Section_05

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1 Continuous Random Variables

X has uncountable/infinite number of values.

1.1 Probability Density function (PDF)

$$f(x)$$

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

$$P(X = x) = 0$$

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

1.2 Cumulative Distribution function (CDF)

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx$$

Notes

$P(a \leq X \leq b) = P(a < X \leq b) = P(a < X < b) = F(b) - F(a)$

$P(X > a) = 1 - P(X \leq a) = 1 - F(a)$

1.3 Expected Value and Variance

$$\mu = E[x] = \int_{-\infty}^{\infty} xf(x)dx$$

$$V(x) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2$$

1.4 Exercises

1.4.1 Example 1

Suppose that

$$f(x) = \frac{x}{8}$$

Where

$$3 < x < 5$$

Determine the following:

(a) $P(X < 4)$

(b) $P(X > 3.5)$

(c) $P(4 < X < 5)$

(d) $P(X > 4.5)$

(e) $P(X < 3.5 \text{ or } x > 4.5)$

1.4.2 EXample 2

Suppose that

$$f(x) = e^{-x}$$

Where

$$x > 0$$

Determine the following:

(a) $P(X > 1)$

(b) $P(1 \leq X < 2.5)$

(c) $P(X = 3)$

(d) $P(X \geq 3)$

(e) Find the value of x where $P(X > x) = 0.1$

1.4.3 EXample 3

Suppose that the cumulative distribution function of the random variable X is

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.25x & 0 \leq x \leq 4 \\ 1 & 4 \leq x \end{cases}$$

Determine the following:

(a) $P(X < 2.8)$

(b) $P(X > 1.5)$

(c) $P(X < -2)$

(d) $P(X > 6)$

1.4.4 Example 4

The probability density function of the time you arrive at a terminal (in minutes after 8:00 a.m.) is

$$f(x) = 0.1e^{-0.1x}$$

for

$$x > 0$$

Determine the probability that

(a) You arrive before 9:00 a.m.

(b) You arrive between 8:15 a.m. and 8:30 a.m.

(c) Determine the cumulative distribution function and use the cumulative distribution function to determine the probability that you arrive between 8:15 a.m. and 8:30 a.m.

1.4.5 Example 5

Determine the probability density function for the following cumulative distribution function.

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.2x & 0 \le x < 4 \\ 0.04x + 0.64 & 4 \le x < 9 \\ 1 & 9 \le x \end{cases}$$

1.4.6 Example 6

Determine the mean and the variance of the random variable X where

$$f(x) = 2x^{-3}$$

for

$$x > 1$$

1.4.7 Example 7

Integration by parts is required.

The probability density function for the diameter of a drilled hole in millimeters

$$f(x) = 10e^{-10(x-5)}$$

for $x > 5$ mm.

Although the target diameter is 5 millimeters, vibrations, tool wear, and other nuisances produce diameters greater than 5 millimeters.

(a) Determine the mean and variance of the diameter of the holes.

(b) Determine the probability that a diameter exceeds 5.1 millimeters.

1.5 Continuous Distributions

1.5.1 Uniform Distribution

$$f(x) = \frac{1}{b-a}$$

where

$$a \leq x \leq b$$

and

$$\mu = E[X] = \frac{a+b}{2}$$

$$\sigma^2 = V(x) = \frac{(b-a)^2}{12}$$

1.5.2 Normal Distribution (Gaussian Dist.)

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

Where

$$-\infty < x < \infty$$

also can be written as $N(\mu, \sigma^2)$

Standard Normal Distribution

$$N(0,1)$$

So $\mu = 0$ and $\sigma^2 = 1$

Normal distribution to Standard normal distribution to convert any normal distribution to a standard normal distribution just apply the following equation

$$Z = \frac{X-\mu}{\sigma}$$

Where μ is the mean and σ is the standard deviation of X

- Standard normal distribution table

  The table represent the cumulative distribution function (CDF) of $N(0,1)$

$$\Phi(z) = P(Z \leq z)$$

1.5.3 Normal approximation

1. Binomal RV The binomial distribution can be approximated to a normal distribution if

$np > 5$ and $n(1-p) > 5$

So

$$Bin(\mu, \sigma^2) \rightarrow N(\mu, \sigma^2)$$

Where $\mu = np$ and $\sigma^2 = np(1-p)$

And the approximated normal distribution can be transformed to a std normal distribution by:

$$Z = \frac{X - np}{\sqrt{np(1-p)}}$$

So we can use the table to get our probabilities

Continuity correction

We need to apply a correction because the binomial distibution is discrete and the normal distribution is continuous.

