Computing the Geometric Category of Finite Topological Spaces

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Topological Complexity was first described by Michael Farber in [Far03]. Informally, the Topological Complexity (TC) of a path-connected space X is the minimal number of continuous motion planning rules required to navigate between any two points in X. Typically, X represents the space of configurations of some mechanical system, such as a robot. More formally, consider the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "c_x"] \arrow[dr, "\Delta"'] & X^{[0,1]} \arrow[d, "\pi"] \\ & X \times X \end{tikzcd}$$

Here, X is our path-connected space of configurations. The map c_x sends a point $x \in X$ to the constant path at x in $X^{[0,1]}$. The diagonal map Δ sends x to $(x, x) \in X \times X$, and the projection map π is the fibrant replacement of Δ. It takes a path $\gamma \in X^{[0,1]}$, and sends it to its start and end points $(\gamma(0), \gamma(1)) \in X \times X$.

The **Schwarz genus** $\mathfrak{g}(p)$ of a fibration $p : E \to B$ is the minimal number k such that there exists an open covering $U_1, \ldots, U_k$ of B where each set U_i admits a local p-section (see [Š58]). That is, each U_i has an associated map s_i such that $p \circ s_i \simeq 1_B$. The **topological complexity** of X is given by $\mathfrak{g}(\pi)$, and it is homotopy-invariant.

Unlike other topological invariants such as homology, there is no algorithm for computing the TC of a space. There are a few spaces whose TC has been manually determined. For the most part, TC is calculated by determining lower- and upper-bounds. The upper-bound of interest to us is the Lusternik-Schnirelmann category of [LS34]. The **Lusternik-Schnirelmann category** of a space X, denoted $LS(X)$, is the minimal number k such that X can be covered in k open sets $U_i \subseteq X$ whose inclusion maps $\iota_i : U_i \hookrightarrow X$ are nullhomotopic. Farber proves in [Far03] that $TC(X) \leq LS(X \times X)$. An obvious upper-bound for the Lusternik-Schnirelmann category is the minimal number of contractible open sets covering a space; in [FTMVMV18], they describe this as the **geometric category** of a space, denoted $\mathrm{gcat}(X)$.

Realistically, a programmer might not be interested in designing a robot that can be in uncountably many positions. This motivates the application of topological complexity to finite topological spaces. When finite spaces are T_0, they yield a partial order, $\leq$. Given two points x and y in a finite space X, we say that $x \leq y$ if the smallest open neighborhood containing x is a subset of the smallest open neighborhood containing y. This partial order allows us to represent finite spaces as Hasse diagrams, which are directed acyclic graphs. A point in a finite space is called **maximal** if its in-degree in the Hasse diagram is 0.

An adaptation of TC for finite topological spaces was described in [Tan18], called Combinatorial Complexity (CC). In that same paper, the author proved that for a finite topological space X, $CC(X) = TC(X)$, and so all the traditional tools of TC can be applied. Tanaka showed that if m is the number of maximal elements in a finite space, we have the inequality $TC(X) \leq m^2$. Combining this bound the the others mentioned above, we have $TC(X) \leq LS(X \times X) \leq gcat(X \times X) \leq m^2$ for any finite path-connected space X. To date, there are no known finite spaces X for which $TC(X) < LS(X \times X)$. In particular, these categorical open sets are either contractible open sets, or disjoint unions of contractible sets. Because of the combinatorial nature of finite topological spaces, we are writing a Python class to detect contractible open subsets of finite spaces, in hopes of establishing a tighter upper-bound on the TC of finite spaces besides m^2.

References

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