Signal Processing 11 (1986) 13-36
North-Holland

THE SPECTRAL CORRELATION THEORY OF CYCLOSTATIONARY TIME-SERIES

William A. GARDNER

Signal Image and Processing Laboratory, Department of Electrical and Computer Engineering, University of California at Davis, Davis, CA 95616, U.S.A.

Received 14 August 1985
Revised 4 February 1986

Abstract. A spectral correlation theory for cyclostationary time-series is introduced. It is established that a time-series is cyclostationary if and only if there exists a quadratic time-invariant transformation that generates spectral lines, and this is so if and only if the time-series exhibits spectral correlation. Fundamental properties of a characterizing spectral correlation function are developed. These include the effects of periodic modulation and periodically time-variant linear filtering. Relationships between the spectral correlation function and the radar ambiguity function and the Wigner-Ville distribution are explained. The spectral correlation properties of Rice's representation for bandpass time-series are derived. A generalization of the Wiener relation from the spectral density function to the spectral correlation function is developed, and generalizations of the aliasing formula for periodic time-sampling, and the frequency conversion formula for amplitude modulation, from the spectral density function to the spectral correlation function are developed.

Zusammenfassung. Vorgestellt wird eine Theorie der spektralen Korrelation zyklisch-stationärer Zeitreihen. Es wird gezeigt, daß eine Zeitreihe dann und nur dann zyklisch-stationär ist, wenn eine zeitinvariant quadratische Transformation existiert, die ein Linienspektrum erzeugt; und dies ist dann und nur dann so, wenn die Zeitreihe eine spektrale Korrelation aufweist. Die grundlegenden Eigenschaften der zugehörigen spektralen Korrelationsfunktion werden entwickelt. Diese beinhalten die Auswirkung periodischer Abtastung, einer Frequenzwandlung sowie periodisch zeitveränderlicher linearer Filterung. Die Beziehungen zwischen der spektralen Korrelationsfunktion, der Mehrdeutigkeitsfunktion für Radar und der Zeit-Frequenzverteilung der Leistungsdichte nach Wigner-Ville werden erklärt. Ebenso werden die Eigenschaften der spektralen Korrelation für bandbegrenzte Zeitreihen in der Darstellung von Rice hergeleitet. Eine Verallgemeinerung der Wiener-Gleichungen von der spektralen Dichtefunktion hin zur spektralen Korrelationsfunktion wird entwickelt. Zusätzlich wird die Formel für Faltungsverzerrungen gleichförmig abgetasteter Signale sowie die Formel für die Frequenzwandlung bei Amplitudenmodulation auf den Fall der spektralen Korrelationsfunktion erweitert.

Résumé. Une théorie de la corrélation spectrale pour des séries temporelles cyclostationnaires est introduite. Il est établi qu'une série temporelle est cyclostationnaire si et seulement si il existe une transformation quadratique invariant dans le temps qui génère des lignes spectrales, et il en est ainsi si et seulement si la série temporelle possède une corrélation spectrale. Les propriétés fondamentales d'une fonction caractéristique de la corrélation spectrale sont développées. Elles comprennent les effets de la modulation périodique et le filtrage périodique variant dans le temps. Les relations entre la fonction de corrélation spectrale, la fonction d'ambiguité en radar et la distribution de Wigner-Ville sont expliquées. Les propriétés de corrélation spectral de la répresentation de Rice pour des séries à bande limitée sont établies. Une généralisation de la relation de Wiener de la fonction de densité spectrale à la fonction de corrélation spectrale est développée, et les généralisations de la relation de repliement pour l'échantillonnage périodique et la formule de conversion de fréquence pour la modulation d'amplitude, de la fonction de densité spectrale à la corrélation spectrale sont développées.

Keywords. Cyclostationary processes, time-series analysis, spectral correlation, periodic phenomena, time-frequency signal representation.

1. Introduction

The subject of this paper is the statistical spectral analysis of empirical time-series from periodic phenomena, which are called *cyclostationary time-series.* The term *empirical* indicates that the time-series represents data from a physical phenomenon; the term *spectral analysis* denotes decomposition of the time-series into sinewave components; and the term *statistical* indicates that averaging is used to reduce random effects in the data that mask the spectral characteristics of the phenomenon under study: in particular, products of pairs of sinewave components are averaged to produce spectral correlations. The purpose of this paper is to introduce a comprehensive theory for spectral correlation analysis of cyclostationary time-series. The motivation for this paper is to foster better understanding of special concepts and special time-series-analysis methods for random data from periodic phenomena. In the approach taken in this paper, the unnecessary abstraction of a probabilistic framework is avoided by extending to periodic phenomena the deterministic approach, based on time-averages, originated by Wiener for constant phenomena [53]. The reason for this is that for many applications the conceptual gap between practice and the deterministic theory presented here is narrower and thus easier to bridge than is the conceptual gap between practice and the more abstract probabilistic theory.[1] Nevertheless, a means for obtaining probabilistic interpretations of the deterministic theory is developed in terms of periodically time-variant fraction-of-time distributions.

To the author's knowledge, essentially all previous theory of random data from periodic phenomena that is comparable to that presented in this paper is based on the probabilistic foundation of *cyclostationary stochastic processes.* This is analogous to the fact that the great majority of theoretical treatments of random data from constant phenomena are based on the probabilistic foundation of stationary stochastic processes. The most comprehensive treatment of the probabilistic theory of cyclostationary stochastic processes to date is given in [24].

In order to avoid unnecessary confusion due to semantics, the terminology used in this paper is explained in the following. Although the terms *statistical* and *probabilistic* are used by many as if they were synonymous, their meanings are quite distinct. According to the Oxford English Dictionary, *statistical* means nothing more than "consisting of or founded on collections of numerical facts". Therefore, an average (e.g., over time) of a collection of measured spectra is a *statistical spectrum.* And this has nothing to do with probability. Thus, there is nothing contradictory in the notion of a deterministic or nonprobabilistic theory of statistical spectral analysis. (An interesting discussion of variations in usage of the term *statistical* is given in [3].) The term *deterministic*[2] is used here as it is commonly used in engineering, as a synonym for *nonprobabilistic.* Nevertheless, the reader should be forewarned that some of the elements of the nonprobabilistic theory presented in this paper are defined by infinite limits of time averages and are therefore no more deterministic in practice than are the elements of the probabilistic theory. In mathematics, deterministic and probabilistic theories, as referred to herein, are sometimes called *functional* and *stochastic* theories, respectively [55]. The term *random* is used here to mean nothing more than erratic unpredictable behavior. Its use is not meant to suggest probabilistic concepts.

Examples of periodic phenomena that give rise to random data abound in engineering and science. For example, in mechanical vibrations monitoring and diagnosis for machinery, periodicity arises

[1] This is explained in considerable detail in a forthcoming book [29]. Basically, the deterministic theory presented here applies to a single time-series, whereas the probabilistic theory only applies to an ensemble of random samples of time-series defined on an abstract probability space.

[2] The meaning of the term *deterministic* used in this paper should not be confused with the meaning of the same term as used in mathematics to describe the singular, or predictable, part of a stochastic process.

from rotation, revolution, and reciprocation of gears, belts, chains, shafts, propellers, bearings, pistons, etc.; in atmospheric science, e.g., for weather forecasting, periodicity arises from seasons caused primarily by rotation and revolution of the Earth; in radio astronomy, periodicity arises from revolution of the moon, rotation and pulsation of the sun, rotation of Jupiter and revolution of its satellite, Io, etc., and can cause strong periodicities in time-series, e.g., pulsar signals; in biology, periodicity in the form of bio-rhythms arises from both internal and external sources, e.g., circadean rhythms; in communications, telemetry, radar, and sonar, periodicity arises from sampling, scanning, modulating, multiplexing, and coding operations, and it can also be caused by rotating reflectors such as helicopter blades, and air- and water-craft propellers. Thus, the potential applications of the theory presented in this paper are diverse (cf. [24, 29]). For example, in the general signal processing field, the relevance of the concept of cyclostationarity is illustrated by recent work in synchronization [17, 18, 36, 39, 40, 41, 42, 43, 46], crosstalk interference and modulation transfer noise [2, 7], transmitter and receiver filter design [14, 31, 37], noise analysis for periodic circuits [50], adaptive filtering and system identification [16, 21], coding [8, 19], queueing [1], detection [22], and digital signal processing algorithms [15, 45]. In addition, the growing role of cyclostationarity in other signal processing areas is illustrated by recent work in biomedical engineering [34], and climatology [32], and by recent developments in basic theory for prediction [38], extraction [31], detection [22], modulation [25, 30], and signal modeling and representation [26, 27]. Other work involving cyclostationarity is cited in [29].

In Section 2, the fundamental idealized statistical parameters of the theory are introduced. These parameters, called the *limit cyclic autocorrelation*, *limit periodic autocorrelation*, *limit cyclic spectrum*, and *limit periodic spectrum*, are generalizations of the conventional limit autocorrelation and limit spectrum, which are the fundamental idealized statistical parameters in the deterministic theory of random data from constant phenomena [53]. The relationships between these fundamental statistical parameters and other well-known data parameters, such as the radar ambiguity function and the Wigner–Ville distribution, are explained, and a means for obtaining probabilistic interpretations is described.

In Section 3, basic properties of the limit cyclic spectrum, which is a spectral correlation function, are described. These include input/output spectral correlation relations for periodic modulators and samplers, linear periodically time-variant filters, and spectral correlation relations for (i) Rice's representation for bandpass time-series, (ii) sampling and aliasing, and (iii) frequency conversion.

2. Fundamental statistical parameters

2.1. Limit cyclic autocorrelation

A time-series $x(t)$ contains a *finite additive sinewave component* with frequency α, say

$$a\cos(2\pi\alpha t+\theta), \quad \alpha \neq 0, \tag{1}$$

if and only if the parameter

$$\hat{M}_x^\alpha \triangleq \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,\mathrm{e}^{-\mathrm{i}2\pi\alpha t}\,\mathrm{d}t \tag{2}$$

exists and is not zero, in which case

$$\hat{M}_x^\alpha = \tfrac{1}{2}a\,\mathrm{e}^{\mathrm{i}\theta}.$$

In this case, the spectral density of $x(t)$ exhibits a spectral line at $f=\alpha$ and its image $f=-\alpha$. That is, the spectral density contains the additive component[3]

$$|\hat{M}_x^\alpha|^2[\delta(f-\alpha)+\delta(f+\alpha)], \tag{3}$$

where $\delta(\cdot)$ is the Dirac delta, or impulse, function. For convenience in the sequel, we shall say that such a time-series contains *first-order periodicity*, with frequency α.

[3] The strength of the spectral line is $|M_x^\alpha|^2$ as indicated in (3) if and only if the limit (2) exists in the temporal mean square sense with respect to the time-parameter u obtained by replacing t with $t+u$ in (2) [29].

Let $x(t)$ be decomposed into the sum of its finite sinewave component, with frequency α, and its residual, say $n(t)$,

$$x(t) = a\cos(2\pi\alpha t + \theta) + n(t), \tag{4}$$

and assume that $n(t)$ is random (erratic). If the strength of the sinewave is weak relative to the random residual, then it is not evident from visual inspection of the time-series that $x(t)$ contains periodicity. Hence, it is said to contain *hidden periodicity*. However, because of the associated spectral lines, hidden periodicity can be detected and otherwise exploited through techniques of spectral analysis.

In this paper, we are concerned with time-series that contain more subtle types of hidden periodicity that do not give rise to spectral lines, but which can be converted into spectral lines with a nonlinear time-invariant transformation of the time-series. In particular, we shall focus on the type of hidden periodicity that can be converted into spectral lines with a *quadratic time-invariant* (QTI) transformation.

A transformation of a time-series $x(t)$ into another time-series $y(t)$ is QTI if and only if there exists a function $k(\cdot,\cdot)$, called the *kernel*, such that $y(t)$ can be expressed in terms of $k(\cdot,\cdot)$ and $x(t)$ by

$$y(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(t-u, t-v)x(u)x(v)\,\mathrm{d}u\,\mathrm{d}v, \tag{5a}$$

which is equivalent (by a change of variables of integration) to

$$y(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(u, v)x(t-u)x(t-v)\,\mathrm{d}u\,\mathrm{d}v. \tag{5b}$$

A QTI transformation is said to be *stable* if and only if the kernel is absolutely integrable,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |k(u, v)|\,\mathrm{d}u\,\mathrm{d}v < \infty. \tag{6}$$

By restricting attention to only those quadratic transformations that are *time-invariant* (as reflected in the dependence of k in (5) on the three variables t, u, v through only the differences $t-u$ and $t-v$) and *stable*, we rule out periodically time-variant and oscillating time-invariant transformations, both of which introduce periodicity into $y(t)$ that is foreign to $x(t)$.

We shall say that a time-series $x(t)$ contains *second-order periodicity* with frequency α if and only if there exists some stable QTI transformation of $x(t)$ into, say, $y(t)$ such that $y(t)$ contains first-order periodicity with frequency α; that is, $y(t)$ exhibits a spectral line at $f = \pm\alpha$. By substitution of (5) into (2), it can be shown that $x(t)$ contains second-order periodicity with frequency α if and only if the parameter[4]

$$\hat{R}_x^\alpha(\tau) \triangleq \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t+\tfrac{1}{2}\tau)(t-\tfrac{1}{2}\tau)\,\mathrm{e}^{-\mathrm{i}2\pi\alpha t}\,\mathrm{d}t \tag{7}$$

exists and is not identically zero as a function of τ. Consequently, we focus our attention in this paper on the class of time-series for which the function $\hat{R}_x^\alpha$ exists and is not identically zero for some nonzero values of α. Also, in order to avoid anomalous time-series,[5] it is assumed that $\hat{R}_x^\alpha(\tau)$ is a continuous function of τ. For $\alpha = 0$, $\hat{R}_x^\alpha$ is the conventional limit autocorrelation, denoted by $\hat{R}_x$, which plays a fundamental role in the theory of conventional spectral analysis (cf. [24, 33, 53]). For $\alpha \neq 0$, $\hat{R}_x^\alpha$ is a generalization of the limit autocorrelation that incorporates a cyclic (sinusoidal) weighting function, and $\hat{R}_x^\alpha$ shall therefore be referred to as the *limit cyclic autocorrelation* (and is sometimes abbreviated to *cyclic autocorrelation*). Whereas $\hat{R}_x(\tau)$, for fixed τ, is the constant (*dc*) component of the time-series

$$z(t) \triangleq x(t+\tfrac{1}{2}\tau)x(t-\tfrac{1}{2}\tau), \tag{8}$$

[4] Much of the theory developed in this paper requires only that (7) converge pointwise in τ and in u (obtained by replacing t with $t+u$). However, in order to include the spectral density of a quadratically transformed version of $x(t)$ in the theory, it is required that (7) converge in temporal mean square with respect to u (obtained by replacng t with $t+u$ in (7)) [29].