$$P(X \leq x) \rightarrow P(X \leq x + 0.5)$$
$$P(X \geq x) \rightarrow P(X \geq x - 0.5)$$

2. Piosson RV If $\lambda > 5$ we can approximate the Piosson RV to $N(\lambda, \sqrt{\lambda})$ and to std. normal dist. using

$$Z = \frac{X - \lambda}{\sqrt{\lambda}}$$

1.5.4 Other Distributions

1. Exponential Dist.

   $$f(x) = \lambda e^{-\lambda x}$$

   for $0 \leq x < \infty$

   where $\mu = E[X] = \frac{1}{\lambda}$ and $\sigma^2 = V(x) = \frac{1}{\lambda^2}$

2. Erlang RV.

3. Gamma RV.

1.5.5 Example 8

Suppose X has a continuous uniform distribution over the interval $[-1, 1]$. Determine the following:

(a) Mean, variance, and standard deviation of X

(b) Value for x such that $P(-x < X < x) = 0.90$

(c) Cumulative distribution function

1.5.6 Example 9

A show is scheduled to start at 9:00 a.m., 9:30 a.m., and 10:00 a.m. Once the show starts, the gate will be closed. A visitor will arrive at the gate at a time uniformly distributed between 8:30 a.m. and 10:00 a.m. Determine the following:

(a) Cumulative distribution function of the time (in minutes) between arrival and 8:30 a.m.

(b) Mean and variance of the distribution in the previous part

(c) Probability that a visitor waits less than 10 minutes for a show

(d) Probability that a visitor waits more than 20 minutes for a show

1.5.7 Example 10

Assume that Z has a standard normal distribution. Use Appendix Table III to determine the value for z that solves each of the following

(a) $P(Z < z) = 0.9$

(b) $P(Z < z) = 0.5$

(c) $P(Z > z) = 0.1$

(d) $P(Z > z) = 0.9$

(e) $P(-1.24 < Z < z) = 0.8$

1.5.8 Example 11

Assume that X is normally distributed with a mean of 5 and a standard deviation of 4. Determine the following:

(a) $P(X < 11)$

(b) $P(X > 0)$

(c) $P(3 < X < 7)$

(d) $P(-2 < X < 9)$

(e) $P(2 < X < 8)$

1.5.9 Example 12

The time until recharge for a battery in a laptop com- puter under common conditions is normally distributed with a mean of 260 minutes and a standard deviation of 50 minutes.

a) What is the probability that a battery lasts more than four hours?

(b) What are the quartiles (the 25% and 75% values) of battery life?

(c) What value of life in minutes is exceeded with 95% probability?

1.5.10 Example 13

The life of a semiconductor laser at a constant power is normally distributed with a mean of 7000 hours and a standard deviation of 600 hours.

(a) What is the probability that a laser fails before 5000 hours?

(b) What is the life in hours that 95% of the lasers exceed?

(c) If three lasers are used in a product and they are assumed to fail independently, what is the probability that all three are still operating after 7000 hours?

1.5.11 Example 14

Suppose that X is a binomial random variable with n = 200 and $p = 0.4$. Approximate the following probabilities:

(a) $P(X \leq 70)$

(b) $P(70 < x < 90)$

(c) $P(X = 80)$

1.5.12 Example 15

The manufacturing of semiconductor chips pro- duces 2% defective chips. Assume that the chips are independent and that a lot contains 1000 chips. Approximate the

following probabilities:

(a) More than 25 chips are defective.

(b) Between 20 and 30 chips are defective.