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LETTER

Motion of photons in time dependent spherical gravitational fields

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Abstract - Schwarzschild's metric is extended to obtain a generalized metric for the gravitational field exterior to time varying spherical distributions of mass. The general relativistic equation of motion for a photon in the gravitational field of a time varying spherical distribution of mass is derived .The second-order differential equation obtained is a modification of the equation of motion in Schwarzschild's field. It introduces a unique dependence of the motion of a photon in this field on Newton's scalar potential exterior to time varying spherical bodies.

Keywords: relativity, photon, motion, Schwarzschild

1. Introduction

There are two kinds of particles in Physics: tardyons and photons (luxons). A tardyon is a particle whose speed never exceeds the speed at which electromagnetic radiation propagates through empty space. Photons (commonly known as light) move with the speed of light in vacuum [1].

The deflection of light (photon) by the Sun was the third prediction of General Relativity that provided the most famous and dramatic test of the theory. Although the effect itself was so small and had no practical implications, the observation of it seized hold of public imagination and cemented Einstein's reputation as a great physicist [2]. Schwarzschild in 1916 constructed the first exact solution of Einstein's gravitational field equations. His solution is the metric tensor due to a static spherically symmetric body situated in empty space such as the Sun or a star. It is well known that using Schwarzschild metric for a space time exterior to a static homogenous spherical body, the General Relativistic equation of motion for a photon [3, 4] is

$$\frac{d^2u}{d\phi^2}+u=\frac{3GM}{c^2}u^2 \tag{1}$$

where u is a radial function defined in terms of r in polar coordinates as

$$u(\phi) = \frac{1}{r(\phi)} \tag{2}$$

ϕ is the angular coordinate ,G the universal gravitational constant , M the mass of the static spherical body and c the speed of light in vacuum. Equation (1) has been solved by the method of successive approximation to obtain satisfactory results for the total deflection of the photon from its original straight line path.

It is also well known that the solar system is not static and isotropic as assumed by Schwarzschild. In Schwarzschild's gravitational field, the Sun is assumed to be a static spherically symmetric body [4]. The Sun varies in position and mass distribution. It is a magnetically active star, supporting a strong magnetic field that varies with time. A recent theory claims that there are magnetic instabilities in the core of the Sun which cause fluctuations with periods of either 41000 or 100000 years [5]. This article presents a standard generalization of Schwarzschild metric to derive the mathematically most simple and astrophysically most satisfactory equation of motion for a photon, moving in a time varying homogenous spherical distribution of mass.

2. Theoretical Analysis

Consider a static spherical mass of radius R and total rest mass M distributed uniformly with density ρ . Schwarzschild metric exterior to such a body is given in polar coordinates [6,7] as

$$g_{00}(t,r,\theta,\phi) = \left[1 + \frac{2}{c^2} f(r)\right] \tag{3}$$

$$g_{11}(t,r,\theta,\phi) = -\left[1 + \frac{2}{c^2} f(r)\right]^{-1} \tag{4}$$

$$g_{22}(t,r,\theta,\phi) = -r^2 \tag{5}$$

$$g_{33}(t,r,\theta,\phi) = -r^2 \sin^2\theta \tag{6}$$

$$g_{\mu\nu}(t,r,\theta,\phi) = 0; otherwise \tag{7}$$

where as in standard notation, the subscripts $\mu,\nu = 0,\ 1,\ 2$ and 3 represent the four space-time coordinates t,r,θ and ϕ respectively; *f(r)* is an arbitrary function determined by the distribution. It is a function of *r* only since the distribution of mass and hence its exterior field possesses spherical symmetry. From the condition that these metric components should reduce to the field of a point mass located at the origin and contain Newton's equations of motion in the gravitational field of the spherical body [4], it follows that *f(r)* is the Newtonian scalar potential in the exterior region of the body. Thus, *f(r)* is determined by the mass or pressure distribution and possesses all the symmetries of the massive body.
The invariant world line element in the exterior region of the static spherical body is thus given as

$$c^2 d\tau^2 = c^2\left[1+\frac{2}{c^2}f(r)\right]dt^2 - \left[1+\frac{2}{c^2}f(r)\right]^{-1}dr^2 - r^2 d\theta^2 - r^2 Sin^2\theta d\phi^2 \tag{8}$$

Now, let the mass distribution inside the sphere vary with time in such a way that its density and total mass remain the same. In other words, let the material inside the sphere experience spherically symmetric radial displacements. In this case, the arbitrary function will not remain same as f(r) but will be transformed to a time dependent function f(t,r) in this gravitational field; where t is the coordinate time. Since our function f(t,r) is solely dependent on the mass distribution, the time variation can be considered to be the effect produced by a radial displacement (explosion) of the matter inside the spherically symmetric body. Thus, if one considers the vacuum gravitational field produced by a spherically symmetric star then the field remains static even if the material in the star experiences a spherically symmetric radial displacement (explosion). The time dependence on the metric will thus describe only the displacement of matter inside the star and the field still remains static and thus adheres to Birkhoff's theorem [6].