[5] For example, if $\hat{R}_x^\alpha(\tau)$ is discontinuous at $\tau = 0$ for $\alpha = 0$, the spectral density can be identically zero even though the average power is nonzero [53].

$\hat{R}_x^\alpha(\tau)$ is the sinewave (ac) component, with frequency α, of the time-series $z(t)$. This interpretation is expanded on in the next subsection.

By comparison of (2) and (7), it can be seen that a time-series contains second-order periodicity if and only if its lag product (8) contains first-order periodicity for some lag values, τ.

2.2. Limit periodic autocorrelation

The limit cyclic autocorrelation (7) can be derived by an alternative means that emphasizes its relationship to the oldest[6] known technique for extracting periodicity from random data, the technique of *synchronized averaging*, which is also referred to as *superposed epoch analysis* [9]. This technique can be viewed graphically as follows. If the period T_0 of periodicity which is hidden in the data is known, the data can be partitioned into disjoint adjacent segments of length T_0, and these horizontally arranged segments can be stacked up vertically as shown in Fig. 1. Then for each point within a period, say t within the first period, an average can be obtained to reduce undesired random effects, by adding the time-samples along the vertical line intersecting the points t, $t \pm T_0$, $t \pm 2T_0$, $t \pm 3T_0, \ldots, t \pm NT_0$, as shown in Fig. 1. In this way, the *time-variant mean*

$$M_x(t)_T \triangleq \frac{1}{2N+1} \sum_{n=-N}^{N} x(t+nT_0), \tag{9}$$

based on a total data-segment length of

$$T = (2N+1)T_0, \tag{10}$$

is obtained.

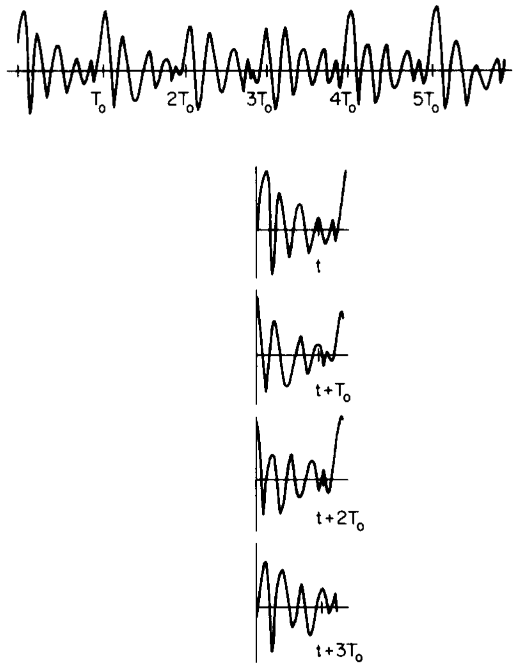

Fig. 1. Superposed epoch analysis.

[6] Carried out with discrete-time data arranged in tabular form, this technique was evidently first used by Buys-Ballot in 1847 [6] (see also [9, Chapter 1]) on meteorological data.

As an alternative approach to implementation, this time-variant mean can be obtained by using a particular linear time-invariant (LTI) transformation, called a *comb filter*, which is equivalent to a sum of *bandpass filters* (BPFs) with center frequencies equal to the harmonic frequencies, $\pm 1/T_0$, $\pm 2/T_0$, $\pm 3/T_0, \ldots$, of the periodicity of interest. To establish this equivalence, we proceed as follows. The time-variant mean (9) can be re-expressed as the convolution

$$M_x(t)_T = \int_{-\infty}^{\infty} g(t-u)x(u)\,\mathrm{d}u \triangleq g(t) \otimes x(t), \tag{11}$$

where the *impulse-response function* g is

$$g(t) = \frac{1}{2N+1} \sum_{n=-N}^{N} \delta(t-nT_0). \tag{12}$$

The corresponding *transfer function* is

$$G(f) = \sum_{m=-\infty}^{\infty} \frac{1}{T} w_{1/T}(f-m/T_0), \tag{13}$$

where

$$w_{1/T}(f) \triangleq \frac{\sin(\pi T f)}{\pi f}. \tag{14}$$

This is the transfer function of a comb filter, which has passbands at all the harmonic frequencies $\{m/T_0\}$, and each passband has width $1/T$ and unity attenuation at band center.

In the limit as the number of time-samples averaged in (9) approaches infinity, $N \to \infty$ (and therefore $T \to \infty$), the limit time-variant mean,

$$\hat{M}_x(t) \triangleq \lim_{T\to\infty} M_x(t)_T, \tag{15}$$

is obtained. In this limit, the bandwidths of the comb filter (13) become infinitesimal so that the filter response $\hat{M}_x(t)$ can contain only frequency components at the discrete frequencies $\{m/T_0\}$. Hence, $\hat{M}_x(t)$ is periodic with period T_0,

$$\hat{M}_x(t+T_0) = \hat{M}_x(t), \tag{16}$$

and will therefore be referred to as the *limit periodic mean.* The individual sinewave components of this periodic function $\hat{M}_x(t)$ can be obtained by using the individual *teeth* of the comb filter, that is, by using individual BPFs from the sum of BPFs that comprise the comb filter (equation (13)). Thus, to obtain the mth sinewave component, the desired filter transfer function is

$$G_m(f) = \frac{1}{T} w_{1/T}(f - m/T_0). \tag{17}$$

The corresponding impulse-response function is obtained by inverse Fourier transformation,

$$g_m(t) = u_T(t)\,\mathrm{e}^{\mathrm{i}2\pi m t/T_0}, \tag{18}$$

where u_T is a unity-area rectangle of width T centered at the origin. Consequently, the desired averaging operation required to obtain the mth sinewave component is, analogous to (11),

$$g_m(t) \otimes x(t),$$

which, upon substitution of (18), becomes

$$M_x^\alpha(t)_T \triangleq \frac{1}{T}\int_{-T/2}^{T/2} x(t+u)\,\mathrm{e}^{-\mathrm{i}2\pi\alpha u}\,\mathrm{d}u, \tag{19}$$

where $\alpha = m/T_0$. In the limit $T \to \infty$, this yields the mth sinewave component of the limit periodic mean

$$\hat{M}_x^\alpha(t) \triangleq \lim_{T\to\infty} M_x^\alpha(t)_T. \tag{20}$$

Comparison of (19)-(20) with (2) reveals that

$$\hat{M}_x^\alpha(t) = \hat{M}_x^\alpha\,\mathrm{e}^{\mathrm{i}2\pi\alpha t}. \tag{21}$$

Summing all the sinewave components yields the limit periodic mean[7]

$$\hat{M}_x(t) = \sum_{m=-\infty}^{\infty} \hat{M}_x^{m/T_0}\,\mathrm{e}^{\mathrm{i}2\pi m t/T_0}. \tag{22}$$

Comparison of (9) and (15) with (19), (21), and (22) reveals the *fundamental identity for synchronized averaging*

$$\begin{aligned}\hat{M}_x(t) &\triangleq \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{N} x(t+nT_0)\\ &= \sum_{m=-\infty}^{\infty} \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t+u)\,\mathrm{e}^{-\mathrm{i}2\pi m u/T_0}\,\mathrm{d}u.\end{aligned} \tag{23}$$

Now, for a time-series $x(t)$ that contains second-order periodicity, but does not contain first-order periodicity, synchronized averaging applied directly to the time-series is of no use, since

$$\hat{M}_x(t) \equiv \text{constant}.$$

However, synchronized averaging applied to the lag-product time-series (8) yields

$$\begin{aligned}\hat{R}_x(t,\tau) \triangleq \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N} & x(t+nT_0+\tfrac{1}{2}\tau)\\ &\times x(t+nT_0-\tfrac{1}{2}\tau),\end{aligned} \tag{24}$$

from which identity (23) yields

$$\hat{R}_x(t,\tau) = \sum_{m=-\infty}^{\infty} \hat{R}_x^{m/T_0}(\tau)\,\mathrm{e}^{\mathrm{i}2\pi m t/T_0}, \tag{25}$$

where

$$\begin{aligned}\hat{R}_x^\alpha(\tau) \triangleq \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} & x(t+\tfrac{1}{2}\tau)x(t-\tfrac{1}{2}\tau)\\ &\times \mathrm{e}^{-\mathrm{i}2\pi\alpha t}\,\mathrm{d}t,\end{aligned} \tag{26}$$

which is recognized as the limit cyclic autocorrelation (7), and which is not identically zero if and only if $x(t)$ contains second-order periodicity with frequency α. By analogy with the terminology for $\hat{M}_x(t)$, the function $\hat{R}_x(t,\tau)$ shall be referred to as

[7] All Fourier series in this paper are assumed to converge in some appropriate sense (such as pointwise or in temporal mean square), which depends on the particular mathematical applications and the corresponding assumptions about the mathematical model for $x(t)$.

the *limit periodic autocorrelation.* In summary, the limit cyclic autocorrelation can be interpreted as a Fourier coefficient in the Fourier series expansion of the limit periodic autocorrelation (25).

2.3. The limit cyclic spectrum

Yet another interpretation of the limit cyclic autocorrelation can be obtained as follows. The generalized limit autocorrelation $\hat{R}_x^\alpha$ defined by (7) is actually the conventional cross-correlation of the two complex-valued frequency-shifted versions,

$$\begin{aligned} u(t) &\triangleq x(t)\,\mathrm{e}^{-\mathrm{i}\pi\alpha t}, \\ v(t) &\triangleq x(t)\,\mathrm{e}^{+\mathrm{i}\pi\alpha t}, \end{aligned} \tag{27}$$

of the real time-series $x(t)$, that is

$$\begin{aligned} \hat{R}_x^\alpha(\tau) &\equiv \hat{R}_{uv}(\tau) \\ &\triangleq \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} u(t+\tfrac{1}{2}\tau)v^*(t-\tfrac{1}{2}\tau)\,\mathrm{d}t. \end{aligned} \tag{28}$$

This is easily verified by substitution of (27) into (28). Consequently, $\hat{R}_x^\alpha$ is the inverse Fourier transform of the cross-spectral density $\hat{S}_{uv}$, of $u(t)$ and $v(t)$ (cf. [24, 33]),

$$\hat{R}_x^\alpha(\tau) = \int_{-\infty}^{\infty} \hat{S}_x^\alpha(f)\,\mathrm{e}^{\mathrm{i}2\pi f\tau}\,\mathrm{d}f, \tag{29}$$

for which the notation

$$\hat{S}_x^\alpha(f) \triangleq \hat{S}_{uv}(f) \tag{30}$$

is introduced. This special limit cross-spectral density shall be referred to as the *limit cyclic spectral density* of $x(t)$ (and is sometimes abbreviated to *cyclic spectrum*). It follows from the definition of the conventional cross-spectral density (cf. [24]) that $\hat{S}_x^\alpha(f)$ is the *limit temporal correlation* of the two spectral components of $x(t)$ with frequencies $f+\frac{1}{2}\alpha$ and $f-\frac{1}{2}\alpha$; that is,[8]

$$\hat{S}_x^\alpha(f) = \lim_{T\to\infty}\lim_{\Delta t\to\infty} S_{uv_T}(t,f)_{\Delta t}, \tag{31}$$

where $S_{uv_T}(t,f)_{\Delta t}$ is the temporal correlation of

$$\frac{1}{\sqrt{T}}\,U_T(t,f) \triangleq \frac{1}{\sqrt{T}}\,X_T(t,f+\tfrac{1}{2}\alpha),$$

and

$$\frac{1}{\sqrt{T}}\,V_T(t,f) \triangleq \frac{1}{\sqrt{T}}\,X_T(t,f-\tfrac{1}{2}\alpha), \tag{32}$$

and $X_T(t,f)$ is the time-variant finite-time complex spectrum[9] of $x(t)$,

$$X_T(t,f) \triangleq \int_{t-T/2}^{t+T/2} x(u)\,\mathrm{e}^{-\mathrm{i}2\pi fu}\,\mathrm{d}u \tag{33}$$

(i.e., $X_T(t,f)$ is the complex envelope of the narrowbandpass component of $x(t)$ with center frequency f and approximate bandwidth $1/T$). The temporal correlation referred to here is defined by

$$S_{uv_T}(y,f)_{\Delta t} \triangleq \frac{1}{\Delta t}\int_{-\Delta t/2}^{\Delta t/2} \frac{1}{T}\,U_T(t+u,f)\,V_T^*(t+u,f)\,\mathrm{d}u. \tag{34}$$

Because of this spectral correlation characterization, the limit cyclic spectral density shall also be called the *spectral correlation function.* Equations (31)-(34) reveal the fundamental result that since any comprehensive statistical theory of second-order periodicity must be based on the limit cyclic autocorrelation (as explained in the two preceding subsections), then such a theory must also be based on cross-spectral analysis of frequency-translated versions of the time-series of interest. In fact, we have just discovered that a time-series $x(t)$ contains second-order periodicity with frequency α (as defined in Section 2.1) if and only if there exists correlation between spectral components of $x(t)$, with frequencies separated by the amount α, namely, frequencies $f+\frac{1}{2}\alpha$ and $f-\frac{1}{2}\alpha$ for appropriate values of f.

[8] Convergence pointwise in both t and f in (31) is adequate for much of the theory developed in this paper. However, in order to include the spectral density of the time-series $S_{uv_T}(t,f)$ in the theory, it is required that (31) converge in temporal mean square with respect to t [29].

[9] Equation (33) is sometimes referred to as the *short-time Fourier transform*; however, the term *short* is relative and not always applicable.

