Intro to Modern Algebra: Homework #2

Due on November 2 at 9:30am

Professor Lorenz 9:30-11:00

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Problem 6

Which of the following subsets of $\mathbb{R}[x]$ are subrings of $\mathbb{R}[x]$.

a All polynomials with constant term of 0_R

b All polynomials of degree 2.

Solution

a) This is a subring. We will show that it is a nonempty subset closed under subtraction and multiplication. First, zero is a polynomial with the constant term 0_R, so it is a nonempty subset. Next, consider polynomials $a, b \epsilon \mathbb{R}[x]$. Then if we subtract them and 0_R, we get $(a + 0_R) - (b + 0_R) = (a - b) + (0_R - 0_R) = a - b$, and since $a, b \epsilon R$, it is closed under subtraction. Similarly for multiplication, we get $(a + 0_R) * (b + 0_R) = (ab) + (0_R * b) + (0_R * a) + 0_R = ab$, so it is similarly closed under multiplication. Therefore, it is a subring of $\mathbb{R}[x]$.

b) This is not a subring because it is not closed under multiplication. Consider the polynomials x^2 and $x^2 + 1$.When we multiply them, $x^2 * (x^2 + 1) = x^4 + x^2$, which is not a degree 2 polynomial, and therefore it is not closed under multiplication and cannot be a ring. Furthermore, 0_R is not of degree 2 and therefore is not in the subset.

Problem 20

Let $D : \mathbb{R}[x] \to \mathbb{R}[x]$ be the derivative map defined by

$$D(a_0 + a_1x + a_2x^2 + ... + a_nx^n) = a_1 + 2a_x + 3a_3x^2 + ... + na_nx^{n-1}$$

Is D a homormorphism of rings? An isomorphism?

Solution

This is neither a homomorphism of rings nor an isomorphism. We will prove this by showing D does not preserve multiplication. Consider $x^2, x^3 \epsilon \mathbb{R}[x]$. When we multiply them, $x^2 * x^3 = x^5$. Now, we will apply D and multiply them. $D(x^2) * D(x^3) = 2x * 3x^2 = 6x^3 \neq x^5$. Therefore, D does not preserve multiplication and cannot be a homomorphic or isomorphic.

Problem 13

Prove Theorem 4.10.
Theorem 4.10
Let $\mathbb{F}$ be a field and $a(x), b(x), c(x) \epsilon \mathbb{F}[x]$. If $a(x)|b(x)c(x)$ and $a(x)$ and $b(x)$ are relatively prime, then $a(x)|c(x)$.

Solution
Assume $\mathbb{F}$ be a field and $a(x), b(x), c(x) \epsilon \mathbb{F}[x]$, $a(x)|b(x)c(x)$, and $a(x)$ and $b(x)$ are relatively prime. Then gcd($a(x), b(x)$)=1, by the definition of relatively prime. But then, by Theorem 4.8, this means that for polynomials $u(x), v(x) \epsilon F[x]$,

$$1 = a(x)v(x) + b(x)u(x).$$

Problem 13 continued on next page. . .

We can multiply this equation by $c(x)$ to get that

$$c(x)a(x)v(x) + c(x)b(x)u(x) = c(x)$$

Since $a(x)|b(x)c(x)$, there exists some $z(x)\epsilon(F)[x]$ such that $a(x)z(x) = b(x)c(x)$. We can substitute this into the above equation to get

$$c(x)a(x)v(x) + a(x)z(x)u(x) = c(x)$$

and we can then factor out $a(x)$ to get

$$a(x)[c(x)v(x) + z(x)u(x)] = c(x)$$

Since $c(x), v(x), z(x), u(x)\epsilon\mathbb{F}[x]$, $a(x)|c(x)$.

Problem 12

Express $x^4 - 4$ as a product of irreducibles in $\mathbb{Q}[x], \mathbb{R}[x]$, and in $\mathbb{C}[x]$.

Solution
First, we will factor this into a product of irreducibles in $\mathbb{C}[x]$. In $\mathbb{C}[x]$,

$$x^4 - 4 = (x - i\sqrt{2})(x + i\sqrt{2})(x - 2)(x + 2)$$

These factors are all irreducible because they are all degree 1, both complex numbers and irrational numbers are in $\mathbb{C}$
In $\mathbb{R}[x]$,

$$x^4 - 4 = (x^2 + 2)(x + \sqrt{2})(x - \sqrt{2})$$

In this case, $(x+\sqrt{2})$ and $(x-\sqrt{2})$ are factors because they are degree 1 with coefficients in the real numbers. $(x + \sqrt{2})$ is only irreducible in $\mathbb{C}[x]$ because it has complex coefficients.
In $\mathbb{Q}[x]$,

$$x^4 - 4 = (x^2 + 2)(x^2 - 2)$$

The first factor is irreducible because if it were not irreducible, it would be reducible in $\mathbb{R}[x]$. The second factor is irreducible because it only factors into polynomials with irrational coefficients, which, by definition, are not in the $\mathbb{Q}$

Problem 4

a For what value of k is $x - 2$ a factor of $x^4 - 5x^3 + 5x^2 + 3x + k$ in $\mathbb{Q}[x]$

b For what value of k is $x + 1$ a factor of $x^4 + 2x^3 - 3x^2 + kx + 1$ in $\mathbb{Z}_5[x]$

Solution
a) For $k = -2$ is $(x - 2)$ a factor of $x^4 - 5x^3 + 5x^2 + 3x + k$. This is simply to solve for because $(x - 2)$ is in the form of a root, so we can evaluate our polynomial at $x = 2$ and solve for k.

$$2^4 - 5x^3 + 5x^2 + 3x - k = 0 \Rightarrow 16 - 40 + 20 + 6 = -k \Rightarrow -2 = k$$

b) For $k = 3$ is $x + 1$ a factor of $x^4 + 2x^3 - 3x^2 + kx + 1$ in $\mathbb{Z}_5[x]$. Like in part a, since $(x + 1)$ is in the form of a root, we can evaluate our polynomial at $x = -1$ and solve for k.

$$(-1)^4 + 2(-1)^3 - 3(-1)^2 + k(-1) + 1 = 0 \Rightarrow 1 - 2 - 3 - k + 1 = 0 \Rightarrow 2 = k + 5 \Rightarrow k = 3$$

Problem 5

Show that $x - 1_F$ divides $a_n x^n + ... + a_2 x^2 + a_1 x + a_0$ in $\mathbb{F}[x]$ if and only if $a_0 + ... + a_n = 0_F$.
Solution

$\Rightarrow$ Assume that $x - 1_F$ divides $a_n x^n + ... + a_2 x^2 + a_1 x + a_0$ in $\mathbb{F}[x]$. Then 1_F is a root of $a_n x^n + ... + a_2 x^2 + a_1 x + a_0$, so this means that $a_n 1^n + ... + a_2 1^2 + a_1 + a_0 = 0$. Since 1 to any power is 1, this simplifies to $a_n + ... + a_2 + a_1 + a_0 = 0$

$\Leftarrow$ This proof is simply the above proof in reverse. Assume that $a_0 + ... + a_n = 0_F$ in $\mathbb{F}[x]$. This can be rewritten as $a_0 1 + ... + a_n 1 = 0_F$, or $a_0 1^0 + ... + a_n 1^n = 0_F$. This is the form of a polynomial root evaluated at 1, so we can rewrite this as $a_n x^n + ... + a_2 x^2 + a_1 x + a_0$

Problem 14

a

Problem 18

Problem 19