Intro to Modern Algebra: Homework #2

Due on November 2 at 9:30am

Professor Lorenz 9:30-11:00

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Problem 6

Which of the following subsets of $\mathbb{R}[x]$ are subrings of $\mathbb{R}[x]$.

a All polynomials with constant term of 0_R

b All polynomials of degree 2.

Solution

a) This is a subring. We will show that it is a nonempty subset closed under subtraction and multiplication. First, zero is a polynomial with the constant term 0_R, so it is a nonempty subset. Next, consider polynomials $a, b \epsilon \mathbb{R}[x]$. Then if we subtract them and 0_R, we get $(a + 0_R) - (b + 0_R) = (a - b) + (0_R - 0_R) = a - b$, and since $a, b \epsilon R$, it is closed under subtraction. Similarly for multiplication, we get $(a + 0_R) * (b + 0_R) = (ab) + (0_R * b) + (0_R * a) + 0_R = ab$, so it is similarly closed under multiplication. Therefore, it is a subring of $\mathbb{R}[x]$.

b) This is not a subring because it is not closed under multiplication. Consider the polynomials x^2 and $x^2 + 1$.When we multiply them, $x^2 * (x^2 + 1) = x^4 + x^2$, which is not a degree 2 polynomial, and therefore it is not closed under multiplication and cannot be a ring. Furthermore, 0_R is not of degree 2 and therefore is not in the subset.

Problem 20

Let $D : \mathbb{R}[x] \to \mathbb{R}[x]$ be the derivative map defined by

$$D(a_0 + a_1x + a_2x^2 + ... + a_nx^n) = a_1 + 2a_x + 3a_3x^2 + ... + na_nx^{n-1}$$

Is D a homormorphism of rings? An isomorphism?

Solution

This is neither a homomorphism of rings nor an isomorphism. We will prove this by showing D does not preserve multiplication. Consider $x^2, x^3 \epsilon \mathbb{R}[x]$. When we multiply them, $x^2 * x^3 = x^5$. Now, we will apply D and multiply them. $D(x^2) * D(x^3) = 2x * 3x^2 = 6x^3 \neq x^5$. Therefore, D does not preserve multiplication and cannot be a homomorphic or isomorphic.

Problem 13

Prove Theorem 4.10.
Theorem 4.10
Let $\mathbb{F}$ be a field and $a(x), b(x), c(x) \epsilon \mathbb{F}[x]$. If $a(x)|b(x)c(x)$ and $a(x)$ and $b(x)$ are relatively prime, then $a(x)|c(x)$.

Solution

Assume $\mathbb{F}$ be a field and $a(x), b(x), c(x) \epsilon \mathbb{F}[x]$, $a(x)|b(x)c(x)$, and $a(x)$ and $b(x)$ are relatively prime. Then gcd($a(x), b(x)$)=1, by the definition of relatively prime. But then, by Theorem 4.8, this means that for polynomials $u(x), v(x) \epsilon F[x]$,

$$1 = a(x)v(x) + b(x)u(x).$$

Problem 13 continued on next page. . .

We can multiply this equation by $c(x)$ to get that

$$c(x)a(x)v(x) + c(x)b(x)u(x) = c(x)$$

Since $a(x)|b(x)c(x)$, there exists some $z(x)\epsilon(F)[x]$ such that $a(x)z(x) = b(x)c(x)$. We can substitute this into the above equation to get

$$c(x)a(x)v(x) + a(x)z(x)u(x) = c(x)$$

and we can then factor out $a(x)$ to get

$$a(x)[c(x)v(x) + z(x)u(x)] = c(x)$$

Since $c(x), v(x), z(x), u(x)\epsilon\mathbb{F}[x]$, $a(x)|c(x)$.

Problem 12

Express $x^4 - 4$ as a product of irreducibles in $\mathbb{Q}[x], \mathbb{R}[x]$, and in $\mathbb{C}[x]$.

Solution

First, we will factor this into a product of irreducibles in $\mathbb{C}[x]$. In $\mathbb{C}[x]$,

$$x^4 - 4 = (x - i\sqrt{2})(x + i\sqrt{2})(x - 2)(x + 2)$$

These factors are all irreducible because they are all degree 1, both complex numbers and irrational numbers are in $\mathbb{C}$

In $\mathbb{R}[x]$,

$$x^4 - 4 = (x^2 + 2)(x + \sqrt{2})(x - \sqrt{2})$$

In this case, $(x + \sqrt{2})$ and $(x - \sqrt{2})$ are factors because they are degree 1 with coefficients in the real numbers. $(x + \sqrt{2})$ is only irreducible in $\mathbb{C}[x]$ because it has complex coefficients.

In $\mathbb{Q}[x]$,

$$x^4 - 4 = (x^2 + 2)(x^2 - 2)$$

The first factor is irreducible because if it were not irreducible, it would be reducible in $\mathbb{R}[x]$. The second factor is irreducible because it only factors into polynomials with irrational coefficients, which, by definition, are not in the $\mathbb{Q}$