Moreover, this *spectral characterization* (29)-(34) of second-order periodicity leads naturally to a particularly convenient and appropriate spectrally decomposed measure of the *strength of second-order periodicity* contained in a time-series, namely, the *limit correlation coefficient* for the two spectral components with frequencies $f+\frac{1}{2}\alpha$ and $f-\frac{1}{2}\alpha$. This is given by the *cross-coherence* between $u(t)$ and $v(t)$, which is defined by (cf. [24, 33]),

$$\hat{C}_{uv}(f) \triangleq \frac{\hat{S}_{uv}(f)}{[\hat{S}_u(f)\hat{S}_v(f)]^{1/2}} \equiv \frac{\hat{S}_x^\alpha(f)}{[\hat{S}_x(f+\frac{1}{2}\alpha)\hat{S}_x(f-\frac{1}{2}\alpha)]^{1/2}} \triangleq \hat{C}_x^\alpha(f), \tag{35}$$

that is,

$$\hat{C}_x^\alpha(f) \equiv \hat{C}_{uv}(f). \tag{36}$$

This special cross-coherence shall be referred to as the *spectral autocoherence* of $x(t)$ at *cycle frequency* α and *spectrum frequency* f. (It should be noted that α is the *separation*, and f the *location*, of the two frequencies $f+\frac{1}{2}\alpha$ and $f-\frac{1}{2}\alpha$ in the autocoherence.) It follows from a fundamental property of the cross-coherence (the correlation coefficient) that the spectral autocoherence is upper-bounded by unity [24],

$$|\hat{C}_x^\alpha(f)| \leq 1, \tag{37}$$

for all time-series containing second-order periodicity. Consequently, $x(t)$ shall be said to be *completely coherent* (contain the maximum amount of second-order periodicity) with cycle frequency α and spectrum frequency f if and only if the spectral autocoherence is unity in magnitude,

$$|\hat{C}_x^\alpha(f)| = 1. \tag{38}$$

Furthermore, $x(t)$ shall be said to be *completely incoherent* [10] (contain no second-order periodicity) with cycle frequency α and spectrum frequency f if and only if the spectral autocoherence is zero,

$$\hat{C}_x^\alpha(f) = 0. \tag{39}$$

[10] A time-series that is completely incoherent for all α and f can contain periodicity of order higher than the second.

In addition to possessing the *normalization property* (37), the spectral autocoherence also is appropriately invariant to linear time-invariant (LTI) transformations of the time-series. That is, it can be shown (using (96)) that if $y(t)$ is a filtered version of $x(t)$,

$$y(t) = h(t) \otimes x(t), \tag{40}$$

and the filter transfer function is nonzero,

$$H(f) \triangleq \int_{-\infty}^{\infty} h(t)\, \mathrm{e}^{-\mathrm{i}2\pi ft}\, \mathrm{d}t \neq 0, \tag{41}$$

then the autocoherence magnitude of $y(t)$ is identical to the autocoherence magnitude of $x(t)$,

$$|\hat{C}_y^\alpha(f)| \equiv |\hat{C}_x^\alpha(f)|. \tag{42}$$

This *invariance property* reveals that the strength of second-order periodicity contained in a time-series is unaffected by LTI transformation, provided that frequency components are not annihilated because the transfer function equals zero at some frequencies.

Alternatives to the definition of the limit cyclic spectrum as the Fourier transform of the limit cyclic autocorrelation,

$$\hat{S}_x^\alpha(f) = \int_{-\infty}^{\infty} \hat{R}_x^\alpha(\tau)\, \mathrm{e}^{-\mathrm{i}2\pi f\tau}\, \mathrm{d}\tau, \tag{43}$$

that are analogous to the empirically motivated definitions of the spectral density ($\alpha = 0$) can be obtained as follows. Let us define the *cyclic periodogram* by

$$S_{x_T}^\alpha(t,f) \triangleq \frac{1}{T} X_T(t, f+\tfrac{1}{2}\alpha) X_T^*(t, f-\tfrac{1}{2}\alpha), \tag{44}$$

where X_T is defined by (33), For $\alpha = 0$, the cyclic periodogram reduces to the conventional periodogram (cf. [24, 33]).The limit cyclic spectrum can be obtained from the cyclic periodogram by either (i) time-averaging as in (31) (using the notation $T = 1/\Delta f$),

$$S_{x_{1/\Delta f}}^\alpha(t,f)_{\Delta t} \triangleq \frac{1}{\Delta t} \int_{t-\Delta t/2}^{t+\Delta t/2} S_{x_{1/\Delta f}}^\alpha(u,f)\, \mathrm{d}u, \tag{45}$$

or (ii) frequency smoothing (using the notation $T=\Delta t$),

$$S^{\alpha}_{x_{\Delta t}}(t,f)_{\Delta f} \triangleq \frac{1}{\Delta f}\int_{f-\Delta f/2}^{f+\Delta f/2} S^{\alpha}_{x_{\Delta t}}(t,v)\,\mathrm{d}v. \tag{46}$$

Specifically, it can be shown (cf. [29]) that, as in (31)-(34),

$$\hat{S}^{\alpha}_x(f) = \lim_{\Delta f\to 0}\lim_{\Delta t\to\infty} S^{\alpha}_{x_{1/\Delta f}}(t,f)_{\Delta t}, \tag{47}$$

and that

$$\hat{S}^{\alpha}_x(f) = \lim_{\Delta f\to 0}\lim_{\Delta t\to\infty} S^{\alpha}_{x_{\Delta t}}(t,f)_{\Delta f}. \tag{48}$$

Furthermore, it can be shown that the cyclic periodogram is the Fourier transform,

$$S^{\alpha}_{x_T}(t,f) = \int_{-\infty}^{\infty} R^{\alpha}_{x_T}(t,\tau)\,\mathrm{e}^{-\mathrm{i}2\pi f\tau}\,\mathrm{d}\tau, \tag{49}$$

of the *cyclic correlogram*, which is defined by

$$R^{\alpha}_{x_T}(t,\tau) \triangleq \frac{1}{T}\int_{t-(T-|\tau|)/2}^{t+(T-|\tau|)/2} x(u+\tfrac{1}{2}\tau)x(u-\tfrac{1}{2}\tau) \times \mathrm{e}^{-\mathrm{i}2\pi\alpha u}\,\mathrm{d}u. \tag{50}$$

For $\alpha=0$, the cyclic correlogram reduces to the conventional correlogram (cf. [24, 33]).

It should be emphasized that the cyclic autocorrelation (7), cyclic spectrum (47)-(48), cyclic periodogram (44), and cyclic correlogram (50), all reduce to conventional statistical parameters for $\alpha=0$. Consequently, the Fourier transform relation (43) shall be called the *cyclic Wiener relation*,[11] as a generalization of the Wiener relation ($\alpha=0$) between the spectral density and the autocorrelation [53], and similarly the Fourier transform relation (49), shall be called the *cyclic-periodogram/cyclic-correlogram relation*, as a generalization of the known periodiogram/correlogram relation ($\alpha=0$) (cf. [33]).

2.4. Limit periodic spectrum

By analogy with (43), the *limit periodic spectrum* is defined to be the Fourier transform of the limit periodic autocorrelation (24),

$$\hat{S}_x(t,f) \triangleq \int_{-\infty}^{\infty} \hat{R}_x(t,\tau)\,\mathrm{e}^{-\mathrm{i}2\pi f\tau}\,\mathrm{d}\tau. \tag{51}$$

It follows from (25), (43), and (51) that the limit cyclic spectra are the Fourier coefficients of the limit periodic spectrum,

$$\hat{S}_x(t,f) = \sum_{m=-\infty}^{\infty} \hat{S}^{m/T_0}_x(f)\,\mathrm{e}^{\mathrm{i}2\pi mt/T_0}. \tag{52}$$

For time-series that contain second-order periodicity with more than one period, the limit periodic autocorrelation and limit periodic spectrum can be generalized to

$$\hat{R}_x(t,\tau) \triangleq \sum_{\alpha} \hat{R}^{\alpha}_x(\tau)\,\mathrm{e}^{\mathrm{i}2\pi\alpha t}, \tag{53}$$

$$\hat{S}_x(t,f) \triangleq \sum_{\alpha} \hat{S}^{\alpha}_x(f)\,\mathrm{e}^{\mathrm{i}2\pi\alpha t}, \tag{54}$$

for which the sums are over all α for which the limit cyclic autocorrelation $\hat{R}^{\alpha}_x$ is not identically zero. These limit functions (53) and (54) are in general *almost periodic functions* in the mathematical sense (cf. [11, 20]).

It follows from (43) and (52) that the limit cyclic autocorrelation and the limit periodic spectrum are related by the Fourier-transform/Fourier-series relation

$$\hat{R}^{\alpha}_x(\tau) = \int_{-\infty}^{\infty}\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \hat{S}_x(t,f)\,\mathrm{e}^{-\mathrm{i}2\pi(\alpha t-\tau f)}\,\mathrm{d}t\,\mathrm{d}f. \tag{55}$$

This is analogous to the double Fourier transform relation

$$\rho_x(\tau,\alpha) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_x(t,f)\,\mathrm{e}^{-\mathrm{i}2\pi(\alpha t-\tau f)}\,\mathrm{d}t\,\mathrm{d}f, \tag{56}$$

where ρ_x is the *symmetric ambiguity function* (with conventional frequency parameter $\nu=-\alpha$) for a real finite-energy waveform $x(t)$ (cf. [51, 56]),

$$\rho_x(\tau,\alpha) \triangleq \int_{-\infty}^{\infty} x(t+\tfrac{1}{2}\tau)x(t-\tfrac{1}{2}\tau)\,\mathrm{e}^{-\mathrm{i}2\pi\alpha t}\,\mathrm{d}t, \tag{57}$$

[11] In the probabilistic theory, the counterpart of the Wiener relation is known as the *Wiener-Khinchine relation* (cf. [24]).

and E_x is the *Wigner-Ville distribution* [12] (cf. [10, 52, 54]) for a real finite-energy waveform $x(t)$,

$$E_x(t,f) \triangleq \int_{-\infty}^{\infty} x(t+\tfrac{1}{2}\tau)x(t-\tfrac{1}{2}\tau)\,\mathrm{e}^{-\mathrm{i}2\pi f\tau}\,\mathrm{d}\tau. \tag{58}$$

More specifically, comparison of (57) and (7) reveals that $\rho_x(\tau,\alpha)$ is the counterpart for finite-energy waveforms of $\hat{R}_x^\alpha(\tau)$, which is for finite-power waveforms. Consequently, the analogy between (55) and (56) suggests that $E_x(t,f)$ is the counterpart for finite-energy waveforms of $\hat{S}_x(t,f)$, which is for finite-power waveforms. However, whereas ρ_x and E_x are of potential use for only finite-energy but otherwise arbitrary waveforms, $\hat{R}_x^\alpha(\tau)$ and $\hat{S}_x(t,f)$ are of use for only finite-power waveforms containing second-order periodicity. Moreover, $\hat{R}_x^\alpha(\tau)$ and $\hat{S}_x(t,f)$ are idealized (limit) statistical parameters in which all randomness has been removed (by averaging), whereas ρ_x and E_x are random (erratic) if the waveform $x(t)$ is random. The difference between the limit cyclic autocorrelation of a real waveform $x(t)$, and the ambiguity function for the *complex envelope* $\gamma(t)$, of $x(t)$ (which is the appropriate ambiguity function for radar ambiguity applications) is even more distinct as explained in the following. The complex envelope of $x(t)$ is defined by

$$\gamma(t) = [x(t) + \mathrm{i}\check{x}(t)]\,\mathrm{e}^{-\mathrm{i}2\pi f_0 t}, \tag{59}$$

where $\check{x}$ is the Hilbert transform of x, and f_0 is typically chosen to be near the center of the spectral band occupied by $x(t)$. It can be shown (cf. Section 3.5) that $\hat{R}_x^\alpha$ cannot in general be recovered from $\hat{R}_\gamma^\alpha$ (except for $\alpha = 0$). Rather, both the limit cyclic autocorrelation of γ (which uses $\gamma(t+\frac{1}{2}\tau)\gamma^*(t-\frac{1}{2}\tau)$) and the limit cyclic cross-correlation of γ and its conjugate γ^* (which uses $\gamma(t+\frac{1}{2}\tau)\gamma(t-\frac{1}{2}\tau)$) are needed to recover $\hat{R}_x^\alpha$. The cross-ambiguity of γ and γ^* plays no role in the conventional theory of radar ambiguity. Thus, the limit cyclic autocorrelation $\hat{R}_x^\alpha$ contains more information about periodicity than that obtainable from the limiting form $\hat{R}_\gamma^\alpha$ of the radar ambiguity function. Similar remarks apply to the Wigner-Ville distribution for the complex envelope $\gamma(t)$.

[12] Equation (58) is sometimes called a *time/frequency energy density*, but is more appropriately interpreted as a time-frequency energy flow rate.

