

# Intro to Modern Algebra

## Homework 3

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### 1 Section 3.3 #8

**Problem** Let  $\mathbb{Q}(\sqrt{2})$  be as in Exercise 39 of Section 3.1. Prove that the function  $f : \mathbb{Q}(\sqrt{2}) \rightarrow \mathbb{Q}(\sqrt{2})$  given by  $f(a + b\sqrt{2}) = a - b\sqrt{2}$  is an isomorphism.

**Solution** In order for  $f$  to be an isomorphism, it must be a bijective homomorphism.

First, to be bijective, it must be both injective and surjective. Take  $f(a), f(b) \in \mathbb{Q}\sqrt{2}$  such that  $f(a) = f(b)$ . Then, for some  $k, l, m, n \in \mathbb{Q}$ ,  $f(k + l\sqrt{2}) = f(m + n\sqrt{2})$ . This implies that  $k - l\sqrt{2} = m - n\sqrt{2}$ , by the definition of  $f$ . We can group the rational and irrational parts together to get  $k - l\sqrt{2} = m - n\sqrt{2} \Rightarrow (k - m) = (l - n)\sqrt{2}$ . But  $(k - m)$  is rational, and  $(l - n)\sqrt{2}$  is irrational. The only way this could be possible is if  $(k - m) = 0$  and  $(l - n) = 0$ , since  $(0 - 0) = (0 - 0)\sqrt{2}$ . This implies that  $k = m$  and  $l = n$ . Therefore,  $k + l\sqrt{2} = m + n\sqrt{2}$  and  $f$  is an injective function.

Now, consider  $c - d\sqrt{2} \in \mathbb{Q}(\sqrt{2})$ .  $c - d\sqrt{2}$  is mapped to from  $c + d\sqrt{2}$ , since  $(c + d\sqrt{2}) = c - d\sqrt{2}$ . Therefore,  $f$  is surjective, and as a result of that, bijective.

Now, we must prove that  $f$  is a homomorphism. Consider  $f((k + l\sqrt{2}) * (m + n\sqrt{2}))$ . This simplifies to  $f(km + kn\sqrt{2} + ml\sqrt{2} + nl)$ , which again simplifies to  $f((km + nl) + (kn + ml)\sqrt{2})$ . By applying the function  $f$ , we get  $f((km + nl) + (kn + ml)\sqrt{2}) = (km + nl) - (kn + ml)\sqrt{2}$ . Now consider  $f(k + l\sqrt{2}) * f(m + n\sqrt{2})$ . By applying  $f$ , we get  $(k - l\sqrt{2}) * (m - n\sqrt{2})$ . We can use distributive laws to get  $km - kn\sqrt{2} - ml\sqrt{2} + 2nl = (km + 2nl) - (kn + ml)\sqrt{2} = f((k + l\sqrt{2}) * (m + n\sqrt{2}))$ . Therefore,  $f$  preserves multiplication.

Now consider  $f((k + l\sqrt{2}) + (m + n\sqrt{2}))$ . This simplifies to  $f((k + m) + (n + l)\sqrt{2})$ . Applying  $f$ ,  $f((k + m) + (n + l)\sqrt{2}) = (k + m) - (n + l)\sqrt{2}$ . Next consider  $f(k + l\sqrt{2}) + f(m + n\sqrt{2})$ . Applying  $f$ ,  $f(k + l\sqrt{2}) + f(m + n\sqrt{2}) = (k - l\sqrt{2}) + (m - n\sqrt{2}) = (k + m) - (n + l)\sqrt{2}$ , by associativity. But  $(k + m) - (n + l)\sqrt{2} = f((k + l\sqrt{2}) + (m + n\sqrt{2}))$  and therefore  $f$  preserves addition. Since  $f$  preserves addition and multiplication, it is a homomorphism, and since it is a bijective homomorphism, it is an isomorphism.

### 2 Section 3.3# 12e

**Problem** Is the following function a homeomorphism or not?

$$f : \mathbb{Z}_{12} \rightarrow \mathbb{Z}_4 \tag{1}$$

defined by  $f([x]_{12}) = [x]_4$ , where  $[u]_4$  denotes the class of the integer  $u$  in  $\mathbb{Z}_n$

**Solution** First, we need to prove that this function is well defined. To do this, consider  $[a]_{12}$  and  $[a + 12]_{12}$  such that  $a, a + 12 \in \mathbb{Z}_{12}$ . It is clear that  $[a]_{12} = [a + 12]_{12}$ , since these are equivalence classes in  $\mathbb{Z}_{12}$ . Now consider  $f([a]_{12})$  and  $f([a + 12]_{12})$ . Then applying  $f$ ,  $f([a]_{12}) = [a]_4$  and  $f([a + 12]_{12}) = [a + 12]_4$ . But  $[a + 12]_4 = [a]_4 + [12]_4 = [a]_4 + [0]_4 = [a]_4$ , since it is an equivalence class. Then  $f([a]_{12}) = f([a + 12]_{12})$ , and  $f$  is well defined.

