143 = 1.0$ $X_{22} = (-0.2143)\times6 + 0.2857\times9 = -1.286 + 2.571 = 1.286$

X = [2.0, 0.857; 1.0, 1.286]

More precisely: $X_{12} = 6/7$, $X_{22} = 9/7$.

(c) Verify AX = B: $(AX)_{11} = 4\times2 + 2\times1 = 10$ ✓ $(AX)_{12} = 4\times(6/7) + 2\times(9/7) = 24/7 + 18/7 = 42/7 = 6$ ✓ $(AX)_{21} = 3\times2 + 5\times1 = 11$ ✓ $(AX)_{22} = 3\times(6/7) + 5\times(9/7) = 18/7 + 45/7 = 63/7 = 9$ ✓

Appendix F — Section F.4: Solving Linear Systems

Practice problems covering Gaussian elimination and Cramer's rule for solving systems of linear equations.

Problem F.4.1

Given: A two-node circuit yields the system: $0.3V_1 - 0.1V_2 = 2$ $-0.1V_1 + 0.2V_2 = 1$

Find: Solve for V_1 and V_2 using Gaussian elimination.

Solution:

Augmented matrix: [0.3, −0.1, 2; −0.1, 0.2, 1]

Multiply row 1 by (0.1/0.3) = 1/3 and add to row 2 to eliminate V_1: $R_2 \rightarrow R_2 + (1/3)R_1$: New R_2: −0.1 + 0.1 = 0, 0.2 + (−0.1/3) = 0.2 − 0.0333 = 0.1667, 1 + 2/3 = 1.6667

Result: [0.3, −0.1, 2; 0, 0.1667, 1.6667]

Back-substitute: $0.1667 \times V_2 = 1.6667 \rightarrow V_2 = 1.6667/0.1667 = 10.0$ V $0.3V_1 - 0.1 \times 10 = 2 \rightarrow 0.3V_1 = 3 \rightarrow V_1 = 10.0$ V

Verification: 0.3(10) − 0.1(10) = 3 − 1 = 2 ✓ and −0.1(10) + 0.2(10) = −1 + 2 = 1 ✓

Problem F.4.2

Given: A three-mesh circuit yields: $20I_1 - 10I_2 = 50$ $-10I_1 + 25I_2 - 5I_3 = 0$ $-5I_2 + 15I_3 = -20$

Find: Solve for I_1, I_2, I_3 using Gaussian elimination.

Solution:

Augmented matrix: [20, −10, 0, 50; −10, 25, −5, 0; 0, −5, 15, −20]

Step 1: Eliminate I_1 from row 2. $R_2 \rightarrow R_2 + (10/20)R_1 = R_2 + 0.5R_1$: [−10+10, 25−5, −5+0, 0+25] = [0, 20, −5, 25]

Matrix: [20, −10, 0, 50; 0, 20, −5, 25; 0, −5, 15, −20]

Step 2: Eliminate I_2 from row 3. $R_3 \rightarrow R_3 + (5/20)R_2 = R_3 + 0.25R_2$: [0, −5+5, 15−1.25, −20+6.25] = [0, 0, 13.75, −13.75]

Matrix: [20, −10, 0, 50; 0, 20, −5, 25; 0, 0, 13.75, −13.75]

Back-substitute: $13.75I_3 = -13.75 \rightarrow I_3 = -1.0$ A $20I_2 - 5(-1) = 25 \rightarrow 20I_2 = 20 \rightarrow I_2 = 1.0$ A $20I_1 - 10(1) = 50 \rightarrow 20I_1 = 60 \rightarrow I_1 = 3.0$ A

Verification: 20(3) − 10(1) = 50 ✓, −10(3) + 25(1) − 5(−1) = −30+25+5 = 0 ✓, −5(1)+15(−1) = −5−15 = −20 ✓

Problem F.4.3

Given: A system of equations from nodal analysis: $5V_1 - 2V_2 = 10$ $-2V_1 + 6V_2 = 8$

Find: Solve using Cramer's rule.