2.5. Probabilistic interpretation

The periodically time-variant infinite synchronized time-average used to define the limit periodic autocorrelation,

$$\hat{R}_x(t,\tau) \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(t+nT_0+\tfrac{1}{2}\tau) \times x(t+nT_0-\tfrac{1}{2}\tau), \tag{60}$$

and thereby the limit cyclic autocorrelation (for $\alpha = m/T_0$)

$$\hat{R}_x^\alpha(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \hat{R}_x(t,\tau)\,\mathrm{e}^{-\mathrm{i}\pi\alpha t}\,\mathrm{d}t, \tag{61}$$

the limit periodic spectrum,

$$\hat{S}_x(t,f) = \int_{-\infty}^{\infty} \hat{R}_x(t,\tau)\,\mathrm{e}^{-\mathrm{i}2\pi f\tau}\,\mathrm{d}\tau, \tag{62}$$

and the limit cyclic spectrum (for $\alpha = m/T_0$),

$$\hat{S}_x^\alpha(f) = \int_{-\infty}^{\infty} \hat{R}_x^\alpha(\tau)\,\mathrm{e}^{-\mathrm{i}2\pi f\tau}\,\mathrm{d}\tau, \tag{63}$$

can be re-interpreted probabilistically in terms of *expected value*. To see this, consider the joint fraction-of-time amplitude distribution for a time-series $x(t)$ defined by (cf. [24])

$$F_{x(t_1)x(t_2)}(y_1,y_2) \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} U[y_1 - x(t_1+nT_0)] \times U[y_2 - x(t_2+nT_0)], \tag{64}$$

in which U is the unit step function. The joint fraction-of-time amplitude density for $x(t)$ is defined by (cf. [24])

$$f_{x(t_1)x(t_2)}(y_1,y_2) = \frac{\partial^2}{\partial y_1\,\partial y_2} F_{x(t_1)x(t_2)}(y_1,y_2). \tag{65}$$

Both $F_{x(t_1)x(t_2)}$ and $f_{x(t_1)x(t_2)}$ are jointly periodic with period T_0 in the two time-variables, t_1 and t_2, e.g.,

$$F_{x(t_1+T_0)x(t_2+T_0)} = F_{x(t_1)x(t_2)}. \tag{66}$$

It can be shown using only (64) and (65) that the probabilistic autocorrelation, which is defined by (cf. [24])

$$\mathbf{E}\{x(t_1)x(t_2)\} \triangleq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y_1 y_2 f_{x(t_1)x(t_2)}(y_1, y_2)\, \mathrm{d}y_1\, \mathrm{d}y_2, \tag{67}$$

is given by

$$\mathbf{E}\{x(t_1)x(t_2)\} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(t_1+nT_0)x(t_2+nT_0). \tag{68}$$

This is verified simply by substitution of (64) into (65) into (67), and interchange of the order of operations.[13] It follows from (60) and (68) that the limit period autocorrelation can be interpreted as the probabilistic autocorrelation,

$$\hat{R}_x(t, \tau) = \mathbf{E}\{x(t+\tfrac{1}{2}\tau)x(t-\tfrac{1}{2}\tau)\}. \tag{69}$$

Moreover, this expected value can be interpreted as an *ensemble average* (at least heuristically) by defining ensemble members (random samples) to be time-translates of $x(t)$. That is, the sth ensemble member is

$$x(t, s) \triangleq x(t - sT_0), \tag{70}$$

for integer values of s. The mapping (70) between an individual time-series and an ensemble of time-series is the basis for an isomorphism between an individual time-series and a cyclostationary stochastic process. This is a generalization of Wold's isomorphism for discrete-time stationary stochastic processes [55]. This isomorphism can be generalized to almost cyclostationary stochastic processes as outlined in [24] (cf. [29]).

[13] Strictly speaking, this interchange of operations must be mathematically justified. The primary requirements are simply that the double integral in (67) exists and the limits in (64) and (68) exist in appropriate senses.

Since it is common practice in the use of the probabilistic theory of stochastic processes to assume that a process has Gaussian distributions, it should be clarified that if either (i) the cyclostationary probabilistic model, (64), or (ii) the stationary probabilistic model

$$F'_{x(t_1)x(t_2)}(y_1, y_2) \triangleq \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} U[y_1 - x(t_1+t)] \times U[y_2 - x(t_2+t)]\, \mathrm{d}t, \tag{71}$$

is Gaussian, then either (ii) or (i), respectively, cannot be Gaussian unless $x(t)$ contains no second-order periodicity, $\hat{R}_x^\alpha \equiv 0$ for $\alpha \neq 0$. This follows (cf. [24]) from the fact that $F'_{x(t_1)x(t_2)}$ is a mixture of $F_{x(t+t_1)x(t+t_2)}$, namely,

$$F'_{x(t_1)x(t_2)} = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} F_{x(t_1+t)x(t_2+t)}\, \mathrm{d}t. \tag{72}$$

3. Properties of the limit cyclic spectrum

3.1. Time dependence

It can be shown (by analogy with the argument for $\alpha = 0$ in [35]) that the limit of the cyclic correlogram is the limit cyclic autocorrelation,

$$\lim_{T\to\infty} R_{x_T}^\alpha(t, \tau) = \hat{R}_x^\alpha(\tau), \tag{73}$$

and this limit is independent of the time-location parameter t of the measurement (cf. (50) in Section 2). Similarly, the limit cyclic spectrum, defined in terms of the cyclic periodogram

$$\lim_{\Delta f\to 0}\lim_{T\to\infty} S_{x_T}^\alpha(t, f)_{\Delta f} = \hat{S}_x^\alpha(f), \tag{74}$$

is also independent of the time-location t, as revealed by (43) in Section 2. However, it should be clarified that when $x(t)$ is translated to, say, $x(t+t')$, then these limit statistics do indeed change, and the variation with t' is sinusoidal. That is, for the time-translated time-series $y(t) \triangleq$

$x(t+t')$ the limit statistics are given by

$$\hat{R}_y^\alpha(\tau) = \hat{R}_x^\alpha(\tau)\, e^{i2\pi\alpha t'}, \tag{75}$$

$$\hat{S}_y^\alpha(f) = \hat{S}_x^\alpha(f)\, e^{i2\pi\alpha t'}. \tag{76}$$

This is consistent with the periodic dependence of the limit periodic autocorrelation and spectrum, for example,

$$\hat{R}_y(t,\tau) = \hat{R}_x(t+t',\tau) = \sum_\alpha \hat{R}_x^\alpha(\tau)\, e^{i2\pi\alpha t'}\, e^{i2\pi\alpha t}. \tag{77}$$

It should be emphasized that this reveals that the limit cyclic spectrum, unlike the conventional limit spectrum, contains phase information.

3.2. Spectrum types and bandwidths

It can be shown that a time-series $x(t)$, for which the lag-product time-series (8) has finite power, can exhibit at most a denumerable set of nonzero cyclic spectra (cf. [20]). Thus, the *cycle spectrum* is discrete, say $\{\alpha_m : m = 0, \pm 1, \pm 2, \ldots\}$. If the cycle spectrum contains only the frequency $\alpha = 0$, then $x(t)$ is said to be *purely stationary*. If the cycle spectrum contains only integer multiples of some *fundamental frequency*, say $\alpha_0 = 1/T_0$, then $x(t)$ is said to be *purely cyclostationary with period* T_0. Otherwise, $x(t)$ is said to be *almost cyclostationary* (cf. [20, 24, 4]) because the Fourier series (54) in Section 2 is an *almost periodic function.*

It can also be shown that if the conventional limit spectrum contains no spectral lines and is therefore a continuous function of f, then the limit cyclic spectrum is also a continuous function of f. On the other hand if there are spectral lines (Dirac deltas in f) in the conventional limit spectrum then there are also Dirac deltas in f in the limit cyclic spectrum.

If $x(t)$ is bandlimited in the temporal mean (time-average) square sense to the band, say, $b < |f| < B$, that is,

$$\hat{S}_x(f) = 0 \quad \text{for } |f| \geq B \text{ or for } |f| \leq b, \tag{78}$$

then it follows from (35)-(37) in Section 2 that the support in the (f, α) plane for the cyclic spectra of $x(t)$ is as shown in Fig. 2, which depicts the constraints

$$\hat{S}_x^\alpha(f) = 0 \quad \text{for } |f| \geq B - \tfrac{1}{2}|\alpha| \quad \text{or for } (|f| \leq b + \tfrac{1}{2}|\alpha| \text{ and } \tfrac{1}{2}|\alpha| \leq b + |f|). \tag{79}$$

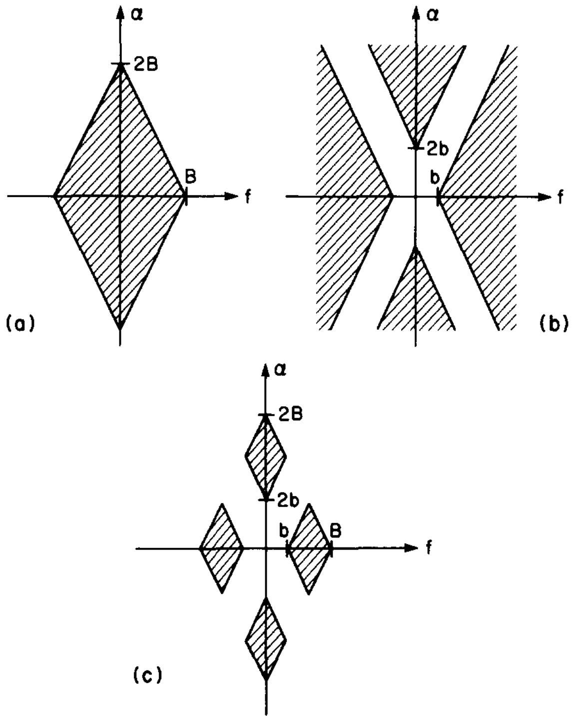

Fig. 2. Bi-frequency support for cyclic spectra of bandlimited time-series. (a) Lowpass. (b) Highpass. (c) Bandpass.

3.3. Real representation

The real and imaginary parts of the limit cyclic autocorrelation can be characterized by conventional auto- and cross-correlations of the two real time-series

$$c(t) \triangleq x(t)\cos(\pi\alpha t), \tag{80}$$

$$s(t) \triangleq x(t)\sin(\pi\alpha t). \tag{81}$$

Specifically, the real part of $\hat{R}_x^\alpha(\tau)$ is given by

$$\hat{R}_x^\alpha(\tau)_r = \hat{R}_c(\tau) - \hat{R}_s(\tau), \tag{82}$$

and the imaginary part is given by

$$\hat{R}_x^\alpha(\tau)_i = -\hat{R}_{cs}(\tau) - \hat{R}_{sc}(\tau). \tag{83}$$

This interesting relationship (82)-(83) is studied more deeply in the discussion of Rice's representation in Section 3.5.

3.4. Linear periodically time-variant transformations

A particularly common situation in which second-order periodicity arises is that for which a purely stationary time-series $x(t)$ is subjected to a *linear periodically time-variant* (LPTV) *transformation.* For example, many modulation systems can be modeled as the scalar response of a multi-input LPTV transformation with purely stationary excitation. This includes amplitude modulation (double sideband, single sideband, vestigial sideband, and with or without suppressed carrier), phase and frequency modulation, quadrature amplitude modulation, pulse-amplitude modulation, pulse-position modulation, and all synchronous digital modulations such as phase-shift keying, frequency-shift keying, etc. (cf. [24]). Consequently, the study of second-order periodicity is facilitated by general formulas that describe limit cyclic spectra, or spectral correlation functions, in terms of the parameters of LPTV transformations. This includes limit cyclic spectra that are *generated* by LPTV transformations of purely stationary time-series, as well as limit cyclic spectra that are transformed by LPTV transformations of cyclostationary time-series.

Let us consider the LPTV transformation

$$y(t)=\int_{-\infty}^{\infty} \boldsymbol{h}(t, u)\boldsymbol{x}(u)\,\mathrm{d}u, \tag{84}$$

for which $\boldsymbol{x}(t)$ is a (column) vector excitation, $y(t)$ is a scalar response, and $\boldsymbol{h}(t, u)=\boldsymbol{h}(t+T_0, u+T_0)$ is the periodically time-variant (row) vector of impulse-response functions that specify the transformation. The function $\boldsymbol{h}(t+\tau, t)$ is periodic in t for each τ, and can therefore be represented by the Fourier series

$$\boldsymbol{h}(t+\tau, t)=\sum_{n=-\infty}^{\infty} \boldsymbol{g}_n(\tau)\,\mathrm{e}^{\mathrm{i}2\pi nt/T_0}, \tag{85}$$

where

$$\boldsymbol{g}_n(\tau)\triangleq\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \boldsymbol{h}(t+\tau, t)\,\mathrm{e}^{-\mathrm{i}2\pi nt/T_0}\,\mathrm{d}t. \tag{86}$$

The *system function*, which is the well-known generalization of the transfer function defined by the Fourier transform [12]

$$\boldsymbol{G}(t,f)\triangleq\int_{-\infty}^{\infty} \boldsymbol{h}(t, t-\tau)\,\mathrm{e}^{-\mathrm{i}2\pi f\tau}\,\mathrm{d}\tau, \tag{87}$$

can therefore also be represented by a Fourier series,

$$\boldsymbol{G}(t,f)=\sum_{n=-\infty}^{\infty} \boldsymbol{G}_n(f+n/T_0)\,\mathrm{e}^{\mathrm{i}2\pi nt/T_0}, \tag{88}$$

where

$$\boldsymbol{G}_n(f)\triangleq\int_{-\infty}^{\infty} \boldsymbol{g}_n(\tau)\,\mathrm{e}^{-\mathrm{i}2\pi f\tau}\,\mathrm{d}\tau. \tag{89}$$

By substitution of (85) into (84) into the definition of the limit cyclic autocorrelation (7), it can be shown that[14]

$$\hat{R}_y^{\alpha}(\tau)=\sum_{n,m=-\infty}^{\infty} \mathrm{tr}\{[\hat{\boldsymbol{R}}_x^{\alpha-(n-m)/T_0}(\tau)\,\mathrm{e}^{-\mathrm{i}\pi(n+m)\tau/T_0}] \otimes \boldsymbol{r}_{nm}^{\alpha}(-\tau)\}, \tag{90}$$

where $\hat{\boldsymbol{R}}_x^{\beta}$ is the matrix of *limit cyclic cross-correlations* of the elements of the vector $\boldsymbol{x}(t)$,

$$\hat{\boldsymbol{R}}_x^{\beta}(\tau)\triangleq\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t+\tfrac{1}{2}\tau)x'(t-\tfrac{1}{2}\tau)\,\mathrm{e}^{-\mathrm{i}2\pi\beta t}\,\mathrm{d}t, \tag{91}$$

and $\boldsymbol{r}_{nm}^{\alpha}$ is the matrix of *finite cyclic cross-correlations*

$$\boldsymbol{r}_{nm}^{\alpha}(\tau)\triangleq\int_{-\infty}^{\infty} \boldsymbol{g}_n'(t+\tfrac{1}{2}\tau)\boldsymbol{g}_m^{*}(t-\tfrac{1}{2}\tau)\,\mathrm{e}^{-\mathrm{i}2\pi\alpha t}\,\mathrm{d}t. \tag{92}$$

Fourier transformation of (90), and application of

[14] In (90), tr{·} is the *trace operation*, and in (91) the superscript prime denotes *matrix transposition.*

the convolution theorem yields

$$\hat{S}_y^\alpha(f) = \sum_{n,m=-\infty}^{\infty} G_n(f+\tfrac{1}{2}\alpha)\hat{S}_x^{\alpha-(n-m)/T_0} \times (f-[n+m]/2T_0)G'_m(f-\tfrac{1}{2}\alpha)^*. \quad (93)$$

Formulas (90)-(93) reveal that the set of limit cyclic autocorrelations and the set of limit cyclic spectra are each *self-determinant characteristics* under an LPTV transformation, in the sense that the only features of the excitation that determine the limit cyclic autocorrelations (spectra) of the response are the limit cyclic autocorrelations (spectra) of the excitation.

