Intro to Modern Algebra
Homework 3

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1 Section 3.3 #8

Problem Let $\mathbb{Q}(\sqrt{2})$ be as in Exercise 39 of Section 3.1 Prove that the function $f : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$ given by $f(a + b\sqrt{2}) = a - b\sqrt{2}$ is an isomorphism.

Solution In order for f to be an isomorphism, it must be a bijective homomorphism.

First, to be bijective, it must be both injective and surjective. Take $f(a), f(b)\epsilon\mathbb{Q}\sqrt{2}$ such that $f(a) = f(b)$. Then, for some $k, l, m, n\epsilon\mathbb{Q}, f(k + l\sqrt{2}) = f(m + n\sqrt{2})$. This implies that $k - l\sqrt{2} = m - n\sqrt{2}$, by the definition of f. We can group the rational and irrational parts together to get $k - l\sqrt{2} = m - n\sqrt{2} \Rightarrow (k - m) = (l - n)\sqrt{2}$. But $(k - m)$ is rational, and $(l - n)\sqrt{2}$ is irrational. The only way this could be possible is if $(k - m) = 0$ and $(l - n) = 0$, since $(0 - 0) = (0 - 0)\sqrt{2}$. This implies that $k = m$ and $l = n$. Therefore, $k + l\sqrt{2} = m + n\sqrt{2}$ and f is an injective function.

Now, consider $c - d\sqrt{2}\epsilon\mathbb{Q}(\sqrt{2})$. $c - d\sqrt{2}$ is mapped to from $c + d\sqrt{2}$, since $(c + d\sqrt{2}) = c - d\sqrt{2}$. Therefore, f is surjective, and as a result of that, bijective.

Now, we must prove that f is a homomorphism. Consider $f((k + l\sqrt{2}) * (m + n\sqrt{2}))$. This simplifies to $f(km + kn\sqrt{2} + ml\sqrt{2} + nl)$, which again simplifies to $f((km + nl) + (kn + ml)\sqrt{2})$. By applying the function f, we get $f((km + nl) + (kn + ml)\sqrt{2}) = (km + nl) - (kn + ml)\sqrt{2}$. Now consider $f(k + l\sqrt{2}) * f(m + n\sqrt{2})$. By appliying f, we get $(k - l\sqrt{2}) * (m - n\sqrt{2})$. We can use distributive laws to get $km - kn\sqrt{2} - ml\sqrt{2} + 2nl = (km + 2nl) - (kn + ml)\sqrt{2} = f((k + l\sqrt{2}) * (m + l\sqrt{2}))$. Therefore, f preserves multiplication

Now consider $f((k + l\sqrt{2}) + (m + n\sqrt{2}))$ This simplifies to $f((k + m) + (n + l)\sqrt{2})$.Applying f, $f((k + m) + (n + l)\sqrt{2}) = (k + m) - (n + l)\sqrt{2}$. Next consider $f(k + l\sqrt{2}) + f(m + n\sqrt{2})$. Applying f, $f(k + l\sqrt{2}) + f(m + n\sqrt{2}) = (k - l\sqrt{2}) + (m - \sqrt{2}) = (k + m) - (n + l)\sqrt{2}$, by associativity. But $(k + m) - (n + l)\sqrt{2} = f((k + l\sqrt{2}) + (m + n\sqrt{2})$ and therefore f preserves addition. Since f preserves addition and multiplication, it is a homomorphism, and since it is a bijective homomorphism, it is an isomorphism.

2 Section 3.3# 12e

Problem Is the following function a homeomorphism or not?

$$f : \mathbb{Z}_{12} \to \mathbb{Z}_4 \tag{1}$$

defined by $f([x]_{12}) = [x]_4$, where $[u]_4$ denotes the class of the integer u in $\mathbb{Z}_n$

Solution First, we need to prove that this function is well defined. To do this, consider $[a]_{12}$ and $[a+12]_{12}$ such that $a, a+12\ epsilon \mathbb{Z}_{12}$. It is clear that $[a]_{12} = [a+12]_{12}$, since these are equivalence classes in $\mathbb{Z}_{12}$. Now consider $f([a]_{12})$ and $f([a+12]_{12})$. Then applying f, $f([a]_{12}) = [a]_4$ and $f([a+12]_{12}) = [a+12]_4$. But $[a+12]_4 = [a]_4 + [12]_4 = [a]_4 + [0]_4 = [a]_4$, since it is an equivalence class. Then $f([a]_{12}) = f([a+12]_{12})$, and f is well defined.

Now, for f to be a homomorphism, it must preserve addition and multiplication. Consider $f([x]_{12} + [y]_{12})$, $x, y \epsilon \mathbb{Z}_{12}$. Then $f([x]_{12} + [y]_{12}) = f([x+y]_{12}) = [x+y]_4$, by the definitions of equivalence classes and f. Now consider $f([x]_{12}) + f([y]_{12}) = [x]_4 + [y]_4 = [x+y]_4 = f([x]_{12} + [y]_{12})$, by the same properties of equivalence classes. Since $f([x]_{12} + [y]_{12}) = f([x]_{12}) + f([y]_{12})$, f preserves addition.

To check if f preserves multiplication, consider $f([x]_{12} * [y]_{12})$. We can use rules of equivalence classes and apply the function f to get $f([x]_{12} * [y]_{12}) = f([xy]_{12}) = [xy]_4$. Now consider $f([x]_{12}) * f([y]_{12})$. By applying f, we get $f([x]_{12}) * f([y]_{12}) = [x]_4 * [y]_4 = [xy]_4 = f([x]_{12} * [y]_{12})$, and therefore f preserves multiplication.

Since f is well-defined and preserves multiplication and addition, it is homomorphism.

3 Section 3.3#30

Problem Let $f : R \to S$ be a homomorphism of rings and let $K = r \epsilon R\, f(r) = 0_r$. Prove that K is a subring of R.

Solution

4 Section 3.3 #38

Problem Let F be a field and $f :\to R$ a homomorphism of rings.

(a) If there is a nonzero element c of F such that $f(c) = 0_R$, prove that f is the zero homomorphism (that is, $f(x) = 0_R$ for every $x \epsilon F$). [Hint: c^{-1} exists (Why?). If $x \epsilon F$, consider $F(xcc^{-1})$.]

(b) Prove that f is either injective of the zero homomorphism. [Hint: If f is not the zero homomorphism and $f(a) = f(b)$, then $f(a-b) = 0_R$.]

Solution

5 Extra Problem

Problem Define an equivalence relation on the set of all rings by defining a ring R to be equivalent to a ring S if there is a ring isomorphism $f : R \to S$, i.e. R is isomorphic to S, $R \simeq S$. Show that this is an equivalence relation by showing

- $R \simeq R$ for all rings R
- If $R \simeq S$ for rings R and S, then $S \simeq R$
- If $R \simeq S$ and $S \simeq T$ for rings R, S, and T, then $R \simeq T$ (See problem #27)

Solution