Solution:

Coefficient matrix A = [5, −2; −2, 6], source vector $B = [10, 8]^T$.

det(A) = 5×6 − (−2)(−2) = 30 − 4 = 26

For V_1: Replace column 1 with B: A_1 = [10, −2; 8, 6] $\det(A_1) = 10\times6 - (-2)\times8 = 60 + 16 = 76$ $V_1 = \det(A_1)/\det(A) = 76/26 = 2.923$ V

For V_2: Replace column 2 with B: A_2 = [5, 10; −2, 8] $\det(A_2) = 5\times8 - 10\times(-2) = 40 + 20 = 60$ $V_2 = \det(A_2)/\det(A) = 60/26 = 2.308$ V

Verification: 5(2.923) − 2(2.308) = 14.615 − 4.615 = 10.0 ✓

Problem F.4.4

Given: A three-node system: $V_1 - 0.5V_2 = 4$ $-0.5V_1 + V_2 - 0.5V_3 = 0$ $-0.5V_2 + V_3 = 2$

Find: Solve using Gaussian elimination with partial pivoting (select the largest pivot at each step).

Solution:

Augmented matrix: [1, −0.5, 0, 4; −0.5, 1, −0.5, 0; 0, −0.5, 1, 2]

Step 1: Pivot element $a_{11} = 1$ (already the largest in column 1). $R_2 \rightarrow R_2 + 0.5R_1$: [0, 0.75, −0.5, 2]

Matrix: [1, −0.5, 0, 4; 0, 0.75, −0.5, 2; 0, −0.5, 1, 2]

Step 2: Pivot column 2. |0.75| > |−0.5|, so no row swap needed. $R_3 \rightarrow R_3 + (0.5/0.75)R_2 = R_3 + (2/3)R_2$: [0, 0, 1−1/3, 2+4/3] = [0, 0, 2/3, 10/3]

Back-substitute: $(2/3)V_3 = 10/3 \rightarrow V_3 = 5.0$ V $0.75V_2 - 0.5(5) = 2 \rightarrow 0.75V_2 = 4.5 \rightarrow V_2 = 6.0$ V $V_1 - 0.5(6) = 4 \rightarrow V_1 = 7.0$ V

Verification: $7-3=4$ ✓, $-3.5+6-2.5=0$ ✓, $-3+5=2$ ✓

Problem F.4.5

Given: The mesh equations for a bridge circuit: $30I_1 - 10I_2 - 20I_3 = 100$ $-10I_1 + 50I_2 - 10I_3 = 0$ $-20I_1 - 10I_2 + 60I_3 = 0$

Find: Solve using Cramer's rule for I_1 only (to determine source current).

Solution:

$A = [30, -10, -20; -10, 50, -10; -20, -10, 60]$

det(A): Expand along row 1: $= 30 \times \det[50,-10;-10,60] - (-10) \times \det[-10,-10;-20,60] + (-20) \times \det[-10,50;-20,-10] = 30 \times (3000 - 100) + 10 \times (-600 - 200) - 20 \times (100 + 1000) = 30 \times 2900 + 10 \times (-800) - 20 \times 1100 = 87{,}000 - 8{,}000 - 22{,}000 = 57{,}000$

For I_1: $A_1 = [100, -10, -20; 0, 50, -10; 0, -10, 60]$ $\det(A_1) = 100 \times \det[50,-10;-10,60] = 100 \times (3000-100) = 100 \times 2900 = 290{,}000$ (The other terms have a factor of 0 in column 1)

$I_1 = \det(A_1)/\det(A) = 290{,}000/57{,}000 = 5.088$ A

The total current drawn from the 100 V source is 5.088 A, giving an equivalent resistance of $R_{eq} = 100/5.088 = 19.65\ \Omega$.

Problem F.4.6

Given: A system with a near-singular coefficient matrix: $100V_1 - 99V_2 = 1$ $99V_1 - 100V_2 = -1$

Find: (a) The solution, (b) the determinant, and (c) a discussion of numerical sensitivity.

Solution:

(a) Using Cramer's rule: $\det(A) = 100\times(-100) - (-99)(99) = -10{,}000 + 9{,}801 = -199$

$\det(A_1) = [1, -99; -1, -100] = 1\times(-100) - (-99)(-1) = -100 - 99 = -199$ $V_1 = -199/(-199) = 1.0$ V

$\det(A_2) = [100, 1; 99, -1] = 100\times(-1) - 1\times 99 = -100 - 99 = -199$ $V_2 = -199/(-199) = 1.0$ V

(b) $\det(A) = -199$. While non-zero, it is small relative to the magnitude of the matrix entries (10,000).