Now, for  $f$  to be a homomorphism, it must preserve addition and multiplication. Consider  $f([x]_{12} + [y]_{12})$ ,  $x, y \in \mathbb{Z}_{12}$ . Then  $f([x]_{12} + [y]_{12}) = f([x + y]_{12}) = [x + y]_4$ , by the definitions of equivalence classes and  $f$ . Now consider  $f([x]_{12}) + f([y]_{12}) = [x]_4 + [y]_4 = [x + y]_4 = f([x]_{12} + [y]_{12})$ , by the same properties of equivalence classes. Since  $f([x]_{12} + [y]_{12}) = f([x]_{12}) + f([y]_{12})$ ,  $f$  preserves addition.

To check if  $f$  preserves multiplication, consider  $f([x]_{12} * [y]_{12})$ . We can use rules of equivalence classes and apply the function  $f$  to get  $f([x]_{12} * [y]_{12}) = f([xy]_{12}) = [xy]_4$ . Now consider  $f([x]_{12}) * f([y]_{12})$ . By applying  $f$ , we get  $f([x]_{12}) * f([y]_{12}) = [x]_4 * [y]_4 = [xy]_4 = f([x]_{12} * [y]_{12})$ , and therefore  $f$  preserves multiplication.

Since  $f$  is well-defined and preserves multiplication and addition, it is homomorphism.

### 3 Section 3.3 #30

**Problem** Let  $f : R \rightarrow S$  be a homomorphism of rings and let  $K = \{r \in R \mid f(r) = 0_S\}$ . Prove that  $K$  is a subring of  $R$ .

**Solution** Since  $f$  is a homomorphism of rings,  $0_R \in R$ ,  $0_S \in S$ , and  $f(0_R) = 0_S$ . Therefore,  $0_R \in K$ , and  $K \neq \emptyset$ .

Now take  $a, b \in K$ . By the definition of  $K$ ,  $f(a) = 0_S$  and  $f(b) = 0_S$ . Subtracting one from the other,  $f(a) - f(b) = 0_S - 0_S = 0_S$ . Since  $f$  is a ring homomorphism, it preserves subtraction, and  $f(a) - f(b) = f(a - b) = 0_S$ . This implies that  $a - b \in K$ , and therefore  $K$  is closed under subtraction.

To prove that  $K$  is closed under multiplication, consider the same  $a, b$  from above. Then  $f(a) * f(b) = 0_S * 0_S = 0_S$ . Again, since  $f$  is a ring homomorphism, it preserves multiplication and  $f(a) * f(b) = f(a * b) = 0_S$ . Therefore,  $a * b \in K$  and  $K$  is closed under multiplication.

Since  $K$  is a nonempty subset closed under subtraction and multiplication, it is a subring of  $R$ .

### 4 Section 3.3 #38

**Problem** Let  $F$  be a field and  $f : F \rightarrow R$  a homomorphism of rings.

(a) If there is a nonzero element  $c$  of  $F$  such that  $f(c) = 0_R$ , prove that  $f$  is the zero homomorphism (that is,  $f(x) = 0_R$  for every  $x \in F$ ). [Hint:  $c^{-1}$  exists (Why?). If  $x \in F$ , consider  $F(xcc^{-1})$ .]

(b) Prove that  $f$  is either injective or the zero homomorphism. [Hint: If  $f$  is not the zero homomorphism and  $f(a) = f(b)$ , then  $f(a - b) = 0_R$ .]

**Solution**

### 5 Extra Problem

**Problem** Define an equivalence relation on the set of all rings by defining a ring  $R$  to be equivalent to a ring  $S$  if there is a ring isomorphism  $f : R \rightarrow S$ , i.e.  $R$  is isomorphic to  $S$ ,

$R \simeq S$ . Show that this is an equivalence relation by showing

- $R \simeq R$  for all rings  $R$
- If  $R \simeq S$  for rings  $R$  and  $S$ , then  $S \simeq R$
- If  $R \simeq S$  and  $S \simeq T$  for rings  $R$ ,  $S$ , and  $T$ , then  $R \simeq T$  (See problem #27)

**Solution**