Intro to Modern Algebra
Homework 3

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October 19th, 2017

1 Section 3.3 #8

Problem Let $\mathbb{Q}(\sqrt{2})$ be as in Exercise 39 of Section 3.1 Prove that the function $f : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$ given by $f(a + b\sqrt{2}) = a - b\sqrt{2}$ is an isomorphism.

Solution In order for f to be an isomorphism, it must be a bijective homomorphism.

First, to be bijective, it must be both injective and surjective. Take $f(a), f(b) \epsilon \mathbb{Q}\sqrt{2}$ such that $f(a) = f(b)$. Then, for some $k, l, m, n \epsilon \mathbb{Q}, f(k + l\sqrt{2}) = f(m+n\sqrt{2})$. This implies that $k - l\sqrt{2} = m - n\sqrt{2}$, by the definition of f.

Now, consider $c - d\sqrt{2}\epsilon\mathbb{Q}(\sqrt{2})$. $c - d\sqrt{2}$ is mapped to from $c + d\sqrt{2}$, since $(c + d\sqrt{2}) = c - d\sqrt{2}$. Therefore, f is surjective, and as a result of that, bijective.

Now, we must prove that f is a homomorphism. Consider $f((k + l\sqrt{2}) * (m + n\sqrt{2}))$. This simplifies to $f(km + kn\sqrt{2} + ml\sqrt{2} + nl)$, which again simplifies to $f((km + nl) + (kn + ml)\sqrt{2})$. By applying the function f, we get $f((km + nl) + (kn + ml)\sqrt{2}) = (km + nl) - (kn + ml)\sqrt{2}$. Now consider $f(k + l\sqrt{2}) * f(m + n\sqrt{2})$. By appliying f, we get $(k - l\sqrt{2}) * (m - n\sqrt{2})$. We can use distributive laws to get $km - kn\sqrt{2} - ml\sqrt{2} + 2nl = (km + 2nl) - (kn + ml)\sqrt{2} = f((k + l\sqrt{2}) * (m + l\sqrt{2}))$. Therefore, f preserves multiplication

Now consider $f((k + l\sqrt{2}) + (m + n\sqrt{2}))$ This simplifies to $f((k + m) + (n + l)\sqrt{2})$.Applying f, $f((k + m) + (n + l)\sqrt{2}) = (k + m) - (n + l)\sqrt{2}$. Next consider $f(k + l\sqrt{2}) + f(m + n\sqrt{2})$. Applying f, $f(k + l\sqrt{2}) + f(m + n\sqrt{2}) = (k - l\sqrt{2}) + (m - \sqrt{2}) = (k + m) - (n + l)\sqrt{2}$, by associativity. But $(k + m) - (n + l)\sqrt{2} = f((k + l\sqrt{2}) + (m + n\sqrt{2})$ and therefore f preserves addition. Since f preserves addition and multiplication, it is a homomorphism, and since it is a bijective homomorphism, it is an isomorphism.