(c) The condition number is approximately $\max|a_{ij}|/|\det/n| \approx 100/1 \approx 100$. More precisely, eigenvalues: $\lambda_1 = 100-99 = 1$ and $\lambda_2 = -(100+99) = -199$, giving condition number $\kappa = 199/1 = 199$. A small perturbation in the right-hand side could shift the solution significantly. For example, changing the RHS from $[1, -1]$ to $[1.01, -0.99]$ shifts V_1 to ≈ 1.01 and V_2 to ≈ 0.99 — the solution change is proportional to κ times the input change, demonstrating ill-conditioning.

Problem F.4.7

Given: A power system load flow equation (linearized): $[B_{11}, B_{12}; B_{21}, B_{22}] \times [\delta_1; \delta_2] = [P_1; P_2]$ where $B = [-20, 10; 10, -15]$ (susceptance matrix, pu) and $P = [1.5, -0.8]^T$ pu.

Find: The voltage angles δ_1 and δ_2 in radians and degrees using Gaussian elimination.

Solution:

Augmented matrix: $[-20, 10, 1.5; 10, -15, -0.8]$

$R_2 \rightarrow R_2 + (10/20)R_1 = R_2 + 0.5R_1$: $[10+(-10), -15+5, -0.8+0.75] = [0, -10, -0.05]$

Back-substitute: $-10\delta_2 = -0.05 \rightarrow \delta_2 = 0.005$ rad $= 0.286°$ $-20\delta_1 + 10(0.005) = 1.5 \rightarrow -20\delta_1 = 1.45 \rightarrow \delta_1 = -0.0725$ rad $= -4.154°$

The negative angle at bus 1 indicates that bus 1 leads the reference (generating power), and bus 2 is nearly at the reference angle (small load).

Problem F.4.8

Given: Ohm's law in matrix form for a resistive network: $V = RI$ where $R = [15, -5, 0; -5, 20, -10; 0, -10, 25]$ Ω and $I = [I_1, I_2, I_3]^T$. The source voltages are $V = [30, 0, -10]^T$ V.

Find: Solve for I_1, I_2, I_3 using Gaussian elimination.

Solution:

Augmented matrix: $[15, -5, 0, 30; -5, 20, -10, 0; 0, -10, 25, -10]$

Step 1: $R_2 \rightarrow R_2 + (5/15)R_1 = R_2 + (1/3)R_1$: $[0, 20-5/3, -10, 0+10] = [0, 18.333, -10, 10]$

Matrix: $[15, -5, 0, 30; 0, 18.333, -10, 10; 0, -10, 25, -10]$

Step 2: $R_3 \rightarrow R_3 + (10/18.333)R_2 = R_3 + 0.5455R_2$: $[0, 0, 25-5.455, -10+5.455] = [0, 0, 19.545, -4.545]$

Back-substitute: $19.545I_3 = -4.545 \rightarrow I_3 = -0.2326$ A $18.333I_2 - 10(-0.2326) = 10 \rightarrow 18.333I_2 = 7.674 \rightarrow I_2 = 0.4186$ A $15I_1 - 5(0.4186) = 30 \rightarrow 15I_1 = 32.093 \rightarrow I_1 = 2.140$ A

Verification: $15(2.14)-5(0.419) = 32.1-2.1 = 30$ ✓

Problem F.4.9

Given: A 2 × 2 system from a transformer equivalent circuit: $(R_1 + jX_1)I_1 - jX_mI_2 = V_1$ $-jX_mI_1 + (R_2 + jX_2 + jX_m)I_2 = 0$

With $R_1 = 2$ Ω, $X_1 = 8$ Ω, $X_m = 200$ Ω, $R_2 = 3$ Ω, $X_2 = 10$ Ω, $V_1 = 240\angle 0°$ V.

Find: Use Cramer's rule to find I_2 (secondary current).