In the special case of a linear time-invariant (LTI) transformation,

$$\boldsymbol{h}(t,u) = \boldsymbol{h}(t-u), \quad (94)$$

formulas (90)-(93) reduce to

$$\hat{R}_y^\alpha(\tau) = \mathrm{tr}\{\hat{\boldsymbol{R}}_x^\alpha(\tau) \otimes \boldsymbol{r}_h^\alpha(-\tau)\} \quad (95)$$

and

$$\hat{S}_y^\alpha(f) = \boldsymbol{H}(f+\tfrac{1}{2}\alpha)\hat{\boldsymbol{S}}_x^\alpha(f)\boldsymbol{H}'(f-\tfrac{1}{2}\alpha)^*, \quad (96)$$

in which

$$\boldsymbol{r}_h^\alpha(\tau) \triangleq \int_{-\infty}^{\infty} \boldsymbol{h}'(t+\tfrac{1}{2}\tau)\boldsymbol{h}(t-\tfrac{1}{2}\tau)\,\mathrm{e}^{-\mathrm{i}2\pi\alpha t}\,\mathrm{d}t \quad (97)$$

and $\boldsymbol{H}(f)$ is the Fourier transform of $\boldsymbol{h}(\tau)$. Also, in the special case for which the excitation $\boldsymbol{x}(t)$ is purely stationary, (90)-(93) reduce to

$$\hat{R}_y^\alpha(\tau) = \begin{cases} \displaystyle\sum_{n=-\infty}^{\infty} \mathrm{tr}\{[\boldsymbol{R}_x(\tau)\,\mathrm{e}^{-\mathrm{i}\pi(2n-p)\tau/T_0}] \otimes \boldsymbol{r}_{n(n-p)}^\alpha(-\tau)\}, & \alpha = p/T_0, \\ 0, & \alpha \neq p/T_0, \end{cases} \quad (98)$$

and

$$\hat{S}_y^\alpha(f) = \begin{cases} \displaystyle\sum_{m=-\infty}^{\infty} \boldsymbol{G}_{m+p}(f+\tfrac{1}{2}\alpha)\hat{\boldsymbol{S}}_x(f-\tfrac{1}{2}\alpha-m/T_0) \times \boldsymbol{G}'_m(f-\tfrac{1}{2}\alpha)^*, & \alpha = p/T_0, \\ 0, & \alpha \neq p/T_0, \end{cases} \quad (99)$$

for all integers p, where

$$\boldsymbol{r}_{nm}^\alpha(\tau) \triangleq \int_{-\infty}^{\infty} \boldsymbol{g}'_n(t+\tfrac{1}{2}\tau)\boldsymbol{g}_m^*(t-\tfrac{1}{2}\tau)\,\mathrm{e}^{-\mathrm{i}2\pi\alpha t}\,\mathrm{d}t. \quad (100)$$

This result reveals that $y(t)$ is purely cyclostationary with period T_0.

For convenient reference, formulae (93), (96), and (99) shall be referred to as *input-output spectral correlation relations.*

By substitution of (85) into (84) into the definition of the limit cyclic cross-correlation, it can be shown that

$$\hat{\boldsymbol{R}}_{xy}^\alpha(\tau) = \sum_{m=-\infty}^{\infty} [\hat{\boldsymbol{R}}_x^{\alpha+m/T_0}(\tau)\,\mathrm{e}^{\mathrm{i}\pi m\tau/T_0}] \otimes [\boldsymbol{g}'_m(-\tau)^*\,\mathrm{e}^{\mathrm{i}\pi\alpha\tau}]. \quad (101)$$

Fourier transformation of (101) and application of the convolution theorem yields

$$\hat{\boldsymbol{S}}_{xy}^\alpha(f) = \sum_{m=-\infty}^{\infty} \hat{\boldsymbol{S}}_x^{\alpha+m/T_0}[f-m/2T_0] \times \boldsymbol{G}'_m(f-\tfrac{1}{2}\alpha)^*. \quad (102)$$

In the special case of an LTI transformation, formulae (101) and (102) reduce to

$$\hat{\boldsymbol{R}}_{xy}^\alpha(\tau) = \hat{\boldsymbol{R}}_x^\alpha(\tau) \otimes [\boldsymbol{h}'(-\tau)\,\mathrm{e}^{\mathrm{i}\pi\alpha\tau}] \quad (103)$$

and

$$\hat{\boldsymbol{S}}_{xy}^\alpha(f) = \hat{\boldsymbol{S}}_x^\alpha(f)\boldsymbol{H}'(f-\tfrac{1}{2}\alpha)^*. \quad (104)$$

Also, in the special case for which the excitation $\boldsymbol{x}(t)$ is purely stationary, (101) and (102) reduce to

$$\hat{\boldsymbol{R}}_{xy}^\alpha(\tau) = \begin{cases} [\hat{\boldsymbol{R}}_x(\tau)\,\mathrm{e}^{-\mathrm{i}\pi\alpha\tau}] \otimes [\boldsymbol{g}'_p(-\tau)\,\mathrm{e}^{\mathrm{i}\pi\alpha\tau}], & \alpha = p/T_0, \\ 0, & \alpha \neq p/T_0, \end{cases} \quad (105)$$

and

$$\hat{\boldsymbol{S}}_{xy}^\alpha(f) = \begin{cases} \hat{\boldsymbol{S}}_x(f+\tfrac{1}{2}\alpha)\boldsymbol{G}'_p(-f+\tfrac{1}{2}\alpha), & \alpha = p/T_0, \\ 0, & \alpha \neq p/T_0, \end{cases} \quad (106)$$

for all integers p.

The general input-output formulas (90), (93), (101), and (102) for periodic transformations are easily generalized for almost periodic transformations, that is, transformations exhibiting more than one incommensurate periodicity, in which case the sum over all harmonically related frequencies $\alpha = n/T_0$ in (85) is generalized to include all values of α for which the Fourier coefficient functions,

$$g_n(\tau) \triangleq \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \boldsymbol{h}(t+\tau, t)\, \mathrm{e}^{-\mathrm{i}2\pi\alpha_n t}\, \mathrm{d}t,$$

are not identically zero. The generalized versions of (90), (93), (101), and (102) simply sum over all corresponding values of frequencies, rather than just the values n/T_0 and m/T_0.

3.5. *Rice's representation*

It is well known that *any* time-series $x(t)$ can be expressed in the *quadrature amplitude modulation* (QAM) form

$$x(t) = c(t)\cos(2\pi f_0 t) - s(t)\sin(2\pi f_0 t), \tag{107}$$

for *any* value of f_0, provided that $c(t)$ and $s(t)$ are given by

$$c(t) = x(t)\cos(2\pi f_0 t) + \check{x}(t)\sin(2\pi f_0 t), \tag{108a}$$

$$s(t) = \check{x}(t)\cos(2\pi f_0 t) - x(t)\sin(2\pi f_0 t), \tag{108b}$$

for *any* auxiliary time-series $\check{x}(t)$. This is easily verified by substitution of (108) into (107) and use of a standard trigonometric identity. The QAM representation is particularly useful when $x(t)$ is a bandpass time-series, with spectrum concentrated near $f = f_0$, because then $c(t)$ and $s(t)$ can both be made low-pass time-series, with spectra concentrated around $f = 0$, by appropriate choice of $\check{x}(t)$. An especially appropriate choice of $\check{x}(t)$ is the Hilbert transform of $x(t)$,

$$\check{x}(t) = h(t) \otimes x(t), \tag{109a}$$

for which

$$h(t) = 1/(\pi t), \tag{109b}$$

$$H(f) = \begin{cases} -i, & f > 0, \\ +i, & f < 0. \end{cases} \tag{109c}$$

In this case, it can be shown that if $x(t)$ is bandlimited to $f \in (f_0 - B, f_0 + B)$ (and the image band $f \in (-f_0 - B, -f_0 + B)$), then $c(t)$ and $s(t)$ are bandlimited to $f \in (-B, B)$. Furthermore, given *any* time-series in the form (107), with $c(t)$ and $s(t)$ bandlimited to $f \in (-B, B)$ for $B < f_0$, it can be shown[15] that $c(t)$ and $s(t)$ are *uniquely* determined by $x(t)$ and are given by (108), with $\check{x}(t)$ defined by (109).

This QAM representation (107)-(109) is often called *Rice's representation*, in honor of Stephen O. Rice's pioneering work [47, 48, 49]. It is valid regardless of the statistical properties of $x(t)$. That is, $x(t)$ can be a finite-energy function, or it can be a finite-power time-series that is purely stationary, purely cyclostationary, or almost cyclostationary. However, the statistical properties of $x(t)$, $c(t)$ and $s(t)$ have evidently been studied only within the probabilistic framework of stationary stochastic processes (cf. [13, 44]), which masks statistical properties associated with second-order periodicity, as explained subsequently.

A complete study of the second-order statistical properties, including the limit cyclic correlations and limit cyclic spectra, for $x(t)$ and its *in-phase* and *quadrature* components $c(t)$ and $s(t)$, as well as their conventional limit correlations and spectra, can be based on one general formula for QAM time-series. Specifically, let us consider a time-series, say $y(t)$, in the QAM form

$$y(t) = z(t)\cos(2\pi f_0 t) + w(t)\sin(2\pi f_0 t). \tag{110}$$

This is a particular LPTV transformation of the two-dimensional vector of time-series $[z(t), w(t)]'$, for which the vector of impulse-response functions is

$$\boldsymbol{h}(t, u) = [\cos(2\pi f_0 t)\delta(t-u), \sin(2\pi f_0 t)\delta(t-u)] \tag{111}$$

[15] This can be verified by substitution of (107) into (108), and use of the fact that, for $B < f_0$, the Hilbert transforms of $c(t)\cos(2\pi f_0 t)$ and $s(t)\sin(2\pi f_0 t)$ are $c(t)\sin(2\pi f_0 t)$ and $-s(t)\cos(2\pi f_0 t)$, respectively.

and the vector of corresponding system functions is

$$G(t,f)=[\cos(2\pi f_0 t), \sin(2\pi f_0 t)]. \tag{112}$$

Application of formula (90) yields[16]

$$\begin{aligned}\hat{R}_y^\alpha(\tau) &= \tfrac{1}{2}[\hat{R}_z^\alpha(\tau)+\hat{R}_w^\alpha(\tau)]\cos(2\pi f_0\tau) \\ &\quad +\tfrac{1}{2}[\hat{R}_{wz}^\alpha(\tau)-\hat{R}_{zw}^\alpha(\tau)]\sin(2\pi f_0\tau) \\ &\quad +\frac{1}{4}\sum_{n=-1,1}[\hat{R}_z^{\alpha+2nf_0}(\tau)-\hat{R}_w^{\alpha+2nf_0}(\tau)] \\ &\qquad +\mathrm{i}n[\hat{R}_{wz}^{\alpha+2nf_0}(\tau)+\hat{R}_{zw}^{\alpha+2nf_0}(\tau)],\end{aligned} \tag{113}$$

and application of formula (93) yields

$$\begin{aligned}\hat{S}_y^\alpha(f) &= \frac{1}{4}\sum_{n=-1,1}[\hat{S}_w^\alpha(f+nf_0)+\hat{S}_z^\alpha(f+nf_0)] \\ &\quad +n\mathrm{i}[\hat{S}_{wz}^\alpha(f+nf_0)-\hat{S}_{zw}^\alpha(f+nf_0)] \\ &\quad +\frac{1}{4}\sum_{n=-1,1}[\hat{S}_z^{\alpha+2nf_0}(f)-\hat{S}_w^{\alpha+2nf_0}(f)] \\ &\qquad +n\mathrm{i}[\hat{S}_{wz}^{\alpha+2nf_0}(f)+\hat{S}_{zw}^{\alpha+2nf_0}(f)].\end{aligned} \tag{114}$$

From formula (113) and its Fourier transform (114), we can determine all cyclic correlations and cyclic spectra[17] for $x(t)$, $c(t)$, and $s(t)$, since each of the three representations, (107), (108a), and (108b), is of the form (110). For example, with the use of $y=x$, $z=c$, and $w=-s$, and selection of $\alpha=0$, (113) yields

$$\begin{aligned}\hat{R}_x(\tau) &= \tfrac{1}{2}[\hat{R}_c(\tau)+\hat{R}_s(\tau)]\cos(2\pi f_0\tau) \\ &\quad +\tfrac{1}{2}[\hat{R}_{cs}(\tau)-\hat{R}_{sc}(\tau)]\sin(2\pi f_0\tau) \\ &\quad +\frac{1}{4}\sum_{n=-1,1}[\hat{R}_c^{2nf_0}(\tau)-\hat{R}_s^{2nf_0}(\tau)] \\ &\qquad -\mathrm{i}n[\hat{R}_{cs}^{2nf_0}(\tau)+\hat{R}_{sc}^{2nf_0}(\tau)].\end{aligned} \tag{115}$$

This result reveals that the conventional formula (e.g., [44]), which omits the terms in the sum over $n=-1,1$, is correct only if $c(t)$ and $s(t)$ contain no second-order periodicity with frequency $\alpha=\pm 2f_0$ (e.g., if $c(t)$ and $s(t)$ are bandlimited to $f\in(-f_0,f_0)$), or the second-order periodicity is *balanced* in the sense that

$$\hat{R}_c^{\pm 2f_0}(\tau)\equiv\hat{R}_s^{\pm 2f_0}(\tau), \tag{116a}$$

$$\hat{R}_{cs}^{\pm 2f_0}(\tau)\equiv-\hat{R}_{cs}^{\pm 2f_0}(-\tau). \tag{116b}$$

As another example, with the use of $y=x$, $z=c$, and $w=-s$, and assuming that $c(t)$ and $s(t)$ are jointly purely stationary, (113) yields

$$\begin{aligned}\hat{R}_x^{\pm 2f_0}(\tau) &= \tfrac{1}{4}[\hat{R}_c(\tau)-\hat{R}_s(\tau)] \\ &\quad \mp\tfrac{1}{4}\mathrm{i}[\hat{R}_{cs}(\tau)+\hat{R}_{sc}(\tau)],\end{aligned} \tag{117a}$$

and also

$$\hat{R}_x^\alpha(\tau)\equiv 0,\quad |\alpha|\neq 2f_0. \tag{117b}$$

This result (117) reveals that $x(t)$ is purely stationary if and only if the correlations of $c(t)$ and $s(t)$ are *balanced* in the sense that

$$\hat{R}_c(\tau)\equiv\hat{R}_s(\tau), \tag{118a}$$

$$\hat{R}_{cs}(\tau)\equiv-\hat{R}_{cs}(-\tau). \tag{118b}$$

Otherwise, $x(t)$ is purely cyclostationary with period $1/(2f_0)$. Similarly, it can be shown through use of $y=c$, $z=x$, and $w=\bar{x}$, and also $y=s$, $z=\bar{x}$, and $w=-x$ in (114) that if $x(t)$ is purely cyclostationary with period $1/(2f_0)$, then $c(t)$ and $s(t)$ are purely stationary if and only if the cyclic spectrum of $x(t)$ is bandlimited,

$$\hat{S}_x^{\pm 2f_0}(f)=0,\quad |f|\geqslant f_0. \tag{119}$$

This necessary and sufficient condition is satisfied if and only if either $x(t)$ is bandlimited such that

$$\hat{S}_x(f)=0,\quad |f|\geqslant 2f_0, \tag{120}$$

or $c(t)$ and $s(t)$ are *balanced out of band* in the sense that

$$\hat{S}_c(f)=\hat{S}_s(f),\qquad |f|>f_0, \tag{121a}$$

$$\hat{S}_{sc}(f)=-\hat{S}_{sc}(-f),\quad |f|>f_0. \tag{121b}$$

[16] The result (113) was obtained by Brown [5], who generalized our formula (90) from the scalar case to the vector case, and then applied the result to generalize our formula (113) from the case $\alpha=0$, $\pm 2f_0$ to the case of arbitrary α.