Hence;

$$f(r) \rightarrow f(t,r) \tag{9}$$

Thus, the metric exterior to a time varying homogenous spherical body becomes

$$g_{00}(t,r,\theta,\phi) = \left[1+\frac{2}{c^2}f(t,r)\right] \tag{10}$$

$$g_{11}(t,r,\theta,\phi) = -\left[1+\frac{2}{c^2}f(t,r)\right]^{-1} \tag{11}$$

$$g_{22}(t,r,\theta,\phi) = -r^2 \tag{12}$$

$$g_{33}(t,r,\theta,\phi) = -r^2\sin^2\theta \tag{13}$$

$$g_{\mu\nu} = 0; otherwise \tag{14}$$

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According to General Relativity, light (photon) moves along a null geodesic [3, 6, 7]; that is

$$c^2 d\tau^2 = 0 \tag{16}$$

Thus, our line element, equation (15) reduces to equation (17) for a photon

$$0 = c^2\left[1+\frac{2}{c^2} f(t,r)\right]dt^2 - \left[1+\frac{2}{c^2} f(t,r)\right]^{-1} dr^2 - r^2 d\theta^2 - r^2 Sin^2\theta d\phi^2 \tag{17}$$

$\theta = \frac{\pi}{2}$ in the equatorial plane of any spherical mass. Thus, in the equatorial plane of the time varying spherical mass, equation (17) becomes;

$$0 = c^2\left[1+\frac{2}{c^2} f(t,r)\right]dt^2 - \left[1+\frac{2}{c^2} f(t,r)\right]^{-1} dr^2 - r^2 d\phi^2 \tag{18}$$

Now, let ϖ be a parameter which may be used to follow the motion of the photon in the gravitational field of the time varying spherical body. Dividing equation (18) by $d\varpi^2$ and denoting differentiation with respect to ϖ by dot$(\,.\,)$ yields;

$$0 = c^2\left[1+\frac{2}{c^2} f(t,r)\right]\ddot{t}^2 - \left[1+\frac{2}{c^2} f(t,r)\right]^{-1} \dot{r}^2 - r^2\dot{\phi}^2 \tag{19}$$

It is well known that $\dot{t}$ is defined in Schwarzschild field$[3,4]$ as

$$\dot{t} = a\left[1+\frac{2}{c^2} f(r)\right]^{-1} \tag{20}$$

where a is a constant of motion. Thus, in a time varying homogenous spherical distribution, it suffices that $\dot{t}$ takes the form

$$\dot{t} = a\left[1+\frac{2}{c^2} f(t,r)\right]^{-1} \tag{21}$$

Also, the azimuthal equation of motion for particles of non-zero rest mass in the equatorial plane of a spherical body is given as [8]

$$\dot{\phi} = \frac{b}{r^2} \tag{22}$$

where b is another constant of motion. Also, $\dot{r}$ can be written in terms of ϕ as follows

$$\dot{r} = \frac{dr}{d\varpi} \text{ , thus } \dot{r} = \frac{dr}{d\phi}\frac{d\phi}{d\omega} \text{ and hence } \dot{r} = \dot{\phi}\frac{dr}{d\phi} \tag{23}$$

Thus, substituting equations (21), (12), (23) in equation (19) and rearrangement gives;

$$\frac{b^2}{r^4}\left(\frac{dr}{d\phi}\right)^2 = a^2c^2 - \left[1+\frac{2}{c^2}f(t,r)\right]\frac{b^2}{r^2} \tag{24}$$

Now, let u be a radial function defined in terms of r in polar coordinates by equation (2) then

$$\frac{dr}{d\phi} = -u^{-2}\frac{du}{d\phi} \tag{25}$$

Substituting equation (25) into equation (24) and simplifying gives

$$\left(\frac{du}{d\phi}\right)^2 = \frac{c^2a^2}{b^2} - \left[1+\frac{2}{c^2}f(t,u)\right]u^2 \tag{26}$$

since f is a function of r and by (2) becomes a function of u . Differentiating both sides of equation (26) with respect to ϕ yields

$$\frac{d^2u}{d\phi^2} + \left[1+\frac{2}{c^2}f(t,u)\right]u = -\frac{3}{c^2}\frac{\partial f(t,u)}{\partial u}u^2 \tag{27}$$

Equation (27) is the general relativistic equation of motion for a photon in gravitational fields exterior to time varying spherical distributions of mass. It differs from equation (1) obtained from Schwarzschild's static spherical field. Thus, the non static nature of the Sun and planets has an effect on the motion of light.

3. Conclusions

The door is thus open for the derivation of a more physically realistic expression for the total deflection of a photon from its original straight line motion, moving in the vicinity of a time varying homogenous spherical distribution of mass. Our metric tensor in this gravitational field can equally be used to study the motion of particles of non-zero rest mass. Also, planetary equations of motion can be obtained with the Sun considered as a time varying homogenous massive spherical body.

Another outstanding consequence of our generalization of the arbitrary function f is that it can be used to investigate theoretically, the existence of gravitational wave radiation and propagation in spherical fields. This can be done by constructing Einstein's field equations for this time varying homogenous field.

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