Solution:

$Z = [2+j8, -j200; -j200, 3+j210]$

$\det(Z) = (2+j8)(3+j210) - (-j200)(-j200) = 6+j420+j24+j^2 1680 - j^2 40{,}000 = 6+j444-1680+40{,}000 = 38{,}326 + j444$

For I_2: Replace column 2 with [240, 0]: $\det(Z_2) = [2+j8, 240; -j200, 0] = (2+j8)(0) - (240)(-j200) = 0 + j48{,}000 = j48{,}000$

$I_2 = \det(Z_2)/\det(Z) = j48{,}000/(38{,}326+j444)$

$|\det(Z)| = \sqrt{(38326^2 + 444^2)} = \sqrt{(1.469\times10^9 + 1.972\times10^5)} \approx 38{,}329$

$I_2 = j48{,}000 \times (38{,}326-j444)/(38{,}329^2) = (j48{,}000\times38{,}326 + 48{,}000\times444)/1.469\times10^9 = (21{,}312+j1.840\times10^9)/1.469\times1$
$= 0.01451 + j1.2526$

Wait — let me recalculate more carefully: $I_2 = j48{,}000/(38{,}326+j444)$ Multiply by conjugate: $(38{,}326-j444)/(38{,}326^2+444^2) = (38{,}326-j444)/1.469\times10^9$

$I_2 = j48{,}000 \times (38{,}326-j444)/1.469\times10^9 = (j\times48{,}000\times38{,}326 - j^2\times48{,}000\times444)/1.469\times10^9 = (48{,}000\times444 + j\times48{,}000\times38{,}326)/1.469\times10^9 = (21{,}312{,}000 + j1{,}839{,}648{,}000)/1{,}469{,}459{,}432 = 0.01450 + j1.2519$

$|I_2| = \sqrt{(0.0145^2 + 1.252^2)} \approx 1.252$ A $\angle I_2 \approx \arctan(1.252/0.0145) \approx 89.3°$

The secondary current lags the applied voltage by nearly 90° (almost purely magnetizing current with a small resistive component). This is expected for a transformer operating near no-load.

Problem F.4.10

Given: A system of four equations from a DC circuit with four node voltages:

$$4V_1 - V_2 - V_3 = 10$$
$$-V_1 + 3V_2 - V_3 = 5$$
$$-V_1 - V_2 + 4V_3 - V_4 = 0$$
$$-V_3 + 2V_4 = -3$$

Find: Solve using Gaussian elimination (forward elimination, then back-substitution).

Solution:

Augmented matrix: [4, −1, −1, 0, 10; −1, 3, −1, 0, 5; −1, −1, 4, −1, 0; 0, 0, −1, 2, −3]

Step 1: Eliminate V_1 from rows 2 and 3. $R_2 \to R_2 + (1/4)R_1$: [0, 2.75, −1.25, 0, 7.5] $R_3 \to R_3 + (1/4)R_1$: [0, −1.25, 3.75, −1, 2.5]

Step 2: Eliminate V_2 from rows 3 and 4. $R_3 \to R_3 + (1.25/2.75)R_2 = R_3 + 0.4545R_2$: [0, 0, 3.75−0.568, −1, 2.5+3.409] = [0, 0, 3.182, −1, 5.909]

Row 4 already has zero in column 2, no change needed.

Step 3: Eliminate V_3 from row 4. $R_4 \to R_4 + (1/3.182)R_3 = R_4 + 0.3143R_3$: [0, 0, 0, 2−0.314, −3+1.857] = [0, 0, 0, 1.686, −1.143]

Back-substitute: $1.686V_4 = -1.143 \to V_4 = -0.678$ V $3.182V_3 - 1\times(-0.678) = 5.909 \to 3.182V_3 = 5.231 \to V_3 = 1.644$ V $2.75V_2 - 1.25\times1.644 = 7.5 \to 2.75V_2 = 9.555 \to V_2 = 3.475$ V $4V_1 - 3.475 - 1.644 = 10 \to 4V_1 = 15.119 \to V_1 = 3.780$ V

Verification of row 4: $-1.644 + 2\times(-0.678) = -1.644 - 1.356 = -3.0$ ✓

Appendix F — Section F.5: Applications in Electrical Engineering

Practice problems covering nodal analysis matrix form, two-port network parameters, and state-space representation.

Problem F.5.1

Given: A four-node circuit with the following conductances: $G_{12} = 0.2$ S, $G_{13} = 0.1$ S, $G_{23} = 0.4$ S, $G_{24} = 0.1$ S, $G_{34} = 0.3$ S, $G_{10} = 0.5$ S (node 1 to ground), $G_{40} = 0.2$ S (node 4 to ground). A 3 A source feeds into node 1, and a 1 A source feeds into node 3.