[17] It is mentioned in passing that by letting $z(t)$ be identically zero, (113) and (114) yield formulas for the limit cyclic autocorrelation and limit cyclic spectrum for an amplitude-modulated time-series (e.g., for frequency conversion).

Moreover, it can be shown that the second-order periodicity of $x(t)$ at cycle frequency α depends on the second-order periodicity of $c(t)$ and $s(t)$ at *only* the cycle frequency α if and only if the cyclic correlations are balanced in the sense that

$$\hat{R}_c^{\alpha\pm 2f_0}(\tau) \equiv \hat{R}_s^{\alpha\pm 2f_0}(\tau), \tag{122a}$$

$$\hat{R}_{cs}^{\alpha\pm 2f_0}(\tau) \equiv -\hat{R}_{cs}^{\alpha\pm 2f_0}(-\tau). \tag{122b}$$

Otherwise, there is dependence on the second-order periodicity of $c(t)$ and $s(t)$ at the cycle frequencies $\alpha \pm 2f_0$, as well as at α.

The only relations needed, in addition to formula (114), to completely determine the cyclic spectra of $c(t)$ and $s(t)$ in terms of the cyclic spectra of $x(t)$ are the following cyclic spectra for Hilbert transforms (which follow from (96), (104), and (109c)):

$$\hat{S}_{\bar{x}}^{\alpha}(f) = \begin{cases} -\hat{S}_x^{\alpha}(f), & |f| < \frac{1}{2}|\alpha|, \\ +\hat{S}_x^{\alpha}(f), & |f| > \frac{1}{2}|\alpha|, \end{cases} \tag{123a}$$

$$\hat{S}_{x\bar{x}}^{\alpha}(f) = \hat{S}_{\bar{x}x}^{\alpha}(-f) = \begin{cases} -\mathrm{i}\hat{S}_x^{\alpha}(f), & f < \frac{1}{2}\alpha, \\ +\mathrm{i}\hat{S}_x^{\alpha}(f), & f > \frac{1}{2}\alpha. \end{cases} \tag{123b}$$

An alternative to the approach based on the general QAM formula (114) for determining explicit formulas for the cyclic spectra of x in terms of c and s, and vice versa, is based on the complex envelope

$$\gamma(t) \triangleq [x(t) + \mathrm{i}\bar{x}(t)]\,\mathrm{e}^{-\mathrm{i}2\pi f_0 t} \tag{124a}$$

$$= c(t) + \mathrm{i}s(t). \tag{124b}$$

This equation is easily solved to obtain

$$x(t) = \tfrac{1}{2}\gamma(t)\,\mathrm{e}^{\mathrm{i}2\pi f_0 t} + \tfrac{1}{2}\gamma^*(t)\,\mathrm{e}^{-\mathrm{i}2\pi f_0 t}, \tag{125a}$$

$$c(t) = \tfrac{1}{2}\gamma(t) + \tfrac{1}{2}\gamma^*(t), \tag{125b}$$

$$s(t) = \frac{1}{2\mathrm{i}}\gamma(t) - \frac{1}{2\mathrm{i}}\gamma^*(t). \tag{125c}$$

Now, it can be shown that

$$\hat{S}_x^{\alpha}(f) = \tfrac{1}{4}[\hat{S}_\gamma^{\alpha}(f-f_0) + \hat{S}_{\gamma^*}^{\alpha}(f+f_0) + \hat{S}_{\gamma\gamma^*}^{\alpha-2f_0}(f) + \hat{S}_{\gamma^*\gamma}^{\alpha+2f_0}(f)], \tag{126a}$$

$$\hat{S}_\gamma^{\alpha}(f) = \hat{S}_x^{\alpha}(f+f_0)\,U(f+f_0-\tfrac{1}{2}|\alpha|), \tag{126b}$$

$$\hat{S}_{\gamma\gamma^*}^{\alpha}(f) = \hat{S}_x^{\alpha+2f_0}(f)\,U(\tfrac{1}{2}\alpha + f_0 - |f|), \tag{126c}$$

$$\hat{S}_c^{\alpha}(f) = \tfrac{1}{4}[\hat{S}_\gamma^{\alpha}(f) + \hat{S}_{\gamma^*}^{\alpha}(f) + \hat{S}_{\gamma\gamma^*}^{\alpha}(f) + \hat{S}_{\gamma^*\gamma}^{\alpha}(f)], \tag{126d}$$

$$\hat{S}_s^{\alpha}(f) = \tfrac{1}{4}[\hat{S}_\gamma^{\alpha}(f) + \hat{S}_{\gamma^*}^{\alpha}(f) - \hat{S}_{\gamma\gamma^*}^{\alpha}(f) - \hat{S}_{\gamma^*\gamma}^{\alpha}(f)], \tag{126e}$$

$$\hat{S}_{cs}^{\alpha}(f) = \tfrac{1}{4}\mathrm{i}[\hat{S}_\gamma^{\alpha}(f) - \hat{S}_{\gamma^*}^{\alpha}(f) - \hat{S}_{\gamma\gamma^*}^{\alpha}(f) + \hat{S}_{\gamma^*\gamma}^{\alpha}(f)]. \tag{126f}$$

Substitution of (126b)-(126c) into (126d)-(126f) yields

$$\begin{aligned}\hat{S}_c^{\alpha}(f) = {} & \hat{S}_x^{\alpha}(f+f_0)\,U(f+f_0-\tfrac{1}{2}|\alpha|) \\ & + \hat{S}_x^{\alpha}(f-f_0)\,U(-f+f_0-\tfrac{1}{2}|\alpha|) \\ & + \hat{S}_x^{\alpha+2f_0}(f)\,U(\tfrac{1}{2}\alpha+f_0-|f|) \\ & + \hat{S}_x^{\alpha-2f_0}(f)\,U(-\tfrac{1}{2}\alpha+f_0-|f|),\end{aligned} \tag{127a}$$

$$\begin{aligned}\hat{S}_s^{\alpha}(f) = {} & \hat{S}_x^{\alpha}(f+f_0)\,U(f+f_0-\tfrac{1}{2}|\alpha|) \\ & + \hat{S}_x^{\alpha}(f-f_0)\,U(-f+f_0-\tfrac{1}{2}|\alpha|) \\ & - \hat{S}_x^{\alpha+2f_0}(f)\,U(\tfrac{1}{2}\alpha+f_0-|f|) \\ & - \hat{S}_x^{\alpha-2f_0}(f)\,U(-\tfrac{1}{2}\alpha+f_0-|f|),\end{aligned} \tag{127b}$$

$$\begin{aligned}\hat{S}_{cs}^{\alpha}(f) = \mathrm{i}[& \hat{S}_x^{\alpha}(f+f_0)\,U(f+f_0-\tfrac{1}{2}|\alpha|) \\ & - \hat{S}_x^{\alpha}(f-f_0)\,U(-f+f_0-\tfrac{1}{2}|\alpha|) \\ & - \hat{S}_x^{\alpha+2f_0}(f)\,U(\tfrac{1}{2}\alpha+f_0-|f|) \\ & + \hat{S}_x^{\alpha-2f_0}(f)\,U(-\tfrac{1}{2}\alpha+f_0-|f|)].\end{aligned} \tag{127c}$$

It should be noted that it follows from (126b) and (126c) that the supports in the (f, α) plane of the four terms in (126a) are disjoint, as shown in Fig. 3. For convenient reference, formulas (126) and (127) will be referred to as the *spectral correlation relations for Rice's representation.*

3.6. Sampling and aliasing

Consider the discrete-time-series $\{x(nT_0)\colon n = 0, \pm 1, \pm 2, \pm 3, \ldots\}$ that is obtained by periodically sampling a continuous-time-series $x(t)$, and let us determine the relationship between the cylic

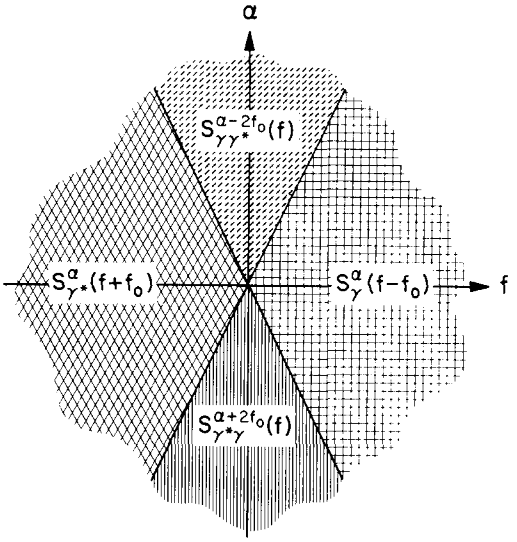

Fig. 3. Bi-frequency support for cyclic spectra of complex envelope.

spectra of $x(t)$ and $\{x(nT_0)\}$. Since the symmetric version of the definition of the limit cyclic autocorrelation (7) cannot be directly extended to discrete-time-series (because the data $\{x(\frac{1}{2}kT_0)\}$ does not exist for odd integers k), then the asymmetric version

$$\hat{R}_x^\alpha(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t+\tau)x(t) \times e^{-i2\pi\alpha(t+\tau/2)}\, dt \qquad (128)$$

is extended. Specifically, the *limit-cyclic autocorrelation* for a discrete-time-series $\{x(nT_0)\}$ is defined by

$$\tilde{R}_x^\alpha(kT_0) \triangleq \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{N} x(nT_0+kT_0)x(nT_0) \times e^{-i2\pi\alpha(n+k/2)T_0}. \qquad (129)$$

Motivated by the cyclic Wiener relation (43), the *limit cyclic spectrum* for $\{x(nT_0)\}$ is defined by

$$\tilde{S}_x^\alpha(f) \triangleq \sum_{k=-\infty}^{\infty} \tilde{R}_x^\alpha(kT_0)\, e^{-i2\pi kT_0 f}. \qquad (130)$$

In order to determine the relationship between the limit cyclic autocorrelations for $x(t)$ and $\{x(nT_0)\}$, the synchronized averaging identity (23) is applied to definition (129) to obtain

$$\tilde{R}_x^\alpha(kT_0) = \sum_{m=-\infty}^{\infty} \hat{R}_x^{\alpha+m/T_0}(kT_0)\, e^{i\pi mk}. \qquad (131)$$

Substitution of relation (131) into definition (130) yields the *spectral correlation aliasing formula*

$$\tilde{S}_x^\alpha(f) = \frac{1}{T_0}\sum_{n,m=-\infty}^{\infty} \hat{S}_x^{\alpha+m/T_0}(f - m/2T_0 - n/T_0). \qquad (132)$$

Thus, $\tilde{S}_x^\alpha(f)$ exhibits the periodicity properties

$$\tilde{S}_x^\alpha(f+1/T_0) = \tilde{S}_x^\alpha(f), \qquad (133a)$$

$$\tilde{S}_x^{\alpha+2/T_0}(f) = \tilde{S}_x^\alpha(f), \qquad (133b)$$

$$\tilde{S}_x^{\alpha+1/T_0}(f-1/2T_0) = \tilde{S}_x^\alpha(f), \qquad (133c)$$

in addition to the symmetry properties

$$\tilde{S}_x^{-\alpha}(f) = \tilde{S}_x^\alpha(f)^*, \qquad (133d)$$

$$\tilde{S}_x^\alpha(-f) = \tilde{S}_x^\alpha(f). \qquad (133e)$$

We see from (132) that the cyclic spectrum of $x(t)$ is in general not obtainable from the cyclic spectrum of $\{x(nT_0)\}$ due to *aliasing effects* in both α and f. However, if $x(t)$ is bandlimited in the temporal mean square sense to the *Nyquist bandwidth*,

$$\hat{S}_x(f) = 0, \quad |f| \geq B < 1/2T_0 \qquad (134)$$

then the bandwidth property (79) applied to (132) reveals that the support in the (f, α) plane of each of the terms in (132) indexed by n and m is disjoint from the support of all other terms, as shown in Fig. 4. Therefore, aliasing does not prevent recovery of $\hat{S}_x^\alpha$ from $\tilde{S}_x^\alpha$, that is,

$$\tilde{S}_x^\alpha(f) = \hat{S}_x^\alpha(f), \quad |f| < |B - \tfrac{1}{2}|\alpha||, \qquad (135a)$$

and

$$\hat{S}_x^\alpha(f) = 0, \quad |f| \geq |B - \tfrac{1}{2}|\alpha||. \qquad (135b)$$

On the other hand, if $x(t)$ is not bandlimited, even the conventional spectrum suffers from aliasing in

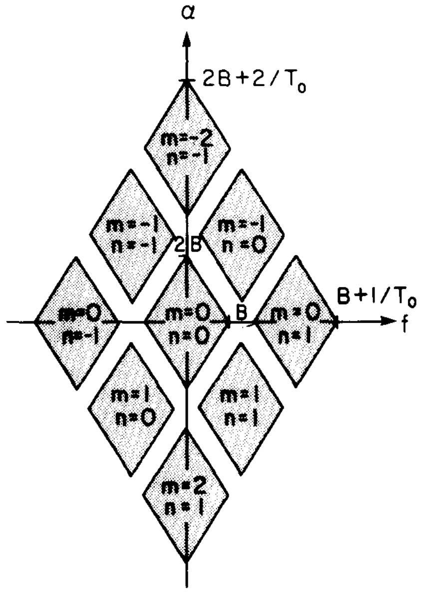

Fig. 4. Bi-frequency support for the first nine terms in the aliasing formula (132) for the cyclic spectrum of a time-sampled time-series for which $\hat{S}_x(f)=0$ when $|f|\geqslant B$ (vertical scale compressed).

