Section_06

April 2, 2022

1 Joint Probability Distribution

1. **X and Y are Discrete Random Variables**
    - Joint pmf
        - $f(x,y) \to P(X = x_i, Y = y_j)$
        - $\sum_x \sum_y f(x,y) = 1$
    - Marginal pmf
        - $f(x) = \sum_y f(x,y)$
        - $f(y) = \sum_x f(x,y)$
    - Conditional pmf
        - $f(x|y) = \frac{f(x,y)}{f(y)}$
        - $f(y|x) = \frac{f(x,y)}{f(x)}$
    - Mean
        - $E[X] = \sum xf(x) = \sum xf(x,y)$
        - $E[Y] = \sum yf(y) = \sum yf(x,y)$
        - $E[XY] = \sum xyf(x,y)$
    - Variance
        - $V(x) = \sum x^2 f(x,y) - \mu_x^2$
        - $V(y) = \sum y^2 f(x,y) - \mu_y^2$
    - Independence $\to f(x,y) = f(x)f(y)$
2. **X and Y are Continuous Random Variables**
    - Joint pdf
        - $f(x,y)$
        - $\int_x \int_y f(x,y) = 1$
    - Marginal pdf
        - $f(x) = \int_y f(x,y)$
        - $f(y) = \int_x f(x,y)$
    - Conditional pmf
        - $f(x|y) = \frac{f(x,y)}{f(y)}$
        - $f(y|x) = \frac{f(x,y)}{f(x)}$
    - Mean
        - $E[X] = \int xf(x)dx = \int_x \int_y xf(x,y)dydx$
        - $E[Y] = \int yf(y)dy = \int_y \int_x yf(x,y)dxdy$
        - $E[XY] = \int_x \int_y xyf(x,y)dydx$
    - Variance

- $V(x) = \int_x x^2 f(x)dx - \mu_x^2$
- $V(y) = \int_y y^2 f(y)dy - \mu_y^2$

- Independence $\rightarrow f(x,y) = f(x)f(y)$

1.1 Examples

1.1.1 Example 1

Show that the following function satisfies the properties of a joint probability mass function

x	y	$f(x,y)$
1	1	$\frac{1}{4}$
1.5	2	$\frac{1}{8}$
1.5	3	$\frac{1}{4}$
2.5	4	$\frac{1}{4}$
3	5	$\frac{1}{8}$

Determine the following:

(a) $P(X < 2.5, Y < 3)$

(b) $P(X < 2.5)$

(c) $P(Y < 3)$

(d) $P(X > 1.8, Y > 4.7)$

(e) $E(X)$, $E(Y)$, $V(X)$, and $V(Y)$.

(f) Marginal probability distribution of X

(g) Conditional probability distribution of Y given that $X = 1.5$

(h) Conditional probability distribution of X given that $Y = 2$

(i) $E(Y|X = 1.5)$

(j) Are X and Y independent?

1.1.2 Example 2

A manufacturing company employs two devices to inspect output for quality control purposes. The first device is able to accurately detect 99.3% of the defective items it receives, whereas the second is able to do so in 99.7% of the cases. Assume that four defective items are produced and sent out for inspection. Let X and Y denote the number of items that will be identified as defective by inspecting devices 1 and 2, respectively. Assume that the devices are independent. Determine:

(a) $f_{X,Y}(x, y)$

(b) $f_x(X)$

(c) $E(X)$

(d) $f_{Y|2}(y)$

(e) $E(Y|X=2)$

(f) $V(Y|X=2)$

(g) Are X and Y independent?

1.1.3 Example 3

Determine the value of c such that the function

$$f(x, y) = cxy$$

for $0 < x < 3$ and $0 < y < 3$

satisfies the properties of a joint probability density function.

Determine the following:

(a) $P(X < 2, Y < 3)$

(b) $P(X < 2.5)$

(c) $P(1 < Y < 2.5)$

(d) $P(X > 1.8, 1 < Y < 2.5)$

(e) $E(X)$

(f) $P(X < 0, Y < 4)$

(g) Marginal probability distribution of X

(h) Conditional probability distribution of Y given $X = 1.5$

(i) $E(Y|X=1.5)$

(j) $P(Y < 2|X=1.5)$

(k) Conditional probability distribution of X given $Y = 2$

1.1.4 Example 4

Two methods of measuring surface smoothness are used to evaluate a paper product. The measurements are recorded as deviations from the nominal surface smoothness in coded units. The joint probability distribution of the two measurements is a uniform distribution over the region $0 < x < 4$, $0 < y$ and $x - 1 < y < x + 1$. That is

$$f_{XY}(x, y) = c$$

for x and y in the region.

Determine the value for c such that $f_{XY}(x, y)$ is a joint probability density function.

Determine the following:

(a) $P(X < 0.5, Y < 0.5)$

(b) $P(X < 0.5)$

(c) $E(X)$

(d) $E(Y)$

(e) Marginal probability distribution of X

(f) Conditional probability distribution of Y given $X = 1$

(g) $E(Y|X = 1)$

(h) $P(Y < 0.5|X = 1)$

2 Convariance

$$Cov(x, y) = E[XY] - E[X]E[Y] = E[XY] - \mu_x \mu_y$$

Correlation

$$\rho xy = \frac{Cov(x, y)}{\sqrt{V(x) + V(y)}} = \frac{Cov(x, y)}{\sigma_x \sigma_y}$$

2.1 Examples

2.1.1 Example 5

Determine the covariance and correlation for the following joint probability distribution

x	1	1	2	4
y	3	4	5	6
$f(x, y)$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{8}$

2.1.2 Example 6

Determine the value for c and the covariance and correlation for the joint probability mass function

$$f_{XY}(x, y) = c(x + y)$$

for $x = 1, 2, 3$

and $y = 1, 2, 3$

2.1.3 Example 7

Determine the covariance and correlation for the joint probability density function

$$f_{XY}(x, y) = e^{-x-y}$$

over the range

$0 < x$ and $0 < y$.