Find: (a) The 4×4 nodal admittance matrix Y, and (b) the current source vector I.

Solution:

(a) Diagonal entries: $Y_{11} = G_{10} + G_{12} + G_{13} = 0.5 + 0.2 + 0.1 = 0.8$ S $Y_{22} = G_{12} + G_{23} + G_{24} = 0.2 + 0.4 + 0.1 = 0.7$ S $Y_{33} = G_{13} + G_{23} + G_{34} = 0.1 + 0.4 + 0.3 = 0.8$ S $Y_{44} = G_{24} + G_{34} + G_{40} = 0.1 + 0.3 + 0.2 = 0.6$ S

Off-diagonal entries: $Y_{12} = Y_{21} = -0.2$, $Y_{13} = Y_{31} = -0.1$, $Y_{14} = Y_{41} = 0$ $Y_{23} = Y_{32} = -0.4$, $Y_{24} = Y_{42} = -0.1$ $Y_{34} = Y_{43} = -0.3$

$Y = [0.8, -0.2, -0.1, 0; -0.2, 0.7, -0.4, -0.1; -0.1, -0.4, 0.8, -0.3; 0, -0.1, -0.3, 0.6]$

(b) Current source vector: $I = [3, 0, 1, 0]^T$ A

The system YV = I can be solved by Gaussian elimination or matrix inversion to find all four node voltages.

Problem F.5.2

Given: A two-port network has Z-parameters: $Z_{11} = 100\ \Omega$, $Z_{12} = Z_{21} = 30\ \Omega$, $Z_{22} = 80\ \Omega$. The network is driven by a 10 V source at port 1 with internal resistance $R_s = 50\ \Omega$, and port 2 is terminated with $R_L = 200\ \Omega$.

Find: (a) The port equations, (b) the port currents I_1 and I_2, and (c) the voltage gain V_2/V_s.

Solution:

(a) Port equations with source and load: $V_1 = Z_{11}I_1 + Z_{12}I_2$ and $V_2 = Z_{21}I_1 + Z_{22}I_2$ Source constraint: $V_1 = V_s - R_sI_1 = 10 - 50I_1$ Load constraint: $V_2 = -R_LI_2 = -200I_2$

(b) Substituting: $10 - 50I_1 = 100I_1 + 30I_2 \rightarrow 150I_1 + 30I_2 = 10$ ... (1) $-200I_2 = 30I_1 + 80I_2 \rightarrow 30I_1 + 280I_2 = 0$ ... (2)

From (2): $I_1 = -280I_2/30 = -9.333I_2$ Substitute into (1): $150(-9.333I_2) + 30I_2 = 10$ $-1400I_2 + 30I_2 = 10$ $-1370I_2 = 10$ $I_2 = -7.30 \times 10^{-3}$ A $= -7.30$ mA (current flows out of port 2 into load) $I_1 = -9.333 \times (-7.30 \times 10^{-3}) = 68.1$ mA

(c) $V_2 = -200 \times (-7.30 \times 10^{-3}) = 1.460$ V Voltage gain: $V_2/V_s = 1.460/10 = 0.146 = -16.7$ dB

Problem F.5.3

Given: Three two-port networks in cascade with ABCD matrices: $M_1 = [1, 0; 0.01, 1]$ (shunt admittance of 0.01 S) $M_2 = [1, 100; 0, 1]$ (series impedance of 100 Ω) $M_3 = [1, 0; 0.02, 1]$ (shunt admittance of 0.02 S)

Find: (a) The overall ABCD matrix $M = M_1 \times M_2 \times M_3$, and (b) the voltage gain V_1/V_2 with $I_2 = 0$ (open-circuit output).