α as well as f,

$$\tilde{S}_x(f)=\frac{1}{T_0}\sum_{n,m=-\infty}^{\infty}\hat{S}_x^{m/T_0}(f-m/T_0-n/T_0). \quad (136)$$

This is often unrecognized in probabilistic treatments. Only in the case for which $x(t)$ is purely stationary and therefore exhibits no spectral correlation do we obtain the known relationship

$$\tilde{S}_x(f)=\frac{1}{T_0}\sum_{n=-\infty}^{\infty}\hat{S}_x(f-n/T_0), \quad (137)$$

for all T_0.

3.7. Frequency conversion

A common signal processing operation is that of *frequency conversion* whereby the spectral content of a time-series is shifted from one band to another. This can be accomplished by multiplying the time-series $x(t)$ by a sinewave to obtain

$$y(t)=x(t)\cos(2\pi f_0t+\theta_0), \quad (138)$$

and then either low-pass filtering or band-pass filtering for *down-conversion* or *up-conversion*, respectively.

Equation (138) is an LPTV transformation with scalar impulse-response function

$$h(t,u)=\cos(2\pi f_0t+\theta_0)\delta(t-u) \quad (139)$$

and corresponding system function

$$G(t,f)=\cos(2\pi f_0t+\theta_0)$$
$$=G_1\,\mathrm{e}^{\mathrm{i}2\pi f_0t}+G_{-1}\,\mathrm{e}^{-\mathrm{i}2\pi f_0t}, \quad (140a)$$

where

$$G_1=G_{-1}^*=\tfrac{1}{2}\,\mathrm{e}^{\mathrm{i}\theta_0}. \quad (140b)$$

Application of formula (93) yields the *spectral correlation frequency-conversion formula*,

$$\hat{S}_y(f)=\tfrac{1}{4}[\hat{S}_x^{\alpha}(f+f_0)+\hat{S}_x^{\alpha}(f-f_0)$$
$$+\hat{S}_x^{\alpha+2f_0}(f)\,\mathrm{e}^{-\mathrm{i}2\theta_0}+\hat{S}_x^{\alpha-2f_0}(f)\,\mathrm{e}^{\mathrm{i}2\theta_0}], \quad (141)$$

which reveals that frequency conversion translates the spectrum frequency f by $\pm f_0$, and translates the cycle frequency α in separate terms by $\pm 2f_0$. Even the conventional spectrum contains cycle-frequency translates,

$$\hat{S}_y(f)=\tfrac{1}{4}[\hat{S}_x(f+f_0)+\hat{S}_x(f-f_0)$$
$$+\hat{S}_x^{2f_0}(f)\,\mathrm{e}^{-\mathrm{i}2\theta_0}+\hat{S}_x^{-2f_0}(f)\,\mathrm{e}^{\mathrm{i}2\theta_0}]. \quad (142)$$

This is often unrecognized in probabilistic treatments. Only in the case for which $x(t)$ is purely stationary and therefore exhibits no spectral correlation do we obtain the known formula

$$\hat{S}_y(f)=\tfrac{1}{4}[\hat{S}_x(f+f_0)+\hat{S}_x(f-f_0)], \quad (143)$$

for all f_0.

3.8. Product modulation

Both periodic time-sampling and frequency conversion are special cases of the more general product modulation operation,

$$y(t)=w(t)x(t), \quad (144)$$

where $x(t)$ and $w(t)$ are arbitrary time-series. For example, $w(t)$ can be a periodic train of narrow pulses, a sine wave, any other periodic or almost periodic function, or a random time-series like

$x(t)$. If $w(t)$ and $x(t)$ are statistically independent (in the sense that their joint fraction-of-time probability distributions factor into the product of the two individual marginal distributions, cf. [29]), then the spectral correlation characteristics of $y(t)$ can be expressed explicitly in terms of the spectral correlation characteristics of $w(t)$ and $x(t)$. It can be shown [5, 29] that every periodic or almost periodic time-series is statistically independent of every other time-series. Given this statistical independence, it can be shown that the *limit almost periodic autocorrelation* (53) of the product (144) is given by the product of limit almost periodic autocorrelations,

$$\hat{R}_y(t, \tau) = \hat{R}_w(t, \tau)\hat{R}_x(t, \tau). \tag{145}$$

Substitution of the Fourier series representations (53) for both $\hat{R}_w(t, \tau)$ and $\hat{R}_x(t, \tau)$ into (145), and use of the formula

$$\hat{R}_y^\alpha(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \hat{R}_y(t, \tau)\,\mathrm{e}^{-\mathrm{i}2\pi\alpha t}\,\mathrm{d}t, \tag{146}$$

yields the discrete convolution

$$\hat{R}_y^\alpha(\tau) = \sum_\beta \hat{R}_x^{\alpha-\beta}(\tau)\hat{R}_w^\beta(\tau). \tag{147}$$

Fourier transformation of (147) yields the double (discrete and continuous) convolution

$$\hat{S}_y^\alpha(f) = \sum_\beta \int_{-\infty}^{\infty} \hat{S}_x^{\alpha-\beta}(f-v)\hat{S}_w^\beta(v)\,\mathrm{d}v, \tag{148}$$

which is the desired result.

As an example, if $w(t)$ is almost periodic,

$$w(t) = \sum_\nu W_\nu\,\mathrm{e}^{\mathrm{i}2\pi\nu t},$$

then

$$S_w^\beta(f) = \sum_\nu W_\nu W_{\beta-\nu}^*\delta(f-\nu+\tfrac{1}{2}\beta), \tag{149}$$

and therefore (148) yields

$$\hat{S}_y^\alpha(f) = \sum_{\nu,\beta} W_\nu W_{\beta-\nu}^*\hat{S}_x^{\alpha-\beta}(f-\nu+\tfrac{1}{2}\beta). \tag{150}$$

As another example, if $x(t)$ is purely stationary, then (148) yields

$$\hat{S}_y^\alpha(f) = \int_{-\infty}^{\infty} \hat{S}_x(f-v)\hat{S}_w^\alpha(v)\,\mathrm{d}v. \tag{151}$$

The *spectral-correlation convolution relation* (148) together with the spectral-correlation input-output relations for time-invariant and periodically time-variant filtering, (96) and (93), provide means for quickly and easily obtaining spectral correlation formulas for many signal processing operations and many modulation types (cf. [24]). As an example, the class of phase-shift-keyed modulation types is briefly considered in the following subsection.

3.9. *Phase-shift-keyed signals*

A particularly important application of the concept of spectral correlation is to the study of modulated signals. As an example, we briefly consider phase-shift keyed signals, and simply present results, that are derived elsewhere [24], for illustrative purposes. Specifically, to show some of the ways in which the spectral correlation function $\hat{S}_x^\alpha(f)$ can be used to characterize different modulation types, the magnitude $|\hat{S}_x^\alpha(f)|$ is graphed as the height of a surface above the bi-frequency (f, α) plane for four different types of phase-shift-keying, namely, BPSK, QPSK, SQPSK, and MSK. It can be seen from these graphs shown in Fig. 5 that the first three types of PSK have identical power spectral density functions ($\alpha = 0$), but have highly distinct spectral correlation functions. In particular, for BPSK, $\hat{S}_x^\alpha(f) \not\equiv 0$ for only $\alpha = pf_d$ and $\alpha = \pm 2f_0 + pf_d$ for all integers p, where f_0 is the carrier frequency and f_d is the data rate; for QPSK, $\hat{S}_x^\alpha(f) \not\equiv 0$ for only $\alpha = pf_d$ for all integers p; for SQPSK, $\hat{S}_x^\alpha(f) \not\equiv 0$ for only $\alpha = pf_d$ for p even and $\alpha = \pm 2f_0 + pf_d$ for p odd, and similarly for MSK.

3.10. *Noise in periodic circuits*

A number of modern signal processing systems employ periodically switched and modulated linear circuits. An important problem in the use of all electrical circuits for signal processing is the analysis and control of noise at the output of the circuit due to thermal noise from the circuit elements. Typically each internal noise source is independent of all other internal noise sources in

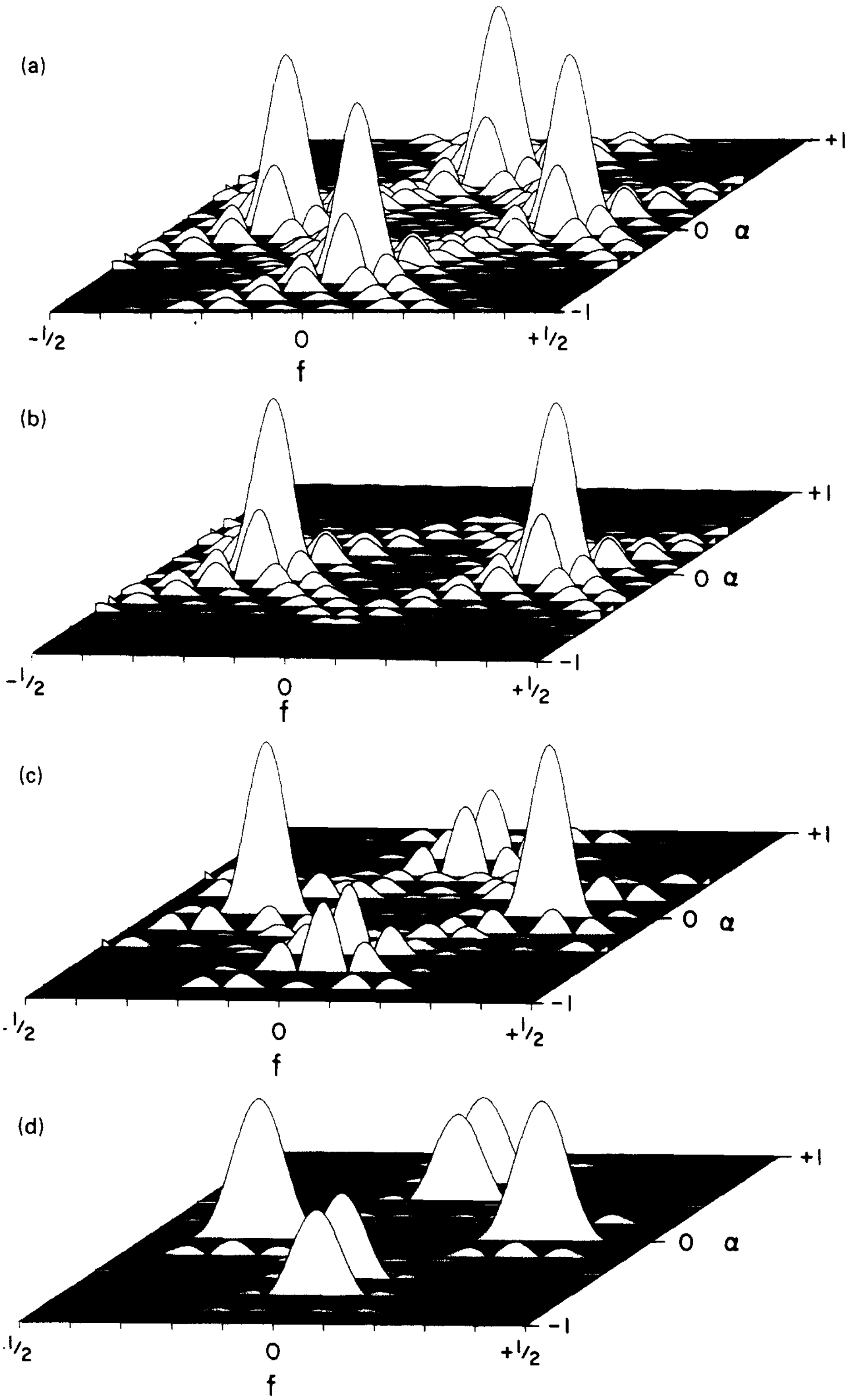

Fig. 5. Cyclic spectrum magnitudes for PSK signals. (a) BPSK. (b) QPSK. (c) SQPSK. (d) MSK.

the circuit. Consequently, the linearity of the circuit results in the spectral density of the total output noise being the sum of spectral densities of the individual output-noise components corresponding to each internal noise source. In order to determine the output-noise spectral density due to a particular internal noise source, we need only the system function that specifies the input–output relation for the particular noise source. If the noise source is stationary, then the output-noise spectral density is, from (99) (generalized for almost periodic transformations), given by

$$\hat{S}_y(f) = \sum_{\alpha} |G_\alpha(f)|^2 \hat{S}_x(f-\alpha), \tag{152}$$

and if the noise source produces white noise (e.g., thermal noise), then (152) reduces to

$$\hat{S}_y(f) = N_0 \sum_{\alpha} |G_\alpha(f)|^2. \tag{153}$$

The output-noise variance in this case is given by the integral of (153),

$$\operatorname{var}\{y(t)\} = N_0 \sum_{\alpha} \int_{-\infty}^{\infty} |G_\alpha(f)|^2 \, \mathrm{d}f \tag{154a}$$

$$= N_0 \sum_{\alpha} \int_{-\infty}^{\infty} |g_\alpha(\tau)|^2 \, \mathrm{d}\tau. \tag{154b}$$

Alternative equivalent formulas for output-noise variance are given by (using (85) and (88))

$$\operatorname{var}\{y(t)\} = N_0 \int_{-\infty}^{\infty} \langle |G(t,f)|^2 \rangle \, \mathrm{d}f \tag{155a}$$

$$= N_0 \int_{-\infty}^{\infty} \langle |h(t+\tau, t)|^2 \rangle \, \mathrm{d}\tau, \tag{155b}$$

where $\langle \cdot \rangle$ denotes average over all t for an almost periodic circuit, or simply average over one period for a periodic circuit.