Solution:

(a) First compute $M_{12} = M_1 \times M_2$: A = 1×1+0×0 = 1 B = 1×100+0×1 = 100 C = 0.01×1+1×0 = 0.01 D = 0.01×100+1×1 = 2

$M_{12} = [1, 100; 0.01, 2]$

Then $M = M_{12} \times M_3$: A = 1×1+100×0.02 = 1+2 = 3 B = 1×0+100×1 = 100 C = 0.01×1+2×0.02 = 0.01+0.04 = 0.05 D = 0.01×0+2×1 = 2

M = [3, 100; 0.05, 2]

Verification: det(M) = 3×2 − 100×0.05 = 6 − 5 = 1 ✓ (reciprocal network)

(b) With $I_2 = 0$: $V_1 = AV_2 + BI_2 = AV_2 = 3V_2$ $V_1/V_2 = A = 3$ (or equivalently, $V_2/V_1 = 1/3 = 0.333 = -9.54$ dB)

Problem F.5.4

Given: A series RLC circuit (R = 4 Ω, L = 0.5 H, C = 0.125 F) driven by voltage source u(t). State variables: x_1 = capacitor voltage, x_2 = inductor current. Output: y = inductor current.

Find: (a) The state-space matrices A, B, C, D, (b) the eigenvalues of A (system poles), and (c) the natural frequency and damping ratio.

Solution:

(a) Circuit equations: $dx_1/dt = (1/C)x_2 = 8x_2$ $dx_2/dt = (1/L)(u - Rx_2 - x_1) = 2(u - 4x_2 - x_1) = -2x_1 - 8x_2 + 2u$ Output: $y = x_2$

A = [0, 8; −2, −8], B = [0; 2], C = [0, 1], D = [0]

(b) Eigenvalues from $\det(A - \lambda I) = 0$: $\det([-\lambda, 8; -2, -8-\lambda]) = -\lambda(-8-\lambda) - 8(-2) = \lambda^2 + 8\lambda + 16 = 0$ $(\lambda + 4)^2 = 0$ $\lambda_1 = \lambda_2 = -4$ (repeated root)

(c) From the characteristic equation $\lambda^2 + 8\lambda + 16 = 0$: Comparing with $\lambda^2 + 2\zeta\omega_0\lambda + \omega_0{}^2 = 0$: $\omega_0{}^2 = 16 \rightarrow \omega_0 = 4$ rad/s $2\zeta\omega_0 = 8 \rightarrow \zeta = 8/(2\times4) = 1.0$ (critically damped)

The system is critically damped — it returns to equilibrium as fast as possible without oscillation.

Problem F.5.5

Given: A DC motor is modeled with state variables x_1 = angular velocity (rad/s), x_2 = armature current (A). Parameters: $J = 0.01$ kg·m² (inertia), $B = 0.1$ N·m·s/rad (friction), $K_t = 0.5$ N·m/A (torque constant), $K_e = 0.5$ V·s/rad (back-EMF constant), $R_a = 2\ \Omega$, $L_a = 0.5$ H. Input: armature voltage u. Output: angular velocity $y = x_1$.

Find: (a) The state equations, (b) the A, B, C, D matrices, and (c) the eigenvalues.

Solution:

(a) Mechanical: $J(dx_1/dt) = K_t x_2 - Bx_1 \rightarrow dx_1/dt = (K_t/J)x_2 - (B/J)x_1$ Electrical: $L_a(dx_2/dt) = u - R_a x_2 - K_e x_1 \rightarrow dx_2/dt = -(K_e/L_a)x_1 - (R_a/L_a)x_2 + (1/L_a)u$

(b) Numerical values: $dx_1/dt = -(0.1/0.01)x_1 + (0.5/0.01)x_2 = -10x_1 + 50x_2$ $dx_2/dt = -(0.5/0.5)x_1 - (2/0.5)x_2 + (1/0.5)u = -x_1 - 4x_2 + 2u$

A = [−10, 50; −1, −4], B = [0; 2], C = [1, 0], D = [0]

(c) Eigenvalues: $\det(A - \lambda I) = (-10-\lambda)(-4-\lambda) - (50)(-1) = \lambda^2 + 14\lambda + 40 + 50 = \lambda^2 + 14\lambda + 90 = 0$ $\lambda = (-14 \pm \sqrt{(196-360)})/2 = (-14 \pm \sqrt{(-164)})/2 = (-14 \pm j12.81)/2$ $\lambda_1 = -7 + j6.40$, $\lambda_2 = -7 - j6.40$

The poles are complex conjugate with negative real parts, indicating a stable, underdamped system with $\omega_n = \sqrt{90} = 9.49$ rad/s and $\zeta = 14/(2\times9.49) = 0.738$.

Problem F.5.6

Given: A two-port network has Y-parameters: $Y_{11} = 0.04$ S, $Y_{12} = Y_{21} = -0.01$ S, $Y_{22} = 0.03$ S.

Find: (a) Convert to Z-parameters using $Z = Y^{-1}$, and (b) verify by computing YZ = I.

Solution:

(a) Y = [0.04, −0.01; −0.01, 0.03] $\det(Y) = 0.04\times0.03 - (-0.01)^2 = 0.0012 - 0.0001 = 0.0011$

$Z = Y^{-1} = (1/0.0011) \times [0.03, 0.01; 0.01, 0.04]$ $Z_{11} = 0.03/0.0011 = 27.27\ \Omega$ $Z_{12} = Z_{21} = 0.01/0.0011 = 9.091\ \Omega$ $Z_{22} = 0.04/0.0011 = 36.36\ \Omega$

$Z = [27.27, 9.091; 9.091, 36.36]\ \Omega$

(b) YZ: $(YZ)_{11} = 0.04\times27.27+(-0.01)\times9.091 = 1.091-0.091 = 1.000$ ✓ $(YZ)_{12} = 0.04\times9.091+(-0.01)\times36.36 = 0.364-0.364 = 0.000$ ✓ $(YZ)_{21} = (-0.01)\times27.27+0.03\times9.091 = -0.273+0.273 = 0.000$ ✓ $(YZ)_{22} = (-0.01)\times9.091+0.03\times36.36 = -0.091+1.091 = 1.000$ ✓

$YZ = I_2$ ✓

Problem F.5.7

Given: A state-space system with: $A = [0, 1; -9, -6]$, $B = [0; 1]$, $C = [9, 0]$, $D = [0]$

Find: (a) The eigenvalues of A, (b) the transfer function $H(s) = C(sI - A)^{-1}B + D$, and (c) the steady-state gain H(0).

Solution:

(a) Eigenvalues: $\det(A - \lambda I) = \det[-\lambda, 1; -9, -6-\lambda] = -\lambda(-6-\lambda)+9 = \lambda^2+6\lambda+9 = (\lambda+3)^2 = 0$ $\lambda_1 = \lambda_2 = -3$ (repeated pole)

(b) Transfer function: $sI - A = [s, -1; 9, s+6]$ $\det(sI - A) = s(s+6)+9 = s^2+6s+9 = (s+3)^2$

$(sI - A)^{-1} = 1/(s+3)^2 \times [s+6, 1; -9, s]$

$C(sI - A)^{-1}B$: $(sI - A)^{-1}B = 1/(s+3)^2 \times [1; s]$ (from matrix-vector multiplication) $C \times [1; s]/(s+3)^2 = [9, 0] \times [1; s]/(s+3)^2 = 9/(s+3)^2$

$H(s) = 9/(s+3)^2 = 9/(s^2 + 6s + 9)$

(c) Steady-state gain: $H(0) = 9/(0+0+9) = 1.0$

The system has unity DC gain and a critically damped response with time constant $\tau = 1/3$ s.

Problem F.5.8

Given: Convert the Y-parameters from Problem F.5.6 to ABCD parameters using the conversion formulas: $A = -Y_{22}/Y_{21}$, $B = -1/Y_{21}$, $C = -\det(Y)/Y_{21}$, $D = -Y_{11}/Y_{21}$

Find: The ABCD matrix and verify det(ABCD) = 1 for a reciprocal network.

Solution:

From Problem F.5.6: $Y_{11} = 0.04$, $Y_{12} = Y_{21} = -0.01$, $Y_{22} = 0.03$, $\det(Y) = 0.0011$.

$A = -Y_{22}/Y_{21} = -0.03/(-0.01) = 3.0$ $B = -1/Y_{21} = -1/(-0.01) = 100\ \Omega$ $C = -\det(Y)/Y_{21} = -0.0011/(-0.01) = 0.11$ S $D = -Y_{11}/Y_{21} = -0.04/(-0.01) = 4.0$

$ABCD = [3.0, 100; 0.11, 4.0]$

Verification: $AD - BC = 3.0\times4.0 - 100\times0.11 = 12.0 - 11.0 = 1.0$ ✓

The determinant equals 1, confirming the network is reciprocal ($Y_{12} = Y_{21}$).