4. Conclusions

By definition, a phenomenon or the time-series it produces is said to exhibit *second-order periodicity* if and only if there exists some quadratic time-invariant transformation of the time-series that gives rise to finite additive periodic components (spectral lines). An introduction to a comprehensive theory of statistical spectral analysis of time-series from phenomena that exhibit second-order periodicity, that does not rely on probabilistic concepts, has been presented. It has been shown that second-order periodicity in the phenomenon is characterized by spectral correlation in the time-series, and that the *degree of coherence* of such a time-series is properly characterized by a spectral correlation coefficient, the *spectral autocoherence function.* A fundamental relationship between superposed epoch analysis (synchronized averaging) of lag products, and spectral correlation, which is based on the *cyclic autocorrelation* and its Fourier transform, the *cyclic spectral density,* has been revealed through a synchronized averaging identity. Relationships to the radar ambiguity function and the Wigner–Ville time-frequency distribution have also been explained. It has been shown that the theory extends and generalizes the concepts associated with such time-frequency representations from finite-energy time-series to finite-power time-series. This complements Wiener's extension and generalization of frequency representations (the Fourier transform and Fourier series) in terms of the theory of spectral density. It has been shown that the deterministic theory can be given a probabilistic interpretation in terms of periodically time-variant fraction-of-time distributions obtained from synchronized time averages. This extends and generalizes Wold's isomorphism from stationary processes to processes that are cyclostationary or almost cyclostationary. Several fundamental properties of the cyclic spectrum, which is a spectral correlation function, including the effects of time sampling, frequency conversion, periodically time-variant linear filtering, and product modulation, and including the spectral correlation properties of Rice's representation for band-pass time-series, have been derived.

It should be emphasized that essentially all the fundamental results of the theory of cyclic spectral analysis presented in this paper are generalizations of results from the conventional theory of spectral analysis, in the sense that the latter are included as the special case of the former for which the cycle frequency α is zero (or the period T_0 is infinite) or the time-series is purely stationary. For example, the *cyclic periodogram/correlogram relation*, the equivalence between time-averaged and spectrally-smoothed cyclic spectra, the *cyclic Wiener relation*, the *periodic Wiener relation*, the input-output *spectral correlation relations for linear periodically time-variant transformations*, the *spectral correlation relations for Rice's representation*, the *spectral correlation aliasing formula* for time-sampling, the *spectral correlation frequency conversion formula*, and the *spectral-correlation convolution relation* for product modulation are all generalizations of results from the conventional theory of spectral analysis, and reduce to the conventional results for $\alpha = 0$ (or $T_0 = \infty$) or purely stationary time-series.

The theory has obvious and immediate applications to the characterization of modulated signals [25, 30], noise analysis for periodically time-variant linear systems [50], and synchronization problems [23]. It also has potentially important applications to problems of identification of periodically time-variant systems [21, 29], and to problems of detection, classification, parameter estimation, and extraction of modulated signals buried in noise and further masked by interference [22, 29]. Many other applications in signal processing are also being pursued [1, 2, 7, 8, 14-19, 21, 22, 29, 31, 32, 34, 36, 37, 39-43, 45, 46, 50].

Further development of the fundamental theory of cyclostationary time-series can be found in [5] and [29].

Acknowledgment

The author gratefully acknowledges the interest of Professor Herschel H. Loomis Jr., and the resultant partial support of this work through the Naval Postgraduate School, Monterey, California.

References

[1] M.H. Ackroyd, "Stationary and cyclostationary finite buffer behaviour computation via Levinson's Method", *AT&T Bell Labs. Tech. J.*, Vol. 63, December 1984, pp. 2159-2170.

[2] J.P.A. Albuquerque, O. Shimbo and L.N. Ngugen, "Modulation transfer noise effects from a continuous digital carrier to FDM/FM carriers in memoryless nonlinear devices", *IEEE Trans. Comm.*, Vol. COM-32, April 1984, pp. 337-353.

[3] V. Barnett, *Comparative Statistical Inference*, Wiley, New York/London, 1973.

[4] R.A. Boyles and W.A. Gardner, "Cycloergodic properties of discrete-parameter non-stationary stochastic processes", *IEEE Trans. Inform. Theory*, Vol. IT-29, 1983, pp. 105-114.

[5] W.A. Brown, "On the theory of cyclostationary signals", Ph.D. Dissertation, Dept. of Electrical and Computer Engineering, Univ. of California at Davis, 1986.

[6] C.H.D. Buys-Ballot, *Les Changements Périodiques de Température*, Utrecht, 1847.

[7] J.C. Campbell, A.J. Gibbs and B.M. Smith, "The cyclostationary nature of crosstalk interference from digital signals in multipair cable—Part I: Fundamentals", *IEEE Trans. Comm.*, Vol. COM-31, No. 5, May 1983, pp. 629-637.

[8] G.L. Cariolaro, G.L. Pierobon and G.P. Tronca, "Analysis of codes and spectra calculations", *Internat. J. Electronics*, Vol. 55, No. 1, 1983, pp. 35-79.

[9] S. Chapman and J. Bartels, *Geomagnetism, Vol. II, Analysis of the Data and Physical Theories*, Oxford University Press (2nd ed., 1951), 1940.

[10] T.A.C.M. Claasen and W.F.G. Mecklenbräuker, "The Wigner distribution—a tool for time-frequency signal analysis, Parts I-III", *Philips J. Res.*, Vol. 35, 1980, pp. 217-250, 276-300, 372-389.

[11] C. Corduneanu, *Almost Periodic Functions*, Wiley, New York, 1961.

[12] H. D'Angelo, *Linear Time-Varying Systems: Analysis and Synthesis*, Allyn & Bacon, Boston, MA, 1970.

[13] W.B. Davenport and W.L. Root, *An Introduction to Random Signals and Noise*, McGraw-Hill, New York, 1958.

[14] T.H.E. Ericson, "Modulation by means of linear periodic filtering", *IEEE Trans. Inform. Theory*, Vol. IT-27, No. 3, May 1981, pp. 322-327.

[15] E.R. Ferrara, Jr., "Frequency domain implementations of periodically time-varying filters", *IEEE Trans Acoustics, Speech, Signal Process.*, Vol. ASSP-33, 1985, pp. 883-892.

[16] E.R. Ferrara, Jr., and B. Widrow, "The time-sequenced adaptive filter", *IEEE Trans. Acoustics, Speech, Signal Process.*, Vol. ASSP-29, No. 3, June 1981, pp. 679-683.

[17] L.E. Franks, "Carrier and bit synchronization in data communication—a tutorial review", *IEEE Trans. Comm.*, Vol. COM-18, 1980, pp. 1107-1121.

[18] L.E. Franks and J. Bubrouski, "Statistical properties of timing jitter in a PAM timing recovery scheme", *IEEE Trans. Comm.*, Vol. COM-22, July 1974, pp. 913-930.

[19] C.A. French and W.A. Gardner, "Spread spectrum despreading without the code," *IEEE Trans. Comm.*, Vol. COM-34, 1986, pp. 404-407.

[20] W.A. Gardner, "Stationarizable random processes", *IEEE Trans. Inform. Theory*, Vol. IT-24, 1978, pp. 8-22.

[21] W.A. Gardner, "Optimization and adaptation of linear periodically time-variant digital systems", Signal and Image Processing Lab., Tech. Rept. No. SIPL-85-9, Dept. of Electrical and Computer Engineering, Univ. of California at Davis, 1985 (see [29]).

[22] W.A. Gardner, "Detection of spread-spectrum signals: A unifying view", Signal and Image Processing Lab., Tech. Rept. No. SIPL-85-6, Dept. of Electrical and Computer Engineering, Univ. of California at Davis, 1985 (see [29]).

[23] W.A. Gardner, "The role of spectral correlation in design and performance analysis of synchronizers", *IEEE Trans. Comm.*, Vol. COM-34, 1986, to appear.

[24] W.A. Gardner, *Introduction to Random Processes with Applications to Signals and Systems*, Macmillan, New York, 1986.

[25] W.A. Gardner, "Spectral correlation of modulated signals, Part I: Analog modulation", *IEEE Trans. Comm.*, Vol. COM-34, 1986, to appear.

[26] W.A. Gardner, "Rice's representation for cyclostationary processes", *IEEE Trans. Comm.*, Vol. COM-34, 1986, to appear.

[27] W.A. Gardner, "Common pitfalls in the application of stationary process theory to time-sampled and modulated signals", *IEEE Trans. Comm.*, Vol. COM-34, 1986, to appear.

[28] W.A. Gardner, "Measurement of spectral correlation", *IEEE Trans. Acoustics, Speech, Signal Process.*, Vol. ASSP-34, 1986, to appear.

[29] W.A. Gardner, *Statistical Spectral Analysis: A Nonprobabilistic Theory*, Prentice-Hall, Englewood Cliffs, NJ, 1986.

[30] W.A. Gardner, W.A. Brown and C.-K. Chen, "Spectral correlation of modulated signals, Part II: Digital modulation", *IEEE Trans. Comm.*, Vol. COM-34, 1986, to appear.

[31] W.A. Gardner and L.E. Franks, "Characterization of cyclostationary random signal processes", *IEEE Trans. Inform. Theory*, Vol. IT-21, 1975, pp. 4-14.

[32] K. Hasselmann and T.P. Barnett, "Techniques of linear prediction for systems with periodic statistics", *J. Atmospheric Sci.*, Vol. 38, 1981, pp. 2275-2283.

[33] G.M. Jenkins and D.G. Watts, *Spectral Analysis and its Applications*, Holden-Day, San Francisco, 1968.

[34] W.K. Johnson, "The dynamic pneumocardiogram: An application of coherent signal processing to cardiovascular measurement", *IEEE Trans. Biomedical Engrg.*, Vol. BME-28, 1981, pp. 471-475.

[35] J. Kampé de Fériet, "Introduction to the statistical theory of turbulence, I and II", *J. Soc. Indust. Appl. Math.*, Vol. 2, Nos. 1, 3, 1954, pp. 1-9 and 143-174.

[36] U. Mengali and E. Pezzani, "Tracking properties of phase-locked loops in optical communication systems", *IEEE Trans. Comm.*, Vol. COM-26, No. 12, December 1978, pp. 1811-1818.

[37] M.F. Mesiya, P.J. McLane and L.L. Campbell, "Optimal receiver filters for BPSK transmission over a bandlimited nonlinear channel", *IEEE Trans. Comm.*, Vol. COM-26, No. 1, January 1978, pp. 12-22.

[38] A.G. Miamee and H. Salihi, "On the prediction of periodically correlated stochastic processes", pp. 167-179, in: P.R. Krishnaiah, ed., *Multivariate Analysis Vol. V*, North-Holland, Amsterdam, 1980.

[39] M. Moenclaey, "Comment on "Tracking performance of the filter and square bit synchronizer" ", *IEEE Trans. Comm.*, Vol. COM-30, No. 2, February 1982, pp. 407-410.

[40] M. Moenclaey, "Linear phase-locked loop theory for cyclostationary input disturbances", *IEEE Trans. Comm.*, Vol. COM-30, No. 10, October 1982, pp. 2253-2259.

[41] M. Moenclaey, "The optimum closed-loop transfer function of a phase-locked loop used for synchronization purposes", *IEEE Trans. Comm.*, Vol. COM-31, No. 4, April 1983, pp. 549-553.

[42] M. Moenclaey, "A fundamental lower bound on the performance of practical joint carrier and bit synchronizers", *IEEE Trans. Comm.*, Vol. COM-32, No. 9, September 1984, pp. 1007-1012.

[43] J.J. O'Reilly, "Timing extraction for baseband digital transmission", in: K.W. Cattermole and J.J. O'Reilly, eds., *Problems of Randomness in Communication Engineering*, Plymouth, London, 1984.

[44] A. Papoulis, *Probability, Random Variables, and Stochastic Processes*, 2nd ed., McGraw-Hill, New York, 1984.

[45] L. Pelkowitz, "Frequency domain wraparound error in fast convolution algorithms", *IEEE Trans. Acoustics*, Speech, Signal Process., Vol. ASSP-29, 1981, pp. 413-422.

[46] S. Pupolin and C. Tomasi, "Spectral analysis of line regenerator time jitter", *IEEE Trans. Comm.*, Vol. COM-32, May 1984, pp. 561-566.

[47] S.O. Rice, "Mathematical analysis of random noise, Parts I and II", *Bell System Tech. J.*, Vol. 23, 1944, pp. 282-332.

[48] S.O. Rice, "Mathematical analysis of random noise, Parts III and IV", *Bell System Tech. J.*, Vol. 24, 1945, pp. 46-156.

[49] S.O. Rice, "Statistical properties of a sine wave plus random noise", *Bell System Tech. J.*, Vol. 27, 1948, pp. 109-157.

[50] T. Strom and S. Signell, "Analysis of periodically switched linear circuits", *IEEE Trans. Circuits and Systems*, Vol. CAS-24, No. 10, October 1977, pp. 531-541.

[51] A.W. Rihaczek, *Principles of High-Resolution Radar*, McGraw-Hill, New York, 1969.

[52] J. Ville, "Theory and application of the notion of the complex signal", *Câbles et Transmission*, Vol. 2, pp. 67-74, 1948; translated into English by I. Selin, Rand Corp. Rept. T-92, 1958.

[53] N. Wiener, "Generalized harmonic analysis", *Acta Mathematica*, Vol. 55, 1930, pp. 117-258.

[54] E. Wigner, "On the quantum correction for thermodynamic equilibrium", *Phys. Rev.*, Vol. 40, 1932, pp. 749-759.

[55] H.O.A. Wold, "On prediction in stationary time-series", *Ann. Math. Statist.*, Vol. 19, 1948, pp. 558-567.

[56] P.M. Woodward, *Probability and Information Theory, With Applications to Radar*, Pergamon Press, New York, 1953.