Problem F.5.9

Given: A discrete-time state-space model for a digital controller: $A = [0, 1; -0.5, 1.2]$, $B = [0; 0.1]$, $C = [1, 0]$, $D = [0]$ Initial state $x[0] = [0, 0]^T$, input $u[k] = 1$ for all $k \geq 0$ (unit step).

Find: (a) x[1], x[2], x[3], and (b) the output y[0], y[1], y[2], y[3].

Solution:

(a) $x[k+1] = Ax[k] + Bu[k]$:

$x[1] = A[0;0] + B\times1 = [0;0] + [0;0.1] = [0, 0.1]^T$

$x[2] = A[0;0.1] + B\times1 = [0\times0+1\times0.1; -0.5\times0+1.2\times0.1] + [0;0.1] = [0.1; 0.12] + [0; 0.1] = [0.1, 0.22]^T$

$x[3] = A[0.1;0.22] + B\times1 = [0\times0.1+1\times0.22; -0.5\times0.1+1.2\times0.22] + [0;0.1] = [0.22; -0.05+0.264] + [0; 0.1] = [0.22; 0.314] = [0.22, 0.314]^T$

(b) $y[k] = Cx[k] + Du[k] = x_1[k]$: $y[0] = 0$, $y[1] = 0$, $y[2] = 0.1$, $y[3] = 0.22$

The output shows a delayed response (one-step delay due to B acting on x_2 only, and C reading x_1 only) followed by an increasing ramp as the step input accumulates through the integrator-like dynamics.

Problem F.5.10

Given: A three-phase power system uses the symmetrical component transformation:

$T = (1/3) \times [1, 1, 1; 1, a, a^2; 1, a^2, a]$

where $a = e^{j120°} = -0.5 + j0.866$ and $a^2 = e^{j240°} = -0.5 - j0.866$.

The unbalanced phase voltages are $V_{abc} = [100\angle0°, 80\angle-125°, 90\angle115°]^T$ V.

Find: (a) The positive-sequence voltage V_1, (b) the negative-sequence voltage V_2, and (c) the zero-sequence voltage V_0 using $V_{012} = T \times V_{abc}$.

Solution:

First, convert phase voltages to rectangular form: $V_a = 100 + j0$ $V_b = 80\angle-125° = 80(\cos(-125°) + j\sin(-125°)) = 80(-0.5736 - j0.8192) = -45.89 - j65.54$ $V_c = 90\angle115° = 90(\cos 115° + j\sin 115°) = 90(-0.4226 + j0.9063) = -38.04 + j81.57$

(a) Zero-sequence: $V_0 = (1/3)(V_a + V_b + V_c) = (1/3)(100 - 45.89 - 38.04 + j(0 - 65.54 + 81.57)) = (1/3)(16.07 + j16.03) = 5.36 + j5.34 = 7.56\angle44.9°$ V

(b) Positive-sequence: $V_1 = (1/3)(V_a + aV_b + a^2V_c)$ $aV_b = (-0.5+j0.866)(-45.89-j65.54) = 22.945+j32.77-j39.74-j^256.76 = 79.71-j6.97$ $a^2V_c = (-0.5-j0.866)(-38.04+j81.57) = 19.02-j40.79+j32.94-j^270.64 = 89.66-j7.85$

$V_1 = (1/3)(100+j0+79.71-j6.97+89.66-j7.85) = (1/3)(269.37-j14.82) = 89.79 - j4.94 = 89.93\angle-3.1°$ V

(c) Negative-sequence: $V_2 = (1/3)(V_a + a^2V_b + aV_c)$ $a^2V_b = (-0.5-j0.866)(-45.89-j65.54) = 22.945+j32.77+j39.74+j^2 56.76 = -33.82+j72.51$ $aV_c = (-0.5+j0.866)(-38.04+j81.57) = 19.02-j40.79-j32.94+j^2 70.64 = -51.62-j73.73$

$V_2 = (1/3)(100+j0-33.82+j72.51-51.62-j73.73) = (1/3)(14.56-j1.22) = 4.85 - j0.41 = 4.87\angle{-4.8°}$ V

The system is nearly balanced: $|V_1| = 89.93$ V dominates, $|V_2| = 4.87$ V (5.4% unbalance), and $|V_0| = 7.56$ V (8.4% zero-sequence, indicating neutral current flow).